

Information Geometric Nonlinear Filtering ^{*}

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Abstract

This paper develops information geometric representations for nonlinear filters in continuous time. The posterior distribution associated with an abstract nonlinear filtering problem is shown to satisfy a stochastic differential equation on a Hilbert information manifold. This supports the Fisher metric as a pseudo-Riemannian metric. Flows of Shannon information are shown to be connected with the quadratic variation of the process of posterior distributions in this metric. Apart from providing a suitable setting in which to study such information-theoretic properties, the Hilbert manifold has an appropriate topology from the point of view of multi-objective filter approximations. A general class of finite-dimensional exponential filters is shown to fit within this framework, and an intrinsic evolution equation, involving Amari's -1 -covariant derivative, is developed for such filters. Three example systems, one of infinite dimension, are developed in detail.

Keywords: Information Geometry, Information Theory, Nonlinear Filtering, Fisher Metric, Quadratic Variation.

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1 Introduction

Let $(X_t \in \mathbb{X}, t \geq 0)$ be a Markov “signal” process taking values in a metric space \mathbb{X} , and let $(Y_t \in \mathbb{R}^d, t \geq 0)$ be an “observation” process defined by

$$Y_t = \int_0^t h_s(X_s) ds + B_t, \quad (1)$$

where $h : [0, \infty) \times \mathbb{X} \rightarrow \mathbb{R}^d$ is a Borel measurable function, and $(B_t \in \mathbb{R}^d, t \geq 0)$ is a d -vector Brownian motion, independent of X . In this context, the problem of nonlinear filtering is that of estimating X_t , at each time t , from the observations available up to that time, $(Y_s, s \in [0, t])$. In order to compute various optimal estimates of X_t , such as the maximum a-posteriori probability estimate (if \mathbb{X} is discrete) or the minimum mean-square-error estimate (if \mathbb{X} is a normed linear space), it is usually necessary to find, or at least to approximate, the entire observation-conditional distribution of X_t . That a *regular* version of such a distribution exists, and can be represented by an abstract version of Bayes’ formula, is one of the important early developments in the subject [11]. However, starting with the work of Wonham [26] and Shiriyayev [24], much of the theory of nonlinear filtering concerns *recursive* filtering equations, in which representations of the posterior distribution are shown to satisfy particular stochastic differential equations. The reader is referred to [6] for a wide range of articles on the theory and current practice of the subject.

Recursive filtering equations are typically expressed in ways that are specific to the nature of the signal space \mathbb{X} . If, for example, \mathbb{X} is discrete, then the filter can be expressed as a stochastic ordinary differential equation for the vector of posterior probabilities of the individual states, $x \in \mathbb{X}$ [26, 24]; whereas, if X is a multidimensional diffusion process, then the filter can be expressed as a stochastic partial differential equation for the posterior density [12]. One of the aims of this paper is to unify such results through the use of a filter “state space” that is based on estimation theoretic constructs rather than the underlying topology of \mathbb{X} . The state space used is a Hilbert manifold of probability measures on \mathbb{X} . This has an appropriate topology for the study of both approximation errors and information theoretic properties. These notions are discussed next in the context of an abstract Bayesian problem, in which the estimand $U : \Omega \rightarrow \mathbb{U}$ and observation $V : \Omega \rightarrow \mathbb{V}$ are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and take values in measure spaces $(\mathbb{U}, \mathcal{U}, \lambda_U)$ and $(\mathbb{V}, \mathcal{V}, \lambda_V)$, respectively. We assume that $P_{UV} \ll P_U \otimes \lambda_V$,

where P_{UV} is the joint distribution of (U, V) , and P_U is the marginal of U . Let $\mathcal{P}(\mathcal{U})$ be the set of probability measures on \mathcal{U} , let $R_V : \mathbb{U} \times \mathbb{V} \rightarrow [0, \infty)$ be a measurable function, for which $dP_{UV} = R_V d(P_U \otimes \lambda_V)$, and let $\Omega' := \{\omega \in \Omega : 0 < \int_{\mathbb{U}} R_V(u, V(\omega)) P_U(du) < \infty\}$. Then $\Omega' \in \mathcal{F}$, $\mathbb{P}(\Omega') = 1$, and $P_{U|V} : \Omega \rightarrow \mathcal{P}(\mathcal{U})$, defined by

$$P_{U|V}(A) = \mathbf{1}_{\Omega'} \frac{\int_A R_V(u, V) P_U(du)}{\int_{\mathbb{U}} R_V(u, V) P_U(du)} + \mathbf{1}_{\Omega \setminus \Omega'} P_U(A), \quad (2)$$

is a *regular V-conditional distribution* for U . (See [15] for details.)

In many applications of Bayesian estimation, including nonlinear filtering, it is not possible to express $P_{U|V}$ in terms of a finite number of statistics, and so it is useful to construct approximations: $\hat{P} : \Omega \rightarrow \mathcal{Q} \subset \mathcal{P}(\mathcal{U})$, where $\hat{P}(A)$ is V -measurable for all A , and \mathcal{Q} is of finite dimension. Single estimation objectives, such as minimum mean-square error in the estimate of a real-valued quantity $f(U)$, induce their own specific measures of approximation error on $\mathcal{P}(\mathcal{U})$. On the other hand, if f is sufficiently regular, a more generic measure of error such as the L^2 metric on densities may be useful. If λ_U is a *probability* measure, and $P_{U|V}(\omega)$ and $\hat{P}(\omega)$ have densities $p_{U|V}(\omega)$ and $\hat{p}(\omega)$ with respect to λ_U , then the difference between the minimum mean-square error estimate of $f(U)$ and the mean of f under $\hat{P}(\omega)$ can be bounded by means of the Cauchy-Schwartz inequality:

$$\left(\mathbf{E}_{P_{U|V}(\omega)} f - \mathbf{E}_{\hat{P}(\omega)} f \right)^2 \leq \mathbf{E}_{\lambda_U} f^2 \mathbf{E}_{\lambda_U} (p_{U|V}(\omega) - \hat{p}(\omega))^2. \quad (3)$$

Although, in this context, the L^2 metric on densities bounds the estimation error, it may still be poor in practice. This is so, for example, if f is the indicator function of a rare, but important, event. Moreover, we often need generic measures of error that are suitable for a *variety* of objectives. This is especially important if the underlying estimation problem is inherently multi-objective.

Multi-objective measures of approximation error are discussed in [21]. One such measure is the Kullback-Leibler (KL) divergence (or “relative entropy”):

$$\mathcal{D}(Q | P) := \begin{cases} \mathbf{E}_P \frac{dQ}{dP} \log \frac{dQ}{dP} & \text{if } Q \ll P \\ +\infty & \text{otherwise.} \end{cases} \quad (4)$$

This is widely used in *variational* Bayesian estimation. (See, for example, [25].) Apart from its use as a measure of approximation error, the KL-divergence plays a central role in Shannon information theory. The *mutual*

information between U and V is defined as follows [5]:

$$I(U; V) := \mathcal{D}(P_{UV} | P_U \otimes P_V) = \mathbb{E}\mathcal{D}(P_{U|V} | P_U). \quad (5)$$

The term $\mathcal{D}(P_{U|V} | P_U)$, here, can be interpreted as the *information gain* of the posterior distribution $P_{U|V}$ over the prior P_U .

Suppose that $W : \Omega \rightarrow \mathbb{W}$ is a second observation taking values in a measure space $(\mathbb{W}, \mathcal{W}, \lambda_W)$ such that V and W are U -conditionally independent and $P_{UW} \ll P_U \otimes \lambda_W$. Let $R_W : \mathbb{U} \times \mathbb{W} \rightarrow [0, \infty)$ be a measurable function for which $dP_{UW} = R_W d(P_U \otimes \lambda_W)$, and let $\Omega'' := \{\omega \in \Omega : 0 < \int_{\mathbb{U}} R_W(u, W(\omega)) P_{U|V}(\omega)(du) < \infty\}$. Then $\Omega'' \in \mathcal{F}$, $\mathbb{P}(\Omega'') = 1$, and $P_{U|VW} : \Omega \rightarrow \mathcal{P}(\mathbb{U})$, defined by

$$P_{U|VW}(A) = \mathbf{1}_{\Omega''} \frac{\int_A R_W(u, W) P_{U|V}(du)}{\int_{\mathbb{U}} R_W(u, W) P_{U|V}(du)} + \mathbf{1}_{\Omega \setminus \Omega''} P_{U|V}(A), \quad (6)$$

is a regular (V, W) -conditional distribution for U . The mutual information $I(U; (V, W))$ can be decomposed in the following way,

$$I(U; (V, W)) = I(U; V) + \mathbb{E}I(U; W|V), \quad (7)$$

where $I(U; W|V)$ is the V -conditional mutual information between U and W ,

$$I(U; W|V) := \mathcal{D}(P_{UW|V} | P_{U|V} \otimes P_{W|V}) = \mathbb{E}(\mathcal{D}(P_{U|VW} | P_{U|V}) | V). \quad (8)$$

(The conditional mutual information is sometimes defined as the *average value* of this quantity [5].) Equation (6) can be used *recursively* in estimation problems having sequences of conditionally independent observations. The information extracted from each observation in the sequence is then associated with a “local” Bayesian problem, in which earlier observations enter only through the “local prior” (the posterior derived from the earlier observations). The decomposition (7), (8) is valid whether or not V and W are U -conditionally independent. However, in the absence of such conditional independence it is not possible to interpret $\mathcal{D}(P_{U|VW} | P_{U|V})$ as a *local* information gain in this way.

