Lowest-Unmatched Price Auctions

Jürgen Eichberger
University of Heidelberg, Germany

Dmitri Vinogradov
University of Essex, UK

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Abstract

Lowest-Unmatched Price Auctions (LUPA) specify that the lowest bid placed by only one participant wins. They are used in internet trading and TV and radio shows. We model LUPAs as games with minimal restrictions, in particular allowing players to place more than one bid, since multiple bids have been observed in most actual LUPAs. Though LUPAs are games for which a closed-form solution does not seem to exist in general, our model generates several testable implications about the type of strategies played in equilibrium and the highest bid in a given LUPA. Our analysis suggests that players follow strategic considerations and arrive at decisions which, at least in the aggregate, are generally consistent with theoretical predictions, yet there are some remarkable deviations.

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Address for Correspondence: Dmitri Vinogradov, Essex Business School, University of Essex, Wivenhoe Park, Colchester, CO4 3SQ, United Kingdom. Email: dvinog@essex.ac.uk

1 Introduction

In standard auctions, the highest bid wins. Depending on the price which the winner pays, the auction may be a First Price (FPA), Second Price (SPA), or a lower price auction. More recently, a special extreme form of auction proved to be popular in different areas. *"Reverse auctions"* or *"backward auctions"*, as they are sometimes called, specify that the lowest bid wins. Without further restrictions this rule would induce all bidders to bunch on the lowest price. The additional rule that only an unmatched (single) bid can win, however, forces participants to predict the bids of other participants. It is this prediction problem, which makes the *L*owest-*U*nmatched *P*rice *A*uction (*LUPA*[1]) worth studying from a strategic point of view.

In 2006, the first 250 LUPAs in the USA awarded prizes worth approximately \$360,000 for bids totalling less than \$1,000[2]. The UK-based company "Auction Air Ltd.". organized more than 500 LUPAs since fall 2004, with a total prize value of more than \$700,000 for about \$12,000 worth of winning bids. LUPAs are occasionally used as a marketing instrument by TV- and radio broadcasters. In fall 2005, in a period of sky-rocketing fuel prices, the German radio station "Radio Brocken" conducted a daily LUPA in which participants could bid for a €500 petrol voucher. "London Capital FM" auctioned off a flat in London, a house in Spain, and a Bentley Continental in LUPAs in 2004. Several other LUPAs were run on German radio and television during the years 2005 and 2006, for which we have obtained micro-level data.[3] More recent examples would include LUPA-sales of houses worth €250 000, with extra €100 000 cash to buy a plot of land, organized by German newspaper Die Bild Zeitung in March 2012 and November 2013.[4]

We suggest here a model which imposes minimal restrictions on the bidding behavior. In particular, players may place as many bids as they wish. This feature is typical for many LUPAs, but distinguishes our study from several other papers on

[1] One can also read the acronym as "lowest unique price auction".
[2] As reported by the *USA Today* on the 25th of October, 2006.
[3] We thank "Legion Telekommunkation GmbH" and "Radio Regenbogen" for cooperation and help.
[4] See http://www.bild.de/geld/immobilien/auktion/ersteigern-sie-dieses-traumhaus-rueckwaertsauktion-23110814.bild.html, and http://www.bild.de/lifestyle/2013/gewinnspiele/ersteigern-sie-ihr-traumhaus-bei-der-bild-auktion-33254352.bild.html.

lowest unmatched bid auctions like Houba et al. (2011) and Ostling, et al. (2011), where players are restricted to placing only one bid. In addition to LUPAs, Rapoport et al. (2007) and Raviv and Virag (2009) consider their counterpart, the highest unique bid auctions, also under a single bid assumption. In Ostling, et al. (2011), the number of players is assumed uncertain, the auction is modelled as a Poisson game, which yields a closed-form solution for the single bid case. A concise review of the most recent literature can be found in Costa-Gomes and Shimoji (2014) who note that (a) most studies so far focused on the case where players place a single bid, (b) a closed-form solution in most cases could not be obtained, and (c) some properties of the equilibrium strategies are consistent across different studies, for example, there are no gaps in the equilibrium support, and the higher the value of the bid, the lower the probability with which this bid is placed in equilibrium. Although most researchers have focused on the frequencies with which bids are placed, some studies also investigate how high players would bid in the optimum. Scarsini et al. (2010) derive an upper bound on bids for a two-player game; Costa-Gomes and Shimoji (2014) obtain this limit for more than two players under the singleton bidding assumption.

Rare exceptions from the one-bid-per-player assumption are Eichberger and Vinogradov (2008) and Scarsini et al. (2010). Both provide closed-form solutions – the latter for a two-player game, and the former for any number of players in the special case of bidders playing "block" strategies in which any bid can only be placed jointly with all lower bids. The two-player case implies "block" bidding because any gap in the strategy of a designated player gives a winning opportunity to the counterpart, and thus makes the designated player strictly worse off by lowering the probability of winning with bids above the gap. In contrast, with three and more players there is a possibility that two or more rivals of the designated player place the same bid and thus the designated player could be better off by allowing gaps in the pure strategy chosen. Still, as players follow mixed strategies, in equilibrium gaps in pure strategies would not imply gaps in the set of bids placed. This strategic limitation of the two-player case implies a need to focus on multiple players, along with multiple bids.

Our analysis in this paper shows that, in general, bidders in a LUPA place more than one bid and that the vast majority of bidders do not bid in blocks, consistent with the theory. Moreover, we derive and test an equilibrium upper bound on bids, related solely to the prize and bidding fee of the LUPA. We also compute the frequency with which individual bids win in observed auctions and compare it with the theoretically predicted probability of winning. Our findings indicate that, in aggregate, players in actual LUPAs behave consistently with the prediction of the Nash equilibrium.

The paper proceeds as follows. In Section 2 we derive some stylized facts of real life LUPAs. Section 3 models LUPAs as a simultaneous-move game. We characterize the solution of the game in Section 4 and derive predictions with regards to the bidding behavior of participants in Section 5. In Section 6, we present the data from several lowest unmatched price auctions and compare it with our theoretical predictions. Section 7 concludes the paper. Appendix A contains auxiliary results used in the proofs gathered in Appendix B.

2 LUPAs

We consider LUPAs with the following rules. The organizer of the game announces the item to be sold (the prize), and the period within which bidding may take place (bidding phase). Bids must be submitted in local currency, e.g., in euros and cents. Bids in non-integer amount of euros and cents are not accepted. Players who wish to take part in the game, can place their bids via a phone call, a text message or the internet. The number of bids which an agent can submit is not restricted. For each bid a fixed fee is charged, which includes the cost of calling or texting. No information about bids placed is provided during the bidding phase of the game.

As soon as the bidding phase is over, the winner is determined from the set of valid bids submitted. The winning bid must meet two conditions:

1. It must be unmatched, i.e., there should be no other participant of the game who has placed the same bid.

2. It must be the lowest bid among all unmatched bids.

The winner is the player, who made the winning bid. The winner pays the winning bid to the auctioneer and receives the prize.

The remainder of this section discusses some variations of actual LUPAs and draws attention to several stylized facts that our theoretical model will aim to explain.

In 2005-2006 several LUPAs took place in Germany for which we could obtain detailed data. Table 1 provides a summary of these auctions. All auctions had a fixed duration and were run in radio or TV shows and in newspapers. In all auctions, providers charged bidders 49 cents per bid.

ID	**Media**	**Prize**	**Bidders**	**Total bids**	**Bids per bidder**	**Duration**	**Winner**
RB	Radio	10000€ monetary	9400	47872	5.09	19 days	14.55€
AMV	Radio	10000€ monetary	10660	52847	4.96	8 days	14.65€
AD1	Radio	1000€ monetary	537	1798	3.35	2 days	0.60€
AD2	Radio	3000€ monetary	916	6732	7.35	4 days	5.82€
AD3	Radio	5000€ monetary	631	6201	9.83	5 days	11.16€
LSR	Newspaper	1099€ bike	437	1272	2.91	17 days	1.51€
BB	TV	20 000€ car	89862	266824	2.96	7 days	20.65€
RTL	Radio	350 000€ house	72588	610104	8.41	23 days	99.82€

Table 1. Summary of some LUPAs run in Germany in 2005-2006.

In Table 1, Auction ID stands for the organizer's acronym. In all cases bidding cost is 49c. Two features distinguish these LUPAs. Firstly, the auctioned "item" is often an amount of money[5], which we refer to as a monetary prize. Hence, we will assume that the valuation of the prize is identical among the participants. Secondly, players are allowed to place as many bids as they wish. This feature makes strategy sets complicated and the analysis of Nash equilibrium difficult.

[5] If prizes are commodities, they were standard products with clearly advertised value. Hence, the common-value assumption appears justified.

A similar mechanism is used by some online traders. Table 2 gives examples of LUPAs organized online by AuctionAir™ in 2004-05. Participants were charged a fee during a short online registration. In these auctions, the maximum number of bidders was predetermined (see the column "Bids required"). Once the maximum number of bidders is reached, the auction is closed. Hence, the duration of the auction is uncertain. Several auctions for the same item were run in a series with the same rules. All participants were informed about the number of times a similar item had been auctioned before (see "preceding auctions"). Data from preceding auctions was available for bidders. Bidders were restricted in the number of bids they could place (see column "Max bids per pers."). Table 2 also provides information about the winning bids. Identification of bidders took place by their credit card number (in our data bidders are identified by the invoice number). These auctions are still running regularly.

Prize	**Cost**	**Bids required**	**Max bids per bidder**	**Preceding auctions**	**Winner**
£259 80 GB iPod	£3.00	120	10	6	£24.00
£1,695 40" LCD TV	£4.00	575	20	28	£6.00
£5,900 7 nights in Mauritius	£12.00	530	20	2	£49.00
£275 Headphones	£5.00	60	5	39	£12.00
£16,900 Mini Cooper	£25.00	945	20	11	£20.00

Table 2. Summary of some LUPAs run by AuctionAir.com

No constraint on the number of bids required to complete the auction is a feature that distinguishes the German LUPAs in Table 1 from those of AcutionAir in Table 2. In both types of LUPA, the potential number of bidders is exogenous and unknown, effectively they are all people who became aware of the auction. The number of actual bidders in German LUPAs is endogenous, whilst it is exogenously constrained in the AuctionAir LUPAs, since the total number of bids and the number of bids per bidder were bounded. Despite this difference, the two types of LUPAs generate qualitatively similar behavior of players due to the strategic similarity.

Our approach is applicable to both types of auctions. It relaxes the time constraint in LUPAs from Table 1 and the constraint on the total number of bids required in LUPAs from Table 2. Modelling the time constraint would require a dynamic version of the game. Modelling the number-of-bids constraint makes the model significantly more complicated, as strategy sets of players become dependent on each other (if one player places more bids, other players' strategy sets shrink). We resolve this theoretical challenge by modelling the LUPAs as an unconstrained one-shot game where players can place as many bids as they like, including the option of submitting no bid. Hence, although there is an exogenously given number of players, the effective number of bidders is endogenously determined as all players have the option of not bidding.

2.1 Bids per participant

In the German LUPAs, there was no constraint on the number of bids that a participant could submit in the above auctions. Table 1 indicates that on average bidders placed between 3 and 10 bids. Figure 1 shows that although placing a single bid is the most frequently chosen option, a majority of bidders prefers placing more than one bid. A few bidders place a large number of bids (the figure only shows numbers of bids up to 100).

In the AuctionAir LUPAs bidders were restricted to a maximum number of bids (typically 5, 10, or 20). Indeed, many bidders submitted the maximal number of bids. For example, in 26 consecutive auctions of a 20GB iPod worth £248 which took place between November 2004 and August 2005 and in which a maximum of five bids was allowed, a total number of 1033 bidders took part of whom 47.24% placed one bid, 17.52% two bids, 8.71% three bids, 2.8% four bids, and 23.71% placed the maximum allowed number of five bids. This observation stands in stark contrast to the monotonically declining number of bids per person observed in unconstrained LUPAs, as in Figure 1. This suggests that the upper limit on the number of bids is binding. Players who may have wished to place more than five bids, seem to have exhausted their limit with one identifier or may have logged in with a different identifier to continue bidding. Using several cards or building coalitions of bidders was a possibility

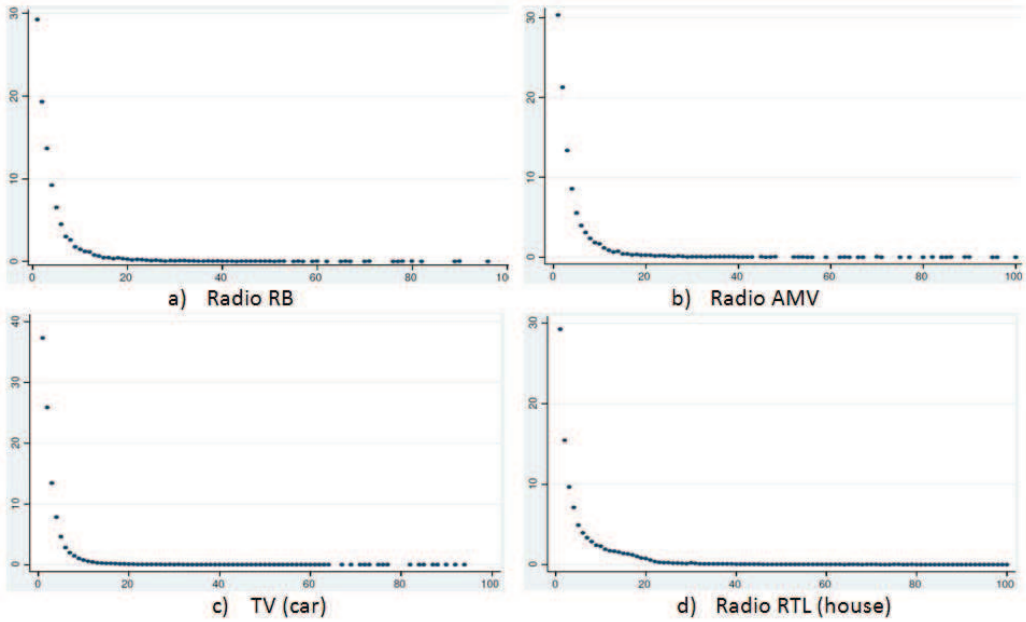

Figure 1. Fraction of players (% of the total sample of bidders, y-axis) in German LUPAs placing various numbers of bids (x-axis).

to bypass the restriction on the number of bids per person. Such behavior, however, cannot be easily identified in the data. In cases where the restriction on the number of bids per person is softer, bidders place more bids, up to the maximum allowed. Figure 2 shows fractions of bidders that placed multiple bids in an auction for a digital sound system worth £1,600 where a maximum of 10 bids per bidder was allowed. [6] Again, there is a noticeable increase at the upper end, suggesting that some bidders would have liked to go beyond the maximum of ten bids if the constraint were relaxed.

We summarize these observations as a stylized fact about LUPAs:

Stylized fact 1. Most players place more than one bid.

Most existing models of LUPAs cannot explain this fact as they usually allow only a single bid per person. In our model, there is no constraint on the number of bids which a participant can place. In Proposition 3 below, we will show that an equilibrium in mixed strategies can accommodate this stylized fact.

[6] It should be noted that it took almost two and half years for this auction to close, with first bid placed on 20.08.2011 and last bid on 28.02.2014.

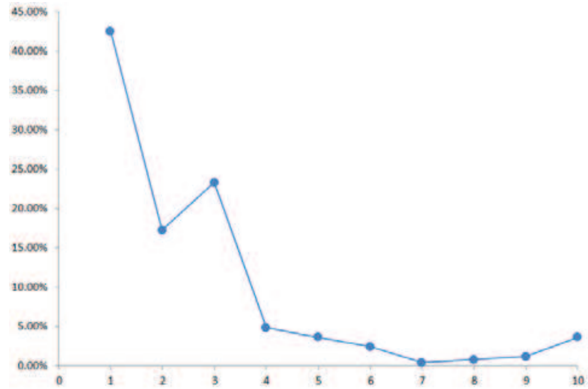

Figure 2. Fraction of players (% of the total sample of bidders, y-axis) in an AuctionAir LUPA placing various numbers of bids (x-axis).

2.2 Frequencies of bids and an upper bound on bids

Some typical distributions of the frequencies of bids for LUPAs from Table 1 are shown in Figure 3. The figure reveals that the frequency of bids declines with the size of bids and that the shape of the distribution depends on the value of the prize. There is no substantial difference in the pattern of these frequencies between the LUPAs with a monetary prize and LUPAs with a commodity prize of comparable values. Note that in the newspaper LUPA the lowest bid was 1€, whereas in the other auctions it was 0,01€.

Figure 4 presents frequencies of bids from three typical AuctionAir auctions for the same item as given in Table 2. Three diagrams show the bidding frequencies of three separate LUPAs while the forth diagram presents the distribution of bidding frequencies aggregated over 26 consecutive auctions. The declining pattern is not always clearly recognizable in the individual auctions, but is clearly visible in the aggregates.

From these observations we deduce a second stylized fact of LUPAs.

Stylized fact 2 Lower bids are placed more frequently than higher bids.

