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Power Brokers: Middlemen in Legislative Bargaining

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Abstract

We study a model of decentralized legislative bargaining over public decisions with transfers. We establish the emergence of middlemen in legislative bargaining as a robust equilibrium phenomenon. We show that legislative intermediation can impact policy outcomes, and can be inefficient. To fulfill this role, the middleman's policy preferences and bargaining position must be such to make his role of intermediary credible. But the political middleman must also directly benefit from policy change. The results highlight fundamental differences between the role of intermediaries in politics and exchange economies. **JEL codes D72, C72, C78.**

Keywords: intermediaries, middlemen, bargaining, legislatures, vote buying.

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1 Introduction

Most significant public policy choices are decided in legislatures and other collective bodies. From health care reform to national defense or regulation of economic activity, enacting new policies requires mutual understanding among committee members with different political views. It also requires, more often than not, a variety of compromises and political exchanges among these legislators.

The use of transfers to secure legislative support in legislatures around the world is widely documented. This is standard operating procedure in multiparty presidential democracies, where small regional or ethnic parties often act as brokers of political deals that require the support of a national coalition. But it is also a common feature in the US Congress. In the passage of the *fiscal cliff* law, for instance, the use of special interest tax breaks was so pervasive that "the law designed to reduce the deficit added $74 billion in spending through changes in the tax law."[1]

This process of legislative bargaining has two readily observable characteristics. First, political exchanges are rarely struck publicly and simultaneously at the time when a proposal is up for a vote. Instead, compromises among members of a legislative coalition are typically made in backroom deals, in a process of *decentralized* and *sequential* bargaining.

Second, whenever there are more than two legislative blocks, this process of decentralized bargaining leads naturally to the emergence of legislative intermediaries. This was fundamental, for example, in the privatization of Argentina's national gas and oil company (YPF) in 1991, when then Governor of Santa Cruz and future President of Argentina Nestor Kirchner brokered a deal that guaranteed the support of the coalition of oil producing provinces in the Senate.[2] The same is true in the US when some issues divide Democrats and Republicans into more than two homogeneous blocks, as was the case during the realignment of the South. In fact, the most notable example of a political broker in american politics is that of Senate Majority leader (then President) Lyndon Johnson (1955-61). As Caro (2002) points out, "From the time he became Majority Leader, Johnson began using talk on the floor as a smoke screen for the maneuvering that was taking place in the cloakrooms, ...as a method of stalling the Senate to give him

[1] CBS evening news, January 2, 2013; *'Fiscal cliff' bill had some hidden pork.*
[2] See https://dl.dropboxusercontent.com/u/954402/YPFKirchner.pdf

time to work out his deals."

Our goal in this paper is to study the dynamics of decentralized legislative bargaining: how private agreements among parties affect subsequent negotiations and policy outcomes, and how parties' conjectures of future negotiations affect agreements in the first place. In particular, we seek to explain the emergence and role of legislative intermediaries. These actors are often crucial in decentralized bargaining, but mostly ignored in the bargaining literature. Can some legislative actors enable political deals by putting together two parties that would not negotiate directly with one another? What do these power brokers bring to the table?

We address these questions within a simple model of decentralized legislative bargaining, which bridges traditional legislative bargaining models with models of competitive market for votes.

To capture the sequential and decentralized nature of bargaining that we observe in political deals, we depart from centralized bargaining models in the Baron and Ferejohn (1989) tradition. Because in these models a proposer makes an offer to all members of a coalition simultaneously, intermediaries are ruled out by fiat. We also depart from the prevailing approach to study decentralized buying and selling of votes in a committee, which assumes a competitive market for votes (Philipson and Snyder (1996), Casella, Llorente-Saguer, and Palfrey (2012)). In these models committee members have the opportunity to buy and sell votes at posted prices, also acting simultaneously. Instead, we assume that parties are matched in bilateral negotiations, and can offer to buy or sell their votes to one another at a price they negotiate, while being forward looking about the implications of their trades on subsequent negotiations and policy outcomes.

We consider a model with three parties, which bargain over policy and rents in an infinite horizon. In each period before a policy is implemented, two parties meet one-on-one according to a stochastic matching process, and can offer to buy or sell their votes to one another in exchange for rents.[3] As in Gul (1989), parties selling their votes relinquish their voting rights to the buyers and are excluded from further negotiations. Here, however, the good exchanged (voting rights) does not produce payoffs directly, but only to the extent that it allows its holder to change policy.

[3]More precisely, we assume, as in Krishna and Serrano (1996), that parties make offers to sell or purchase the right to represent the accepting player in any future negotiations.

While in this context no party has private information or superior commitment power, we show that the emergence of a political intermediary is a robust equilibrium phenomenon.

Our approach is to pose this problem as one of rationalizability of a *broker equilibrium*: a Markov Perfect equilibrium in which one of the parties (B) becomes a broker of a deal between two other parties, A and C, by which A transfers voting power to C via the broker. In particular, we ask whether for given matching parameters (discount factor and matching probabilities), there exist preference profiles for which we can support a broker equilibrium. A key advantage of this formulation is that the equilibrium conditions can be written as a set of linear inequalities $\Lambda u \leq \alpha$, where the unknowns are the payoffs $u_i(z_j)$ of party i for implementing policy z_j, and Λ is a matrix of matching parameters. We can then use basic duality results from convex analysis (Farkas' Lemma) to obtain necessary and sufficient conditions for the existence of a solution to this problem.

