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NEW TRAVELLING WAVE SOLUTIONS OF TWO NONLINEAR PHYSICAL MODELS BY USING A MODIFIED TANH-COTH METHOD

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Abstract

In this work, a modified tanh − coth method is used to derive travelling wave solutions for (2 + 1)-dimensional Zakharov-Kuznetsov (ZK) equation and (3 + 1)-dimensional Burgers equation. A new variable is used to solve these equations and established new travelling wave solutions.

Keywords: tanh-coth method; travelling wave solution; (2 + 1)-dimensional Zakharov-Kuznetsov equation; (3 + 1)-dimensional Burgers equation .

1. INTRODUCTION

The tanh − coth method is a powerful technique to solve nonlinear wave and evolution equations for travelling solutions. Nonlinear wave phenomena appears in many areas of the natural sciences, such as fluid dynamics [1], chemical kinetics involving reactions [2], population dynamics [3], solid state physics [4], etc... In the recent years, such problems has increased interest. As a result of this, a whole range of solution methods was developed [5-8] and the tanh − coth method is one of these solution methods. This technique was used by Huibin and Kelin first [8] and then developed by Malfliet and Hereman [9,12], Senthilvelan [13], Fan [14], Wazwaz [15] and others [16-19].

Generally, in the tanh − coth method, tanh function is used as a new variable, since all derivatives of tanh are represented by tanh itself. Also, Senthilvelan used tan and cot functions as a new variable [13]. As is well known, these are particular solutions of the Riccati equations $Y' = 1 - Y^2$ and $Y' = 1 + Y^2$.

One could extend the tanh method to solve nonlinear wave equations depending on more than two variables. When solving these equations, Malfliet suggested the new coordinate $\eta = kx + ly + mz - Vt$ in 3-dimensional space, where k, l, m are nonzero real numbers and $Y = \tanh \eta$ [10]. Also this was modified to $\xi = x + y + z - Vt$ and $Y = \tanh(\mu\xi)$ by Wazwaz [15], where μ is the wave number.

The Zakharov-Kuznetsov (ZK) equation is given by

$$u_t + au_x + \left(\nabla^2 u\right)_x = 0 \tag{1}$$

and it is a generalization of the KdV equation. The ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [20,21]. Wazwaz employed the tanh − coth method to solve the Zakharov-Kuznetsov equation in the (2+1) dimensions, two spatial and one time variables, and established solitary wave, travelling wave, solitons and periodic solutions [22].

(3 + 1)-dimensional integrable Burgers equation is given by

$$\begin{aligned} u_t &= u_{xx} + u_{yy} + u_{zz} + \alpha u u_y + \beta v u_x + \gamma w u_x \\ u_x &= v_y \\ u_z &= w_y \end{aligned} \tag{2}$$

where α,β and γ are nonzero constants. This class of equations may describe the flow of particles in a lattice fluid past an impenetrable obstacle [23,24] and it has applications in gas dynamics and in plasma dynamics [25]. For more details we refer the reader to [26-29] and references therein.

Wazwaz considered solutions in a moving coordinate frame, so that the PDEs considered become ODEs. Independent variable

$$Y = \tanh(\mu\xi), \qquad \xi = x + y + z - Vt \tag{3}$$

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leads to the change of derivatives

$$\frac{d}{d\xi} = \mu\left(1-Y^2\right)\frac{d}{dY} \tag{4}$$

$$\frac{d^2}{d\xi^2} = \mu^2\left(1-Y^2\right)\left(-2Y\frac{d}{dY}+\left(1-Y^2\right)\frac{d^2}{dY^2}\right) \tag{5}$$

and so on other derivatives. The tanh − coth method admits the use of a finite expansion of tanh function

$$U\left(\mu\xi\right) = S(Y) = \sum_{k=0}^{N} a_k Y^k + \sum_{k=1}^{N} b_k Y^{-k} \tag{6}$$

where M is a positive integer that will be determined by using the balancing procedure [15].

