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Forward and Inverse Kinematics Models for a 5-dof Pioneer 2 Robot Arm

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Abstract**:** Manipulator control is one of the main research areas in robotics, requiring in the first instance the manipulator model. Using the Denavit-Hartenberg methodology this paper develops both forward and inverse kinematics models for a 5-dof Pioneer 2 robot arm. This paper also includes background information about robot arms, especially the Pioneer 2 robot arm, and discusses some implementation issues.

I. INTRODUCTION

In the development of manipulator controllers, the first step is to build manipulator models: both kinematics and dynamics models. The kinematics model relates the position of the arm end-effector to the joint variables, whereas the dynamics model relates the motor torques with those joint variables. There are forward and inverse kinematics models as well as forward and inverse dynamics models. This report concentrates only on the development of the kinematics models for a 5-dof Pioneer 2 robot arm.

The report is composed of the following sections: section II introduces some basic concepts in order to understand section III which describes the development of the forward and inverse kinematics models using the Denavit-Hartenberg methodology. Section IV discusses some implementation issues.

II. BASIC CONCEPTS

There exist various definitions of a robot and a robot arm. The most accepted one was given by the Robot Institute of America:

> *"A programmable multi-function manipulator designed to move material, parts, or specialized devices through variable programmable motions for the performance of a variety of tasks." [McKerrow; 1991:8]*

There is, however, according to the French Normalization Association (AFNOR), a difference between a robot arm and a robot. In fact, a robot may, or may not have, an arm but another kind of end-effector. For the AFNOR, a robot arm is defined in terms of manipulators as follows:

> *"A manipulator is a mechanism generally composed of a series of links, jointed between them, which aims to grasp and move objects. It is multifunctional and it can be governed directly by a human operator or through a logic device." [Barrientos et al; 1997:10]*

and a robot is defined as:

> *"An automatic servo controlled manipulator, reprogrammable and polyvalent capable of positioning and orientating material, parts, or special devices through variable reprogrammable motions/trajectories for the performance of a variety of tasks." [Barrientos et al; 1997:10]*

Informally, a robot can be regarded as any device capable of sensing and reacting to its environment by automatic means; and a robot arm can be seen as a part of a robot or another kind of robot. Due to the previous definitions, a robot arm can be conceptualised as:

"A mechanism generally composed of jointed links. It can be controlled directly by a human operator or through an electronic, electrical or logical device".

2.1 Robotic Manipulator Elements

To properly understand what needs to be controlled, the morphology of a robotic system should be described. Although the mechanical, electrical and computational structure of robots can vary, most have the following four major components in common: (1) a manipulator or arm (the mechanical unit), (2) one or more sensors (the sensorial system), (3) a controller (the 'brain'), and (4) a power supply. Some configurations of robots may include in the mechanical unit end-effectors, which basically are tools to perform specific tasks.

2.1.1 Mechanical unit

A robotic manipulator is composed of links connected by joints to form an open-loop kinematic chain that permits a relative movement between two consecutive links. The movement of each joint can be a displacement, a rotation, or a combination of both. From the six possible movements (spherical, planar, screw, prismatic, rotational, and cylindrical) only three of them are normally used: prismatic, rotational, and spherical. The independent movement of a joint with respect to another is called a degree of freedom (dof), thus the number of joints gives the degrees of freedom of a robot. The use of different combination of joints in a robot gives different configurations. Figure 2.1 shows the most common configurations [Barrientos et al; 1997:18].

The manipulator defined by the joint-link structure generally contains three main structural elements: the arm, the wrist, and the hand that is sometimes called end-effector. These devices are referred to as actuators and may be pneumatic, hydraulic or electric in nature. They are invariable coupled to the various mechanical links or joints (axes) of the arm either directly or indirectly. In order to position and orientate an object in any place inside a workspace six parameters need to be defined: three for the position and three for the orientation. Therefore, 6-dof are required in general. However, many industrial robots have only 4-dof or 5-dof because these dof are enough to perform the tasks they were designed for.

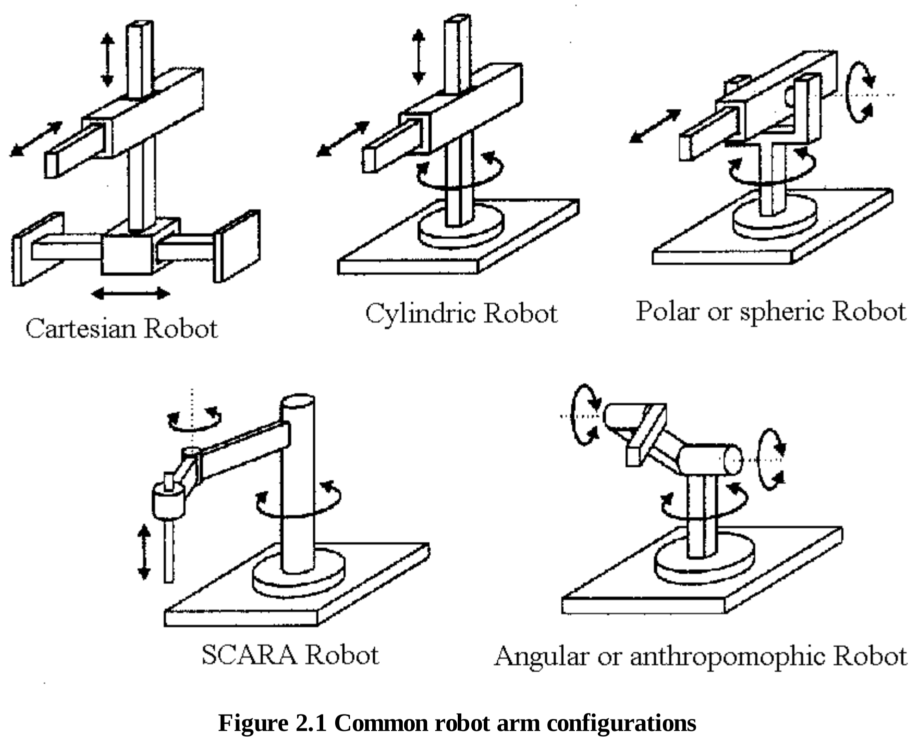

Figure 2.1 Common robot arm configurations

2.1.2 Sensorial unit

For a robot to perform its task precisely, quickly and intelligently, knowledge of its internal and external state is required. This information is obtained from sensors; internal sensors give information about the position, velocity of the joints while external sensors gather information from the environment. Position sensors, such as encoders, and velocity sensors, tachometers, are the most common used sensors, whereas ultrasound, infrared sensors, and laser scanners give information about the position of objects in the environment nearby the robot.

2.1.3 Control unit

This unit orchestrates the other mechanisms to perform the specified task. Robot controllers generally perform three functions: a) Initiate and terminate the motion of individual components of the manipulator in a desired sequence and at specified points; b) Store position and sequence data in their memory; and c) Permit the robot to be interfaced to the ‘outside’ world via sensors mounted in the area where work is being performed. In order to carry out these tasks, controllers must perform the necessary arithmetic computations for determining the correct manipulator path, speed, and position. They must also send signals to the joint-actuating devices and utilize the information provided by the sensors. Finally, they must permit communication between peripheral devices and the manipulator.

2.1.4 Power supply

The purpose of this component of the robot is to provide the necessary energy to the manipulator’s actuators. It can take the form of a power amplifier in the case of servomotor-actuated systems.

2.2 Spatial Localisation

Manipulation is the skilful handling and treating of objects: picking them up, moving them, fixing them to one another, and working on them with tools [McKerrow; 1991:132]. The manipulation of objects carried out by a robot arm implies the spatial movement of its extreme element. In order to move the end-effector so that it gets near the object, it is necessary to know the position and orientation of the object with respect to the robot’s base. This will allow programming a method of specifying where the object is relative to the robot’s gripper and a way of controlling the motion of the gripper.

Controlling the motion of an end-effector is complicated due to the fact that there may be several arm configurations that will place it on the object. On the other hand, a joint in a robot arm usually has one degree of freedom, but it can move over a much greater range than human joints can. An end-effector location can be achieved with several configurations, thus a method for choosing the optimal one is needed. Orientations from which a hand can approach an object depend not only on the object but also on the environment, e.g. the presence of obstacles. The number of available paths or tracks is limited by restrictions imposed in the environment. Moreover, if the object is moving, not only its location but also a possible future position has to be computed so that the manipulator would be directed to that estimated position.

Commonly, a robot computes the position and orientation of the end-effector using the kinematics model of its arm. Thus, mathematical tools are needed to compute the position and orientation of not only the objects to manipulate but also of the end-effector. Besides, these tools should be powerful enough to allow obtaining easily spatial relationship among distinct objects and especially between objects and the manipulator.

A description of various coordinates is given in this section. Also, a description of some methods to transform the position and orientation is presented. The aim is to answer the question: how to locate the end effector of a robot arm?. The following sub-sections come from [Barrientos et al; 1997] and [McKerrow; 1991].

2.2.1 Spatial representation

The spatial representation of an end-effector or a joint consists of two elements: its position and orientation. The first one tells the position of an object in an n-dimensional space and the orientation gives the rotation of the object with respect to a reference system.

a) Position

The representation of an object's position depends on the dimension of the space. The common systems to represent the position of an object in a two-dimensional space are the Cartesian and polar coordinates; and for a three-dimensional space are Cartesian, cylindrical and spherical coordinates, which are described as follows:

- Cartesian coordinates. In $\Re^2$ the position of an object **p** is expressed as **p**(x, y), where x is the displacement from the origin on the X-axis, and y is the displacement from the origin on the Y-axis. In $\Re^3$ the position of an object **p** is expressed as **p**(x, y, z), where similar to $\Re^2$, x and y are displacements from the origin on the X- and Y-axis, and z is the displacement from the origin on the Z-axis. Both representations are depicted in Figure 2.2.

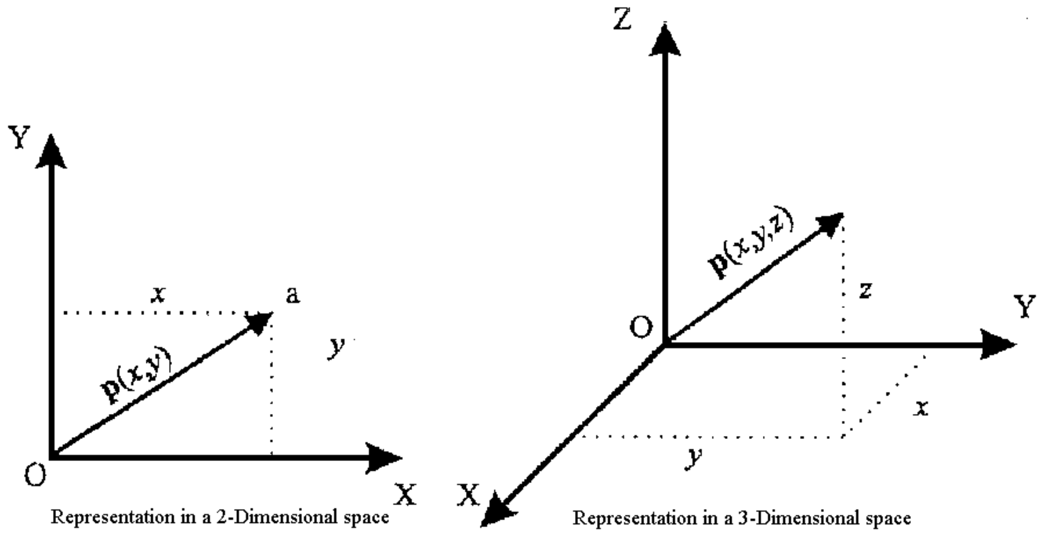

Figure 2.2 Vector representations in Cartesian coordinates in $\Re^2$ and **$\Re^3$**

- Polar coordinates. In $\Re^2$ the position of an object **p** is expressed as **p**(r, θ), where r is the distance from the origin to the point **p**, and θ is the angle that the vector **p** forms with the X-axis, as depicted in the following figure.

