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Decay of bound states in a sine-Gordon equation with double-well potentials

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We consider a spatially inhomogeneous sine-Gordon equation with a double-well potential, describing long Josephson junctions with phase-shifts. We discuss the interactions of symmetric and antisymmetric bound states in the system. Using a multiple scale expansion, we show that the modes decay algebraically in time due to the energy transfer from the discrete to the continuous spectrum. In particular, exciting the two modes at the same time yields an increased decay rate. An external time-periodic drive is shown to sustain symmetric state, while it damps the antisymmetric one. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4917284]

I. INTRODUCTION

We consider the inhomogeneous sine-Gordon equation

$$\phi_{xx}(x,t) - \phi_{tt}(x,t) = \sin\left(\phi(x,t) + \theta(x)\right) + h\cos(\Omega t), \tag{1}$$

which is driven by a time-periodic drive with amplitude h and frequency Ω, which is below the plane wave cut-off frequency $\Omega^2 < 1$. The spatial and temporal coordinates are $x \in \mathbb{R}$ and $t > 0$. Here, $\phi(x,t)$ is a one-dimensional wave field. The internal phase shift is taken as

$$\theta(x) = \begin{cases} 0, & |x| > L + a, \\ \pi, & L < |x| < L + a, \\ 0, & 0 < |x| < L, \end{cases} \tag{2}$$

which introduces a double well potential in the system. At the position of the jumps in the phase, the field ϕ is naturally subject to the continuity conditions

$$\lim_{x\to\pm\{L,L+a\}^+} \phi(x,t) = \lim_{x\to\pm\{L,L+a\}^-} \phi(x,t), \tag{3}$$

$$\lim_{x\to\pm\{L,L+a\}^+} \phi_x(x,t) = \lim_{x\to\pm\{L,L+a\}^-} \phi_x(x,t). \tag{4}$$

Equation (1) with (2) models the quantum phase difference between two superconductors in the so-called Josephson junction systems with phase-shifts, which have been successfully fabricated experimentally, e.g., by installing magnetic impurities[19] or Abrikosov vortices,[20] using multilayer junctions with controlled thicknesses over the ferromagnetic barrier,[21] pairs of current injectors,[22] and junctions with unconventional order parameter symmetry.[2,23–25]

Due to the double-well potential (2), Eq. (1) with $h = 0$ admits bound states or localized modes. Depending on the parameters, multiple bound states may exist.[18] Interesting features of localized

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mode interactions in the system, such as mode tunneling and envelope solitons, have been considered in Ref. 18, where the mode dynamics were described by two dynamic variables using a two mode approximation. However, the nonlinear coupling of the bound states and extended modes, which yields energy transfer from the localized modes to the continuous spectrum, is removed by approximation. Such a transfer in Eq. (1) with a single-well potential was discussed in Ref. 1. The energy transfer causes dissipation from the bound state and makes it decay at a rate of at most $O\left(t^{-1/4}\right)$ or $O\left(t^{-1/2}\right)$, depending on the ground state of the system (note that Eq. (1) can have a nonuniform ground state when the trivial state is unstable[18]).

Localized mode dispersions in Klein-Gordon-type equations were first discussed in Ref. 16. A similar dispersion was also reported in the study of wobbling kinks of a ϕ^4-equation.[4] Because of the integrability, wobbling kinks in the sine-Gordon equation do not radiate continuous waves.[7–9,15] Radiation does not exist in the integrable system due to the absence of internal modes.[6,10] Radiation damping due to resonances in discrete Klein-Gordon equations was discussed in Ref. 11. However, in most of the previous studies, there is only one eigenvalue causing resonances with the continuous waves. Kirr and Weinstein[13] obtained dispersive estimates of nonlinear Klein-Gordon equations with a number of localized modes, which were extended by Bambusi and Cuccagna[3] to higher dimensions.

In this paper, we extend our previous work[1] and study the decay rate of two localized modes of the trivial ground state, in the same spirit as Refs. 3 and 13. However, it is important to mention that our governing equation (1) does not immediately belong to the class of equations considered before. Using multiple scale expansions, we obtain the decay rate of the multiple bound states. In particular, we show that interactions of the modes will increase the decay of the bound states.

In addition to the decay of freely oscillating, i.e., undriven, localised modes, we also consider the influence of an external drive to the system. Previously, ac-driven localised modes in Klein-Gordon equations were considered in, e.g., Refs. 13, 14, and 17. It was shown that a direct ac-drive sustains localized modes,[14] while parametric drives destroy them.[13,17] As for the governing equation (1) studied herein, the direct ac-drive is motivated by the experimental work in Refs. 5 and 12, where the drive is a magnetic field radiation applied to Josephson junctions with phase-shifts. In single potentials, such a drive was previously considered in Ref. 1. Here, we will show that the drive sustains the first bound state, but dampens the second.

The paper is outlined as follows. In Sec. II, we construct a perturbation expansion to solve the undriven sine-Gordon equation for the localized modes to obtain equations for the slow time evolution of oscillation amplitude in the system. In Sec. III, the method of multiple scales is applied to obtain the amplitude of oscillation in the presence of driving. Section IV compares the analytical results obtained in Secs. II and III with numerical computations, which confirm our asymptotic calculations. Algebraic expressions of the functions appearing in Secs. II–III are listed in Appendices A and B.

II. FREELY OSCILLATING BREATHING MODES

In this section, we consider the dynamics of long Josephson junctions governed by the sine-Gordon equation (1), with $h = 0$, and $\theta(x)$ given by (2), which represents a double well potential comprising two so-called π-junctions of length a separated by a 0–junction of length $2L$. We apply a perturbation expansion to Eq. (1) by writing

$$\phi = \phi_0 + \epsilon\,\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \cdots, \tag{5}$$

where ϵ is a small parameter. Furthermore, we use multiple scales expansions by introducing the slow time and space variables

$$X_n = \epsilon^n x, \quad T_n = \epsilon^n t, \quad n = 0,1,2,\ldots. \tag{6}$$

We use the notation

$$\partial_n = \frac{\partial}{\partial X_n}, \quad D_n = \frac{\partial}{\partial T_n}, \tag{7}$$

so that the derivatives with respect to the original variables in terms of the scaled variables using the chain rule are given by

$$\frac{\partial}{\partial x} = \partial_0 + \epsilon\,\partial_1 + \epsilon^2\partial_2 + \epsilon^3\partial_3 + \cdots, \tag{8}$$

$$\frac{\partial}{\partial t} = D_0 + \epsilon\,D_1 + \epsilon^2 D_2 + \epsilon^3 D_3 + \cdots. \tag{9}$$

Inserting (6) into Eq. (1) and equating like powers of ϵ, we find a system of partial differential equations for the functions of the time and space variables X_n, T_n.

Due to the symmetry of the functions, i.e., they are either spatially symmetric or antisymmetric, in the following, we will denote them in the positive region only for the sake of presentation.

A. Leading order correction equations

At leading orders, we obtain

$$O(1) : \partial_0^2\phi_0 - D_0^2\phi_0 = \sin(\theta + \phi_0), \tag{10}$$

$$O(\epsilon) : \partial_0^2\phi_1 - D_0^2\phi_1 = \cos(\theta + \phi_0)\phi_1 + 2D_0D_1\phi_0 - 2\partial_0\partial_1\phi_0. \tag{11}$$

A stable solution representing a uniform background for Eq. (10) is

$$\phi_0(X_0, T_0) = 0, \tag{12}$$

while the solution of Eq. (11) is given by

$$\phi_1 = B_1\Phi_1(X_0)e^{i\,\lambda_1 T_0} + B_2\Phi_2(X_0)e^{i\,\lambda_2 T_0} + c.c., \tag{13}$$

where $B_1 = B_1(T_1, T_2, \ldots)$, $B_2 = B_2(T_1, T_2, \ldots)$ are as yet unknown time-dependent complex amplitudes of oscillation and throughout the proceeding work $c.c.$ stands for the complex conjugate of the preceding terms. By linearizing equation (1) around the uniform solution (12), we find that bounded solutions satisfying the boundary conditions (3) and (4) are[18]

$$\Phi_1(X_0) = \begin{cases} e^{-\sqrt{1-\lambda_1^2}(X_0-L-a)}, & X_0 > L + a, \\ \cos\left(\sqrt{1+\lambda_1^2}(X_0 - L - a)\right) + C_1 \sin\left(\sqrt{1+\lambda_1^2}(X_0 - L - a)\right), & L < X_0 < L + a, \\ K_1 \cosh\left(\sqrt{1-\lambda_1^2}\, X_0\right), & 0 < X_0 < L, \end{cases} \tag{14}$$

$$\Phi_2(X_0) = \begin{cases} e^{-\sqrt{1-\lambda_2^2}(X_0-L-a)}, & X_0 > L + a, \\ \cos\left(\sqrt{1+\lambda_2^2}(X_0 - L - a)\right) + C_2 \sin\left(\sqrt{1+\lambda_2^2}(X_0 - L - a)\right), & L < X_0 < L + a, \\ K_2 \sinh\left(\sqrt{1-\lambda_2^2}\, X_0\right), & 0 < X_0 < L, \end{cases} \tag{15}$$

with the oscillation frequencies λ_1 and λ_2, satisfying

$$\lambda_j^2 = \frac{\sqrt{1-\lambda_j^4}}{\tan\left(\sqrt{1+\lambda_j^2}\, a\right)} \mp e^{-2\sqrt{1-\lambda_j^2}L}, \tag{16}$$

$$K_j = \frac{2\,e^{-\sqrt{1-\lambda_j^2}L}\sin\left(\sqrt{1+\lambda_j^2}\, a\right)}{\sqrt{1-\lambda_j^4}}, \quad C_j = -\sqrt{\frac{1-\lambda_j^2}{1+\lambda_j^2}}, \tag{17}$$

for $j = 1, 2$. The two eigenvalues λ_j are functions of a and L, and $\Phi_{1,2}$, given by (14) and (15), satisfy the conditions, $\Phi_1(-X_0) = \Phi_1(X_0)$ and $\Phi_2(-X_0) = -\Phi_2(X_0)$. To derive an effective equation for the complex mode amplitudes B_1, B_2, we continue the perturbation expansion order by order and proceed to find the solvability conditions for the obtained equations.

B. Equations at $O(\epsilon^2)$

The terms at order $O(\epsilon^2)$ give

$$\partial_0^2\phi_2 - D_0^2\phi_2 - \cos(\theta)\phi_2 = 2D_0D_1\phi_1 - 2\partial_0\partial_1\phi_1. \tag{18}$$

Substituting the spectral ansatz

$$\phi_2 = \widetilde{\phi}_{21}(X_0)e^{i\lambda_1 T_0} + c.c. + \widetilde{\phi}_{22}(X_0)e^{i\lambda_2 T_0} + c.c., \tag{19}$$

we obtain the corresponding set of ordinary differential equations

$$\partial_0^2\widetilde{\phi}_{21} - (\cos(\theta) - \lambda_1^2)\widetilde{\phi}_{21} = 2\,i\lambda_1 D_1 B_1 \Phi_1, \tag{20}$$

$$\partial_0^2\widetilde{\phi}_{22} - (\cos(\theta) - \lambda_2^2)\widetilde{\phi}_{22} = 2\,i\lambda_2 D_1 B_2 \Phi_2. \tag{21}$$

To find a bounded solution for $\widetilde{\phi}_{21}$, $\widetilde{\phi}_{22}$, Eqs. (20) and (21) generate constraints on the right hand sides that are solvability conditions which lead to an important equation for the amplitudes B_1, B_2 as well as to equations at higher order when the expansion is continued further. We write (20) and (21) as $\mathrm{T}\psi\,(X_0) = f\,(X_0)$ where T is a self-adjoint operator ($\mathrm{T} = \mathrm{T}^\dagger$) given by the left hand side of the above system. The necessary and sufficient conditions for Eqs. (20) and (21) to be solvable are then that the right hand side be orthogonal to the null space of the corresponding adjoint operator, where the inner product is defined as

$$\langle a, b\rangle = \int_{-\infty}^{\infty} a(\xi)b(\xi)dx. \tag{22}$$

The span of this space can be written as ψ where

$$\mathrm{T}\psi\,(X_0) = 0. \tag{23}$$

By applying the theorem, known as the Fredholm alternative, we find that the solvability conditions for the above system are

$$D_1B_1 = 0, \quad D_1B_2 = 0. \tag{24}$$

Hence, B_j are independent of T_1.

By putting the solvability conditions (24) in Eqs. (20) and (21), we obtain the result which is similar to that at $O(\epsilon)$, that is, Eq. (11). Due to uniformity in the perturbation expansion, we conclude that $\phi_2(X_0, T_0) = 0$.

C. Equations at $O(\epsilon^3)$

Equating terms at $O(\epsilon^3)$, we obtain an equation of the form

$$\partial_0^2\phi_3 - D_0^2\phi_3 - \cos(\theta + \phi_0)\phi_3 = 2(D_0D_2 - \partial_0\partial_2)\phi_1 + (D_1^2 - \partial_1^2)\phi_1 - \frac{1}{6}\phi_1^3\cos(\theta). \tag{25}$$

Calculating the right hand side of Eq. (25), we obtain

$$\begin{aligned}\partial_0^2\phi_3 - D_0^2\phi_3 - \cos(\theta + \phi_0)\phi_3 \;=\;& 2\,i\lambda_1 D_2 B_1 \Phi_1 e^{i\lambda_1 T_0} + 2\,i\lambda_2 D_2 B_2 \Phi_2 e^{i\lambda_2 T_0} - \frac{1}{6}\Big[B_1^3\Phi_1^3 e^{3\,i\lambda_1 T_0} + B_2^3\Phi_2^3 e^{3\,i\lambda_2 T_0}\\ &+3\,B_1|B_1|^2\Phi_1^3 e^{i\lambda_1 T_0} + 3\,B_2|B_2|^2\Phi_2^3 e^{i\lambda_2 T_0} + 3\,B_1^2\overline{B}_2\,\Phi_1^2\,\Phi_2 e^{(2\,\lambda_1-\lambda_2)T_0\,i} + 3\,\overline{B}_1 B_2^2\Phi_1\Phi_2^2 e^{(2\,\lambda_2-\lambda_1)T_0\,i}\\ &+6\,|B_1|^2 B_2\Phi_1^2\Phi_2 e^{\lambda_2 T_0\,i} + 6\,B_1|B_2|^2\Phi_1\Phi_2^2 e^{\lambda_1 T_0\,i} + 3\,B_1^2 B_2\Phi_1^2\Phi_2 e^{(2\,\lambda_1+\lambda_2)T_0\,i} + 3\,B_1 B_2^2\Phi_1\Phi_2^2 e^{(2\,\lambda_2+\lambda_1)T_0\,i}\Big]\cos(\theta)\\ &+c.c.\end{aligned} \tag{26}$$

Equation (26) is linear, so its solution can be written as the linear combination of solutions with frequencies present in the forcing terms; therefore, the solution will consist of the harmonics present in Eq. (26), that is

$$\phi_3 = \phi_{311}e^{i\lambda_1 T_0} + \phi_{312}e^{i\lambda_2 T_0} + \phi_{321}e^{(2\lambda_1+\lambda_2)T_0\, i} + \phi_{322}e^{(2\lambda_1-\lambda_2)T_0\, i} + \phi_{331}e^{3i\,\lambda_1 T_0} + \phi_{332}e^{3i\,\lambda_2 T_0} + \phi_{341}e^{(2\,\lambda_2+\lambda_1)T_0\, i} + \phi_{342}e^{(2\,\lambda_2-\lambda_1)T_0\, i} + c.c. \tag{27}$$