In this article, we used $Y = \frac{ke^{2\mu\xi}-1}{ke^{2\mu\xi}+1}$ instead of $Y = \tanh\left(\mu\xi\right)$ as a new variable to establish new travelling wave solutions for nonlinear two physical models (2 + 1)-dimensional Zakharov-Kuznetsov (ZK) equation and (3 +1)-dimensional Burgers equation using a modified tanh − coth method. Mathematica and Maple facilitate the tedious algebraic calculations.

2. OUTLINE OF THE METHOD

We want to investigate one or more dimensional nonlinear wave and evolution equations. This kind of equations is commonly written as

$$u_t = [u, u_x, u_{xx}, ...] \quad \text{or} \quad u_{tt} = [u, u_x, u_{xx}, ...] \tag{7}$$

A pde like (7) can be converted to an ODE

$$-V\frac{dU}{d\xi} = \left[U, \frac{dU}{d\xi}, \frac{d^2U}{d\xi^2}, ...\right] \quad \text{or} \quad V^2\frac{dU}{d\xi} = \left[U, \frac{dU}{d\xi}, \frac{d^2U}{d\xi^2}, ...\right] \tag{8}$$

upon using a wave variable $\xi = x + y + \cdots - Vt$ so that $u(x, y, ..., t) = U\left(\xi\right)$. Here V represents the velocity of the travelling wave. Eq. (8) is then integrated as long as all terms contain derivatives where integration constants are considered zeros. Introducing a new independent variable

$$Y = \frac{ke^{2\mu\xi}-1}{ke^{2\mu\xi}+1}, \quad \xi = x + y + \cdots - Vt \tag{9}$$

and the quantity $u(x, y, ..., t)$ is replaced by $U(\xi)$ leads to derivatives

$$\frac{dU}{d\xi} = \frac{4k\mu e^{2\mu\xi}}{\left(ke^{2\mu\xi}+1\right)^2} = \mu\left(1-Y^2\right)\frac{dU}{dY} \tag{10}$$

$$\frac{d^2U}{d\xi^2} = \mu^2\left(1-Y^2\right)\left(-2Y\frac{dU}{dY}+\left(1-Y^2\right)\frac{d^2U}{dY^2}\right) \tag{11}$$

$$\frac{d^3U}{d\xi^3} = 2\mu^3\left(1-Y^2\right)\left(3Y^2-1\right)\frac{dU}{dY} - 6\mu^3 Y\left(1-Y^2\right)^2\frac{d^2U}{dY^2} + \mu^3 Y\left(1-Y^2\right)^3\frac{d^3U}{dY^3} \tag{12}$$

$$\vdots$$

μ represents the wave number and it is inversely proportional to the width of the wave. Depending on the problem under study, V and μ will be determined or will remain a free and arbitrary parameters. The derivatives that obtained by us above by introducing a new independent variable in (9) are the same as those found by Hereman [9], Malfliet [12], Wazwaz [15] and others [13,14]. The tanh − coth method admits the use of the finite expansion

$$U\left(\xi\right) = S(Y) = \sum_{k=0}^{N} a_k Y^k + \sum_{k=1}^{N} b_k Y^{-k} \tag{13}$$

where N is a positive integer and $0 \leq k \leq N$. Substituting (13) into the ODE (8) results in an algebraic equation in powers of Y. To determine N, we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. We then collect all coefficients of powers of Y in the resulting equation where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters a_k and b_k $(0 \leq k \leq N)$, μ and V. Having determined these parameters we obtain an analytic solution $u(x, y, ..., t)$ in a closed form.

In the following, we will apply the described method to two examples.