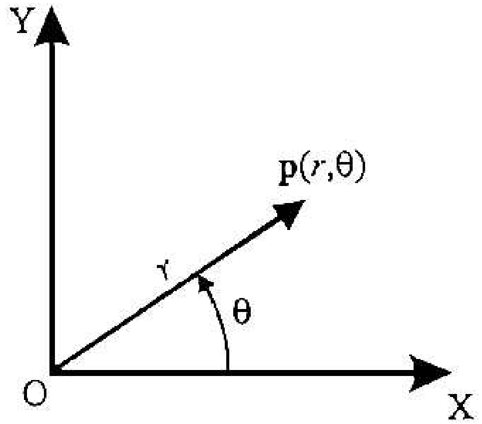

Figure 2.3 Representation of polar coordinates

- Cylindrical coordinates. In $\Re^3$ the position of an object **p** is expressed as **p**(r, θ, z), where r and θ are the distance from the origin to the point **p** and the angle that the vector forms with the X-axis, respectively; and z is the projection of the vector **p** over the Z-axis, as shown in Figure 2.4.

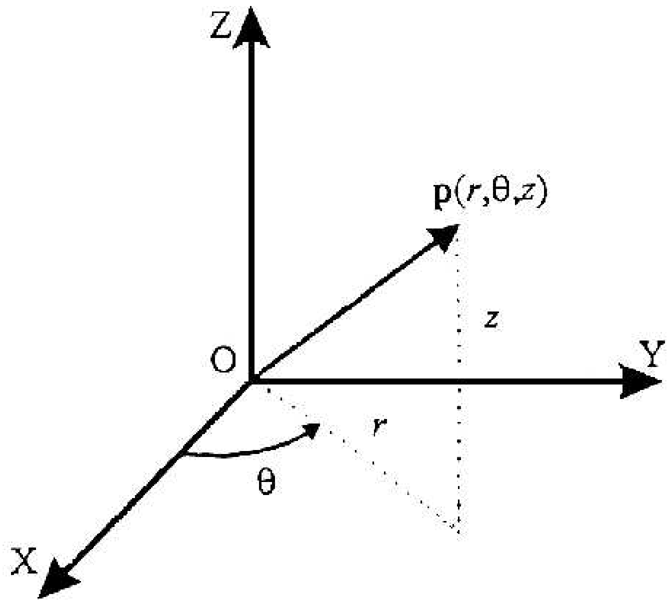

Figure 2.4 Representation of cylindrical coordinates

- Spherical coordinates. As shown in Figure 2.5, the position of an object **p** is expressed as **p**(r, θ, ϕ) where r is the distance from the origin to the point **p**, θ is the angle of the projection of **p** over the plane OXY, and ϕ is the angle that the vector **p** forms with the Z-axis.

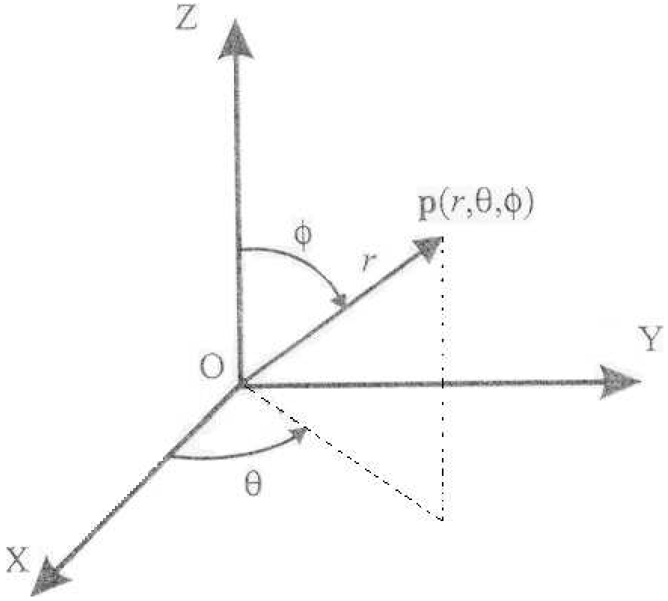

Figure 2.5 Representation of spherical coordinates

b) Orientation

A solid object is located in a space not only through its position but also through its orientation with respect to a reference system. In $\Re^2$ a rotation with respect to the X-axis can be made whereas in $\Re^3$ a rotation with respect to X-axis, a rotation with respect to Y-axis and a rotation with respect to Z-axis can be performed. The common methods to represent orientation include rotational matrices, Euler's angles, rotational pair, and quaternion, which are described as follows.

- Rotational matrices. Rotational matrices are the most extended method for describing orientation due to the utility of the algebra of matrices. Consider two reference systems, a fixed reference system OXY and the object's system OUV with the same origin. Vectors $\mathbf{i}_x$ and $\mathbf{j}_y$ are unitary vectors of the fixed reference system, whereas $\mathbf{i}_u$ and $\mathbf{j}_v$ are the respective unitary vectors of the object's reference system, as shown in Figure 2.6. A vector **p** can be represented as

$$\begin{aligned} \mathbf{p}_{xy} &= \left[p_x, p_y \right]^t = p_x * i_x + p_y * j_y \\ \mathbf{p}_{uv} &= \left[p_u, p_v \right]^t = p_u * i_u + p_v * j_v \end{aligned} \tag{2.1}$$

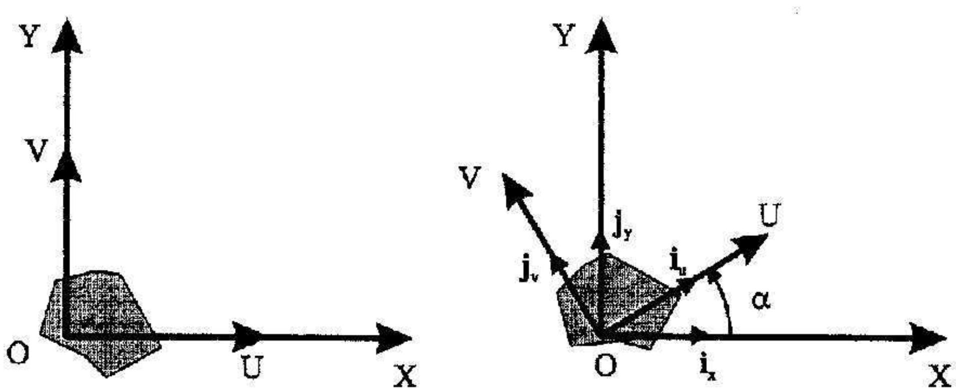

Figure 2.6 Orientation of an OUV system with respect to an OXY system

By applying linear transformation, the following equivalence can be found:

$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} = \mathbf{R} \begin{bmatrix} p_u \\ p_v \end{bmatrix}, \quad \text{where } \mathbf{R} = \begin{bmatrix} i_x i_u & i_x j_v \\ j_y i_u & j_y j_u \end{bmatrix} \tag{2.2}$$

which is the rotational matrix defining the orientation of the OUV system with respect to the fixed OXY reference system, and is used to transform the coordinates of a vector in the object's reference system to the coordinates of another system, such as the fixed reference system OXY. The orientation in $\Re^2$ is given by a unique parameter, α, the angle of rotation over OXY. The rotational matrix is given by

$$\mathbf{R} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \tag{2.3}$$

In $\Re^3$, similarly to $\Re^2$, the orientation of an object **p** can be expressed as

$$\begin{aligned} \mathbf{p}_{uvw} &= [p_u, p_v, p_w]^t = p_u * i_u + p_v * j_v + p_w * k_w \\ \mathbf{p}_{xyz} &= \left[p_x, p_y, p_z\right]^t = p_x * i_x + p_y * j_y + p_z * k_z \end{aligned} \tag{2.4}$$

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \mathbf{R} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} \tag{2.5}$$

When the rotation of an object's system OUVW is over the OXYZ system with the U-axis coinciding with the X-axis, the rotational matrix will be

$$\mathbf{R}(X,\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \tag{2.6}$$

If the V-axis coincides with the Y-axis after the rotation, the rotational matrix will be

$$\mathbf{R}(Y,\gamma) = \begin{bmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{bmatrix} \tag{2.7}$$

If the W-axis coincides with the Z-axis after the rotation, the rotational matrix will be

$$\mathbf{R}(Z,\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.8}$$

The above matrices are called basic rotational matrices for a three-dimensional spatial system.

- Euler's angles. The rotation of the OUVW reference system with respect to the fixed OXYZ reference system can be defined through three rotational angles α, γ, ϕ, which are called Euler's angles. It is necessary to know not only the values of the angles but also the rotational axes. There are 24 defined possibilities for rotating a coordinate system, among which the following are the most common:

  - Euler's angles ZYX: It is the result of applying the consecutive rotations about *z-axis*, *y-axis*, and *x-axis*, respectively.
  - Roll, pitch and yaw. It is the application of the consecutive rotations about *x-axis*, the yaw; *y-axis*, the pitch; and *z-axis*, the roll.

- Rotational pair. Another representation for the orientation of an OUVW coordinate system with respect to a reference OXYZ is the rotational pair, which is defined by a vector $\mathbf{K}=(k_x, k_y, k_z)$ and an angle θ. The application of a rotational pair that will rotate a vector $\mathbf{p}$, an angle θ over $\mathbf{K}$ is expressed by:

$$Rot(\mathbf{K},\theta)\mathbf{p} = \mathbf{p}\cos\theta - (\mathbf{K}\times\mathbf{p})\sin\theta + \mathbf{K}(\mathbf{K}*\mathbf{p})(1-\cos\theta) \tag{2.9}$$

- Quaternion. Quaternion is a versatile mathematical tool to work with rotations and orientations. A quaternion is formed by four elements (q_0, q_1, q_2, q_3), which represent the coordinates of a base quaternion {e, i, j, k}. The element e:q_0 is a scalar value and (i, j, k) is a vector. The equivalence between the rotational pair and the quaternion is:

$$\begin{aligned} \mathbf{Q} &= [q_o, q_1, q_2, q_3] = [s, \mathbf{v}] \\ \mathbf{Q} &= Rot(\mathbf{K},\theta) = (\cos\frac{\theta}{2}, \mathbf{K}\sin\frac{\theta}{2}) \end{aligned} \tag{2.10}$$

2.2.2 Transformation

The position and orientation of an arm's end-effector with respect to a fixed reference system are needed in a unique representation, and the above methods are not sufficient for this end. In order to get a conjunct representation of position and orientation, homogeneous coordinates (HC) can be used. The representation through homogeneous coordinates for the localisation of solid objects in an n-dimensional space can be expressed through the coordinates of a $(n+1)$-dimensional space. This means that an n-dimensional space can be represented with HC by $(n+1)$ dimensions, for example, a vector $\mathbf{p}(x, y, z)$ will be expressed by $\mathbf{p}(w_x, w_y, w_z, w)$ where w has an arbitrary value and represents a scale value. Generally, a vector $p = ai + bj + ck$, where i, j, k are unitary vectors of X, Y, and Z axes, is represented in homogeneous coordinates by

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} aw \\ bw \\ cw \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ 1 \end{bmatrix} \text{ (if } w=1\text{)} \tag{2.11}$$

From the definition of homogeneous coordinates, the concept of homogeneous matrix arises. The homogeneous transformation matrix, $\mathbf{T}$, is a 4x4 matrix that represents the transformation of a vector of homogeneous coordinates from one system to another reference system. Mathematically, this is expressed as:

$$\mathbf{T} = \begin{bmatrix} \mathbf{R}_{3\times3} & \mathbf{P}_{3\times1} \\ \mathbf{f}_{1\times3} & \mathbf{W}_{1\times1} \end{bmatrix} = \begin{bmatrix} \text{Rotation} & \text{Translation} \\ \text{Perspective} & \text{Scale} \end{bmatrix} \tag{2.12}$$

In robotics only the rotational matrix **R**, and the translation vector **p** are relevant to the control and manipulation, and the vectors of perspective and scale are set as **f**=**0**, **W**=1.