The functions ϕ_{311}, ϕ_{312} are functions of the space variable X_0 which satisfy the following linear inhomogeneous equations:

$$\left[\partial_0^2 - \cos(\theta) + \lambda_1^2\right]\phi_{311} = \begin{cases} E_1, & X_0 > L + a, \\ E_2, & L < X_0 < L + a, \\ E_3, & 0 < X_0 < L, \end{cases} \tag{28}$$

$$\left[\partial_0^2 - \cos(\theta) + \lambda_2^2\right]\phi_{312} = \begin{cases} F_1, & X_0 > L + a, \\ F_2, & L < X_0 < L + a, \\ F_3, & 0 < X_0 < L, \end{cases} \tag{29}$$

with

$$E_1 = 2\,i\lambda_1 D_2 B_1\,\Phi_1 - \frac{1}{2}B_1|B_1|^2\,\Phi_1^3 - B_1|B_2|^2\,\Phi_1\Phi_2^2,$$
$$E_2 = 2\,i\lambda_1 D_2 B_1\,\Phi_1 + \frac{1}{2}B_1|B_1|^2\,\Phi_1^3 + B_1|B_2|^2\,\Phi_1\Phi_2^2,$$
$$E_3 = 2\,i\lambda_1 D_2 B_1\,\Phi_1 - \frac{1}{2}B_1|B_1|^2\,\Phi_1^3 - B_1|B_2|^2\,\Phi_1\Phi_2^2,$$
$$F_1 = 2\,i\lambda_2 D_2 B_2\,\Phi_2 - \frac{1}{2}B_2|B_2|^2\,\Phi_2^3 - B_2|B_1|^2\,\Phi_2\Phi_2^1,$$
$$F_2 = 2\,i\lambda_2 D_2 B_2\,\Phi_2 + \frac{1}{2}B_2|B_2|^2\,\Phi_2^3 + B_2|B_1|^2\,\Phi_2\Phi_2^1,$$
$$F_3 = 2\,i\lambda_2 D_2 B_2\,\Phi_2 - \frac{1}{2}B_2|B_2|^2\,\Phi_2^3 - B_2|B_1|^2\,\Phi_2\Phi_2^1.$$

The homogeneous solutions of these equations are given by the eigenfunctions (14) and (15). Using the Fredholm alternative, the solvability conditions for (28) and (29) are

$$D_2 B_1 = \alpha_1\,B_1|B_1|^2\,i + \alpha_2\,B_1|B_2|^2\,i, \tag{30}$$
$$D_2 B_2 = \alpha_3\,B_2|B_2|^2\,i + \alpha_4\,B_2|B_1|^2\,i, \tag{31}$$

where

$$\alpha_1 = -\frac{p_2}{p_1}, \quad \alpha_2 = -\frac{p_3}{p_1}, \quad \alpha_3 = -\frac{p_5}{p_4}, \quad \alpha_4 = -\frac{p_6}{p_4}, \tag{32}$$

and the quantities p_j $(1 \le j \le 6)$ are given in Appendix A. Putting the conditions (30) and (31) into (28) and (29), respectively, and solving these equations, we obtain a bounded solution of the form

$$\phi_{311} = B_1 \begin{cases} \psi_1, & X_0 > L + a, \\ \psi_2, & L < X_0 < L + a, \\ \psi_3, & 0 < X_0 < L, \end{cases} \tag{33}$$

$$\phi_{312} = B_2 \begin{cases} \psi_4, & X_0 > L + a, \\ \psi_5, & L < X_0 < L + a, \\ \psi_6, & 0 < X_0 < L, \end{cases} \tag{34}$$

where ψ_j, for $j = 1, 2, .., 6$, can be seen in Appendix A. To obtain the final amplitude equations, we have to find bounded solutions for other harmonics present in (26), as these will appear in the next stage. To do this, we assume that

$$(3\lambda_1)^2 > 1, \tag{35}$$

i.e., the third harmonics lie in the continuous (phonon) spectrum. For $\lambda_2 > \lambda_1$ and with assumption (35), we have

$$(2\lambda_1 + \lambda_2)^2 > 1, \qquad (2\lambda_2 + \lambda_1)^2 > 1, \tag{36}$$

which also lie in the continuous spectrum. The equations for the harmonics $(2\lambda_1 + \lambda_2)$, $(2\lambda_1 - \lambda_2)$ are

$$\partial_0^2\phi_{321} + (2\lambda_1 + \lambda_2)^2\phi_{321} - \cos(\theta + \phi_0)\phi_{321} = -\frac{1}{2} B_1^2 B_2 \Phi_1^2 \Phi_2 \cos(\theta), \tag{37}$$

$$\partial_0^2\phi_{322} + (2\lambda_1 - \lambda_2)^2\phi_{322} - \cos(\theta + \phi_0)\phi_{322} = -\frac{1}{2} B_1^2 \overline{B}_2 \Phi_1^2 \Phi_2 \cos(\theta), \tag{38}$$

with bounded solutions

$$\phi_{321} = B_1^2 B_2 \begin{cases} \psi_7, & X_0 > L + a, \\ \psi_8, & L < X_0 < L + a, \\ \psi_9, & 0 < X_0 < L, \end{cases} \tag{39}$$

$$\phi_{322} = B_1^2 \overline{B_2} \begin{cases} \psi_{10}, & X_0 > L + a, \\ \psi_{11}, & L < X_0 < L + a, \\ \psi_{12}, & 0 < X_0 < L, \end{cases} \tag{40}$$

where $\psi_7, \ldots, \psi_{12}$ are given in Appendix A. The equations for the third harmonics are

$$\partial_0^2\phi_{331} + 9\lambda_1^2\phi_{331} - \cos(\theta + \phi_0)\phi_{331} = -\frac{1}{6} B_1^3 \Phi_1^3 \cos(\theta),$$

$$\partial_0^2\phi_{332} + 9\lambda_2^2\phi_{332} - \cos(\theta + \phi_0)\phi_{332} = -\frac{1}{6} B_2^3 \Phi_2^3 \cos(\theta),$$

with solutions

$$\phi_{331} = B_1^3 \begin{cases} \psi_{13}, & X_0 > L + a, \\ \psi_{14}, & L < X_0 < L + a, \\ \psi_{15}, & 0 < X_0 < L, \end{cases} \tag{41}$$

$$\phi_{332} = B_2^3 \begin{cases} \psi_{16}, & X_0 > L + a, \\ \psi_{17}, & L < X_0 < L + a, \\ \psi_{18}, & 0 < X_0 < L, \end{cases} \tag{42}$$

where $\psi_{13}, \ldots, \psi_{18}$ are given in Appendix A. The equations for the harmonics $(\lambda_1 + 2\lambda_2)$, $(\lambda_1 - 2\lambda_2)$ are

$$\partial_0^2\phi_{341} + (\lambda_1 + 2\lambda_2)^2\phi_{341} - \cos(\theta + \phi_0)\phi_{341} = -\frac{1}{2} B_1 B_2^2 \Phi_1 \Phi_2^2 \cos(\theta), \tag{43}$$

$$\partial_0^2\phi_{342} + (\lambda_1 - 2\lambda_2)^2\phi_{342} - \cos(\theta + \phi_0)\phi_{342} = -\frac{1}{2} \overline{B}_1 B_2^2 \Phi_1 \Phi_2^2 \cos(\theta), \tag{44}$$

with solutions

$$\phi_{341} = B_1 B_2^2 \begin{cases} \psi_{19}, & X_0 > L + a, \\ \psi_{20}, & L < X_0 < L + a, \\ \psi_{21}, & 0 < X_0 < L, \end{cases} \tag{45}$$

$$\phi_{342} = B_2^2 \overline{B_1} \begin{cases} \psi_{22}, & X_0 > L + a, \\ \psi_{23}, & L < X_0 < L + a, \\ \psi_{24}, & 0 < X_0 < L. \end{cases} \tag{46}$$

The functions $\psi_{19}, \ldots, \psi_{24}$ are given in Appendix A. With the assumption (35), i.e., $\lambda_1 > 1/3$, we see that solutions ϕ_{321}, ϕ_{311}, ϕ_{332}, ϕ_{341} describe the right moving radiation in $X_0 > L + a$ and left moving radiation in $X_0 < L$, which are responsible for the energy loss in the final amplitude equations.

D. Equation at $O(\epsilon^4)$

The terms at order $O(\epsilon^4)$ give

$$D_0{}^2\phi_4 - \partial_0{}^2\phi_4 - \cos(\theta)\,\phi_4 = 2\,(D_1D_2 - \partial_1\partial_2)\,\phi_1 + 2\,(D_0D_3 - \partial_0\partial_3)\,\phi_1 + 2\,(D_0D_1 - \partial_0\partial_1)\,\phi_3 + \left(\frac{1}{24}\phi_1^4 - \phi_3\phi_1\right)\sin(\theta)\,. \tag{47}$$

Using the same procedure as in Sec. II B, the solvability conditions for the above equations are

$$D_3B_1 = 0, \quad D_3B_2 = 0, \tag{48}$$

and hence we impose that $\phi_4 = 0$, as we did for ϕ_2. This implies that $B_j = B_j(T_2, T_4, \ldots)$ are independent of T_3 for $j = 1, 2$.

E. Equation at (ϵ^5)

Equating terms at $O(\epsilon^5)$ gives the equation

$$\partial_0{}^2\phi_5 - D_0{}^2\phi_5 - \cos(\theta)\phi_5 = 2(D_0D_4 - \partial_0\partial_4)\phi_1 + 2(D_3D_1 - \partial_3\partial_1)\phi_1 + (D_2^2 - \partial_2^2)\phi_1 + (D_1^2 - \partial_1^2)\phi_3 + 2(D_2D_0 - \partial_2\partial_0)\phi_3 - \left(\frac{1}{2}\,\phi_1^2\phi_3 - \frac{1}{120}\phi_1^5\right)\cos(\theta). \tag{49}$$

It should be noted that for the sake of simplicity, we have not quoted the terms involving ϕ_0, ϕ_2, and ϕ_4 as these vanish. Having calculated the right hand side using the known functions ϕ_1, ϕ_3, we again split the solution into components proportional to the simple harmonics as above, and calculate the fundamental modes, since we expect to obtain the leading order amplitude equation. The equations for these harmonics are

$$\left[\partial_0^2 - \cos(\theta) + \lambda_1^2\right]\phi_{511} = \begin{cases} G_1, & X_0 > L + a, \\ G_2, & L < X_0 < L + a, \\ G_3, & 0 < X_0 < L, \end{cases} \tag{50}$$

$$\left[\partial_0^2 - \cos(\theta) + \lambda_2^2\right]\phi_{512} = \begin{cases} H_1, & X_0 > L + a, \\ H_2, & L < X_0 < L + a, \\ H_3, & 0 < X_0 < L, \end{cases} \tag{51}$$

where the functions G_j, H_j are given in Appendix A.

We do not calculate the other harmonics as we expect to obtain oscillatory behaviour over the long time scale of the localised mode here. Using the Fredholm theorem, the solvability conditions for the lowest harmonics are

$$D_4B_1 = \beta_1B_1|B_1|^4 + \beta_2B_1|B_2|^4 + \beta_3B_1|B_1|^2|B_2|^2, \tag{52}$$

$$D_4B_2 = \gamma_1B_2|B_2|^4 + \gamma_2B_2|B_1|^4 + \gamma_3B_2|B_1|^2|B_2|^2, \tag{53}$$

where numerical values for β_j, γ_j for some parameter values are given in Sec. IV.

We do not extend the perturbation expansion to higher orders, as we have obtained the equations governing the decaying oscillatory behaviour of the localized modes in a double-well potential.

F. Amplitude equations

Having solved the system (1)–(4) at various orders of ϵ, we now combine the solutions to obtain expressions which can be compared with numerical simulations of (1)–(2). By noting that

$$\frac{d}{dt}B_1 = \epsilon D_1B_1 + \epsilon^2D_2B_1 + \epsilon^3D_3B_1 + \epsilon^4\,D_4B_1, \tag{54}$$

$$\frac{d}{dt}B_2 = \epsilon D_1B_2 + \epsilon^2D_2B_2 + \epsilon^3D_3B_2 + \epsilon^4\,D_4B_2, \tag{55}$$

and combining the solvability conditions (24), (30), (31), (48), (52), and (53), we obtain the system of two coupled equations

$$\frac{d|b_1|^2}{dt} = 2\left(\tilde{\beta}_1|b_1|^6 + \tilde{\beta}_2|b_1|^2|b_2|^4 + \tilde{\beta}_3|b_1|^4|b_2|^2\right), \tag{56}$$

$$\frac{d|b_2|^2}{dt} = 2\left(\tilde{\gamma}_1|b_2|^6 + \tilde{\gamma}_2|b_2|^2|b_1|^4 + \tilde{\gamma}_3|b_2|^4|b_1|^2\right), \tag{57}$$

where $b_j(t)$ is the oscillation amplitude of the bound states. By assuming that $\tilde{\beta}_j = \mathrm{Re}(\beta_{\mathrm{j}}), \tilde{\gamma}_j = \mathrm{Re}(\gamma_{\mathrm{j}}) < 0$, for $j = 1,2,3$ as will be shown later in Sec. IV, Eqs. (56) and (57) describe the gradual decrease in the amplitude of coupled oscillations due to energy emission in the form of radiation.

When $b_2 = 0$, and $b_1 \neq 0$, Eq. (56) is satisfied by

$$|b_1(t)| = \left(\frac{|b_1(0)|^4}{1 - 4\,\mathrm{Re}\,(\gamma_1)\,|b_1(0)|^4 t}\right)^{1/4}. \tag{58}$$

Since $\mathrm{Re}(\gamma_1) < 0$, this describes algebraic decay of b_1 with increasing time. Similarly, when $b_1 = 0$, and $b_2 \neq 0$, from (57), we obtain

$$|b_2(t)| = \left(\frac{|b_2(0)|^4}{1 - 4\,\mathrm{Re}\,(\beta_1)\,|b_2(0)|^4 t}\right)^{1/4}. \tag{59}$$

The decay in both cases is due to the assumptions (35) and (36), that is, the third harmonics lie in the continuous spectrum and so nonlinearity causes the formation of phonons which radiate energy to $x \to \pm\infty$.

G. Resonance condition: $(3\lambda_1)^2 < 1 < (3\lambda_2)^2$

In Subsections II C–II F (Eq. (35)) we considered the case when $(3\lambda_2)^2 > (3\lambda_1)^2 > 1$, i.e., the third harmonics of the two modes are in the continuous spectrum. It is interesting to determine what will happen if only one of the harmonics is in the continuous spectrum. Here, we consider values of L and a such that

$$(3\lambda_1)^2 < 1 < (3\lambda_2)^2. \tag{60}$$

By using the same perturbation expansion as at the start of this section, the solvability conditions at $O(\epsilon^2)$ are the same as Eq. (24) and hence $B_j = B_j(T_2,T_3,\ldots)$. Solving Eq. (25), we obtain the solvability condition

$$D_2B_1 = d_{11}B_1|B_1|^2 i + d_{12}B_1|B_2|^2 i, \tag{61}$$

$$D_2B_2 = d_{21}B_2|B_2|^2 i + d_{22}B_2|B_1|^2 i, \tag{62}$$

where values for d_{ij} are given in Sec. IV.

With the assumption (60), we observe that the solutions ϕ_{332}, ϕ_{321}, and ϕ_{341} describe the right moving radiation for $X_0 > L + a$ and left moving radiation for $X_0 < L$. Similarly, from Eq. (47) at $O(\epsilon^3)$, we obtain

$$D_3B_1 = 0, \quad D_3B_2 = 0. \tag{63}$$

Hence, $B_j = B_j(T_2,T_4,\ldots)$. Solving Eq. (49) and using (A1)–(A6), the solvability conditions at $O(\epsilon^5)$ are

$$D_4B_1 = \left(e_{11}|B_1|^4 i + e_{12}|B_2|^4 + e_{13}|B_1|^2|B_2|^2\right) B_1, \tag{64}$$

$$D_4B_2 = \left(e_{21}|B_2|^4 + e_{22}|B_1|^4 + e_{23}|B_1|^2|B_2|^2\right) B_2, \tag{65}$$

with e_{ij} given in Sec. IV. Combining Eqs. (61), (62), (64), and (65), we obtain

$$\frac{d}{dt}|b_1|^2 = 2\left(\mathrm{Re}(e_{12})|b_1|^2|b_2|^4 + \mathrm{Re}(e_{13})|b_1|^4|b_2|^2\right), \tag{66}$$

$$\frac{d}{dt}|b_2|^2 = 2\Big(\mathrm{Re}(e_{21})|b_2|^6 + \mathrm{Re}(e_{22})|b_2|^2|b_1|^4 + \mathrm{Re}(e_{23})|b_2|^4|b_1|^2\Big). \tag{67}$$

Hence, it can be noted that even though $(3\lambda_1)^2 < 1$, the coupled Eqs. (66) and (67) still show that ϕ_1 decays in time. Note that e_{12} and e_{21} are also determined by the "external" radiation function due to the harmonic $(3\lambda_2)^2 > 1$.