3. THE ZAKRAROV-KUZNETSOV EQUATION

The (2+1)-dimensional Zakharov-Kuznetsov (ZK) equation is given by

$$u_t + auu_x + b\left(u_{xx} + u_{yy}\right)_x = 0, \tag{14}$$

where a and b are constants. Using the wave variable $\xi = x + y - Vt$ transforms the PDE (14) into the ODE

$$-VU' + \frac{a}{2}\left(U^2\right)' + 2bU''' = 0 \tag{15}$$

where by integrating (15) and neglecting the constant of integration we obtain

$$-VU + \frac{a}{2}U^2 + 2bU'' = 0. \tag{16}$$

Balancing U^2 with U'' in (16) gives $N = 2$. The tanh-coth method admits the use of the finite expansion

$$U\left(\xi\right) = S\left(Y\right) = \sum_{k=0}^{2} a_k Y^k + \sum_{k=1}^{2} b_k Y^{-k} \tag{17}$$

where $Y = \frac{ke^{2\mu\xi}-1}{ke^{2\mu\xi}+1}$. Substituting (17) into (16), collecting the coefficients of Y and setting it equal to zero we find system of equation

$$\begin{aligned}
Y^8 &: \; aa_2^2 + 24ba_2\mu^2 = 0, \\
Y^7 &: \; 8ba_1\mu^2 + 2aa_1a_2 = 0, \\
Y^6 &: \; aa_1^2 - 32ba_2\mu^2 - 2Va_2 + 2aa_0a_2 = 0, \\
Y^5 &: \; 2ab_1a_2 - 2Va_1 - 8ba_1\mu^2 + 2aa_0a_1 = 0, \\
Y^4 &: \; aa_0^2 - 2Va_0 + 8bb_2\mu^2 + 8ba_2\mu^2 + 2ab_1a_1 + 2ab_2a_2 = 0, \\
Y^3 &: \; 2ab_1a_0 - 2Vb_1 - 8bb_1\mu^2 + 2ab_2a_1 = 0, \\
Y^2 &: \; ab_1^2 - 32bb_2\mu^2 - 2Vb_2 + 2ab_2a_0 = 0, \\
Y^1 &: \; 8bb_1\mu^2 + 2ab_1b_2 = 0, \\
Y^0 &: \; ab_2^2 + 24bb_2\mu^2 = 0.
\end{aligned} \tag{18}$$

Solving this system by using Maple or Mathematica we find the following sets of solutions

$$a_0 = -\frac{16b\mu^2}{a}, \quad a_1 = 0, \quad a_2 = -\frac{24b\mu^2}{a}, \quad b_1 = 0, \quad b_2 = -\frac{24b\mu^2}{a}, \quad V = -32b\mu^2, \tag{19}$$

$$a_0 = \frac{48b\mu^2}{a}, \quad a_1 = 0, \quad a_2 = -\frac{24b\mu^2}{a}, \quad b_1 = 0, \quad b_2 = -\frac{24b\mu^2}{a}, \quad V = 32b\mu^2, \tag{20}$$

$$a_0 = \frac{8b\mu^2}{a}, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = -\frac{24b\mu^2}{a}, \quad V = -8b\mu^2, \tag{21}$$

$$a_0 = \frac{24b\mu^2}{a}, \quad a_1 = 0, \quad a_2 = 0, \quad b_1 = 0, \quad b_2 = -\frac{24b\mu^2}{a}, \quad V = 8b\mu^2, \tag{22}$$

$$a_0 = \frac{8b\mu^2}{a}, \quad a_1 = 0, \quad a_2 = -\frac{24b\mu^2}{a}, \quad b_1 = 0, \quad b_2 = 0, \quad V = -8b\mu^2, \tag{23}$$

$$a_0 = \frac{24b\mu^2}{a}, \quad a_1 = 0, \quad a_2 = -\frac{24b\mu^2}{a}, \quad b_1 = 0, \quad b_2 = 0, \quad V = 8b\mu^2, \tag{24}$$

where μ is left as a free parameter. These sets give the solutions respectively

$$u_1 = -\frac{16b\mu^2}{a} - \frac{24b\mu^2}{a}\left(\frac{ke^{2\mu\left(x+y+32b\mu^2t\right)}-1}{ke^{2\mu(x+y+32b\mu^2t)}+1}\right)^2 - \frac{24b\mu^2}{a}\left(\frac{ke^{2\mu\left(x+y+32b\mu^2t\right)}+1}{ke^{2\mu(x+y+32b\mu^2t)}-1}\right)^2, \tag{25}$$

