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Hengxu Wang, John G. O'Hara, Nick Constantinou

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A path-independent approach to integrated variance under the CEV model

Hengxu Wang[*†] John G. O'Hara [*‡] Nick Constantinou[§]

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Abstract

In this paper, a closed form path-independent approximation of the fair variance strike for a variance swap under the constant elasticity of variance (CEV) model is obtained by applying the small disturbance asymptotic expansion. The realized variance is sampled continuously in a risk-neutral market environment. With the application of a Brownian bridge, we derive a theorem for the conditionally expected product of a Brownian motion at two different times for arbitrary powers. This theorem enables us to provide a conditional Monte-Carlo scheme for simulating the fair variance strike. Compared with results in the recent literature, the method outlined in our paper leads to a simplified approach for pricing variance swaps. The method may also be applied to other more sophisticated volatility derivatives.

JEL classification: C02, C63, G13

Keywords

CEV process, Realized variance, Small disturbance asymptotic expansion, Brownian bridge, Conditional Monte-Carlo simulation

1 Introduction

There has been significant interest in the valuation of variance swaps both from the academic and practitioner communities. The value of a variance swap depends on the realized variance of the underlying asset over its lifetime. Zhu and Lian [19] presented a closed form formula for pricing a variance swap under the Heston model [10] and they defined the realized variance with discrete sampling times. Demeterfi et al. [9] argued that the fair variance strike (annualized variance of the underlying asset accumulated over the lifetime of the contract) can be expressed in terms of a log-contract, maturity and risk free interest rate if the realized variance is sampled continuously in a risk-neutral market environment. The value of a log-contract is the discounted expectation of the underlying's log-return (continuously compounded rate of return) over the lifetime of the contract. It can be obtained by applying Monte-Carlo simulation

[*]Centre for Computational Finance and Economic Agents, University of Essex, Wivenhoe Park, Colchester CO4 3SQ
[†]hwangd@essex.ac.uk
[‡]Corresponding author, johara@essex.ac.uk
[§]Essex Business School, University of Essex, Wivenhoe Park, Colchester, CO4 3SQ, nconst@essex.ac.uk

to the distribution of the underlying at maturity. The log-contract made the fair variance strike path-independent and simplifies the pricing of variance swaps whose underlying is strictly non-zero. However, for those variance swaps whose underlying may possibly reach zero, the value of a log-contract is difficult to compute by standard Monte-Carlo simulation because the natural logarithm of zero is negative infinity.

As discussed by Davydov and Linetsky [8], the CEV model allows the underlying asset to reach zero when model parameters fall in a certain range. Jordan and Tier [11] used asymptotic methods to expand the probability of the underlying asset attaining zero and obtained an approximation of a log-contract under the CEV model. In this paper, the realized variance is defined as a integral of the instantaneous variance of the CEV process. We determine a closed-form approximation of the fair variance strike in terms of the initial variance, time to maturity, risk free interest rate and model parameters under the CEV model by applying the small disturbance asymptotic expansion, of Kunitomo and Takahashi [13], to the instantaneous variance. Our method overcomes the computation difficulties associated with the underlying having value zero. We show that the approximation is very accurate using numerical methods. In addition, corollary 4.4 provides a conditional Monte-Carlo simulation scheme for computing fair variance strike of a variance swap by applying theorem 4.2. Compared with other simulation methods which need to generate the entire path of the underlying asset to compute the realized variance, our approach automatically simulates the distribution of the realized variance at maturity. This new simulation method is more time efficient and also generates less bias during the simulation procedure.

The paper is structured as follows. In section 2, we give a stochastic differential equation for instantaneous variance of a CEV process by applying Itô's lemma. In section 3, we derive the small disturbance asymptotic expansion of the instantaneous variance and provide a closed form approximation of the fair variance strike. In section 4, we give the closed form approximation of conditionally expected realized variance by using a Brownian bridge and this approximation can be implemented by a conditional Monte Carlo simulation. Finally in section 5, the fair variance strike of a variance swap is computed to demonstrate the correctness our methods. Comparison with a log-Euler Monte-Carlo simulation are also carried out to indicate that our methods are more efficient for computing the fair variance strike.

2 CEV variance process

In this section we concentrate on the CEV process introduced by Cox and Ross [7]

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \delta S_t^{\beta+1}\,\mathrm{d}W_t, \qquad t \in [0, T]. \tag{1}$$

Equation (1) gives the dynamics of the underlying asset. In the above, S_t denotes the asset price at time $t \in [0, T]$ and $S_0 > 0$, δ is a positive constant and β is in the range $[-1, 0)$, r is the constant riskless interest rate, and W_t is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$.

We chose the CEV process for three reasons. First, the process is consistent with one of well-known stylized facts that volatility changes are negatively correlated with asset returns in stock markets for negative β, i.e., local volatility δS_t^{β} is a decreasing function of S_t for $\beta < 0$ and $\delta > 0$. Second, as shown by Davydov and Linetsky [8], the process is able to capture the observed volatility skew for both equity and index options. Third, the process can be readily calibrated to the market prices of European options using closed form option formulas provided in the papers of Schroder [16] as well as Davydov and Linetsky [8].

Cox and Ross [7] restricted β in the region $\beta \in [-1, 0]$ whilst Schroder [16] relaxed the range of β to

$\beta \leq 0$. In this paper, the range of β will be restricted in $[-1, 0)$ for three reasons. First, $\beta > 0$ does not guarantee the model to obey the stylized fact as indicated in the previous paragraph. Second, the CEV model recovers the Black Scholes model [2] for $\beta = 0$, which will not be concentrated on in this paper. Third, as suggested by Lindsay and Brecher [14], the CEV process admits three distinct types of solution with respect to $\beta < -0.5$, $-0.5 \leq \beta < 0$ and $\beta \geq 0$. We let $\beta \in [-1, 0)$ because this restriction allows two types of solution with respect to $\beta < -0.5$, $-0.5 \leq \beta < 0$ and more importantly, it simplifies our numerical experiments in section 5 without loss of generality of the CEV model. In addition, research based on the CEV process with β in the Cox and Ross [7] range can also be found in papers such as [1, 6, 18]

When $-0.5 \leq \beta < 0$, boundary value $S_t = 0$ is attainable and absorbing[1]. When $-1 \leq \beta < -0.5$, the boundary $S_t = 0$ is attainable, and can be absorbing or reflecting[2]. Introducing an absorbing boundary is an appropriate choice because it excludes any arbitrage opportunity after S_t attains 0. If a reflecting boundary is chosen for $-1 \leq \beta < -0.5$, an investor purchasing the asset when S_t hits 0 would never lose money, since the asset price will always be positive.

By taking $r > 0$, $-1 \leq \beta < 0$ and $\delta > 0$ with an absorbing boundary we have the instantaneous variance of S_t given by Davydov and Linetsky [8] as

$$V_t = \delta^2 S_t^{2\beta}, \qquad t \in [0, T], \qquad V_0 > 0. \tag{2}$$

By Itô's lemma [17], we obtain a stochastic differential equation for V_t

$$\mathrm{d}V_t = \beta \left[(2\beta - 1) V_t^2 + 2rV_t \right] \mathrm{d}t + 2\beta V_t^{\frac{3}{2}} \mathrm{d}W_t, \tag{3}$$

where its parameters are defined as in equation (1). From Davydov and Linetsky [8], we also have the probability of $V_t \to \infty$ given by

$$\mathbb{Q}(V_t \to \infty) = \Gamma(-\frac{1}{2\beta}, \zeta), \tag{4}$$

where $\Gamma(x, y)$ is the complementary Gamma distribution function and $\zeta = r \left[\beta V_0 (1 - e^{-2\beta r t}) \right]^{-1}$.

3 Small disturbance asymptotic expansion of CEV variance process

In this section, we apply the small disturbance asymptotic expansion first proposed by Kunitomo and Takahashi [13] for the CEV Variance process (3).

The small disturbance asymptotic expansion is closely related to the Taylor expansion. In order to implement this technique, we introduce a small parameter $0 < \epsilon \ll 1$ and a negative variable $\bar{\beta}$ such that $\beta = \epsilon \bar{\beta}$. From equation (3), we have

$$V_t = V_0 + \epsilon \bar{\beta} \int_0^t \left[(2\epsilon\bar{\beta} - 1) V_u^2 + 2rV_u \right] \mathrm{d}u + 2\epsilon\bar{\beta} \int_0^t V_u^{\frac{3}{2}} \, \mathrm{d}W_u, \tag{5}$$

and $V_t|_{\epsilon=0} = V_0$. The small disturbance asymptotic expansion requires us to construct a Taylor expansion

[1] Here absorbing means that S_t remains at 0 if it hits 0.
[2] Here reflecting means that values of S_t may increase after it hits 0.

of V_t about the point $\epsilon = 0$:

$$V_t = V_0 + \epsilon \left.\frac{\partial V_t}{\partial \epsilon}\right|_{\epsilon=0} + \frac{\epsilon^2}{2!}\left.\frac{\partial^2 V_t}{\partial \epsilon^2}\right|_{\epsilon=0} + \frac{\epsilon^3}{3!}\left.\frac{\partial^3 V_t}{\partial \epsilon^3}\right|_{\epsilon=0} + \frac{\epsilon^4}{4!}\left.\frac{\partial^4 V_t}{\partial \epsilon^4}\right|_{\epsilon=0} + O(\epsilon^5). \tag{6}$$

From equation (5), V_t depends on V_t, V_t^2 and $V_t^{\frac{3}{2}}$. Equation (6) requires us to compute the derivative of V_t with respect to ϵ up to the fourth order, which means we also require higher order derivatives of V_t^2 and $V_t^{\frac{3}{2}}$. The following lemma provides a general iteration formula for the nth order derivative of V_t^m.

Lemma 3.1. *The nth order derivative of V_t^m is given by*

$$\begin{aligned}\frac{\partial^n(V_t^m)}{\partial \epsilon^n} &= mn\bar{\beta}\int_0^t \left[2m(n-1)\bar{\beta}\frac{\partial^{n-2}(V_u^{m+1})}{\partial \epsilon^{n-2}} + (2m\epsilon\bar{\beta}-1)\frac{\partial^{n-1}(V_u^{m+1})}{\partial \epsilon^{n-1}} + 2r\frac{\partial^{n-1}(V_u^{m})}{\partial \epsilon^{n-1}}\right]\mathrm{d}u \\ &\quad + m\epsilon\bar{\beta}\int_0^t \left[2mn\bar{\beta}\frac{\partial^{n-1}(V_u^{m+1})}{\partial \epsilon^{n-1}} + (2m\epsilon\bar{\beta}-1)\frac{\partial^{n}(V_u^{m+1})}{\partial \epsilon^{n}} + 2r\frac{\partial^{n}(V_u^{m})}{\partial \epsilon^{n}}\right]\mathrm{d}u \\ &\quad + 2mn\bar{\beta}\int_0^t \frac{\partial^{n-1}(V_u^{m+\frac{1}{2}})}{\partial \epsilon^{n-1}}\mathrm{d}W_u + 2m\epsilon\bar{\beta}\int_0^t \frac{\partial^{n}(V_u^{m+\frac{1}{2}})}{\partial \epsilon^{n}}\mathrm{d}W_u, \end{aligned} \tag{7}$$

with $m \geq 0$ and $n \in \mathbb{N}$.