Another expression for **T** is:

$$\mathbf{T} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.13}$$

where (**n**, **o**, **a**) is an orthonormal triplet representing the orientation and **p** is the vector reflecting the position.

A complex homogeneous matrix of transformation can be formed by consecutive applications of simple transformations:

$$T = T_1 T_2 \cdots T_N \tag{2.14}$$

2.2.3 Comparison among methods

All the above spatial representation methods are useful. Depending on the application one method would be more suitable than others. The criterions of selection are based on a) the capacity of joint representation of position and orientation, b) the representation of position and orientation of a rotated and translated system OUVW with respect to a fixed reference system OXYZ, c) transformation of a vector expressed in coordinates with respect to a system OUVW to a vector expressed in coordinates with respect to a reference system OXYZ, and d) the rotation and translation of a vector with respect to a fixed reference system OXYZ.

The homogeneous matrix of transformation meets all the mentioned criterions. Besides, it is easy to manipulate the matrix, though its redundancy. It will be used in the robot arm kinematics.

2.3 Kinematics of a Robot Arm

Kinematics is the study of motion with respect to a reference system without regard to forces or other factors that influence the motion. The analytical description of the robot arm's spatial motion as a function of time is the main concern of the kinematics, particularly the relation between the position and orientation of the arm's extreme with the values of the joints' coordinates. The kinematics is also concerned with the relation between the velocities of the motion of the joints and those from the arm's extreme. There are two fundamental problems to solve by kinematics: direct and inverse kinematics.

2.3.1 Direct kinematics problem

A robot arm, as mentioned before, consists of a group of "rigid bodies" called links connected together by joints. The links and joints of the manipulator form a kinematics chain which is open at one end and connected to ground of the other. The end-effector, or hand, or gripper, is connected to the free end, the arm's extreme, and the control objective of the robot system is to position the end-effector at a desired location.

The direct kinematics problem (DKP) is to determine the position and orientation of the arm's extreme, with respect to a fixed reference coordinate system, given its joints' angles. That is to find a transformation matrix **T** that relates the position and orientation of the arm's extreme with joints' coordinates, which can be obtained by applying homogeneous matrices of transformations and the conventional numerical algorithm developed by Denavit and Hartenberg (D-H). According to the D-H representation, when choosing appropriately a coordinates system associated to each link, it will be possible to transform from one link to the next through the application of four basic transformations. The transformations are rotations and translations, which relate the coordinates system of the *k-th* link with the coordinates system of the *(k-1)-th* link.

2.3.2 Inverse kinematics problem

The inverse kinematics problem (IKP) can be stated as: Given a required location of the arm's extreme, find the values of the joints' variables that will achieve the required position and orientation. It can be mathematically expressed as follows:

$$\begin{aligned} &\theta_k = f_k(x, y, z, \alpha, \gamma, \phi), \\ &k = 1, \cdots, N \end{aligned} \tag{2.15}$$

where θ_k are joint angles and ($x, y, z, \alpha, \gamma, \phi$) represents the position and orientation.

There are numerous techniques and approaches for finding the inverse solution. A common attribute is that the difficulty increases with the complexity of the kinematics chain. An optimal situation is when a closed arm solution is founded. However, there exist various problems in the IKP, which will be discussed in the derivation of the IKP for the Pioneer 2 robot arm.

III. THE PIONEER 2 ROBOT ARM AND ITS KINEMATICS

In this section, the Pioneer 2 robot arm (P2Arm) is introduced and its kinematics models will be developed using the Denavit-Hartenberg methodology mentioned above.

3.1 P2Arm Specification

The study object is a 5-dof robot arm, which in a first stage will be used to formulate a control scheme for trajectory tracking under the assumption that the arm is in a fixed position. Then, in a second stage, the mobility will be considered, thus the pioneer mobile robot will be used as well. Kinematics and dynamics models of both platforms will be derived.

The specification of the Pioneer 2 Arm is described in Table 3.1.

Table 3.1 Pioneer 2 Arm Specification

Pioneer 2 Arm

Description:	The Pioneer 2 Arm is a relatively low-cost arm for use in research and teaching. It's a 5 degree-of-freedom robotic arm that holds a gripper with foam-lined fingers for firm grasp. Driven by six, reversible 5v DC open-loop servomotors, the arm can reach up to 50 cm from the centre of its rotating base to the tip of its closed fingers. This allows P2-DXE's and P2-AT's to pick up objects from the floor.
Key Features	Nose mounted 5-axis arm allows grippers to handle objects 1 to 8 cm wide 6 degrees of freedom, including gripper Reach of 50cm RS-232 compatible Compatible with Pioneer P2-DXE, DX or P2-AT. *(P2-AT requires additional BatPak and 3 additional batteries.)*
Arm Specifications	**CONSTRUCTION:** anodized, CNC fabricated, and painted aluminium, and plastic, with foam-covered gripper fingers **MOTION:** 5 dof arm and 1 dof gripper **POWER:** +5 and +12 VDC supplied by Pioneer robot **ARM RANGE:** 50 cm fully extended **GRIPPER RANGE:** Grippers part to 5 cm **PAYLOAD:** 150 gm (5 oz.) lift capability **SPEED:** 1 second from fully extended to fully relaxed position

The measures of the P2Arm links are shown in Figure 3.1, which also shows the status of the arm when it is fully extended.

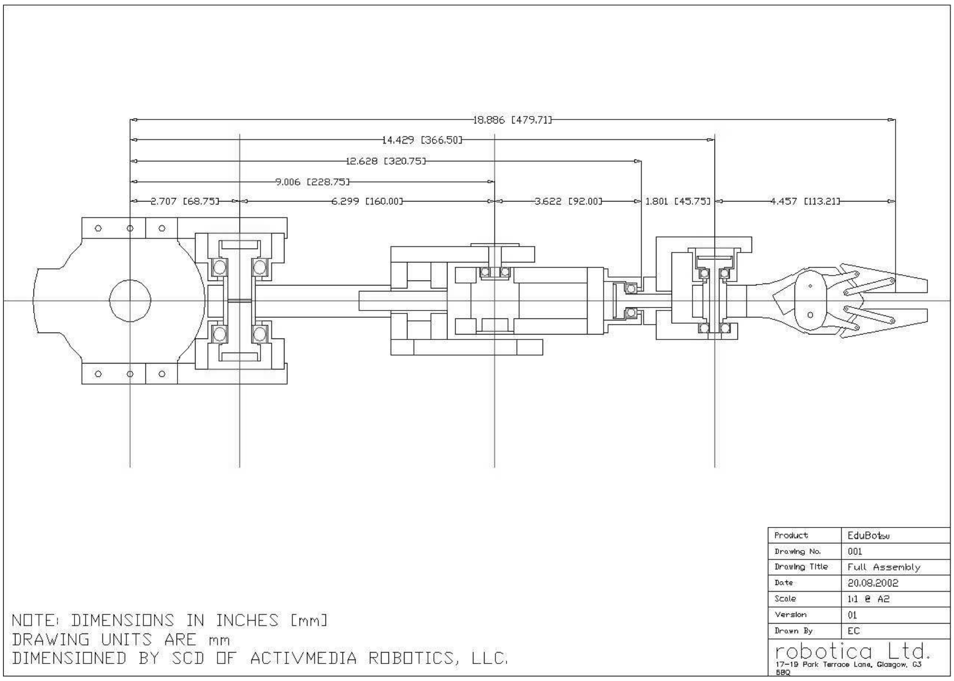

Figure 3.1 Measures for the Pioneer 2 Arm

3.2 P2Arm Kinematics Models

3.2.1 Forward kinematics model

To derive the forward kinematics, according to the Denavit-Hartenberg (DH) convention, the following steps should be done:

a) Obtaining the DH parameters:

There is a theory behind the DH parameters. These parameters need to be obtained by positioning the arm in its zero position that is when the joints values are zero; then, for each joint a reference frame is assigned according to its link type and joint type.

There are two types of joints: revolute and prismatic. Figure 3.2 shows both types of joints with some variants. The rotational angle about the joint axis is denoted as ϕ, corresponding to the joint variable θ in the Denavit-Hartenberg notation.

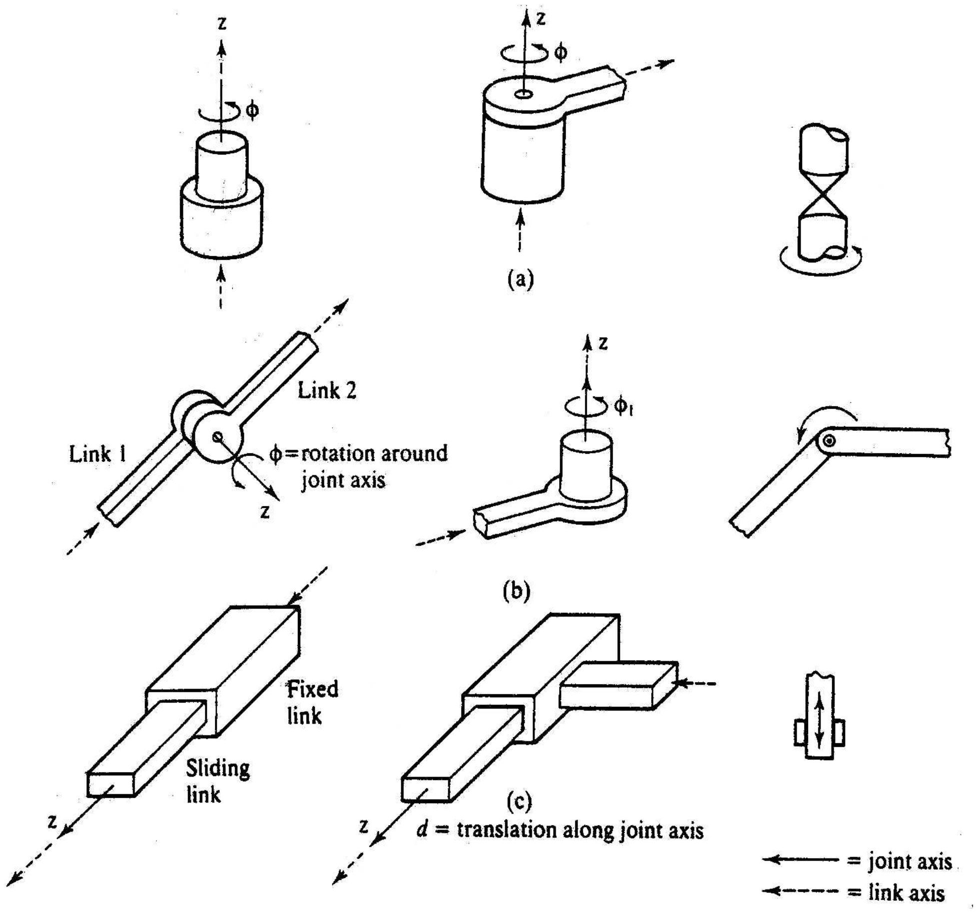

Figure 3.2 Types of joints

There are several types of links. Shown in Figures. 3.3-3.7 are some types related to the Pioneer 2 arm.

Type I: This is the simplest link. It has two parallel revolute joints with no twist between the joint axes, which are parallel and separated by a distance a_n, known as the length of the link. The joint variable is the rotational angle θ_{n+1}. The z_{n-1}-axis is assigned to be coincident with the axis of the joint, and the x_{n-1}-axis to be coincident with the centre line of the link; the y_{n-1}-axis is found by the right-hand rule.