Our results show that the triangulation of political agreements implemented by the broker equilibrium can impact outcomes and welfare, and yield a host of empirical implications (we expand on both sets of results in the conclusions). At a broad level, however, the main result of the paper is to explain the role of middlemen in legislative bargaining and how this differs from what we know about middlemen in exchange economies.

In exchange economies with private goods and no externalities, intermediaries can only benefit from mediation by making a monetary profit. Their ability to generate this profit must then come from some initial advantage they are endowed with. The intermediary can be the only link between the buyer and the seller (Spulber (1996a), Spulber (1996b), Rust and Hall (2003)), it can exploit economies of scale in the use of a monitoring technology (Biglaiser (1993), Li (1998)), it can reduce the cost of matching/search (Rubinstein and Wolinsky (1987), Yavaş (1992), Gehrig (1993)), or it can have an advantageous position in a network and a low valuation for the good (Condorelli, Galeotti, and Renou (2015)). With public decisions, instead, the broker cares about the final policy outcome, and thus about the identity of the party buying votes. This introduces two substantial changes in the nature of mediation.

First, the political broker has to be trusted to (i.e., has to have incentives to) carry out the mediated transaction. In particular, the party selling its votes to the broker must anticipate that once the broker is in a position of power, it will keep on negotiating, and will not use this power to implement its preferred policy. This condition arises endogenously in equilibrium for appropriate preferences and matching parameters, and does not require endowing the broker with a differential advantage or a superior commitment ability.

Second, because the political broker cares about policy outcomes, it can get part of its retribution in policy gains. In fact, we show that in order to be able to fulfill this role, the broker *must* have a stake in the policy outcome. In particular, the broker must prefer the final policy outcome to both the status quo and the preferred policy of the party whom he initially transacts with. Because of this, the initial seller can extract rents from the broker to partially finance the trade. The political broker, hence, is not a two-sided platform that can charge both sides for its services, but an agent who derives surplus from facilitating a beneficial policy change.

The rest of the paper is organized as follows. We review the literature in Section 2 and present the model in Section 3. In Section 4 we analyze the final bargaining stage, in which only two parties control voting rights. In Section 5 we present our main results. We begin with the case of a dominant majority party in Section 5.1, and extend the analysis to all initial vote allocations and dominance relations in Sections 5.2 and 5.3. Section 6 concludes. The proofs of the main results are in the Appendix. All other proofs can be found in online appendices A and B accompanying this paper.

2 Related Literature

In this paper, we build on Gul (1989) to write down a simple model of decentralized political bargaining that bridges models of legislative bargaining and competitive market for votes.

The dominant approach to study bargaining in collective bodies follows the seminal paper by Baron and Ferejohn (1989), with a heavy emphasis on bargaining over distribution. The closest paper to ours is Jackson and Moselle (2002), where the policy space consists of both an ideological dimension over which legislators have single peaked preferences, and a purely distributive dimension. In this environment, Jackson and Moselle show that the policy outcome will generally not consist of a median decision on policy together with some distribution of spending.[4] All papers in the Baron-Ferejohn tradition are

[4]Banks and Duggan (2000) establishes existence of stationary equilibria in a generalized version of the Baron and Ferejohn (1989) model that includes public decisions with transfers. For unidimensional

models of *centralized bargaining*, where a proposer makes an offer to all members of a coalition simultaneously. We depart from this assumption because it fails to capture the decentralized sequential bargaining process that we see as fundamental to backroom political deals, and rule out intermediaries by fiat.

The prevailing approach to study decentralized bargaining in committees is to model exchanges as occurring in a competitive market for votes (Philipson and Snyder (1996), Casella, Llorente-Saguer, and Palfrey (2012)). The analysis of vote trading differs from that of a typical exchange economy because vote buying has externalities on non-traders (Riker and Brams (1973), Philipson and Snyder (1996)). Because of these externalities on non-traders and the discontinuity in payoffs associated with majority rule, existence of an equilibrium is not a trivial problem. To address this issue, both papers use rationing rules and stochastic elements (randomly choosing among suppliers when there is excess supply in Philipson and Snyder (1996); allowing mixed demands in Casella, Llorente-Saguer, and Palfrey (2012)). While modeling *decentralized* trading, the models of competitive markets for votes do not allow sequential transactions, and as a result, also rule out intermediaries by default. The price-taking assumption, moreover, can be restrictive in this setup.

Instead, we assume that parties are matched in bilateral negotiations, and can offer to buy or sell their votes to one another at a price they negotiate, while being forward looking about the implications of their trades on subsequent negotiations and policy outcomes.[5]

Our model of decentralized bargaining builds on Gul (1989). The key difference in our model is that agents bargain over a public decision. This introduces externalities on non-traders, which Gul (1989) does not allow. Gul shows that given a condition on payoffs that guarantees that value functions are superadditive, then as bargaining frictions vanish there is a unique efficient equilibrium, and players' equilibrium payoffs converge to the Shapley value (under uniform matching). This assumption is not satisfied in our model of bargaining over public decisions, and neither is the result on efficiency of equilibria as the time between offers goes to zero (See Section 5.1.1).

policy space, they show that as $\delta \to 1$ the equilibrium outcomes converge to the ideal point of the median voter, providing a noncooperative foundation of Black's median voter theorem. (Note that this result does not apply to bargaining over public decisions with transfers).