This monotonicity has been observed also for LUPAs where participants were constrained to a single bid. We remove the singleton bidding constraint and, in Proposition 6 below, we provide estimates for the probabilities of bids in a mixed strategy

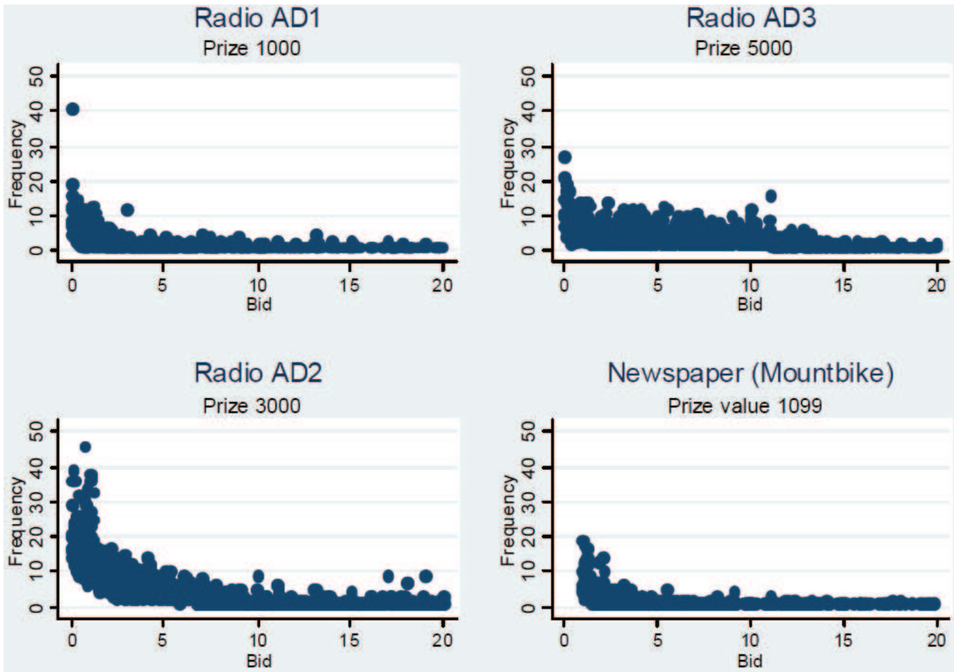

Figure 3. Frequencies of bids in four LUPAs with comparable prizes, costs and number of participants.

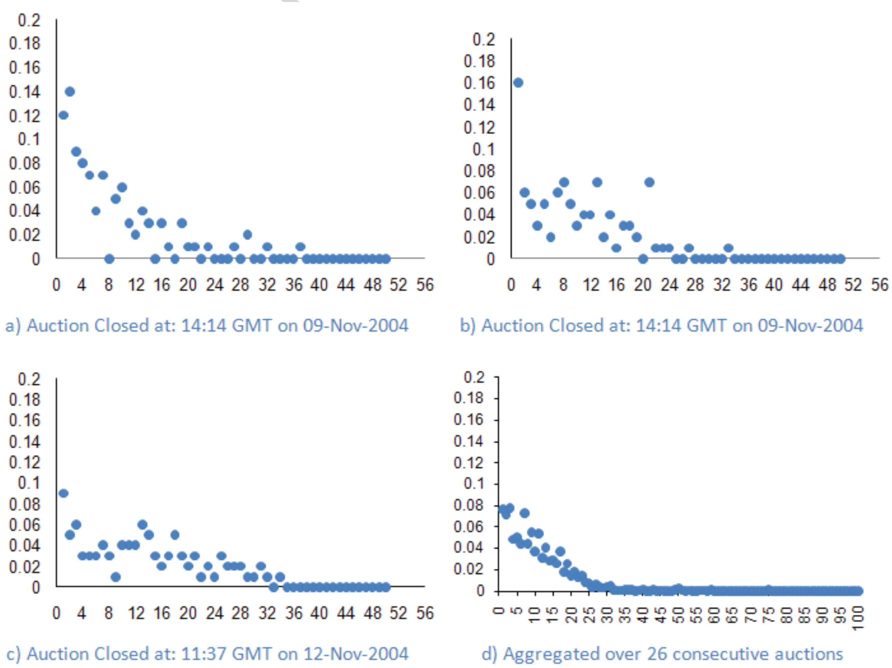

Figure 4. AuctionAir Lot: R005 - 20GB Apple iPOD (with docking station).

equilibrium. Using these probabilities as a proxy for the observed frequencies of bids confirms this stylized fact.

Figure 4 also indicates that players do place extremely high bids, e.g., over £100 for an item worth £209). Figure 3 only presents bids up to €20.00, yet individual bidders place bids well above €200 and, for high value items like a car or a house, even bids above €2 000 and €20 000 were observed (see Section 6 for more details). This suggests a third stylized fact about LUPAs.

Stylized fact 3 Some bidders place high bids relative to the value of the object.

While it seems natural that bidders bid higher for high-value prizes, it remains puzzling that in AuctionAir bidders go beyond 50% of the prize value, whereas in German LUPAs the upper bound as a fraction of the prize value is noticeably lower. This raises a question about whether there is a reasonable upper bound for bids. Proposition 5 below will suggest a theoretical upper bound for bids that is consistent with the behavior of about 95% of bidders.

3 The model

Denote by $I = \{1, .., N\}$ the set of potential bidders (the number of active bidders is uncertain and will be endogenously determined by the mixed equilibrium below). We will assume that players are risk-neutral with respect to monetary prizes. Bids are natural numbers $\mathbb{N}$. For example, a bid of €12.34 corresponds to the number 1234, or, if bids in fractions of the basic currency unit are not allowed, a bid of £3 would correspond to the number 3.

During the bidding phase, each player i can place an arbitrary number of bids. There exists a bidding fee of c per bid, measured in bidding increments (e.g. cents, dollars, or pounds). If player i wins with some bid, he obtains the prize $A \in \mathbb{R}_+$ and pays his winning bid. Obviously, a player cannot achieve a payoff greater than $(A-1)-c$ and thus bidding above this value would make a player strictly worse off compared to the option of not bidding. Hence, without loss of generality, we can restrict the set of bids to the interval $[1,\overline{b}]$ of natural numbers between 1 and $\overline{b}:=(A-1)-c$. A strategy s_i is a simultaneous choice[7] of the set of multiple bids, which we will represent with a binary vector

$$s_i=\left(\underset{1}{1},\underset{2}{0},....,\underset{b-1}{0},\underset{b}{1},\underset{b+1}{0},...,\underset{\overline{b}}{0}\right).$$

Here the strategy of player i corresponds to "placing the bid b" if there is a "1" at position b and "not placing the bid b" if there is a "0" at position b. In the above example, strategy s_i corresponds to player i placing bids 1 and b. We denote $s^0=(0,...,0)$ the decision to place no bids. There are in total $2^{\overline{b}}$ pure strategies. All players have the same set of strategies $S_i=S=\{s^t\}_{t=0}^{2^{\overline{b}}-1}$. For any $s^t\in S$, we will denote by $s^t(b)$ the component b of vector s^t, i.e. $s^t(b)$ equals either one or zero, depending on whether, according to strategy s^t, bid b is placed or not. For notational convenience, we will associate each s^t with a reversed binary notation for number t, e.g. $s^4=(0,0,1)$, which is a reversal of "100", the binary representation of number "4".

Denote by S^N the set of all possible strategy combinations and by $\mathbf{s}=(s_1,s_2,..s_N)$ a combination of pure strategies $s_i\in S$ for players $i=1,...,N$. Given a strategy combination $\mathbf{s}$, we define the set of unmatched bids $B(\mathbf{s})=\left\{b:\sum_{j=1}^{N}s_j(b)=1\right\}$ and the function $\mu:S^N\rightarrow\mathbb{N}$ selecting the smallest unmatched bid,

$$\mu(\mathbf{s})=\begin{cases}\min B(\mathbf{s}) & \text{if } \; B(\mathbf{s})\neq\varnothing\\ 0 & \text{if } \; B(\mathbf{s})=\varnothing\end{cases}. \tag{1}$$

If $\mu(\mathbf{s})>0$ then bid $b=\mu(\mathbf{s})$ is *the lowest unmatched bid* (winning bid). By construction, for any $\mathbf{s}$ such that $\mu(\mathbf{s})>0$, there exists a unique player $i\in I$ for whom $s_i(\mu(\mathbf{s}))=1$ holds. We will adopt the convention, $s_i(0)=0$. Note that a player can guarantee himself a payoff of zero by not participating in the game, i.e., by choosing $s_i=s^0$.

The set of mixed strategies on S is $\Delta(S):=\{\pi\in\mathbb{R}_+^{2^{\overline{b}}}|\sum_{t=0}^{2^{\overline{b}}-1}\pi^t=1\}$. For player i, we will denote by $\pi_i^t=\pi_i(s^t)$ the probability of strategy s^t in his mixed strategy π_i. We will write $\pi(\mathbf{s})$ for the probability of the pure strategy combination $\mathbf{s}\in S^N$

[7] In most LUPAs a participant can make bids sequentially. Since no information about other participants' bidding behavior is released during the bidding phase, however, one can treat sequentially placed bids of a player as a simultaneous choice of a set of bids.

according to the mixed strategy combination $\boldsymbol{\pi} = (\pi_1, .., \pi_N) \in \Delta(S)^N$. Similarly $\pi_{-i}(\mathbf{s}_{-i})$ denotes the probabilities of the pure strategies $\mathbf{s}_{-i} \in S^{N-1}$ according to the mixed strategies played by player i's rivals, $\pi_{-i} \in \Delta(S)^{N-1}$.

With this notation, we can define the probability $w_b(s_i, \pi_{-i})$ that b is the winning bid if player i plays some pure strategy s_i and all his rivals follow mixed strategy π:[8]

$$w_b(s_i, \pi_{-i}) = \sum_{\mathbf{s}_{-i} \in \{S^{N-1} : \mu(s_i, \mathbf{s}_{-i}) = b\}} \pi_{-i}(\mathbf{s}_{-i}). \tag{2}$$

Now one can represent the expected payoff function of player i as the weighted sum of the expected gains obtained from placing individual bids.

For a given profile $\pi = (\pi_i, \pi_{-i})$ of mixed strategies, the expected payoff of player i is

$$P_i(\pi) = \sum_{t=1}^{2^{\bar{b}}-1} \pi_i^t \sum_{b=1}^{\bar{b}} s^t(b) \left[(A-b)\, w_b\left(s^t, \pi_{-i}\right) - c\right]. \tag{3}$$

For the special case when player i chooses a pure strategy s_i, we obtain his expected payoff of

$$P_i(s_i, \pi_{-i}) = \sum_{b=1}^{\bar{b}} s_i(b)\left[(A-b)\, w_b(s_i, \pi_{-i}) - c\right]. \tag{4}$$

A *Nash equilibrium in mixed strategies* is a combination of mixed strategies $\boldsymbol{\pi}^* = \left(\pi_i^*, \pi_{-i}^*\right)$ such that

$$P_i\left(\pi_i^*, \pi_{-i}^*\right) \geq P_i\left(\pi_i, \pi_{-i}^*\right) \quad \text{for all } \pi_i \in \Delta(S) \quad \text{and all } i \in I.$$

Since the strategy set S is finite, existence of a Nash equilibrium in mixed strategies follows immediately from Nash's theorem (Nash, 1950).

It is not difficult to show[9] that, for sufficiently high prizes A and low bidding fees c, there are no Nash equilibria in pure strategies.

Proposition 1 *A LUPA with $N \geq 3$ and $c < \frac{A}{2} - 1$ has no Nash equilibrium in pure strategies.*

In most real life LUPAs, the number of players N and the value A are large, while the bidding cost c is low. Moreover, we can treat players symmetrically, since

[8] A more detailed discussion of $w_b(s_i, \pi_{-i})$ can be found in Appendix A.
[9] Eichberger and Vinogradov (2008).

there are no distinguishing features of players in our model. Consequently, we will focus on symmetric mixed strategy equilibria.

In order to simplify notation, we will omit individual players' indices and, in a slight abuse of notation, denote by π both the common mixed strategy of players $\pi := \left(\pi^0, \pi^1, ..\pi^t, ..\pi^{2^{\bar{b}}-1}\right)$ and the mixed strategy combination $\boldsymbol{\pi} := (\underbrace{\pi, .., \pi}_{N \text{ times}})$.

Note that, for a given symmetric mixed strategy π, the probabilities with which individual bids are placed, are determined by

$$q_b = \sum_{s \in S} \pi(s) \cdot s(b). \tag{5}$$

4 The structure of symmetric mixed strategy equilibrium

In this section we will characterize the symmetric mixed strategy equilibrium π. Since the strategy space is very large, one cannot hope for a complete characterization. We can, however, deduce certain features of the symmetric equilibrium mixed strategy which will allow us to draw conclusions on the observed stylized facts.

The basic idea is to construct an increasing sequence of special LUPAs which will include the actual LUPA. In this sequence we can show by induction that some structural features observed in simple cases carry over to the general case. We will study the special case of a LUPA where only bids up to the level k are allowed, i.e., with the additional constraint $b \leq k$. Since there is a finite upper bound $\bar{b}$ to the undominated bids in every LUPA, the sequence of LUPAs with $k = 1, 2, ...$ includes the actual LUPA under consideration.

Consider a LUPA with parameters A and c in which players can place only bids b up to k, for some $k \leq \bar{b}$. Denote by $\Gamma_k = \left(I, S_k, \{p_i\}_{i \in I}\right)$ a LUPA with a constrained strategy set

$$S_k = \{s \in S : s(b) = 0, \text{ for all } b > k\},$$

and payoffs

$$p_i(\mathbf{s}) := (A - \mu(\mathbf{s}))\, s_i(\mu(\mathbf{s})) - c \sum_{b=1}^{k} s_i(b).$$

For the induction argument, we need to establish the link between equilibria in Γ_k and Γ_{k+1}. Denote the symmetric equilibrium mixed strategy of any game Γ_k by $\pi_k := (\pi_k^0, \pi_k^1, ..., \pi_k^{\overline{k}})$ with $\overline{k} := 2^k - 1$.

First, we find equilibria in Γ_1 and Γ_2. Straightforward computation[10] yields the equilibrium for Γ_1.

Lemma 1 *Γ_1 has a unique symmetric equilibrium in mixed strategies:*

$$\pi_1^0 = \sqrt[N-1]{\tfrac{c}{A-1}}, \qquad \pi_1^1 = 1 - \sqrt[N-1]{\tfrac{c}{A-1}}.$$

Depending on parameters A, c and N, there are four possible equilibrium constellations for Γ_2. We will show that pure strategies s^0 and s^1 will always be played with positive probabilities, while strategies s^2 and/or s^3 may be played with probability zero. The following expressions help characterize the equilibrium. Let

$$K(N) = \frac{(N-1)^{N-1}}{N^{N-1} - (N-1)^{N-1}},$$

and denote by

$$\begin{aligned}
z_1(A,c,N) &:= \tfrac{2-K(N)}{1-K(N)} - A, \\
z_2(A,c,N) &:= \tfrac{(A-1)(A-2)}{2A-3} - c, \\
z_3(A,c,N) &:= 1 - \tfrac{c}{A-1} - \tfrac{c}{A-2} - (N-1)\left(\tfrac{c}{A-1}\right)^{\frac{N-2}{N-1}} + N\tfrac{c}{A-1}, \\
z_4(A,c,N) &:= c - \left(\sqrt[N-1]{\tfrac{1}{A-1}} + \tfrac{1}{N-1}\sqrt[N-1]{K(N)\cdot\tfrac{1}{A-2}}\right)^{1-N}.
\end{aligned}$$

[10] One only needs to note that bid 1 in strategy s^1 wins with probability $\left(\pi_1^0\right)^{N-1}$, and that the expected payoff is zero since s^0 is played with positive probability.

Lemma 2 Γ_2 *has a* unique symmetric equilibrium in mixed strategies $\pi_2 = (\pi_2^0, \pi_2^1, \pi_2^2, \pi_2^3)$*:*

$$(i) \quad \left\{ \begin{array}{lcl} \pi_2^0 & = & \sqrt[N-1]{K(N)\frac{c}{A-2}}, \\ \pi_2^1 & = & \frac{1}{N-1}\sqrt[N-1]{K(N)\frac{c}{A-2}}, \\ \pi_2^2 & = & \sqrt[N-1]{\frac{c}{A-1}} - \sqrt[N-1]{K(N)\frac{c}{A-2}}, \\ \pi_2^3 & = & 1 - \left(\sqrt[N-1]{\frac{c}{A-1}} + \frac{1}{N-1}\sqrt[N-1]{K(N)\frac{c}{A-2}}\right). \end{array} \right\} \quad \textit{iff} \quad \begin{array}{l} z_1(A,c,N) \leq 0 \textit{ and} \\ z_4(A,c,N) \leq 0. \end{array}$$

$$(ii) \quad \left\{ \begin{array}{lcl} \pi_2^0 & = & \sqrt[N-1]{\frac{c}{A-1}}, \\ \pi_2^1 & = & \sqrt[N-1]{\frac{c}{A-1} + \frac{c}{A-2}} - \sqrt[N-1]{\frac{c}{A-1}}, \\ \pi_2^2 & = & 0, \\ \pi_2^3 & = & 1 - \sqrt[N-1]{\frac{c}{A-1} + \frac{c}{A-2}}. \end{array} \right\} \quad \textit{iff} \quad \begin{array}{l} z_1(A,c,N) \geq 0 \textit{ and} \\ z_2(A,c,N) \geq 0. \end{array}$$

$$(iii) \quad \left\{ \begin{array}{lcl} \pi_2^0 & = & \widehat{\pi}^0(A,c,N), \\ \pi_2^1 & = & 1 - \sqrt[N-1]{\frac{c}{A-1}}, \\ \pi_2^2 & = & \sqrt[N-1]{\frac{c}{A-1}} - \widehat{\pi}^0(A,c,N), \\ \pi_2^3 & = & 0. \end{array} \right\} \quad \textit{iff} \quad \begin{array}{l} z_3(A,c,N) \geq 0 \textit{ and} \\ z_4(A,c,N) \geq 0. \end{array}$$

with $\widehat{\pi}^0(A,c,N)$ *implicitly defined by*

$$\left(\widehat{\pi}^0(A,c,N) + 1 - \sqrt[N-1]{\frac{c}{A-1}}\right)^{N-1} - (N-1)\left(1 - \sqrt[N-1]{\frac{c}{A-1}}\right)\left(\widehat{\pi}^0(A,c,N)\right)^{N-2} = \frac{c}{(A-2)}.$$

$$(iv) \quad \left\{ \begin{array}{lcl} \pi_2^0 & = & \sqrt[N-1]{\frac{c}{A-1}}, \\ \pi_2^1 & = & 1 - \sqrt[N-1]{\frac{c}{A-1}}, \\ \pi_2^2 & = & 0, \\ \pi_2^3 & = & 0. \end{array} \right\} \quad \textit{iff} \quad \begin{array}{l} z_2(A,c,N) \leq 0 \textit{ and} \\ z_3(A,c,N) \leq 0 \end{array}$$

For a given number of players N, it depends on the values of the prize A and the bidding fee c which of the four types of equilibria in Γ_2 will obtain. For a high bidding fee c and low values of A no player will choose strategies s_2^2 or s_2^3. In these cases, no player will place the bid $b = 2$ and the same mixed strategy will be played as in Γ_1. For high prizes A and low bidding costs c, however, all pure strategies will be played in equilibrium with positive probabilities. There are intermediate parameter constellations where either s_2^2 or s_2^3 will be played with probability zero. The parameter regions are depicted in Figure 5.