Let $((X_t, Y_t), t \geq 0)$ be as described at the start of this section, and consider the problem of estimating the *path* of X from Y . For any $0 \leq t \leq s < \infty$, let

$$Y_t^s := (Y_r - Y_t, r \in [t, s]). \quad (9)$$

Then Y_0^t and Y_t^s are X -conditionally independent, and we can use the above methodology to identify the Y_0^t -conditional mutual information between X and Y_t^s , $I(X; Y_t^s | Y_0^t)$. This is shown in section 3.1 to be related to the quadratic variation of the posterior distribution in the Fisher metric. The latter is defined in terms of the mixed second derivative of the KL-divergence, and so an appropriate state space for the nonlinear filter in this context is a subset of $\mathcal{P}(\mathcal{X})$ having a differentiable structure with respect to which the KL-divergence admits such a derivative. This is also desirable for the assessment of approximation errors. *Information Geometry* is the study of sets of probability measures having such structures.

Information geometry is applied to nonlinear filtering in [2] and the references therein. The posterior distributions for diffusion signal processes are assumed, there, to have densities with respect to Lebesgue measure, whose square-roots satisfy stochastic differential equations in $L^2(\mathbb{R}^m, \mathcal{B}^m, \text{Leb})$. The induced distance function between probability measures is the *Hellinger distance*. The coefficients of the filtering equation are projected in this sense onto the tangent spaces of finite-dimensional exponential models, in order to obtain approximations to filters. Information theoretic justification is given (when restricted to tangent vectors corresponding to differentiable curves of square-root probability densities, the L^2 norm corresponds to the Fisher metric), and comparisons are made with other methods such as moment matching. Although suitable for this purpose, the Hellinger space cannot be used as an infinite-dimensional statistical manifold since the KL-divergence is discontinuous at every point of it. (See the discussion at the end of section 2 in [20].) Furthermore, in common with the L^2 space of densities, it has a boundary, which can create problems with numerical methods.

The local information gain of a nonlinear filter is connected with the notion of *entropy production* in nonequilibrium statistical mechanics [16, 18, 19]. The information geometric properties of nonlinear filters are also, therefore, of interest in this context.

The remainder of the paper is structured as follows. Section 2 outlines the main ingredients of information geometry and reviews the Hilbert manifold M , which is used extensively in the sequel. Section 3 outlines a general nonlinear filtering problem, and expresses the associated process of conditional distributions as an Itô process on M . This allows the study of the quadratic variation of the filter in the Fisher metric. An M -valued evolution equation is derived for a class of finite-dimensional exponential filters in section 4. The results of sections 3 and 4 are formulated in terms of a set of hypotheses,

some of which are not especially ripe. Section 5 develops three examples in which they are satisfied. Finally, section 6 makes some concluding remarks.

2 Information Geometry

We review the main ingredients of information geometry, by outlining the classical finite-dimensional exponential model. This is also used in section 4. Let $(\mathbb{X}, \mathcal{X}, \mu)$ be a probability space on which are defined random variables $(\tilde{\xi}_i; i = 1, \dots, n)$ with the following properties: (i) the random variables $(1, \tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n)$ represent linearly independent elements of $L^0(\mu)$, i.e. $\mu(\alpha + \sum_i y^i \tilde{\xi}_i = 0) = 1$ if and only if $\alpha = 0$ and $\mathbb{R}^n \ni y = 0$; (ii) $\mathbf{E}_\mu \exp(\sum_i y^i \tilde{\xi}_i) < \infty$ for all y in a non-empty open subset $G \subseteq \mathbb{R}^n$. For each $y \in G$, let P_y be the probability measure on \mathcal{X} with density

$$\frac{dP_y}{d\mu} = \exp\left(\sum_i y^i \tilde{\xi}_i - c(y)\right),$$

where $c(y) = \log \mathbf{E}_\mu \exp(\sum_i y^i \tilde{\xi}_i)$, and let $N := \{P_y : y \in G\}$. It follows from (i) that the map $G \ni y \mapsto P_y \in N$ is a bijection. Let $\theta : N \rightarrow G$ be its inverse; then (N, θ) is an *exponential statistical manifold* with an atlas comprising the single chart θ . We can think of a *tangent vector* at $P \in N$ as being an equivalence class of differentiable curves passing through P : two curves (expressed in coordinates), $(\mathbf{y}(t) \in G : t \in (-\epsilon, \epsilon))$ and $(\mathbf{z}(t) \in G : t \in (-\epsilon, \epsilon))$ being equivalent at P if $\mathbf{y}(0) = \mathbf{z}(0) = \theta(P)$ and $\dot{\mathbf{y}}(0) = \dot{\mathbf{z}}(0)$. The *tangent space* at P , $T_P N$, is the linear space of all such tangent vectors, and is spanned by the vectors $(\partial_i; i = 1, \dots, n)$, where ∂_i is the equivalence class containing the curve $(\mathbf{y}_i(t) := \theta(P) + t\mathbf{e}_i, t \in (-\epsilon, \epsilon))$, and \mathbf{e}_i^j is equal to the Krönecker delta. The *tangent bundle* is the disjoint union $TN := \cup_{P \in N} (P, T_P N)$, and admits the global chart $\Theta : TN \rightarrow G \times \mathbb{R}^n$, where $\Theta^{-1}(y, u) := (\theta^{-1}(y), u^i \partial_i)$. If a function $f : N \rightarrow \mathbb{R}^k$ is differentiable, and $U \in T_P N$, then we write

$$Uf = u^i \partial_i f := u^i \left. \frac{d}{dt} (f \circ \theta^{-1})(\mathbf{y}_i(t)) \right|_{t=0} = u^i \frac{\partial (f \circ \theta^{-1})}{\partial y^i}(y),$$

where $(y, u) = \Theta(P, U) = (\theta(P), U\theta)$, and we have used the Einstein summation convention, that indices appearing once as a superscript and once as a subscript are summed out.

According to the Eguchi relations [9], the mixed second derivative of the KL-divergence defines the *Fisher metric* as a Riemannian metric on N : for any $P \in N$ and any $U, V \in T_P N$,

$$\langle U, V \rangle_P := -UV\mathcal{D} = g(P)_{i,j}u^i v^j,$$

where U and V act on the first and second argument of \mathcal{D} , respectively, and

$$g(P)_{i,j} := \langle \partial_i, \partial_j \rangle_P = \mathbf{E}_P(\tilde{\xi}_i - \mathbf{E}_P \tilde{\xi}_i)(\tilde{\xi}_j - \mathbf{E}_P \tilde{\xi}_j), \quad (10)$$

is the matrix form of the Fisher metric [1]. The mixed third derivatives of the KL-divergence define a pair of *covariant derivatives* on N [1]. These give rise to notions of *curvature* of statistical manifolds, which are important in the theory of asymptotic statistics [1].

The literature on information geometry is dominated by the study of finite-dimensional manifolds of probability measures such as (N, θ) . The reader is referred to [1, 3] and the references therein for further information. In order to extend these ideas to infinite-dimensions we need to choose a system of charts with respect to which the KL-divergence admits a suitable number of derivatives. It is clear from (4) that the smoothness properties of this divergence are closely connected with those of the density dQ/dP and its log (considered as elements of dual spaces of functions). In the series of papers [4, 10, 22, 23], G. Pistone and his co-workers developed an infinite-dimensional exponential statistical manifold on an abstract probability space $(\mathbb{X}, \mathcal{X}, \mu)$. Probability measures in the manifold are mutually absolutely continuous with respect to the reference measure μ , and the manifold is covered by the charts $s_P(Q) = \log dQ/dP - \mathbf{E}_P \log dQ/dP$ for different “patch-centric” probability measures P . These readily give $\log dQ/dP$ the desired regularity, but require ranges that are subsets of exponential Orlicz spaces in order to do the same for dQ/dP . The exponential Orlicz manifold is a natural extension of the finite-dimensional manifold (N, θ) described above; it has a strong topology, under which the KL-divergence is of class C^∞ .

However, this approach is technically demanding and leads to manifolds that are larger than needed in many applications. Furthermore, the exponential Orlicz space is less suited to the theory of stochastic differential equations than Hilbert space; the latter is the natural setting for the L^2 theory of stochastic integration [7]. An infinite-dimensional Hilbert manifold of “finite-entropy” probability measures, on which the KL-divergence is twice differentiable, is developed in [20]. This uses a chart involving both

the density, $dP/d\mu$, and its log. The finite entropy condition is natural in estimation problems where the mutual information between the estimand and the observation is finite. Banach manifolds of finite-entropy measures, on which the KL-divergence admits higher derivatives, are developed in [21]. That reference also develops Hilbert and Banach manifolds of *finite measures* suitable for the “un-normalised” equations of nonlinear filtering. We shall make extensive use of the Hilbert manifold of [20] in this paper; it is reviewed next.

2.1 The Hilbert Manifold M

For a probability space $(\mathbb{X}, \mathcal{X}, \mu)$, M is the set of probability measures on \mathcal{X} satisfying the following conditions:

- (M1) P is mutually absolutely continuous with respect to μ ;
- (M2) $\mathbf{E}_\mu p^2 < \infty$;
- (M3) $\mathbf{E}_\mu \log^2 p < \infty$.