[5]In our model vote trading is done *internally*, by members of the committee. This complements the literature on vote buying of inside members by *outsiders* (Myerson (1993), Dixit and Londregan (1996), Groseclose and Snyder (1996), Lizzeri and Persico (2001), Dal Bo (2007), Dekel, Jackson, and Wolinsky (2008, 2009), and Iaryczower and Oliveros (2015)). Importantly, in these models vote buyers are precluded from forming coalitions among them, or from reselling their votes to members of the committee.

Bloch (1996), Ray and Vohra (1999, 2001), Gomes (2005), and Gomes and Jehiel (2005) study non-cooperative coalitional bargaining games with externalities. These papers address several general properties of equilibria for games with an arbitrary number of players (existence, efficiency, uniqueness), but generally do not provide a more detailed characterization of equilibrium behavior. A central assumption in this literature is that coalition members play cooperatively within the coalition, but that coalitions play non-cooperatively against other coalitions (see Ray and Vohra (2013)).[6] In our model, instead, we assume that a party i buying votes from j has full control of the votes of i and j, and therefore does not necessarily choose transactions that maximize the payoffs of $\{i, j\}$.

Our paper also relates to the literature on intermediaries. In addition to the work on middlemen in exchange economies that we pointed to in the Introduction, there is also a literature on middlemen in cheap talk games, which is more distant from our paper. This literature shows that by adding noise, randomizing over recommendations or collapsing information a mediator can, under some conditions, improve the efficiency of the interaction between a sender and a receiver with conflicts of interests (see for example Ivanov (2010), Goltsman, Hörner, Pavlov, and Squintani (2009), Hörner, Morelli, and Squintani (2015)).

3 The Model

There are three parties, $i \in N = \{A, B, C\}$, and an infinite number of periods, $\tau = 1, 2, \ldots$. Each party $i \in N$ has an ideal policy $z_i \in X$, a discount factor δ, and is endowed with $k_i > 0$ votes.[7] The vector $\mathbf{k} \equiv (k_A, k_B, k_C)$ denotes a generic vote allocation. Parties participate in a process of bilateral transactions to enact a policy. Let $\mathcal{N}_\tau$ denote the set of parties holding voting rights in period τ. In each period τ in which at least two parties hold voting rights, two parties $i, j \in \mathcal{N}_\tau$ are randomly matched to negotiate with one another, and one of them is randomly selected to make an offer. We let ρ_{ij} and p_{ij} denote the probability that i and j are matched and i is selected to make an offer when

[6]As it is the case in this paper, most papers in this literature assume that agreements are binding. For exceptions, see Seidmann and Winter (1998) (in games with no externalities) and Gomes and Jehiel (2005), or Gomes (2005) (in games with externalities).

[7]Later on we define formally the role of broker, which will be endogenously taken up by one of the parties in equilibrium. All of our results hold unchanged if the party acting as broker has no initial voting power (see our remark in the Concluding Section).

$\mathcal{N}_\tau = \{i, j\}$ and $\mathcal{N}_\tau = N$ respectively.

The proposer i can offer to buy or sell voting rights, or choose not to make an offer. A feasible transaction is an exchange of a party's voting rights for rents. If i sells its votes to j, i votes as instructed by j, and is excluded from further negotiations. We let $t_{ij}(\mathbf{k})$ denote the net transfer from i to j that follows a deal when i and j are matched and i proposed to j given voting rights $\mathbf{k}$. We say that i makes a relevant offer to j when i makes an offer to j that j will accept. In any period τ in which a party i has a majority of the votes after trade ($k_i' \geq r \equiv \sum_i k_i/2$), party i can choose whether to implement its preferred policy z_i or extend negotiations. When a party chooses to implement its preferred policy, the game ends immediately and the policy z_i is implemented forever. In any period τ prior to the implementation of a new policy, the outcome is the status quo Q. Party i's preferences are represented by the utility function

$$V_i = \sum_{\tau=0}^{\infty} \delta^\tau \left[(1-\delta)u_i(y_\tau) - t_i^\tau\right],$$

where y_τ denotes the policy implemented in period τ, and t_i^τ denotes the τ period net transfer from i to others. The function $u_i(\cdot)$ is uniquely maximized at z_i. We normalize $u_i(Q) = 0$ for all i and let $u_i^* \equiv u_i(z_i)$. We say that i *dominates* j ($i \gg j$) if i's willingness to pay for implementing z_i instead of z_j exceeds j's willingness to pay for implementing z_j instead of z_i; i.e., if $u_i^* - u_i(z_j) \geq u_j^* - u_j(z_i)$. Equivalently, letting $S_{ij}(y) \equiv u_i(y) + u_j(y)$ denote the aggregate surplus for i and j of implementing a policy y, we say that $i \gg j$ if $S_{ij}(z_i) > S_{ij}(z_j)$.

An equilibrium is a Markov Perfect Equilibrium (MPE). A trading state is a pair $((i,j),\mathbf{k})$, where (i,j) denotes that i is matched with j and i is selected to propose, and $\mathbf{k}$ denotes the pre-trade allocation of voting rights. We let $W_{ij}^i(\mathbf{k}, buy)$ and $W_{ij}^i(\mathbf{k}, sell)$ denote i's equilibrium payoff from her best relevant buy and sell offers in trading state $((i,j),\mathbf{k})$, and $W_{ij}^i(\mathbf{k}, wait)$ i's equilibrium payoff from not making a relevant offer. Then $W_{ij}^i(\mathbf{k}) \equiv \max_a W_{ij}^i(\mathbf{k}, a)$, for $a \in \{buy, sell, wait\}$, denotes i's equilibrium payoff in trading state $((i,j),\mathbf{k})$. We also let $W^i(\mathbf{k}) \equiv E[W_{ij}^i(\mathbf{k})]$, where the expectation is taken over all possible realizations of matches and proposing power. Finally, because a party with a majority of the votes after trading can *choose* to implement its preferred policy or extend negotiations, we also need to consider i's post-trade equilibrium payoff after trade opportunities resulted in a vote endowment $\mathbf{k}$, which we denote by $w^i(\mathbf{k})$.