III. DRIVEN BOUND STATES

We now consider the dynamics of the sine-Gordon equation (1) perturbed by a time-dependent external force modelling an externally driven Josephson junction with $h \neq 0$ and $\Omega = \lambda_1(1+\rho)$. For notational compactness, we make the transformation

$$\Omega t = \lambda_1 \tau. \tag{68}$$

Equation (1) then becomes

$$\phi_{xx}(x,\tau) - (1+\rho)^2\phi_{\tau\tau}(x,\tau) = \sin(\phi+\theta) + \frac{1}{2}h\left(e^{i\lambda_1\tau} + c.c.\right). \tag{69}$$

Here, we assume that the driving amplitude h is small and the driving frequency is close to resonance with the fundamental mode of the homogeneous system. In this case, we consider

$$h = \epsilon^3 H, \quad \rho = \epsilon^3 R, \tag{70}$$

with $H, R \sim O(1)$. Due to the time rescaling above, our slow temporal variables are now defined as

$$X_n = \epsilon^n x, \quad T_n = \epsilon^n \tau, \quad n = 0,1,2,\ldots, \tag{71}$$

with the shorthand notation (7). Performing the perturbation expansion order by order as in Sec. II, we obtain the same perturbation expansion up to $O(\epsilon^2)$.

A. Equation at $O(\epsilon^3)$

The terms at order of $O(\epsilon^3)$ give

$$\begin{aligned}\partial_0^2\phi_3 - D_0^2\phi_3 - \cos(\theta+\phi_0)\phi_3 &= 2(D_0D_2 - \partial_0\partial_2)\phi_1\\ &+(D_1^2 - \partial_1^2)\phi_1 - \frac{1}{6}\phi_1^3\cos(\theta) + \frac{1}{2}H(e^{i\lambda_1\tau} + c.c.).\end{aligned} \tag{72}$$

Calculating the right hand side, we obtain various harmonics, namely,

$$\begin{aligned}\partial_0^2\phi_3 - D_0^2\phi_3 - \cos(\theta+\phi_0)\phi_3 &= 2\,i\lambda_1 D_2B_1\Phi_1 e^{i\lambda_1T_0} + 2\,i\lambda_2 D_2B_2\Phi_2 e^{i\lambda_2T_0} - \frac{1}{6}\Big[B_1^3\Phi_1^3e^{3\,i\lambda_1T_0} + B_2^3\Phi_2^3e^{3\,i\lambda_2T_0}\\ &+3\,B_1|B_1|^2\Phi_1^3e^{i\lambda_1T_0} + 3\,B_2|B_2|^2\Phi_2^3e^{i\lambda_2T_0} + 3\,B_1^2B_2\Phi_1^2\Phi_2e^{(2\,\lambda_1+\lambda_2)T_0\,i} + 3\,B_1B_2^2\Phi_1\Phi_2^2e^{(2\,\lambda_2+\lambda_1)T_0\,i}\\ &+3\,B_1^2\overline{B}_2\,\Phi_1^2\,\Phi_2e^{(2\,\lambda_1-\lambda_2)T_0\,i} + 6\,B_1|B_2|^2\Phi_1\Phi_2^2e^{\lambda_1T_0\,i} + 3\,\overline{B}_1B_2^2\Phi_1\Phi_2^2e^{(2\,\lambda_2-\lambda_1)T_0\,i} + 6\,|B_1|^2B_2\Phi_1^2\Phi_2e^{\lambda_2T_0\,i}\Big]\cos(\theta)\\ &+\frac{1}{2}H\mathrm{e}^{i\lambda_1\tau} + c.c.\end{aligned} \tag{73}$$

Using (27), we split the harmonics as in Sec. II. Using the Fredholm alternative, the solvability condition for the first harmonic is

$$D_2B_1 = \alpha_1B_1|B_1|^2 i + \alpha_2B_1|B_2|^2 i + \mu_1H\,i, \tag{74}$$

where α_j, μ_1 are given in Appendix A. The solvability condition D_2B_2 is the same as (31). With (74), the solution for the first harmonic is obtained in the form

$$\tilde{\phi}_{311} = \begin{cases} Z_1\mathrm{e}^{-\sqrt{1-\lambda_1^2}(X_0-L-a)} + B_1\Psi_1 + H\,\Psi_2 & X_0 > L+a,\\ Z_2\cos\left(\sqrt{1+\lambda_1^2}(X_0-L-a)\right) + B_1\Psi_3 + Z_3\sin\left(\sqrt{1+\lambda_1^2}(X_0-L-a)\right) + H\Psi_4, & L < X_0 < L+a,\\ Z_4\cosh\left(\sqrt{1-\lambda_1^2}X_0\right) + B_1\Psi_5 + H\,\Psi_6, & 0 < X_0 < L,\end{cases} \tag{75}$$

where $\Psi_j = \Psi_j\left(|B_1|^2, |B_2|^2, \lambda_1, \lambda_2\right)$ for $j = 1,2,3,4$ that appears in $\tilde{\phi}_{311}$ can be seen in Appendix B. The constants of integration $Z_j = Z_j\left(B_1, H, |B_1|^2, |B_2|^2, \lambda_1, \lambda_2\right)$ are found by applying the continuity conditions (3) and (4) at the discontinuity points. We do not calculate the other harmonics that appear in (73) as these are identical to the undriven case considered in Sec. II.

B. Equation at $O(\epsilon^4)$

Equating terms at $O(\epsilon^4)$, we obtain

$$D_0{}^2\phi_4 - \partial_0{}^2\phi_4 - \cos\left(\Phi_0 + \theta\right)\phi_4 = 2\left(D_1D_2 - \partial_1\partial_2\right)\phi_1 + 2\left(D_0D_3 - \partial_0\partial_3\right)\phi_1 + 2\,RD_0^2\phi_1$$
$$+2\left(D_0D_1 - \partial_0\partial_1\right)\phi_3 + \left(\frac{1}{24}\phi_1^4 - \phi_3\phi_1\right)\sin\left(\theta\right). \tag{76}$$

Calculating the right hand side and applying the Fredholm alternative, the solvability conditions for the first harmonics are

$$D_3B_1 = -\lambda_1\, B_1\, R\, i, \quad D_3B_2 = -\lambda_2\, B_2\, R\, i. \tag{77}$$

Since $\lambda_1, \lambda_2, R \in \mathbb{R}$, these modes are purely oscillatory, being given by

$$B_1 = B_1(T_2, T_4, \ldots)e^{-\lambda_1 RT_3 i}, \tag{78}$$

$$B_2 = B_2(T_2, T_4, \ldots)e^{-\lambda_2 RT_3 i}. \tag{79}$$

At this stage, we solve (76) imposing $\phi_4 = 0$.

C. Equation at $O(\epsilon^5)$

The terms at order $O(\epsilon^5)$ give

$$\partial_0^2\phi_5 - D_0^2\phi_5 - \phi_5\cos\left(\theta\right) = 2(D_0D_4 - \partial_0\partial_4)\phi_1 + 2(D_3D_1 - \partial_3\partial_1)\phi_1 + 4\,R\,D_0D_1\phi_1 + (D_2^2 - \partial_2^2)\phi_1$$
$$+(D_1^2 - \partial_1^2)\phi_3 + 2(D_2D_0 - \partial_2\partial_0)\phi_3 - \left(\frac{1}{2}\,\phi_1^2\phi_3 - \frac{1}{120}\phi_1^5\right)\cos(\theta). \tag{80}$$

In calculating the right hand side, we consider only the first harmonics as our main aim is to obtain the amplitude equation at this stage, that is,

$$\left[\partial_0^2 - \cos(\theta) + \lambda_1^2\right]\phi_{511} = \begin{cases} L_1, & X_0 > L + a, \\ L_2, & L < X_0 < L + a, \\ L_3, & 0 < X_0 < L, \end{cases} \tag{81}$$

$$\left[\partial_0^2 - \cos(\theta) + \lambda_1^2\right]\phi_{512} = \begin{cases} M_1, & X_0 > L + a, \\ M_2, & L < X_0 < L + a, \\ M_3, & 0 < X_0 < L, \end{cases} \tag{82}$$

where L_j, M_j are given in Appendix B.

Using the Fredholm alternative, the solvability conditions for Eqs. (81) and (82) are

$$D_4B_1 = a_1B_1|B_1|^4 + a_2B_1|B_2|^4 + a_3B_1|B_1|^2|B_2|^2 + \left(a_4|B_1|^2 + a_5|B_2|^2 + a_6B_1^2\right)Hi, \tag{83}$$

$$D_4B_2 = c_1B_2|B_2|^4 + c_2B_2|B_1|^4 + c_3B_1|B_2|^2|B_2|^2 + c_4B_2\left(B_1 + \overline{B_1}\right)\,H\,i, \tag{84}$$

where a_j, c_j for some parameter values are given in Sec. IV.

D. Amplitude equations

Equations (83) and (84) are the leading order equations for the coupled mode oscillations. Combining all the solvability conditions (30), (74), (77), (83), and (84), and considering $b_j = \epsilon B_j$ for

$j = 1,2$, we obtain

$$\frac{\Omega}{\lambda_1}\frac{db_1}{dt} = \alpha_1 b_1|b_1|^2 i + \alpha_2 b_1|b_2|^2 i + \mu_1 h\, i - \lambda_1 b_1\, \rho\, i + a_1 b_1|b_1|^4 + a_2 b_1|b_2|^4 + a_3 b_1|b_1|^2|b_2|^2 + \left(a_4|b_1|^2 + a_5|b_2|^2 + a_6 b_1^2\right) h\, i, \quad (85)$$

$$\frac{\Omega}{\lambda_1}\frac{db_2}{dt} = \alpha_3 b_2|b_2|^2 i + \alpha_4 b_2|b_1|^2 i - \lambda_2 b_2\, \rho\, i + c_1 b_2|b_2|^4 + c_2 b_2|b_1|^4 + c_3 b_2|b_1|^2|b_2|^2 + c_4\, b_2\left(b_1 + \overline{b_1}\right) h\, i, \quad (86)$$

where α_j are given by (32) with p_j in Appendix A.

From the above equations, we expect that the presence of a non-zero external drive will induce mode oscillations. Note that in Eqs. (85) and (86), there is a solution with $b_2 = 0$ and $b_1 \neq 0$ as well as one with $b_1 \neq 0$ and $b_2 \neq 0$, but $b_1 = 0$ and $b_2 \neq 0$ are, in general, impossible (it would require b_2 to satisfy both (86) and $|b_2|^2 = -\mu_1/a_5$).

E. Resonance condition: $(3\lambda_1)^2 < 1 < (3\lambda_2)^2$ in the driven case

Now, we consider the same case as in Sec. II G, but in the presence of a driving term. Repeating the same procedure as above, the solvability conditions at $O(\epsilon^2)$ and $O(\epsilon^4)$ are the same as Eqs. (30) and (77).

The solvability condition at $O(\epsilon^3)$ from Eq. (72) gives

$$D_2B_1 = d_{11}B_1|B_1|^2 i + d_{12}B_1|B_2|^2 i + d_{13}H\, i, \quad (87)$$

$$D_2B_2 = d_{21}B_2|B_2|^2 i + d_{22}B_2|B_1|^2 i, \quad (88)$$

where the parameters $d_{i,j}$ are given in Sec. IV. Similarly, from Eq. (80), the solvability conditions at $O(\epsilon^5)$ yield

$$D_4B_1 = \zeta_{11}B_1|B_1|^4 i + \zeta_{12}B_1|B_2|^4 + \zeta_{13}B_1|B_1|^2|B_2|^2 + \left(\zeta_{14}|B_1|^2 + \zeta_{15}|B_2|^2 + \zeta_{16}B_1^2\right) H\, i, \quad (89)$$

$$D_4B_2 = \zeta_{21}B_2|B_2|^4 + \zeta_{22}B_2|B_1|^4 + \zeta_{23}B_2|B_1|^2|B_2|^2 + \zeta_{24}(B_1 + \bar{B}_1)B_2H\, i, \quad (90)$$

where $\zeta_{i,j}$ are given in Sec. IV.

Combining (30), (77), and (88)–(91) and considering $b_i = \epsilon B_i$ for $i = 1,2$, we obtain amplitude equations of the form

$$\frac{\Omega}{\lambda_1}\frac{db_1}{dt} = d_{11}b_1|b_1|^2 i + d_{12}b_1|b_2|^2 i + d_{13}h\, i - \lambda_1 b_1\, \rho\, i + \zeta_{11}b_1|b_1|^4 i + \zeta_{12}b_1|b_2|^4 + \zeta_{13}b_1|b_1|^2|b_2|^2 + \left(\zeta_{14}|b_1|^2 + \zeta_{15}|b_2|^2 + \zeta_{16}b_1^2\right) h\, i, \quad (91)$$

$$\frac{\Omega}{\lambda_1}\frac{db_2}{dt} = d_{21}b_2|b_2|^2 i + d_{22}b_2|b_1|^2 i - \lambda_2 b_2\, \rho\, i + \zeta_{21}b_2|b_2|^4 + \zeta_{22}b_2|b_1|^4 + \zeta_{23}b_2|b_1|^2|b_2|^2 + \zeta_{24}\left(b_1 + \overline{b_1}\right) b_2\, h\, i. \quad (92)$$

As in (85) and (86), from the above equations, we expect that the non-zero external drive amplitude ($h > 0$) induces coupled mode oscillations.

IV. NUMERICAL CALCULATIONS

To check the analytical results obtained in Secs. II–III, we have numerically solved the governing Eq. (1), with $\theta(x)$ given by (2). We discretise the Laplacian operator using central differences and integrate the resulting system of differential equations using a fourth-order Runge–Kutta method, with spatial and temporal discretizations of $\Delta x = 0.01$ and $\Delta t = 0.002$, respectively. The computational domain is $x \in (-M, M)$, with $M = 50$. At the boundaries, we use a periodic boundary condition. To model an infinitely long junction, we apply an increasing damping at the boundaries to reduce reflected continuous waves incoming from the boundaries. To ensure that the numerical results are not influenced by the choice of the parameter values, we have taken different values (Δx, Δt, M) as well as different boundary conditions and damping, and we obtained quantitatively similar results.

In this section, for the Josephson junction with the double-well potential (2), we fix the facet length $a = 1$ and $L = 2$, which implies that we are in the case $\lambda_2 > \lambda_1 > 1/3$, since

$$\lambda_1 \approx 0.599\,41, \quad K_1 \approx 0.397\,34, \quad C_1 \approx -0.686\,55,$$
$$\lambda_2 \approx 0.642\,47, \quad K_2 \approx 0.440\,02, \quad C_2 \approx -0.644\,71.$$

For the choice of parameters above, we obtain the coefficients in the analytic approximations (56) and (57) and (85) and (86) as

$$\alpha_1 = 0.158\,64, \quad \alpha_2 = 0.323\,26, \quad \alpha_3 = 0.167\,53,$$
$$\alpha_4 = 0.340\,44, \quad \mu_1 = 0.551\,68, \quad a_4 = 0.291\,91,$$
$$a_5 = -0.212\,75, \quad a_6 = 1.553\,08, \quad c_4 = 0.021\,64,$$

$$\beta_1 = -0.008\,32 - 0.141\,02\,i, \quad \beta_2 = -0.012\,72 - 0.085\,09\,i,$$
$$\beta_3 = -0.162\,95 + 7.786\,99\,i, \quad \gamma_1 = -0.029\,67 - 0.106\,55\,i,$$
$$\gamma_2 = -0.064\,74 - 1.776\,12\,i, \quad \gamma_3 = -0.026\,80 - 1.521\,20\,i,$$
$$a_1 = -0.008\,32 + 0.450\,10\,i, \quad a_2 = -0.012\,72 - 0.085\,11\,i,$$
$$a_3 = -0.122\,95 - 3.234\,90\,i, \quad c_1 = -0.029\,57 - 0.206\,50\,i,$$
$$c_2 = -0.079\,74 - 1.825\,00\,i, \quad c_3 = -0.046\,80 + 2.145\,72\,i.$$

Note that $\beta_j, \gamma_j \in \mathbb{C} \setminus \mathbb{R}$ and hence have real parts as noted after Eq. (57).