Lemmata 1 and 2 reveal a relationship between equilibria in Γ_1 and Γ_2:

Observation 1 The equilibrium probability of bid 1, $q_1 = 1 - \sqrt[N-1]{\frac{c}{A-1}}$ remains the same in both Γ_1 and Γ_2.

Observation 2 Strategies s^0 and s^1 belong to the support of the equilibrium mixed strategy both in Γ_1 and Γ_2.

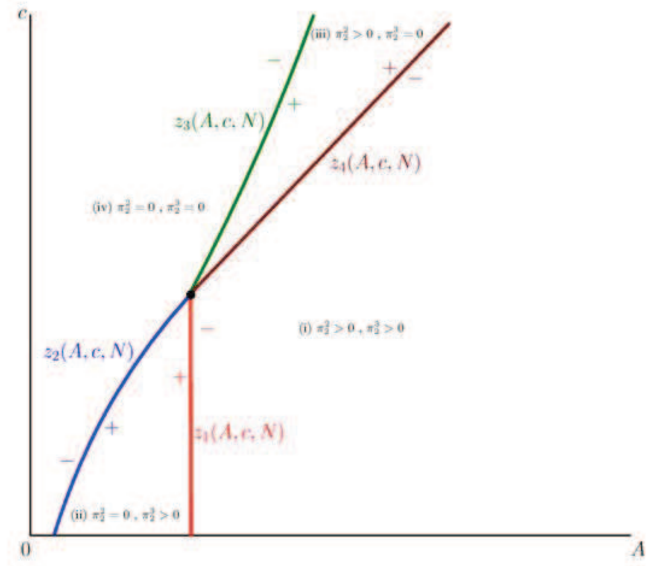

Figure 5. Equilibria in Γ_2

Observation 3 Equilibria π_1 from Γ_1 and π_2 from Γ_2 are related to each other by

$$\begin{aligned} &\bullet \quad \pi_1^0 = \pi_2^0 + \pi_2^2 \quad \text{and} \\ &\bullet \quad \pi_1^1 = \pi_2^1 + \pi_2^3. \end{aligned} \tag{6}$$

We will prove that these observations hold in general. Observation 1 suggests that equilibrium frequencies of bids q_b in game Γ_k do not change if one considers a "larger" game Γ_{k+1}. Generalizing Observation 2, the supports of equilibria in two "consecutive" games Γ_k and Γ_{k+1} have a subset of pure strategies in common. Observation 3 points to a relationship that links equilibria in two consecutive games.

We will first show by induction over the finite sequence $k = 1, ..., \overline{b}$, that the link between equilibria in Γ_k and Γ_{k+1} suggested by Observation 3 holds in general. For any strategy $s \in S_{k+1}$ and any l, $1 \leq l \leq k+1$, denote by $\sigma(s, l)$ the pure strategy with elements

$$\sigma(b|s, l) = \begin{cases} s(b) & \text{for} \quad b \neq l \\ 1 - s(b) & \text{for} \quad b = l \end{cases} \quad \text{for all } b = 1, ..., k+1. \tag{7}$$

Strategy $\sigma(s, l)$ differs from strategy s only in bid l. For example, for strategy s^2, one obtains $\sigma(s^2, 1) = s^3$ and $\sigma(s^2, 2) = s^0$. For an arbitrary $s \in S_k \subset S_{k+1}$, one has $\sigma(s, k+1) \in S_{k+1} \setminus S_k$, because bid $k+1$ is not placed in s but placed according to strategy $\sigma(s, k+1)$.

Given an equilibrium strategy π_{k+1}, define the mixed strategy $\widetilde{\pi}_k(\pi_{k+1})$ with support in S_k by the following probabilities $\widetilde{\pi}_k(s|\pi_{k+1})$ of pure strategies $s \in S_{k+1}$:

$$\widetilde{\pi}_k(s|\pi_{k+1}) := \begin{cases} \pi_{k+1}(s) + \pi_{k+1}(\sigma(s, k+1)) & \text{if} \quad s \in S_k, \\ 0 & \text{if} \quad s \in S_{k+1} \setminus S_k. \end{cases} \tag{8}$$

Equation (8) generalizes the link in Observation 3: for $s = s^0$ and $k = 1$ one has

$$\widetilde{\pi}_1(s^0) = \pi_{k+1}(s) + \pi_{k+1}(\sigma(s, k+1)) = \pi_2(s^0) + \pi_2(s^2).$$

Similarly, for $s = s^1$ one deduces $\widetilde{\pi}_1(s^1) = \pi_2(s^1) + \pi_2(s^3)$, and for s^2 and s^3 one obtains $\widetilde{\pi}_1(s^2) = \widetilde{\pi}_1(s^3) = 0$. Hence $\widetilde{\pi}_1$ is a mixed strategy in Γ_1. We only need to show that it is also an equilibrium mixed strategy. The following proposition shows that $\widetilde{\pi}_k(s|\pi_{k+1})$ maps the equilibrium mixed strategy π_{k+1} to an equilibrium mixed strategy in Γ_k.

Proposition 2 *If π_{k+1} is an equilibrium mixed strategy of Γ_{k+1}, then $\widetilde{\pi}_k$ is an equilibrium mixed strategy of Γ_k.*

Proposition 2 generalizes Observation 3. It immediately follows that any bid $b \leq k$ which is placed in Γ_{k+1} with probability q_b will be placed with the same probability in an equilibrium of Γ_k:

$$\begin{aligned} q_b(\pi_{k+1}) &= \sum_{s \in S_{k+1}} \pi_{k+1}(s) \cdot s(b) = \sum_{s \in S_k} \pi_{k+1}(s) \cdot s(b) + \sum_{s \in S_{k+1} \setminus S_k} \pi_{k+1}(s) \cdot s(b) \\ &= \sum_{s \in S_k} \pi_{k+1}(s) \cdot s(b) + \sum_{s \in S_k} \pi_{k+1}(\sigma(s, k+1)) \cdot \underbrace{\sigma(b|s, k+1)}_{=s(b)} \\ &= \sum_{s \in S_k} (\pi_{k+1}(s) + \pi_{k+1}(\sigma(s, k+1))) \cdot s(b) = q_b(\widetilde{\pi}_k). \end{aligned}$$

This generalizes Observation 1. It follows that if the equilibrium in Γ_k is unique, and hence probabilities q_b of bids $b \leq k$ are uniquely determined, they are also uniquely determined and have the same values in any equilibrium in Γ_{k+1} (and, by induction, in any equilibrium Γ_{k+n}, $1 \leq n \leq b-k$).

Proposition 2 also generalizes Observation 2. First, the support of the equilibrium in Γ_{k+1} contains strategies from S_k. i.e. $\operatorname{supp} \pi_{k+1} \cap S_k \neq \varnothing$.[11] Second,

[11] This is because if $\pi_{k+1}(s) = 0$ for any $s \in S_k$, then all players place bid $k+1$ with certainty, hence bid

$\pi_{k+1}(s) > 0$ for some $s \in S_k$ in equilibrium in Γ_{k+1} implies $\widetilde{\pi}_k(s|\pi_{k+1}) > 0$, hence the strategy s is also in the support of the equilibrium mixed strategy of Γ_k.

In combination with Lemmata 1 and 2, Proposition 2 delivers an important result about the bidding pattern. In equilibrium, bidders place on average more than one bid (this rules out equilibria in singletons) and do not necessarily bid in blocks where a higher bid is placed jointly with all lower bids, i.e. where $s(b) = 1$ implies $s(b-1) = 1$.

Proposition 3 *Let π be a symmetric equilibrium in mixed strategies in $\Gamma = \Gamma_{\overline{b}}$. If $c < \min\left[\frac{A}{2} - 1, (A-1)\frac{K(N)}{1+K(N)}\right]$ then*

1. *There exists $s \in \operatorname{supp}\pi$ such that $\sum_{b=1}^{\overline{b}} s(b) > 1$ and*

2. *There exists $s' \in \operatorname{supp}\pi$ such that $s'(b) = 1$ but $s'(b-1) = 0$ for some $b \leq \overline{b}$.*

For $A \geq 10$ condition $c < \frac{A}{2} - 1$ is always satisfied[12]. The second requirement $c < (A-1)\frac{K(N)}{1+K(N)}$ is also met in most real life LUPAs[13].

5 Bidding behavior and probability of winning

The symmetric equilibrium mixed strategy studied in the previous section depends on the number of players N, the *potential* bidders, which is an unobservable variable. In the data one observes only the number of actual bidders, which is determined endogenously in a mixed strategy equilibrium. For comparing theoretical predictions with actual data, it is useful to derive results that do not depend on the number of players. In this section, we will show that the win probability of a bid b introduced in Equation (2) has this property. Moreover, bids' winning probabilities determine an upper bound on bids that does not depend on the number of players either. Finally, we will also provide an estimate of the probabilities with which individual bids

$k+1$ wins with probability zero and all players are better off by dropping bid $k+1$, a contradiction to π_{k+1} being an equilibrium.

[12] Condition $c < (A-1)\frac{K(N)}{1+K(N)}$ implies $c < \frac{A}{2} - 1$ for $A \geq 10$, see proof of Proposition 3 in Appendix B.

[13] Note that $\frac{K(N)}{1+K(N)} = \frac{(N-1)^{N-1}}{N^{N-1}} > \frac{1}{e}$ with $e = 2.71$. Hence, it suffices to require $c < \frac{A-1}{e}$ to satisfy $c < (A-1)\frac{K(N)}{1+K(N)}$, a condition met in all LUPAs considered in this paper.

are placed in equilibrium, dependent on N.

Proposition 4 *If π is an equilibrium mixed strategy, then for any b*

$$w_b(s,\pi) = \tfrac{c}{A-b}, \quad \textit{for all} \quad s \in \operatorname{supp}\pi \ \textit{such that}\ s(b) = 1. \tag{9}$$

The proof of the proposition uses the fact that the expected marginal contributions of bids, $(A-b)\,w_b - c$, must be equal to zero in equilibrium, since players have the option not to bid at all, which yields a payoff of zero. In Γ_1 the strategy of not bidding, s^0, is in the support of the equilibrium mixed strategy and yields an expected payoff of zero, which for bid 1 translates into $(A-1)\,w_1(s^1,\pi) - c = 0$. In Γ_2 one considers marginal contribution of bid 2 depending on whether s^2 or s^3 or both are played in equilibrium. In all cases, if $(A-2)\,w_2 - c$ was strictly positive, players would be strictly better off by not playing s^1 in equilibrium; if it was strictly negative, they would not play s^2 or/and s^3. Both are contradictions and imply $(A-2)\,w_2 - c = 0$. By induction, this argument can be shown to hold in general.

We have shown before that bids over $A-c$ are strictly dominated by not bidding. In fact, the upper bound on bids in a symmetric mixed strategy equilibrium is lower since the cumulative probability of winning with a bid below b cannot exceed unity. It is easy to prove that bids above $\widehat{b} := \inf\left\{b' : \sum_{b=1}^{b'} w_b \geq 1\right\}$ are never placed in equilibrium because some bid less or equal to $\widehat{b}$ wins with certainty. Effectively, $\widehat{b}$ is the highest bid that can optimally be placed in a LUPA. By definition it does not depend on N, since it is given by the winning probabilities of bids, independent of N by proposition 4. Proposition 5 derives a convenient approximation of the value $\widehat{b}$.

Proposition 5 *For A large enough, the upper bound on bids $\widehat{b}$ is given by*

$$\widehat{b} \approx (A-1)\cdot\left(1 - e^{-\frac{1}{c}}\right).$$

By choosing the optimal mixed strategy π, players decide on probabilities with which individual bids are placed, $q_b = \sum\limits_{s\in S} \pi(s)\cdot s(b)$. If all players follow the equilibrium mixed strategy π, in a realization of an equilibrium with a large number of players we would expect strategy s, and correspondingly the bids that s contains, to occur with the frequency reflecting the equilibrium probability $\pi(s)$.

Proposition 6 *If π is a symmetric equilibrium then for any bid $b \leq \widehat{b}$ its probability q_b satisfies*

$$1 - \sqrt[N-1]{\frac{c}{A-b}} \geq q_b \geq 1 - \sqrt[N-1]{\sum_{k=1}^{b} \frac{c}{A-k}}. \tag{10}$$

For $b \leq \widehat{b}$, $\sum_{k=1}^{b} \frac{c}{A-k} \leq 1$ holds for the right-hand side. For $b = 1$ the left-hand side and the right-hand side in (10) coincide, yielding $q_1 = 1 - \sqrt[N-1]{\frac{c}{A-1}}$. For larger bids the left inequality is strict, reflecting the observation that the probability of winning with bid b is strictly smaller than the probability that bid b is unmatched.

6 Empirical observations

In this section we extend Section 2 by providing more empirical observations from LUPAs. The main purpose of this section is to illustrate main variables observable in LUPAs and to enable a better understanding of players' behavior in this sort of games. We do not consider these examples to be comprehensive formal tests of our theoretical predictions, although some comparisons of the data against the theory prove useful and informative. The data from eight German LUPAs were provided by *Legion Telekommunikation GmbH*. Bidders can place bids either by making a phone call or by sending a text message. Bidders are identified with their phone numbers. Theoretically, it is possible that one bidder uses several phones to bid, or several bidders form a coalition and bid from different telephone numbers. In the first case, the data would identify one player as two distinct bidders. In the second case the data would record one coalition as several distinct bidders. Bosch-Domènech et al. (2002) also note that in experiments through mass media coalition forming is possible. In some cases, it is possible to identify or to suspect coalitions in a LUPA[14] but they are rare.

[14] For example, two different players are suspected to build a coalition, if bidding from two different phone numbers represents two complementing parts of one "block" strategy. Say, if bidder A places all bids from 1 to 1000, and bidder B places all bids from 1001 to 2000, we might suspect the two are in a coalition. The winner of one LUPA admitted in an interview after the game that she played in a coalition with a friend.

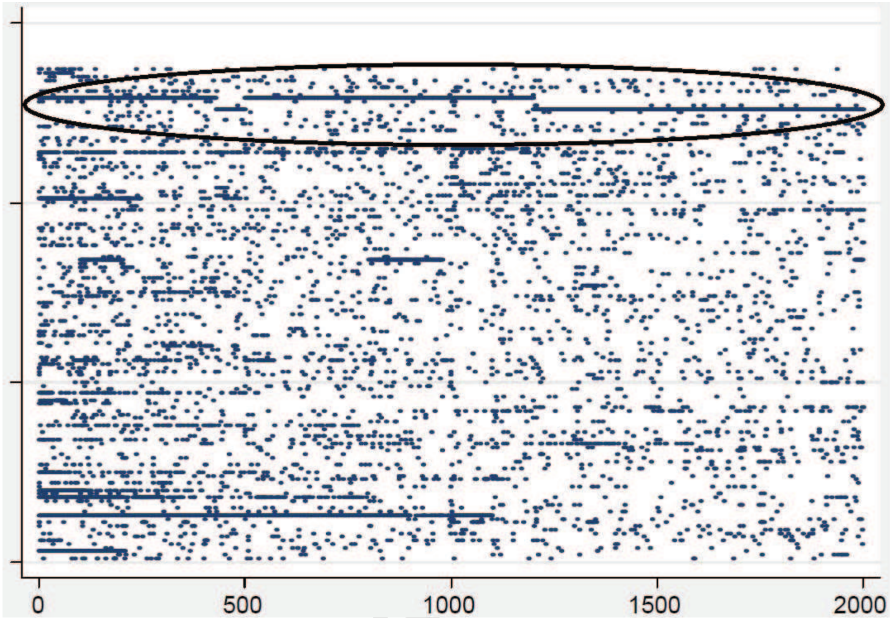

Figure 6. Realized pure strategies (Auction RB, 10000€).

6.1 Observed strategies

In this section we illustrate the actual bidding strategies in a diagram. To represent strategies which were actually played we identify bidders i with their unique identification number on a vertical axis and bids b with natural numbers on the horizontal axis. Whenever bidder i placed bid b we mark the corresponding combination (b, i) with a dot. Each horizontal sequence of dots thus represents pure strategy s_i actually played by player i. To improve the readability of the figures, we only show strategies that contain 20 bids and more. For the same reason, we only show bids from 1 cent to 20€. Solid lines in the strategy space correspond to "sequential" strategies, in which a bidder covers a whole range of bids. We present here a selection of typical strategy plots, to illustrate a typical bidding pattern. As in Proposition 3, bidders do not limit themselves to singleton or sequential strategies, although some sequences are clearly identifiable in the figures. Figure 6 shows examples of sequential bidding (solid lines) as well as an example of a coalition (complementing segments of solid lines in the upper part, marked with an oval).[15]

[15] We actually know from the auction organizer that the winner was a couple who used two mobile phones to bid by text messages.