4 Majority-Minority Bargaining

We begin by analyzing the final bargaining stage, in which only two parties, say B and C, control voting rights (see Figure 1). Because of simple majority rule, one of these parties, say B, has a majority of the votes; i.e. $\mathbf{k} = (0, k_B, k_C)$, with $k_B > k_C$. We call B the *majority party* and C the *minority party*.

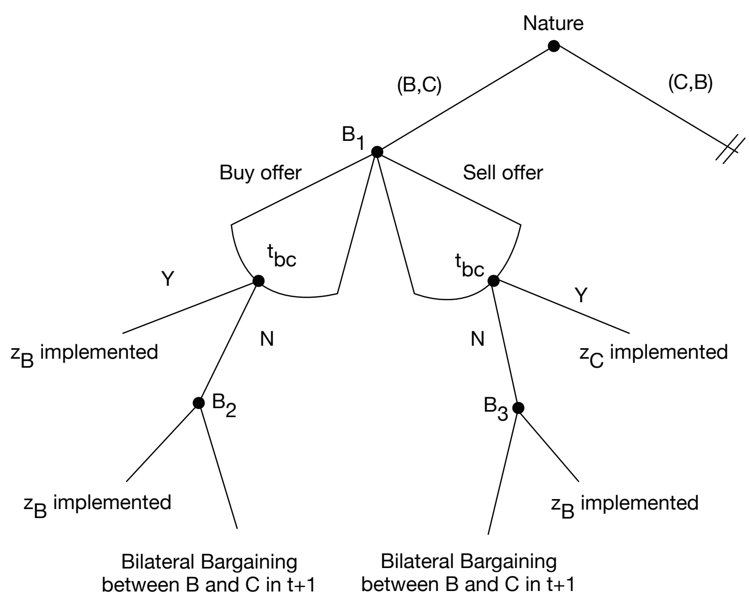

Figure 1: Bilateral Bargaining. Here B is assumed to be the majority party, C the minority. The figure illustrates the full sequence of play if B is selected to propose (state (B, C)). B makes a buy or a sell offer (node B_1), which C can accept (Y) or reject (N). If C rejects, B decides whether to implement its preferred policy or extends negotiations (nodes B_2 and B_3).

Equilibrium behavior in the majority-minority bargaining game relies on two key factors. The first is parties' relative intensity of preferences for the majority and minority policies z_B and z_C. This part of this analysis is standard. When $B \gg C$, total surplus is higher if the majority alternative is implemented. As a result, there is no transfer that C would be willing to offer that would compensate B for not implementing its preferred policy z_B. In this case, there is a MPE in which there is no trade and B implements its preferred policy.[8]

When instead $C \gg B$, there are gains from trade. Whether these gains from trade are realized, and if so how they are distributed, depends on parties' perception of their relative bargaining power. The key factor here is that B has the *option* to implement its preferred policy without C's consent. Differently to a standard bilateral bargaining game (where negotiations are automatically extended after disagreement), the majority party can either reject an offer and extend negotiations, in which case $w^j(\mathbf{k}) = \delta W^j(\mathbf{k})$, *or* reject the offer and implement its preferred policy, in which case $w^j(\mathbf{k}) = u_j(z_B)$. The threat of implementing its preferred policy after disagreement, however, is not always credible. In fact, B has incentives to implement its preferred policy after disagreement only if $u_B^* \geq \delta W^B(\mathbf{k})$, and otherwise prefers to extend negotiations for an additional period.[9]

This off-equilibrium-path choice has important consequences for equilibrium behavior and the distribution of rents in majority-minority bargaining. Consider the problem of the majority party when it has an opportunity to propose. B can buy or sell votes to C, generating payoffs $S_{BC}(z_B) - w^C(\mathbf{k})$ and $S_{BC}(z_C) - w^C(\mathbf{k})$ respectively, or it can choose not to make C a relevant offer, yielding $w^B(\mathbf{k})$. The key here is that B's payoffs for waiting and trading votes depend on the reservation values $w^B(\mathbf{k})$ and $w^C(\mathbf{k})$, which in turn depend on whether B prefers to implement its preferred policy or extend negotiations after disagreement.

Given equilibrium beliefs about play after disagreement we can characterize parties' optimal actions in each decision node as a function of the continuation values, and then equilibria of the majority-minority bargaining game.

Proposition 4.1 *Suppose at time τ two parties, i and j, have voting rights, where $k_i > k_j$*

[8] If in addition $u_B^* < \delta\, \rho_{BF}\, S_{BC}(z_B)$, there also exists a MPE in which C pays B so that it implements z_B immediately. In this equilibrium B offers to buy from C at a negative price (or accepts only a negative price offer), threatening C with maintaining the status quo after disagreement. This is interesting in itself, but largely irrelevant for our main argument, with the exception of the uniqueness claim in Theorem 5.4. We return to this point in the proof of this theorem.