$$u_2 = \frac{48b\mu^2}{a} - \frac{24b\mu^2}{a}\left(\frac{ke^{2\mu\left(x+y-32b\mu^2t\right)}-1}{ke^{2\mu(x+y-32b\mu^2t)}+1}\right)^2 - \frac{24b\mu^2}{a}\left(\frac{ke^{2\mu\left(x+y-32b\mu^2t\right)}+1}{ke^{2\mu(x+y-32b\mu^2t)}-1}\right)^2, \tag{26}$$

$$u_3 = \frac{8b\mu^2}{a} - \frac{24b\mu^2}{a}\left(\frac{ke^{2\mu\left(x+y+8b\mu^2t\right)}+1}{ke^{2\mu(x+y+8b\mu^2t)}-1}\right)^2, \tag{27}$$

$$u_4 = \frac{24b\mu^2}{a} - \frac{24b\mu^2}{a}\left(\frac{ke^{2\mu\left(x+y-8b\mu^2t\right)}+1}{ke^{2\mu(x+y-8b\mu^2t)}-1}\right)^2, \tag{28}$$

$$u_5 = \frac{8b\mu^2}{a} - \frac{24b\mu^2}{a}\left(\frac{ke^{2\mu\left(x+y+8b\mu^2t\right)}-1}{ke^{2\mu(x+y+8b\mu^2t)}+1}\right)^2, \tag{29}$$

$$u_6 = \frac{24b\mu^2}{a} - \frac{24b\mu^2}{a}\left(\frac{ke^{2\mu\left(x+y-8b\mu^2t\right)}-1}{ke^{2\mu(x+y-8b\mu^2t)}+1}\right)^2. \tag{30}$$

The (2+1)-dimensional Zakharov-Kuznetsov (ZK) equation was solved by Wazwaz [15] using the tanh-coth method and he obtained solutions

$$u_1(x,y,t) = \frac{3V}{a}\operatorname{sech}^2\left[\frac{1}{6}\sqrt{\frac{3V}{b}}(x+y-Vt)\right], \quad \frac{V}{b}>0, \tag{31}$$

$$u_2(x,y,t) = -\frac{V}{a}\left\{1-3\tanh^2\left[\frac{1}{6}\sqrt{-\frac{3V}{b}}(x+y-Vt)\right]\right\}, \quad \frac{V}{b}<0, \tag{32}$$

$$u_3(x,y,t) = -\frac{V}{a}\left\{1-3\coth^2\left[\frac{1}{6}\sqrt{\frac{3V}{b}}(x+y-Vt)\right]\right\}, \quad \frac{V}{b}>0, \tag{33}$$

$$u_4(x,y,t) = \frac{3V}{4a}\left\{2+\tanh^2\left[\frac{1}{12}\sqrt{-\frac{3V}{b}}(x+y-Vt)\right]+\coth^2\left[\frac{1}{12}\sqrt{-\frac{3V}{b}}(x+y-Vt)\right]\right\}, \quad \frac{V}{b}<0, \tag{34}$$

$$u_5(x,y,t) = \frac{3V}{4a}\left\{2-\tanh^2\left[\frac{1}{12}\sqrt{\frac{3V}{b}}(x+y-Vt)\right]-\coth^2\left[\frac{1}{12}\sqrt{\frac{3V}{b}}(x+y-Vt)\right]\right\}, \quad \frac{V}{b}>0, \tag{35}$$

$$u_6(x,y,t) = \frac{3V}{2a}\left\{1-2\coth^2\left[\frac{1}{6}\sqrt{\frac{3V}{b}}(x+y-Vt)\right]\right\}, \quad \frac{V}{b}>0. \tag{36}$$

These results can be obtained by setting $k=1$ in (25-30). So comparing our results with Wazwaz's results, it can be seen easily that our solutions are more general.