Proof. By Itô's lemma, we obtain a stochastic differential equation for V_t^m given by

$$V_t^m = V_0^m + m\beta\int_0^t \left[(2m\beta - 1)V_u^{m+1} + 2rV_u^m\right]\mathrm{d}u + 2m\beta\int_0^t V_u^{m+\frac{1}{2}}\mathrm{d}W_u.$$

Since $\beta = \epsilon\bar{\beta}$ for $0 < \epsilon \ll 1$, we have

$$V_t^m = V_0^m + m\epsilon\bar{\beta}\int_0^t \left[(2m\epsilon\bar{\beta} - 1)V_u^{m+1} + 2rV_u^m\right]\mathrm{d}u + 2m\epsilon\bar{\beta}\int_0^t V_u^{m+\frac{1}{2}}\mathrm{d}W_u. \tag{8}$$

Lemma 3.1 is proved by induction. We see the result follows in the case $n = 1$, since

$$\begin{aligned}\frac{\partial V_t^m}{\partial \epsilon} &= m\bar{\beta}\int_0^t \left[(2m\epsilon\bar{\beta}-1)V_u^{m+1} + 2rV_u^m\right]\mathrm{d}u + m\epsilon\bar{\beta}\int_0^t \left[2m\bar{\beta}V_u^{m+1} + (2m\epsilon\bar{\beta}-1)\frac{\partial V_u^{m+1}}{\partial \epsilon} + 2r\frac{\partial V_u^m}{\partial \epsilon}\right]\mathrm{d}u \\ &\quad + 2m\bar{\beta}\int_0^t V_u^{m+\frac{1}{2}}\mathrm{d}W_u + 2m\epsilon\bar{\beta}\int_0^t \frac{\partial V_u^{m+\frac{1}{2}}}{\partial \epsilon}\mathrm{d}W_u.\end{aligned}$$

Suppose that equation (7) holds for $n = l$. Then the derivative of V_t^m with respect to ϵ of order $l+1$ is given by

$$\begin{aligned}\frac{\partial}{\partial \epsilon}\left(\frac{\partial^l(V_t^m)}{\partial \epsilon^l}\right) &= m(l+1)\bar{\beta}\int_0^t \left[2ml\bar{\beta}\frac{\partial^{l-1}(V_u^{m+1})}{\partial \epsilon^{l-1}} + (2m\epsilon\bar{\beta}-1)\frac{\partial^{l}(V_u^{m+1})}{\partial \epsilon^{l}} + 2r\frac{\partial^{l}(V_u^{m})}{\partial \epsilon^{l}}\right]\mathrm{d}u \\ &\quad + m\epsilon\bar{\beta}\int_0^t \left[2m(l+1)\bar{\beta}\frac{\partial^{l}(V_u^{m+1})}{\partial \epsilon^{l}} + (2m\epsilon\bar{\beta}-1)\frac{\partial^{l+1}(V_u^{m+1})}{\partial \epsilon^{l+1}} + 2r\frac{\partial^{l+1}(V_u^{m})}{\partial \epsilon^{l+1}}\right]\mathrm{d}u \\ &\quad + 2m(l+1)\bar{\beta}\int_0^t \frac{\partial^{l}(V_u^{m+\frac{1}{2}})}{\partial \epsilon^{l}}\mathrm{d}W_u + 2m\epsilon\bar{\beta}\int_0^t \frac{\partial^{l+1}(V_u^{m+\frac{1}{2}})}{\partial \epsilon^{l+1}}\mathrm{d}W_u,\end{aligned}$$

hence equation (7) also holds for $n = l + 1$. □

Remark 3.2. When $n = 1$, the derivative of the function V_u^m, $V_u^{m+\frac{1}{2}}$ and V_u^{m+1} to the order $n-1$ in equation (7) are actually the functions themselves.

Then by applying lemma 3.1 to equation (6), we have the small disturbance asymptotic expansion of V_t.

Lemma 3.3. *The small disturbance asymptotic expansion of V_t up to fourth order is*

$$V_t = V_0 + \beta G + \beta^2 H + \beta^3 L + \beta^4 M + O(\epsilon^5), \tag{9}$$

where

$$\begin{aligned}
G &= V_0 \left[(2r - V_0)t + 2V_0^{\frac{1}{2}} W_t\right], \\
H &= V_0 \Big[(V_0^2 - 3rV_0 + 2r^2)t^2 - V_0 t + 3V_0^{\frac{1}{2}}(2r - V_0)tW_t + 3V_0 W_t^2 \\
&\quad - V_0^{\frac{1}{2}}(2r + V_0)\int_0^t W_u \mathrm{d}u\Big], \\
L &= V_0 \Big[(-V_0^3 + 4rV_0^2 - \frac{14}{3}r^2 V_0 + \frac{4}{3}r^3)t^3 + 3V_0^{\frac{1}{2}}(\frac{5}{4}V_0^2 - 4rV_0 + 3r^2)t^2 W_t \\
&\quad - V_0(3r - 2V_0)t^2 + 6V_0(2r - V_0)tW_t^2 - 3V_0^{\frac{3}{2}} tW_t + 4V_0^{\frac{3}{2}} W_t^3 + 2V_0(3r + V_0)\int_0^t W_u^2 \mathrm{d}u \\
&\quad - 3V_0(2r + V_0)W_t \int_0^t W_u \mathrm{d}u + V_0^{\frac{1}{2}}(\frac{5}{2}V_0^2 - 2rV_0 - 6r^2)\int_0^t uW_u \mathrm{d}u \\
&\quad + 2V_0^{\frac{1}{2}}(V_0^2 + rV_0 - 2r^2)\int_0^t \int_0^u W_s \mathrm{d}s\mathrm{d}u - V_0^{\frac{3}{2}}\int_0^t W_u \mathrm{d}u\Big], \\
M &= V_0 \Big[(V_0^4 - 5rV_0^3 + \frac{25}{3}r^2 V_0^2 - 5r^3 V_0 + \frac{2}{3}r^4)t^4 - V_0(3V_0^2 - 8rV_0 - \frac{14}{3}r^2)t^3 \\
&\quad - V_0^{\frac{1}{2}}(\frac{35}{8}V_0^3 - \frac{75}{4}rV_0^2 + \frac{49}{2}r^2 V_0 - 9r^3)t^3 W_t + 6V_0(\frac{3}{2}V_0^2 - 5rV_0 + 4r^2)t^2 W_t^2 \\
&\quad - 3V_0^{\frac{3}{2}}(4r - \frac{5}{2}V_0)t^2 W_t + V_0^2 t^2 + 10V_0^{\frac{3}{2}}(2r - V_0)tW_t^3 - 6V_0^2 tW_t^2 + 5V_0^2 W_t^4 \\
&\quad + V_0^{\frac{3}{2}}(5V_0 - 2r)\int_0^t uW_u \mathrm{d}u - V_0^2 \int_0^t W_u^2 \mathrm{d}u + 2V_0^{\frac{3}{2}}(r + 2V_0)\int_0^t \int_0^u W_s \mathrm{d}s\mathrm{d}u \\
&\quad - V_0^{\frac{1}{2}}(\frac{35}{8}V_0^3 - \frac{45}{4}rV_0^2 + \frac{1}{2}r^2 V_0 + 9r^3)\int_0^t u^2 W_u \mathrm{d}u + 3V_0^2(V_0 - 2r)\int_0^t uW_u^2 \mathrm{d}u \\
&\quad + 3V_0(3V_0^2 - 2rV_0 - 8r^2)W_t \int_0^t uW_u \mathrm{d}u - 4V_0(2V_0^2 + 4rV_0 - 3r^2)\int_0^t \int_0^u W_s^2 \mathrm{d}s\mathrm{d}u \\
&\quad + 3V_0^{\frac{3}{2}}(8r + 3V_0)W_t \int_0^t W_u^2 \mathrm{d}u + 3V_0(\frac{5}{2}V_0^2 - 4rV_0 + 6r^2)W_t \int_0^t \int_0^u W_s \mathrm{d}s\mathrm{d}u \\
&\quad + V_0(\frac{5}{2}V_0^2 + 26rV_0 - 30r^2)\int_0^t W_u \int_0^u W_s \mathrm{d}s\mathrm{d}u - V_0^{\frac{3}{2}}(24r + 13V_0)\int_0^t W_u^3 \mathrm{d}u \\
&\quad - V_0^{\frac{1}{2}}(7V_0^3 - 9rV_0^2 - 16r^2 V_0 + 12r^3)\int_0^t \int_0^u sW_s \mathrm{d}s\mathrm{d}u - 6V_0^{\frac{3}{2}}(2r + V_0)W_t^2 \int_0^t W_u \mathrm{d}u \\
&\quad - 2V_0^{\frac{1}{2}}(3V_0^3 - 10r^2 V_0 + 4r^3)\int_0^t \int_0^u \int_0^s W_a \mathrm{d}a\mathrm{d}s\mathrm{d}u - 3V_0^2 W_t \int_0^t W_u \mathrm{d}u\Big].
\end{aligned}$$

Proof. From lemma 3.1, we have

$$\begin{aligned}
\frac{\partial V_t}{\partial \epsilon} &= \bar{\beta}\int_0^t \left[(2\epsilon\bar{\beta} - 1)V_u^2 + 2rV_u\right]\mathrm{d}u + \epsilon\bar{\beta}\int_0^t \left[2\bar{\beta}V_u^2 + (2\epsilon\bar{\beta} - 1)\frac{\partial V_u^2}{\partial \epsilon} + 2r\frac{\partial V_u)}{\partial \epsilon}\right]\mathrm{d}u \\
&\quad + 2\bar{\beta}\int_0^t V_u^{\frac{3}{2}}\mathrm{d}W_u + 2\epsilon\bar{\beta}\int_0^t \frac{\partial V_u^{\frac{3}{2}}}{\partial \epsilon}\mathrm{d}W_u
\end{aligned} \tag{10}$$

by taking $m = n = 1$. The second and the last terms on the right hand side of equation (10) can be ignored since $0 < \epsilon \ll 1$, and from equation (6), we have $V_t \to V_0$ when $\epsilon \to 0$. So we have

$$\begin{aligned} \left.\frac{\partial V_t}{\partial \epsilon}\right|_{\epsilon=0} &= \bar{\beta} \int_0^t \left[-V_0^2 + 2rV_0\right] \mathrm{d}u + 2\bar{\beta} \int_0^t V_0^{\frac{3}{2}} \mathrm{d}W_u \\ &= \bar{\beta} V_0 \left[(2r - V_0)t + 2V_0^{\frac{1}{2}} W_t\right]. \end{aligned} \tag{11}$$

Defining $G = V_0 \left[(2r - V_0)t + 2V_0^{\frac{1}{2}} W_t\right]$, we have $\epsilon \left.\frac{\partial V_t}{\partial \epsilon}\right|_{\epsilon=0} = \beta G$. By by repeating this procedure we can obtain the other terms with higher order derivatives in equation (6). See appendix A for a detailed proof. □

The value of a volatility derivative depends on its underlying's annualized realized variance over its lifetime. For example, the value of a fair variance strike of a variance swap, P, at time $t = 0$ is defined as

$$P := E^{\mathbb{Q}}\left[V_{(0,T)}\right], \tag{12}$$

where $V_{(0,T)} := \frac{1}{T}\int_0^T V_t \mathrm{d}t$, T is the maturity, V_t is the instantaneous variance of the underlying asset and $E^{\mathbb{Q}}[\cdot]$ represents expectation under risk-neutral measure $\mathbb{Q}$. $V_{(0,T)}$ depends on the value of V_t for $t \in [0, T]$ and hence it is a path-dependent random variable, which makes it computationally expensive to evaluate by Monte-Carlo simulation, since the value of V_t requires to be computed for every element in a discrete time interval from time 0 to T. However, lemma 3.3 enables us to obtain a closed form approximation for the fair variance strike, which is path-independent, by substituting equation (9) into (12).