[9] This is similar to bargaining games with *outside options* (see for example Muthoo (1999)). However, in bargaining games with outside options it is assumed that the party receiving the offer can reject the offer and take her outside option. This counterbalances the proposal power of the other party. In our game, instead, it is only the majority party who can implement its preferred policy after disagreement, independently of whether it is the proposer or the receiver of the offer. This difference in the sequence introduces relatively large changes in the equilibrium of the game.

but $j \gg i$. Then there exists a MPE in which, independently of who has the opportunity to propose, j buys i's votes and implements its preferred policy z_j. The majority party i extends negotiations after disagreement with probability one if and only if $u_i^* \leq \delta\rho_{ij}S_{ij}(z_j)$.

Proposition 4.1 follows immediately from Proposition 7.2 in the Appendix, which characterizes the equilibrium of the majority-minority bargaining game in more detail.[10] A key takeaway from Proposition 4.1 is that conjectures of equilibrium play after disagreement are fundamental for the analysis of equilibria with intermediaries. If the joint payoff of implementing the minority policy is not large enough, or the majority can't appropriate a large fraction of this surplus ($\delta\, \rho_{BC}\, S_{BC}(z_C) < u_B^*$), the majority implements its preferred policy after disagreement with positive probability, and in equilibrium $u_B^* \geq \delta W^B(\mathbf{k})$. But in this case the majority-minority bargaining node wouldn't be reached in the first place. This is because the decision problem of the majority party after disagreement in bilateral bargaining is strategically equivalent to its decision problem after acquiring the majority when all parties have voting rights. It follows that a necessary condition for the existence of an equilibrium with intermediaries is that the broker's relevant threat after disagreement in the majority-minority bargaining game is *not* to implement its preferred policy, and hence that $u_B^* \leq \delta\, \rho_{BC}\, S_{BC}(z_C)$.

5 Intermediaries in Legislative Bargaining

In this section we present our main results. We establish the existence of MPE in which one of the parties serves as a legislative intermediary for an open, non-empty, and fully dimensional subset of matching parameters and preference profiles. We then discuss the implications of this result for welfare and policy outcomes, and characterize conditions on preference profiles under which an equilibrium with mediated trade can arise.

From here on, we refer to the stage in which all parties have uncommitted voting rights as the decentralized bargaining stage (see Figure 2). We refer to equilibria with mediated trade as *broker equilibria*. In a broker equilibrium, a party i buys votes from party

[10]As Proposition 7.2 makes clear, establishing existence of equilibrium in the majority-minority bargaining game requires using mixed strategies. This is because the majority's option to extend negotiations or implement its preferred policy after disagreement creates a discontinuity in payoffs that leads to nonexistence of a MPE in pure strategies. Mixing after disagreement smoothes out this discontinuity in equilibrium payoffs and restores existence.

j in decentralized bargaining only to sell its votes to party $\ell \neq i, j$ in majority-minority bargaining. In addition to this core feature, we impose two additional requirements. First, to assure that the broker is not merely replicating indirectly a trade that would also occur directly, we require that parties j and ℓ do not trade when they meet in decentralized bargaining. Second, we ask that the trade enabled by the broker occurs on the equilibrium path independently of the realization of meetings. This requires that party ℓ does not trade with i or j in decentralized bargaining, and that if a party initially has a majority of the votes, that party extends negotiations after disagreement in any bilateral meeting.

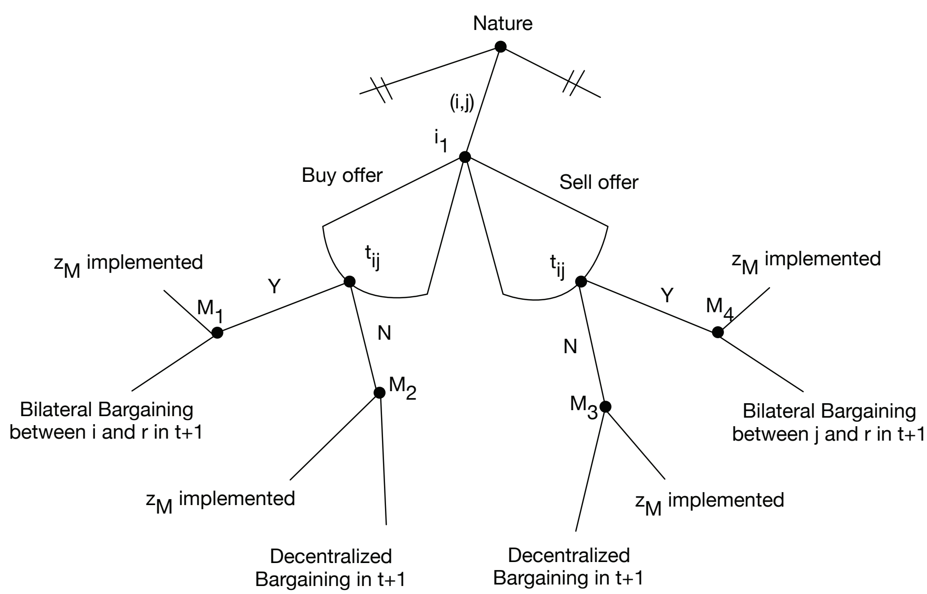

Figure 2: Decentralized Bargaining: In each decision node M_k, the party $M \in \{A, B, C, \emptyset\}$ with a majority of the votes decides whether to implement its preferred policy z_M or extend negotiations. If $M = \emptyset$, negotiations are extended by default.