To illustrate the case $\lambda_1 < 1/3 < \lambda_2$, we choose $L = 0.5$, $a = 1.1$, that is,

$$\lambda_1 \approx 0.274\,31, \quad K_1 \approx 1.127\,09, \quad C_1 \approx -0.927\,38,$$
$$\lambda_2 \approx 0.821\,48, \quad K_2 \approx 2.015\,78, \quad C_2 \approx -0.440\,62.$$

In this case, we obtain the coefficients in the analytically obtained approximations (66) and (67) and (91) and (92) as

$$d_{11} = 0.536\,42, \quad d_{12} = 0.761\,04, \quad d_{13} = 0.920\,35,$$
$$d_{21} = 0.092\,43, \quad d_{22} = 0.386\,62, \quad e_{11} = -1.797\,63,$$
$$\zeta_{11} = 0.101\,20, \quad \zeta_{14} = -1.2231, \quad \zeta_{15} = -2.026\,15,$$
$$\zeta_{16} = 2.273\,24, \quad \zeta_{24} = -1.212\,83,$$

$$e_{12} = -0.001\,62 + 0.334\,06\,i, \quad e_{13} = -0.156\,74 + 0.502\,24\,i,$$
$$e_{21} = -0.002\,52 + 0.012\,12\,i, \quad e_{22} = -0.046\,99 - 0.927\,83\,i,$$
$$e_{23} = -0.046\,19 - 0.058\,82\,i, \quad \zeta_{12} = -0.035\,62 + 0.340\,56\,i,$$
$$\zeta_{13} = -0.365\,24 + 1.472\,70\,i, \quad \zeta_{21} = -0.002\,519 - 0.0558\,i,$$
$$\zeta_{22} = -0.062\,29 + 0.437\,52\,i, \quad \zeta_{23} = -0.036\,19 + 0.603\,35\,i.$$

A. The undriven case

First, we consider the undriven case, $h = 0$. In Fig. 1, we show the typical dynamics of the bound states of the system. The initial condition is that both modes are excited with similar amplitudes (see below for the details). The presence of radiation moving to the boundaries can be seen clearly. The traveling wave is responsible for the decay of the bound states discussed in Secs. I–III.

To analyze further and compare the analytical results obtained previously with numerical computations, using the expansion (5) and (12) and (13), we note that

$$\phi(x,t) \approx A_1(t)\Phi_1(x) + A_2(t)\Phi_2(x). \tag{93}$$

Analytically,

$$A_j(t) = b_j e^{i\lambda_i t} + \overline{b_j} e^{-i\lambda_i t}, \tag{94}$$

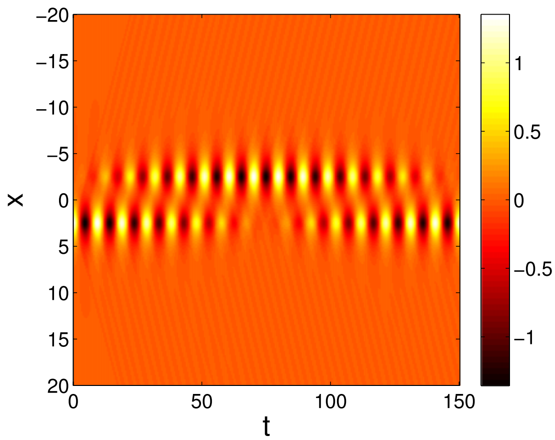

FIG. 1. Time-dynamics of the system's bound states for $a = 1$ and $L = 2$, when both modes are excited initially with relatively the same amplitude. Elevation plot of $\phi(x,t)$.

and b_j satisfies the amplitude equations (56) and (57) or (66) and (67). To determine the oscillation amplitude $A_j(t)$ of the two modes from the full Eq. (1), we approximate it by

$$A_j(t) = \frac{\int_{-L}^{L} \phi(x,t)\Phi_j(x)dx}{\int_{-L}^{L} \Phi_j^2(x)dx}. \tag{95}$$

We consider two sets of initial conditions

$$A_1(0) = 0.6, \quad A_2(0) = 0.6, \ \ 0.3. \tag{96}$$

The initial condition in Fig. 1 is $A_1(0) = A_2(0) = 0.6$.

In Figures 2 and 3, we plot the *envelope* of the oscillation amplitudes $A_j(t)$ from the governing equation (1) as red and black curves, respectively. One can see that the coupled mode oscillation amplitude decreases in time, i.e., it experiences damping due to higher harmonic excitation with frequency in the dispersion relation.

To compare the numerical results with our analytical calculations, we solve the coupled (56) and (57) or (66) and (67) numerically using a fourth-order Runge–Kutta method with a relatively fine time-discretization parameter, as general analytical solutions may not be available. With the initial condition

$$|b_j(0)|^2 = \frac{A_j^2(0)}{4F^2}, \tag{97}$$

the analytical approximation is then given by $2F|b_1(t)|$ and $2F|b_2(t)|$, with F being a free real-valued parameter. In general, the factor is simply $F = 1$. Yet, by treating F as a fitting parameter, we observed that the best fit is not given by the aforementioned value. For the initial conditions (96), we found that optimum fits are, respectively, provided by $F = 1.1,\ 1.05$ for Figure 2 and $F = 1.03,\ 0.97$ for Figure 3, respectively. The differences may be explained by the fact that our asymptotic approximation is only valid for long times; thus, there is a short initial transient, which can be better described by allowing $F \neq 1$. Our approximations are shown as green and blue solid lines in Figures 2 and 3 where one can see good agreement with the numerically obtained oscillation.

Comparing the panels in Figs. 2 and 3, we observe that exciting the two modes at the same time increases the decay of the bound states. This is due to higher harmonic excitation and coupling of the oscillation amplitudes $b_j(t)$, that can be seen in the analytically obtained approximation. It is also due to the coupling that the second mode in Fig. 3 decays at the same rate as the first mode, while in the absence of the first mode, it will decay slower.

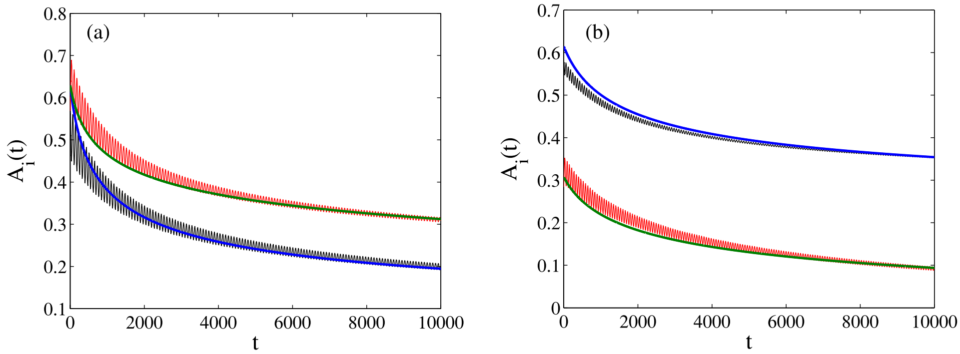

FIG. 2. Oscillation amplitude of the breathing coupled modes. The case $\lambda_2 > \lambda_1 > 1/3$, with no driving ($h = 0$). (a) $A_2(0) = 0.6$, (b) $A_2(0) = 0.3$, while in all cases, $A_1(0) = 0.6$. The black oscillation curves (lower oscillatory curve in the top panel and upper oscillatory curve in lower panel) are from the oscillation amplitude $A_1(t)$, and red for $A_2(t)$ (upper oscillatory curve in top panel and lower oscillatory curve in bottom panel) is obtained from the original governing equation (1). These curves clearly indicate the decay of the coupled mode oscillation. Analytical approximations (56) and (57) are shown as $A_1(t)$ for blue curves (bottom smooth curve in top panel and upper smooth curve in lower panel) and $A_2(t)$ as green solid curves (upper smooth curve in top panel and lower smooth curve in bottom panel).

B. The driven case

Next, we consider the case of driven Josephson junctions, i.e., Eq. (1) with $h \neq 0$. In this case, the initial condition to the governing equation (1) is the same as before. Throughout the subsection, we take $\Omega = \lambda_1$.

Figure 4 repeats Figure 1 but with a nonzero drive amplitude. One clearly observes that while the mode tunneling between the wells in the undriven case is preserved, in the current case, it is destroyed by the drive. Therefore, it is interesting to study the mode destruction further.

In Figures 5 and 6, we present the dynamics of the mode amplitude $A_j(t)$ of the driven system with $h = 0.006$ and $h = 0.008$. The initial amplitudes are $A_j(0) = 0.3$.

For all the cases, we obtain that the envelope of $A_1(t)$ oscillates and tends slowly to a constant amplitude while the envelope of $A_2(t)$ vanishes. Hence, it is important to note that the drive acts as a damping to the antisymmetric mode. In other words, we have a synchronized oscillation between localized modes in the two potential wells.

Considering the panels in Figures 5 and 6, we observe that the modes do not oscillate with an unbounded or growing amplitude. After a while, there is a balance between the energy input into the breathing mode due to the external drive and the radiative damping.

To assess the accuracy of the asymptotic analysis, we have solved the amplitude equations (85) and (86) and (91) and (92) numerically. The analytical approximation is again given by $2F|b_j(t)|$,

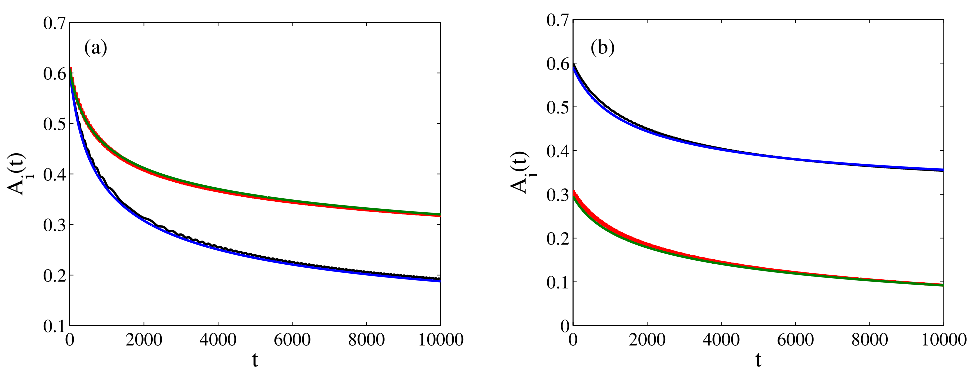

FIG. 3. As in Fig. 2, but for the case of $\lambda_2 > 1/3 > \lambda_1$. The analytical approximations (66) and (67) in the two panels are shown as green and blue lines.

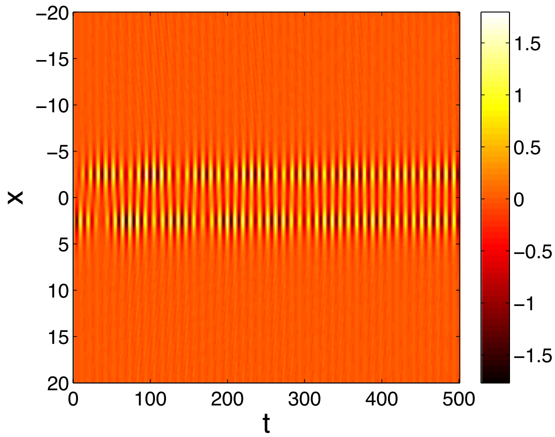

FIG. 4. The same time-dynamics of the system's bound states as in Fig. 1, but with nonvanishing drive amplitude, that is slowly increasing in time to $h = 0.03$.

where F in this case is taken as $F = 1.15$ for Figure 5, while $F = 1.05$ and $F = 1.035$ for Figure 6. In the panels of the figures, the approximations (85) and (86) and (91) and (92) are shown in green and blue lines, respectively, using $\rho = 0$, where one can see that our approximations are in good agreement.

V. CONCLUSIONS

We have considered a spatially inhomogeneous sine-Gordon equation with a double well potential and a time-periodic drive, modeling long Josephson junctions with phase-shifts. Using multiple scale expansions, we have shown that due to the energy transfer from the discrete to continuous modes, the system's localized modes will decay in time. The complex decay mechanism leads to an algebraic rate of decrease in the amplitude of the oscillations, with the exponent depending on whether only one or both the modes are excited. In the presence of an ac-drive with small amplitude, there is an energy balance for the symmetric bound state between the energy input given by the external drive and the energy output due to radiative damping experienced by the coupled mode. As for the antisymmetric state, the drive acts as a damping.

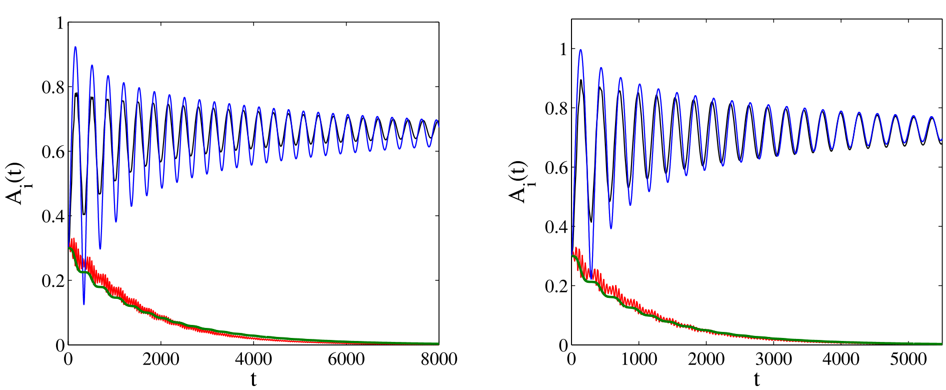

FIG. 5. The same as in Figure 2, but for nonzero driving amplitude. The initial condition is the same as in Fig. 2(a). The two panels correspond to $h = 0.006$ and $h = 0.008$, respectively. Analytical approximations (85) and (86) are shown as green and blue lines. In both panels, the lower smooth curve is the asymptotic approximation of A_2, the lower oscillatory curve is the numerical simulation of A_2, the smaller amplitude upper curve is the numerical simulation of A_1, and the larger amplitude upper curve is the asymptotic approximation of A_1.

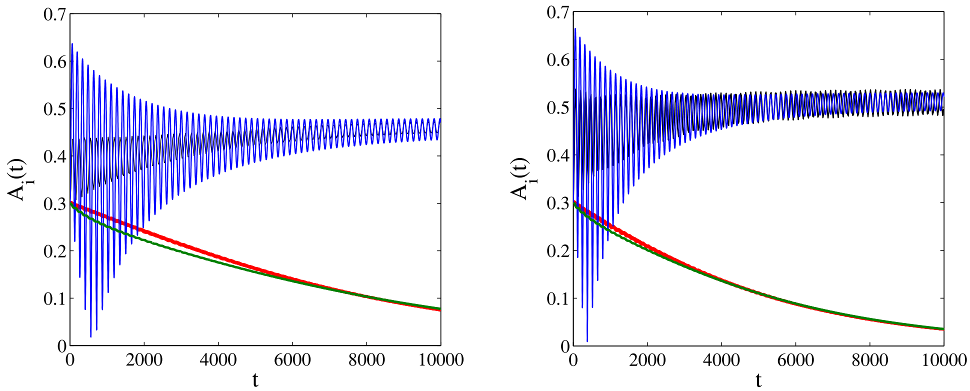

FIG. 6. The same as in Figure 3(a), but for nonzero driving amplitude. The panels correspond to $h = 0.006$ and $h = 0.008$. Analytical approximations (91) and (92) are shown as green and blue lines (in both panels, the larger amplitude oscillatory curve corresponds to the asymptotic solution of A_1, the smaller amplitude curve to the numerical simulation of A_1, the lowest curve shows the asymptotic solution of A_2, whilst the upper decaying curve illustrates the numerical solution of A_2).