Lemma 3.4. *The fair variance strike of a variance swap is given by*

$$E^{\mathbb{Q}}[V_{(0,T)}] = V_0 + \beta G' + \beta^2 H' + \beta^3 L' + \beta^4 M' + O(\epsilon^5), \tag{13}$$

where

$$\begin{aligned} G' &= \frac{1}{T} E^{\mathbb{Q}}\left[\int_0^T G\mathrm{d}t\right] = \frac{1}{2} V_0(2r - V_0)T, \\ H' &= \frac{1}{T} E^{\mathbb{Q}}\left[\int_0^T H\mathrm{d}t\right] = \frac{1}{3} V_0(V_0^2 - 3rV_0 + 2r^2)T^2 + V_0^2 T, \\ L' &= \frac{1}{T} E^{\mathbb{Q}}\left[\int_0^T L\mathrm{d}t\right] = V_0\left[\frac{1}{4}(-V_0^3 + 4rV_0^2 - \frac{14}{3} r^2 V_0 + \frac{4}{3} r^3)T^3 + V_0(3r - \frac{3}{2} V_0)T^2\right], \\ M' &= \frac{1}{T} E^{\mathbb{Q}}\left[\int_0^T M\mathrm{d}t\right] = V_0\Bigg[\frac{1}{5}(V_0^4 - 5rV_0^3 + \frac{25}{3} r^2 V_0^2 - 5r^3 V_0 + \frac{2}{3} r^4)T^4 + V_0(\frac{31}{12} V_0^2 - \frac{79}{12} rV_0 + \frac{31}{6} r^2)T^3 \\ &\qquad + \frac{8}{3} V_0^2 T^2\Bigg]. \end{aligned}$$

Proof. Substituting equation (9) into (12) gives

$$\begin{aligned} E^{\mathbb{Q}}\left[V_{(0,T)}\right] &= \frac{1}{T} E^{\mathbb{Q}}\left[\int_0^T \left(V_0 + \beta G + \beta^2 H + \beta^3 L + \beta^4 M + O(\epsilon^5)\right) \mathrm{d}t\right] \\ &= V_0 + \frac{\beta}{T} E^{\mathbb{Q}}\left[\int_0^T G\mathrm{d}t\right] + \frac{\beta^2}{T} E^{\mathbb{Q}}\left[\int_0^T H\mathrm{d}t\right] + \frac{\beta^3}{T} E^{\mathbb{Q}}\left[\int_0^T L\mathrm{d}t\right] \\ &\quad + \frac{\beta^4}{T} E^{\mathbb{Q}}\left[\int_0^T M\mathrm{d}t\right] + O(\epsilon^5). \end{aligned}$$

Then the lemma is proved by substituting G, H, L and M of equation (9) into the above formula. $\square$

4 Expectation of annualized integrated variance

In this section we apply the results from section 3 to obtain an efficient Monte-Carlo method to compute the fair variance strike by transforming the value of $V_{(0,T)}$ to a path-independent random variable. This method allows us to simulate the conditional distribution of the realized variance. In essence, the fair variance may be calculated by the methods outlined in section 3, however, for those more sophisticated volatility derivatives whose payoff is not linear on the realized variance, for example, options on realized variance, a closed-form solution may be difficult to be obtained. Using an appropriate adjustment, the Monte-Carlo method provided in this section is able to simulate the distribution of the realized variance and hence can be applied to evaluate other more sophisticated volatility derivatives.

It is well known that

$$E[V_{(0,T)}] = E[E[V_{(0,T)}|\mathcal{F}]] \tag{14}$$

and by Jensen's inequality we have

$$Var[E[V_{(0,T)}|\mathcal{F}]] \leq Var[V_{(0,T)}]$$

holding for any condition $\mathcal{F}$. This implies that simulating the conditional expectation $E^{\mathbb{Q}}[V_{(0,T)}|\mathcal{F}]$ is more efficient than simulating $V_{(0,T)}$ itself. So in this section we provide a framework to compute the conditional expectation of the annualized integrated variance given a Brownian motion with fixed end point at time T, such that $W_T = b$. The path dependent random variable $V_{(0,T)}$ is transformed to a path independent random variable $E^{\mathbb{Q}}[V_{(0,T)}|W_T = b]$, via a Brownian bridge. This is one of the key results of our paper.

A Brownian bridge is a continuous stochastic process whose probability distribution is the distribution of a Brownian motion with a fixed initial and end point. Next we give one of its propositions.

Proposition 4.1. *(See Karatzas and Sherve[12]) Let W_t, $0 \leq t \leq T$ be a Brownian motion, then a Brownian bridge*

$$X_t = \begin{cases} \frac{b}{T}t + (T-t)\int_0^t \frac{\mathrm{d}W_u}{T-u}; & \text{for} \quad 0 \leq t < T, \\ b; & \text{for} \quad t = T, \end{cases} \tag{15}$$

is Gaussian with almost surely continuous paths, with mean $\mu = E[X_t] = \frac{b}{T}t$ and variance $\sigma^2 = Var[X_t] = t - \frac{t^2}{T}$ for a constant b.

Lemma 3.3 expanded V_t into a function of Brownian motion. So with this lemma, the calculation of $E^{\mathbb{Q}}[V_{(0,T)}|W_T = b]$ involves the computation of some conditional expectation such as $E^{\mathbb{Q}}[W_t|W_T = b]$, $E^{\mathbb{Q}}[W_uW_t|W_T = b]$, $E^{\mathbb{Q}}[W_uW_t^2|W_T = b]$ and so on. We provide a theorem to compute these conditional expectations in a more general form, $E^{\mathbb{Q}}[W_u^m W_t^n|W_T = b]$, for $0 \leq u \leq t \leq T$. For simplicity, $E^{\mathbb{Q}}[\,\cdot|W_T = b]$ is denoted by $E_b^{\mathbb{Q}}[\cdot]$.

Theorem 4.2. *Let W_t for $t \in [0,T]$ be a Brownian motion. Then we have the conditional expectation*

$$E_b^{\mathbb{Q}}[W_u^m W_t^n] = m!n! \sum_{\substack{0 \leq j \leq m \\ 0 \leq l \leq k \leq n}} \left[\delta_\nu \times 2^{-\frac{k+j}{2}} \binom{l+j}{j} \frac{(T-u)^{\frac{j-k}{2}}(T-t)^{\frac{k+l}{2}}(t-u)^{\frac{k-l}{2}}\ t^{n-k}\ b^{m+n-j-k}}{(m-j)!(n-k)!(\frac{k-l}{2})!(\frac{l+j}{2})!\ u^{-m+\frac{j-l}{2}} T^{m+n-k-\frac{j-l}{2}}} \right], \tag{16}$$

where m, n are non-negative integers, $0 \le u < t \le T$, and δ_ν is a Kronecker delta function such that

$$\delta_\nu = \begin{cases} 0, & \text{if } \nu \neq 0 \\ 1, & \text{if } \nu = 0 \end{cases} \quad \text{for } \nu = \frac{k+j}{2} - \left\lfloor \frac{l+j}{2} \right\rfloor - \left\lfloor \frac{k-l}{2} \right\rfloor,$$

and the floor function $\lfloor x \rfloor = \max\{p \le x | p \in \mathbb{N}\}$ for any $x \ge 0$.

Proof. Since W_t for $t \in [0, T]$ is a Brownian motion,

$$E_b^{\mathbb{Q}} \left[W_u^m W_t^n \right] = E^{\mathbb{Q}} \left[X_u^m X_t^n \right].$$

Take $Y_{u,t} = \int_u^t \frac{\mathrm{d}W_s}{T-s}$ for $0 \le u \le t \le T$. Then $Y_{u,t}$ is a Gaussian random variable with mean $\mu = E^{\mathbb{Q}}[Y_{u,t}] = 0$, and variance $\sigma_{u,t}^2 = E^{\mathbb{Q}}\left[\left(\int_u^t \frac{\mathrm{d}W_s}{T-s}\right)^2\right] = E^{\mathbb{Q}}\left[\int_u^t \left(\frac{1}{T-s}\right)^2 \mathrm{d}s\right] = \frac{t-u}{(T-t)(T-u)}$. From equation (15), we have

$$\begin{aligned}
E^{\mathbb{Q}} \left[X_u^m X_t^n \right] &= E^{\mathbb{Q}} \left[\left(\frac{b}{T} u + (T-u) Y_{0,u} \right)^m \left(\frac{b}{T} t + (T-t) Y_{0,t} \right)^n \right] \\
&= m!n! \sum_{\substack{0 \le j \le m \\ 0 \le k \le n}} \frac{(T-u)^j (T-t)^k u^{m-j} t^{n-k}}{j!k!(m-j)!(n-k)!} \left(\frac{b}{T} \right)^{m+n-j-k} E^{\mathbb{Q}} \left[Y_{0,u}^j (Y_{0,u} + Y_{u,t})^k \right] \\
&= m!n! \sum_{\substack{0 \le j \le m \\ 0 \le k \le n}} \frac{(T-u)^j (T-t)^k u^{m-j} t^{n-k}}{j!k!(m-j)!(n-k)!} \left(\frac{b}{T} \right)^{m+n-j-k} E^{\mathbb{Q}} \left[Y_{0,u}^j \sum_{0 \le l \le k} \frac{k!}{l!(k-l)!} Y_{0,u}^l Y_{u,t}^{k-l} \right] \\
&= m!n! \sum_{\substack{0 \le j \le m \\ 0 \le l \le k \le n}} \frac{(T-u)^j (T-t)^k u^{m-j} t^{n-k}}{j!l!(m-j)!(n-k)!(k-l)!} \left(\frac{b}{T} \right)^{m+n-j-k} E^{\mathbb{Q}} \left[Y_{0,u}^{l+j} Y_{u,t}^{k-l} \right] \\
&= m!n! \sum_{\substack{0 \le j \le m \\ 0 \le l \le k \le n}} \frac{(T-u)^j (T-t)^k u^{m-j} t^{n-k}}{j!l!(m-j)!(n-k)!(k-l)!} \left(\frac{b}{T} \right)^{m+n-j-k} E^{\mathbb{Q}} \left[Y_{0,u}^{l+j} \right] E^{\mathbb{Q}} \left[Y_{u,t}^{k-l} \right]. \qquad (17)
\end{aligned}$$

According to Miller and Childers [15], we have

$$E^{\mathbb{Q}} \left[Y_{u,t}^{k-l} \right] = \delta_i 2^{-\frac{k-l}{2}} \frac{(k-l)!}{(\frac{k-l}{2})!} \left(\frac{t-u}{(T-t)(T-u)} \right)^{\frac{k-l}{2}}, \qquad (18)$$

$$E^{\mathbb{Q}} \left[Y_{0,u}^{l+j} \right] = \delta_h 2^{-\frac{l+j}{2}} \frac{(l+j)!}{(\frac{l+j}{2})!} \left(\frac{u}{T(T-u)} \right)^{\frac{l+j}{2}}, \qquad (19)$$

for $i = \frac{k-l}{2} - \lfloor \frac{k-l}{2} \rfloor$, $h = \frac{l+j}{2} - \lfloor \frac{l+j}{2} \rfloor$. δ_Δ is a Kronecker delta function, where $\Delta \in \{i,\ h,\ \nu\}$ and $\nu = i + h$, such that

$$\delta_\Delta = \begin{cases} 0, & \text{if } \Delta \neq 0 \\ 1, & \text{if } \Delta = 0 \end{cases}.$$

Then we have $\delta_i \cdot \delta_h = \delta_{i+h} = \delta_\nu$ and the theorem is proved by substituting equations (18) and (19) into equation (17). □

Corollary 4.3. *Theorem 4.2 holds for some special values of m and n, such as when $m = n = 1$,*

$$E_b^{\mathbb{Q}}[W_u W_t] = \frac{b^2}{T^2}ut + u - \frac{ut}{T}, \tag{20}$$

or when $n = 0$,

$$E_b^{\mathbb{Q}}[W_u^m] = m! \sum_{0 \le j \le m} \delta_\nu \times 2^{-\frac{j}{2}} \frac{b^{m+j}(T-u)^{\frac{j}{2}}}{(m-j)!(\frac{j}{2})!} \frac{u^{m-\frac{j}{2}}}{T^{m-\frac{j}{2}}}, \tag{21}$$

where $\nu = \frac{j}{2} - \lfloor \frac{j}{2} \rfloor$.

Theorem 4.2 allows us to compute the value of $E_b^{\mathbb{Q}}[V_{(0,T)}]$, given by the following corollary.