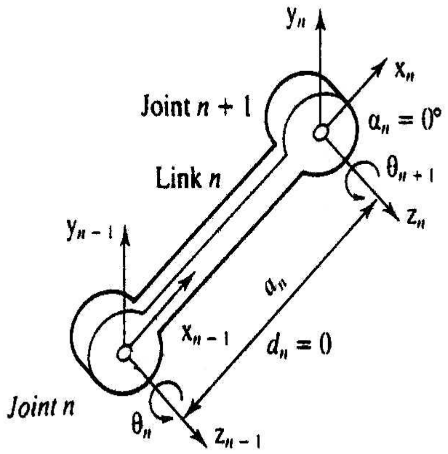

Figure 3.3 Link type I

Type II: This type results from type I by twisting the link about its centre line (axis x_{n-1}), by an angle α_n.

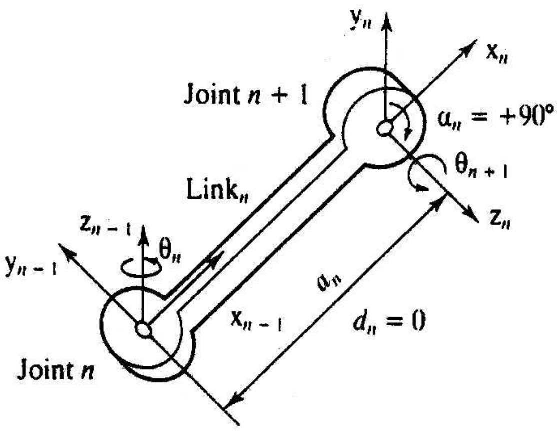

Figure 3.4 Link type II

Type III: This link is shown in Figure 3.5. In this configuration, joint axes intersect, making the length of the link be zero since the distance between the joint axes (z_{n-1} and z_n) is zero. However, there is a translation of distance d_n between the two joints, measured between the common normals to the joints axes (usually the x_{n-1} and x_n axes).

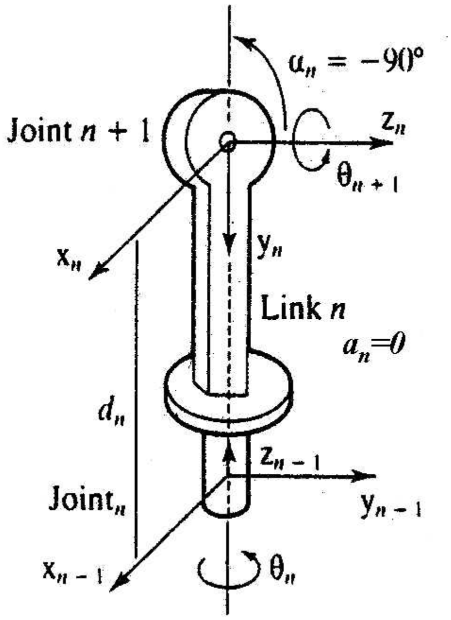

Figure 3.5 Link type III

Type IV: This link is similar to type III, but it has the joints the other way around, resulting in different values for the link parameters. First of all, the origins of the axes for the two joints coincide making the length of the link and the distance between the links zero. The physical distance between the joints is included in the next link.

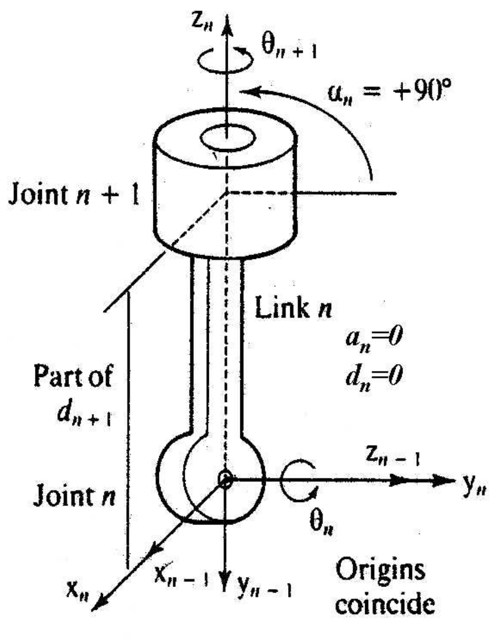

Figure 3.6 Link type IV

Type V: This link consists of a revolute joint whose axis is orthogonal to the link. In this link joint axes (z_{n-1} and z_n) intersect. This type has the angle θ_{n+1} the joint variable.

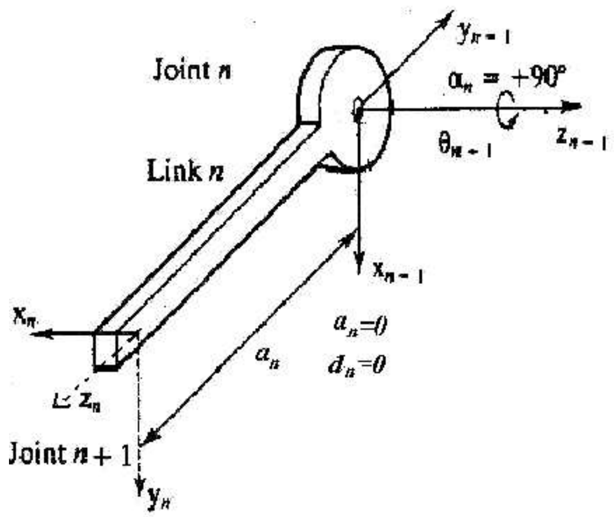

Figure 3.7 Link type V

The reference frame to each joint is applied by using the following guidelines:

i) After positioning the arm in its zero position, starting at the base, number the joints from 0 to *N*.
ii) The base coordinate frame is assigned with its axes parallel to the world coordinate frame.
iii) The origin of the frame is located at the intersection of the common normal (to the joints axes), and the axis of the distal joint. If the axes of the joints are parallel, then the position of the origin is chosen to make the distance between the links (d_n or a_n) zero. If the joint axes intersect, the origin is placed in the intersection of the axes.
iv) The *z-axis* is coincident with the joint axis. For a revolute joint, the direction of the *z-axis* is determined from the positive direction of rotation around the *z-axis*.
v) The *x-axis* is parallel to the common normal between the joint axes of the link. In the case of parallel axes, the *x-axis* is coincident with the centre line of the link. If the axes intersect, there is no unique common normal, and the axis is parallel, or anti-parallel, to the vector cross product of the *z-axes* for the preceding link and this link (z_{n-1} and z_n). In many cases this results in the *x-axis* being in the same direction as the *x-axis* for the previous link. At this point, it is necessary to check that the selected zero position is consistent with the allocation of the *x-axis*.
vi) The direction of the *y-axis* can be found by using the right-hand rule.
vii) A coordinate frame is attached to the end of the final link (*N*), usually within the end effector or tool. If the robot has an articulated hand, or changes end effector regularly, it may be necessary to locate this coordinate frame at the tool plate, and have a separate hand transformation.

The frame assignation process leads to the frames shown in Figure 3.8.

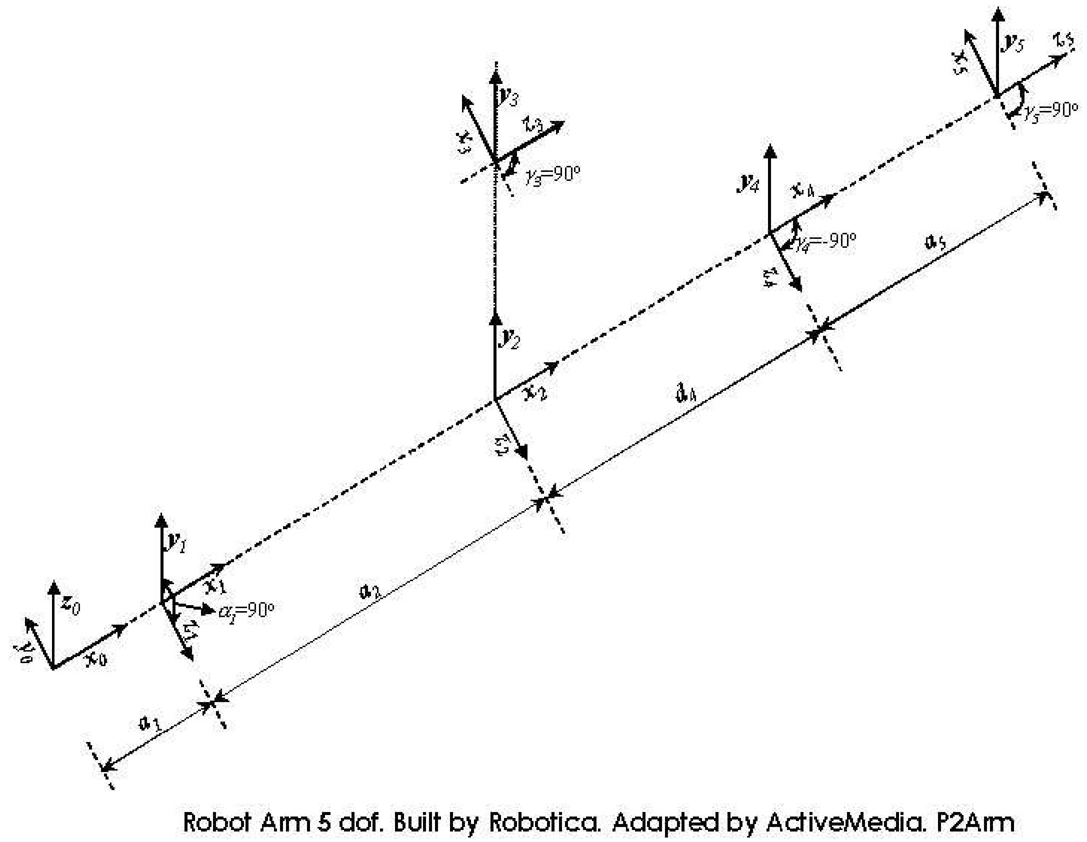

Figure 3.8 Frame assignment

After the coordinate frame assignation, the DH parameters are defined as follows::

i) θ_{n+1}: Angle of rotation about the z_n-axis.
ii) d_{n+1}: Distance from x_n to x_{n+1} (or y_n to y_{n+1}), along the z_n-axis.
iii) a_{n+1}: Distance from z_n to z_{n+1}, along the x_n-axis (or y_n-axis).
iv) α_{n+1}: Angle of rotation about the x_{n+1}-axis.
v) γ_{n+1}: Angle of rotation about the y_{n+1}-axis.

The following table summarizes the values of the DH parameters for the P2Arm.

Table 3.2 Denavit-Hartenberg parameters for the Pioneer 2 Arm

Link	Joint	Type	θ	d	a	α	γ
1	0-1	II	θ_1	0	a_1=6.875	90°	0°
2	1-2	I	θ_2	0	a_2=16	0	0°
3	2-3	IV	θ_3	0	0	0	90°
4	3-4	III	θ_4	d_4=13.775	0	0	-90°
5	4-5	V	θ_5	0	a_5=11.321	0	90°

b) Obtaining link transformation matrices $^{n}A_{n+1}$ (A matrices)

Based on the DH convention, the matrix of transformation from joint n to joint $n+1$ can be calculated by:

$$
{}^{n}A_{n+1} = \begin{bmatrix} \cos\theta_{n+1}\cos\gamma_{n+1} - \sin\theta_{n+1}\sin\alpha_{n+1}\sin\gamma_{n+1} & -\sin\theta_{n+1}\cos\alpha_{n+1} & \cos\theta_{n+1}\sin\gamma_{n+1} + \sin\theta_{n+1}\sin\alpha_{n+1}\cos\gamma_{n+1} & a_{n+1}\cos\theta_{n+1} \\ \sin\theta_{n+1}\cos\gamma_{n+1} + \cos\theta_{n+1}\sin\alpha_{n+1}\sin\gamma_{n+1} & \cos\theta_{n+1}\cos\alpha_{n+1} & \sin\theta_{n+1}\sin\gamma_{n+1} - \cos\theta_{n+1}\sin\alpha_{n+1}\cos\gamma_{n+1} & a_{n+1}\sin\theta_{n+1} \\ -\cos\alpha_{n+1}\sin\gamma_{n+1} & \sin\alpha_{n+1} & \cos\alpha_{n+1}\cos\gamma_{n+1} & d_{n+1} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

The link transformation matrices (A matrices) of the P2Arm are given in Table 3.3.