APPENDIX A: FUNCTIONS IN SEC. II

$$
\begin{aligned}
G_1 &= 2\,i\lambda_1 D_4 B_1\Phi_1 + B_1\left(2\,i\lambda_1\left(\alpha_1|B_1|^2 i + \alpha_2|B_2|^2 i\right) - \frac{3}{2}|B_1|^2\Phi_1^2 - |B_2|^2{\Phi_2}^2\right)\psi_1\\
&\quad + B_1\left(\alpha_1|B_1|^2 i + \alpha_2|B_2|^2 i\right)^2\Phi_1 + \frac{1}{12}B_1|B_1|^4{\Phi_1}^5 + \frac{1}{2}B_1|B_1|^2|B_2|^2\Phi_1^3\Phi_2^2\\
&\quad + \frac{1}{4}B_1|B_2|^4\Phi_1\Phi_2^4 - 2\,B_1|B_2|^2\Phi_1\Phi_2\psi_4 - B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_7 - \frac{1}{2}B_1|B_1|^4\Phi_1^2\psi_{13}\\
&\quad - B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{10} - \frac{1}{2}B_1|B_2|^4{\Phi_2}^2\psi_{19} - \frac{1}{2}B_1|B_2|^4{\Phi_2}^2\psi_{22}, \qquad \text{(A1)}\\
G_2 &= 2\,i\lambda_1 D_4 B_1\Phi_1 + B_1\left(2\,i\lambda_1\left(\alpha_1|B_1|^2 i + \alpha_2|B_2|^2 i\right) + \frac{3}{2}|B_1|^2\Phi_1^2 + |B_2|^2{\Phi_2}^2\right)\psi_2\\
&\quad + B_1\left(\alpha_1|B_1|^2 i + \alpha_2|B_2|^2 i\right)^2\Phi_1 - \frac{1}{12}B_1|B_1|^4{\Phi_1}^5 - \frac{1}{2}B_1|B_1|^2|B_2|^2\Phi_1^3\Phi_2^2\\
&\quad - \frac{1}{4}B_1|B_2|^4\Phi_1\Phi_2^4 + 2\,B_1|B_2|^2\Phi_1\Phi_2\psi_5 + B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_8 + \frac{1}{2}B_1|B_1|^4\Phi_1^2\psi_{14}\\
&\quad + B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{11} + \frac{1}{2}B_1|B_2|^4{\Phi_2}^2\psi_{20} + \frac{1}{2}B_1|B_2|^4{\Phi_2}^2\psi_{23}, \qquad \text{(A2)}\\
G_3 &= 2\,i\lambda_1 D_4 B_1\Phi_1 + B_1\left(2\,i\lambda_1\left(\alpha_1|B_1|^2 i + \alpha_2|B_2|^2 i\right) - \frac{3}{2}|B_1|^2\Phi_1^2 - |B_2|^2{\Phi_2}^2\right)\psi_3\\
&\quad + B_1\left(\alpha_1|B_1|^2 i + \alpha_2|B_2|^2 i\right)^2\Phi_1 + \frac{1}{12}B_1|B_1|^4{\Phi_1}^5 + \frac{1}{2}B_1|B_1|^2|B_2|^2\Phi_1^3\Phi_2^2\\
&\quad + \frac{1}{4}B_1|B_2|^4\Phi_1\Phi_2^4 - 2\,B_1|B_2|^2\Phi_1\Phi_2\psi_6 - B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_9 - \frac{1}{2}B_1|B_1|^4\Phi_1^2\psi_{15}\\
&\quad - B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{12} - \frac{1}{2}B_1|B_2|^4{\Phi_2}^2\psi_{21} - \frac{1}{2}B_1|B_2|^4{\Phi_2}^2\psi_{24}, \qquad \text{(A3)}\\
H_1 &= 2\,i\lambda_2 D_4 B_2\Phi_2 + B_2\left(2\,i\lambda_2\left(\alpha_3|B_2|^2 i + \alpha_4|B_1|^2 i\right) - \frac{3}{2}|B_2|^2{\Phi_2}^2 - |B_1|^2{\Phi_1}^2\right)\psi_4\\
&\quad + B_2\left(\alpha_3|B_2|^2 i + \alpha_4|B_1|^2 i\right)^2\Phi_2 + \frac{1}{12}B_2|B_2|^4\Phi_2^5 + \frac{1}{2}B_2|B_1|^2|B_2|^2\Phi_1^2{\Phi_2}^3\\
&\quad + \frac{1}{4}B_2|B_1|^4{\Phi_1}^4\Phi_2 - 2\,B_2|B_1|^2\Phi_1\Phi_2\psi_1 - \frac{1}{2}B_2|B_1|^4{\Phi_1}^2\psi_7 - \frac{1}{2}B_2|B_1|^4{\Phi_1}^2\psi_{10}\\
&\quad - \frac{1}{2}B_2|B_2|^4{\Phi_2}^2\psi_{16} - B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{19} - B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{22}, \qquad \text{(A4)}
\end{aligned}
$$

$$H_2 = 2\,i\lambda_2 D_4 B_2 \Phi_2 + B_2\left(2\,i\lambda_2\left(\alpha_3|B_2|^2 i + \alpha_4|B_1|^2 i\right) + \frac{3}{2}|B_2|^2{\Phi_2}^2 + |B_1|^2{\Phi_1}^2\right)\psi_5$$
$$+B_2\left(|B_2|^2\alpha_3\,i + |B_1|^2\alpha_4\,i\right)^2\Phi_2 - \frac{1}{12}B_2|B_2|^4\Phi_2^5 - \frac{1}{2}B_2|B_1|^2|B_2|^2\Phi_1^2{\Phi_2}^3$$
$$-\frac{1}{4}B_2|B_1|^4{\Phi_1}^4\Phi_2 + 2\,B_2|B_1|^2\Phi_1\Phi_2\psi_2 + \frac{1}{2}B_2|B_1|^4{\Phi_1}^2\psi_8 + \frac{1}{2}B_2|B_1|^4{\Phi_1}^2\psi_{11}$$
$$+\frac{1}{2}B_2|B_2|^4{\Phi_2}^2\psi_{17} + B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{20} + B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{23}, \tag{A5}$$

$$H_3 = 2\,i\lambda_2 D_4 B_2 \Phi_2 + B_2\left(2\,i\lambda_2\left(\alpha_3|B_2|^2 i + \alpha_4|B_1|^2 i\right) - \frac{3}{2}|B_2|^2{\Phi_2}^2 - |B_1|^2{\Phi_1}^2\right)\psi_6$$
$$+B_2\left(\alpha_3|B_2|^2 i + \alpha_4|B_1|^2 i\right)^2\Phi_2 + \frac{1}{12}B_2|B_2|^4\Phi_2^5 + \frac{1}{2}B_2|B_1|^2|B_2|^2\Phi_1^2{\Phi_2}^3$$
$$+\frac{1}{4}B_2|B_1|^4{\Phi_1}^4\Phi_2 - 2\,B_2|B_1|^2\Phi_1\Phi_2\psi_3 - \frac{1}{2}B_2|B_1|^4{\Phi_1}^2\psi_9 - \frac{1}{2}B_2|B_1|^4{\Phi_1}^2\psi_{12}$$
$$-\frac{1}{2}B_2|B_2|^4{\Phi_2}^2\psi_{18} - B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{21} - B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{24}, \tag{A6}$$

$$p_1 = \frac{\lambda_1\left[(1-C_1^2)\sin(2\sqrt{1+{\lambda_1}^2}a) + 2(1+{C_1}^2)\sqrt{1+{\lambda_1}^2}a - 4C_1\sin^2(\sqrt{1+{\lambda_1}^2}a)\right]}{2\sqrt{1+\lambda_1^2}}$$
$$+\frac{\lambda_1\left(2 + K_1^2\left(\sinh\left(2\,\sqrt{1-{\lambda_1}^2}L\right) + 2\,\sqrt{1-{\lambda_1}^2}L\right)\right)}{2\,\sqrt{1-\lambda_1^2}}, \tag{A7}$$

$$p_2 = \frac{3\,L{K_1}^4 - 6\,{C_1}^2 a - 3\,\left({C_1}^4+1\right)a}{16} + \frac{8\,C_1\left(1 + {C_1}^2 - 2\,{C_1}^2\cos^2\left(\sqrt{1+{\lambda_1}^2}a\right)\right)}{16\,\sqrt{1+\lambda_1^2}}$$
$$+\frac{8\,C_1\left(C_1^2-1\right)\cos^4\left(\sqrt{1+{\lambda_1}^2}a\right)}{16\,\sqrt{1+{\lambda_1}^2}} + \frac{\left(5\,{C_1}^4 - 6\,C_1^2 - 3\right)\sin\left(2\,\sqrt{1+{\lambda_1}^2}a\right)}{32\,\sqrt{1+{\lambda_1}^2}}$$
$$-\frac{2\,\left({C_1}^4 - 6\,{C_1}^2 + 1\right)\cos^3\left(\sqrt{1+{\lambda_1}^2}a\right)\sin\left(\sqrt{1+{\lambda_1}^2}a\right)}{16\,\sqrt{1+\lambda_1^2}} + \frac{1}{8\,\sqrt{1-\lambda_1^2}}$$
$$+\frac{{K_1}^4\sinh\left(2\,\sqrt{1-{\lambda_1}^2}L\right)\left(2\cosh^2\left(\sqrt{1-{\lambda_1}^2}L\right) + 3\right)}{32\,\sqrt{1-{\lambda_1}^2}}, \tag{A8}$$

$$p_3 = -\frac{\sqrt{1-{\lambda_1}^2} - \sqrt{1-{\lambda_2}^2}}{2\,({\lambda_1}^2 - {\lambda_2}^2)} - \frac{\left(\int_L^{L+a} A_1(X_0)\,dX_0\lambda_1^2 - \int_L^{L+a} A_1(X_0)\,dX_0\lambda_2^2\right)}{\lambda_1^2 - \lambda_2^2}$$
$$+\frac{K_1^2\,K_2^2\left(\int_0^L A_2(X_0)\,dX_0\lambda_1^2 - \int_0^L A_2(X_0)\,dX_0\lambda_2^2\right)}{\lambda_1^2 - \lambda_2^2}, \tag{A9}$$

$$p_4 = \frac{\lambda_2\left(2\left(C_2^2+1\right)\sqrt{1+\lambda_2^2}a - \left(C_2^2-1\right)\sin(2\sqrt{1+{\lambda_2}^2}a) - 4C_2\sin^2(\sqrt{1+\lambda_2^2}a)\right)}{2\,\sqrt{1+{\lambda_2}^2}}$$

$$+\frac{\lambda_2\left(2+K_2{}^2\sinh\left(2\sqrt{1-\lambda_2{}^2}L\right)-2K_2{}^2\sqrt{1-\lambda_2{}^2}L\right)}{2\sqrt{1-\lambda_2{}^2}}, \tag{A10}$$

$$p_5=\frac{\sin\left(2\sqrt{1+\lambda_2{}^2}a\right)\left(5C_2{}^4-6C_2{}^2-3\right)-6a\sqrt{1+\lambda_2{}^2}\left(C_2{}^4+2C_2{}^2+1\right)}{32\sqrt{1+\lambda_2{}^2}}$$

$$-\frac{C_2{}^3\cos^2\left(\sqrt{1+\lambda_2{}^2}a\right)\left(16+C_2\sin\left(2\sqrt{1+\lambda_2{}^2}a\right)\right)}{16\sqrt{1+\lambda_2{}^2}}+\frac{C_2\left(C_2{}^2+1\right)}{2\sqrt{1+\lambda_2{}^2}}$$

$$+\frac{\cos^3(\sqrt{1+\lambda_2^2}a)\left[(6C_2^2-1)\sin(\sqrt{1+\lambda_2^2}a)+4C_2(C_2^2-1)\cos(\sqrt{1+\lambda_2{}^2}a)\right]}{8\sqrt{1+\lambda_2{}^2}}$$

$$+\frac{K_2{}^4\sinh\left(2\sqrt{1-\lambda_2{}^2}L\right)\left(2\cosh^2\left(\sqrt{1-\lambda_2{}^2}L\right)-5\right)}{32\sqrt{1-\lambda_2^2}}$$

$$+\frac{2+3K_2{}^4\sqrt{1-\lambda_2{}^2}L}{16\sqrt{1-\lambda_2{}^2}}, \tag{A11}$$

$$p_6=-\frac{\sqrt{1-\lambda_1{}^2}-\sqrt{1-\lambda_2{}^2}}{2\left(\lambda_1{}^2-\lambda_2{}^2\right)}-\frac{\left(\int_L^{L+a}A_1(X_0)\,dX_0\lambda_1^2-\int_L^{L+a}A_1(X_0)\,dX_0\lambda_2^2\right)}{\lambda_1^2-\lambda_2^2}$$

$$+\frac{K_1^2K_2^2\left(\int_0^L A_2(X_0)\,dX_0\lambda_1^2-\int_0^L A_2(X_0)\,dX_0\lambda_2^2\right)}{\lambda_1^2-\lambda_2^2}, \tag{A12}$$

$$\psi_1=z_1\mathrm{e}^{-\sqrt{1-\lambda_1{}^2}(X_0-L-a)}-\frac{|B_1|^2\mathrm{e}^{-3\sqrt{1-\lambda_1{}^2}(\mathrm{X_0-L-a})}}{16\left(1-\lambda_1{}^2\right)}$$

$$+\frac{\lambda_1(\alpha_1|B_1|^2+\alpha_2|B_2|^2)\left(1+2\sqrt{1-\lambda_2{}^2}X_0\right)\mathrm{e}^{-\sqrt{1-\lambda_1{}^2}(X_0-L-a)}}{2\sqrt{1-\lambda_1{}^2}\left(\sqrt{1-\lambda_1{}^2}+\sqrt{1-\lambda_2{}^2}\right)}$$

$$+\frac{\lambda_1(\alpha_1|B_1|^2+\alpha_2|B_2|^2)\left(\sqrt{1-\lambda_2{}^2}+2X_0\left(1-\lambda_1{}^2\right)\right)\mathrm{e}^{-\sqrt{1-\lambda_1{}^2}(X_0-L-a)}}{2\left(1-\lambda_1{}^2\right)\left(\sqrt{1-\lambda_1{}^2}+\sqrt{1-\lambda_2{}^2}\right)}$$

$$-\frac{|B_2|^2\mathrm{e}^{-\left(\sqrt{1-\lambda_1^2}+2\sqrt{1-\lambda_2^2}\right)(X_0-L-a)}}{4\sqrt{1-\lambda_2^2}\left(\sqrt{1-\lambda_1^2}+\sqrt{1-\lambda_2^2}\right)}, \tag{A13}$$

$$\psi_2=z_2\cos\left(\sqrt{1+\lambda_1{}^2}\,(X_0-L-a)\right)+z_3\sin\left(\sqrt{1+\lambda_1{}^2}\,(X_0-L-a)\right)$$

$$+\frac{1}{2\sqrt{1+\lambda_1^2}}\left[\int F_1(X_0)\,dX_0\sin(\sqrt{1+\lambda_1^2}X_0)+\int G_1(X_0)\,dX_0\cos(\sqrt{1+\lambda_1^2}X_0)\right], \tag{A14}$$

$$\psi_3 = z_4 \cosh\left(\sqrt{1-\lambda_1{}^2}X_0\right) + \frac{|B_1|^2 K_1^3}{64\,(1-\lambda_1^2)} \times$$
$$\Big[\left(8\sinh(2\sqrt{1-\lambda_1^2}X_0) + \sinh(4\sqrt{1-\lambda_1^2}X_0) + 12\sqrt{1-\lambda_1^2}X_0)\right)$$
$$+4\cosh(2\sqrt{1-\lambda_1{}^2}X_0) + \cosh(4\sqrt{1-\lambda_1{}^2}X_0)\cosh(\sqrt{1-\lambda_1{}^2}X_0)\Big]$$
$$-\frac{1}{2}|B_2|^2 K_1 K_2^2 \Big[\frac{\cosh\left(2\sqrt{1-\lambda_1{}^2}X_0\right)\cosh\left(\sqrt{1-\lambda_1{}^2}X_0\right)}{8\,(1-\lambda_1{}^2)} + \frac{X_0 \sinh\left(\sqrt{1-\lambda_1{}^2}X_0\right)}{4\sqrt{1-\lambda_1^2}}$$
$$+\frac{\cosh\left(\sqrt{1-\lambda_1{}^2}X_0\right)\left(\sqrt{1-\lambda_1{}^2}+\sqrt{1-\lambda_2{}^2}\right)\cosh\left(2\left(\sqrt{1-\lambda_1{}^2}-\sqrt{1-\lambda_2{}^2}\right)X_0\right)}{16\sqrt{1-\lambda_1{}^2}\left(\lambda_1{}^2-\lambda_2{}^2\right)}$$
$$+\frac{\cosh\left(\sqrt{1-\lambda_1{}^2}X_0\right)\left(\sqrt{1-\lambda_1{}^2}-\sqrt{1-\lambda_2{}^2}\right)\cosh\left(2\left(\sqrt{1-\lambda_1{}^2}+\sqrt{1-\lambda_2{}^2}\right)X_0\right)}{16\sqrt{1-\lambda_1{}^2}\left(\lambda_1{}^2-\lambda_2{}^2\right)}$$
$$-\frac{\sinh\left(\sqrt{1-\lambda_1{}^2}X_0\right)\left(\sqrt{1-\lambda_1{}^2}+\sqrt{1-\lambda_2{}^2}\right)\sinh\left(2\left(\sqrt{1-\lambda_1{}^2}-\sqrt{1-\lambda_2{}^2}\right)X_0\right)}{16\sqrt{1-\lambda_1{}^2}\left(\lambda_1{}^2-\lambda_2{}^2\right)}$$
$$-\frac{\sinh\left(\sqrt{1-\lambda_1{}^2}X_0\right)\left(\sqrt{1-\lambda_1{}^2}-\sqrt{1-\lambda_2{}^2}\right)\sinh\left(2\left(\sqrt{1-\lambda_1{}^2}+\sqrt{1-\lambda_2{}^2}\right)X_0\right)}{16\sqrt{1-\lambda_1{}^2}\left(\lambda_1{}^2-\lambda_2{}^2\right)}$$
$$+\frac{\left(\sqrt{1-\lambda_2^2}\sinh(2\sqrt{1-\lambda_1^2}X_0)-\sqrt{1-\lambda_1^2}\sinh(2\sqrt{1-\lambda_2^2}X_0)\right)\sinh\left(\sqrt{1-\lambda_1^2}X_0\right)}{8\sqrt{1-\lambda_2^2}(1-\lambda_1^2)}\Big]$$
$$+\left(\alpha_1|B_1|^2+\alpha_2|B_2|^2\right)\Big[\frac{\lambda_1\cosh\left(2\sqrt{1-\lambda_1{}^2}X_0\right)\cosh\left(\sqrt{1-\lambda_1{}^2}X_0\right)}{2\,(1-\lambda_1{}^2)}$$
$$+\frac{\lambda_1\left(\sinh\left(2\sqrt{1-\lambda_1{}^2}X_0\right)+2\sqrt{1-\lambda_1{}^2}X_0\right)\sinh\left(\sqrt{1-\lambda_1{}^2}X_0\right)}{2\,(1-\lambda_1^2)}\Big], \tag{A15}$$