Corollary 4.4. *The conditional expectation of the annualized realized variance over the lifetime of a variance swap is given by*

$$E_b^{\mathbb{Q}}[V_{(0,T)}] = V_0 + \beta G_b^{'} + \beta^2 H_b^{'} + \beta^3 L_b^{'} + \beta^4 M_b^{'} + O(\epsilon^5), \tag{22}$$

where

$$\begin{aligned}
G_b^{'} &= V_0\Big[\frac{1}{2}(2r - V_0)T + V_0^{\frac{1}{2}}b\Big], \\
H_b^{'} &= V_0\Big[\frac{1}{3}(V_0^2 - 3rV_0 + 2r^2)T^2 + \frac{1}{6}V_0^{\frac{1}{2}}(10r - 7V_0)bT + V_0 b^2\Big], \\
L_b^{'} &= V_0\Big[\frac{1}{4}(-V_0^3 + 4rV_0^2 - \frac{14}{3}r^2V_0 + \frac{4}{3}r^3)T^3 + V_0^{\frac{1}{2}}(\frac{59}{48}V_0^2 - \frac{37}{12}rV_0 + \frac{19}{12}r^2)bT^2 + \frac{1}{4}V_0(r + \frac{5}{6}V_0)T^2 \\
&\quad + V_0(\frac{11}{4}r - \frac{41}{24}V_0)b^2T - \frac{1}{6}V_0^{\frac{3}{2}}bT + V_0^{\frac{3}{2}}b^3\Big], \\
M_b^{'} &= V_0\Big[\frac{1}{5}(V_0^4 - 5rV_0^3 + \frac{25}{3}r^2V_0^2 - 5r^3V_0 + \frac{2}{3}r^4)T^4 + V_0^{\frac{1}{2}}(-\frac{121}{96}V_0^3 + \frac{357}{80}rV_0^2 - \frac{183}{40}r^2V_0 + \frac{13}{12}r^3)bT^3 \\
&\quad + V_0(-\frac{7}{48}V_0^2 + \frac{2}{15}rV_0 + \frac{23}{12}r^2)T^3 + V_0(\frac{131}{48}V_0^2 - \frac{403}{60}rV_0 + \frac{13}{4}r^2)b^2T^2 + V_0^{\frac{3}{2}}(\frac{1}{30}V_0 - \frac{71}{60}r)bT^2 \\
&\quad + \frac{1}{8}V_0^2T^2 + V_0^{\frac{3}{2}}(\frac{16}{5}r - \frac{53}{20}V_0)b^3T - \frac{11}{24}V_0^2b^2T + V_0^2b^4\Big].
\end{aligned}$$

Proof. See Appendix B. □

Remark 4.5. Given equation (14), we have $E^{\mathbb{Q}}[V_{(0,T)}] = E^{\mathbb{Q}}\left[E_b^{\mathbb{Q}}[V_{(0,T)}]\right]$. Then lemma 3.4 may also be proved using corollary 4.4 combined with the fact that $E^{\mathbb{Q}}[W_T] = E^{\mathbb{Q}}[W_T^3] = 0$, $E^{\mathbb{Q}}[W_T^2] = T$ and $E^{\mathbb{Q}}[W_T^4] = 3T^2$. Substituting these values into equation (22) gives the value of $E^{\mathbb{Q}}[V_{(0,T)}]$ directly.

5 Numerical experiments

In this section we will perform a numerical study of the approximation given by the corollary 4.4 and lemma 3.4. Section 5.1 gives the log-Euler Monte Carlo scheme to simulate the instantaneous variance and compute $E^{\mathbb{Q}}[V_{(0,T)}]$. We also illustrate why this method may not be suitable for some model parameters. Section 5.2 computes $E^{\mathbb{Q}}[V_{(0,T)}]$ by the small disturbance asymptotic expansion as described in lemma 3.4 and corollary 4.4.

5.1 Log-Euler Monte-Carlo scheme for the CEV variance process

In the literature, some methods are used to simulate the underlying price, S_t, of the CEV process (1). For example, Lindsay and Brecher [14] provided a quasi Monte-Carlo method to simulate the underlying price from a non-central Chi square distribution. Chen et.al. [5] also suggested using a moment-matched quadratic Gaussian approximation method and a direct inversion scheme to simulate the underlying price S_t. However, since our aim is primarily to obtain some benchmark values of the fair variance strike, we will not focus our attention on the use of these techniques. In our Monte-Carlo simulations, we have employed the standard log-Euler scheme for the CEV variance process (3).

Given equation (3), we have a stochastic differential equation of $\ln V_t$ by Itô's lemma, such that

$$d\ln V_t = \beta(2r - V_t)dt + 2\beta\sqrt{V_t}dW_t,$$

for $\beta < 0$ and $t \in [0, T]$. Then applying the first-order Taylor approximation scheme for $\ln V_t$ gives:

$$\ln V_{t_{n+1}} = \ln V_{t_n} + \beta(2r - V_{t_n})h + 2\beta\sqrt{V_{t_n}}Z\sqrt{h} + O(h^2),$$

with Z a standard normal random variable, $t_n = \frac{n}{N}T$ for $n = 0, 1, 2, \cdots, N$ and $h = \frac{T}{N}$. By taking exponentials on both side, we have

$$V_{t_{n+1}} = V_{t_n} \cdot \exp\left[\beta(2r - V_{t_n})h + 2\beta\sqrt{V_{t_n} \cdot h}Z\right] + O(h^2), \tag{23}$$

and $V_{t_{n+1}} > 0$ for all $V_{t_n} > 0$, i.e., $V_{t_n} > 0$ if $V_0 > 0$. However, this numerical scheme becomes unstable, i.e., $V_{t_n} = \delta^2 S_{t_n}^{2\beta} \to \infty$ when $S_{t_n} \to 0$ for some model parameters.

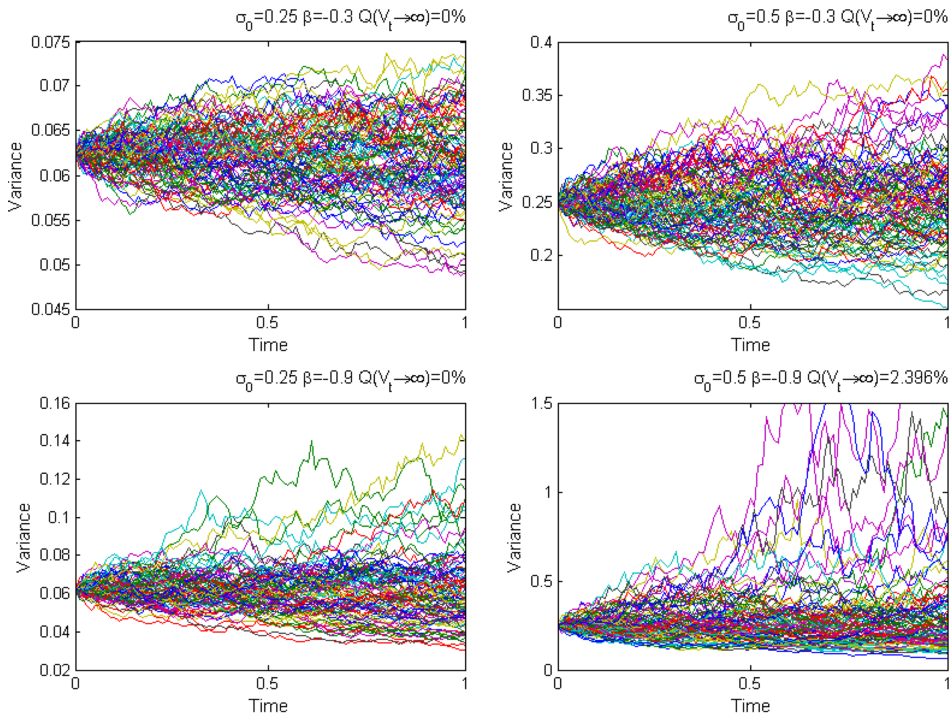

Figure 1: Sample paths of V_t by the log-Euler Monte-Carlo simulation for different values of β and initial volatilities $\sigma_0 = \sqrt{V_0}$. $r = 0.1$, $T = 1$. Model parameters are shown on the top of each plot. The value of $Q(V_t \to \infty)$ represents the probability of instantaneous variance V_t reaching infinity with corresponding model parameters.

Figure (1) shows sample paths of $\{V_t\}_{t\in[0,T]}$ generated by equation (23) with different initial volatilities and value of β. Each subplot contains 100 sample paths and those in the left column shows paths with initial volatility $\sigma_0 = \sqrt{V_0} = 25\%$, $\beta = -0.3$ and $\beta = -0.9$ from top to the bottom respectively. Plots in the right column also indicates paths with initial volatility $\sigma_0 = 50\%$, $\beta = -0.3$ and $\beta = -0.9$ from top to the bottom respectively. It is observed that the number of unstable paths increases with $|\beta|$ for both high and low initial volatilities σ_0.

By applying the log-Euler scheme, we obtain the square root of the fair variance strikes and the standard deviations of this numerical scheme shown in table 1. The square root of the fair variance strike is computed for different initial volatilities and values of β. We choose nine values of initial volatility equally spaced in interval $[10\%, 50\%]$ and ten values of β equally spaced in interval $[-1, -0.1]$[3]. We set time steps at 252 and generate $2^{16} - 1$ sample paths by using a Sobol sequence. It is clear that there are several infinity at the upper right corner of the table. This is due to the result shown by equation (4) that the probability of $V_t \to \infty$ increases with initial variance V_0. This behaviour could also be seen in table 2, which gives the probabilities of $V_t \to \infty$ before maturity for different values of the initial volatility and β. When this probability is large enough to ensure one sample path can reach infinity, the average realized variance will be infinity. For example, with $\beta = -0.9$ and initial volatility increasing from 25% to 30%, the probability of $V_t \to \infty$ increased from 0.0004% to 0.0135%. When initial volatility is 25%, there might be $0.0004\% \times (2^{16} - 1)$, i.e., 0.26 paths reaching infinity before T. In other words, the realized variance for every sample path does not reach infinity since the instantaneous variance at every step before maturity is less than infinity, so does the expected realized variance. But when the initial volatility increase to 30% with the same value of β, the increasing probability of $V_t \to \infty$ makes $0.0135\% \times (2^{16} - 1)$, approximately 9 sample paths to reach infinity. As a result, the expected realized variance diverges. It follows that the log-Euler scheme is not always stable when computing the fair variance strike of a variance swap under the CEV model.

Standard deviations are shown in the brackets as percentage numbers. We can see that this value increases with initial volatility σ_0 and $|\beta|$. There are $2^{16} - 1$ sample paths generated by the Sobol sequence ensuring the standard deviation less than 0.03%. Using a computer with a 3.33GHz Intel Core Duo CPU and 4GB memory, it takes 250.3495 seconds to compute all the values in this table, 2.7817 seconds for each value.

5.2 Small disturbance asymptotic expansion of the CEV variance process

As indicated in section 4, small disturbance asymptotic expansion of the CEV variance process gives us two distinct methods to price the fair variance strike of a variance swap. By applying lemma 3.4, the fair variance strike is obtained explicitly as a deterministic function of the initial volatility, interest rate and maturity. Table 3 shows the square roots of fair variance strikes, as percentages, with different values of parameters. Initial volatility σ_0 and β are sampled in a similar way as in the log-Euler simulation. The values shown in the brackets are the absolute value of relative difference between the square roots of the fair variance strike calculated by the log-Euler simulation of equation (23) and the closed form approximation of lemma13 respectively. These values are also expressed in percentage.

It is interesting to point out that by using the small disturbance asymptotic expansion we solve the problem of the aberration of infinities for the fair variance strike in our calculation, when the probability of $V_t \to \infty$ increases. Although these differences increase with the initial volatility σ_0 and $|\beta|$, they are

[3]As mentioned in section 2, the CEV model recovers the Black Scholes model when $\beta = 0$. We ignore this well known problem in this paper.