Table 3.3 Link transformation matrices

${}^{0}A_1 = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & a_1\cos\theta_1 \\ \sin\theta_1 & 0 & -\cos\theta_1 & a_1\sin\theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	${}^{1}A_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
${}^{2}A_3 = \begin{bmatrix} 0 & -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & \cos\theta_3 & \sin\theta_3 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	${}^{3}A_4 = \begin{bmatrix} 0 & -\sin\theta_4 & -\cos\theta_4 & 0 \\ 0 & \cos\theta_4 & -\sin\theta_4 & 0 \\ 1 & 0 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
${}^{4}A_5 = \begin{bmatrix} 0 & -\sin\theta_5 & \cos\theta_5 & a_5\cos\theta_5 \\ 0 & \cos\theta_5 & \sin\theta_5 & a_5\sin\theta_5 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	

c) Obtaining the manipulator transformation matrix ${}^{0}T_N$ (T matrix)

This is obtained by multiplying all the transformation matrices as follows:

$$
{}^{0}T_5 = {}^{0}A_1 \times {}^{1}A_2 \times {}^{2}A_3 \times {}^{3}A_4 \times {}^{4}A_5 = {}^{R}T_H
$$

By using the following trigonometric identities: $\sin\theta_i\cos\theta_j + \cos\theta_i\sin\theta_j = \sin(\theta_i + \theta_j) = \mathrm{s}_{ij}$, $\cos\theta_i\cos\theta_j - \sin\theta_i\sin\theta_j = \cos(\theta_i + \theta_j) = \mathrm{c}_{ij}$ the following matrix is obtained:

$$
{}^{0}T_5 = \begin{bmatrix} -s_1c_4 - c_1s_{23}s_4 & -c_1c_{23}s_5 + s_1s_4c_5 - c_1s_{23}c_4c_5 & c_1c_{23}c_5 + s_1s_4s_5 - c_1s_{23}c_4s_5 & \begin{matrix} a_5c_1c_{23}c_5 + a_5s_1s_4s_5 - a_5c_1s_{23}c_4s_5 + \\ c_1(d_4c_{23} + a_2c_2 + a_1) \end{matrix} \\ c_1c_4 - s_1s_{23}s_4 & -s_1c_{23}s_5 - c_1s_4c_5 - s_1s_{23}c_4c_5 & s_1c_{23}c_5 - c_1s_4s_5 - s_1s_{23}c_4s_5 & \begin{matrix} a_5s_1c_{23}c_5 - a_5c_1s_4s_5 - a_5s_1s_{23}c_4s_5 + \\ s_1(d_4c_{23} + a_2c_2 + a_1) \end{matrix} \\ c_{23}s_4 & -s_{23}s_5 + c_{23}c_4c_5 & +s_{23}c_5 + c_{23}c_4s_5 & a_5s_{23}c_5 + a_5c_{23}c_4s_5 + d_4s_{23} + a_2s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.1)
$$

d) Calculating the position and orientation of the end-effector

Representing the orientation by $[\mathbf{n}\ \mathbf{o}\ \mathbf{a}]_{3x3}$ and the position by vector **p** and using Euler's angles ZYX convention**,** we have:

$$\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\gamma & \cos\phi\sin\gamma\sin\alpha - \sin\phi\cos\alpha & \cos\phi\sin\gamma\cos\alpha + \sin\phi\sin\alpha & p_x \\ \sin\phi\cos\gamma & \sin\phi\sin\gamma\sin\alpha + \cos\phi\cos\alpha & \sin\phi\sin\gamma\cos\alpha - \cos\phi\sin\alpha & p_y \\ -\sin\gamma & \cos\gamma\sin\alpha & \cos\gamma\cos\alpha & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^0T_5 \quad (3.2)$$

The position and orientation of the end-effector can be calculated as follows:

$$\begin{aligned}
p_x &= a_5c_1c_{23}c_5 + a_5s_1s_4s_5 - a_5c_1s_{23}c_4s_5 + c_1(d_4c_{23} + a_2c_2 + a_1) \\
p_y &= a_5s_1c_{23}c_5 - a_5c_1s_4s_5 - a_5s_1s_{23}c_4s_5 + s_1(d_4c_{23} + a_2c_2 + a_1) \\
p_z &= a_5s_{23}c_5 + a_5c_{23}c_4s_5 + d_4s_{23} + a_2s_2 \\
n_x &= -s_1c_4 - c_1s_{23}s_4 \\
n_y &= c_1c_4 - s_1s_{23}s_4 \\
n_z &= c_{23}s_4 \\
o_x &= -c_1c_{23}s_5 + s_1s_4c_5 - c_1s_{23}c_4c_5 \\
o_y &= -s_1c_{23}s_5 - c_1s_4c_5 - s_1s_{23}c_4c_5 \\
o_z &= -s_{23}s_5 + c_{23}c_4c_5 \\
a_z &= s_{23}c_5 + c_{23}c_4s_5
\end{aligned}$$

$$\text{if}(n_x = 0 \wedge n_y = 0) \Rightarrow \begin{cases} \alpha = \operatorname{atan2}(o_x, o_y) \\ \gamma = 90 \\ \phi = 0 \end{cases}$$

$$\text{if}(n_x \neq 0 \vee n_y \neq 0) \Rightarrow \begin{cases} \alpha = \operatorname{atan2}(o_z, a_z) \\ \phi = \operatorname{atan2}(n_y, n_x) \\ \gamma = \operatorname{atan2}(-n_z, \sqrt{n_x^2 + n_y^2}) \end{cases} \quad (3.3)$$

where p_x, p_y, p_z are the coordinates indicating the spatial position, and α, γ, and ϕ representation the orientation in terms of roll, pitch and yaw respectively; *i.e.*, α is the rotation about the *x*-axis, γ is the rotation about the *y*-axis, and ϕ is the rotation about the *z*-axis.

Note: Another way for finding the manipulator transformation matrix is to consider virtual joints. That is, besides the normal D-H transformation matrices, joints 2, 3 and 4 will have an extra transformation - the rotation about the *y*-axis. The parameters are as follows:

Table 3.4 DH parameters when having virtual joints

Link	Joint	θ	*D*	*a*	α	λ
1	0-1	θ_1	0	a_1=6.875	90^o	0^o
2	1-2	θ_2	0	a_2=16	0	0^o
3'	2-3'	θ_3	0	a_3=9.2	0	0^o
3	3'-3	NA	NA	NA	NA	90^o
4'	3-4'	θ_4	d_4=4.575	0	0	0^o
4	4'-4	NA	NA	NA	NA	-90^o
5'	4-5'	θ_5	0	a_5=11.321	0	0^o
5	5'-5	NA	NA	NA	NA	90^o

In Table 3.4, the pairs of joints 3'-3, 4'-4 and 5'-5 are used to get the real transformation matrix for joints 2-3, 3-4 and 4-5; and they are calculated just by multiplying transformations ${}^{2}A_{3'}{}^{3'}A_3$, ${}^{3}A_{4'}{}^{4'}A_4$ and ${}^{4}A_{5'}{}^{5'}A_5$, respectively. The link transformation matrices for the real links are as follows:

${}^{0}A_1 = \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 & a_1\cos\theta_1 \\ \sin\theta_1 & 0 & -\cos\theta_1 & a_1\sin\theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	${}^{1}A_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
${}^{2}A_3 = \begin{bmatrix} 0 & -\sin\theta_3 & \cos\theta_3 & a_3\cos\theta_3 \\ 0 & \cos\theta_3 & \sin\theta_3 & a_3\sin\theta_3 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	${}^{3}A_4 = \begin{bmatrix} 0 & -\sin\theta_4 & -\cos\theta_4 & 0 \\ 0 & \cos\theta_4 & -\sin\theta_4 & 0 \\ 1 & 0 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
${}^{4}A_5 = \begin{bmatrix} 0 & -\sin\theta_5 & \cos\theta_5 & a_5\cos\theta_5 \\ 0 & \cos\theta_5 & \sin\theta_5 & a_5\sin\theta_5 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	

It is noticed that the orientation matrix [**n o a**] in the manipulator transformation matrices from the two methods are the same, while the position vector **p** has different terms. The terms in the vector **p** calculated by the second method are as follows:

$$\begin{aligned} p_x &= a_5c_1c_{23}c_5 + a_5s_1s_4s_5 - a_5c_1s_{23}c_4s_5 + c_1(d_4c_{23} + a_3c_{23} + a_2c_2 + a_1) \\ p_y &= a_5s_1c_{23}c_5 - a_5c_1s_4s_5 - a_5s_1s_{23}c_4s_5 + s_1(d_4c_{23} + a_3c_{23} + a_2c_2 + a_1) \\ p_z &= a_5s_{23}c_5 + a_5c_{23}c_4s_5 + (d_4 + a_3)s_{23} + a_2s_2 \end{aligned} \tag{3.4}$$

By noticing that d_4 in (3.3) is equivalent to d_4+a_3 in (3.4), we find that the position vectors in (3.3) and (3.4) are equivalent. In the following sections, the notation in (3.3) will be used.

3.2.2 Inverse kinematics

The steps used to derive the inverse kinematics model are as follows:

a) Given a position and orientation (p_x, p_y, p_z, α, γ, ϕ), compute the general transformation matrix as:

$$\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi\cos\gamma & \cos\phi\sin\gamma\sin\alpha - \sin\phi\cos\alpha & \cos\phi\sin\gamma\cos\alpha + \sin\phi\sin\alpha & p_x \\ \sin\phi\cos\gamma & \sin\phi\sin\gamma\sin\alpha + \cos\phi\cos\alpha & \sin\phi\sin\gamma\cos\alpha - \cos\phi\sin\alpha & p_y \\ -\sin\gamma & \cos\gamma\sin\alpha & \cos\gamma\cos\alpha & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

b) Apply the following algorithm, called the inverse kinematics heuristic, which uses the Denavit-Hartenberg representation (DH-parameters).
   i) Equate the general transformation matrix to the manipulator's transformation matrix:

$$\begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{R}T_H \tag{3.6}$$

ii) Look at both matrices for
   1) Elements that contain only one joint variable.
   2) Pairs of elements, which will produce an expression in function of only one joint variable. In particular, look for divisions that result in the tangent function.
   3) Elements, or combinations of elements, which can be simplified using trigonometric identities.

iii) After selecting an element, equate it to the corresponding element in the other matrix to produce an equation. Solve this equation to find a description of one joint variable in terms of the elements of the general transformation matrix.

iv) Repeat step iii) until all the identified elements in step ii) have been used.

v) If any of these solutions suffer from inaccuracy problem, set them aside and look for better solutions.

vi) If there are more joint angles to be found, more equations with one or a couple of joint variables can be obtained by multiplying both sides of (3.6) by the inverse of the A matrix for a certain link.

vii) Repeat steps ii) through vi) until solutions to all the joint variables have been found.

viii) If a suitable solution cannot be found for a joint variable, choose one of those discarded solutions found in step v), taking note of regions where problems may occur.

ix) If a solution cannot be found for a joint variable in terms of the elements of the manipulator's transformation matrix, it may be that the arm cannot achieve the specified position and orientation: the position is outside the manipulator's workspace.

The application of this heuristic implies the calculation the inverse of the A matrices, which are shown in Table 3.5.