Definition 5.1 *A strategy profile σ is a broker equilibrium if (i) σ is a MPE, and (ii) there is a party i and a party j such that (ii.a) i buys j's votes in any trading meeting $((i, j), \mathbf{k})$ in decentralized bargaining and sells its votes to $\ell \neq i, j$ in majority-minority bargaining, (ii.b) there is no trade in decentralized bargaining in nodes $((j, \ell), \mathbf{k})$ or $((i, \ell), \mathbf{k})$, and (ii.c) if $k_m \geq r$ for some $m \in \{i, j, \ell\}$, then m extends negotiations after disagreement in decentralized bargaining.*

We organize our analysis in two parts. In Section 5.1 we develop our analysis of broker equilibria with a *dominant* majority party A; i.e., we let $k_A \geq r$ and assume that $A \gg B$

and $A \gg C$. (We then fix $C \gg B$ without loss of generality.) Doing this allows us to simplify the discussion considerably. We then present general results for all dominance relations and initial vote allocations in Sections 5.2 and 5.3.

The analysis of broker equilibria with a dominant majority party is particularly interesting for two reasons. First, since $k_A \geq r$, the existence of a broker equilibrium can be surprising, because A has the power to implement her preferred policy without engaging in negotiations with other parties, or incurring any delay. Moreover, the fact that A is a dominant majority party implies that in two-party bargaining with either B or C, A would implement her preferred policy without trading (see Proposition 7.2). Thus, whenever it exists, a broker equilibrium enables a trade that would not have occurred in the absence of the third party, causing a change in policy outcomes.

5.1 Broker Equilibrium with a Dominant Majority Party

In this section, we analyze broker equilibria with a *dominant* majority party; i.e., we let $k_A \geq r$ and fix the dominance relation $(A \gg C, C \gg B, A \gg B)$.[11] Given the equilibrium in section 4, a necessary condition for trade in the continuation is that the party who has a majority of the votes in majority-minority bargaining is dominated by the minority party. Since A is dominant, this excludes the cases in which B sells to/buys from C in decentralized bargaining, and similarly excludes the cases in which A acts as a broker. Thus, the only possible broker equilibrium with a dominant majority party A is one in which B acts as a broker, so that A sells to B in decentralized bargaining, and B sells to C in the majority-minority bargaining stage.

Implementing a broker equilibrium introduces equilibrium incentive constraints. First, having the majority of the votes, A must have incentives to extend negotiations after disagreement in decentralized bargaining; i.e.

$$u_A^* \leq \delta W^A(\mathbf{k}) \tag{1}$$

Second, after buying A out in decentralized bargaining, B has to prefer to wait in order to broker a deal with C rather than implementing its preferred policy right away. As

[11] The discussion in this section is relatively informal to favor intuition. Equilibrium conditions (1)-(6) follow from Propositions B.2 and B.3. in Online Appendix B.

we discussed in Section 4, this is in fact the same strategic problem faced by B after disagreement in the majority-minority bargaining game with C. Thus after B acquires the majority from A in decentralized bargaining, it will extend negotiations if and only if $C \gg B$ (as we are assuming throughout) *and*

$$u_B^* \leq \delta\, \rho_{BC}\, S_{BC}(z_C) \tag{2}$$

The remaining equilibrium conditions come from parties' best responses in each trading node. First, whenever A and B meet, A has to be willing to sell its votes to B at a price B is willing to accept. Given $C \gg B$ and condition (2), we know that if B were to buy A's votes, it would go on to broker a deal with C. Thus if A were to sell to B, it could appropriate their joint value $\Pi(AB, C) \equiv \delta[u_A(z_C) + \rho_{BC} S_{BC}(z_C)]$ from transferring all voting rights to B and letting it negotiate the sale of their votes to C, net of B's discounted continuation payoff $\delta W^B(\mathbf{k})$. If instead A were to buy B's votes, A could only appropriate their joint surplus from implementing its own preferred policy, $S_{AB}(z_A)$ (again, net of B's discounted continuation payoff $\delta W^B(\mathbf{k})$). This is because given $A \gg C$, there are no gains from trade between A and C in majority-minority bargaining. Thus A prefers selling to buying from B if and only if

$$S_{AB}(z_A) \leq \Pi(AB, C), \tag{3}$$

and prefers selling to B rather than extending negotiations if and only if

$$\delta\, [W^A(\mathbf{k}) + W^B(\mathbf{k})] \leq \Pi(AB, C). \tag{4}$$

Second, when A and C meet, the broker equilibrium requires that they do not trade. Now, given that $A \gg B$ and $C \gg B$, neither A nor C has a further gain from trading with B in majority-minority bargaining. Thus $W^A_{AC}(\mathbf{k}; buy) = S_{AC}(z_A) - \delta W^C(\mathbf{k})$ and $W^A_{AC}(\mathbf{k}; sell) = S_{AC}(z_C) - \delta W^C(\mathbf{k})$. And since $A \gg C$, it follows that A would rather buy than sell to C. Moreover, A prefers to extend negotiations rather than making C a relevant buy offer if and only if A and C's aggregate discounted continuation value is larger than their joint payoff of implementing z_A, i.e.,

$$S_{AC}(z_A) \leq \delta\, [W^A(\mathbf{k}) + W^C(\mathbf{k})]. \tag{5}$$

Finally, the broker equilibrium requires that C and B choose not to trade when they meet in decentralized bargaining. But given $A \gg B$ and $A \gg C$, the analysis is similar to the case above. Here A implements its preferred policy immediately after either C sells to B *or* after B sells to C, but extends negotiations if B and C fail to reach an agreement. Then $W^C_{CB}(\mathbf{k}; sell) = W^C_{CB}(\mathbf{k}; buy) = S_{CB}(z_A) - \delta W^B(\mathbf{k})$, and C prefers not to make a relevant offer than to sell or buy from B if and only if

$$S_{BC}(z_A) \leq \delta\, [W^B(\mathbf{k}) + W^C(\mathbf{k})]. \tag{6}$$

Conditions (1)-(6) are necessary and sufficient for a broker equilibrium, given the continuation values. Continuation values, in turn, are determined by equilibrium strategies, and can be easily computed for a broker equilibrium.