(We denote probability measures in M by the upper-case letters P, Q , etc., and their densities with respect to μ by the corresponding lower case letters, p, q , etc.) Let $\mathcal{L}^0(\mathbb{X}, \mathcal{X})$ be the set of real-valued random variables on \mathbb{X} , and $\mathcal{L}^2(\mathbb{X}, \mathcal{X}, \mu)$ the subset of square-integrable random variables. Let H be the Hilbert space of equivalence classes of centred elements of $\mathcal{L}^2(\mathbb{X}, \mathcal{X}, \mu)$ (those having zero mean), and let $\Lambda : \mathcal{L}^0(\mathbb{X}, \mathcal{X}) \rightarrow H$ be defined by

$$\begin{aligned} \Lambda f &\ni f - \mathbf{E}_\mu f && \text{if } f \in \mathcal{L}^2(\mathbb{X}, \mathcal{X}, \mu), \\ \Lambda f &= 0 && \text{otherwise.} \end{aligned} \tag{11}$$

Let $m, e : M \rightarrow H$ be defined as follows:

$$m(P) = \Lambda p \quad \text{and} \quad e(P) = \Lambda \log p. \tag{12}$$

Variants of these are used in finite-dimensional information geometry as coordinate maps for *mixture* and *exponential* models, respectively [1]. However, in the present context their images $m(M)$ and $e(M)$ are typically not open subsets of H [20]; so, even though they are injective, m and e cannot be used as charts for M . On the other hand the map $\phi : M \rightarrow H$ defined by their sum,

$$\phi(P) = m(P) + e(P) = \Lambda(p + \log p), \tag{13}$$

is a bijection [20]. (M, ϕ) is a Hilbert manifold with an atlas comprising a single chart.

Although not themselves charts, the maps m and e provide useful representations of elements of H since they are *bi-orthogonal*. The simplest manifestation of this property is the identity

$$\mathcal{D}(P | Q) + \mathcal{D}(Q | P) = \langle m(P) - m(Q), e(P) - e(Q) \rangle_H. \quad (14)$$

It thus follows from the non-negativity of the KL-divergence that

$$\|m(P) - m(Q)\|_H^2 + \|e(P) - e(Q)\|_H^2 \leq \|\phi(P) - \phi(Q)\|_H^2; \quad (15)$$

in particular, $m \circ \phi^{-1}$ and $e \circ \phi^{-1}$ are Lipschitz continuous. Furthermore

$$\mathcal{D}(P | Q) + \mathcal{D}(Q | P) \leq \frac{1}{2} \|\phi(P) - \phi(Q)\|_H^2. \quad (16)$$

The inverse map $\phi^{-1} : H \rightarrow M$ is given by

$$\frac{d\phi^{-1}(a)}{d\mu}(x) = \psi(\tilde{a}(x) + Z(a)),$$

where $\psi : \mathbb{R} \rightarrow (0, \infty)$ is the inverse of the function $(0, \infty) \ni z \mapsto z + \log z \in \mathbb{R}$, \tilde{a} is any function in the equivalence class a , and $Z : H \rightarrow \mathbb{R}$ is the unique function for which $\mathbf{E}_\mu \psi(\tilde{a} + Z(a)) = 1$ for all $a \in H$. Z is (Fréchet) differentiable with derivative

$$DZ_a u = - \frac{\mathbf{E}_\mu \psi'(\tilde{a} + Z(a)) \tilde{u}}{\mathbf{E}_\mu \psi'(\tilde{a} + Z(a))}, \quad (17)$$

where \tilde{u} is any function in the equivalence class u [20].

A tangent vector U at $P \in M$ is an equivalence class of differentiable curves at P . We denote the tangent space at P by $T_P M$, and the tangent bundle by TM . The latter admits the global chart $\Phi : TM \rightarrow H \times H$, where $\Phi(P, U) = (\mathbf{a}(0), \dot{\mathbf{a}}(0))$, and $(\mathbf{a}(t), t \in (-\epsilon, \epsilon))$ is any differentiable curve in the equivalence class U (expressed in terms of the chart ϕ). If $f : M \rightarrow Y$ is a map with range Y (a Banach space) and the map $f \circ \phi^{-1} : H \rightarrow Y$ is (Fréchet) differentiable, then we write

$$Uf := \left. \frac{d}{dt} (f \circ \phi^{-1})(\mathbf{a}(t)) \right|_{t=0} = D(f \circ \phi^{-1})_a u,$$

where $(a, u) = \Phi(P, U) = (\phi(P), U\phi)$. A weaker notion of *d-differentiability* is defined in [20]. The map $f : M \rightarrow Y$ is *d-differentiable* if, for any $P \in M$, there exists a continuous linear map $d(f \circ \phi^{-1})_a : H \rightarrow Y$ such that

$$\left. \frac{d}{dt}(f \circ \phi^{-1})(\mathbf{a}(t)) \right|_{t=0} = d(f \circ \phi^{-1})_a u,$$

for all differentiable curves \mathbf{a} in the equivalence class U . We then write $Uf = d(f \circ \phi^{-1})_a u$.

The KL-divergence, $\mathcal{D} : M \times M \rightarrow [0, \infty)$, is Fréchet differentiable in each argument, and both derivatives are *d-differentiable* in the remaining argument [20]. We can use this fact, together with the Eguchi relations [9], to define the Fisher metric on $T_P M$: for any $P \in M$ and $U, V \in T_P M$,

$$\langle U, V \rangle_P := -UV\mathcal{D} = \mathbf{E}_\mu \frac{p}{(1+p)^2} (\tilde{u} + DZ_a u)(\tilde{v} + DZ_a v), \quad (18)$$

where U and V act on the first and second arguments of \mathcal{D} , respectively, $a = \phi(P)$, $u = U\phi$, $v = V\phi$, \tilde{u} is any function in the equivalence class u , and \tilde{v} is any function in the equivalence class v . $(T_P M, \langle \cdot, \cdot \rangle_P)$ is an *inner product space*, whose norm admits the bound: $\|U\|_P \leq \|u\|_H$. However, since the Fisher norm is not (in general) equivalent to the model space norm, $(T_P M, \langle \cdot, \cdot \rangle_P)$ is not a Hilbert space. $(M, \langle \cdot, \cdot \rangle_P)$ is a *pseudo-Riemannian manifold* rather than a Riemannian manifold.

H -valued stochastic processes play a major role in what follows. In order to ensure that they have suitable measurability properties, we introduce the following additional hypothesis. This is satisfied, for example, if \mathbb{X} is a complete separable metric (Polish) space, and \mathcal{X} is its Borel σ -algebra.

(M4) H is separable.

Lemma 2.1. *If $(\mathbb{Z}, \mathcal{Z})$ is a measurable space, and $f : \mathbb{Z} \times \mathbb{X} \rightarrow \mathbb{R}$ is jointly measurable, then the map $\mathbb{Z} \ni z \mapsto \Lambda f(z, \cdot) \in H$ is \mathcal{Z} -measurable.*

Proof. Let $B := \{z \in \mathbb{Z} : f(z, \cdot) \in \mathcal{L}^2(\mathbb{X}, \mathcal{X}, \mu)\}$ then, according to Tonelli's theorem, $B \in \mathcal{Z}$. Fubini's theorem shows that, for any $g \in \mathcal{L}^2(\mathbb{X}, \mathcal{X}, \mu)$, the function $\mathbb{Z} \ni z \mapsto \mathbf{1}_B(z) \mathbf{E}_\mu f(z, \cdot) g \in \mathbb{R}$ is \mathcal{Z} -measurable, i.e. the map $\mathbb{Z} \ni z \mapsto \Lambda f(z, \cdot) \in H$ is *weakly* \mathcal{Z} -measurable. The statement of the lemma follows from (M4) and Pettis's theorem. \square

Under (M4), H is of countable dimension, and so admits a complete orthonormal basis $(\eta_i \in H, i \in \mathbb{N})$. An element $(P, U) \in TM$ thus has the coordinate representation $((a^i, u^j), i, j \in \mathbb{N})$ where $a^i = \langle \phi(P), \eta_i \rangle_H$ and $u^j = \langle U\phi, \eta_j \rangle_H$. In particular, any $U \in T_P M$ admits the representation $U = u^j D_j$, where $(P, D_j) = \Phi^{-1}(\phi(P), \eta_j)$. For any $P \in M$ and $i, j \in \mathbb{N}$, let

$$G(P)_{i,j} := \langle D_i, D_j \rangle_P. \quad (19)$$

Then it follows from the domination of $\|U\|_P$ by $\|u\|_H$ that, for any $U, V \in T_P M$,

$$\langle U, V \rangle_P = G(P)_{i,j} u^i v^j, \quad (20)$$

in the sense that both series are absolutely convergent, and the result does not depend on the order in which the limits are taken.

3 The M -Valued Nonlinear Filter

We consider a general nonlinear filtering problem as outlined in section 1, in which all random quantities are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The signal space \mathbb{X} is a complete separable metric space, \mathcal{X} is its Borel σ -algebra, and μ is a reference probability measure on \mathcal{X} . (M, ϕ) is the associated Hilbert manifold, as described in section 2.1. We shall assume that X has right-continuous sample paths with left limits at all $t \in (0, \infty)$, and that the distribution of X_t, P_t , has a density with respect to μ satisfying the Kolmogorov forward equation

$$\frac{\partial p_t}{\partial t} = \mathcal{A}_t p_t \quad \text{for } t \in [0, \infty), \quad (21)$$

where $(\mathcal{A}_t, t \geq 0)$ is a family of linear operators on an appropriate class of functions $f : \mathbb{X} \rightarrow \mathbb{R}$.

Example 3.1. $\mathbb{X} = \{1, 2, \dots, m\}$, X is a time-homogeneous Markov jump process with rate matrix A , μ is mutually absolutely continuous with respect to the counting measure (with Radon-Nikodym derivative r), and $h_t (= h)$ does not depend on t . In this case

$$(\mathcal{A}_t p)(x) = (A p)(x) = \frac{1}{r(x)} \sum_{\tilde{x} \in \mathbb{X}} A_{\tilde{x}x} r p(\tilde{x}) \quad \text{for all } t.$$

Example 3.2. $\mathbb{X} = \mathbb{R}^m$, X is a time-homogeneous multidimensional diffusion process with suitably regular drift vector b and diffusion matrix a , μ is mutually absolutely continuous with respect to Lebesgue measure (with Radon-Nikodym derivative r), and $h_t (= h)$ does not depend on t . In this case

$$\mathcal{A}_t p = \mathcal{A} p = \frac{1}{2r} \frac{\partial^2}{\partial x^i \partial x^j} (a^{ij} r p) - \frac{1}{r} \frac{\partial}{\partial x^i} (b^i r p) \quad \text{for all } t.$$

The set-up is sufficiently general to include *path estimators*.