Table 3.5 Inverse of link transformation matrices

${}^0A_1^- = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 & -a_1 \\ 0 & 0 & 1 & 0 \\ \sin\theta_1 & -\cos\theta_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	${}^1A_2^- = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 & -a_2 \\ -\sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
${}^2A_3^- = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 & 0 \\ \cos\theta_3 & \sin\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	${}^3A_4^- = \begin{bmatrix} 0 & 0 & 1 & -d_4 \\ -\sin\theta_4 & \cos\theta_4 & 0 & 0 \\ -\cos\theta_4 & -\sin\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
${}^4A_5^- = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -\sin\theta_5 & \cos\theta_5 & 0 & 0 \\ \cos\theta_5 & \sin\theta_5 & 0 & -a_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	

The details for deriving the inverse kinematics of the P2Arm are given in the following. First, by multiplying equation (3.6) by the inverse of ${}^0\mathbf{A}_1$ in both sides, i.e., ${}^0A_1^-[n \quad o \quad a \quad p] = {}^1A_2\,{}^2A_3\,{}^3A_4\,{}^4A_5$, the following equation is obtained:

$$\begin{bmatrix} n_x\cos\theta_1+n_y\sin\theta_1 & o_x\cos\theta_1+o_y\sin\theta_1 & a_x\cos\theta_1+a_y\sin\theta_1 & p_x\cos\theta_1+p_y\sin\theta_1-a_1 \\ n_z & o_z & a_z & p_z \\ n_x\sin\theta_1-n_y\cos\theta_1 & o_x\sin\theta_1-o_y\cos\theta_1 & a_x\sin\theta_1-a_y\cos\theta_1 & p_x\sin\theta_1-p_y\cos\theta_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -s_{23}s_4 & -c_{23}s_5-s_{23}c_4c_5 & c_{23}c_5-s_{23}c_4s_5 & a_5c_{23}c_5-a_5s_{23}c_4s_5+d_4c_{23}+a_2c_2 \\ c_{23}s_4 & -s_{23}s_5+c_{23}c_4c_5 & s_{23}c_5+c_{23}c_4s_5 & a_5s_{23}c_5+a_5c_{23}c_4s_5+d_4s_{23}+a_2s_2 \\ -c_4 & s_4c_5 & s_4s_5 & a_5s_4s_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.7}$$

From equation (3.7) the following relationships can be found:

$$n_x\sin\theta_1-n_y\cos\theta_1=-\cos\theta_4 \tag{3.8}$$

$$o_x\sin\theta_1-o_y\cos\theta_1=\sin\theta_4\cos\theta_5 \tag{3.9}$$

$$a_x\sin\theta_1-a_y\cos\theta_1=\sin\theta_4\sin\theta_5 \tag{3.10}$$

$$p_x\sin\theta_1-p_y\cos\theta_1=a_5\sin\theta_4\sin\theta_5 \tag{3.11}$$

From (3.10) and (3.11), we have

$$\begin{aligned} \sin\theta_1 &= \frac{a_5a_y-p_y}{a_yp_x-a_xp_y}\sin\theta_4\sin\theta_5 \\ \cos\theta_1 &= \frac{a_5a_x-p_x}{a_yp_x-a_xp_y}\sin\theta_4\sin\theta_5 \\ \tan\theta_1 &= \frac{\dfrac{a_5a_y-p_y}{a_yp_x-a_xp_y}\sin\theta_4\sin\theta_5}{\dfrac{a_5a_x-p_x}{a_yp_x-a_xp_y}\sin\theta_4\sin\theta_5} \end{aligned} \tag{3.12}$$

in which the values of θ_4 and θ_5 are unknown yet, but we can assume that if $\sin\theta_4\sin\theta_5>0$ then:

$$\theta_1=a\tan2((a_5a_y-p_y)\cdot sign(a_yp_x-a_xp_y),(a_5a_x-p_x)\cdot sign(a_yp_x-a_xp_y)) \tag{3.13}$$

and if $\sin\theta_4\sin\theta_5<0$ then:

$$\theta_1=a\tan2((p_y-a_5a_y)\cdot sign(a_yp_x-a_xp_y),(p_x-a_5a_x)\cdot sign(a_yp_x-a_xp_y)) \tag{3.14}$$

From (3.9) and (3.10), we have

$$
\begin{aligned}
\sin\theta_5 &= \frac{a_x \sin\theta_1 - a_y \cos\theta_1}{\sin\theta_4} \\
\cos\theta_5 &= \frac{o_x \sin\theta_1 - o_y \cos\theta_1}{\sin\theta_4} \\
\tan\theta_5 &= \frac{\dfrac{a_x \sin\theta_1 - a_y \cos\theta_1}{\sin\theta_4}}{\dfrac{o_x \sin\theta_1 - o_y \cos\theta_1}{\sin\theta_4}} \\
\tan\theta_5 &= \frac{(a_x \sin\theta_1 - a_y \cos\theta_1)\sin\theta_4}{(o_x \sin\theta_1 - o_y \cos\theta_1)\sin\theta_4}
\end{aligned}
\tag{3.15}
$$

in which θ_4 is unknown but, in a similar way, if $\sin\theta_4 > 0$, then

$$
\theta_5 = \operatorname{atan2}(a_x \sin\theta_1 - a_y \cos\theta_1, o_x \sin\theta_1 - o_y \cos\theta_1) \tag{3.16}
$$

And if $\sin\theta_4 < 0$, then

$$
\theta_5 = \operatorname{atan2}(a_y \cos\theta_1 - a_x \sin\theta_1, o_y \cos\theta_1 - o_x \sin\theta_1) \tag{3.17}
$$

In case the above solutions suffer from inaccuracy problems (when both terms in the atan2 function approach to zero), other solutions should be available as alternatives. For example, from (3.9) and (3.11) we obtain an alternative solution to θ_5:

$$
\begin{aligned}
\sin\theta_5 &= \frac{p_x \sin\theta_1 - p_y \cos\theta_1}{a_5 \sin\theta_4} \\
\cos\theta_5 &= \frac{o_x \sin\theta_1 - o_y \cos\theta_1}{\sin\theta_4} \\
\tan\theta_5 &= \frac{\dfrac{p_x \sin\theta_1 - p_y \cos\theta_1}{a_5 \sin\theta_4}}{\dfrac{o_x \sin\theta_1 - o_y \cos\theta_1}{\sin\theta_4}} \\
\tan\theta_5 &= \frac{(p_x \sin\theta_1 - p_y \cos\theta_1)\sin\theta_4}{(a_5 o_x \sin\theta_1 - a_5 o_y \cos\theta)\sin\theta_4}
\end{aligned}
\tag{3.18}
$$

Since θ_4 is unknown yet, if $\sin\theta_4 > 0$, then

$$
\theta_5 = \operatorname{atan2}(p_x \sin\theta_1 - p_y \cos\theta_1, a_5(o_x \sin\theta_1 - o_y \cos\theta_1)) \tag{3.19}
$$

And if $\sin\theta_4 < 0$, then:

$$
\theta_5 = \operatorname{atan2}(p_y \cos\theta_1 - p_x \sin\theta_1, a_5(o_y \cos\theta_1 - o_x \sin\theta_1)) \tag{3.20}
$$

Now making use of (3.8) and (3.9) we have

$$\cos\theta_4 = n_y \cos\theta_1 - n_x \sin\theta_1$$
$$\sin\theta_4 = \frac{o_x \sin\theta_1 - o_y \cos\theta_1}{\cos\theta_5} \qquad (3.21)$$
$$\tan\theta_4 = \frac{\dfrac{o_x \sin\theta_1 - o_y \cos\theta_1}{\cos\theta_5}}{n_y \cos\theta_1 - n_x \sin\theta_1}$$

which leads to a solution to θ_4:

$$\theta_4 = \operatorname{atan2}(\frac{o_x \sin\theta_1 - o_y \cos\theta_1}{\cos\theta_5}, n_y \cos\theta_1 - n_x \sin\theta_1) \qquad (3.22)$$

Another solution to θ_4 can be obtained by combining (3.8) with (3.10), giving:

$$\theta_4 = \operatorname{atan2}(\frac{a_x \sin\theta_1 - a_y \cos\theta_1}{\sin\theta_5}, n_y \cos\theta_1 - n_x \sin\theta_1) \qquad (3.23)$$

And another solution for θ_4 can be derived from (3.8) and (3.11), giving:

$$\theta_4 = \operatorname{atan2}(\frac{p_x \sin\theta_1 - p_y \cos\theta_1}{a_5 \sin\theta_5}, n_y \cos\theta_1 - n_x \sin\theta_1) \qquad (3.24)$$

Due to the lack of more helpful relationships in (3.7), its both sides are then multiplied by the inverse of ${}^1\mathbf{A}_2$, i.e., ${}^1A_2^{-}\,{}^0A_1^{-}[n \quad o \quad a \quad p] = {}^2A_3\,{}^3A_4\,{}^4A_5$, producing the following equation:

$$\begin{bmatrix} c_2(n_x c_1 + n_y s_1) + n_z s_2 & c_2(o_x c_1 + o_y s_1) + o_z s_2 & c_2(a_x c_1 + a_y s_1) + a_z s_2 & c_2(p_x c_1 + p_y s_1 - a_1) + p_z s_2 - a_2 \\ -s_2(n_x c_1 + n_y s_1) + n_z c_2 & -s_2(o_x c_1 + o_y s_1) + o_z c_2 & -s_2(a_x c_1 + a_y s_1) + a_z c_2 & -s_2(p_x c_1 + p_y s_1 - a_1) + p_z c_2 \\ n_x s_1 - n_y c_1 & o_x s_1 - o_y c_1 & a_x s_1 - a_y c_1 & p_x s_1 - p_y c_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \qquad (3.25)$$
$$\begin{bmatrix} -s_3 s_4 & -c_3 s_5 - s_3 c_4 c_5 & c_3 c_5 - s_3 c_4 s_5 & a_5 c_3 c_5 - a_5 s_3 c_4 s_5 + d_4 c_3 \\ c_3 s_4 & -s_3 s_5 + c_3 c_4 c_5 & s_3 c_5 + c_3 c_4 s_5 & a_5 s_3 c_5 + a_5 c_3 c_4 s_5 + d_4 s_3 \\ -c_4 & s_4 c_5 & s_4 s_5 & a_5 s_4 s_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From (3.25), the following relationships can be found:

$$c_2(n_x c_1 + n_y s_1) + n_z s_2 = -s_3 s_4 \qquad (3.26)$$
$$-s_2(n_x c_1 + n_y s_1) + n_z c_2 = c_3 s_4 \qquad (3.27)$$
$$c_2(o_x c_1 + o_y s_1) + o_z s_2 = -c_3 s_5 - s_3 c_4 c_5 \qquad (3.28)$$
$$-s_2(o_x c_1 + o_y s_1) + o_z c_2 = -s_3 s_5 + c_3 c_4 c_5 \qquad (3.29)$$
$$c_2(a_x c_1 + a_y s_1) + a_z s_2 = c_3 c_5 - s_3 c_4 s_5 \qquad (3.30)$$
$$-s_2(a_x c_1 + a_y s_1) + a_z c_2 = s_3 c_5 + c_3 c_4 s_5 \qquad (3.31)$$
$$-s_2(p_x c_1 + p_y s_1 - a_1) + p_z c_2 = a_5 s_3 c_5 + a_5 c_3 c_4 s_5 + d_4 s_3 \qquad (3.32)$$

From (3.26), we obtain:

$$s_3 = \frac{c_2(n_x c_1 + n_y s_1) + n_z s_2}{-s_4} \tag{3.33}$$

and from (3.27), we have:

$$c_3 = \frac{-s_2(n_x c_1 + n_y s_1) + n_z c_2}{s_4} \tag{3.34}$$

Substituting (3.33) for sinθ_3 and (3.34) for cosθ_3 in (3.32), we obtain a solution to θ_2 when $\sin\theta_4 > 0$ as:

$$\begin{aligned}\theta_2 = \operatorname{atan2}(&(a_5c_5 + d_4)(n_x c_1 + n_y s_1) + p_z s_4 - a_5 n_z c_4 s_5, \\ &s_4(p_x c_1 + p_y s_1 - a_1) - n_z(a_5 c_5 + d_4) - a_5 c_4 s_5(n_x c_1 + n_y s_1))\end{aligned} \tag{3.35}$$