β	Initial Volatilities (%)								
	10	15	20	25	30	35	40	45	50
-1	9.6052	14.5400	19.6604	25.1258	∞	∞	∞	∞	∞
	(4.1)	(14.6)	(38.6)	(119.9)	∞	∞	∞	∞	∞
-0.9	9.6415	14.5731	19.6553	24.9836	∞	∞	∞	∞	∞
	(3.7)	(13.1)	(33.7)	(76.9)	∞	∞	∞	∞	∞
-0.8	9.6789	14.6098	19.6619	24.9006	30.4572	∞	∞	∞	∞
	(3.3)	(11.6)	(29.3)	(63.4)	(161.9)	∞	∞	∞	∞
-0.7	9.7174	14.6501	19.6787	24.8493	30.2300	∞	∞	∞	∞
	(2.9)	(10.1)	(25.2)	(53.0)	(102.8)	∞	∞	∞	∞
-0.6	9.7570	14.6939	19.7050	24.8228	30.0899	35.5695	∞	∞	∞
	(2.5)	(8.7)	(21.4)	(44.0)	(81.8)	(145.7)	∞	∞	∞
-0.5	9.7978	14.7412	19.7402	24.8174	30.0001	35.3230	40.8357	∞	∞
	(2.1)	(7.3)	(17.7)	(35.9)	(65.0)	(110.2)	(180.7)	∞	∞
-0.4	9.8397	14.7919	19.7838	24.8308	29.9500	35.1615	40.4901	45.9676	51.6406
	(1.7)	(5.8)	(14.1)	(28.3)	(50.4)	(83.3)	(130.7)	(198.8)	(299.5)
-0.3	9.8829	14.8460	19.8357	24.8615	29.9338	35.0640	40.2650	45.5516	50.9414
	(1.3)	(4.4)	(10.6)	(21.0)	(37.0)	(60.1)	(92.4)	(136.1)	(194.6)
-0.2	9.9273	14.9037	19.8956	24.9085	29.9479	35.0198	40.1304	45.2866	50.4958
	(0.9)	(3.0)	(7.1)	(13.9)	(24.3)	(39.2)	(59.3)	(85.9)	(120.2)
-0.1	9.9729	14.9649	19.9636	24.9713	29.9902	35.0227	40.0713	45.1383	50.2264
	(0.4)	(1.5)	(3.6)	(7.0)	(12.1)	(19.3)	(29.0)	(41.6)	(57.5)

Table 1: Square root of fair variance strikes (%) obtained by log-Euler Monte Carlo scheme of equation (23) with $\beta \in [-1, -0.1]$, initial volatilities $\sigma_0 \in [0.1, 0.5]$, $r = 10\%$ and $T = 1$. Values shown in brackets are the standard deviations ($\times 10^{-6}$) of this scheme with respect to the corresponding model parameters.

β	Initial Volatilities (%)								
	10	15	20	25	30	35	40	45	50
-1	0	3×10^{-10}	2×10^{-05}	0.0027	0.0463	0.2690	0.8640	1.9585	3.5659
-0.9	0	1×10^{-12}	8×10^{-07}	0.0004	0.0135	0.1111	0.4469	1.1806	2.3960
-0.8	0	0	1×10^{-08}	3×10^{-05}	0.0024	0.0324	0.1785	0.5833	1.3763
-0.7	0	0	4×10^{-11}	9×10^{-07}	0.0002	0.0054	0.0470	0.2091	0.6140
-0.6	0	0	0	3×10^{-09}	4×10^{-06}	0.0003	0.0060	0.0430	0.1770
-0.5	0	0	0	2×10^{-13}	7×10^{-09}	4×10^{-06}	0.0002	0.0031	0.0223
-0.4	0	0	0	0	6×10^{-14}	7×10^{-10}	4×10^{-07}	2×10^{-05}	0.0005
-0.3	0	0	0	0	0	0	4×10^{-13}	6×10^{-10}	1×10^{-07}
-0.2	0	0	0	0	0	0	0	0	0
-0.1	0	0	0	0	0	0	0	0	0

Table 2: Probability (%) of $V_t \to \infty$ with $\beta \in [-1, -0.1]$, initial volatilities $\sigma_0 \in [0.1, 0.5]$, $r = 10\%$ and $T = 1$.

β	Initial Volatilities (%)								
	10	15	20	25	30	35	40	45	50
-1.0	9.5831	14.4999	19.5860	24.9292	30.6442	36.8781	43.8153	51.6789	60.7313
	(0.2301)	(0.2758)	(0.3784)	(0.7825)	–	–	–	–	–
-0.9	9.6200	14.5358	19.5914	24.8554	30.4155	36.3828	42.8962	50.1250	58.2708
	(0.2230)	(0.2560)	(0.3251)	(0.5131)	–	–	–	–	–
-0.8	9.6580	14.5748	19.6057	24.8033	30.2333	35.9777	42.1379	48.8380	56.2265
	(0.2159)	(0.2396)	(0.2858)	(0.3908)	(0.7351)	–	–	–	–
-0.7	9.6970	14.6170	19.6283	24.7709	30.0927	35.6527	41.5227	47.7899	54.5590
	(0.2099)	(0.2259)	(0.2561)	(0.3155)	(0.4542)	–	–	–	–
-0.6	9.7369	14.6622	19.6589	24.7565	29.9895	35.3988	41.0335	46.9524	53.2254
	(0.2060)	(0.2157)	(0.2340)	(0.2671)	(0.3337)	(0.4799)	–	–	–
-0.5	9.7780	14.7105	19.6971	24.7589	29.9200	35.2078	40.6545	46.2977	52.1818
	(0.2021)	(0.2083)	(0.2183)	(0.2357)	(0.2670)	(0.3261)	(0.4437)	–	–
-0.4	9.8200	14.7619	19.7427	24.7772	29.8812	35.0726	40.3714	45.8004	51.3856
	(0.2002)	(0.2028)	(0.2077)	(0.2159)	(0.2297)	(0.2528)	(0.2932)	(0.3637)	(0.4938)
-0.3	9.8632	14.8164	19.7957	24.8106	29.8710	34.9877	40.1724	45.4378	50.7983
	(0.1993)	(0.1994)	(0.2017)	(0.2047)	(0.2098)	(0.2176)	(0.2300)	(0.2498)	(0.2809)
-0.2	9.9076	14.8741	19.8561	24.8588	29.8878	34.9488	40.0480	45.1918	50.3870
	(0.1984)	(0.1986)	(0.1985)	(0.1995)	(0.2007)	(0.2027)	(0.2053)	(0.2093)	(0.2155)
-0.1	9.9532	14.9353	19.9241	24.9218	29.9308	34.9533	39.9918	45.0486	50.1264
	(0.1975)	(0.1978)	(0.1979)	(0.1982)	(0.1981)	(0.1982)	(0.1984)	(0.1987)	(0.1991)

Table 3: Square root of the fair variance strikes (%) obtained by the closed form approximation of lemma 3.4, when initial volatilities $\sigma_0 \in [0.1, 0.5]$, $\beta \in [-1, -0.1]$, $r = 10\%$ and $T = 1$. Values shown in brackets are the relative differences (%) between the fair variance strikes obtained by this closed form approximation and the log-Euler scheme with respect to the corresponding model parameters. Symbol '$-$' is used for those variance swaps whose fair variance strike is ∞ in table 1.

β	Initial Volatilities (%)								
	10	15	20	25	30	35	40	45	50
-1.0	9.5831	14.5000	19.5862	24.9295	30.6447	36.8791	43.8167	51.6811	60.7345
	(0.035)	(0.124)	(0.320)	(0.704)	(1.412)	(2.677)	(4.869)	(8.562)	(14.601)
-0.9	9.6201	14.5358	19.5915	24.8557	30.4160	36.3836	42.8973	50.1267	58.2733
	(0.031)	(0.111)	(0.284)	(0.611)	(1.194)	(2.201)	(3.892)	(6.666)	(11.111)
-0.8	9.6580	14.5749	19.6058	24.8035	30.2337	35.9783	42.1388	48.8392	56.2284
	(0.028)	(0.099)	(0.249)	(0.526)	(1.005)	(1.802)	(3.097)	(5.158)	(8.377)
-0.7	9.6970	14.6170	19.6284	24.7711	30.0930	35.6532	41.5234	47.7909	54.5604
	(0.025)	(0.087)	(0.216)	(0.449)	(0.839)	(1.466)	(2.450)	(3.965)	(6.262)
-0.6	9.7370	14.6622	19.6590	24.7567	29.9897	35.3991	41.0340	46.9531	53.2264
	(0.022)	(0.075)	(0.184)	(0.377)	(0.691)	(1.181)	(1.922)	(3.024)	(4.638)
-0.5	9.7780	14.7105	19.6972	24.7591	29.9201	35.2080	40.6548	46.2982	52.1824
	(0.018)	(0.063)	(0.153)	(0.309)	(0.557)	(0.934)	(1.485)	(2.276)	(3.393)
-0.4	9.8201	14.7619	19.7428	24.7773	29.8814	35.0728	40.3716	45.8007	51.3861
	(0.015)	(0.051)	(0.122)	(0.244)	(0.435)	(0.716)	(1.117)	(1.673)	(2.431)
-0.3	9.8633	14.8164	19.7957	24.8106	29.8711	34.9879	40.1726	45.4381	50.7986
	(0.011)	(0.038)	(0.092)	(0.182)	(0.320)	(0.521)	(0.799)	(1.174)	(1.670)
-0.2	9.9076	14.8741	19.8561	24.8588	29.8878	34.9489	40.0481	45.1919	50.3871
	(0.008)	(0.026)	(0.061)	(0.121)	(0.211)	(0.340)	(0.515)	(0.746)	(1.043)
-0.1	9.9532	14.9353	19.9241	24.9218	29.9308	34.9534	39.9918	45.0487	50.1265
	(0.004)	(0.013)	(0.031)	(0.061)	(0.105)	(0.168)	(0.252)	(0.361)	(0.499)

Table 4: Square roots of the fair variance strikes (%) and their standard deviations obtained by the conditional Monte-Carlo simulation of corollary 4.4, when initial volatilities $\sigma_0 \in [0.1, 0.5]$, $\beta \in [-1, -0.1]$, $r = 10\%$ and $T = 1$. Values shown in brackets are the standard deviations ($\times 10^{-6}$) of this scheme with respect to the corresponding model parameters.

still in fairly low level, that all of them are less than 1%. We use the symbol '−' to denote when a value may not be returned, because the result of log-Euler scheme is infinity.

The alternative approach is to apply corollary 4.4 by implementing a conditional Monte-Carlo simulation. The value of the conditional expected realized variance only depends on the initial volatility, interest rate, maturity and a sample of Brownian motion W_T, which is a path independent random variable and is straightforwardly simulated by quasi-Monte Carlo method. The first step is to generate samples of a Brownian motion W_T via a Sobol sequence. Next, we compute the value of $E_b^{\mathbb{Q}}[V_{(0,T)}]$ by taking b as a sample of W_T and substitute b to equation (22). Then the value of fair variance strike is the mean of $E_b^{\mathbb{Q}}[V_{(0,T)}]$ computed by repeating the second step for all samples of W_T.

Table 4 shows the square root of fair variance strikes in percent by applying this conditional Monte-Carlo simulation with different values of parameters. The model parameters σ_0 and β are the same as those used in table 1. We generate $2^{16} - 1$ elements in a Sobol sequence to simulate the distribution of $E_b^{\mathbb{Q}}[V_{(0,T)}]$ for all the parameter values, and also compute its the standard deviations, which are shown in the brackets as a percentage value. As is shown, the fair variance strikes in this table are vary close to the values in table 3. The standard deviation also increases with initial volatility σ_0 and $|\beta|$, which agrees with the trend shown in table 1. However, the level of standard derivations generated by the small disturbance asymptotic expansion is approximately 100 times smaller than by the log-Euler simulation when they both use $2^{16} - 1$ paths. In addition, the conditional Monte-Carlo simulation with small disturbance asymptotic expansion is shown to be very fast in our case. It takes 2.8105 seconds to compute all the values in table 4, i.e., 0.0312 seconds for each of them, an approximate improvement of a factor of 90. This method also has the potential to be applied to price more complicated volatility derivatives, such as corridor variance swaps and option on realized variance. See Carr and Madan, Carr and Lewis [3, 4] for a detailed discussion.