Example 3.3. $\mathbb{X} = C([0, \infty); \mathbb{R}^m)$, $X_t = (\tilde{X}_s, s \geq 0)$ for all t , where \tilde{X} is the diffusion process of Example 3.2, and $h_t(x) = \tilde{h}(x_t)$ for some $\tilde{h} : \mathbb{R}^m \rightarrow \mathbb{R}^d$. In this case $\mathcal{A}_t = 0$ for all t .

Let $(\mathcal{Y}_t \subset \mathcal{F}, t \geq 0)$ be the filtration generated by the observation process Y , augmented by the \mathbb{P} -null sets of \mathcal{F} , and let \mathcal{P}_Y be the σ -algebra of (\mathcal{Y}_t) -predictable subsets of $[0, \infty) \times \Omega$. We assume that:

- (F1) $P_0 \in M$.
- (F2) For any $T < \infty$, $\int_0^T \mathbb{E} |h_t(X_t)|^2 dt < \infty$.
- (F3) There exists, on the product space $\Omega \times \mathbb{X}$, a $\mathcal{P}_Y \times \mathcal{X}$ -measurable, $(0, \infty)$ -valued process $(\pi_t, t \geq 0)$, for which

$$\mathbb{P} \left(\int_{\mathbb{X}} \pi_t(\cdot, x) \mu(dx) = 1 \text{ for all } t \geq 0 \right) = 1.$$

For any t and any $A \in \mathcal{X}$, $\mathbb{P}(X_t \in A | \mathcal{Y}_t) = \Pi_t(A)$, where

$$\Pi_t(A) := \int_A \pi_t(\cdot, x) \mu(dx). \quad (22)$$

- (F4) $\mathbb{P}(\pi_t \in \text{Dom} \mathcal{A}_t \text{ for all } t \geq 0) = 1$, the process $(\mathcal{A}_t \pi_t, t \geq 0)$ is $\mathcal{P}_Y \times \mathcal{X}$ -measurable and, for any $T < \infty$,

$$\mathbb{P} \left(\int_0^T \sqrt{\mathbf{E}_\mu (1 + \pi_t^{-1})^2 (\mathcal{A}_t \pi_t)^2} dt < \infty \right) = 1. \quad (23)$$

(F5) For any $T < \infty$,

$$\mathbb{P} \left(\int_0^T \sqrt{\mathbf{E}_\mu |h_t - \bar{h}_t|^4} dt < \infty \right) = 1, \quad (24)$$

$$\mathbb{P} \left(\int_0^T \mathbf{E}_\mu (\pi_t + 1)^2 |h_t - \bar{h}_t|^2 dt < \infty \right) = 1, \quad (25)$$

where

$$\bar{h}_t := \begin{cases} \mathbf{E}_\mu \pi_t h_t & \text{if } \mathbf{E}_\mu \pi_t |h_t| < \infty \\ 0 & \text{otherwise.} \end{cases} \quad (26)$$

(F6) For almost all x , $(\pi_t, t \geq 0)$ satisfies the following Itô equation on Ω :

$$\pi_t = p_0 + \int_0^t \mathcal{A}_s \pi_s ds + \int_0^t \pi_s (h_s - \bar{h}_s)^* d\nu_s, \quad (27)$$

where $(\nu_t, t \geq 0)$ is the *innovations process*,

$$\nu_t := Y_t - \int_0^t \bar{h}_s ds. \quad (28)$$

Remark 3.1. (i) Because of (F2), $(\nu_t, t \geq 0)$ is a d -dimensional (\mathcal{Y}_t) -Brownian motion [13].

(ii) In the context of Example 3.1, (27) becomes a system of stochastic ordinary differential equations derived independently by Wonham [26] and Shiriyayev [24]. In the context of Example 3.2, it becomes a stochastic partial differential equation known as the Kushner-Stratonovich equation [12]. See [6] for a variety of conditions under which nonlinear filters admit the representation (27).

The intention here is to develop an M -valued representation for the process Π of (22). With this in mind, we introduce the following H -valued processes

$$u_t := \Lambda(1 + \pi_t^{-1}) \mathcal{A}_t \pi_t \quad \text{and} \quad \zeta_t := \frac{1}{2} \Lambda |h_t - \bar{h}_t|^2, \quad (29)$$

where Λ is as defined in (11), and the following $L(\mathbb{R}^d, H)$ -valued process

$$v_t := \Lambda(\pi_t + 1)(h_t - \bar{h}_t)^*. \quad (30)$$

Proposition 3.1. *Suppose that (X, Y) satisfies (F1–F6), and Π , u , ζ and v are as defined in (22), (29) and (30). Then*

(i) $\mathbb{P}(\Pi_t \in M \text{ for all } t \geq 0) = 1$;

(ii) $(\phi(\Pi_t), t \geq 0)$ satisfies the following (infinite-dimensional) Itô equation

$$\phi(\Pi_t) = \phi(P_0) + \int_0^t (u_s - \zeta_s) ds + \int_0^t v_s d\nu_s. \quad (31)$$

Remark 3.2. (i) *The first integral in (31) is a Bochner integral, and the second is an Itô integral. The stochastic calculus of Hilbert-space-valued semimartingales is developed pedagogically in [7]. In the general case, the stochastic integral is defined for a Hilbert-space-valued Wiener process and the stochastic integrand is a Hilbert-Schmidt-operator-valued process. In the present context ν is of finite dimension, and so, if \mathbb{R}^d is equipped with the Euclidean inner product, any element of $L(\mathbb{R}^d, H)$ is a Hilbert-Schmidt operator. For any $\rho, \sigma \in L(\mathbb{R}^d, H)$, the associated inner product is*

$$\langle \rho, \sigma \rangle_{HS} := \sum_{k=1}^d \langle \rho \mathbf{e}_k, \sigma \mathbf{e}_k \rangle_H, \quad (32)$$

where $(\mathbf{e}_k, 1 \leq k \leq d)$ is any orthonormal basis in \mathbb{R}^d .

(ii) *Although natural and inclusive, (F3–F6) are not particularly ripe. We develop some examples in which they are satisfied in section 5.*

(iii) *The case $h = 0$ provides an M -valued representation for the marginal distribution $(P_t, t \geq 0)$.*

Proof. According to (F6) there exists an $F \in \mathcal{X}$ with $\mu(F) = 1$ such that $\pi(\cdot, x)$ satisfies (27) on Ω for all $x \in F$. Itô's rule shows that, for any such x , $(\pi_t + \log \pi_t, t \geq 0)$ satisfies the following Itô equation on Ω :

$$\pi_t + \log \pi_t = p_0 + \log p_0 + I_t + J_t, \quad (33)$$

where

$$\begin{aligned} I_t &:= \int_0^t (\tilde{u}_s - \tilde{\zeta}_s) ds, & \tilde{u}_t &:= \mathbf{1}_F(1 + \pi_t^{-1})\mathcal{A}_t\pi_t, & \tilde{\zeta}_t &:= \frac{1}{2}\mathbf{1}_F|h_t - \bar{h}_t|^2, \\ J_t &:= \int_0^t \tilde{v}_s d\nu_s, & \tilde{v}_t &:= \mathbf{1}_F(\pi_t + 1)(h_t - \bar{h}_t)^*. \end{aligned} \quad (34)$$

The Fubini theorem and Cauchy-Schwartz inequality show that, for any $T < \infty$,

$$\begin{aligned} \mathbf{E}_\mu \left(\int_0^T |\tilde{u}_t - \tilde{\zeta}_t| dt \right)^2 &= \int_0^T \int_0^T \mathbf{E}_\mu |\tilde{u}_t - \tilde{\zeta}_t| |\tilde{u}_s - \tilde{\zeta}_s| ds dt \\ &\leq \left(\int_0^T \sqrt{\mathbf{E}_\mu (\tilde{u}_t - \tilde{\zeta}_t)^2} dt \right)^2, \end{aligned}$$

and so it follows from (23) and (24) that

$$\mathbb{P}(I_t \in \mathcal{L}^2(\mathbb{X}, \mathcal{X}, \mu) \text{ for all } t \geq 0) = 1. \quad (35)$$

For any $m \in \mathbb{N}$ and any $T < \infty$, let

$$\tau_m := \inf\{t > 0 : K_t \geq m\} \wedge T \quad \text{where} \quad K_t := \int_0^t \mathbf{E}_\mu |\tilde{v}_s|^2 ds. \quad (36)$$

According to (25), $\mathbb{P}(K_T < \infty) = 1$, τ_m is a (\mathcal{Y}_t) -stopping time, and $(J_{t \wedge \tau_m}, t \geq 0)$ is a continuous martingale on Ω , for almost all x . According to the stochastic Fubini theorem (Theorem 4.18 in [7]), $J_{t \wedge \tau_m}$ is $\mathbb{P} \otimes \mu$ -measurable for each t , and so $\sup_{[0, T]} J_{t \wedge \tau_m}^2$ is also $\mathbb{P} \otimes \mu$ -measurable. Applying Doob's L^2 inequality and then integrating with respect to μ , we obtain

$$\mathbf{E}_\mu \mathbb{E} \sup_{t \in [0, T]} J_{t \wedge \tau_m}^2 \leq 4 \mathbf{E}_\mu \mathbb{E} J_{\tau_m}^2 = 4 \mathbb{E} K_{\tau_m} \leq 4m.$$