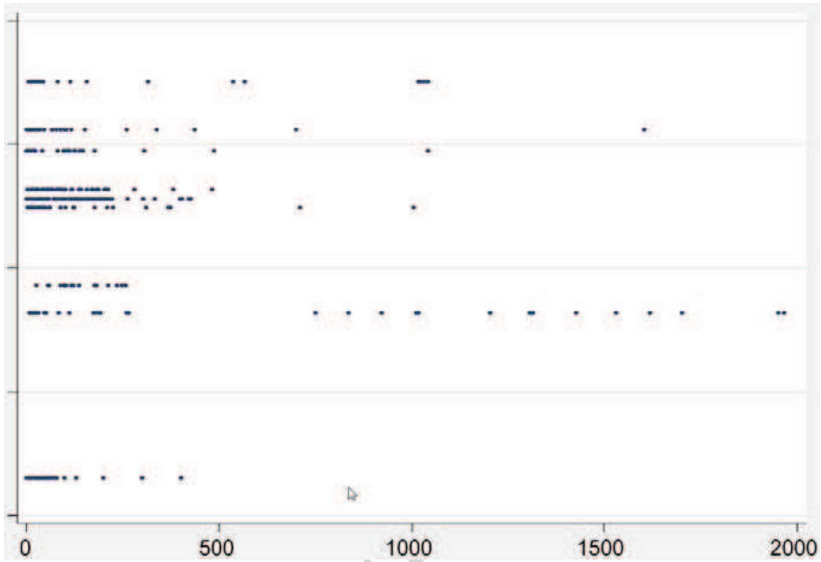

Figure 7. Realized pure strategies (Auction AD1, 1000€).

Figures 7, 8 and 9 show strategies played in the LUPAs run by the same broadcaster in a sequence, one after another. Two observations stand out from these LUPAs: (1) The number of bidders who place more than 20 bids is low in Figure 7, but substantially higher in the two other games, and (2) the number of "block" strategies played in Figure 7 is lower than in the two other games. This may indicate a learning effect: players play more sophisticated strategies once they are more familiar with the game. Yet, at the same time, the effect of an increase in the value of the prize from AD1 to AD3 is not ruled out.

6.2 Winning bids

From a large enough number of identical auctions with the same prize and the same bidding fee, one can compute the probability that a bid wins by observing the number of times this bid wins relative to the number of times this bid has been placed,

$$\frac{n\,(b \text{ wins})}{n\,(b \text{ placed})} = \widetilde{w}_b.$$

Note that we only *observe* that bid b wins if it is placed *and* it is the winning bid. We expect $\widetilde{w}_b = w_b = \frac{c}{A-b}$.

The AuctionAir data offer an opportunity to estimate $\widetilde{w}_b$, although the number of long enough series of perfectly identical auctions is not large. We use data from

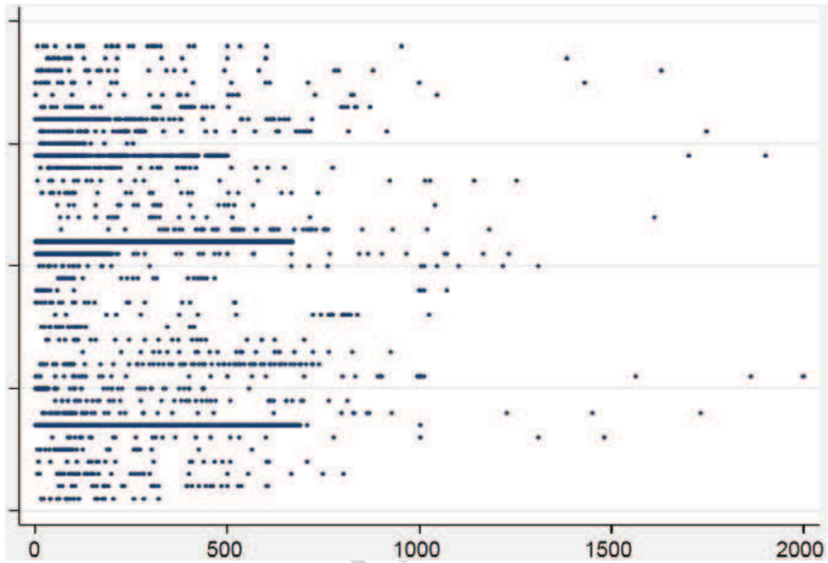

Figure 8. Realized pure strategies (Auction AD2, 3000€).

Figure 9. Realized pure strategies (Auction AD3, 5000€).

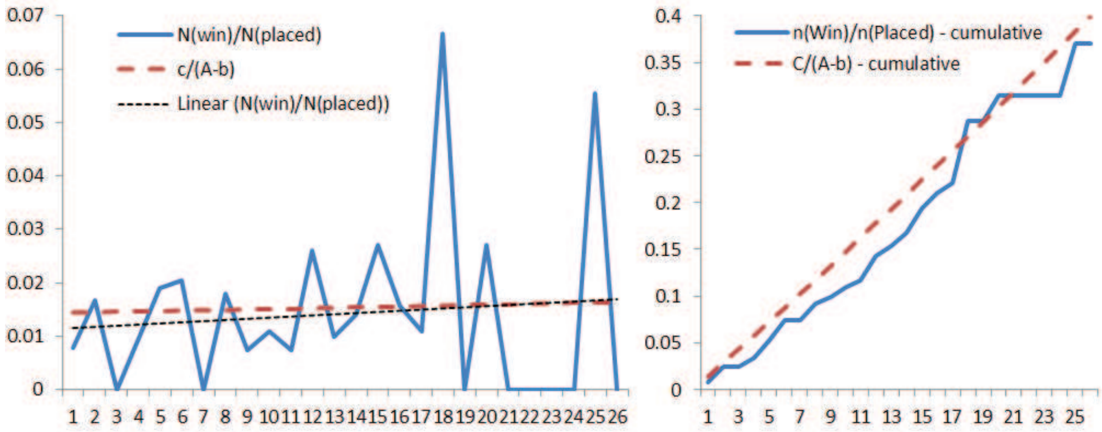

Figure 10. Empirically derived win probabilities for bids 1-25 in 26 identical LUPAs, with a trend line, compared with the theoretical prediction (left panel) and their respective cumulative values (right panel).

26 identical auctions for 20Gb iPods shown in Figure 4, to construct $\widetilde{w}_b$ as described above, see Figure 10. The theoretically predicted win probability matches the one empirically observed relatively well for this number of observations. Note that both the theoretical and the estimated win probabilities imply that the overall chance of winning with some bid between 1 and 25 in this auction is 37-38%, leaving 62-63% chance to win with one of the higher bids. Proposition 5 suggests that it might be optimal for bidders in this auction to go up to bid 73, however in our sample of 26 auctions (and hence 26 winners) these higher bids did not win, although many of them were actually placed, see Figure 4 (some bidders went as high as to place bids 100, 101 and 125, yet 99% of them did not bid above 55). This underlines the difficulty with the empirical estimation of the win probability: since it is only one bid per auction that actually wins, to actually observe some of bids up to 73 winning, one would require at least hundreds of identical auctions, which is hardly feasible in reality.

6.3 Highest bids

Based on equilibrium values of win probabilities w_b, Proposition 5 predicts the upper bound (UB) on bids as a fraction $\lambda = \frac{\bar{b}}{A-1}$ of the prize: $\lambda = 1 - e^{-\frac{1}{c}}$. Although this value may be fairly small (it is about 2% of the prize value in German LUPAs considered in this paper[16]), it captures over 90% of bids placed (only one auction falls below the 90% threshold, with 88.93% bids placed below UB; in five auctions over 95% bids are below UB), see Table 3. The table lists absolute upper bounds (UB) for auctions with different prizes, the percentage of bids above UB, as well as the distribution of bids above the upper bound in percentiles of the space between the upper bound and the monetary value of the prize.[17]

ID	Prize, €	*N.bids*	UB, €	above UB	0-10%	10-50%	50-90%	90-99%	99-100%
RB	10 000	47 872	202.01	9.04%	6.22%	2.48%	0.22%	0.03%	0.08%
AMV	10 000[a]	52 847	202.01	4.16%	4.16%	0.00%	0.00%	0.00%	0.00%
AD1	1 000	1 798	20.20	11.07%	7.56%	2.39%	0.94%	0.06%	0.11%
AD2	3 000	6 732	60.60	2.73%	2.38%	0.29%	0.04%	0.01%	0.00%
AD3	5 000	6 201	101.01	0.97%	0.71%	0.26%	0.00%	0.00%	0.00%
LSR	1 099	1 272	22.20	6.13%	5.35%	0.79%	0.00%	0.00%	0.00%
BB	20 000[b]	266 824	404.02	2.22%	1.97%	0.20%	0.02%	0.00%	0.00%
RTL	350 000	610 104	7 070.46	1.63%	1.38%	0.21%	0.03%	0.00%	0.00%

Table 3. Theoretical upper bound (UB) on bids in German LUPAs, and the actual bidding above the UB.

Importantly, the upper bound depends on the bidding fee as measured in bidding "units" (bidding increments). The reason for this is that we represent bids, fees and the prize value on the grid of natural numbers, without gaps. This makes a bid of 1 cent in a LUPA where bids in fractions of the basic monetary unit are allowed, numerically equivalent to a bid of £1 in a LUPA where only bids in basic units are accepted. As a result, a fee of £3 yields an upper bound of $\lambda(3) = 28.35\%$, whilst a lower (in nominal terms) fee of €0.49 yields a lower upper bound of $\lambda(49) = 2.02\%$. This explains why bidders in the LUPAs from Table 3 bid relatively low (bids in cents allowed), whilst bidders in AuctionAir auctions (only bids in whole pounds accepted) place bids relatively high compared to the value of the prize. Within the same bidding system, an increase in the fee would be associated with a decrease in the upper bound.

[16] For comparison, upper bounds reported by Costa-Gomes and Shimoji (2014) are in the range 20-90% of the value of the prize, depending on the prize value and the number of participants, seemingly increasing in both (Table 1 in Costa-Gomes and Shimoji, 2014). Yet the authors do not report their estimates for the upper bound for prizes and numbers of participants above 1000, which is the case in the auctions reported here.

[17] For the AMV auction, the description provided by the organizers indicates the prize of €10 000, yet the highest bid placed is €1000. As in all other aspects this auction resembles RB, which has the same prize, it is likely that bids above €1000 were not allowed in AMV. For the BB auction, the auction description estimates the value of the prize at €20 000, however the data show 44 participants with bids above 20 000, 38 of which have placed bid €49 999,99 and 6 placed €50 000.

6.4 Observed frequencies of bids

Our analysis suggests an upper bound

$$\overline{q}_b = 1 - \sqrt[N-1]{\frac{c}{A-1}} \tag{11}$$

and a lower bound

$$\underline{q}_b = 1 - \sqrt[N-1]{\sum_{k=1}^{b} \frac{c}{A-k}}. \tag{12}$$

for the probability q_b with which bids $b \leq \widehat{b}$ are placed in equilibrium, see (10).

Östling et al (2011) derived a closed form solution for LUPAs in which players are only allowed to play single bids. Their solution is reproduced here in our notation for the sake of comparability[18]:

$$q_b = q_{b-1} - \frac{1}{N-1} \ln \left(1 - (N-1)\, q_{b-1} e^{-(N-1)q_{b-1}}\right). \tag{13}$$

The above equation defines the probability of placing bid b iteratively and needs an initial condition. Proposition 6 provides $q_1 = 1 - \sqrt[N-1]{\frac{c}{A-1}}$ as the solution for $b = 1$ and, thus, serves as an exact initial condition. Solution (13) incorporates the bidding fee through the initial condition. As the actual number of players N is unknown and we only observe the number of active bidders $N_{bid} = N\left(1 - \pi^0\right) < N)$ we substitute the number of bidders for the number of players.[19]

In Fig. 11-14 we plot the upper limit (Equation 11) as a dotted line, the lower limit (Equation 12) as a solid line, and the (exact) Östling et al (2011) solution (Equation 13) as a dashed line against the actual frequencies of bids for four selected games. These figures show the frequencies of bids only up to the bid level of 20€ (2000 cent). For smaller prizes A, Figures 11 and 12, this covers almost the full range of bids observed, while for large prizes A, Figures 13 and 14, the diagrams show only the lower range of bids observed. Hence, the levelling off of frequencies as bids get higher

[18] To do this we also use the Environmental Equivalence they apply in their Poisson game: any player sees the total number of players in the game equal to the number of his rivals.

[19] Some broadcasters report total estimated audience, ranging from 94 000 to 281 000. For the newspaper (LSR) the number of printed copies was 94 000 and the estimated number of readers reported at 280 000. Using these numbers as the number of players, and the actual number of bidders we are able to estimate π^0 for some of the auctions (96.65% for RB, 88.67% for AMV, over 99% for the newspaper). Using these numbers for the number of players resulted only in scaling effects on the range of bids presented in figures. We also tried a range of possible values of N up to 1000 times the number of bidders for auctions with unknown audience, with the same effect.

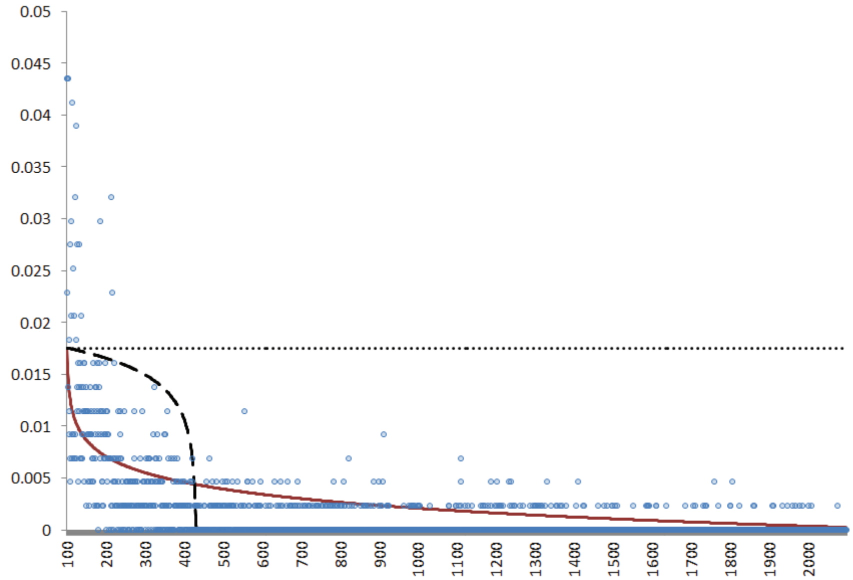

Figure 11. Frequencies of bids (Auction LRS, 1099€: $A = 109900$, bidding fee $c = 49$, number of players substituted with $N_{bid} = 437$).

is less pronounced in these figures.

Figures 11 and 12 reveal that the observed decrease in bidding frequencies as bids get higher is approximated quite well by the convex shape of the lower bounds. The decreasing rate of the fall in frequencies is also visible in the data of LUPAs with high prizes in Figures 13 and 14, though less pronounced because the range of bids displayed represents only a small fraction of the full range of bids observed.

The Östling et al (2011) solution predicts an accelerating decrease in the frequency of bids, resulting in reaching the upper bound on bids much faster than observed in the data. This pattern in bidding frequencies appears to be a consequence of the restriction to place only single bids. Placing a lower bid jointly with a higher one is a "defensive" strategy as it prevents others from winning with the lower bid. Without this "defensive" option, players place lower bids more frequently than they would have done in the unconstrained case, and do not go high in their bidding. With multiple bids, players can place higher bids by combining them with defensive lower ones. This allows players to place relatively high bids, yet this comes at the cost of reducing the winning probability of lower bids. Due to this, players play lower bids

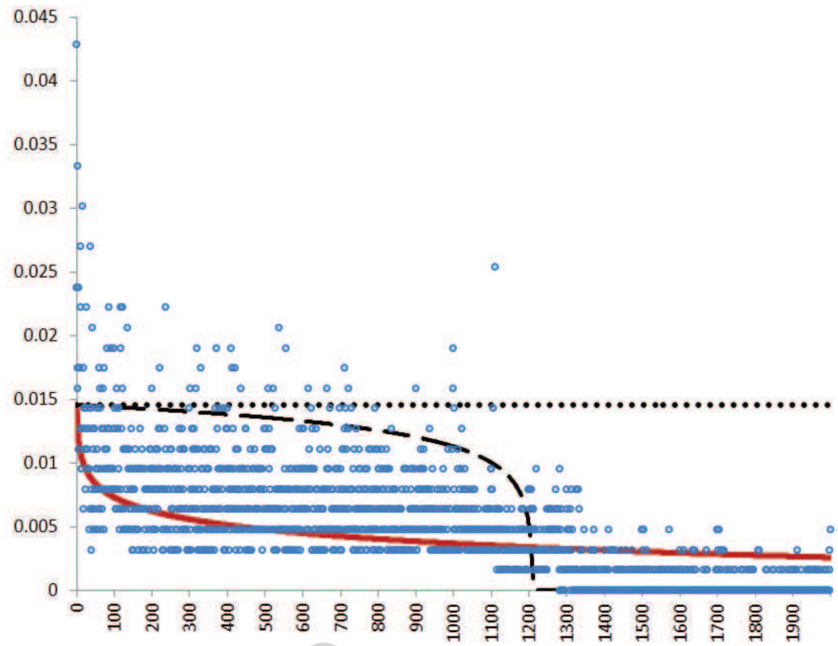

Figure 12. Frequencies of bids (Auction AD3, 5000€: $A = 500000$, $c = 49$, number of players substituted with $N_{bid} = 631$).