And when $\sin\theta_4 < 0$, the value of θ_2 is:

$$\begin{aligned}\theta_2 = \operatorname{atan2}(&a_5 n_z c_4 s_5 - (a_5 c_5 + d_4)(n_x c_1 + n_y s_1) - p_z s_4, \\ &n_z(a_5 c_5 + d_4) + a_5 c_4 s_5(n_x c_1 + n_y s_1) - s_4(p_x c_1 + p_y s_1 - a_1))\end{aligned} \tag{3.36}$$

To avoid inaccuracy problems, other solutions to θ_2 can also be derived as follows. Substituting (3.33) for sinθ_3 and (3.34) for cosθ_3 in (3.28), we obtain the value of θ_2 when $\sin\theta_4 > 0$:

$$\begin{aligned}\theta_2 = \operatorname{atan2}(&s_4(o_x c_1 + o_y s_1) + n_z s_5 - c_4 c_5(n_x c_1 + n_y s_1), \\ &s_5(n_x c_1 + n_y s_1) + n_z c_4 c_5 - o_z s_4)\end{aligned} \tag{3.37}$$

And when $\sin\theta_4 < 0$ the value of θ_2 can be obtained multiplying the arguments of (3.37) by -1. Another solution is derived by substituting (3.33) for sinθ_3 and (3.34) for cosθ_3 in (3.29), we have the value of θ_2 when $\sin\theta_4 > 0$ as:

$$\begin{aligned}\theta_2 = \operatorname{atan2}(&s_5(n_x c_1 + n_y s_1) + o_z s_4 - n_z c_4 c_5, \\ &s_4(o_x c_1 + o_y s_1) - c_4 c_5(n_x c_1 + n_y s_1) - n_z s_5)\end{aligned} \tag{3.38}$$

And the value of θ_2 when $\sin\theta_4 < 0$ can be obtained by multiplying the arguments of (3.38) by -1. Another solution can be achieved by substituting (3.33) for sinθ_3 and (3.34) for cosθ_3 in (3.30), which when $\sin\theta_4 > 0$ leads to:

$$\begin{aligned}\theta_2 = \operatorname{atan2}(&s_4(a_x c_1 + a_y s_1) - c_4 s_5(n_x c_1 + n_y s_1) - n_z s_5, \\ &n_z c_4 s_5 - c_5(n_x c_1 + n_y s_1) - a_z s_4)\end{aligned} \tag{3.39}$$

The value of θ_2 when $\sin\theta_4 > 0$ is obtained by multiplying the arguments of (3.39) by -1. Another solution for θ_2 when $\sin\theta_4 > 0$ is obtained by substituting (3.33) for sinθ_3 and (3.34) for cosθ_3 in (3.31), which results in:

$$\begin{aligned}\theta_2 = \operatorname{atan2}(&c_5(n_x c_1 + n_y s_1) + a_z s_4 - n_z c_4 s_5, \\ &s_4(a_x c_1 + a_y s_1) - c_4 s_5(n_x c_1 + n_y s_1) - n_z s_5)\end{aligned} \tag{3.40}$$

And when $\sin\theta_4 < 0$ the value of θ_2 is computed by multiplying the arguments of (3.40) by -1.

Now let's look for the solution to θ_3. From (3.33) and (3.34), we have

$$\tan\theta_3 = \frac{\dfrac{c_2(n_x c_1 + n_y s_1) + n_z s_2}{-s_4}}{\dfrac{-s_2(n_x c_1 + n_y s_1) + n_z c_2}{s_4}} \tag{3.41}$$
$$\tan\theta_3 = \frac{(-c_2(n_x c_1 + n_y s_1) - n_z s_2)s_4}{(-s_2(n_x c_1 + n_y s_1) + n_z c_2)s_4}$$

Therefore,

$$\theta_3 = a\tan 2((-n_x c_1 c_2 - n_y s_1 c_2 - n_z s_2)\cdot sign(s_4), (-n_x c_1 s_2 - n_y s_1 s_2 + n_z c_2)\cdot sign(s_4)) \tag{3.42}$$

Various inverse kinematics equations have been derived under different assumptions. How to choose the correct equations is a problem. There are 4 possible situations, which are:

a) When $\sin(\theta_4)\sin(\theta_5)>0$ and $\sin(\theta_4)>0$.
b) When $\sin(\theta_4)\sin(\theta_5)>0$ and $\sin(\theta_4)<0$.
c) When $\sin(\theta_4)\sin(\theta_5)<0$ and $\sin(\theta_4)>0$.
d) When $\sin(\theta_4)\sin(\theta_5)<0$ and $\sin(\theta_4)<0$.

Given the position and orientation of the end-effector, there are 4 possible solutions, corresponding to the 4 possible situations:

a) Assuming $\sin(\theta_4)\sin(\theta_5)>0$ and $\sin(\theta_4)>0$:

$$\begin{aligned}
\theta_1 &= a\tan 2((a_5 a_y - p_y)\cdot sign(a_y p_x - a_x p_y), (a_5 a_x - p_x)\cdot sign(a_y p_x - a_x p_y)) \\
\theta_5 &= a\tan 2(a_x s_1 - a_y c_1, o_x s_1 - o_y c_1) \\
\theta_4 &= a\tan 2(\frac{o_x s_1 - o_y c_1}{c_5}, n_y c_1 - n_x s_1) \\
\theta_2 &= \mathrm{atan2}((a_5 c_5 + d_4)(n_x c_1 + n_y s_1) + p_z s_4 - a_5 n_z c_4 s_5, \\
&\quad s_4(p_x c_1 + p_y s_1 - a_1) - n_z(a_5 c_5 + d_4) - a_5 c_4 s_5(n_x c_1 + n_y s_1)) \\
\theta_3 &= a\tan 2((-n_x c_1 c_2 - n_y s_1 c_2 - n_z s_2)\cdot sign(s_4), (-n_x c_1 s_2 - n_y s_1 s_2 + n_z c_2)\cdot sign(s_4))
\end{aligned} \tag{3.43}$$

b) Assuming $\sin(\theta_4)\sin(\theta_5)>0$ and $\sin(\theta_4)<0$:

$$\begin{aligned}
&\theta_1 = a\tan 2((a_5 a_y - p_y)\cdot sign(a_y p_x - a_x p_y), (a_5 a_x - p_x)\cdot sign(a_y p_x - a_x p_y))\\
&\theta_5 = a\tan 2(a_y c_1 - a_x s_1, o_y c_1 - o_x s_1)\\
&\theta_4 = a\tan 2(\frac{o_x s_1 - o_y c_1}{c_5}, n_y c_1 - n_x s_1)\\
&\theta_2 = \mathrm{atan}\,2(a_5 n_z c_4 s_5 - (a_5 c_5 + d_4)(n_x c_1 + n_y s_1) - p_z s_4,\\
&\quad n_z(a_5 c_5 + d_4) + a_5 c_4 s_5 (n_x c_1 + n_y s_1) - s_4(p_x c_1 + p_y s_1 - a_1))\\
&\theta_3 = a\tan 2((-n_x c_1 c_2 - n_y s_1 c_2 - n_z s_2)\cdot sign(s_4), (-n_x c_1 s_2 - n_y s_1 s_2 + n_z c_2)\cdot sign(s_4))
\end{aligned} \tag{3.44}$$

c) Assuming $\sin(\theta_4)\sin(\theta_5)<0$ and $\sin(\theta_4)>0$:

$$\begin{aligned}
&\theta_1 = a\tan 2((p_y - a_5 a_y)\cdot sign(a_y p_x - a_x p_y), (p_x - a_5 a_x)\cdot sign(a_y p_x - a_x p_y))\\
&\theta_5 = a\tan 2(a_x s_1 - a_y c_1, o_x s_1 - o_y c_1)\\
&\theta_4 = a\tan 2(\frac{o_x s_1 - o_y c_1}{c_5}, n_y c_1 - n_x s_1)\\
&\theta_2 = \mathrm{atan}\,2((a_5 c_5 + d_4)(n_x c_1 + n_y s_1) + p_z s_4 - a_5 n_z c_4 s_5,\\
&\quad s_4(p_x c_1 + p_y s_1 - a_1) - n_z(a_5 c_5 + d_4) - a_5 c_4 s_5 (n_x c_1 + n_y s_1))\\
&\theta_3 = a\tan 2((-n_x c_1 c_2 - n_y s_1 c_2 - n_z s_2)\cdot sign(s_4), (-n_x c_1 s_2 - n_y s_1 s_2 + n_z c_2)\cdot sign(s_4))
\end{aligned} \tag{3.45}$$

d) Assuming $\sin(\theta_4)\sin(\theta_5)<0$ and $\sin(\theta_4)<0$:

$$\begin{aligned}
&\theta_1 = a\tan 2((p_y - a_5 a_y)\cdot sign(a_y p_x - a_x p_y), (p_x - a_5 a_x)\cdot sign(a_y p_x - a_x p_y))\\
&\theta_5 = a\tan 2(a_y c_1 - a_x s_1, o_y c_1 - o_x s_1)\\
&\theta_4 = a\tan 2(\frac{o_x s_1 - o_y c_1}{c_5}, n_y c_1 - n_x s_1)\\
&\theta_2 = \mathrm{atan}\,2(a_5 n_z c_4 s_5 - (a_5 c_5 + d_4)(n_x c_1 + n_y s_1) - p_z s_4,\\
&\quad n_z(a_5 c_5 + d_4) + a_5 c_4 s_5 (n_x c_1 + n_y s_1) - s_4(p_x c_1 + p_y s_1 - a_1))\\
&\theta_3 = a\tan 2((-n_x c_1 c_2 - n_y s_1 c_2 - n_z s_2)\cdot sign(s_4), (-n_x c_1 s_2 - n_y s_1 s_2 + n_z c_2)\cdot sign(s_4))
\end{aligned} \tag{3.46}$$

It should be noted that there is no analytical solution for the situations when $\sin\theta_4=0$ or $\sin\theta_5=0$. Other approaches, such as neural networks, should be explored. It should also be noted that only the position and orientation of the end-effector are given in the inverse kinematics problem, some of the above assumptions about the sign of $\sin(\theta_4)$ and $\sin(\theta_5)$ must be wrong. How to choose the correct solution from the 4 possible sets of solusions? Our strategy is to accept a solution only if its joint angles are within the constrained range and then to check its correctness using the forward kinematics. Based on extensive experiments on a huge data set generated by the forword kinematics equations, we find that only one solution among the 4 give reasonable joing angle values which produce the expected position and orientation. Therefore, whether the solution gives reasonable joint angle values can be used as a criterion for choosing the correct solution.

It should also be noted that the alternative solutions to θ_5, θ_4, and θ_2 could be very useful when the arguments of some *atan2(.,.)* function approach to zero. The following table summarizes the use of the derived equations for the inverse kinematics problem.

Table 3.6 Selection of equations to solve the inverse kinematics problem

CASE	Joint Angles	Equations and alternative equations to use
a) $\sin(\theta_4)\sin(\theta_5)>0$ and $\sin(\theta_4)>0$	1	3.13
	5	3.16/3.19
	4	3.22/3.23/3.24
	2	3.35/3.37/3.38/3.39/3.40
	3	3.42
b) $\sin(\theta_4)\sin(\theta_5)>0$ and $\sin(\theta_4)<0$	1	3.13
	5	3.17/3.20
	4	3.22/3.23/3.24
	2	3.36/3.37*/3.38*/3.39*/3.40*
	3	3.42
c) $\sin(\theta_4)\sin(\theta_5)<0$ and $\sin(\theta_4)>0$	1	3.14
	5	3.16/3.19
	4	3.22/3.23/3.24
	2	3.35/3.37/3.38/3.39/3.40
	3	3.42
d) $\sin(\theta_4)\sin(\theta_5)<0$ and $\sin(\theta_4)<0$	1	3.14
	5	3.17/3.20
	4	3.22/3.23/3.24
	2	3.36/3.37*/3.38*/3.39*/3.40*
	3	3.42

In real-world applications, the inverse kinematics model may bring about problems. For instance, the calculated joint angles could be unreachable due to some physical constraints; and the calculated joint angles could be incorrect due to the inaccuracy problem caused by both terms in the *atan2* function approaching to zero. Some implementation issues are discussed in the next section.