Now $\tau_m = T$ for all $m \geq K_T$, and so

$$\mathbb{P} \left(\mathbf{E}_\mu \sup_{t \in [0, T]} J_t^2 < \infty \right) = 1; \quad (37)$$

in particular,

$$\mathbb{P}(J_t \in \mathcal{L}^2(\mathbb{X}, \mathcal{X}, \mu) \text{ for all } t \geq 0) = 1. \quad (38)$$

Now $\inf_{y \in (0, \infty)} y \log y = -1/e$, and so $\pi_t^2 + \log^2 \pi_t \leq (\pi_t + \log \pi_t)^2 + 2/e$. Part (i) thus follows from (F1), (33), (35) and (38), as does the fact that

$$\mathbb{P}(\phi(\Pi_t) = \phi(P_0) + \Lambda I_t + \Lambda J_t \text{ for all } t) = 1.$$

According to (F3), (F4) and Lemma 2.1, $\phi(\Pi)$, \bar{h} , u , ζ and v are all \mathcal{P}_Y -measurable. Furthermore, it follows from (23–25) that

$$\mathbb{P} \left(\int_0^t \|u_s - \zeta_s\|_H ds + \int_0^t \|v_s\|_{HS}^2 ds < \infty \text{ for all } t \geq 0 \right) = 1,$$

and so the integrals on the right-hand side of (31) are well defined. (See section 4.2 in [7].) It remains to show that the operator Λ commutes with the operations of ordinary and stochastic integration in (34).

Let $(a_t, t \geq 0)$ be the (continuous) H -valued process on the right-hand side of (31), let $(\eta_i, i \in \mathbb{N})$ be a complete orthonormal basis for H and, for each i , let $\tilde{\eta}_i \in \mathcal{L}^2(\mathbb{X}, \mathcal{X}, \mu)$ be a function in the equivalence class η_i . It follows from the infinite-dimensional Itô rule (Theorem 4.17 in [7]) that

$$\begin{aligned} \langle \eta_i, a_t \rangle_H &= \langle \eta_i, \phi(P_0) \rangle_H + \int_0^t \langle \eta_i, u_s - \zeta_s \rangle_H ds + \int_0^t \langle \eta_i, v_s d\nu_s \rangle_H \\ &= \mathbf{E}_\mu \tilde{\eta}_i(p_0 + \log p_0) + \int_0^t \mathbf{E}_\mu \tilde{\eta}_i(\tilde{u}_s - \tilde{\zeta}_s) ds + \int_0^t \mathbf{E}_\mu \tilde{\eta}_i \tilde{v}_s d\nu_s, \end{aligned} \quad (39)$$

where, in the second step, we have used the fact (derived from (23–25)) that $\tilde{u}_t, \tilde{\zeta}_t, \tilde{v}_t y \in \mathcal{L}^2(\mathbb{X}, \mathcal{X}, \mu)$ for all $y \in \mathbb{R}^d$, for almost all (t, ω) . The Cauchy-Schwartz inequality shows that, for any $m \in \mathbb{N}$ and any $T < \infty$,

$$\mathbf{E}_\mu \left(\mathbb{E} \int_0^{\tau_m} |\tilde{\eta}_i \tilde{v}_s|^2 ds \right)^{1/2} = \mathbf{E}_\mu |\tilde{\eta}_i| \left(\mathbb{E} \int_0^{\tau_m} |\tilde{v}_s|^2 ds \right)^{1/2} \leq \sqrt{\mathbb{E} K_{\tau_m}} \leq \sqrt{m},$$

where τ_m and K_t are as defined in (36), and so, according to the stochastic Fubini theorem (Theorem 4.18 in [7])

$$\int_0^{t \wedge \tau_m} \mathbf{E}_\mu \tilde{\eta}_i \tilde{v}_s d\nu_s = \mathbf{E}_\mu \tilde{\eta}_i J_{t \wedge \tau_m}.$$

Since m and T are arbitrary, this is also true if $t \wedge \tau_m$ is replaced by any $t \geq 0$. According to (37) and the dominated convergence theorem ΛJ is continuous; so

$$\mathbb{P} \left(\int_0^t \mathbf{E}_\mu \tilde{\eta}_i \tilde{v}_s d\nu_s = \mathbf{E}_\mu \tilde{\eta}_i J_t \text{ for all } t \geq 0 \right) = 1.$$

Applying this and the Fubini theorem to (39), we obtain

$$\begin{aligned} \langle \eta_i, a_t \rangle_H &= \mathbf{E}_\mu \tilde{\eta}_i(p_0 + \log p_0) + \mathbf{E}_\mu \tilde{\eta}_i I_t + \mathbf{E}_\mu \tilde{\eta}_i J_t \\ &= \mathbf{E}_\mu \tilde{\eta}_i(\pi_t + \log \pi_t) \\ &= \langle \eta_i, \phi(\Pi_t) \rangle_H, \end{aligned}$$

where we have used (33) in the second step, and part (i) in the third step. This completes the proof of part (ii). \square

3.1 Quadratic Variation

Suppose that (F1–F6) hold, and let $(\phi(\Pi_t)^i, i \in \mathbb{N})$ be the coordinate representation for Π_t in terms of a complete orthonormal basis, $(\eta_i, i \in \mathbb{N})$, for H . Then $\phi(\Pi_t)^i$ is equal to the right-hand side of (39). The components $\{(\phi(\Pi_t)^i, \mathcal{Y}_t), i = 1, 2, \dots\}$ form a system of real-valued semimartingales with quadratic co-variations

$$[\phi(\Pi)^i, \phi(\Pi)^j]_t = \int_0^t \sum_k \langle \eta_i, v_s e_k \rangle_H \langle \eta_j, v_s e_k \rangle_H ds. \quad (40)$$

We define the *M-intrinsic quadratic variation* of the process Π (its quadratic variation in the Fisher metric) as follows:

$$\begin{aligned} [\Pi]_t &:= \int_0^t G(\Pi_s)_{i,j} d[\phi(\Pi)^i, \phi(\Pi)^j]_s \\ &= \int_0^t \sum_k G(\Pi_s)_{i,j} \langle \eta_i, v_s e_k \rangle_H \langle \eta_j, v_s e_k \rangle_H ds \\ &= \int_0^t \sum_k \mathbf{E}_\mu \frac{\pi_s}{(1 + \pi_s)^2} (\tilde{v}_s e_k + DZ_{\phi(\Pi_s)} v_s e_k)^2 ds \\ &= \int_0^t \mathbf{E}_\mu \pi_s |h_s - \bar{h}_s|^2 ds, \end{aligned} \quad (41)$$

where G is as defined in (19), and we have used (20) and (18) in the third step, and (17) in the final step. The final integrand here is the \mathcal{Y}_s -conditional mean-square error for the filter's estimate of $h_s(X_s)$. The average value of the final integral is known to be related to the mutual information $I(X; Y_0^t)$ [8]. We refine this result here, characterising the \mathcal{Y}_t -conditional variant of this mutual information, as defined in (8).

Proposition 3.2. *For any $0 \leq t \leq s < \infty$,*

$$I(X; Y_t^s | Y_0^t) = \frac{1}{2} \mathbb{E}([\Pi]_s - [\Pi]_t | \mathcal{Y}_t), \quad (42)$$

where Y_t^s is as defined in (9).

Proof. Let $D([0, \infty); \mathbb{X})$ be the Skorohod space of right-continuous, left-limit maps $\theta : [0, \infty) \rightarrow \mathbb{X}$, and let $H : [0, s] \times D([0, \infty); \mathbb{X}) \rightarrow \mathbb{R}^d$ be defined by

$$H_t(\theta) = \begin{cases} h_t(\theta_t) & \text{if } \int_0^s |h_t(\theta_t)|^2 dt < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Fubini's theorem and (F2) show that $\mathbb{P}(H_r(X) = h_r(X_r))$ for all $r \in [0, s]$ = 1. Let $\rho : [0, s] \times \Omega \times D([0, \infty); \mathbb{X}) \rightarrow [0, \infty)$ be defined by

$$\rho_t(\cdot, \theta) = \exp \left(\int_0^t (H_r(\theta) - \bar{h}_r)^* dB_r + \frac{1}{2} \int_0^t |H_r(\theta) - \bar{h}_r|^2 dr \right),$$

let \mathcal{S} be the Borel σ -algebra on $D([0, \infty); \mathbb{X})$, and let $P_X \in \mathcal{P}(\mathcal{S})$ be the distribution of X . Theorem 7.23 of [13] shows that there exists a regular Y_0^t -conditional distribution $P_{X|Y_0^t} : \Omega \rightarrow \mathcal{P}(\mathcal{S})$, whose density with respect to P_X is ρ_t . (The filtration (\mathcal{F}_t) of Theorem 7.23 is here $\mathcal{F}_t = X^{-1}(\mathcal{S}) \vee \mathcal{Y}_t$.) So, from (8),

$$\begin{aligned} I(X; Y_t^s | \mathcal{Y}_t) &= \mathbb{E} \left(\mathcal{D}(P_{X|Y_0^s} | P_{X|Y_0^t}) | \mathcal{Y}_t \right) \\ &= \mathbb{E} \left(\log \frac{\rho_s(\cdot, X)}{\rho_t(\cdot, X)} | \mathcal{Y}_t \right) \\ &= \frac{1}{2} \mathbb{E} \left(\int_t^s |H_r(X) - \bar{h}_r|^2 dr | \mathcal{Y}_t \right) \\ &= \frac{1}{2} \mathbb{E} \left(\int_t^s \int_{D([0, \infty); \mathbb{X})} |h_r(\theta_r) - \bar{h}_r|^2 P_{X|Y_0^r}(d\theta) dr | \mathcal{Y}_t \right), \end{aligned}$$

which completes the proof. \square

The solutions of stochastic differential equations driven by continuous semimartingales are typically non-differentiable. This is an important property of nonlinear filters in continuous time. Over a *short* time interval $[t, s]$ the observation process Y introduces the small quantity of new information (42) to the filter. If the filter process Π were differentiable, then the filter would “know” Π_s to first-order accuracy at time t , thereby contradicting the novelty of the information. Proposition 3.2 takes this intuition further by connecting the infinitesimal information gain with the quadratic variation of the filter.