4. (3+1) DIMENSIONAL BURGERS EQUATION

Now we will apply the tanh − coth method to the (3+1)-dimensional Burgers equation:

$$\begin{aligned} u_t &= u_{xx}+u_{yy}+u_{zz}+\alpha uu_y+\beta vu_x+\gamma wu_x \\ u_x &= v_y \\ u_z &= w_y \end{aligned} \tag{37}$$

where α, β, θ are nonzero constants. Using the wave variable $\xi = x+y+z-Vt$ transforms the system (37) into a system of ODEs given by

$$\begin{aligned} VU'+\alpha UU'+\beta VU'+\gamma WU'+3U'' &= 0 \\ U' &= V' \\ U' &= W'. \end{aligned} \tag{38}$$

Integrating (38) and neglecting the constant of integration we obtain

$$U = V = W. \tag{39}$$

So the first equation in (38) can be written as

$$VU' + \frac{\alpha+\beta+\gamma}{2}\left(U^2\right)' + 3U'' = 0. \tag{40}$$

Integrating (40) and neglecting the constant of integration again we obtain

$$VU + \frac{\alpha+\beta+\gamma}{2}U^2 + 3U' = 0. \tag{41}$$

Balancing U^2 with U' in (41) gives $N = 1$. The $\tanh - \coth$ method admits the use of the finite expansion

$$U(\xi) = S(Y) = \sum_{k=0}^{1} a_k Y^k + \sum_{k=1}^{1} b_k Y^{-k} \tag{42}$$

where $Y = \frac{ke^{2\mu\xi}-1}{ke^{2\mu\xi}+1}$. Substituting (42) into (41), we obtain

$$V\left(a_0 + a_1 Y + b_1\frac{1}{Y}\right) + \frac{\alpha+\beta+\gamma}{2}\left(a_0 + a_1 Y + b_1\frac{1}{Y}\right)^2 + 3\mu\left(1-Y^2\right)\frac{d}{dY}\left(a_0 + a_1 Y + b_1\frac{1}{Y}\right) = 0. \tag{43}$$

Collecting the coefficients of Y and setting it equal to zero we find system of equation

$$\begin{array}{lcl} Y^4 & : & a_1^2\alpha - 6a_1\mu + a_1^2\beta + a_1^2\gamma = 0 \\ Y^3 & : & 2Va_1 + 2a_0a_1\alpha + 2a_0a_1\beta + 2a_0a_1\gamma = 0 \\ Y^2 & : & 6a_1\mu + 6b_1\mu + a_0^2\alpha + a_0^2\beta + a_0^2\gamma + 2Va_0 + 2a_1b_1\alpha + 2a_1b_1\beta + 2a_1b_1\gamma = 0 \\ Y^1 & : & 2Vb_1 + 2a_0b_1\alpha + 2a_0b_1\beta + 2a_0b_1\gamma = 0 \\ Y^0 & : & b_1^2\alpha - 6b_1\mu + b_1^2\beta + b_1^2\gamma = 0 \end{array} \tag{44}$$

and solving this system we find the following sets of solutions

$$a_0 = \frac{12\mu}{\alpha+\beta+\gamma}, \quad a_1 = b_1 = \frac{6\mu}{\alpha+\beta+\gamma}, \quad V = -12\mu, \tag{45}$$

$$a_0 = \frac{-12\mu}{\alpha+\beta+\gamma}, \quad a_1 = b_1 = \frac{6\mu}{\alpha+\beta+\gamma}, \quad V = 12\mu, \tag{46}$$

$$a_0 = a_1 = \frac{6\mu}{\alpha+\beta+\gamma}, \quad b_1 = 0, \quad V = 6\mu, \tag{47}$$

$$a_0 = b_1 = \frac{6\mu}{\alpha+\beta+\gamma}, \quad a_1 = 0, \quad V = -6\mu\ , \tag{48}$$

$$a_0 = \frac{-6\mu}{\alpha+\beta+\gamma}, \quad a_1 = 0, b_1 = \frac{6\mu}{\alpha+\beta+\gamma}, \quad V = 6\mu, \tag{49}$$

$$a_0 = \frac{6\mu}{\alpha+\beta+\gamma}, \quad a_1 = 0, b_1 = \frac{6\mu}{\alpha+\beta+\gamma}, \quad V = -6\mu, \tag{50}$$