6 Conclusion

In this paper we apply the small disturbance asymptotic expansion to the CEV variance process and compute the fair variance strike of a variance swap by giving a path-independent closed from approximation. A conditional Monte Carlo simulation is also applied to obtain the fair variance strike by deriving a theorem for the conditionally expected product of a Brownian motion at two different times with arbitrary powers. The results presented in the paper have three main advantages. First, the fair variance strike is expressed as a deterministic function of interest rate, expiry time and initial volatility and can be easily implemented numerically. Second, the conditional expectation of the realized variance only depends on interest rate, expiry time, initial volatility and a Brownian motion at expiry, which can be applied by a conditional Monte-Carlo simulation scheme. This method is compared with the first order log-Euler scheme. The results demonstrate that the speed and accuracy of our conditional Monte-Carlo simulation is improved significantly compared with the first order log-Euler scheme. Finally, ignoring the pathological case when the variance goes to infinity, we gained reasonable and accurate results.

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Appendix A

Proof of lemma 3.3.

Proof. By lemma 3.1, we have

$$\begin{aligned}
\frac{\partial^2 V_t}{\partial \epsilon^2} &= 2\bar{\beta}\int_0^t \left[2\bar{\beta}V_u^2 + (2\epsilon\bar{\beta}-1)\frac{\partial V_u^2}{\partial \epsilon} + 2r\frac{\partial V_u}{\partial \epsilon}\right]\mathrm{d}u + 4\bar{\beta}\int_0^t \frac{\partial V_u^{\frac{3}{2}}}{\partial \epsilon}\mathrm{d}W_u \\
&\quad + \epsilon\bar{\beta}\int_0^t \left[4\bar{\beta}\frac{\partial V_u^2}{\partial \epsilon} + (2\epsilon\bar{\beta}-1)\frac{\partial^2 (V_u^2)}{\partial \epsilon^2} + 2r\frac{\partial^2 V_u}{\partial \epsilon^2}\right]\mathrm{d}u + 2\epsilon\bar{\beta}\int_0^t \frac{\partial^2 (V_u^{\frac{3}{2}})}{\partial \epsilon^2}\mathrm{d}W_u; \\
\frac{\partial^3 V_t}{\partial \epsilon^3} &= 3\bar{\beta}\int_0^t \left[4\bar{\beta}\frac{\partial V_u^2}{\partial \epsilon} + (2\epsilon\bar{\beta}-1)\frac{\partial^2 (V_u^2)}{\partial \epsilon^2} + 2r\frac{\partial^2 V_u}{\partial \epsilon^2}\right]\mathrm{d}u + 6\bar{\beta}\int_0^t \frac{\partial^2 (V_u^{\frac{3}{2}})}{\partial \epsilon^2}\mathrm{d}W_u \\
&\quad + \epsilon\bar{\beta}\int_0^t \left[6\bar{\beta}\frac{\partial^2 (V_u^2)}{\partial \epsilon^2} + (2\epsilon\bar{\beta}-1)\frac{\partial^3 (V_u^2)}{\partial \epsilon^3} + 2r\frac{\partial^3 V_u}{\partial \epsilon^3}\right]\mathrm{d}u + 2\epsilon\bar{\beta}\int_0^t \frac{\partial^3 (V_u^{\frac{3}{2}})}{\partial \epsilon^3}\mathrm{d}W_u; \text{ and} \\
\frac{\partial^4 V_t}{\partial \epsilon^4} &= 4\bar{\beta}\int_0^t \left[6\bar{\beta}\frac{\partial^2 (V_u^2)}{\partial \epsilon^2} + (2\epsilon\bar{\beta}-1)\frac{\partial^3 (V_u^2)}{\partial \epsilon^3} + 2r\frac{\partial^3 V_u}{\partial \epsilon^3}\right]\mathrm{d}u + 8\bar{\beta}\int_0^t \frac{\partial^3 (V_u^{\frac{3}{2}})}{\partial \epsilon^3}\mathrm{d}W_u \\
&\quad + \epsilon\bar{\beta}\int_0^t \left[8\bar{\beta}\frac{\partial^3 (V_u^2)}{\partial \epsilon^3} + (2\epsilon\bar{\beta}-1)\frac{\partial^4 (V_u^2)}{\partial \epsilon^4} + 2r\frac{\partial^4 V_u}{\partial \epsilon^4}\right]\mathrm{d}u + 2\epsilon\bar{\beta}\int_0^t \frac{\partial^4 (V_u^{\frac{3}{2}})}{\partial \epsilon^4}\mathrm{d}W_u.
\end{aligned}$$

Taking $\epsilon \to 0$ gives us the following expressions.

$$\left.\frac{\partial^2 V_t}{\partial \epsilon^2}\right|_{\epsilon=0} = 2\bar{\beta}\int_0^t \left[2\bar{\beta}V_0^2 - \left.\frac{\partial V_u^2}{\partial \epsilon}\right|_{\epsilon=0} + 2rG\right]\mathrm{d}u + 4\bar{\beta}\int_0^t \left.\frac{\partial V_u^{\frac{3}{2}}}{\partial \epsilon}\right|_{\epsilon=0}\mathrm{d}W_u, \tag{24}$$

$$\left.\frac{\partial^3 V_t}{\partial \epsilon^3}\right|_{\epsilon=0} = 3\bar{\beta}\int_0^t \left[4\bar{\beta}\left.\frac{\partial V_u^2}{\partial \epsilon}\right|_{\epsilon=0} - \left.\frac{\partial^2 (V_u^2)}{\partial \epsilon^2}\right|_{\epsilon=0} + 2rH\right]\mathrm{d}u + 6\bar{\beta}\int_0^t \left.\frac{\partial^2 (V_u^{\frac{3}{2}})}{\partial \epsilon^2}\right|_{\epsilon=0}\mathrm{d}W_u, \tag{25}$$

$$\left.\frac{\partial^4 V_t}{\partial \epsilon^4}\right|_{\epsilon=0} = 4\bar{\beta}\int_0^t \left[6\bar{\beta}\left.\frac{\partial^2 (V_u^2)}{\partial \epsilon^2}\right|_{\epsilon=0} - \left.\frac{\partial^3 (V_u^2)}{\partial \epsilon^3}\right|_{\epsilon=0} + 2rL\right]\mathrm{d}u + 8\bar{\beta}\int_0^t \left.\frac{\partial^3 (V_u^{\frac{3}{2}})}{\partial \epsilon^3}\right|_{\epsilon=0}\mathrm{d}W_u. \tag{26}$$

The derivatives of $V_t^{\frac{3}{2}}$ and V_t^2 needs to be computed up to the third order. They will be computed in three steps.

Step 1: the first derivative

$$\left.\frac{\partial V_t^m}{\partial \epsilon}\right|_{\epsilon=0} = m\bar{\beta}V_0^m\left[(2r - V_0)t + 2V_0^{\frac{1}{2}}W_t\right] \text{ for } m \geq 0. \tag{27}$$

Step 2: the second derivative

$$\begin{aligned}
\left.\frac{\partial^2 (V_t^{\frac{3}{2}})}{\partial \epsilon^2}\right|_{\epsilon=0} &= 6\bar{\beta}\int_0^t \left[\frac{3}{2}\bar{\beta}V_0^{\frac{5}{2}} - \frac{1}{2}\left.\frac{\partial V_u^{\frac{5}{2}}}{\partial \epsilon}\right|_{\epsilon=0} + r\left.\frac{\partial V_u^{\frac{3}{2}}}{\partial \epsilon}\right|_{\epsilon=0}\right]\mathrm{d}u + 6\bar{\beta}\int_0^t \left.\frac{\partial V_u^2}{\partial \epsilon}\right|_{\epsilon=0}\mathrm{d}W_u \\
&= 6\bar{\beta}^2 V_0^{\frac{3}{2}}\int_0^t \left[\frac{3}{2}V_0 - \frac{5}{2}V_0(r - \frac{1}{2}V_0)u - \frac{5}{2}V_0^{\frac{3}{2}}W_u + 3r\left((r - \frac{1}{2}V_0)u + V_0^{\frac{1}{2}}W_u\right)\right]\mathrm{d}u \\
&\quad + 12\bar{\beta}^2 V_0^2\int_0^t \left[(2r - V_0)u + 2V_0^{\frac{1}{2}}W_u\right]\mathrm{d}W_u.
\end{aligned} \tag{28}$$

$$\begin{aligned}
\left.\frac{\partial^2 (V_t^2)}{\partial \epsilon^2}\right|_{\epsilon=0} &= 4\bar{\beta}\int_0^t \left[4\bar{\beta}V_0^3 - \left.\frac{\partial V_u^3}{\partial \epsilon}\right|_{\epsilon=0} + 2r\left.\frac{\partial V_u^2}{\partial \epsilon}\right|_{\epsilon=0}\right]\mathrm{d}u + 8\bar{\beta}\int_0^t \left.\frac{\partial V_u^{\frac{5}{2}}}{\partial \epsilon}\right|_{\epsilon=0}\mathrm{d}W_u \\
&= 4\bar{\beta}^2 V_0^2\int_0^t \left[4V_0 - 3V_0(2r - V_0)u - 6V_0^{\frac{3}{2}}W_u + 4r\Big((2r - V_0)u + 2V_0^{\frac{1}{2}}W_u\Big)\right]\mathrm{d}u \\
&\quad + 40\bar{\beta}^2 V_0^{\frac{5}{2}}\int_0^t \left[(r - \frac{1}{2}V_0)u + V_0^{\frac{1}{2}}W_u\right]\mathrm{d}W_u.
\end{aligned} \tag{29}$$

$$\left.\frac{\partial^2 (V_t^{\frac{5}{2}})}{\partial \epsilon^2}\right|_{\epsilon=0} = 10\bar{\beta}\int_0^t \left[\frac{5}{2}\bar{\beta}V_0^{\frac{7}{2}} - \frac{1}{2}\left.\frac{\partial V_u^{\frac{7}{2}}}{\partial \epsilon}\right|_{\epsilon=0} + r\left.\frac{\partial V_u^{\frac{5}{2}}}{\partial \epsilon}\right|_{\epsilon=0}\right]\mathrm{d}u + 10\bar{\beta}\int_0^t \left.\frac{\partial V_u^3}{\partial \epsilon}\right|_{\epsilon=0}\mathrm{d}W_u$$

$$= 10\bar{\beta}^2 V_0^{\frac{5}{2}} \int_0^t \left[\frac{5}{2} V_0 - \frac{7}{2} V_0 (r - \frac{1}{2} V_0) u - \frac{7}{2} V_0^{\frac{3}{2}} W_t + 5r \left((r - \frac{1}{2} V_0) u + V_0^{\frac{1}{2}} W_u \right) \right] \mathrm{d}u$$

$$+ 30\bar{\beta}^2 V_0^3 \int_0^t \left[(2r - V_0) u + 2V_0^{\frac{1}{2}} W_u \right] \mathrm{d}W_u. \tag{30}$$

$$\left. \frac{\partial^2 (V_t^3)}{\partial \epsilon^2} \right|_{\epsilon=0} = 6\bar{\beta} \int_0^t \left[6\bar{\beta} V_0^4 - \left. \frac{\partial V_u^4}{\partial \epsilon} \right|_{\epsilon=0} + 2r \left. \frac{\partial V_u^3}{\partial \epsilon} \right|_{\epsilon=0} \right] \mathrm{d}u + 12\bar{\beta} \int_0^t \left. \frac{\partial V_u^{\frac{7}{2}}}{\partial \epsilon} \right|_{\epsilon=0} \mathrm{d}W_u$$

$$= 12\bar{\beta}^2 V_0^3 \int_0^t \left[3V_0 - 2V_0 (2r - V_0) u - 4V_0^{\frac{3}{2}} W_u + 3r \left((2r - V_0) u + 2V_0^{\frac{1}{2}} W_u \right) \right] \mathrm{d}u$$

$$+ 84\bar{\beta}^2 V_0^{\frac{7}{2}} \int_0^t \left[(r - \frac{1}{2} V_0) u + V_0^{\frac{1}{2}} W_u \right] \mathrm{d}W_u. \tag{31}$$