Lemma 5.2 *Consider a MPE in which party B brokers a deal between A and C. Then*

$$W^C(\mathbf{k}) = \frac{\delta\,(p_{AB} + p_{BA})\,\rho_{CB}\,S_{BC}(z_C)}{(1-\delta) + \delta(p_{AB} + p_{BA})},$$

$$W^A(\mathbf{k}) = \frac{p_{AB}\,\Pi(AB, C)}{(1-\delta) + \delta(p_{AB} + p_{BA})} \quad \textit{and} \quad W^B(\mathbf{k}) = \frac{p_{BA}\,\Pi(AB, C)}{(1-\delta) + \delta(p_{AB} + p_{BA})}$$

Note in particular that A's share in the joint equilibrium payoff of A and B, $W^A/(W^A + W^B)$, is given by the conditional probability that A proposes to B whenever A and B meet in decentralized bargaining, $p_{AB}/(p_{AB} + p_{BA})$. Moreover, C's equilibrium payoff relative to the joint equilibrium payoff of A and B is increasing in the probability that C can propose to B in majority-minority bargaining, ρ_{CB}, and in the ratio $u_B(z_C)/u_A(z_C)$.

Substituting the values from Lemma 5.2 in the equilibrium conditions (1)-(6) we obtain a homogeneous system of inequalities that is linear in the payoffs $u_i(z_j)$ (see (1b)-(6b) in Lemma 7.3 in the Appendix). Since this is a homogeneous system, it has a solution when all parties are indifferent between all alternatives, i.e., $u_i(z_j) = 0$ for all $i, j \in N$. However, we want to know if there can be a broker equilibrium when each party has a strict preference for its own ideal policy. Formally, we ask that $u \in \mathcal{U}$, where

$$\mathcal{U} \equiv \{u \in \mathbb{R}^9 : -u_i^* < 0, -u_i^* + u_i(z_j) < 0\ \forall i = A, B, C, j \neq i\}$$

The system (1b)-(6b), together with the requirement that $u \in \mathcal{U}$, and the dominance relations ($A \gg C \gg B, A \gg B$), still form a system of linear inequalities in the unknowns $u_i(z_j)$, which can be written as $\Lambda u \leq \alpha$ for a matrix of coefficients Λ, where

$$u^T = \begin{pmatrix} u_A^* & u_C(z_A) & u_B(z_A) & u_A(z_C) & u_C^* & u_B(z_C) & u_A(z_B) & u_C(z_B) & u_B^* \end{pmatrix}, \ \alpha \equiv \begin{bmatrix} \mathbf{0}_9 \\ \mathbf{b}_9 \end{bmatrix}$$

for some $b > 0$, and Λ is an $m \times 9$ matrix whose elements are functions of the *matching parameters* $\omega \equiv (p, \rho, \delta) \in \Omega$. Thus, proving that there exists $u \in \mathcal{U}$ that admits a broker equilibrium boils down to proving that the system of linear inequalities $\Lambda u \leq \alpha$ has a solution. At this point, the following result, known as Farkas' Lemma (see Rockafellar (1970), Theorem 22.1), is useful:

Lemma 5.3 (Farkas' Lemma) *Let Λ be an $m \times n$ matrix, and $\alpha \in \mathbf{R}^m$. Then one and only one of the following alternatives holds:*

1. *There exists a vector $u \in \mathbf{R}^n$, such that $\Lambda u \leq \alpha$, or*
2. *There exists a non-negative vector $\lambda \in \mathbf{R}^m$ such that $\lambda^T \Lambda = \mathbf{0}$ and $\lambda^T \alpha < 0$.*

Lemma 5.3 is useful because showing that the linear system of equalities $\lambda^T \Lambda = \mathbf{0}_n$ does not have a nonnegative solution $\lambda \in \mathbf{R}^m$ such that $\lambda^T \alpha < 0$ is considerably simpler (algebraically) than proving that the system of linear inequalities $\Lambda u \leq \alpha$ has a solution.

With this simplification, we can prove our first main result. For convenience, we define

$$\upsilon \equiv (1 - \delta) + \delta(p_{AB} + p_{BA}), \quad \theta \equiv p_{AB} + p_{BA}\rho_{CB}, \quad \mu \equiv p_{BA} + p_{AB}\rho_{CB}$$

Theorem 5.4 *Suppose there is a dominant majority party A. For any $\omega \in \Omega$, there exists a compact set of preference profiles $U_\omega \subset \mathcal{U}$ such that for any $u \in U_\omega$, the legislative bargaining game with parameters (ω, u) admits a broker equilibrium if and only if $\omega \in \Omega^* \equiv \{\omega \in \Omega : (1 - \delta\, \rho_{BC})\, \upsilon \leq \delta^2\, \theta\}$.*

Note that increasing ρ_{BC} expands Ω^*, and thus broadens the conditions under which there is a broker equilibrium. The intuition is as follows. Increasing the likelihood that the broker has agenda setting power in majority-minority bargaining has the direct effect of increasing its bargaining power, and therefore the share of the surplus it can obtain when negotiating with the ultimate buyer. As a result, B is now more inclined to negotiate with C instead of implementing its preferred policy after obtaining A's votes, in line with equilibrium. Increasing ρ_{BC} also has indirect effects on bargaining incentives in decentralized bargaining. In particular, since B is now more able to extract surplus from C, reaching the majority minority stage is not as desirable for C ($5b$-$6b$). The condition $(1-\delta\,\rho_{BC})\,\upsilon \leq \delta^2\,\theta$, characterizing Ω^*, therefore says that the tightening of the constraints ($5b$) and ($6b$) (the indirect effect), never overpowers the loosening of the constraints ($1b$)-($4b$) (the direct effect).