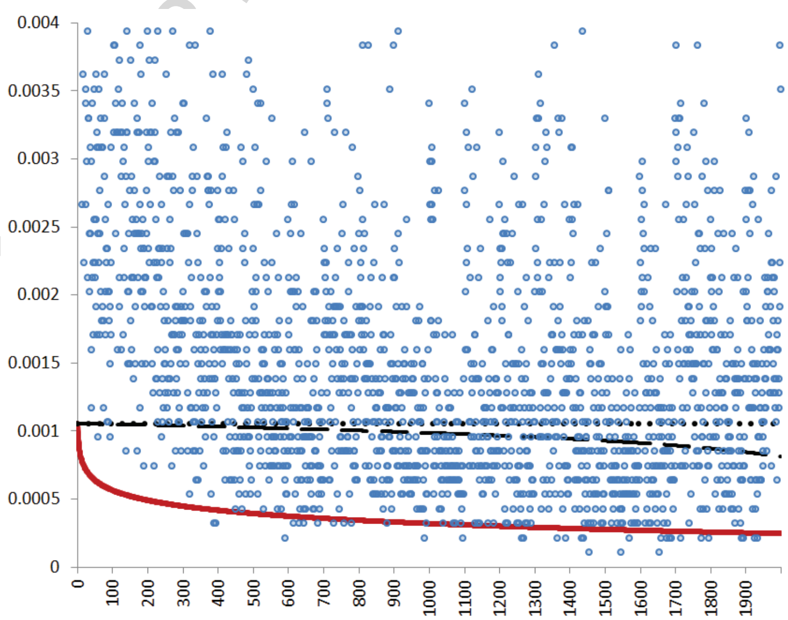

Figure 13. Frequencies of bids (Auction RB, 10 000€: $A = 1000000$, $c = 49$, number of players substituted with $N_{bid} = 9400$).

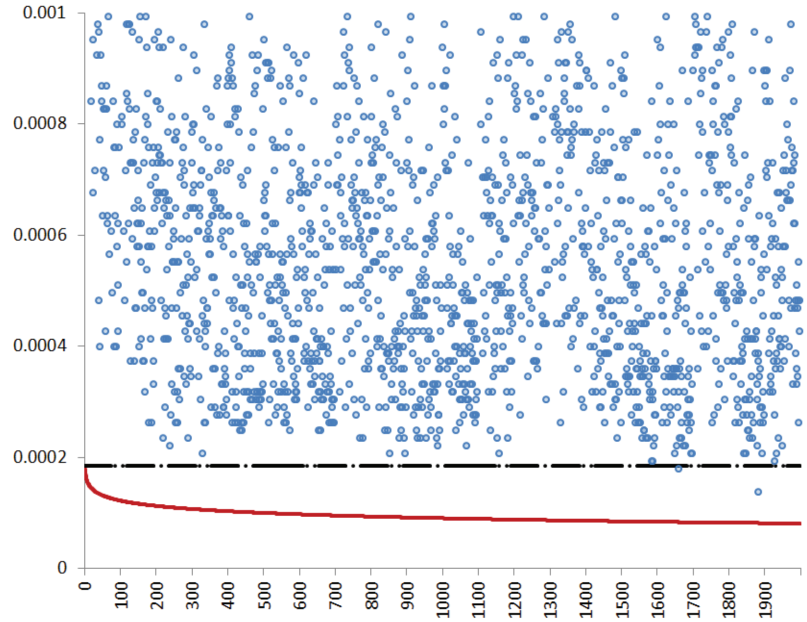

Figure 14. Frequencies of bids (Auction RTL, 350 000€: $A = 35000000$, $c = 49$, number of players substituted with $N_{bid} = 72588$).

less frequently than in the singleton bidding.

The theoretically observed upper bound of bidding frequencies $\overline{q}_b$, on the other hand, seems to be too low, in particular for the LUPAs with high prizes in Figures 13 and 14. This upward bias in frequencies at low levels of bids deserves further study.

7 Conclusion

Our paper begins with three stylized facts about Lowest-Unmatched Price Auctions (LUPAs): (1) bidders place on average more than one bid, (2) lower bids are placed more frequently than higher ones, and (3) bidders tend to place rather high bids. The theoretical model proposed in the paper views LUPAs as strategic games where bidders can place several bids. In contrast to existing approaches which restrict strategies to singleton bids or block bidding, the more general bidding strategies allowed for in this paper suggests a rational explanation for these stylized facts.

On the whole our model captures the main features of bidders' behavior in actual LUPAs quite well. Comparing our data with the model reveals however some other interesting features. There appears to be overbidding both in the sense of too many players placing the same bid and of some players placing bids which are much higher than predicted by the model. Incorporating these effects in the model and further testing them, in particular experimentally, are promising directions of further research.

Appendix A. Probability of winning and expected payoff: auxiliary results

In Section 3 the expected payoff (3) was defined with the help of win probabilities (2). Here we will show that this definition is equivalent to the standard definition of expected payoffs in games in strategic form. We will also derive properties of the probability of winning that are later used in the proofs of the main propositions.

Consider player i. For strategy combination $\mathbf{s} \in S^N$, player i's payoff is

$$p_i(\mathbf{s}) := (A - \mu(\mathbf{s}))\, s_i(\mu(\mathbf{s})) - c \sum_{k=1}^{\bar{b}} s_i(k). \tag{A-1}$$

If player i plays $s_i = s^0$ then $p_i(s^0, \mathbf{s}_{-i}) = 0$ for any strategy combination of his rivals $\mathbf{s}_{-i}$. This notation assumes the usual convention of splitting the strategy combination $\mathbf{s}$ into strategy $s_i \in S_i$ of player i, and the opponents' strategy combination, $\mathbf{s}_{-i} \in S^{N-1}$. Denote $\boldsymbol{\pi}(\mathbf{s}) := \prod_{i=1}^{N} \pi_i(s_i)$ the probability of pure strategy combination $\mathbf{s} = (s_1, ..., s_N) \in S^N$ according to the mixed strategy combination $\boldsymbol{\pi} := (\pi_1, .., \pi_N) \in \Delta(S)^N$. The probability of the combination of pure strategies played by player i's rivals, $\pi_{-i}(\mathbf{s}_{-i})$, is defined similarly. The expected payoff of player i is

$$P_i(\boldsymbol{\pi}) = \sum_{\mathbf{s} \in S^N} \boldsymbol{\pi}(\mathbf{s})\, p_i(\mathbf{s}) = \sum_{s_i \in S} \pi_i(s_i) \left(\sum_{\mathbf{s}_{-i} \in S^{N-1}} \pi_{-i}(\mathbf{s}_{-i})\, p_i(s_i, \mathbf{s}_{-i}) \right). \tag{A-2}$$

On the set of all strategy combinations define the indicator function $1_b : S^N \to \{0; 1\}$ which returns unity if bid b is unmatched in a particular strategy combination $\mathbf{s} = (s_i, \mathbf{s}_{-i})$:

$$1_b(s_i, \mathbf{s}_{-i}) = \begin{cases} 1 & \text{if} \quad b \in B(\mathbf{s}), \\ 0 & \text{otherwise.} \end{cases} \tag{A-3}$$

We can characterize whether bid b is the lowest unmatched bid in a strategy combination $\mathbf{s} = (s_i, \mathbf{s}_{-i})$ as follows:

$$\prod_{l=1}^{b-1} (1 - 1_l (s_i, \mathbf{s}_{-i})) \cdot 1_b (s_i, \mathbf{s}_{-i}) = 1 \Leftrightarrow b = \mu (s_i, \mathbf{s}_{-i}) . \tag{A-4}$$

To see this it suffices to notice that the set $B (s_i, \mathbf{s}_{-i})$ contains bid b if and only if $1_b (s_i, \mathbf{s}_{-i}) = 1$, and it contains no lower bids if and only if $1_l (s_i, \mathbf{s}_{-i}) = 0$ for all $l < b$. Given a pure strategy s_i of player i and a mixed strategy π played by his rivals, the probability that bid b is *the lowest unmatched* bid**,** defined in (2), can be equivalently written as:

$$w_b (s_i, \pi) := \sum_{\mathbf{s}_{-i} \in S^{N-1}} \pi_{-i} (\mathbf{s}_{-i}) \cdot \prod_{l=1}^{b-1} (1 - 1_l (s_i, \mathbf{s}_{-i})) \cdot 1_b (s_i, \mathbf{s}_{-i}) . \tag{A-5}$$

The following lemma establishes the equivalence between the definitions of expected payoff functions (3) and (A-2).

Lemma A.1

$$\sum_{\mathbf{s} \in S^N} \boldsymbol{\pi} (\mathbf{s}) p_i (\mathbf{s}) = \sum_{s_i \in S} \pi_i (s_i) \sum_{b=1}^{\bar{b}} s_i (b) [(A - b) w_b (s_i, \pi_{-i}) - c] . \tag{A-6}$$

Proof. From (A-1) and (A-2) obtain

$$P_i (\pi_i, \pi_{-i}) = \sum_{s_i \in S} \pi_i (s_i) \sum_{s_{-i} \in S^{N-1}} \pi_{-i} (s_{-i}) \left[(A - \mu (s_i, s_{-i})) s_i (\mu (s_i, s_{-i})) - c \sum_{b=1}^{\mathbf{b}} s_i (b) \right] \tag{A-7}$$

Player i wins if only if $s_i (\mu (s_i, s_{-i})) = 1$. By using the indicator function $\delta_b (s_i, s_{-i}) = \begin{cases} 1 & \text{if} \quad b = \mu (s_i, s_{-i}) \\ 0 & \text{if} \quad b \neq \mu (s_i, s_{-i}) \end{cases}$, we can write: $(A - \mu (s_i, s_{-i})) s_i (\mu (s_i, s_{-i})) \equiv \sum_{b=1}^{\mathbf{b}} (A - b) \cdot s_i (b) \cdot \delta_b (s_i, s_{-i})$. Rearrange terms in (A-7):

$$P_i (\pi_i, \pi_{-i}) = \sum_{s_i \in S} \pi_i (s_i) \sum_{b=1}^{\mathbf{b}} s_i (b) \left[(A - b) \sum_{s_{-i} \in S^{N-1}} \pi_{-i} (s_{-i}) \cdot \delta_b (s_i, s_{-i}) - c \sum_{s_{-i} \in S^{N-1}} \pi_{-i} (s_{-i}) \right] . \tag{A-8}$$

The last summand in the square brackets turns to c because $\sum_{s_{-i} \in S^{N-1}} \pi_{-i} (s_{-i}) = 1$.

Recall from (A-4) that $b = \mu(s_i, s_{-i}) \Leftrightarrow \prod_{l=1}^{b-1} (1 - 1_l(s_i, s_{-i})) \cdot 1_b(s_i, s_{-i}) = 1$.and thus

$$\sum_{s_{-i} \in S^{N-1}} \pi_{-i}(s_{-i}) \cdot \delta_b(s_i, s_{-i}) = \sum_{s_{-i} \in S^{N-1}} \pi_{-i}(s_{-i}) \cdot \prod_{l=1}^{b-1} (1 - 1_l(s_i, s_{-i})) \cdot 1_b(s_i, s_{-i}) = w_b(s_i, \pi).$$

Substituting this to (A-8) delivers the result. ■

The following two properties of the win probability are useful for the proofs of main results. First, win probability of bid b is invariant to any changes in the bidder's strategic choice with regards to bids above b, as long as the choice with regards to bids below b is unchanged. Intuitively this is because these changes do not affect the likelihood that any of the lower bids wins. Second, if players are not allowed to play any bid above b and adjust their mixed strategies by merging probabilities of strategies that only differed in that one forbidden bid (and therefore are identical now that this bid is forbidden), win probability of bid b does not change. Both properties are formalized in the following lemmata.

Lemma A.2 (Invariance I)

$$w_b(s_i, \pi) = w_b(s_i', \pi) \ \forall s_i, s_i' \in S : s_i(l) = s_i'(l), \forall l \leq b. \tag{A-9}$$

Proof. To prove the lemma we will show that $\prod_{l=1}^{b-1} (1 - 1_l(s_i, \mathbf{s}_{-i})) \cdot 1_b(s_i, \mathbf{s}_{-i}) = \prod_{l=1}^{b-1} (1 - 1_l(s_i', \mathbf{s}_{-i})) \cdot 1_b(s_i', \mathbf{s}_{-i})$ for any strategy combination $\mathbf{s}_{-i}$. This is obvious if $1_b(s_i, \mathbf{s}_{-i}) = 1_b(s_i', \mathbf{s}_{-i}) = 0$. The equality $1_b(s_i, \mathbf{s}_{-i}) = 1_b(s_i', \mathbf{s}_{-i})$ is due to $s_i(b) = s_i'(b)$. Now assume $1_b(s_i, \mathbf{s}_{-i}) = 1_b(s_i', \mathbf{s}_{-i}) = 1$. Since $\prod_{l=1}^{b-1} (1 - 1_l(s_i, \mathbf{s}_{-i})) \cdot 1_b(s_i, \mathbf{s}_{-i}) = 1$ if and only if $b = \mu(s_i, \mathbf{s}_{-i})$ (see A-4) we only need to establish that $\mu(s_i, \mathbf{s}_{-i}) = \mu(s_i', \mathbf{s}_{-i})$ for any strategy combination $\mathbf{s}_{-i}$. For the given s_i and s_i' and an arbitrary $\mathbf{s}_{-i}$ consider sets $B(s_i, \mathbf{s}_{-i}) = \left\{ b : \sum_{\substack{j=1, \\ j \neq i}}^{N} s_j(b) + s_i(b) = 1 \right\}$ and $B(s_i', \mathbf{s}_{-i}) = \left\{ b : \sum_{\substack{j=1, \\ j \neq i}}^{N} s_j(b) + s_i'(b) = 1 \right\}$ which contain all unmatched bids in respective strategy combinations. Since by assumption $s_i(l) = s'_i(l)$ for all $l \le b$, we obtain $B(s_i, \mathbf{s}_{-i}) \cap \{1, .., b\} = B(s'_i, \mathbf{s}_{-i}) \cap \{1, .., b\}$. Since by assumption $1_b(s_i, \mathbf{s}_{-i}) = 1_b(s'_i, \mathbf{s}_{-i}) = 1$, both intersections are non-empty (at least bid b is unmatched) and have a well-defined minimum: $\min B(s_i, \mathbf{s}_{-i}) = \min B(s'_i, \mathbf{s}_{-i})$. This yields tautologically $\mu(s_i, \mathbf{s}_{-i}) = \mu(s'_i, \mathbf{s}_{-i})$. ■

Lemma A.3 (Invariance II) *For an arbitrary $1 < k \le \bar{b}$ and mixed strategy profiles π and $\widetilde{\pi}$ such that*

$$\widetilde{\pi}(s) = \begin{cases} \pi(s) + \pi(\sigma(s, k+1)) & \text{if } s(k+1) = 0 \\ 0 & \text{if } s(k+1) = 1 \end{cases} \tag{A-10}$$

the following holds:

$$w_b(s_i, \pi) = w_b(s_i, \widetilde{\pi}), \forall b \le k. \tag{A-11}$$

Proof. First, we are interested in winning probabilities of bids below or equal to k, which by Lemma A.2 are invariant to higher bids. We therefore disregard higher bids and without loss of generality restrict consideration to strategy sets $S_k = \{s \in S : s(b) = 0, \forall b > k\}$ and $S_{k+1} = \{s \in S : s(b) = 0, \forall b > k+1\}$. Strategies in the former do not contain bid $k+1$, and are a subset of the latter. To shorten notation denote $\delta_b(s_i, \mathbf{s}_{-i}) = \prod_{l=1}^{b-1}(1 - 1_l(s_i, \mathbf{s}_{-i})) \cdot 1_b(s_i, \mathbf{s}_{-i})$. By definition (A-5) and by restricting the set of strategies to $S = S_{k+1}$,

$$\begin{aligned} w_b(s_i, \pi) &= \sum_{\mathbf{s}_{-i} \in S^{N-1}} \pi(\mathbf{s}_{-i}) \cdot \delta_b(s_i, \mathbf{s}_{-i}) = \sum_{\mathbf{s}_{-i} \in S_{k+1}^{N-1}} \pi(\mathbf{s}_{-i}) \cdot \delta_b(s_i, \mathbf{s}_{-i}) \\ &= \sum_{\mathbf{s}_{-i} \in S_k^{N-1}} \pi(\mathbf{s}_{-i}) \cdot \delta_b(s_i, \mathbf{s}_{-i}) + \sum_{\mathbf{s}_{-i} \in S_{k+1}^{N-1} \setminus S_k^{N-1}} \pi(\mathbf{s}_{-i}) \cdot \delta_b(s_i, \mathbf{s}_{-i}) \end{aligned} \tag{A-12}$$

For any $\mathbf{s}_{-i} \in S_k^{N-1}$ let $\sigma^{N-1}(\mathbf{s}_{-i}, k+1)$ be the set of all strategy combinations $\mathbf{s}'_{-i}$ identical to $\mathbf{s}_{-i}$ except for bid $k+1$:

$$\sigma^{N-1}(\mathbf{s}_{-i}, k+1) = \left\{ s'_{-i} = \{s_j\}_{j \ne i} \in S_{k+1}^{N-1} \setminus S_k^{N-1} \,\middle|\, s'_j = \sigma\left(s'_j, k+1\right) \text{ for some } j \ne i \right\}.$$

By construction, any strategy combination $\mathbf{s}'_{-i} \in S_{k+1}^{N-1} \setminus S_k^{N-1}$ belongs to exactly one set $\sigma^{N-1}(\mathbf{s}_{-i}, k+1)$ formed by a particular matching strategy combination $\mathbf{s}_{-i} \in S_k^{N-1}$. Since for any player $j \ne i$ the above strategies s_j and s'_j are identical in bids $b \ne k+1$ it follows that $\delta_b(s_i, s_{-i}) = \delta_b\left(s_i, s'_{-i}\right), \forall b \le k$ for any $s'_{-i} \in \sigma^{N-1}(\mathbf{s}_{-i}, k+1)$. Equation