$$\psi_4 = z_1 e^{-\sqrt{1-\lambda_2^2}(X_0-L-a)} - \frac{|B_1|^2 e^{-\left(2\sqrt{1-\lambda_1^2}+\sqrt{1-\lambda_2^2}\right)(X_0-L-a)}}{4\sqrt{1-\lambda_1^2}\left(\sqrt{1-\lambda_2{}^2}+\sqrt{1-\lambda_1{}^2}\right)} - \frac{|B_2|^2 e^{-3\sqrt{1-\lambda_2^2}(X_0-L-a)}}{16(1-\lambda_2^2)}$$
$$+\frac{\lambda_2\left(\alpha_3|B_2|^2+\alpha_4|B_1|^2\right)\left(1+2\sqrt{1-\lambda_2^2}X_0\right)e^{-\sqrt{1-\lambda_2^2}(X_0-L-a)}}{2\,(1-\lambda_2^2)}, \tag{A16}$$

$$\psi_5 = z_2\cos\left(\sqrt{1+\lambda_2^2}\,(X_0-L-a)\right) + z_3\sin\left(\sqrt{1+\lambda_2^2}\,(X_0-L-a)\right)$$
$$-\frac{1}{2\sqrt{1+\lambda_2^2}}\Big[\int F_0(X_0)dX_0\sin(\sqrt{1+\lambda_2^2}X_0) - \int G_0(X_0)dX_0\cos(\sqrt{1+\lambda_2^2}X_0)\Big], \tag{A17}$$

$$
\begin{aligned}
\psi_6 = {} & z_4 \sinh\left(\sqrt{1-{\lambda_2}^2}X_0\right) - \frac{1}{2}K_1^2 K_2 |B_1|^2 \times \\
& [\frac{\cosh\left(\sqrt{1-\lambda_2^2}X_0\right)\left(\sqrt{1-\lambda_1^2}+\sqrt{1-\lambda_2^2}\right)\sinh\left(2(\sqrt{1-\lambda_1^2}-\sqrt{1-\lambda_2^2})X_0\right)}{8\sqrt{1-{\lambda_2}^2}\left({\lambda_1}^2-{\lambda_2}^2\right)} \\
& + \frac{\cosh\left(\sqrt{1-{\lambda_2}^2}X_0\right)\left(\sqrt{1-{\lambda_1}^2}-\sqrt{1-{\lambda_2}^2}\right)\sinh\left(2(\sqrt{1-{\lambda_1}^2}+\sqrt{1-{\lambda_2}^2})X_0\right)}{8\sqrt{1-{\lambda_2}^2}\left({\lambda_1}^2-{\lambda_2}^2\right)} \\
& + \frac{\sinh\left(\sqrt{1-{\lambda_2}^2}X_0\right)\left(\sqrt{1-{\lambda_1}^2}+\sqrt{1-{\lambda_2}^2}\right)\cosh\left(2(\sqrt{1-{\lambda_1}^2}-\sqrt{1-{\lambda_2}^2})X_0\right)}{8\sqrt{1-{\lambda_2}^2}\left({\lambda_1}^2-{\lambda_2}^2\right)} \\
& - \frac{\sinh\left(\sqrt{1-{\lambda_2}^2}X_0\right)\left(\sqrt{1-{\lambda_1}^2}-\sqrt{1-{\lambda_2}^2}\right)\cosh\left(2(\sqrt{1-{\lambda_1}^2}+\sqrt{1-{\lambda_2}^2})X_0\right)}{8\sqrt{1-{\lambda_2}^2}\left({\lambda_1}^2-{\lambda_2}^2\right)} \\
& - \frac{\left(\sqrt{1-\lambda_1^2}\sinh(2\sqrt{1-\lambda_2^2}X_0)-\sqrt{1-\lambda_2^2}\sinh(2\sqrt{1-\lambda_1^2}X_0)\right)\cosh\left(\sqrt{1-\lambda_2^2}X_0\right)}{16\sqrt{1-\lambda_1^2}\,(1-\lambda_2^2)} \\
& + \frac{\cosh\left(2\sqrt{1-{\lambda_2}^2}X_0\right)\sinh\left(\sqrt{1-{\lambda_2}^2}X_0\right)}{4\,(1-{\lambda_2}^2)} + \frac{\cosh\left(\sqrt{1-{\lambda_2}^2}X_0\right)X_0}{8\sqrt{1-{\lambda_2}^2}}] \\
& - \frac{|B_2|^2K_2^3}{64(1-\lambda_2^2)}\left[\left(4\cosh(2\sqrt{1-\lambda_2^2}X_0)-\cosh(4\sqrt{1-\lambda_2^2}X_0)\right)\sinh\left(2\sqrt{1-\lambda_2^2}X_0\right)\right. \\
& - \left(\sinh\left(4\sqrt{1-{\lambda_2}^2}X_0\right) - 8\sqrt{1-{\lambda_1}^2}\sinh\left(2\sqrt{1-{\lambda_2}^2}X_0\right) + 12\sqrt{1-{\lambda_2}^2}X_0\right) \\
& \left.\times\cosh\left(\sqrt{1-{\lambda_2}^2}X_0\right)\right] \\
& - \frac{K_2(\alpha_3|B_2|^2+\alpha_4|B_1|^2)}{2(1-\lambda_2^2)}\left[\lambda_2\cosh\left(2\sqrt{1-{\lambda_2}^2}X_0\right)\sinh\left(\sqrt{1-{\lambda_2}^2}X_0\right)\right. \\
& \left.+\cosh\left(2\sqrt{1-{\lambda_2}^2}X_0\right)\lambda_2\left(2\sqrt{1-{\lambda_2}^2}X_0-\sinh\left(2\sqrt{1-{\lambda_2}^2}X_0\right)\right)\right],
\end{aligned}
\tag{A18}
$$

$$
\begin{aligned}
\psi_7 = {} & z_1 e^{-\sqrt{1-(2\lambda_1+\lambda_2)^2}(X_0-L-a)} \\
& - \frac{\left(1+\lambda_1\lambda_2-\sqrt{1-\lambda_1^2}\sqrt{1-{\lambda_2}^2}\right)e^{-\left(2\sqrt{1-{\lambda_1}^2}+\sqrt{1-{\lambda_2}^2}\right)(X_0-L-a)}}{8\,(\lambda_1+\lambda_2)^2},
\end{aligned}
\tag{A19}
$$

$$
\begin{aligned}
\psi_8 = {} & z_2\cos(\sqrt{1+(2\lambda_1+\lambda_2)^2}(X_0-L-a)) + z_3\sin(\sqrt{1+(2\lambda_1+\lambda_2)^2}\,(X_0-L-a)) \\
& + \frac{\int\cos\left(\sqrt{1+(2\,\lambda_1+\lambda_2)^2}X_0\right)G_2(X_0)\,dX_0\sin\left(\sqrt{1+(2\,\lambda_1+\lambda_2)^2}X_0\right)}{2\sqrt{1+(2\,\lambda_1+\lambda_2)^2}}
\end{aligned}
$$

$$-\frac{\int \sin\left(\sqrt{1+(2\,\lambda_1+\lambda_2)^2}X_0\right) G_2(X_0)\,dX_0 \cos\left(\sqrt{1+(2\,\lambda_1+\lambda_2)^2}X_0\right)}{2\sqrt{1+(2\,\lambda_1+\lambda_2)^2}}, \tag{A20}$$

$$\psi_9 = z_4\cosh\left(\sqrt{1-(2\,\lambda_1+\lambda_2)^2}X_0\right) - \frac{K_1^2 K_2}{2\sqrt{(2\lambda_1+\lambda_2)^2-1}}$$
$$\times\Big[\int F_4(X_0)\,dX_0 \sin\left(\sqrt{(2\,\lambda_1+\lambda_2)^2-1}X_0\right) - \int G_4(X_0)\,dX_0\cos\left(\sqrt{(2\,\lambda_1+\lambda_2)^2-1}X_0\right)\Big], \tag{A21}$$

$$\psi_{10} = z_1 \mathrm{e}^{-\sqrt{1-(2\,\lambda_1-\lambda_2)^2}(X_0-L-a)}$$
$$+\frac{\left(\lambda_1\lambda_2+\sqrt{1-\lambda_1^2}\sqrt{1-{\lambda_2}^2}-1\right)\mathrm{e}^{-\left(2\sqrt{1-{\lambda_1}^2}+\sqrt{1-{\lambda_2}^2}\right)(X_0-L-a)}}{8\,(\lambda_1-\lambda_2)^2}, \tag{A22}$$

$$\psi_{11} = z_2\cos(\sqrt{1+(2\,\lambda_1-\lambda_2)^2}\,(X_0-L-a)) + z_3\sin(\sqrt{1+(2\,\lambda_1-\lambda_2)^2}\,(X_0-L-a))$$
$$+\frac{\int \cos\left(\sqrt{1+(2\,\lambda_1-\lambda_2)^2}X_0\right) G_2(X_0)\,dX_0 \sin\left(\sqrt{1+(2\,\lambda_1-\lambda_2)^2}X_0\right)}{2\sqrt{1+(2\,\lambda_1-\lambda_2)^2}}$$
$$-\frac{\int \sin\left(\sqrt{1+(2\,\lambda_1-\lambda_2)^2}X_0\right) G_2(X_0)\,dX_0 \cos\left(\sqrt{1+(2\,\lambda_1-\lambda_2)^2}X_0\right)}{2\sqrt{1+(2\,\lambda_1-\lambda_2)^2}}, \tag{A23}$$

$$\psi_{12} = z_4\cosh\left(\sqrt{1-(2\,\lambda_1-\lambda_2)^2}X_0\right) - \frac{K_1^2 K_2}{4\sqrt{1-(2\,\lambda_1-\lambda_2)^2}}\times$$
$$\Big[\int F_5(X_0)\,dX_0\,\mathrm{e}^{\sqrt{1-(2\,\lambda_1-\lambda_2)^2}\,X_0} - \int G_5(X_0)dX_0\,\mathrm{e}^{-\sqrt{1-(2\,\lambda_1+\lambda_2)^2}\,X_0}\Big], \tag{A24}$$

$$\psi_{13} = z_1 e^{-\sqrt{1-9\,\lambda_1^2}(X_0-L-a)} - \frac{1}{48}\mathrm{e}^{-3\sqrt{1-\lambda_1^2}(X_0-L-a)}, \tag{A25}$$

$$\psi_{14} = z_2\sin\left(\sqrt{1+9\,\lambda_1^2}\,(X_0-L-a)\right) + z_3\cos\left(\sqrt{1+9\,{\lambda_1}^2}\,(X_0-L-a)\right)$$
$$+\frac{3\left(C_1^2+1\right)\left(C_1\sin\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)-\cos\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)\right)}{192\,\lambda_1^2}$$
$$+\frac{\left(3C_1^2-1\right)\cos(3\sqrt{1+\lambda_1^2}\,(X_0-L-a))}{192} + \frac{C_1\left(C_1^2-3\right)\sin(3\sqrt{1+\lambda_1^2}\,(X_0-L-a))}{192}, \tag{A26}$$

$$\psi_{15} = z_4\cosh\left(\sqrt{1-9\,{\lambda_1}^2}X_0\right) - \frac{K_1^3\cosh\left(\sqrt{1-9\,\lambda_1^2}X_0\right)}{384\,\lambda_1^2\sqrt{1-9\,\lambda_1^2}}\times$$
$$\Big[3\left(\sqrt{1-9\,{\lambda_1}^2}+\sqrt{1-{\lambda_1}^2}\right)\cosh\left((\sqrt{1-9\,{\lambda_1}^2}-\sqrt{1-{\lambda_1}^2})X_0\right)$$
$$+{\lambda_1}^2\left(\sqrt{1-9\,{\lambda_1}^2}+3\sqrt{1-{\lambda_1}^2}\right)\cosh\left((\sqrt{1-9\,{\lambda_1}^2}-3\sqrt{1-{\lambda_1}^2})X_0\right)$$

$$+\lambda_1{}^2\left(\sqrt{1-9\,\lambda_1{}^2}-3\,\sqrt{1-\lambda_1{}^2}\right)\cosh\left((\sqrt{1-9\,\lambda_1{}^2}+3\,\sqrt{1-\lambda_1{}^2})X_0\right)$$
$$+3\,\left(\sqrt{1-9\,\lambda_1{}^2}-\sqrt{1-\lambda_1{}^2}\right)\cosh\left(\left(\sqrt{1-9\,\lambda_1{}^2}+\sqrt{1-\lambda_1{}^2}\right)X_0\right)\Big]$$
$$-\frac{K_1^3\sinh\left(\sqrt{1-9\,\lambda_1^2}X_0\right)}{384\,\lambda_1^2\sqrt{1-9\,\lambda_1^2}}\times$$
$$\Big[3\,\left(\sqrt{1-9\,\lambda_1{}^2}+\sqrt{1-\lambda_1{}^2}\right)\sinh\left(\left(\sqrt{1-9\,\lambda_1{}^2}-\sqrt{1-\lambda_1{}^2}\right)X_0\right)$$
$$-\lambda_1{}^2\left(\sqrt{1-9\,\lambda_1{}^2}-3\,\sqrt{1-\lambda_1{}^2}\right)\sinh\left(\left(\sqrt{1-9\,\lambda_1{}^2}+3\,\sqrt{1-\lambda_1{}^2}\right)X_0\right)$$
$$-3\,\left(\sqrt{1-9\,\lambda_1{}^2}-\sqrt{1-\lambda_1{}^2}\right)\sinh\left(\left(\sqrt{1-9\,\lambda_1{}^2}+\sqrt{1-\lambda_1{}^2}\right)X_0\right)$$
$$-\lambda_1^2\left(\sqrt{1-9\lambda_1^2}+3\sqrt{1-\lambda_1^2}\right)\sinh\left((\sqrt{1-9\lambda_1^2}-3\sqrt{1-\lambda_1^2})X_0\right)\Big],\tag{A27}$$

$$\psi_{16}=z_1\mathrm{e}^{-\sqrt{1-9\,\lambda_2^2}(X_0-L-a)}-\frac{1}{48}\mathrm{e}^{-3\,\sqrt{1-\lambda_2^2}(X_0-L-a)},\tag{A28}$$

$$\psi_{17}=z_2\cos\left(\sqrt{1+9\,\lambda_2{}^2}\,(X_0-L-a)\right)+z_3\sin\left(\sqrt{1+9\,\lambda_2{}^2}\,(X_0-L-a)\right)$$
$$+\frac{3\,(C_2^2+1)\left(C_2\sin\left(\sqrt{1+\lambda_2^2}\,(X_0-L-a)\right)-\cos\left(\sqrt{1+\lambda_2^2}\,(X_0-L-a)\right)\right)}{192\,\lambda_2^2}$$
$$+\frac{(3\,C_2^2-1)\cos\left(3\,\sqrt{1+\lambda_2^2}\,(X_0-L-a)\right)}{192}+\frac{C_2\,(C_2^2-3)\sin\left(3\,\sqrt{1+\lambda_2^2}\,(X_0-L-a)\right)}{192},\tag{A29}$$