IV. IMPLEMENTATION ISSUES

4.1 About the P2Arm.

The P2Arm presents some inconsistencies when comparing the direction of rotation of some joints with respect to the kinematics model, and the real zero position is different from that assumed in the development of the kinematics model. The inconsistencies and solutions are discussed in detail in the following.

a) About θ_1

The real positive rotation of this joint, when observing the arm form the top, is from left to right, giving the positive z-axis downwards. To overcome this problem, the sign of this joint angle should be changed in programming. Figure 4.1 shows this change.

* Multiply the arguments in this equation by -1.

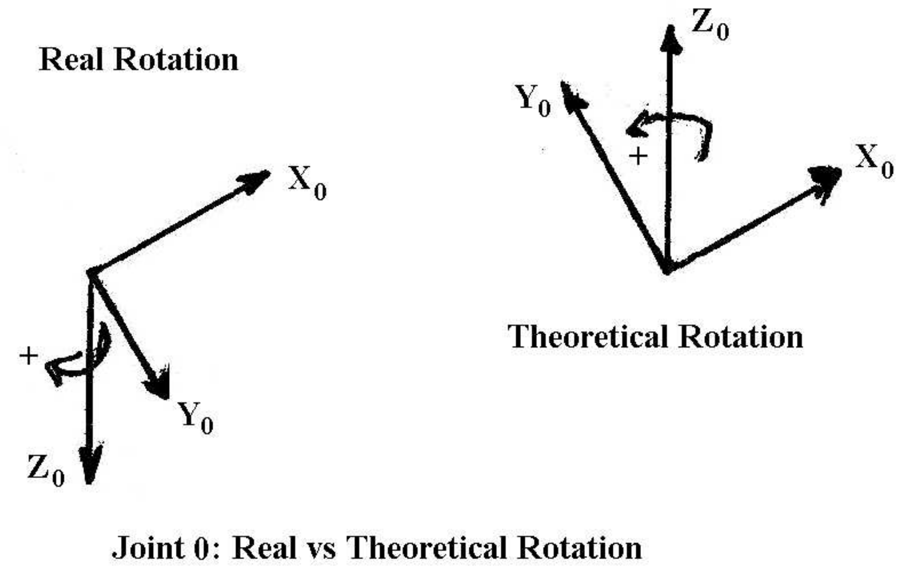

Figure 4.1 Change of sign for the rotation of joint 0

b) About θ_2

The real zero value of θ_2 is not corresponding to the situation when the arm is extended horizontally. There is a deviation of -9^o, that is, when the theoretical value is 0^o the corresponding real value is 9^o. Figure 4.2 shows this problem. Calibration should be done in the program.

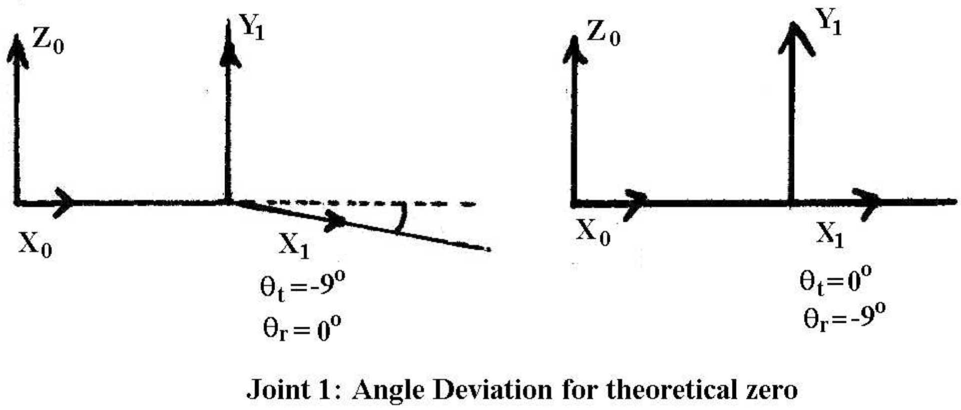

Figure 4.2 Adjustment in the angle of joint 1

c) About θ_4

The real positive rotation is clockwise when it should be counter clockwise. The solution is to change the sign of the values. The same procedure applied to θ_2 should be used for θ_4.

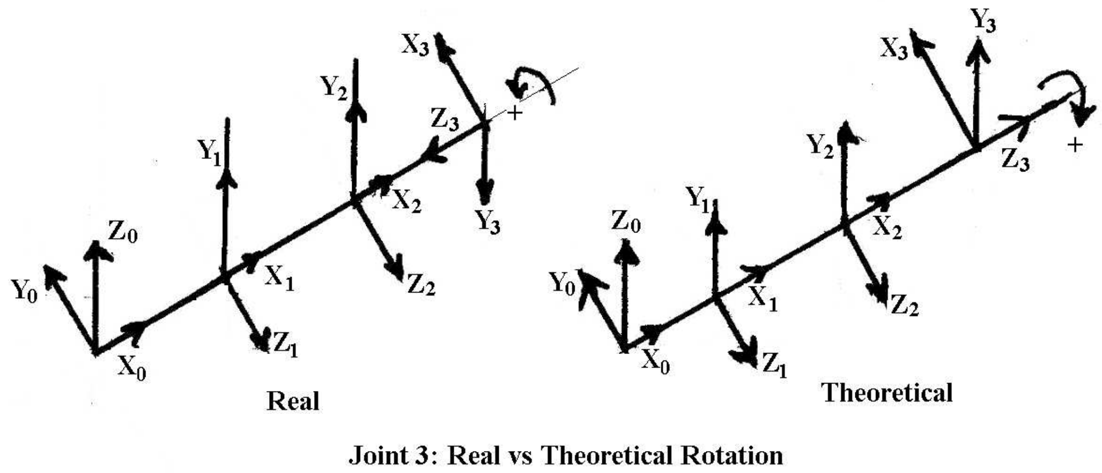

Figure 4.3 Change of sign for the rotation of joint 3

d) About θ_5

There is a deviation of 45° in θ_5, corresponding to the fully extended arm position, as shown in Fig. 4.4. A similar calibration as conducted for θ_2 should be done for θ_5.

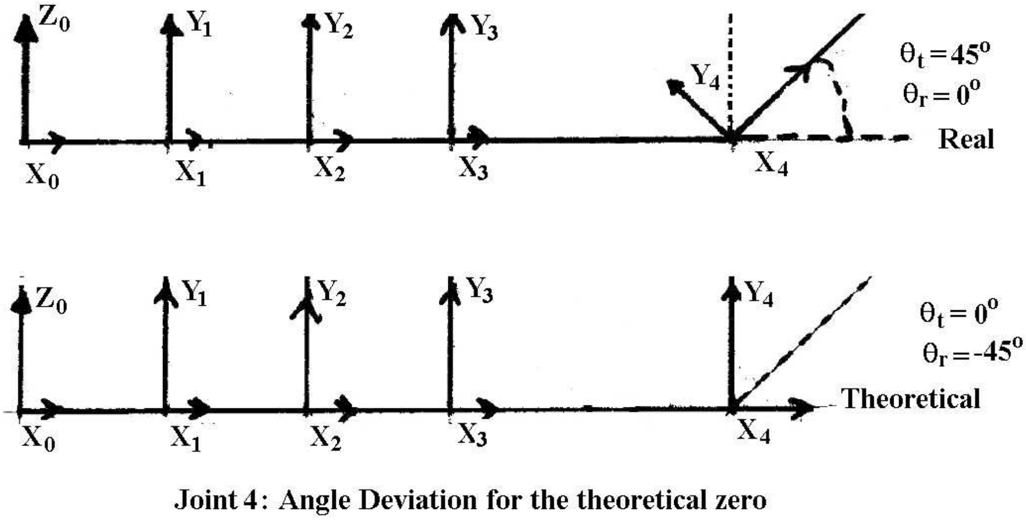

Figure 4.4 Adjustment in the angle of joint 4

Also, care should be taken when positioning the arm to certain positions since the arm is mounted on a mobile robot and some positions can damage the arm, though being allowed positions for the whole joint space. Therefore, positions should be limited to those that are safe for the arm. Table 4.1 shows the normal and safe ranges for each joint for both the real values and the adjusted ones.

Table 4.1 Operational ranges for the P2Arm

	Normal Ranges				Safe Ranges			
Joint	Real min	Real max	Adjusted min	Adjusted max	Real min	Real Max	Adjusted min	Adjusted max
0	-93	101	-93	101	-68	80	-68	80
1	-77	142	-86	133	-29	142	-38	133
2	-102	94	-102	94	-77	94	-77	94
3	-96	111	-96	111	-96	111	-96	111
4	-130	69	-85	114	-130	69	-85	114

4.2 About the Program

4.2.1 The atan2(.,.) function

The inverse kinematics model shows that the value of any joint variable is computed in terms of the function *atan2(y,x)*, which gives the angle according not only to the *arctang(y,x)* but also to the signs of *y* and *x*. Nowadays, there are libraries that computes this function without the need to implement it; however for clarity reasons the sketch of the function is depicted next.

atan2(y,x)

```
//returns φ for -π<=φ<=π
if (x=0) or (y=0) then
     if (x=0) and (y is positive) then
               φ=+π/2
     else
               φ=-π/2
     if (y=0) and (x is positive) then
               φ=0
     else
               φ=-π
else
```

$$sign = \frac{x \times y}{abs(x \times y)}$$

$$\phi = sign \times \operatorname{atan}(abs(y/x))$$

```
return φ
```

4.2.2 Precision of position and joint angles

When a command, expressed in degrees, is sent to the arm, the built-in program for the arm control converts the angle to ticks position so that the low-level control is to move each servomotor to a specific tick position. A tick position is represented by an integer value, and thus the conversion from the degrees to ticks gives only integer values; therefore there is a change from the float type to an integer type. The same happens when converting the ticks position to degrees: joint values are rounded to integer values, e.g. if the value of any joint variable is 10.5 (or greater) the value is rounded to 11; if the value of any joint variable is 10.4 (or less) the value is rounded to 10. Thus, when computing the joint variables, they should be rounded accordingly.

In order to simplify computations, the position and orientation values will also be rounded considering a precision of 4 decimal digits.

4.2.3 Selection of equations for inverse kinematics

The derived inverse kinematics equations are based on assumptions about the sign of sin(θ_4) and sin(θ_5). The program will compute all possible solutions if necessary and only accept the result that is inside the constrained range, which is described as follows:

Compute Joint angles

```
Compute joint angles using equations for Case a) or alternative equations if necessary
If joint angles are inside range
  Return
Else
  Compute angles using equations for Case b) or alternative equations if necessary
  If joint angles are inside range
        Return
  Else
        Compute angles using equations for Case c) or alternative equations if necessary
        If joint angles are inside range
        Return
     Else
        Compute angles using equations for Case d) or alternative equations if necessary
        If joint angles are inside range
              Return
        Else
              Mark error
        End_if
     End_if
  End_if
End_if
```

V. CONCLUSIONS

This technical report gives a detailed explanation of the theory for kinematics modelling, which forms the basis for developing the forward and inverse kinematics for the P2Arm, a 5-dof robot arm mounted on a Pioneer 2 mobile robot.

Forward and inverse kinematics models for the P2Arm have been derived with some guides for the model implementation. As it was mentioned, inaccuracy problem may arise in the model when arguments of the atan2 function approach to zero, producing incorrect positions and possible joint angles out of range. Care should be taken under these circumstances and other methods should be sought to compute the joints variables.

As part of the research, a neural network called CMAC is being investigated to model the inverse kinematics, aiming to produce a more reliable arm controller.

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