(A-12) therefore transforms into

$$w_b(s_i,\pi)=\sum_{\mathbf{s}_{-i}\in S_k^{N-1}}\left(\pi(\mathbf{s}_{-i})+\sum_{\mathbf{s}'_{-i}\in\sigma^{N-1}(\mathbf{s}_{-i},k+1)}\pi\left(\mathbf{s}'_{-i}\right)\right)\cdot\delta_b(s_i,\mathbf{s}_{-i}),\forall b\le k. \quad \text{(A-13)}$$

It remains to show that $\pi(\mathbf{s}_{-i})+\sum_{\mathbf{s}'_{-i}\in\sigma^{N-1}(\mathbf{s}_{-i},k+1)}\pi\left(\mathbf{s}'_{-i}\right)=\widetilde{\pi}(\mathbf{s}_{-i})$ for any $\mathbf{s}_{-i}\in S_k^{N-1}$. For a given combination of pure strategies $\mathbf{s}_{-i}=\{s_j\}_{j\neq i}$ one obtains

$$\widetilde{\pi}(\mathbf{s}_{-i})=\prod_{s_j\in\mathbf{s}_{-i}}\widetilde{\pi}(s_j)=\prod_{s_j\in\mathbf{s}_{-i}}\left[\pi(s_j)+\pi\left(s'_j\right)\right].$$

The product of all $\pi(s_j)$ above results in the probability of the strategy combination $\mathbf{s}_{-i}$ such that no player $j\neq i$ places bid $k+1$, which is $\pi(\mathbf{s}_{-i})$. The remainder of the product on the right-hand side represents probabilities of strategy combinations in which at least one player j plays s'_j such that $s'_j(k)=1$ and $s'_j(l)=s_j(l),\forall l\neq k$. By construction all these strategy combinations constitute the set $S_k^{N-1}(\mathbf{s}_{-i})$. Formally, if we denote $a_{0j}:=\pi(s_j)$ and $a_{1j}:=\pi\left(s'_j\right)$ and without loss of generality renumber players so that $i=N$ then the following decomposition can be used to see the result:

$$\prod_{j=1}^{N-1}\left[a_{0j}+a_{1j}\right]=\prod_{j=1}^{N-1}\sum_{n=0}^{1}a_{nj}=\sum_{n_1=0}^{1}\sum_{n_2=0}^{1}..\sum_{n_{N-1}=0}^{1}\prod_{j=1}^{N-1}a_{n_jj}.$$

Now consider an arbitrary $\mathbf{s}_{-i}\in S_k^{N-1}$ and related set of strategy combinations $\sigma^{N-1}(\mathbf{s}_{-i},k+1)$. In the above formula the term that corresponds to all $n_j=0$ is $\prod_{j=1}^{N-1}a_{0j}=\prod_{j=1}^{N-1}\pi(s_j)=\pi(\mathbf{s}_{-i})$. For any $\mathbf{s}'_{-i}\in\sigma^{N-1}(\mathbf{s}_{-i},k+1)$ assign $n_j:=0$ if for player's j strategy holds $s'_j(k+1)=0$ and $n_j:=1$ if $s'_j(k+1)=1$. The resulting array $(n_1,n_2..n_{N-1})$ produces the term $\prod_{j=1}^{N-1}a_{n_jj}=\prod_{j:n_j=0}a_{0j}\prod_{j:n_j=1}a_{1j}=\prod_{j:n_j=0}\pi(s_j)\prod_{j:n_j=1}\pi\left(s'_j\right)=\pi\left(\mathbf{s}'_{-i}\right)$. It follows that

$$\widetilde{\pi}(\mathbf{s}_{-i})=\pi(\mathbf{s}_{-i})+\sum_{\mathbf{s}'_{-i}\in\sigma^{N-1}(\mathbf{s}_{-i},k+1)}\pi\left(\mathbf{s}'_{-i}\right).$$

Combined with (A-13) this delivers the result. ■

The last lemma of this appendix extends the above result to show that if player i does not place bids above k, his expected payoff is invariant to whether his opponents play mixed strategy profile π or $\widetilde{\pi}$.

Lemma A.4 *Consider $s\in S$ such that $s(b)=0$ for any $b\geq k$. For a mixed strategy*

π and $\widetilde{\pi}$ as defined in (8), one has

$$P_i(s,\widetilde{\pi}) = P_i(s,\pi) \quad \text{for all} \quad s \in S.$$

Proof. From Lemma A.1, using condition $s(b) = 0$ for all $b \geq k$, obtain $P_i(s,\pi) = \sum_{l=1}^{k} s(l)\left[(A-b)\, w_l(s,\pi) - c\right]$. By lemma A.3 $w_l(s,\pi) = w_l(s,\widetilde{\pi}), \forall s \in S, \forall l \leq k$. It follows that $P_i(s,\pi) = P_i(s,\widetilde{\pi})$. ■

Appendix B. Proofs

Proof of Proposition 1

Proof. First, note that $s_1^* = .. = s_I^* = s^0$ is not an equilibrium as soon as $c < A - 1$ since each player has incentives to deviate to strategy s^1 with a strictly positive payoff of $A - 1 - c$.

Assume $\mathbf{s}^* = (s_1^*, ..s_I^*)$ is an equilibrium, and player i wins. This implies that all his rivals play $s_j^* = s^0$, $\forall j \in I : j \neq i$ (otherwise rivals of i obtain negative payoff). If all rivals of i play s^0, player i maximizes his payoff by playing $s_i^* = s^1$, which is his best response. If player i plays s^1 and other players play s^0 then some player $j \neq i$ can win with $b = 2$ by playing strategy $s_j = s^3$, which yields a payoff of $p_j = A - 2 - 2c$. If $c < \frac{A}{2} - 1$ then player j profitably deviates from s^0 to s^3 in which case the best response of i is to play $s_i^* = s^0$. Now all rivals of j play s^0, and hence the best response of j is $s_j^* = s^1$ but as just has been shown for player i, strategy combination $\left(s_j^*, \mathbf{s}_{-j}^*\right) = (s^1, s^0..s^0)$ is not an equilibrium. Therefore no Nash equilibrium in pure strategies exists if $c < \frac{A}{2} - 1$. If $c \geq \frac{A}{2} - 1$, no player j has incentives to deviate from s^0, and hence strategy combination $(s_i^*, \mathbf{s}_{-i}) = (s^1, s^0..s^0)$ is an equilibrium. ■

Proof of Lemma 2

Proof. In Γ_2 the strategy set is $S_2 = \{s^0, s^1, s^2, s^3\} = \{(00), (10), (01), (11)\}$. Let

$\pi_2 = (\pi_2^0, \pi_2^1, \pi_2^2, \pi_2^3)$ be a symmetric mixed strategy equilibrium in Γ_2. For an arbitrary player i expected payoffs are

$$\begin{aligned} P_i\left(s^0, \pi_2\right) &= 0, \\ P_i\left(s^1, \pi_2\right) &= (A-1)\, w_1\left(s^1, \pi_2\right) - c, \\ P_i\left(s^2, \pi_2\right) &= (A-2)\, w_2\left(s^2, \pi_2\right) - c, \\ P_i\left(s^3, \pi_2\right) &= (A-1)\, w_1\left(s^3, \pi_2\right) - c + (A-2)\, w_2\left(s^3, \pi_2\right) - c \\ &= P_i\left(s^1, \pi_2\right) + (A-2)\, w_2\left(s^3, \pi_2\right) - c. \end{aligned}$$

The latter is due to Lemma A.2. For easy reference, we repeat the list of all relevant win probabilities:

$$\begin{aligned} w_1\left(s^1, \pi_2\right) &= w_1\left(s^3, \pi_2\right) = \left(\pi_2^0 + \pi_2^2\right)^{N-1}, \\ w_2\left(s^2, \pi_2\right) &= \left(\pi_2^0 + \pi_2^1\right)^{N-1} - (N-1)\, \pi_2^1 \left(\pi_2^0\right)^{N-2}, \\ w_2\left(s^3, \pi_2\right) &= \left(\pi_2^0 + \pi_2^1\right)^{N-1} - \left(\pi_2^0\right)^{N-1}. \end{aligned}$$

First, we establish that $\pi_2^0 > 0$ and $\pi_2^1 > 0$. Next we check whether $\pi_2^2 > 0$ or $\pi_2^3 > 0$, or both, and determine the equilibrium mixed strategy.

CLAIM 1. $\pi_2^0 > 0$.

PROOF OF CLAIM 1: Assume $\pi_2^0 = 0$. For any $s \in \operatorname{supp} \pi_2$ holds $P_i(s, \pi_2) > P_i(s^0, \pi_2) = 0$. There are four cases to consider:

1) $\pi_2^1 > 0, \pi_2^2 > 0$ implies $P_i(s^1, \pi_2) = P_i(s^2, \pi_2) > 0$. Assumption $\pi_2^0 = 0$ implies $w_2(s^2, \pi_2) = w_2(s^3, \pi_2)$ and thus $P_i(s^3, \pi_2) = P_i(s^1, \pi_2) + P_i(s^2, \pi_2)$. It follows that $P_i(s^3, \pi_2) > P_i(s^1, \pi_2)$ and $P_i(s^3, \pi_2) > P_i(s^2, \pi_2)$, thus yielding $\pi_2^3 = 1$, a contradiction.

2) $\pi_2^1 = 0, \pi_2^2 > 0$ implies $w_2(s^2, \pi_2) = 0$, and therefore $P_i(s^2, \pi_2) < 0$, a contradiction.

3) $\pi_2^1 > 0, \pi_2^2 = 0$ implies $w_1(s^1, \pi_2) = 0$, and therefore $P_i(s^1, \pi_2) < 0$, a contradiction.

4) $\pi_2^1 = 0, \pi_2^2 = 0$ implies $\pi_1^3 = 1$, a contradiction.□

CLAIM 2. $\pi_2^1 > 0$.

PROOF OF CLAIM 2: If $\pi_2^1 = 0$ then $w_2(s^3, \pi_2) = 0$ and thus $P_i(s^3, \pi_2) <$

$P_i(s^1,\pi_2) \leq 0$, which implies $\pi_2^3 = 0$. Moreover, $w_2(s^2,\pi_2) = w_1(s^1,\pi_2)$ and thus $P_i(s^2,\pi_2) < P_i(s^1,\pi) \leq 0$, which implies $\pi_2^2 = 0$. As a result $\pi_2^0 = 1$, a contradiction.□

Claim 1 guarantees that $\pi_2^0 > 0$, and hence $P_i(s,\pi_2) = P_i(s^0,\pi_2) = 0$ for all $s \in \operatorname{supp}\pi_2$. Claim 2 implies that $\pi_2^1 > 0$ and, hence, $P_i(s^1,\pi_2) = 0$. The latter condition yields

$$\pi_2^0 + \pi_2^2 = \sqrt[N-1]{\frac{c}{A-1}}. \tag{B-1}$$

We will consider the following four exhaustive cases:

(i) $P_i(s^2,\pi_2) = 0$, $P_i(s^3,\pi_2) = 0$ and $\pi_2^2 \geq 0$, $\pi_2^3 \geq 0$.
(ii) $P_i(s^2,\pi_2) \leq 0$, $P_i(s^3,\pi_2) = 0$ and $\pi_2^2 = 0$, $\pi_2^3 \geq 0$.
(iii) $P_i(s^2,\pi_2) = 0$, $P_i(s^3,\pi_2) \leq 0$ and $\pi_2^2 \geq 0$, $\pi_2^3 = 0$.
(iv) $P_i(s^2,\pi_2) \leq 0$, $P_i(s^3,\pi_2) \leq 0$ and $\pi_2^2 = 0$, $\pi_2^3 = 0$.

(i) From $P_i(s^2,\pi_2) = P_i(s^3,\pi_2) = 0$ one obtains $w_2(s^2,\pi_2) = w_2(s^3,\pi_2)$ and $(N-1)\cdot\pi_2^1 = \pi_2^0$. Moreover, from $P_i(s^3,\pi_2) = 0$, we have $w_2(s^3,\pi_2) = \frac{c}{A-2}$ and, substituting $\pi_2^0 = (N-1)\cdot\pi_2^1$, $(N\cdot\pi_2^1)^{N-1} - (N-1)^{N-1}(\pi_2^1)^{N-1} = \frac{c}{A-2}$. Solving the latter equation for π_2^1 and denoting $K(N) := \frac{(N-1)^{N-1}}{N^{N-1}-(N-1)^{N-1}}$ yields $\pi_2^1 = \frac{1}{N-1}\sqrt[N-1]{K(N)\frac{c}{A-2}}$ and $\pi_2^0 = \sqrt[N-1]{K(N)\frac{c}{A-2}}$. From $w_1(s^1,\pi_2) = \frac{c}{A-1}$ we obtain $\pi_2^2 = \sqrt[N-1]{\frac{c}{A-1}} - \sqrt[N-1]{K(N)\frac{c}{A-2}}$. Finally, $\pi_2^3 = 1 - \pi_2^0 - \pi_2^1 - \pi_2^2 = 1 - \left(\sqrt[N-1]{\frac{c}{A-1}} + \frac{1}{N-1}\sqrt[N-1]{K(N)\frac{c}{A-2}}\right)$.

Condition $\pi_2^2 \geq 0$ holds iff $z_1(A,c,N) := \frac{2-K(N)}{1-K(N)} - A \leq 0$:

$$\begin{aligned}
\sqrt[N-1]{\frac{c}{A-1}} - \sqrt[N-1]{K(N)\frac{c}{A-2}} &\geq 0\\
\frac{A-2}{A-1} &\geq K(N)\\
A-2 &\geq A\cdot K(N) - K(N),\\
&\quad \{\text{Note that } K(N) < 1 \text{ for any } N \geq 3\}\\
A(1-K(N)) &\geq 2 - K(N)\\
0 &\geq \frac{2-K(N)}{1-K(N)} - A =: z_1(A,c,N).
\end{aligned}$$

Condition $\pi_2^3 \geq 0$ is equivalent to $1 \geq \sqrt[N-1]{\frac{c}{A-1}} + \frac{1}{N-1}\sqrt[N-1]{K(N)\frac{c}{A-2}} \Leftrightarrow$

$$\begin{aligned}
1 &\geq \sqrt[N-1]{c}\left(\sqrt[N-1]{\frac{c}{A-1}} + \frac{1}{N-1}\sqrt[N-1]{K(N)\frac{c}{A-2}}\right)\\
c &\leq \left(\sqrt[N-1]{\frac{c}{A-1}} + \frac{1}{N-1}\sqrt[N-1]{K(N)\frac{c}{A-2}}\right)^{(1-N)}.
\end{aligned}$$

Hence, $\pi_2^3 \geq 0$ iff $z_4(A,c,N) := c - \left(\sqrt[N-1]{\frac{c}{A-1}} + \frac{1}{N-1} \sqrt[N-1]{K\left(N\right)\frac{c}{A-2}} \right)^{(1-N)} \leq 0.$

(ii) From Equation (B-1) and $\pi_2^2 = 0$ one obtains $\pi_2^0 = \sqrt[N-1]{\frac{c}{A-1}}$. From $P_i(s^3, \pi_2) = (A-2)\, w_2\left(s^3, \pi_2\right) - c = 0$ one deduces $\left(1-\pi_2^3\right)^{N-1} - \left(\pi_2^0\right)^{N-1} = \frac{c}{A-2}$and, hence, $\pi_2^3 = 1 - \sqrt[N-1]{\frac{c}{A-1} + \frac{c}{A-2}}$. Finally, $\pi_2^0 + \pi_2^1 + \pi_2^3 = 1$ implies $\pi_2^1 = \sqrt[N-1]{\frac{c}{A-1} + \frac{c}{A-2}} - \sqrt[N-1]{\frac{c}{A-1}} \geq 0$.

Note that $\pi_2^3 = 1 - \sqrt[N-1]{\frac{c}{A-1} + \frac{c}{A-2}} \geq 0$ iff $\frac{c}{A-1} + \frac{c}{A-2} \leq 1$ or

$$z_2(A,c,N) := \frac{\left(A-1\right)\left(A-2\right)}{2A-3} - c \geq 0.$$

Condition $P_i(s^2, \pi_2) \leq P_i(s^3, \pi_2)$ is equivalent to $\left(N-1\right)\pi_2^1 \geq \pi_2^0$ or $\left(N-1\right) \sqrt[N-1]{\frac{c}{A-1} + \frac{c}{A-2}} \geq N \sqrt[N-1]{\frac{c}{A-1}}$. The last inequality yields $\frac{1}{A-1} + \frac{1}{A-2} \geq \left(\frac{N}{N-1}\right)^{N-1} \frac{1}{A-1}$ or $\frac{2A-3}{A-2} \geq \left(\frac{N}{N-1}\right)^{N-1} = \frac{K(N)+1}{K(N)}$. Hence,

$$z_1(A,c,N) := \frac{2-K\left(N\right)}{1-K(N)} - A \geq 0.$$

In the latter derivation recall that $K\left(N\right) < 1$ for any $N \geq 3$.