$$\psi_{18}=z_4\sinh\left(\sqrt{1-9\,\lambda_2{}^2}X_0\right)+\frac{K_2^3\sinh\left(2\,\sqrt{1-9\,\lambda_2{}^2}X_0\right)}{384\,\lambda_2{}^2\sqrt{1-9\,\lambda_2{}^2}}\times$$
$$\Big[3\left(\sqrt{1-9\,\lambda_2{}^2}+\sqrt{1-\lambda_2{}^2}\right)\cosh\left(\left(\sqrt{1-9\,\lambda_2{}^2}-\sqrt{1-\lambda_2^2}\right)X_0\right)$$
$$-\lambda_2{}^2\left(\sqrt{1-9\,\lambda_2{}^2}+3\,\sqrt{1-\lambda_2{}^2}\right)\cosh\left(\left(\sqrt{1-9\,\lambda_2{}^2}-3\,\sqrt{1-\lambda_2{}^2}\right)X_0\right)$$
$$+\lambda_2{}^2\left(\sqrt{1-9\,\lambda_2{}^2}-3\,\sqrt{1-\lambda_2{}^2}\right)\cosh\left(\left(\sqrt{1-9\,\lambda_2{}^2}+3\,\sqrt{1-\lambda_2{}^2}\right)X_0\right)$$
$$-3\,\left(\sqrt{1-9\,\lambda_2{}^2}-\sqrt{1-\lambda_2{}^2}\right)\cosh\left(\left(\sqrt{1-9\,\lambda_2{}^2}+\sqrt{1-\lambda_2{}^2}\right)X_0\right)\Big]$$
$$-\frac{K_2^3\cosh\left(2\,\sqrt{1-9\,\lambda_2{}^2}X_0\right)}{384\,\lambda_2{}^2\sqrt{1-9\,\lambda_2{}^2}}\times$$
$$\Big[3\,\left(\sqrt{1-9\,\lambda_2{}^2}+\sqrt{1-\lambda_2{}^2}\right)\sinh\left(\left(\sqrt{1-9\,\lambda_2{}^2}-\sqrt{1-\lambda_2{}^2}\right)X_0\right)$$
$$+\lambda_2{}^2\left(\sqrt{1-9\,\lambda_2{}^2}-3\,\sqrt{1-\lambda_2{}^2}\right)\sinh\left(\left(\sqrt{1-9\,\lambda_2{}^2}+3\,\sqrt{1-\lambda_2{}^2}\right)X_0\right)$$

$$-3\left(\sqrt{1-9\lambda_2{}^2}-\sqrt{1-\lambda_2{}^2}\right)\sinh\left(\left(\sqrt{1-9\lambda_2{}^2}+\sqrt{1-\lambda_2{}^2}\right)X_0\right)$$

$$-\lambda_2^2\left(\sqrt{1-9\lambda_2^2}+3\sqrt{1-\lambda_2^2}\right)\sinh\left((\sqrt{1-9\lambda_2^2}-3\sqrt{1-\lambda_2^2})X_0\right)\Big], \quad \text{(A30)}$$

$$\psi_{19}=z_1\mathrm{e}^{-\sqrt{1-(\lambda_1+2\lambda_2)^2}(X_0-L-a)}$$

$$-\frac{\left(1+\lambda_1\lambda_2-\sqrt{(1-\lambda_2^2)(1-\lambda_1^2)}\right)\mathrm{e}^{-\left(2\sqrt{1-\lambda_2^2}+\sqrt{1-\lambda_1^2}\right)(X_0-L-a)}}{8\,(\lambda_1+\lambda_2)^2}, \quad \text{(A31)}$$

$$\psi_{20}=z_2\cos\left(\sqrt{1+(\lambda_1+2\lambda_2)^2}\,(X_0-L-a)\right)$$

$$+z_3\sin\left(\sqrt{1+(\lambda_1+2\lambda_2)^2}\,(X_0-L-a)\right)$$

$$+\frac{\int\cos\left(\sqrt{1+(\lambda_1+2\lambda_2)^2}X_0\right)F_2(X_0)\,dX_0\sin\left(\sqrt{1+(\lambda_1+2\lambda_2)^2}X_0\right)}{2\sqrt{1+(\lambda_1+2\lambda_2)^2}}$$

$$-\frac{\int\sin\left(\sqrt{1+(\lambda_1+2\lambda_2)^2}X_0\right)F_2dX_0\cos\left(\sqrt{1+(\lambda_1+2\lambda_2)^2}X_0\right)}{2\sqrt{1+(\lambda_1+2\lambda_2)^2}}, \quad \text{(A32)}$$

$$\psi_{21}=z_4\cosh\left(\sqrt{1-(\lambda_1+2\lambda_2)^2}X_0\right)-\frac{K_1K_2^2}{4\,(\lambda_1+2\lambda_2)^2}\times$$

$$\Big[\int F_3(X_0)\,dX_0\,\mathrm{e}^{\sqrt{1-(\lambda_1+2\lambda_2)^2}X_0}-\int G_3(X_0)\,dX_0\,\mathrm{e}^{-\sqrt{1-(\lambda_1+2\lambda_2)^2}X_0}\Big], \quad \text{(A33)}$$

$$\psi_{22}=z_1\mathrm{e}^{-\sqrt{1-(\lambda_1-2\,\lambda_2)^2}(X_0-L-a)}+$$

$$\frac{\left(\sqrt{1-\lambda_1{}^2}\sqrt{1-\lambda_2^2}-1+\lambda_1\lambda_2\right)\mathrm{e}^{-\left(\sqrt{1-\lambda_1^2}+2\sqrt{1-\lambda_2^2}\right)(X_0-L-a)}}{8\,(\lambda_1-\lambda_2)^2}, \quad \text{(A34)}$$

$$\psi_{23}=z_2\cos\left(\sqrt{1+(\lambda_1-2\,\lambda_2)^2}\,(X_0-L-a)\right)$$

$$+z_3\sin\left(\sqrt{1+(\lambda_1-2\,\lambda_2)^2}\,(X_0-L-a)\right)$$

$$+\frac{\int\cos\left(\sqrt{1+(\lambda_1-2\lambda_2)^2}X_0\right)F_2(X_0)\,dX_0\sin\left(\sqrt{1+(\lambda_1-2\lambda_2)^2}X_0\right)}{2\sqrt{1+(\lambda_1-2\lambda_2)^2}}-$$

$$\frac{\int\sin\left(\sqrt{1+(\lambda_1-2\lambda_2)^2}X_0\right)F_2(X_0)dX_0\cos\left(\sqrt{1+(\lambda_1-2\lambda_2)^2}X_0\right)}{2\sqrt{1+(\lambda_1-2\lambda_2)^2}}, \quad \text{(A35)}$$

$$\psi_{24}=z_4\cosh\left(\sqrt{1-(\lambda_1-2\lambda_2)^2}X_0\right)-\frac{K_1K_2^2}{4\,(\lambda_1-2\lambda_2)^2}\times$$

$$\Big[\int F_{41}(X_0)\,dX_0\,\mathrm{e}^{\sqrt{1-(\lambda_1-2\lambda_2)^2}X_0}-\int G_{41}(X_0)\,dX_0\,\mathrm{e}^{-\sqrt{1-(\lambda_1-2\lambda_2)^2}X_0}\Big], \quad \text{(A36)}$$

$$F_0(X_0)=2|B_1|^2\left(\cos(\sqrt{1+\lambda_2{}^2}\,(X_0-L-a))+C_2\sin(\sqrt{1+\lambda_2{}^2}\,(X_0-L-a))\right)$$

$$\times\Big[(C_1^2-1)\cos^2(\sqrt{1+\lambda_1^2}(X_0-L-a))-C_1\sin(2\sqrt{1+\lambda_1^2}(X_0-L-a))-C_1^2\Big]$$
$$\times\cos\left(\sqrt{1+\lambda_2{}^2}X_0\right)+|B_2|^2\Big(\big(3\,C_2^2-1\big)\cos^3\left(\sqrt{1+\lambda_2^2}(X_0-L-a)\right)$$
$$+C_2\big(C_2^2-3\big)\sin\left(\sqrt{1+\lambda_2^2}(X_0-L-a)\right)\cos^2\left(\sqrt{1+\lambda_2{}^2}(X_0-L-a)\right)$$
$$-3C_2^2\cos\left(\sqrt{1+\lambda_2^2}(X_0-L-a)\right)-C_2^3\sin\left(\sqrt{1+\lambda_2^2}+(X_0-L-a)\right)\Big)$$
$$\times\cos\left(\sqrt{1+\lambda_2^2}X_0\right)+4\lambda_2\big(\alpha_3|B_2|^2+\alpha_4|B_1|^2\big)\Big(\cos(\sqrt{1+\lambda_2^2}(X_0-L-a))$$
$$+C_2\sin(\sqrt{1+\lambda_2^2}(X_0-L-a))\Big)\cos\left(\sqrt{1+\lambda_2{}^2}X_0\right), \tag{A37}$$

$$G_0(X_0)=2|B_1|^2\left(\cos(\sqrt{1+\lambda_2{}^2}(X_0-L-a))+C_2\sin(\sqrt{1+\lambda_2{}^2}(X_0-L-a))\right)$$
$$\times\Big[(C_1^2-1)\cos^2(\sqrt{1+\lambda_1^2}(X_0-L-a))-C_1\sin(2\sqrt{1+\lambda_1^2}(X_0-L-a))-C_1^2\Big]$$
$$\times\sin\left(\sqrt{1+\lambda_2{}^2}X_0\right)+|B_2|^2\Big(\big(3C_2^2-1\big)\cos^3\left(\sqrt{1+\lambda_2^2}(X_0-L-a)\right)$$
$$+C_2\big(C_2^2-3\big)\sin\left(\sqrt{1+\lambda_2^2}(X_0-L-a)\right)\cos^2\left(\sqrt{1+\lambda_2{}^2}(X_0-L-a)\right)$$
$$-3\,C_2^2\cos\left(\sqrt{1+\lambda_2^2}(X_0-L-a)\right)-C_2^3\sin\left(\sqrt{1+\lambda_2^2}+(X_0-L-a)\right)\Big)$$
$$\times\sin\left(\sqrt{1+\lambda_2^2}X_0\right)+4\lambda_2\big(\alpha_3|B_2|^2+\alpha_4|B_1|^2\big)\Big(\cos\left(\sqrt{1+\lambda_2^2}(X_0-L-a)\right)$$
$$+C_2\sin\left(\sqrt{1+\lambda_2^2}(X_0-L-a)\right)\Big)\sin\left(\sqrt{1+\lambda_2{}^2}X_0\right),$$

$$A_1(X_0)=\Big[\cos(\sqrt{1+\lambda_1{}^2}(X_0-L-a))+C_1\sin(\sqrt{1+\lambda_1{}^2}(X_0-L-a))\Big]^2$$
$$\times\Big[\cos(\sqrt{1+\lambda_2{}^2}(X_0-L-a))+C_2\sin(\sqrt{1+\lambda_2{}^2}(X_0-L-a))\Big]^2,$$
$$A_2(X_0)=\cosh^2\left(\sqrt{1-\lambda_1^2}X_0\right)\sinh^2\left(\sqrt{1-\lambda_2^2}X_0\right), \tag{A38}$$

$$F_1(X_0)=|B_1|^2\cos\left(\sqrt{1+\lambda_1^2}X_0\right)\Big[(3C_1{}^2-1)\cos^3\left(\sqrt{1+\lambda_1{}^2}(X_0-L-a)\right)$$
$$+C_1(C_1^2-3)\sin\left(\sqrt{1+\lambda_1{}^2}(X_0-L-a)\right)\cos^2\left(\sqrt{1+\lambda_1{}^2}(X_0-L-a)\right)$$
$$-3C_1^2\cos\left(\sqrt{1+\lambda_1{}^2}(X_0-L-a)\right)-C_1^3\sin\left(\sqrt{1+\lambda_1{}^2}(X_0-L-a)\right)\Big]$$
$$-2|B_2|^2\cos\left(\sqrt{1+\lambda_1^2}X_0\right)\Big[\cos\left(\sqrt{1+\lambda_1{}^2}(X_0-L-a)\right)$$
$$+C_1\sin\left(\sqrt{1+\lambda_1^2}(X_0-L-a)\right)\Big]\Big[(C_2^2-1)\cos^2\left(\sqrt{1+\lambda_2^2}(X_0-L-a)\right)$$
$$-C_2\sin(2\sqrt{1+\lambda_2{}^2}(X_0-L-a))-C_2^2\Big]$$
$$-4\lambda_1\big(\alpha_1|B_1|^2+\alpha_2B_2|^2\big)\cos\left(\sqrt{1+\lambda_1{}^2}X_0\right)\Big[\cos\left(\sqrt{1+\lambda_1{}^2}(X_0-L-a)\right)$$
$$+C_1\sin\left(\sqrt{1+\lambda_1{}^2}(X_0-L-a)\right)\Big], \tag{A39}$$

$$
\begin{aligned}
G_1(X_0) =& -|B_1|^2 \sin\left(\sqrt{1+\lambda_1{}^2}X_0\right)\Big[\left(3C_1{}^2-1\right)\cos^3\left(\sqrt{1+\lambda_1{}^2}\,(X_0-L-a)\right)\\
&+C_1(C_1{}^2-3)\sin\left(\sqrt{1+\lambda_1{}^2}(X_0-L-a)\right)\cos^2\left(\sqrt{1+\lambda_1{}^2}(X_0-L-a)\right)\\
&-\left(3\,C_1{}^2-4\,\lambda_1\alpha_1\right)\cos\left(\sqrt{1+\lambda_1{}^2}\,(X_0-L-a)\right)-C_1\left(C_1{}^2-4\,\lambda_1\alpha_1\right)\\
&\times\sin\left(\sqrt{1+\lambda_1{}^2}\,(X_0-L-a)\right)\Big]-2|B_2|^2\sin\left(\sqrt{1+\lambda_1{}^2}X_0\right)\\
&\left(\cos(\sqrt{1+\lambda_1{}^2}\,(X_0-L-a))+C_1\sin(\sqrt{1+\lambda_1{}^2}\,(X_0-L-a))\right)\\
&\times\Big[(C_2{}^2-1)\cos^2\left(\sqrt{1+\lambda_2^2}(X_0-L-a)\right)-2C_2\sin\left(\sqrt{1+\lambda_2^2}(X_0-L-a)\right)\\
&\times\cos\left(\sqrt{1+\lambda_2{}^2}\,(X_0-L-a)\right)-C_2{}^2+2\,\lambda_1\alpha_2\Big],
\end{aligned}
\tag{A40}
$$

$$
\begin{aligned}
F_2(X_0) =& \left[\cos\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)+C_1\sin\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)\right]\\
&\times\left[\cos\left(\sqrt{1+\lambda_2^2}\,(X_0-L-a)\right)+C_2\sin\left(\sqrt{1+\lambda_2}^2\,(X_0-L-a)\right)\right]^2,
\end{aligned}
\tag{A41}
$$

$$
\begin{aligned}
G_2(X_0) =& \left[\cos\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)+C_1\sin\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)\right]^2\\
&\times\left[\cos\left(\sqrt{1+\lambda_2^2}\,(X_0-L-a)\right)+C_2\sin\left(\sqrt{1+\lambda_2^2}\,(X_0-L-a)\right)\right],
\end{aligned}
\tag{A42}
$$

$$F_3(X_0) = \mathrm{e}^{-\sqrt{1-(\lambda_1+2\lambda_2)^2}X_0}\cosh(\sqrt{1-\lambda_1^2}\,X_0)\sinh^2(\sqrt{1-\lambda_2^2}\,X_0), \tag{A43}$$

$$G_3(X_0) = \mathrm{e}^{\sqrt{1-(\lambda_1+2\lambda_2)^2}X_0}\cosh(\sqrt{1-\lambda_1^2}\,X_0)\sinh^2(\sqrt{1-\lambda_2^2}\,X_0), \tag{A44}$$

$$F_4(X_0) = \mathrm{e}^{-\sqrt{1-(\lambda_1-2\lambda_2)^2}X_0}\cosh(\sqrt{1-\lambda_1^2}\,X_0)\sinh^2(\sqrt{1-\lambda_2^2}\,X_0), \tag{A45}$$

$$G_4(X_0) = \mathrm{e}^{\sqrt{1-(\lambda_1-2\lambda_2)^2}X_0}\cosh(\sqrt{1-\lambda_1^2}\,X_0)\sinh^2(\sqrt{1-\lambda_2^2}\,X_0), \tag{A46}$$

$$F_{41}(X_0) = \cos\left(\sqrt{(2\,\lambda_1+\lambda_2)^2-1}\;X_0\right)\cosh^2(\sqrt{1-\lambda_1^2}\,X_0)\sinh(\sqrt{1-\lambda_2^2}\,X_0), \tag{A47}$$

$$G_{41}(X_0) = \sin\left(\sqrt{(2\,\lambda_1+\lambda_2)^2-1}\;X_0\right)\cosh^2(\sqrt{1-\lambda_1^2}\,X_0)\sinh(\sqrt{1-\lambda_2^2}\,X_0), \tag{A48}$$

$$F_5(X_0) = \mathrm{e}^{-\sqrt{1-(2\,\lambda_1-\lambda_2)^2}\;X_0}\cosh^2(\sqrt{1-\lambda_1^2}\,X_0)\sinh(\sqrt{1-\lambda_2^2}\,X_0), \tag{A49}$$

$$G_5(X_0) = \mathrm{e}^{\sqrt{1-(2\,\lambda_1-\lambda_2)^2}\;X_0}\cosh^2(\sqrt{1-\lambda_1^2}\,X_0)\sinh(\sqrt{1-\lambda_2^2}\,X_0). \tag{A50}$$