4 Finite-Dimensional Exponential Filters

Let (N, θ) be the finite-dimensional exponential manifold outlined in section 2, with the stronger conditions that $\mathbf{E}_\mu \xi_i^2 < \infty$ for all i , and $\mathbf{E}_\mu \exp(2 \sum_i y^i \tilde{\xi}_i) < \infty$ for all $y \in G$; let $\xi_i := \Lambda \tilde{\xi}_i$ for $i = 1, \dots, n$. Theorem 5.1 in [20] shows

that N is a C^∞ -embedded submanifold of M . We consider a nonlinear filtering problem, as developed in section 3, fulfilling (F1–F6) and the following additional hypotheses.

(F7) $\mathbb{P}(\Pi_t \in N, \text{ for all } t \geq 0) = 1.$

(F8) For any t and any $P \in N$, there exists at least one measurable function $p : \mathbb{X} \rightarrow (0, \infty)$ lying in the domain of \mathcal{A}_t for which $dP/d\mu = p$. For any p with this property, $p^{-1}\mathcal{A}_t p \in \mathcal{L}^2(\mathbb{X}, \mathcal{X}, \mu)$ and $\Lambda p^{-1}\mathcal{A}_t p \in \text{span}\{\xi_i\}$. For any p, \tilde{p} with this property $\mu(\mathcal{A}_t p = \mathcal{A}_t \tilde{p}) = 1.$

(F9) For any t and any k , $\Lambda h_t^k \in \text{span}\{\xi_i\}$ and $\Lambda|h_t|^2 \in \text{span}\{\xi_i\}.$

Let $\mathbf{U}, \mathbf{V}_k : [0, \infty) \times N \rightarrow TN$ be vector fields on N with e -representations

$$\mathbf{u}_{e,t}(P) := \Lambda p^{-1}\mathcal{A}_t p = \mathbf{u}_{\theta,t}^i(P)\xi_i \quad (43)$$

$$\mathbf{v}_{e,k,t}(P) := \mathbf{v}_{e,k,t} = \Lambda h_t^k = \mathbf{v}_{\theta,k,t}^i \xi_i, \quad (44)$$

where p is as in (F8), and $\mathbf{u}_{\theta,t}(P)$ and $\mathbf{v}_{\theta,k,t}$ are the θ -representations. Since $\mathbf{v}_{e,k,t}$ does not depend on P , $\mathbf{V}_{k,t} \in C^\infty(N; TN)$ for all t . We assume, further, that

(F10) $\mathbf{U}_t \in C^1(N; TN)$ for all t .

Consider the following intrinsic Stratonovich equation on N :

$$\circ dQ_t = \left(\mathbf{U}_t(Q_t) - \frac{1}{2} \sum_{k=1}^d \nabla_{\mathbf{v}_{k,t}}^{(-1)} \mathbf{V}_{k,t}(Q_t) \right) dt + \mathbf{V}_{k,t}(Q_t) \circ d\tilde{B}_t^k, \quad (45)$$

where $\nabla^{(-1)}$ is Amari's -1 -covariant derivative [1], and $(\tilde{B}_t, t \geq 0)$ is a d -vector Brownian motion.

Proposition 4.1. *Suppose that (X, Y) satisfies (F1–F6), and (F7–F10) with respect to N , and that \mathbf{U} and \mathbf{V}_k are as defined in (43,44). Then (45) has a strong solution $\Psi : N \times C([0, \infty); \mathbb{R}^d) \rightarrow C([0, \infty); N)$, and $\Pi = \Psi(P_0, \nu)$.*

Proof. As in the proof of Proposition 3.1, we can apply Itô's rule to (27) to show that, for all $x \in F$, $(\log \pi_t, t \geq 0)$ satisfies the following Itô equation on Ω :

$$\log \pi_t = \log p_0 + \int_0^t \mathbf{1}_F(\pi_s^{-1}\mathcal{A}_s \pi_s - \tilde{\zeta}_s) ds, + \int_0^t \mathbf{1}_F(h_s - \bar{h}_s)^* d\nu_s,$$

where $\tilde{\zeta}$ is as defined in (34). This can be “lifted” to an H -valued equation in the same way that (33) was lifted to (31) in the proof of Proposition 3.1. The resulting equation is:

$$\begin{aligned} e(\Pi_t) &= e(P_0) + \int_0^t (\Lambda\pi_s^{-1}\mathcal{A}\pi_s - \zeta_s) ds + \int_0^t \Lambda h_s^* d\nu_s \\ &= e(P_0) + \int_0^t (\mathbf{u}_{e,s}(\Pi_s) - \mathbf{w}_{e,s}(\Pi_s)) ds + \int_0^t \mathbf{v}_{e,k,s} d\nu_s^k, \end{aligned} \quad (46)$$

where $\mathbf{w}_{e,t}(P) (:= \Lambda|h_t - \mathbf{E}_P h_t|^2/2)$ is the e -representation of a time-dependent vector field $\mathbf{W} : [0, \infty) \times N \rightarrow TN$. For any t such that $|h_t| \in \mathcal{L}^2(\mathbb{X}, \mathcal{X}, \mu)$, $\mathbf{E}_P h_t^k = \langle m(P), \Lambda h_t^k \rangle_H + \mathbf{E}_\mu h_t^k$, and so, according to Theorem 5.1 in [20], $W_t \in C^\infty(N; TN)$ for all t .

The Christoffel symbols for $\nabla^{(-1)}$ can be found from the Eguchi relations [1, 9]:

$$\begin{aligned} \Gamma_{ij}^{(-1),l}(P) &= -g^{lm}(P) \frac{\partial^3}{\partial y^i \partial y^j \partial \tilde{y}^m} \mathcal{D}(\theta^{-1}(y) | \theta^{-1}(\tilde{y}))|_{\tilde{y}=y=\theta(P)} \\ &= g^{lm}(P) \mathbf{E}_P(\tilde{\xi}_i - \mathbf{E}_P \tilde{\xi}_i)(\tilde{\xi}_j - \mathbf{E}_P \tilde{\xi}_j)(\tilde{\xi}_m - \mathbf{E}_P \tilde{\xi}_m), \end{aligned}$$

where $g^{lm}(P)$ is the (l, m) element in the inverse of the Fisher matrix in θ -coordinates, g , as defined in (10). So

$$\begin{aligned} \nabla_{\mathbf{V}_{k,t}}^{(-1)} \mathbf{V}_{k,t}(P) &= \Gamma_{ij}^{(-1),l}(P) \mathbf{v}_{\theta,k,t}^i \mathbf{v}_{\theta,k,t}^j \partial_l \\ &= g^{lm}(P) \mathbf{E}_P \mathbf{v}_{\theta,k,t}^i (\tilde{\xi}_i - \mathbf{E}_P \tilde{\xi}_i) \mathbf{v}_{\theta,k,t}^j (\tilde{\xi}_j - \mathbf{E}_P \tilde{\xi}_j) (\tilde{\xi}_m - \mathbf{E}_P \tilde{\xi}_m) \partial_l \\ &= g^{lm}(P) \mathbf{E}_P (h_t^k - \mathbf{E}_P h_t^k)^2 (\tilde{\xi}_m - \mathbf{E}_P \tilde{\xi}_m) \partial_l \\ &= g^{lm}(P) \mathbf{E}_P ((h_t^k - \mathbf{E}_P h_t^k)^2 - \mathbf{E}_P (h_t^k - \mathbf{E}_P h_t^k)^2) (\tilde{\xi}_m - \mathbf{E}_P \tilde{\xi}_m) \partial_l, \end{aligned}$$

and

$$\sum_k \nabla_{\mathbf{V}_{k,t}}^{(-1)} \mathbf{V}_{k,t}(P) = 2g^{lm}(P) \langle \mathbf{W}_t(P), \partial_m \rangle_P \partial_l,$$

which shows that $\sum_k \nabla_{\mathbf{V}_{k,t}}^{(-1)} \mathbf{V}_{k,t} = 2\mathbf{W}_t \in C^\infty(N; TN)$. So (45) has a strong solution, Ψ . The fact that $\Pi = \Psi(P_0, \nu)$ follows from (46), which is the e representation of (45). \square

5 Examples

5.1 An infinite-dimensional diffusion filter

This example is developed from that in section 8.6.2 of [13]. The signal is a special case of the diffusion process of Example 3.2 in section 3, in which $m = d = 1$, $a \equiv 1$, and b and h satisfy

$$\begin{aligned} |b(x)| + |b'(x)| + |b''(x)| + |b'''(x)| + |h(x)| + |h'(x)| + |h''(x)| &\leq C, \\ |b'''(x) - b'''(y)| + |h''(x) - h''(y)| &\leq C|x - y|, \end{aligned} \quad (47)$$

for all $x, y \in \mathbb{R}$ and some $C < \infty$, $P_0 = N(0, R)$ (the Gaussian measure with mean zero and variance $R > 0$), and μ has density $2^{-1} \exp(-|x|)$ with respect to Lebesgue measure.