(iii) From $\pi_2^3 = 0$, we obtain $\pi_2^1 = 1 - (\pi_2^0 + \pi_2^2)$. Substituting from Equation (B-1) $\pi_2^0 + \pi_2^2 = \sqrt[N-1]{\frac{c}{A-1}}$ yields $\pi_2^1 = 1 - \sqrt[N-1]{\frac{c}{A-1}} > 0$. Condition $P_i(s^2, \pi_2) = (A-2)\, w_2\left(s^2, \pi_2\right) - c = 0$ yields $\left(\pi_2^0 + \pi_2^1\right)^{N-1} - \left(N-1\right)\pi_2^1 \left(\pi_2^0\right)^{N-2} = \frac{c}{A-2}$. Substituting for π_2^1 results in the equation

$$\left(\pi_2^0 + 1 - \sqrt[N-1]{\frac{c}{A-1}}\right)^{N-1} - \left(N-1\right)\left(1 - \sqrt[N-1]{\frac{c}{A-1}}\right)\left(\pi_2^0\right)^{N-2} = \frac{c}{A-2} \qquad \text{(B-2)}$$

which implicitly defines $\widehat{\pi}^0(A,c,N)$. The left-hand side positively depends on $\widehat{\pi}^0(A,c,N)$. To see this, substitute the binomial expansion of $\left(\pi_2^0 + \pi_2^1\right)^{N-1} = \sum_{k=0}^{N-1} \begin{pmatrix} N-1 \\ k \end{pmatrix} \left(\pi_2^0\right)^{N-1-k} \left(\pi_2^1\right)^k = \left(\pi_2^0\right)^{N-1} + \left(N-1\right)\left(\pi_2^0\right)^{N-2}\left(\pi_2^1\right) + ...$ into the left-hand side of Equation (B-2),

$$\begin{aligned} f(\pi_2^0, \pi_2^1) \quad &: \quad = \left(\pi_2^0 + \pi_2^1\right)^{N-1} - \left(N-1\right)\pi_2^1 \left(\pi_2^0\right)^{N-2} \\ &= \quad \left(\pi_2^0\right)^{N-1} + \sum_{k=2}^{N-1} \begin{pmatrix} N-1 \\ k \end{pmatrix} \left(\pi_2^0\right)^{N-1-k} \left(\pi_2^1\right)^k, \end{aligned}$$

which shows that $f(\pi_2^0, \pi_2^1)$ is a strictly increasing function of π_2^0.

From (B-1), $\pi_2^2 = \sqrt[N-1]{\frac{c}{A-1}} - \widehat{\pi}^0(A,c,N)$. Condition $\pi_2^2 \geq 0$ is satisfied iff $\sqrt[N-1]{\frac{c}{A-1}} \geq \widehat{\pi}^0(A,c,N)$. Using monotonicity of the left-hand side in (B-2), we can write

$$\begin{aligned}
\frac{c}{A-2} &= \left(\widehat{\pi}^0(A,c,N)+1-\sqrt[N-1]{\frac{c}{A-1}}\right)^{N-1}-(N-1)\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)\left(\widehat{\pi}^0(A,c,N)\right)^{N-2} \\
&= f\left(\widehat{\pi}^0(A,c,N),1-\sqrt[N-1]{\frac{c}{A-1}}\right)\le f\left(\sqrt[N-1]{\frac{c}{A-1}}),1-\sqrt[N-1]{\frac{c}{A-1}}\right) \\
&= \left(\sqrt[N-1]{\frac{c}{A-1}}+1-\sqrt[N-1]{\frac{c}{A-1}}\right)^{N-1}-(N-1)\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)\left(\sqrt[N-1]{\frac{c}{A-1}}\right)^{N-2} \\
&= 1-(N-1)\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)\left(\sqrt[N-1]{\frac{c}{A-1}}\right)^{N-2}
\end{aligned}$$

or, equivalently,

$$z_3\left(A,c,N\right):=1-\frac{c}{A-2}-(N-1)\left(\frac{c}{A-1}\right)^{\frac{N-2}{N-1}}+(N-1)\frac{c}{A-1}\ge 0$$

Condition $P_i(s^3,\pi_2)\le P_i(s^2,\pi_2)$ is equivalent to $w_2\left(s^3,\pi_2\right)\le w_2\left(s^2,\pi_2\right)$ or $(N-1)\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)\le\widehat{\pi}^0(A,c,N)$. Hence,

$$\begin{aligned}
\frac{c}{A-2} &= \left(\widehat{\pi}^0(A,c,N)+1-\sqrt[N-1]{\frac{c}{A-1}}\right)^{N-1}-(N-1)\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)\left(\widehat{\pi}^0(A,c,N)\right)^{N-2} \\
&= f\left(\widehat{\pi}^0(A,c,N),1-\sqrt[N-1]{\frac{c}{A-1}}\right)\ge f\left((N-1)\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right),1-\sqrt[N-1]{\frac{c}{A-1}}\right) \\
&= \left((N-1)\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)+\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)\right)^{N-1} \\
&\quad -(N-1)\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)\left((N-1)\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)\right)^{N-2} \\
&= \left(N\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)\right)^{N-1}-\left((N-1)\left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)\right)^{N-1} \\
&= \left(1-\sqrt[N-1]{\frac{c}{A-1}}\right)^{N-1}\left[N^{N-1}-(N-1)^{N-1}\right].
\end{aligned}$$

The inequality $\frac{c}{A-2} \geq \left(1 - \sqrt[N-1]{\frac{c}{A-1}}\right)^{N-1} \left[N^{N-1} - (N-1)^{N-1}\right]$ is equivalent to

$$
\begin{aligned}
\frac{c}{A-2} K(N) \frac{1}{(N-1)^{N-1}} &\geq \left(1 - \sqrt[N-1]{\frac{c}{A-1}}\right)^{N-1} \\
\frac{1}{N-1} \sqrt[N-1]{K(N) \frac{c}{A-2}} &\geq 1 - \sqrt[N-1]{\frac{c}{A-1}} \\
\sqrt[N-1]{c} \left(\sqrt[N-1]{\frac{1}{A-1}} + \frac{1}{N-1} \sqrt[N-1]{K(N) \frac{1}{A-2}} \right) &\geq 1 \\
c \geq \left(\sqrt[N-1]{\frac{1}{A-1}} + \frac{1}{N-1} \sqrt[N-1]{K(N) \frac{1}{A-2}} \right)^{-(N-1)} & \quad \text{, i.e.} \\
z_4(A, c, N) := c - \left(\sqrt[N-1]{\frac{1}{A-1}} + \frac{1}{N-1} \sqrt[N-1]{K(N) \frac{1}{A-2}} \right)^{(1-N)} & \geq 0,
\end{aligned}
$$

(iv) From Equation (B-1) and $\pi_2^2 = 0$ we have $\pi_2^0 = \sqrt[N-1]{\frac{c}{A-1}}$ and from $\pi_2^3 = 0$ follows $\pi_2^1 = 1 - \sqrt[N-1]{\frac{c}{A-1}}$.

Condition $P_i(s^2, \pi_2) = (A-2) w_2(s^2, \pi_2) - c \leq 0$ implies

$$
\begin{aligned}
0 &\geq w_2\left(s^2, \pi_2\right) - \frac{c}{A-2} \\
&= \left(\pi_2^0 + \pi_2^1\right)^{N-1} - (N-1)\, \pi_2^1 \left(\pi_2^0\right)^{N-2} - \frac{c}{A-2} \\
&= 1 - (N-1) \left(1 - \sqrt[N-1]{\frac{c}{A-1}}\right) \left(\sqrt[N-1]{\frac{c}{A-1}}\right)^{N-2} - \frac{c}{A-2} \\
&= 1 - (N-1) \left(\sqrt[N-1]{\frac{c}{A-1}}\right)^{N-2} + (N-1) \frac{c}{A-1} - \frac{c}{A-2} \\
&= 1 - \frac{c}{A-1} - \frac{c}{A-2} - (N-1) \left(\frac{c}{A-1}\right)^{\frac{N-2}{N-1}} + N \frac{c}{A-1} \\
&= : z_3(A, c, N).
\end{aligned}
$$

Condition $P_i(s^3,\pi_2) = (A-2)\,w_2\left(s^3,\pi_2\right) - c \leq 0$ yields

$$\begin{aligned}
0 &\geq w_2\left(s^3,\pi_2\right) - \frac{c}{A-2} \\
&= \left(\pi_2^0+\pi_2^1\right)^{N-1} - \left(\pi_2^0\right)^{N-1} - \frac{c}{A-2} \\
&= 1 - \left(\sqrt[N-1]{\frac{c}{A-1}}\right)^{N-1} - \frac{c}{A-2} \\
&= 1 - \frac{c}{A-1} - \frac{c}{A-2} \\
&\Leftrightarrow \\
c\frac{2A-3}{(A-1)(A-2)} &\geq 1 \Leftrightarrow c \geq \frac{(A-1)(A-2)}{2A-3} \\
&\Leftrightarrow 0 \geq \frac{(A-1)(A-2)}{2A-3} - c =: z_2\left(A,c,N\right).
\end{aligned}$$

Implicit functions $c(A)$ given by conditions $z_1\left(A,c,N\right)=0$, $z_2\left(A,c,N\right)=0$, $z_3\left(A,c,N\right)=0$ and $z_4\left(A,c,N\right)=0$ for any fixed $N\geq 3$ have a remarkable property: they all intersect at one point $A^* = \frac{2-K(N)}{1-K(N)}$, $c^* = \frac{(A^*-1)(A^*-2)}{2A^*-3} = \frac{K(N)}{1-K^2(N)}$. Obviously, $z_1\left(A^*,c^*,N\right) = z_2\left(A^*,c^*,N\right) = 0$. For $z_3\left(A^*,c^*,N\right)$ we obtain

$$\begin{aligned}
z_3\left(A^*,c^*,N\right) &= 1 - c^*\left(\frac{1}{A^*-2}+\frac{1}{A^*-1}\right) - (N-1)\left(\left(\frac{c^*}{A^*-1}\right)^{\frac{N-2}{N-1}} - \frac{N}{N-1}\frac{c^*}{A^*-1}\right) = \\
&= \underbrace{1 - c^*\cdot\frac{2A^*-3}{(A^*-1)(A^*-2)}}_{=0} - (N-1)\left(\frac{c^*}{A^*-1}\right)^{\frac{N-2}{N-1}}\left(1-\frac{N}{N-1}\sqrt[N-1]{\frac{c^*}{A^*-1}}\right) = \\
&= -(N-1)\left(\frac{c^*}{A^*-1}\right)^{\frac{N-2}{N-1}}\left(1-\frac{N}{N-1}\sqrt[N-1]{\frac{A^*-2}{2A^*-3}}\right) = 0,
\end{aligned}$$

with the latter due to

$$\begin{aligned}
\frac{A^*-2}{2A^*-3} &= \left(\frac{N-1}{N}\right)^{N-1} = \frac{K(N)}{K(N)+1} \Leftrightarrow \\
A^*K(N) + A^* - 2K(N) - 2 &= 2A^*K(N) - 3K(N) \Leftrightarrow A^* = \frac{2-K(N)}{1-K(N)}.
\end{aligned}$$

For $z_4(A^*, c^*, N)$ substitute for $A^* - 1 = \frac{1}{1-K(N)}$ and $A^* - 2 = \frac{K(N)}{1-K(N)}$

$$\begin{aligned}
c^* - \left(\sqrt[N-1]{\frac{1}{A^*-1}} + \frac{1}{N-1}\sqrt[N-1]{K(N)\cdot\frac{1}{A^*-2}}\right)^{1-N} &= \\
c^* - \left(\sqrt[N-1]{1-K(N)} + \frac{1}{N-1}\sqrt[N-1]{1-K(N)}\right)^{1-N} &= \\
c^* - \frac{1}{1-K(N)}\cdot\left(\frac{N-1}{N}\right)^{N-1} &= \\
c^* - (A^*-1)\cdot\frac{A^*-2}{2A^*-3} &= 0.
\end{aligned}$$

■

Proof of Proposition 2

Proof. Let π_{k+1} be equilibrium in $\Gamma_{k+1} = \left(I, S_{k+1}, \{p_i\}_{i\in I}\right) : P_i(\pi_{k+1}, \pi_{k+1}) \geq P_i\left(\pi'_{k+1}, \pi_{k+1}\right), \forall \pi'_{k+1}$. Consider $\widetilde{\pi}_k$ with elements

$$\widetilde{\pi}_k(s) := \begin{cases} \pi_{k+1}(s) + \pi_{k+1}(\sigma(s, k+1)) & \text{if} \quad s(k+1) = 0, \\ 0 & \text{if} \quad s(k+1) = 1. \end{cases} \tag{B-3}$$

Note that $\sum_{s\in S_k}\widetilde{\pi}_k(s) = \sum_{s\in S_k}\pi_{k+1}(s) + \sum_{s\in S_k}\pi_{k+1}(\sigma(s,k+1)) = \sum_{s\in S_{k+1}}\pi_{k+1}(s) = 1$, hence $\widetilde{\pi}_k$ is a well defined probability distribution with support on S_k. We will show that $\widetilde{\pi}_k$ is equilibrium in $\Gamma_k = \left(I, S_k, \{p_i\}_{i\in I}\right)$. Assume player i plays some π'_{k+1} in Γ_{k+1} and his expected payoff is therefore $P_i\left(\pi'_{k+1}, \pi_{k+1}\right)$:

$$\begin{aligned}
P_i\left(\pi'_{k+1}, \pi_{k+1}\right) &= \sum_{s\in S_{k+1}} \pi'_{k+1}(s) \sum_{l=1}^{k+1} s(l)\left[(A-b)\, w_l(s, \pi_{k+1}) - c\right] = \\
&= \sum_{s\in S_k} \pi'_{k+1}(s) \sum_{l=1}^{k+1} s(l)\left[(A-l)\, w_l(s, \pi_{k+1}) - c\right] + \\
&\quad + \sum_{s\in S_{k+1}\setminus S_k} \pi'_{k+1}(s) \sum_{l=1}^{k+1} s(l)\left[(A-l)\, w_l(s, \pi_{k+1}) - c\right].
\end{aligned}$$

For any $s \in S_k$ there exists a unique strategy $\sigma(s, k+1) \in S_{k+1} \setminus S_k$. Therefore

$$\begin{aligned} P_i\left(\pi'_{k+1}, \pi_{k+1}\right) &= \sum_{s \in S_k} \pi'_{k+1}(s) \sum_{l=1}^{k+1} s(l)\left[(A-l)\, w_l\left(s, \pi_{k+1}\right) - c\right] \\ &\quad + \sum_{s \in S_k} \pi'_{k+1}(\sigma(s, k+1)) \sum_{l=1}^{k+1} \sigma(l \mid s, k+1)\left[(A-l)\, w_l\left(\sigma(s, k+1), \pi_{k+1}\right) - c\right]. \end{aligned}$$

By lemma A.2 $w_l(s, \pi_{k+1}) = w_l(\sigma(s, k+1), \pi_{k+1}), \forall l \leq k$, thus yielding

$$\begin{aligned} P_i\left(\pi'_{k+1}, \pi_{k+1}\right) &= \sum_{s \in S_k} \left(\pi'_{k+1}(s) + \pi'_{k+1}(\sigma(s, k+1))\right) \sum_{l=1}^{k} s(l)\left[(A-l)\, w_l\left(s, \pi_{k+1}\right) - c\right] \\ &\quad + \sum_{s \in S_k} \pi'_{k+1}(\sigma(s, k+1))\left[(A-(k+1))\, w_{k+1}\left(\sigma(s, k+1), \pi_{k+1}\right) - c\right]. \end{aligned}$$

By lemma A.3 $w_l(s, \pi_{k+1}) = w_l(s, \widetilde{\pi}_k), \forall l \leq k$ and $w_{k+1}(\sigma(s, k+1), \pi_{k+1}) \geq w_{k+1}(\sigma(s, k+1), \widetilde{\pi}_k)$ since $\sigma(k+1 \mid s, k+1) = 1$. As a result, we obtain

$$\begin{aligned} P_i\left(\pi'_{k+1}, \pi_{k+1}\right) &\geq \sum_{s \in S_k} \left(\pi'_{k+1}(s) + \pi'_{k+1}(\sigma(s, k+1))\right) \sum_{l=1}^{k} s(l)\left[(A-l)\, w_l\left(s, \widetilde{\pi}_k\right) - c\right] \\ &\quad + \sum_{s \in S_k} \pi'_{k+1}(\sigma(s, k+1))\left[(A-(k+1))\, w_{k+1}\left(\sigma(s, k+1), \widetilde{\pi}_k\right) - c\right] \\ &= P_i\left(\pi'_{k+1}, \widetilde{\pi}_k\right). \end{aligned} \tag{B-4}$$

If π_{k+1} is equilibrium then for any strategy $s \in \operatorname{supp} \pi_{k+1}$ holds $P_i(s, \pi_{k+1}) \geq P_i(s', \pi_{k+1}), \forall s' \in S_{k+1}$. Denote $P^*(\pi_{k+1})$ the equilibrium payoff of any pure strategy in equilibrium π_{k+1}: $P_i(s, \pi_{k+1}) = P_i(s'', \pi_{k+1}) = P^*(\pi_{k+1}), \forall s, s'' \in \operatorname{supp} \pi_{k+1}$. Equilibrium payoff of player i is

$$P_i(\pi_{k+1}, \pi_{k+1}) = P^*(\pi_{k+1}) \cdot \sum_{s \in S_k} \left(\pi_{k+1}(s) + \pi_{k+1}(\sigma(s, k+1))\right).$$

By lemma A.4, $P^*(\pi_{k+1}) = P^*(\widetilde{\pi}_k)$. It follows that

$$P_i(\pi_{k+1}, \pi_{k+1}) = P^*(\widetilde{\pi}_k) \cdot \sum_{s \in S_k} \widetilde{\pi}_k(s) = P_i(\widetilde{\pi}_k, \widetilde{\pi}_k). \tag{B-5}$$

By combining condition $P_i(\pi_{k+1}, \pi_{k+1}) \geq P_i\left(\pi'_{k+1}, \pi_{k+1}\right), \forall \pi'_{k+1}$ with (B-4) and (B-5) we obtain

$$P_i(\widetilde{\pi}_k, \widetilde{\pi}_k) = P_i(\pi_{k+1}, \pi_{k+1}) \geq P_i\left(\pi'_{k+1}, \pi_{k+1}\right) \geq P_i\left(\pi'_{k+1}, \widetilde{\pi}_k\right), \forall \pi'_{k+1} \in \Delta(S_{k+1}).$$

It only remains to note that any $\pi'_k \in \Delta(S_k)$ is a special case of probability distribution from $\Delta(S_{k+1})$, as it assigns probability of zero to all strategies s such that $s(k+1) = 1$.