By Itô's lemma, we have $W_t^2 = 2\int_0^t W_u \mathrm{d}W_u + t$ and $tW_t = \int_0^t W_u \mathrm{d}u + \int_0^t u \mathrm{d}W_u$. So equation (28) to equation (31) may be written as

$$\left. \frac{\partial^2 (V_t^{\frac{3}{2}})}{\partial \epsilon^2} \right|_{\epsilon=0} = 3\bar{\beta}^2 V_0^{\frac{3}{2}} \Big[(\frac{5}{4} V_0^2 - 4rV_0 + 3r^2) t^2 - V_0 t + 4V_0^{\frac{1}{2}} (2r - V_0) t W_t$$

$$+ 4V_0 W_t^2 - V_0^{\frac{1}{2}} (2r + V_0) \int_0^t W_u \mathrm{d}u \Big]. \tag{32}$$

$$\left. \frac{\partial^2 (V_t^2)}{\partial \epsilon^2} \right|_{\epsilon=0} = 4\bar{\beta}^2 V_0^2 \Big[(\frac{3}{2} V_0^2 - 5rV_0 + 4r^2) t^2 - V_0 t + 5V_0^{\frac{1}{2}} (2r - V_0) t W_t$$

$$+ 5V_0 W_t^2 - V_0^{\frac{1}{2}} (2r + V_0) \int_0^t W_u \mathrm{d}u \Big]. \tag{33}$$

$$\left. \frac{\partial^2 (V_t^{\frac{5}{2}})}{\partial \epsilon^2} \right|_{\epsilon=0} = 5\bar{\beta}^2 V_0^{\frac{5}{2}} \Big[(\frac{7}{4} V_0^2 - 6rV_0 + 5r^2) t^2 - V_0 t + 6V_0^{\frac{1}{2}} (2r - V_0) t W_t$$

$$+ 6V_0 W_t^2 - V_0^{\frac{1}{2}} (2r + V_0) \int_0^t W_u \mathrm{d}u \Big]. \tag{34}$$

$$\left. \frac{\partial^2 (V_t^3)}{\partial \epsilon^2} \right|_{\epsilon=0} = 6\bar{\beta}^2 V_0^3 \Big[(2V_0^2 - 7rV_0 + 6r^2) t^2 - V_0 t + 7V_0^{\frac{1}{2}} (2r - V_0) t W_t$$

$$+ 7V_0 W_t^2 - V_0^{\frac{1}{2}} (2r + V_0) \int_0^t W_u \mathrm{d}u \Big]. \tag{35}$$

Step 3: the third derivative

$$\left. \frac{\partial^3 (V_t^{\frac{3}{2}})}{\partial \epsilon^3} \right|_{\epsilon=0} = 9\bar{\beta} \int_0^t \left[3\bar{\beta} \left. \frac{\partial V_u^{\frac{5}{2}}}{\partial \epsilon} \right|_{\epsilon=0} - \frac{1}{2} \left. \frac{\partial^2 (V_u^{\frac{5}{2}})}{\partial \epsilon^2} \right|_{\epsilon=0} + r \left. \frac{\partial^2 (V_u^{\frac{3}{2}})}{\partial \epsilon^2} \right|_{\epsilon=0} \right] \mathrm{d}u + 9\bar{\beta} \int_0^t \left. \frac{\partial^2 (V_u^2)}{\partial \epsilon^2} \right|_{\epsilon=0} \mathrm{d}W_u$$

$$= 9\bar{\beta}^3 V_0^{\frac{3}{2}} \int_0^t \Big[\frac{15}{2} V_0 ((2r - V_0) u + 2V_0^{\frac{1}{2}} W_u) - \frac{5}{2} V_0 \Big((\frac{7}{4} V_0^2 - 6rV_0 + 5r^2) u^2 - V_0 u$$

$$+ 6V_0^{\frac{1}{2}} (2r - V_0) u W_u + 6V_0 W_u^2 - V_0^{\frac{1}{2}} (2r + V_0) \int_0^u W_s \mathrm{d}s \Big) + 3r \Big((\frac{5}{4} V_0^2 - 4rV_0 + 3r^2) u^2$$

$$- V_0 u + 4V_0^{\frac{1}{2}} (2r - V_0) u W_u + 4V_0 W_u^2 - V_0^{\frac{1}{2}} (2r + V_0) \int_0^u W_s \mathrm{d}s \Big) \Big] \mathrm{d}u$$

$$+ 36\bar{\beta}^3 V_0^2 \int_0^t \Big[(\frac{3}{2} V_0^2 - 5rV_0 + 4r^2) u^2 - V_0 u + 5V_0^{\frac{1}{2}} (2r - V_0) u W_u + 5V_0 W_u^2$$

$$- V_0^{\frac{1}{2}} (2r + V_0) \int_0^u W_s \mathrm{d}s \Big] \mathrm{d}W_u. \tag{36}$$

$$\left. \frac{\partial^3 (V_t^2)}{\partial \epsilon^3} \right|_{\epsilon=0} = 6\bar{\beta} \int_0^t \left[8\bar{\beta} \left. \frac{\partial V_u^3}{\partial \epsilon} \right|_{\epsilon=0} - \left. \frac{\partial^2 (V_u^3)}{\partial \epsilon^2} \right|_{\epsilon=0} + 2r \left. \frac{\partial^2 (V_u^2)}{\partial \epsilon^2} \right|_{\epsilon=0} \right] \mathrm{d}u + 12\bar{\beta} \int_0^t \left. \frac{\partial^2 (V_u^{\frac{5}{2}})}{\partial \epsilon^2} \right|_{\epsilon=0} \mathrm{d}W_u$$

$$= 12\bar{\beta}^3 V_0^2 \int_0^t \Big[12V_0 \left((2r - V_0) u + 2V_0^{\frac{1}{2}} W_u \right) - 3V_0 \Big((2V_0^2 - 7rV_0 + 6r^2) u^2 - V_0 u$$

$$+ 7V_0^{\frac{1}{2}} (2r - V_0) u W_u + 7V_0 W_u^2 - V_0^{\frac{1}{2}} (2r + V_0) \int_0^u W_s \mathrm{d}s \Big) + 4r \Big((\frac{3}{2} V_0^2 - 5rV_0 + 4r^2) u^2$$

$$- V_0 u + 5V_0^{\frac{1}{2}} (2r - V_0) u W_u + 5V_0 W_u^2 - V_0^{\frac{1}{2}} (2r + V_0) \int_0^u W_s \mathrm{d}s \Big) \Big] \mathrm{d}u$$

$$+ 60\bar{\beta}^3 V_0^{\frac{5}{2}} \int_0^t \Big[(\frac{7}{4}V_0^2 - 6rV_0 + 5r^2)u^2 - V_0 u + 6V_0^{\frac{1}{2}}(2r - V_0)uW_u + 6V_0 W_u^2$$
$$- V_0^{\frac{1}{2}}(2r + V_0) \int_0^u W_s \mathrm{d}s \Big] \mathrm{d}W_u. \tag{37}$$

Again, by Itô's lemma, we have $W_t^3 = 3\int_0^t W_u^2 \mathrm{d}W_u + 3\int_0^t W_u \mathrm{d}u$, $tW_t^2 = \int_0^t W_u^2 \mathrm{d}u + 2\int_0^t uW_u \mathrm{d}W_u + \frac{1}{2}t^2$, $t^2 W_t = 2\int_0^t uW_u \mathrm{d}u + \int_0^t u^2 \mathrm{d}W_u$ and $\int_0^t \int_0^s W_s \mathrm{d}s \mathrm{d}W_u = W_t \int_0^t W_u \mathrm{d}u - 2\int_0^t W_u^2 \mathrm{d}u$. After substituting these equations to equation (36) and (37), we derive that

$$\frac{\partial^3 (V_t^{\frac{3}{2}})}{\partial \epsilon^3}\Big|_{\epsilon=0} = 9\bar{\beta}^3 V_0^{\frac{3}{2}} \Big[(-\frac{35}{24}V_0^3 + \frac{25}{4}rV_0^2 - \frac{49}{6}r^2 V_0 + 3r^3)t^3 + 4V_0^{\frac{1}{2}}(\frac{3}{2}V_0^2 - 5rV_0 + 4r^2)t^2 W_t$$
$$- V_0(4r - \frac{5}{2}V_0)t^2 + 10V_0(2r - V_0)tW_t^2 - 4V_0^{\frac{3}{2}}tW_t + \frac{20}{3}V_0^{\frac{3}{2}}W_t^3 + V_0(8r + 3V_0)\int_0^t W_u^2 \mathrm{d}u$$
$$+ V_0^{\frac{1}{2}}(3V_0^2 - 2rV_0 - 8r^2)\int_0^t uW_u \mathrm{d}u - 4V_0(2r + V_0)W_t \int_0^t W_u \mathrm{d}u$$
$$+ V_0^{\frac{1}{2}}(\frac{5}{2}V_0^2 + 2rV_0 - 6r^2)\int_0^t \int_0^u W_s \mathrm{d}s \mathrm{d}u - V_0^{\frac{3}{2}} \int_0^t W_u \mathrm{d}u \Big]. \tag{38}$$

$$\frac{\partial^3 (V_t^2)}{\partial \epsilon^3}\Big|_{\epsilon=0} = 12\bar{\beta}^3 V_0^2 \Big[(-2V_0^3 + 9rV_0^2 - \frac{38}{3}r^2 V_0 + \frac{16}{3}r^3)t^3 + 5V_0^{\frac{1}{2}}(\frac{7}{4}V_0^2 - 6rV_0 + 5r^2)t^2 W_t$$
$$- V_0(5r - 3V_0)t^2 + 15V_0(2r - V_0)tW_t^2 - 5V_0^{\frac{3}{2}}tW_t + 10V_0^{\frac{3}{2}}W_t^3 + 2V_0(5r + 2V_0)\int_0^t W_u^2 \mathrm{d}u$$
$$+ 2V_0^{\frac{1}{2}}(\frac{7}{4}V_0^2 - rV_0 - 5r^2)\int_0^t uW_u \mathrm{d}u - 5V_0(2r + V_0)W_t \int_0^t W_u \mathrm{d}u$$
$$+ V_0^{\frac{1}{2}}(3V_0^2 + 2rV_0 - 8r^2)\int_0^t \int_0^u W_s \mathrm{d}s \mathrm{d}u - V_0^{\frac{3}{2}} \int_0^t W_u \mathrm{d}u \Big]. \tag{39}$$

Now, we are able to derive H, L and M from equations (24) to (26). First, substituting equation (11) and (27) to equation (24) derives

$$\frac{\partial^2 V_t}{\partial \epsilon^2}\Big|_{\epsilon=0} = 2\bar{\beta}^2 V_0 \Big[\int_0^t \Big(2V_0 - 2V_0 \Big((2r - V_0)u + 2V_0^{\frac{1}{2}} W_u \Big) + 2r \Big((2r - V_0)u + 2V_0^{\frac{1}{2}} W_u \Big) \Big) \mathrm{d}u$$
$$+ 3V_0^{\frac{1}{2}} \int_0^t \Big((2r - V_0)u + 2V_0^{\frac{1}{2}} W_u \Big) \mathrm{d}W_u \Big]$$
$$= 2\bar{\beta}^2 V_0 \Big[(V_0^2 - 3rV_0 + 2r^2)t^2 - V_0 t + 3V_0^{\frac{1}{2}}(2r - V_0)tW_t + 3V_0 W_t^2 - V_0^{\frac{1}{2}}(2r + V_0)\int_0^t W_u \mathrm{d}u \Big]. \tag{40}$$