Proposition 5.1. *The diffusion process (X, Y) defined above satisfies (F1–F6).*

Proof. (F1) is easily verified and, since h is bounded, (F2) is immediate. According to Lemma 8.5 in [13], X_t admits the following \mathcal{Y}_t -conditional density with respect to Lebesgue measure:

$$\tilde{\pi}_t(x) = \int \tilde{\mathbb{E}} \exp(B(x) - B(y) + \Gamma_t(x, y)) n(y, t)(x) n(0, R)(y) dy,$$

where $B(x) := \int_0^x b(y) dy$, $n(m, v)$ is the mean m , variance v Gaussian density,

$$\Gamma_t(x, y) := \int_0^t (h - \bar{h}_s)(\tilde{X}_s^{y,t,x}) d\nu_s - \frac{1}{2} \int_0^t ((h - \bar{h}_s)^2 + b^2 + b')(\tilde{X}_s^{y,t,x}) ds,$$

and $(\tilde{X}_s^{y,t,x}, s \in [0, t])$ is a Brownian motion on an auxiliary probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, pinned to the values y at $s = 0$ and x at $s = t$. $\tilde{X}^{y,t,x}$ can be expressed in terms of a Brownian bridge process $(\tilde{W}_s^t, s \in [0, t])$ as $\tilde{X}_s^{y,t,x} = sx/t + (t-s)y/t + \tilde{W}_s^t$. (See Corollary 8.6 in [13]. NB. Equations (8.97) and (8.108) in [13] contain some typographical errors, which are corrected in the above.) The \mathcal{Y}_t -conditional distribution of X_t thus admits the strictly positive density $\pi_t := \tilde{\pi}_t/r$ with respect to μ , where $r = 2^{-1} \exp(-|x|)$.

Theorem 8.7 in [13] shows that π satisfies (F6). In particular, π is (\mathcal{Y}_t) -adapted for each x and continuous in (t, x) , and hence $\mathcal{P}_Y \times \mathcal{X}$ -measurable.

It thus satisfies (F3). Equations (8.123) and (8.124) in [13] enable the explicit calculation of $\mathcal{A}\pi_t$; straightforward calculations (involving integration by parts) show that

$$(\mathcal{A}\pi_t)(x) = \frac{1}{2r} \int \tilde{\mathbb{E}}\gamma_t(x, y) \exp(B(x) - B(y) + \Gamma_t(x, y)) n(y, t)(x) n(0, R)(y) dy,$$

where

$$\begin{aligned} \gamma_t(x, y) := & -b^2(x) + \left(b(y) - \frac{\partial\Gamma_t}{\partial x} - \frac{\partial\Gamma_t}{\partial y} + \frac{y}{R} \right)^2 \\ & - b'(x) - b'(y) + \frac{\partial^2\Gamma_t}{\partial x^2} + 2\frac{\partial^2\Gamma_t}{\partial x\partial y} + \frac{\partial^2\Gamma_t}{\partial y^2} - \frac{1}{R}. \end{aligned}$$

(A proof of the existence and continuity of the derivatives of Γ is contained in the proof of Lemma 8.8 in [13].) In particular, $\mathcal{A}\pi$ is (\mathcal{Y}_t) -adapted for each x and continuous in (t, x) , and hence $\mathcal{P}_Y \times \mathcal{X}$ -measurable. The derivatives of Γ can be computed in closed form; for example

$$\frac{\partial\Gamma_t}{\partial x} = \int_0^t \frac{s}{t} h'(\tilde{X}_s^{y,t,x}) d\nu_s - \int_0^t \frac{s}{t} ((h - \bar{h}_s)h' + bb' + b''/2)(\tilde{X}_s^{y,t,x}) ds.$$

The joint density $n(y, t)(x)n(0, R)(y)$ can be written in the x -marginal/ y -conditional form, $n(\alpha_t x, \sigma_t^2)(y)n(0, R+t)(x)$, where $\alpha_t := R/(R+t)$ and $\sigma_t^2 := Rt/(R+t)$, and so

$$\begin{aligned} \pi_t(x) = & \frac{n(0, R+t)}{r}(x) \int \tilde{\mathbb{E}} \exp(B(x) - B(y) + \Gamma_t(x, y)) \\ & \times n(\alpha_t x, \sigma_t^2)(y) dy, \end{aligned} \quad (48)$$

$$\begin{aligned} (\mathcal{A}\pi_t)(x) = & \frac{n(0, R+t)}{2r}(x) \int \tilde{\mathbb{E}}\gamma_t(x, y) \exp(B(x) - B(y) + \Gamma_t(x, y)) \\ & \times n(\alpha_t x, \sigma_t^2)(y) dy. \end{aligned} \quad (49)$$

Since $|b|, |b'|, |h| \leq C$, for any $k \in \mathbb{N}$ and any $x, y \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E} \tilde{\mathbb{E}} \exp(k\Gamma_t(x, y)) & \leq \mathbb{E} \tilde{\mathbb{E}} \Xi_t(x, y) \exp((2k(k-1)C^2 + k(C^2 + C)/2)t) \\ & = \exp((2k(k-1)C^2 + k(C^2 + C)/2)t), \end{aligned} \quad (50)$$

where $\Xi(x, y)$ is the exponential martingale

$$\Xi_t(x, y) := \exp \left(k \int_0^t (h - \bar{h}_s)(\tilde{X}_s^{y,t,x}) d\nu_s - \frac{k^2}{2} \int_0^t (h - \bar{h}_s)^2 (\tilde{X}_s^{y,t,x}) ds \right).$$

Furthermore, since $|B(y)| \leq C|y|$,

$$\begin{aligned} \int \exp(-kB(y))n(\alpha_t x, \sigma_t^2)(y) dy &\leq 2 \int \cosh(kCy)n(\alpha_t x, \sigma_t^2)(y) dy \\ &= 2 \exp(k^2 C^2 \sigma_t^2 / 2) \cosh(kC\alpha_t x) \\ &\leq 2 \cosh(kCx) \exp(k^2 C^2 t / 2). \end{aligned} \quad (51)$$

Applying Jensen's inequality to (48), we obtain

$$\pi_t^2(x) \leq \frac{n(0, R+t)^2}{r^2}(x) \int \tilde{\mathbb{E}} \exp(2(B(x) - B(y) + \Gamma_t(x, y)))n(\alpha_t x, \sigma_t^2)(y) dy,$$

and so it follows from (50) and (51) with $k = 2$ that

$$\begin{aligned} \mathbb{E} \mathbf{E}_\mu \pi_t^2 &\leq 4 \exp((7C^2 + C)t) \int \cosh^2(2Cx) \frac{n(0, R+t)^2}{r}(x) dx \\ &\leq K_T < \infty \quad \text{for all } t \in [0, T] \text{ and any } T < \infty. \end{aligned} \quad (52)$$

Together with the boundedness of h , this shows that (F5) is satisfied. Applying Jensen's inequality and the Cauchy-Schwartz inequality to (49), we obtain

$$\begin{aligned} \mathbb{E}(\mathcal{A}\pi_t)^2(x) &\leq \frac{n(0, R+t)^2}{4r^2}(x) \int \sqrt{\mathbb{E}\tilde{\mathbb{E}}\gamma_t(x, y)^4} \\ &\quad \times \sqrt{\mathbb{E}\tilde{\mathbb{E}} \exp(4(B(x) - B(y) + \Gamma_t(x, y)))n(\alpha_t x, \sigma_t^2)(y) dy}. \end{aligned}$$

It follows from the bounds in (47), and standard properties of the Lebesgue and Itô integrals that, for any $x, y \in \mathbb{R}$,

$$\mathbb{E}\tilde{\mathbb{E}}\gamma_t(x, y)^4 \leq K(1+t^8)(1+y^8) \quad \text{for some } K < \infty.$$

Following the same steps as were used in the proof of (52) (but using $k = 4$ in (50) and (51)) we now conclude that

$$\mathbb{E} \mathbf{E}_\mu (\mathcal{A}\pi_t)^2 \leq \tilde{K}_T < \infty \quad \text{for all } t \in [0, T] \text{ and any } T < \infty. \quad (53)$$

A further application of Jensen's inequality to (48) yields

$$\pi_t(x)^{-1} \leq \frac{r}{n(0, R+t)}(x) \int \tilde{\mathbb{E}} \exp(-B(x) + B(y) - \Gamma_t(x, y))n(\alpha_t x, \sigma_t^2)(y) dy,$$

which, together with (49), shows that

$$\begin{aligned} |\pi_t(x)^{-1} \mathcal{A} \pi_t(x)| &\leq \frac{1}{2} \int \tilde{\mathbb{E}} |\gamma_t(x, y)| \exp(-B(y) + \Gamma_t(x, y)) n(\alpha_t x, \sigma_t^2)(y) dy \\ &\quad \times \int \tilde{\mathbb{E}} \exp(B(y) - \Gamma_t(x, y)) n(\alpha_t x, \sigma_t^2)(y) dy, \end{aligned}$$

and the bound,

$$\mathbb{E} \mathbf{E}_\mu (\pi_t^{-1} \mathcal{A} \pi_t)^2 \leq \exp(K(1+t)) \quad \text{for some } K < \infty, \quad (54)$$

easily follows. The bound (23), for any $T < \infty$, follows from (53) and (54), and this establishes (F4). \square