It follows that $P_i(\widetilde{\pi}_k, \widetilde{\pi}_k) \geq P_i(\pi'_k, \widetilde{\pi}_k)$, for any $\pi'_k \in \Delta(S_k)$ and thus $\widetilde{\pi}_k$ is equilibrium in Γ_k. ■

Proof of Proposition 3

Proof. Consider three types of equilibrium in Γ_2 in Lemma 2. From the proof of the lemma, equilibrium (iii) requires $P_i(s^3, \pi_2) \leq P_i(s^2, \pi_2)$, or, equivalently, $w_2(s^3, \pi_2) \leq w_2(s^2, \pi_2)$, which holds if and only if $\pi_2^0 \geq (N-1)\pi_2^1$, i.e $\pi_2^0 + \pi_2^1 \geq N\pi_2^1$. At the same time, $\pi_2^3 = 0$ implies $\pi_2^2 = 1 - (\pi_2^0 + \pi_2^1)$. Combining the two yields $\pi_2^2 \leq 1 - N\pi_2^1 = 1 - N\left(1 - \sqrt[N-1]{\frac{1}{A-1}}\right)$. From $\pi_2^2 \geq 0$ we obtain the necessary condition for equilibrium (iii):

$$
\begin{aligned}
1 - N\left(1 - \sqrt[N-1]{\frac{c}{A-1}}\right) &\geq 0 \Leftrightarrow \sqrt[N-1]{\frac{c}{A-1}} \geq 1 - \frac{1}{N} \Leftrightarrow \\
c \cdot \left(\frac{N}{N-1}\right)^{N-1} &\geq A - 1 \Leftrightarrow A \leq 1 + c \cdot \frac{1+K(N)}{K(N)}.
\end{aligned}
$$

Assumption $c < (A-1)\frac{K(N)}{1+K(N)}$ violates this necessary condition and rules out equilibrium (iii).

A necessary condition for equilibrium (ii) is $z_1 \geq 0$, i.e. $A \leq \frac{2-K(N)}{1-K(N)}$, which, combined with the above, implies $\frac{2-K(N)}{1-K(N)} > 1 + c \cdot \frac{1+K(N)}{K(N)}$, equivalent to $\frac{1}{1-K(N)} > c \cdot \frac{1+K(N)}{K(N)}$, or $K(N) > c \cdot (1 - K^2(N))$. From here $K(N) > \frac{1+\sqrt{1+4c^2}}{2c} > 1$ for any $c > 0$, which contradicts to $K(N) = \frac{1}{\left(\frac{N}{N-1}\right)^{N-1} - 1} < 1$ for any $N \geq 3$. It follows that $A > 1 + c \cdot \frac{1+K(N)}{K(N)}$ implies $A > \frac{2-K(N)}{1-K(N)}$ and thus rules out equilibrium (ii).

Equilibrium (iv) requires $z_2 \leq 0$, i.e. $c \geq \frac{(A-1)(A-2)}{2A-3} = \left(\frac{1}{A-1} + \frac{1}{A-2}\right)^{-1} \geq \left(\frac{1}{A-2} + \frac{1}{A-2}\right)^{-1} = \frac{A}{2} - 1$, which is a contradiction to $c < \frac{A}{2} - 1$.

Note that if $A > 10$ then $c < \frac{A}{2} - 1$ is implied by $A > 1 + c \cdot \frac{1+K(N)}{K(N)}$. This is because $\frac{K(N)}{1+K(N)} = \frac{(N-1)^{N-1}}{N^{N-1}} = \frac{1}{\left(1+\frac{1}{N-1}\right)^{N-1}}$ equals $\frac{1}{e} = \frac{1}{2.71}$ in the limit $N \to \infty$, and has an upper bound of $\frac{4}{9}$ with $N = 3$. For this reason, $A > 1 + c \cdot \frac{1+K(N)}{K(N)}$ implies $c < \frac{4}{9}(A-1) \leq \frac{A}{2} - 1$ as soon as $A \geq 10$.

We are left with equilibrium (i) in which both s^2 and s^3 are played with positive probabilities. Thus the proposition holds for Γ_2.

Now assume that an equilibrium in Γ contains only singletons. By Proposition 2 and by construction of $\widetilde{\pi}_k$, there exists an equilibrium in Γ_k that contains only singletons in the support, for any $k \geq 1$. This contradicts the fact that s^3 is played with positive probability in Γ_2. Similarly for sequential bidding: assume that for any $s' \in \operatorname{supp} \pi$ holds that $s'(b-1) = 1$ if $s'(b) = 1$. Proposition 2 implies that this also holds in Γ_2, which is a contradiction: s^2 is played in equilibrium with positive probability. ■

Proof of Proposition 4

Proof. The proof is by induction. The first step follows from lemma 1 which guarantees that $s(b)[(A-b)w_b(s,\pi_1) - c] = 0, \forall s \in \operatorname{supp} \pi_1$ in Γ_1. Now assume that the proposition holds for Γ_k and consider equilibrium π_{k+1} in Γ_{k+1}. Construct $\widetilde{\pi}_k$ as in (8): $\widetilde{\pi}_k(s) = \pi_{k+1}(s) + \pi_{k+1}(\sigma(s,k+1)), \forall s \in S_k$. Proposition 2 ensures that $\widetilde{\pi}_k$ is equilibrium in Γ_k. By inductive hypothesis, for any $s \in \operatorname{supp} \widetilde{\pi}_k$ holds $s(b)[(A-b)w_b(s,\widetilde{\pi}_k) - c] = 0$, for any $b \leq k$, and hence $P_i(s,\widetilde{\pi}_k) = 0$.

Now let $s \in \operatorname{supp} \pi_{k+1}$. If $s(k+1) = 0$ then from $\pi_{k+1}(s) > 0$ straightforwardly follows $\widetilde{\pi}_k(s) > 0$ and thus $s \in \operatorname{supp} \widetilde{\pi}_k$, and the proposition holds.

Now assume $s(k+1) = 1$. From $\pi_{k+1}(s) > 0$ follows $\widetilde{\pi}_k(\sigma(s,k+1)) > 0$ and therefore proposition holds in Γ_k for $\sigma(s,k+1)$. The latter strategy coincides with s in all bids below $k+1$. Invariance $w_b(s,\pi_{k+1}) = w_b(s,\widetilde{\pi}_k)$ for $b \leq k$ guarantees $s(b)[(A-b)w_b(s,\pi_{k+1}) - c] = 0$ in Γ_{k+1}.

It only remains to show that the proposition holds for bid $k+1$. The expected payoff of player i can be written as

$$P_i(s,\pi_{k+1}) = \sum_{b=1}^{k} s(b)[(A-b)w_b(s,\pi_{k+1}) - c] + (A-(k+1))w_{k+1}(s,\pi_{k+1}) - c. \tag{B-6}$$

By Lemma A.4, $P_i(s,\pi_{k+1}) = P_i(s,\widetilde{\pi}_k) = 0$.

Assume $(A-(k+1))w_{k+1}(s,\pi_{k+1}) - c < 0$ and consider strategy $\sigma(s,k+1) \in S_k$. By Lemma A.2 holds $w_b(s,\pi_{k+1}) = w_b(\sigma(s,k+1),\pi_{k+1})$ and hence $P_i(\sigma(s,k+1),\pi_{k+1}) =$

$\sum_{b=1}^{k} s(b)\left[(A-b)\, w_b(s,\pi_{k+1})-c\right] > P_i(s,\pi_{k+1})$, which is a contradiction to $s \in \operatorname{supp} \pi_{k+1}$: player i is better off by dropping bid $k+1$ from his optimal strategy.

Assume now that $(A-(k+1))\, w_{k+1}(s,\pi_{k+1}) - c > 0$, which implies $\sum_{b=1}^{k} s(b)\left[(A-b)\, w_b(s,\pi_{k+1})-c\right] < 0$ due to $P_i(s,\pi_{k+1}) = 0$. Since $s \in \operatorname{supp} \pi_{k+1}$, for strategy $\sigma(s,k+1) \in S_k$ we obtain $\widetilde{\pi}_k(\sigma) = \pi_{k+1}(\sigma) + \pi_{k+1}(s) > 0$ and thus $\sigma \in \operatorname{supp} \widetilde{\pi}_k$, which contradicts to

$$\sum_{b=1}^{k} \sigma(b)\left[(A-b)\, w_b(\sigma,\widetilde{\pi}_k)-c\right] = \sum_{b=1}^{k} s(b)\left[(A-b)\, w_b(s,\pi_{k+1})-c\right] < 0.$$

The equality above is due to $w_b(\sigma,\widetilde{\pi}_k) = w_b(s,\widetilde{\pi}_k)$ by Lemma A.2 and $w_b(s,\widetilde{\pi}_k) = w_b(s,\pi_{k+1})$ by lemma A.3.

We therefore obtain that $(A-(k+1))\, w_{k+1}(s,\pi_{k+1}) - c = 0$. This completes the induction and the proof. ■

Proof of Proposition 5

Proof. By proposition 4, $w_b = \frac{c}{A-b}$ holds for *any pure strategy* s played in equilibrium π with positive probability, i.e. for any $s \in \operatorname{supp} \pi$, such that $s(b) = 1$. There are no gaps in the set of bids b placed in equilibrium, i.e. if b is placed in some strategy $s \in \operatorname{supp} \pi$, then there exists strategy $s' \in \operatorname{supp} \pi$ such that $s'(b-1) = 1$ (otherwise one is strictly better off by replacing bid b in strategy s with bid $b-1$). It follows that $w_b = \frac{c}{A-b}$ holds for *any bid* placed in equilibrium.
Bids above $\overline{b} = \inf\left\{b' : \sum_{b=1}^{b'} w_b \geq 1\right\}$ are never placed in equilibrium because with probability 1 some bid below or equal to $\overline{b}$ wins. Assume that strict equality holds $\sum_{b=1}^{\overline{b}} w_b = 1$ (we will relax this later) and solve $\sum_{b=1}^{\overline{b}} \frac{c}{A-b} = 1$ for $\overline{b}$. To do this re-arrange:

$$\sum_{b=1}^{\overline{b}} \frac{c}{A-b} = c \sum_{k=A-\overline{b}}^{A-1} \frac{1}{k} = c \left(\sum_{k=1}^{A-1} \frac{1}{k} - \sum_{k=1}^{A-\overline{b}-1} \frac{1}{k} \right).$$

For the two harmonic series in the brackets employ the Euler equation for the sum $H_n = \sum_{k=1}^{n} \frac{1}{k}$ of the first n summands of a harmonic series, $H_n \approx \ln n + \gamma + \varepsilon_n$, where

$\gamma = .577$ is the Euler-Mascheroni constant, and ε_n is the error term converging to zero with large n. With this in mind obtain[20]

$$\sum_{b=1}^{\bar{b}} \frac{c}{A-b} = c\left(\ln\left(A-1\right) + \varepsilon_{A-1} - \ln\left(A-\bar{b}-1\right) - \varepsilon_{A-\bar{b}-1}\right) = 1 + \left(\varepsilon_{A-1} - \varepsilon_{A-\bar{b}-1}\right),$$

where $\varepsilon_{A-1} - \varepsilon_{A-\bar{b}-1}$ converges to zero (as a sum of two converging to zero terms) for large A and $A-\bar{b}$, which yields $\frac{\bar{b}}{A-1} \approx 1 - e^{-\frac{1}{c}}$. From here, $\bar{b}$ is at most 63.21% of $A-1$ (the value of $1-e^{-\frac{1}{c}}$ with $c=1$, strictly decreasing in c), hence for large A the difference $A-\bar{b}$ is also large indeed. If there is no such $\bar{b}$ that the strict equality $\sum_{b=1}^{\bar{b}} w_b = 1$ holds, one needs to consider $\sum_{b=1}^{\bar{b}} w_b \leq 1$ and $\sum_{b=1}^{\bar{b}+1} w_b \geq 1$, which by the same reasoning as above yields $\frac{\bar{b}}{A-1} \lesssim 1 - e^{-\frac{1}{c}}$ and $\frac{\bar{b}+1}{A-1} \gtrsim 1 - e^{-\frac{1}{c}}$, yielding $1 - e^{-\frac{1}{c}} \gtrsim \frac{\bar{b}}{A-1} \gtrsim 1 - e^{-\frac{1}{c}} - \frac{1}{A-1}$. With large A the last summand is negligible, leading to the same expression as above. ■

Proof of Proposition 6

Proof. For any strategy s such that $s(b) = 1$, win probability of bid b is

$$\begin{aligned} w_b(s,\pi) &= \sum_{\mathbf{s}_{-i}\in S^{N-1}} \pi(\mathbf{s}_{-i}) \prod_{l=1}^{b-1} \left(1 - 1_l(s,\mathbf{s}_{-i})\right) 1_b(s,\mathbf{s}_{-i}) = \\ &= \sum_{\mathbf{s}_{-i}\in S^{N-1}} \pi(\mathbf{s}_{-i}) 1_b(s,\mathbf{s}_{-i}) - \sum_{\mathbf{s}_{-i}\in S^{N-1}} \pi(\mathbf{s}_{-i}) \left(1 - \prod_{l=1}^{b-1}\left(1 - 1_l(s,\mathbf{s}_{-i})\right)\right) 1_b(s,\mathbf{s}_{-i}). \end{aligned}$$

By induction we can show that $1 - \prod_{l=1}^{b-1}(1-1_l) = \sum_{k=1}^{b-1}\prod_{l=1}^{k-1}(1-1_l)1_k$. This is obvious for $b=2$. Assume it holds for $b=n$ and show for $b=n+1 : 1 - \prod_{l=1}^{n}(1-1_l) = 1 - (1-1_n)\prod_{l=1}^{n-1}(1-1_l) = 1 - \prod_{l=1}^{n-1}(1-1_l) + \prod_{l=1}^{n-1}(1-1_l)1_n$ and the rest is straightforward.

Moreover, $s(b) = 1$ implies $\sum_{\mathbf{s}_{-i}\in S^{N-1}} \pi(\mathbf{s}_{-i}) 1_b(s,\mathbf{s}_{-i}) = (1-q_b)^{N-1}$ as it is the probability that no rival of i plays any strategy containing bid b.

[20] The same can be obtained by approximating the series with an integral $\int_1^{\bar{b}+1} \frac{c}{A-b} db = c \ln \frac{A-1}{A-\bar{b}-1}$, again, valid only for large enough $\bar{b}$ so that $db = 1$ can be regarded as a small enough increment.

We can thus write

$$w_b(s,\pi) = (1-q_b)^{N-1} - \sum_{k=1}^{b-1} \sum_{\mathbf{s}_{-i}\in S^{N-1}} \pi(\mathbf{s}_{-i}) \prod_{l=1}^{k-1} (1 - 1_l(s,\mathbf{s}_{-i}))\, 1_k(s,\mathbf{s}_{-i})\, 1_b(s,\mathbf{s}_{-i}).$$

Note that $1_b(s,\mathbf{s}_{-i}) \leq 1$ in any strategy combination $(s,\mathbf{s}_{-i})$ and hence[21]

$$\begin{aligned} w_b(s,\pi) &\geq (1-q_b)^{N-1} - \sum_{k=1}^{b-1} \sum_{\mathbf{s}_{-i}\in S^{N-1}} \pi(\mathbf{s}_{-i}) \prod_{l=1}^{k-1} (1 - 1_l(s,\mathbf{s}_{-i}))\, 1_k(s,\mathbf{s}_{-i}) \\ &= (1-q_b)^{N-1} - \sum_{k=1}^{b-1} w_k(s,\pi), \end{aligned}$$

which implies $\sum_{k=1}^{b} w_k(s,\pi) \geq (1-q_b)^{N-1}$. The right-hand side does not depend on s, and the inequality holds for any $s \in S$. By proposition 4, if π is equilibrium and $s \in \operatorname{supp}\pi$ then $s(k) = 1$ implies $w_k(s,\pi) = \frac{c}{A-k}$. Now consider a "block" strategy s', in which $s'(k) = 1$, for all $k \leq b$. Note that only player i can win with bids $k \leq b$ (if any of them wins). One cannot guarantee that $s' \in \operatorname{supp}\pi$, therefore $w_k(s',\pi) \leq \frac{c}{A-k}$ for any $k \leq b$: a suboptimal strategy cannot deliver a higher probability of winning than an optimal one. As a result,

$$\sum_{k=1}^{b} \frac{c}{A-k} \geq \sum_{k=1}^{b} w_k(s',\pi) \geq (1-q_b)^{N-1},$$

yielding $q_b \geq \sqrt[N-1]{\sum_{k=1}^{b} \frac{c}{A-k}}$.

By using $w_b(s,\pi) \leq (1-q_b)^{N-1}$ one straightforwardly obtains $q_b \leq \sqrt[N-1]{\frac{c}{A-b}}$. ■

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[21] In Eichberger and Vinogradov (2008) only block strategies are allowed, hence for any strategy s that contains bid b and therefore also contains bid $k \leq b$ (as bids are placed jointly) $1_k(s,\mathbf{s}_{-i}) = 1$ implies $1_b(s,\mathbf{s}_{-i}) = 1$ (if no rival plays k, then no rival also plays $b \geq k$). The inequality in this proof thus turns into an equality.

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Highlights:

1. In LUPAs only the lowest bid placed by only one participant wins. The paper characterizes equilibrium in a LUPA in terms of bidding behavior of players.
2. In equilibrium bidders place on average more than one bid.
3. About 5% of bidders place bids above the theoretical upper bound.
4. Players seem to place lower bids more frequently than theoretically predicted.