Second, substituting equations (27), (32), (33) and (40) into equation (25) gives us that

$$\frac{\partial^3 V_t}{\partial \epsilon^3}\Big|_{\epsilon=0} = 3\bar{\beta}^3 V_0 \int_0^t \Big[8V_0 \Big((2r - V_0)u + 2V_0^{\frac{1}{2}} W_u \Big) - 4V_0 \Big((\frac{3}{2}V_0^2 - 5rV_0 + 4r^2)u^2 + 5V_0^{\frac{1}{2}}(2r - V_0)uW_u$$
$$- V_0 u + 5V_0 W_u^2 - V_0^{\frac{1}{2}}(2r + V_0)\int_0^u W_s \mathrm{d}s \Big) + 4r \Big((V_0^2 - 3rV_0 + 2r^2)u^2 - V_0 u + 3V_0 W_u^2$$
$$+ 3V_0^{\frac{1}{2}}(2r - V_0)uW_u - V_0^{\frac{1}{2}}(2r + V_0)\int_0^u W_s \mathrm{d}s \Big) \Big] \mathrm{d}u + 18\bar{\beta}^3 V_0^{\frac{3}{2}} \int_0^t \Big[(\frac{5}{4}V_0^2 - 4rV_0 + 3r^2)u^2$$
$$- V_0 u + 4V_0^{\frac{1}{2}}(2r - V_0)uW_u + 4V_0 W_u^2 - V_0^{\frac{1}{2}}(2r + V_0)\int_0^u W_s \mathrm{d}s \Big] \mathrm{d}W_u$$
$$= 6\bar{\beta}^3 V_0 \Big[(-V_0^3 + 4rV_0^2 - \frac{14}{3}r^2 V_0 + \frac{4}{3}r^3)t^3 + 3V_0^{\frac{1}{2}}(\frac{5}{4}V_0^2 - 4rV_0 + 3r^2)t^2 W_t - V_0(3r - 2V_0)t^2$$
$$+ 6V_0(2r - V_0)tW_t^2 - 3V_0^{\frac{3}{2}}tW_t + 4V_0^{\frac{3}{2}}W_t^3 + 2V_0(3r + V_0)\int_0^t W_u^2 \mathrm{d}u - 3V_0(2r + V_0)W_t \int_0^t W_u \mathrm{d}u$$
$$+ V_0^{\frac{1}{2}}(\frac{5}{2}V_0^2 - 2rV_0 - 6r^2)\int_0^t uW_u \mathrm{d}u + 2V_0^{\frac{1}{2}}(V_0^2 + rV_0 - 2r^2)\int_0^t \int_0^u W_s \mathrm{d}s \mathrm{d}u - V_0^{\frac{3}{2}} \int_0^t W_u \mathrm{d}u \Big]. \tag{41}$$

Next, we derive $\left.\frac{\partial^4 V_t}{\partial \epsilon^4}\right|_{\epsilon=0}$, but we firstly give the following equations by applying Itô's lemma.

$$W_t^4 = 4\int_0^t W_u^3 \mathrm{d}W_u + 6\int_0^t W_u^2 \mathrm{d}u, \tag{42}$$

$$tW_t^3 = \int_0^t (W_u^3 + 3uW_u)\mathrm{d}u + 3\int_0^t uW_u^2 \mathrm{d}W_u, \tag{43}$$

$$t^3 W_t = 3\int_0^t u^2 W_u \mathrm{d}u + \int_0^t u^3 \mathrm{d}W_u, \tag{44}$$

$$t^2 W_t^2 = 2\int_0^t uW_u^2 \mathrm{d}u + 2\int_0^t u^2 W_u \mathrm{d}W_u + \frac{1}{3}t^3, \tag{45}$$

$$\int_0^t \int_0^u W_s^2 \mathrm{d}s\mathrm{d}W_u = W_t \int_0^t W_u^2 \mathrm{d}u - \int_0^t W_u^3 \mathrm{d}u, \tag{46}$$

$$\int_0^t \int_0^u sW_s \mathrm{d}s\mathrm{d}W_u = W_t \int_0^t uW_u \mathrm{d}u - \int_0^t uW_u^2 \mathrm{d}u, \tag{47}$$

$$\int_0^t \int_0^u \int_0^s W_a \mathrm{d}a\mathrm{d}s\mathrm{d}W_u = W_t \int_0^t \int_0^u W_s \mathrm{d}s\mathrm{d}u - \int_0^t W_u \int_0^u W_s \mathrm{d}s\mathrm{d}u, \tag{48}$$

$$\int_0^t W_u \int_0^u W_s \mathrm{d}s\mathrm{d}W_u = \frac{1}{2}\left(W_t^2 \int_0^t W_u \mathrm{d}u - \int_0^t W_u^3 \mathrm{d}u - \int_0^t \int_0^u W_s \mathrm{d}s\mathrm{d}u\right). \tag{49}$$

Then substituting equations (33), (38), (39), (41) and (42) to (49) into equation (26), we obtain

$$\begin{aligned}
\left.\frac{\partial^4 V_t}{\partial \epsilon^4}\right|_{\epsilon=0} &= 24\bar{\beta}^4 V_0 \Big[(V_0^4 - 5rV_0^3 + \frac{25}{3}r^2V_0^2 - 5r^3V_0 + \frac{2}{3}r^4)t^4 - V_0^{\frac{1}{2}}(\frac{35}{8}V_0^3 - \frac{75}{4}rV_0^2 + \frac{49}{2}r^2V_0 - 9r^3)t^3W_t \\
&- V_0(3V_0^2 - 8rV_0 - \frac{14}{3}r^2)t^3 + 6V_0(\frac{3}{2}V_0^2 - 5rV_0 + 4r^2)t^2W_t^2 - 3V_0^{\frac{3}{2}}(4r - \frac{5}{2}V_0)t^2W_t + V_0^2t^2 \\
&+ 10V_0^{\frac{3}{2}}(2r - V_0)tW_t^3 - 6V_0^2tW_t^2 + 5V_0^2W_t^4 + V_0^{\frac{3}{2}}(5V_0 - 2r)\int_0^t uW_u\mathrm{d}u - V_0^2\int_0^t W_u^2\mathrm{d}u \\
&+ 2V_0^{\frac{3}{2}}(r + 2V_0)\int_0^t\int_0^u W_s\mathrm{d}s\mathrm{d}u - V_0^{\frac{1}{2}}(\frac{35}{8}V_0^3 - \frac{45}{4}rV_0^2 + \frac{1}{2}r^2V_0 + 9r^3)\int_0^t u^2W_u\mathrm{d}u \\
&+ 3V_0^2(V_0 - 2r)\int_0^t uW_u^2\mathrm{d}u - V_0^{\frac{3}{2}}(24r + 13V_0)\int_0^t W_u^3\mathrm{d}u - 4V_0(2V_0^2 + 4rV_0 - 3r^2)\int_0^t\int_0^u W_s^2\mathrm{d}s\mathrm{d}u \\
&- V_0^{\frac{1}{2}}(7V_0^3 - 9rV_0^2 - 16r^2V_0 + 12r^3)\int_0^t\int_0^u sW_s\mathrm{d}s\mathrm{d}u + V_0(\frac{5}{2}V_0^2 + 26rV_0 - 30r^2)\int_0^t W_u\int_0^u W_s\mathrm{d}s\mathrm{d}u \\
&- 2V_0^{\frac{1}{2}}(3V_0^3 - 10r^2V_0 + 4r^3)\int_0^t\int_0^u\int_0^s W_a\mathrm{d}a\mathrm{d}s\mathrm{d}u + 3V_0^{\frac{3}{2}}(8r + 3V_0)W_t\int_0^t W_u^2\mathrm{d}u \\
&+ 3V_0(3V_0^2 - 2rV_0 - 8r^2)W_t\int_0^t uW_u\mathrm{d}u - 6V_0^{\frac{3}{2}}(2r + V_0)W_t^2\int_0^t W_u\mathrm{d}u \\
&+ 3V_0(\frac{5}{2}V_0^2 - 4rV_0 + 6r^2)W_t\int_0^t\int_0^u W_s\mathrm{d}s\mathrm{d}u - 3V_0^2W_t\int_0^t W_u\mathrm{d}u\Big].
\end{aligned} \tag{50}$$

Finally, H, L and M can be derived by substituting equations (40), (41) and (50) into equation (6). □

Appendix B

Proof of lemma 4.4.

Proof. From lemma (3.3), we have the expectation of integrated variance with condition $\{W_T = b\}$ in the form

$$\begin{aligned}
E_b[V_{(0,T)}] &= \frac{1}{T}E_b\left[\int_0^T V_t\mathrm{d}t\right] \\
&= \frac{1}{T}E_b\left[\int_0^T (V_0 + \beta G + \beta^2 H + \beta^3 L + \beta^4 M + O(\epsilon^5))\mathrm{d}t\right] \\
&= V_0 + \frac{\beta}{T}E_b\left[\int_0^T G\mathrm{d}t\right] + \frac{\beta^2}{T}E_b\left[\int_0^T H\mathrm{d}t\right] + \frac{\beta^3}{T}E_b\left[\int_0^T L\mathrm{d}t\right] + \frac{\beta^4}{T}E_b\left[\int_0^T M\mathrm{d}t\right] + O(\epsilon^5).
\end{aligned} \tag{51}$$

Then by corollary 4.4, we derive the conditional expectation $E_b\left[\int_0^T Gdt\right]$ as follows:

$$E_b\left[\int_0^T Gdt\right] = V_0(2r - V_0)E_b\left[\int_0^T t\mathrm{d}t\right] + 2V_0^{\frac{3}{2}}E_b\left[\int_0^T W_t\mathrm{d}t\right] = V_0T\Big[\frac{1}{2}(2r - V_0)T + V_0^{\frac{1}{2}}b\Big]. \tag{52}$$

Likewise, by repeating this calculation, we have the $E_b\left[\int_0^T H\mathrm{d}t\right]$, $E_b\left[\int_0^T L\mathrm{d}t\right]$, and $E_b\left[\int_0^T M\mathrm{d}t\right]$ as

$$E_b\left[\int_0^T H\mathrm{d}t\right] = V_0T\Big[\frac{1}{3}(V_0^2 - 3rV_0 + 2r^2)T^2 + \frac{1}{6}V_0^{\frac{1}{2}}(10r - 7V_0)bT + V_0b^2\Big]. \tag{53}$$

$$\begin{aligned} E_b\left[\int_0^T L\mathrm{d}t\right] &= V_0T\Big[\frac{1}{4}(-V_0^3 + 4rV_0^2 - \frac{14}{3}r^2V_0 + \frac{4}{3}r^3)T^3 + V_0^{\frac{1}{2}}(\frac{59}{48}V_0^2 - \frac{37}{12}rV_0 + \frac{19}{12}r^2)bT^2 \\ &\quad + \frac{1}{4}V_0(r + \frac{5}{6}V_0)T^2 + V_0(\frac{11}{4}r - \frac{41}{24}V_0)b^2T - \frac{1}{6}V_0^{\frac{3}{2}}bT + V_0^{\frac{3}{2}}b^3\Big]. \end{aligned} \tag{54}$$

$$\begin{aligned} E_b\left[\int_0^T M\mathrm{d}t\right] &= V_0T\Big[\frac{1}{5}(V_0^4 - 5rV_0^3 + \frac{25}{3}r^2V_0^2 - 5r^3V_0 + \frac{2}{3}r^4)T^4 + V_0^{\frac{1}{2}}(-\frac{121}{96}V_0^3 + \frac{357}{80}rV_0^2 \\ &\quad - \frac{183}{40}r^2V_0 + \frac{13}{12}r^3)bT^3 + V_0(-\frac{7}{48}V_0^2 + \frac{2}{15}rV_0 + \frac{23}{12}r^2)T^3 + V_0(\frac{131}{48}V_0^2 \\ &\quad - \frac{403}{60}rV_0 + \frac{13}{4}r^2)b^2T^2 + V_0^{\frac{3}{2}}(\frac{1}{30}V_0 - \frac{71}{60}r)bT^2 + \frac{1}{8}V_0^2T^2 + V_0^{\frac{3}{2}}(\frac{16}{5}r - \frac{53}{20}V_0)b^3T \\ &\quad - \frac{11}{24}V_0^2b^2T + V_0^2b^4\Big]. \end{aligned} \tag{55}$$

Then the lemma is proved by substituting equation (52) to (55) into equation (51). □