5.2 A Kalman-Bucy Filter

This is another example in which the signal is a diffusion process. Here $b(x) = Bx$, $a(x) = A$, $h(x) = Cx$ and $P_0 = N(m_0, R_0)$, where B is an $m \times m$ matrix, A is a positive semi-definite $m \times m$ matrix, C is a $d \times m$ matrix, and R_0 is a positive definite $m \times m$ matrix. The posterior distribution is $\Pi_t = N(\bar{X}_t, R_t)$, where the mean vector, \bar{X}_t , and covariance matrix, R_t , satisfy the Kalman-Bucy filtering equations [13]:

$$\bar{X}_t = m_0 + \int_0^t B \bar{X}_s ds + \int_0^t R_s C^* d\nu_s \quad (55)$$

$$R_t = R_0 + \int_0^t (BR_s + R_s B^* + A - R_s C^* C R_s) ds. \quad (56)$$

It is well known that such Gaussian measures belong to finite-dimensional exponential statistical manifolds. In order to apply the results of sections 3 and 4, we construct such a manifold as a C^∞ -embedded submanifold of $M(\mathbb{R}^m, \mu)$, where μ has density $2^{-m} \exp(-\sum_j |x^j|)$ with respect to Lebesgue measure. Let \mathbb{S} be the set of symmetric positive definite $m \times m$ real matrices, and let $n := m(m+3)/2$. For any $y \in \mathbb{R}^n$, let $\alpha(y)$ be the m -vector comprising the first m elements of y , and let $\beta(y)$ be the symmetric $m \times m$ matrix whose lower triangle contains the elements $y^{m+1}, y^{m+2}, \dots, y^n$ in some fixed arrangement. Then $G := (\alpha, \beta)^{-1}(\mathbb{R}^m \times \mathbb{S})$ is an open subset of \mathbb{R}^n , and the map $(\alpha, \beta) : G \rightarrow \mathbb{R}^m \times \mathbb{S}$ is a linear bijection. Let $\tilde{e} : \mathbb{X} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$\tilde{e}(x, y, z) := -\frac{1}{2} x^* \beta(y) x + \alpha(y)^* x + z \sum_{j=1}^m |x^j|;$$

then $\Lambda\tilde{e}(\cdot, y, z) \in e(M)$ for all $(y, z) \in G \times \mathbb{R}$. Let $\xi_i := \Lambda\tilde{e}(\cdot, \mathbf{e}_i, 0)$, where $(\mathbf{e}_i, 1 \leq i \leq n)$ is the coordinate orthonormal basis in \mathbb{R}^n , let $\xi_{n+1} := \Lambda\tilde{e}(\cdot, 0, 1)$, and let $\tilde{N} := e^{-1} \circ \gamma(G \times \mathbb{R})$, where $\gamma(y, z) := y^i \xi_i + z \xi_{n+1}$. \tilde{N} is an $n + 1$ -dimensional instance of the exponential manifold discussed in section 4. The n -dimensional submanifold $N := e^{-1} \circ \gamma(G \times \{1\})$ comprises all the non-singular Gaussian measures on \mathbb{R}^m .

Proposition 5.2. *The diffusion process (X, Y) defined above satisfies (F1–F6), and (F7–F10) with respect to the n -dimensional submanifold N .*

Proof. (F7) (and hence (F1)) follows from the fact that $R_t \in \mathbb{S}$ for all t ; (F2) and (F9) are obvious. Straightforward calculations show that, for any $P \in N$,

$$\begin{aligned} \frac{\mathcal{A}p}{p}(x) &= \frac{1}{2}x^* \beta(y)(A\beta(y) + 2B)x - \alpha(y)^*(A\beta(y) + B)x \\ &\quad + \frac{1}{2}(\alpha(y)^* A \alpha(y) - \text{tr}(A\beta(y) + 2B)), \end{aligned}$$

where $y = \theta(P)$, and (F8) and (F10) readily follow. (F3–F6) are easily verified from (55,56). \square

5.3 Wonham’s Filter

In this, \mathbb{X} and \mathcal{A} are as defined in Example 3.1 of section 3, X is a Markov jump process for which $\mathbb{P}(X_0 = x) > 0$ for all x , and μ is the uniform probability measure. M is itself an n ($= m - 1$)-dimensional exponential statistical manifold. In the set-up of section 4, appropriate choices are $G = \mathbb{R}^n$ and $\xi_i := \mathbf{1}_{\{i\}} - n^{-1} \sum_{j \neq i} \mathbf{1}_{\{j\}}$. (F1–F10) are easily verified.

6 Concluding Remarks

This paper developed information geometric representations for nonlinear filters in continuous time, and studied their properties. Information manifolds are natural state spaces for the posterior distributions of Bayesian estimation problems where many statistics are required. They clarify information-theoretic properties of estimators, and their metrics are appropriate “multi-objective” measures of approximation error. The results also have bearing

on the theory of non-equilibrium statistical mechanics, in which rates of entropy production can be associated with rates of information supply [19], and hence with the quadratic variation of a process of “mesoscopic states” in a particular pseudo-Riemannian metric.

The development of approximations is beyond the scope of this paper. However, we conclude with a few remarks on this issue. One approach is to first define an appropriate differential equation (evolution equation) to which numerical methods might be applied. Equations (31) and (45) are expressed in terms of the innovations process ν in order to emphasise their information theoretic properties. Substituting for ν in (31), we obtain an H -valued Itô equation for the nonlinear filter in terms of the observation process Y :

$$\phi(\Pi_t) = \phi(P_0) + \int_0^t (u_s - z_s) ds + \int_0^t v_s dY_s, \quad (57)$$

where $z_t := \zeta_t + v_t \bar{h}_t$. If h is bounded, then z and ve_k can be expressed in terms of the time-dependent, locally Lipschitz vector fields $\mathbf{z}, \mathbf{v}_k : [0, \infty) \times H \rightarrow H$, where

$$\mathbf{z}_t(a) = \Lambda \left(\frac{1}{2} |h_t - \mathbf{E}_P h_t|^2 + (p+1)(h_t - \mathbf{E}_P h_t)^* \mathbf{E}_P h_t \right) \quad (58)$$

$$\mathbf{v}_{k,t}(a) = \Lambda(p+1)(h_t^k - \mathbf{E}_P h_t^k), \quad (59)$$

and $P = \phi^{-1}(a)$. However, except in special cases such as the exponential filters of section 4, u is more problematic since the *infinitesimal* characterisation of P_t in (21) is dependent on the topology of the signal space \mathbb{X} . The topology of M arises from purely measure-theoretic constructs, and is not dependent on the existence of a topology on \mathbb{X} . ([20] assumes only that $(\mathbb{X}, \mathcal{X}, \mu)$ is a probability space.) This is quite natural in the context of Bayesian estimation and, in particular, nonlinear filtering: Bayes’ formula and Shannon’s information quantities are measure-theoretic in nature, as is the Markov property in its most general form (a property of conditional independence). It may be possible to overcome this problem by strengthening the topology of M in some way (for example, by the use of Sobolev space techniques in the case of filters for diffusion signals). However, this is not necessarily the best approach; for the purposes of approximation, it suffices to solve a simpler evolution equation for an approximate filter. This idea is developed in [2], where approximations to Π are constrained to remain on finite-dimensional exponential statistical manifolds, on which projections of

the processes u , z and v_k can be represented in terms of locally Lipschitz vector fields. The manifold M contains a rich variety of smoothly embedded submanifolds, to which this method could be generalised [20]. The selection of a good submanifold for a particular problem, together with a suitable coordinate system, would be critical to this approach.

Hilbert-space-valued filtering equations based on the *Zakai equation* may be more suitable for these purposes. Manifolds of finite (un-normalised) measures, to which the Zakai equation could be lifted, are developed in [21]. These avoid the normalisation constant $Z(a)$ in the computation of the density.

Another approach to the problem of approximation would be to switch to a discrete-time model “up front”, replacing Π by a time sampled version. This would replace the Itô equation (57) by a difference equation, on which approximations could be based. This would avoid the Kolmogorov forward equation (21), replacing it by an integral equation (the Chapman-Kolmogorov equation) over each time step, and thereby eliminating problems concerning the topology of \mathbb{X} . (In the case of diffusion signal processes, for example, the transition measure over a short time step could be approximated by an appropriate Gaussian.)

Time reversal is used in [17, 18, 19] to construct *dual* filtering problems, in which the primal signal and filter processes exchange roles. In the notation of this article, the dual filter computes the process of posterior distributions for the primal filter Π (regarded as a dual signal) in reverse time, based on a dual observation process. Such posterior distributions take values in the set of probability measures on M . Since M is itself a complete, separable metric space, one can easily use the construction of section 2.1 to define a (dual) Hilbert manifold of such probability measures. However, a striking feature of the dual filter is that it is parametrised by the primal signal process X , reversed in time. In this way, the topology of the primal signal space \mathbb{X} is connected with the information topology of the dual problem. It may be possible to exploit this fact in filter approximations.

Notions of information *supply* and *dissipation* for nonlinear filters are defined in [18, 19]. The supply at time t is the mutual information $I(X; Y_0^t)$, and the dissipation is the X_t -conditional variant, $\mathbb{E}I(X; Y_0^t | X_t)$. Modulo initial conditions, the supply of the primal filter is the dissipation of its dual, and vice-versa [19]. The quantity $\mathbb{E}I(X; Y_0^t | X_t)$ was studied in [14] in the context of filters for diffusion processes, and shown to be connected with the

Fisher metric in the sense that

$$\mathbb{E}I(X; Y_0^t | X_t) = \frac{1}{2} \int_0^t \mathbb{E} \left(\nabla \log \frac{\pi_s}{p_s} \right)^* a \left(\nabla \log \frac{\pi_s}{p_s} \right) (X_s) ds,$$

where p and π are the prior and posterior densities, and a is the diffusion matrix for the signal. The integral here is the average quadratic variation of the dual filter in the Fisher metric, and the integrand is the mean-square error for the dual observation function $h^d(X_s, p) := (\sigma^* \nabla \log p)(X_s)$, where σ is a matrix square-root of a [19].

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