APPENDIX B: FUNCTIONS IN SEC. III

$$\Psi_1 = |B_1|^2\Big[\frac{\lambda_1\alpha_1\left(-2\,X_0\lambda_2{}^2+\sqrt{1-\lambda_2{}^2}+2\,X_0\right)\mathrm{e}^{\sqrt{1-\lambda_1{}^2}(L+a-X_0)}}{2\sqrt{1-\lambda_1{}^2}\sqrt{1-\lambda_2{}^2}\left(\sqrt{1-\lambda_1{}^2}+\sqrt{1-\lambda_2{}^2}\right)}$$

$$-\frac{e^{3\sqrt{1-\lambda_1{}^2}(L+a-X_0)}}{16(1-\lambda_1{}^2)}-\frac{\lambda_1\alpha_1\left(2X_0\lambda_1{}^2-2X_0-\sqrt{1-\lambda_2{}^2}\right)e^{\sqrt{1-\lambda_1{}^2}(L+a-X_0)}}{2\left(1-\lambda_1{}^2\right)\left(\sqrt{1-\lambda_1{}^2}+\sqrt{1-\lambda_2{}^2}\right)}\Bigg]$$

$$+|B_2|^2\Bigg[\frac{\lambda_1\alpha_2\left(-2X_0\lambda_2{}^2+\sqrt{1-\lambda_2{}^2}+2X_0\right)e^{\sqrt{1-\lambda_1{}^2}(L+a-X_0)}}{2\sqrt{1-\lambda_1{}^2}\sqrt{1-\lambda_2{}^2}\left(\sqrt{1-\lambda_1{}^2}+\sqrt{1-\lambda_2{}^2}\right)}$$

$$-\frac{\lambda_1\alpha_2\left(2X_0\lambda_1{}^2-2X_0-\sqrt{1-\lambda_2{}^2}\right)e^{\sqrt{1-\lambda_1{}^2}(L+a-X_0)}}{2\left(1-\lambda_1^2\right)\left(\sqrt{1-\lambda_1^2}+\sqrt{1-\lambda_2^2}\right)}-\frac{e^{\left(\sqrt{1-\lambda_1{}^2}+2\sqrt{1-\lambda_2{}^2}\right)(L-X_0+a)}}{4(\sqrt{1-\lambda_2^2}\sqrt{1-\lambda_1^2}+1-\lambda_2^2)}\Bigg], \tag{B1}$$

$$\Psi_2=\frac{\lambda_1\mu_1\left(\left(x\lambda_1{}^2-x\right)\sqrt{1-\lambda_2{}^2}-\frac{1}{2}+\frac{1}{2}\lambda_2{}^2\right)e^{\sqrt{1-\lambda_1{}^2}(-x+a+L)}}{\left(1-\lambda_1{}^2\right)\left(-\sqrt{1-\lambda_2{}^2}\sqrt{1-\lambda_1{}^2}-1+\lambda_2{}^2\right)}$$
$$+\frac{\mu_1\lambda_1\left(x\lambda_2{}^2-1/2\sqrt{1-\lambda_2{}^2}-x\right)e^{\sqrt{1-\lambda_1{}^2}(-x+a+L)}}{\sqrt{1-\lambda_1{}^2}\left(-\sqrt{1-\lambda_2{}^2}\sqrt{1-\lambda_1{}^2}-1+\lambda_2{}^2\right)}-\frac{1}{2(1-\lambda_1^2)}, \tag{B2}$$

$$\Psi_3=\frac{\int F_6(X_0)\cos\left(\sqrt{1+\lambda_1^2}X_0\right)dX_0\sin\left(\sqrt{1+\lambda_1^2}X_0\right)}{\sqrt{1+\lambda_1^2}}$$
$$-\frac{\int F_6(X_0)\sin\left(\sqrt{1+\lambda_1^2}X_0\right)dX_0\cos\left(\sqrt{1+\lambda_1^2}X_0\right)}{\sqrt{1+\lambda_1^2}}, \tag{B3}$$

$$\Psi_4=\frac{\int G_6(X_0)\cos\left(\sqrt{1+\lambda_1^2}X_0\right)dX_0\sin\left(\sqrt{1+\lambda_1^2}X_0\right)}{\sqrt{1+\lambda_1^2}}$$
$$-\frac{\int G_6(X_0)\sin\left(\sqrt{1+\lambda_1^2}X_0\right)dX_0\cos\left(\sqrt{1+\lambda_1^2}X_0\right)}{\sqrt{1+\lambda_1^2}}, \tag{B4}$$

$$\Psi_5=\frac{\int e^{-\sqrt{1-\lambda_1^2}X_0}F_7(X_0)d\,X_0e^{\sqrt{1-\lambda_1^2}X_0}-\int e^{\sqrt{1-\lambda_1^2}X_0}F_7(X_0)dX_0e^{-\sqrt{1-\lambda_1^2}X_0}}{4\sqrt{1-\lambda_1^2}}, \tag{B5}$$

$$\Psi_6=\frac{\int e^{-\sqrt{1-\lambda_1^2}X_0}G_7(X_0)dX_0e^{\sqrt{1-\lambda_1^2}X_0}-\int e^{\sqrt{1-\lambda_1^2}X_0}G_7(X_0)dX_0e^{-\sqrt{1-\lambda_1^2}X_0}}{4\sqrt{1-\lambda_1^2}}, \tag{B6}$$

$$\begin{aligned}F_6(X_0)=&\ 8|B_1|^2\left(1-3C_1^2\right)\cos^3\left(\sqrt{1+\lambda_1{}^2}\,(X_0-L-a)\right)\\&-\frac{1}{2}|B_1|^2C_1\left(C_1^2-3\right)\sin\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)\cos^2\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)\\&+\frac{1}{2}|B_1|^2C_1^2\left(3\cos\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)+C_1\sin\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)\right)\\&-\Big[|B_2|^2\left(C_2^2-1\right)\cos^2\left(\sqrt{1+\lambda_2^2}\,(X_0-L-a)\right)\\&-C_2|B_2|^2\sin\left(2\sqrt{1+\lambda_2^2}\,(X_0-L-a)\right)-|B_2|^2C_2^2+2\lambda_1\left(\alpha_1|B_1|^2+\alpha_2|B_2|^2\right)\Big]\\&\times\left[\cos\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)+C_1\sin\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)\right],\end{aligned}$$

$$G_6(X_0) = \frac{1}{2} - \lambda_1\mu_1\left[2\cos\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right) + C_1\sin\left(\sqrt{1+\lambda_1^2}\,(X_0-L-a)\right)\right], \tag{B7}$$

$$\begin{aligned} F_7(X_0) &= 2K_1\left(|B_2|^2K_2^2 - 2\lambda_1\left(\alpha_1|B_1|^2+\alpha_2|B_2|^2\right)\right)\cosh\left(\sqrt{1-\lambda_1{}^2}X_0\right) \\ &\quad -2|B_2|^2K_1\cosh\left(\sqrt{1-\lambda_1{}^2}X_0\right)K_2^2\cosh^2\left(\sqrt{1-\lambda_2^2}X_0\right) - |B_1|^2K_1^3\cosh^3\left(\sqrt{1-\lambda_1^2}X_0\right), \end{aligned} \tag{B8}$$

$$G_7(X_0) = 1 - 4\,\lambda_1\mu_3K_1\cosh\left(\sqrt{1-\lambda_1{}^2}X_0\right), \tag{B9}$$

$$\begin{aligned} L_1 &= 2\,i\lambda_1D_4B_1\Phi_1 - \left(\alpha_1|B_1|^2+\alpha_2|B_2|^2\right)\left(\alpha_1B_1|B_1|^2+\alpha_2B_1|B_2|^2+\mu_1H\right)\Phi_1 \\ &\quad -2\lambda_1\left(\alpha_1B_1|B_1|^2+\alpha_2B_1|B_2|^2+\mu_1H\right)\Psi_1 + \frac{1}{12}B_1|B_1|^4\Phi_1^5 + \frac{1}{2}B_1|B_1|^2|B_2|^2\Phi_1^3\Phi_2^2 \\ &\quad +\frac{1}{4}B_1|B_2|^4\Phi_1\Phi_2^4 - \frac{3}{2}B_1|B_1|^2\Phi_1^2\Psi_1 - |B_2|^2B_1\Phi_2^2\Psi_1 - |B_1|^2H\Phi_1^2\Psi_2 - |B_2|^2H\Phi_2^2\Psi_2 \\ &\quad -\frac{1}{2}B_1{}^2H\Phi_1{}^2\Psi_2 - 2\,B_1|B_2|^2\Phi_1\Phi_2\psi_4 - \frac{1}{2}B_1|B_1|^4\Phi_1{}^2\psi_{13} - \frac{1}{2}B_1|B_2|^4\Phi_2^2\psi_{22} \\ &\quad -\frac{1}{2}B_1|B_2|^4\Phi_2{}^2\psi_{19} - B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_7 - B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{10}, \end{aligned} \tag{B10}$$

$$\begin{aligned} L_2 &= 2\,i\lambda_1D_4B_1\Phi_1 - \left(\alpha_1|B_1|^2+\alpha_2|B_2|^2\right)\left(\alpha_1B_1|B_1|^2+\alpha_2B_1|B_2|^2+\mu_1H\right)\Phi_1 \\ &\quad -2\lambda_1\left(\alpha_1B_1|B_1|^2+\alpha_2B_1|B_2|^2+\mu_1H\right)\Psi_3 - \frac{1}{12}B_1|B_1|^4\Phi_1^5 - \frac{1}{2}B_1|B_1|^2|B_2|^2\Phi_1^3\Phi_2^2 \\ &\quad -\frac{1}{4}B_1|B_2|^4\Phi_1\Phi_2^4 + \frac{3}{2}B_1|B_1|^2\Phi_1^2\Psi_3 + |B_2|^2B_1\Phi_2^2\Psi_3 + |B_1|^2H\Phi_1^2\Psi_4 + |B_2|^2H\Phi_2^2\Psi_4 \\ &\quad +\frac{1}{2}B_1{}^2H\Phi_1{}^2\Psi_4 + 2\,B_1|B_2|^2\Phi_1\Phi_2\psi_5 + \frac{1}{2}B_1|B_1|^4\Phi_1{}^2\psi_{14} + \frac{1}{2}B_1|B_2|^4\Phi_2{}^2\psi_{23} \\ &\quad +\frac{1}{2}B_1|B_2|^4\Phi_2{}^2\psi_{20} + B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_8 + B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{11}, \end{aligned} \tag{B11}$$

$$\begin{aligned} L_3 &= 2\,i\lambda_1D_4B_1\Phi_1 - \left(\alpha_1|B_1|^2+\alpha_2|B_2|^2\right)\left(\alpha_1B_1|B_1|^2+\alpha_2B_1|B_2|^2+\mu_1H\right)\Phi_1 \\ &\quad -2\lambda_1\left(\alpha_1B_1|B_1|^2+\alpha_2B_1|B_2|^2+\mu_1H\right)\Psi_5 + \frac{1}{12}B_1|B_1|^4\Phi_1^5 + \frac{1}{2}B_1|B_1|^2|B_2|^2\Phi_1^3\Phi_2^2 \\ &\quad +\frac{1}{4}B_1|B_2|^4\Phi_1\Phi_2^4 - \frac{3}{2}B_1|B_1|^2\Phi_1^2\Psi_5 - |B_2|^2B_1\Phi_2^2\Psi_5 - |B_1|^2H\Phi_1^2\Psi_6 - |B_2|^2H\Phi_2^2\Psi_6 \\ &\quad -\frac{1}{2}B_1{}^2H\Phi_1{}^2\Psi_6 - 2\,B_1|B_2|^2\Phi_1\Phi_2\psi_6 - \frac{1}{2}B_1|B_1|^4\Phi_1{}^2\psi_{15} - \frac{1}{2}B_1|B_2|^4\Phi_2{}^2\psi_{24} \\ &\quad -\frac{1}{2}B_1|B_2|^4\Phi_2{}^2\psi_{21} - B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_9 - B_1|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{12}, \end{aligned} \tag{B12}$$

$$\begin{aligned} M_1 &= 2\,i\lambda_2D_4B_2\Phi_2 - B_2\left(\alpha_3|B_2|^2+\alpha_4|B_1|^2\right)^2\Phi_2 - 2\lambda_2B_2\left(\alpha_3|B_2|^2+\alpha_4|B_1|^2\right)\psi_4 \\ &\quad +\frac{1}{12}B_2|B_2|^4\Phi_2^5 + \frac{1}{2}B_2|B_1|^2|B_2|^2\Phi_1^2\Phi_2^3 + \frac{1}{4}B_2|B_1|^4\Phi_1^4\Phi_2 \\ &\quad -2\,B_2|B_1|^2\Phi_1\Phi_2\Psi_1 - B_1B_2H\Phi_1\Phi_2\Psi_2 - B_2\overline{B_1}\Phi_1\Phi_2H\Psi_2 - \frac{3}{2}B_2|B_2|^2\Phi_2{}^2\psi_4 \\ &\quad -\frac{1}{2}B_2|B_2|^4\Phi_2{}^2\psi_{16} - \frac{1}{2}B_2|B_1|^4\Phi_1{}^2\psi_{10} - \frac{1}{2}B_2|B_1|^4\Phi_1{}^2\psi_7 - B_2|B_1|^2\Phi_1{}^2\psi_4 \\ &\quad -B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{19} - B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{22}, \end{aligned} \tag{B13}$$

$$\begin{aligned} M_2 &= 2\,i\lambda_2D_4B_2\Phi_2 - B_2\left(\alpha_3|B_2|^2+\alpha_4|B_1|^2\right)^2\Phi_2 - 2\lambda_2B_2\left(\alpha_3|B_2|^2+\alpha_4|B_1|^2\right)\psi_5 \\ &\quad -\frac{1}{12}B_2|B_2|^4\Phi_2^5 - \frac{1}{2}B_2|B_1|^2|B_2|^2\Phi_1^2\Phi_2^3 - \frac{1}{4}B_2|B_1|^4\Phi_1^4\Phi_2 \\ &\quad +2\,B_2|B_1|^2\Phi_1\Phi_2\Psi_3 + B_1B_2H\Phi_1\Phi_2\Psi_4 + B_2\overline{B_1}\Phi_1\Phi_2H\Psi_4 + \frac{3}{2}B_2|B_2|^2\Phi_2{}^2\psi_5 \\ &\quad +\frac{1}{2}B_2|B_2|^4\Phi_2{}^2\psi_{17} + \frac{1}{2}B_2|B_1|^4\Phi_1{}^2\psi_{11} + \frac{1}{2}B_2|B_1|^4\Phi_1{}^2\psi_8 + B_2|B_1|^2\Phi_1{}^2\psi_5 \\ &\quad +B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{20} + B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{23}, \end{aligned} \tag{B14}$$

$$\begin{aligned} M_3 &= 2\,i\lambda_2D_4B_2\Phi_2 - B_2\left(\alpha_3|B_2|^2+\alpha_4|B_1|^2\right)^2\Phi_2 - 2\lambda_2B_2\left(\alpha_3|B_2|^2+\alpha_4|B_1|^2\right)\psi_6 \\ &\quad +\frac{1}{12}B_2|B_2|^4\Phi_2^5 + \frac{1}{2}B_2|B_1|^2|B_2|^2\Phi_1^2\Phi_2^3 + \frac{1}{4}B_2|B_1|^4\Phi_1^4\Phi_2 \\ &\quad -2\,B_2|B_1|^2\Phi_1\Phi_2\Psi_5 - B_1B_2H\Phi_1\Phi_2\Psi_6 - B_2\overline{B_1}\Phi_1\Phi_2H\Psi_6 - \frac{3}{2}B_2|B_2|^2\Phi_2{}^2\psi_6 \\ &\quad -\frac{1}{2}B_2|B_2|^4\Phi_2{}^2\psi_{18} - \frac{1}{2}B_2|B_1|^4\Phi_1{}^2\psi_{12} - \frac{1}{2}B_2|B_1|^4\Phi_1{}^2\psi_9 - B_2|B_1|^2\Phi_1{}^2\psi_6 \\ &\quad -B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{21} - B_2|B_1|^2|B_2|^2\Phi_1\Phi_2\psi_{24}. \end{aligned} \tag{B15}$$

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