where μ is left as a free parameter. These sets give the solutions respectively

$$u_1 = \frac{6\mu}{\alpha+\beta+\gamma}\left(2 + \frac{ke^{2\mu(x+y+z+12\mu t)}-1}{ke^{2\mu(x+y+z+12\mu t)}+1} + \frac{ke^{2\mu(x+y+z+12\mu t)}+1}{ke^{2\mu(x+y+z+12\mu t)}-1}\right), \tag{51}$$

$$u_2 = \frac{6\mu}{\alpha+\beta+\gamma}\left(-2 + \frac{ke^{2\mu(x+y+z-12\mu t)}-1}{ke^{2\mu(x+y+z-12\mu t)}+1} + \frac{ke^{2\mu(x+y+z-12\mu t)}+1}{ke^{2\mu(x+y+z-12\mu t)}-1}\right), \tag{52}$$

$$u_3 = \frac{6\mu}{\alpha+\beta+\gamma}\left(1 + \frac{ke^{2\mu(x+y+z-6\mu t)}-1}{ke^{2\mu(x+y+z-6\mu t)}+1}\right), \tag{53}$$

$$u_4 = \frac{6\mu}{\alpha+\beta+\gamma}\left(1 + \frac{ke^{2\mu(x+y+z+6\mu t)}+1}{ke^{2\mu(x+y+z+6\mu t)}-1}\right), \tag{54}$$

$$u_5 = \frac{6\mu}{\alpha+\beta+\gamma}\left(-1 + \frac{ke^{2\mu(x+y+z-6\mu t)}+1}{ke^{2\mu(x+y+z-6\mu t)}-1}\right), \tag{55}$$

$$u_6 = \frac{6\mu}{\alpha+\beta+\gamma}\left(1 + \frac{ke^{2\mu(x+y+z+6\mu t)}+1}{ke^{2\mu(x+y+z+6\mu t)}-1}\right). \tag{56}$$

In [15], Wazwaz investigated the (3 + 1)-dimensional Burgers equation and obtained single kink solutions by using the $\tanh - \coth$ method as follows :

$$\begin{aligned} u_{1,2} &= \pm\lambda\,[1 \pm \tanh\mu\,(x + y + z \mp \lambda\,(\alpha + \beta + \gamma)\,t)] & (57) \\ u_{3,4} &= \pm\lambda\,[1 \pm \coth\mu\,(x + y + z \mp \lambda\,(\alpha + \beta + \gamma)\,t)] & (58) \\ u_{5,6} &= \pm\lambda\,[1 \pm \tanh\mu\,(x + y + z \mp 2\lambda\,(\alpha + \beta + \gamma)\,t)] \pm \lambda\,[1 \pm \coth\mu\,(x + y + z \mp 2\lambda\,(\alpha + \beta + \gamma)\,t)] & (59) \end{aligned}$$

As can be seen easily, if one set $k = 1$ and $\lambda = \frac{6\mu}{\alpha+\beta+\gamma}$ in (51-56) then the solutions in (57-59) are obtained. For instance, if we consider $\alpha + \beta + \gamma = 6$, $\mu = 1$ in (57) we find $\lambda = 1$ and $u_1 = 1 + \tanh(x + y + z - 6t)$. For the same values, in (53) we obtain $u_3 = 1 + \frac{ke^{2(x+y+z-6t)}-1}{ke^{2(x+y+z-6t)}+1}$. Using the fact that $\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$, we can say for $k = 1$ these are the same solutions. However, different solutions will be obtained for different k values.

5. CONCLUSION

The (2+1)-dimensional Zakharov-Kuznetsov (ZK) equation and the (3 + 1)-dimensional Burgers equation have been investigated. $Y = \frac{ke^{2\mu\xi}-1}{ke^{2\mu\xi}+1}$ was introduced as a new variable to obtain travelling wave solutions of these equations. With the help of this ansatz can be obtained exact solutions of many other equations.

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