Reducing bargaining frictions also unambiguously expands the conditions under which there is a broker equilibrium. In fact, increasing δ not only expands Ω^*, but also weakly relaxes each of the equilibrium constraints ($1b$)-($6b$).[12] In fact, the condition $(1-\delta\,\rho_{BC})\,\upsilon \leq \delta^2\,\theta$ is always satisfied in the limit as $\delta \to 1$. Thus, as bargaining frictions vanish, there is always a preference profile for which there is a broker equilibrium; i.e.,

$$\lim_{\delta\to 1}\Omega^* = \Omega.$$

This result has two implications. First, it shows that the existence of broker equilibria does not require that A and C have few opportunities to trade. It also shows that with no bargaining frictions, no matching environment can be ruled out as incompatible with brokers; i.e., that it is not possible to find necessary conditions for the existence of a broker equilibrium by looking only at the matching environment.

The next two results address two immediate possible concerns regarding Theorem 5.4. A first possible concern is that the conditions for existence of a broker equilibrium are *knife-edge*. Proposition 5.5 shows that this is not the case.

Proposition 5.5 *For every $\omega \in int(\Omega^*) \neq \emptyset$, there is an open subset $P \subset \Omega^*$ containing ω, and an open subset $V \subset \mathcal{U}$, such that for any $(\omega', u) \in P \times V$, (ω', u) admits a broker equilibrium.*

A second possible concern is that the result in Theorem 5.4 might be a curiosity arising

[12] As we discuss in Section 5.1.2, this is due to the fact that the value of the transfers between players increases as bargaining frictions vanish.

from a large multiplicity of equilibria in which *everything goes*. Proposition 5.6 shows that this is not the case either.

Proposition 5.6 *Suppose there is a dominant majority party and $\omega \in \Omega^*$, then whenever a broker equilibrium exists, (1) it is the unique equilibrium in which z_C is the policy outcome, and (2) there is no equilibrium in which z_B is the policy outcome. Moreover, (3) there exists a nonempty subset of parameters $O \subset \Omega^*$ such that for any $\omega \in O$ there exists u in compact set of preference profiles $U_\omega \subset \mathcal{U}$ such that the broker equilibrium is the unique pure strategy MPE of the legislative bargaining game with parameters (ω, u).*

5.1.1 Outcomes and Welfare

As we discussed earlier, the case of a dominant majority party is interesting because A would not trade with B if C were not present, and similarly A would not trade with C if B were not present. Thus, in this counterfactual comparison across games, the presence of the broker unambiguously changes policy outcomes vis-à-vis a two-party legislature. Furthermore, we have defined broker equilibria so that A and C do not trade directly when they meet. Thus *in this equilibrium* the broker is creating a trade that would not occur without it. In addition, Proposition 5.6 establishes that *there is no equilibrium* in which A and C trade directly. We conclude that the broker is creating a trade that would not have occurred without it, in this or any other equilibrium.

The fact that brokers enable transactions that wouldn't have occurred in their absence does not imply that A and C are better off with than without brokers. Note that the values of the game in which only A and C are present are given by $\hat{W}^A(\mathbf{k}) = u_A^*$ and $\hat{W}^C(\mathbf{k}) = u_C(z_A)$. Since in a broker equilibrium $u_A^* < \delta W^A(\mathbf{k})$, it is immediate to verify that the majority party benefits from the existence of the broker. However, from Lemma 5.2, $\Delta W^C(\mathbf{k}) = (\delta/\upsilon)(p_{AB} + p_{BA})\,\rho_{CB}\,S_{BC}(z_C) - u_C(z_A)$, which in general can be positive or negative. Thus the ultimate buyer might prefer that no trades were set in motion in the first place.

In fact, the broker equilibrium is not generally efficient, even as frictions vanish. Suppose first that $\delta < 1$. Note that in all strategy profiles in which the majority-minority game is reached, one of the players meets all others. Thus, in any such strategy profile, we can always transfer rents from any player to any other player, and efficiency coincides with maximizing aggregate policy payoffs. But then the individual payoffs of all players can be improved from what they obtain in a broker equilibrium if A cedes its votes to C whenever they meet in decentralized bargaining (this reduces delay). Now, in the limit with $\delta \to 1$, it is still possible that equilibrium payoffs approach efficiency. We show however that this is not the case generically. Note that from Lemma 5.2, parties' aggregate welfare in an equilibrium with brokers is given by

$$\sum_i W^i(\mathbf{k}) = \left(\frac{\delta(p_{AB} + p_{BA})}{(1-\delta) + \delta(p_{AB} + p_{BA})} \right) \sum_i u_i(z_C)$$

so that $\sum W^i(\mathbf{k}) \to u_A(z_C) + u_C^* + u_B(z_C)$ as $\delta \to 1$. It is then enough to show that there is a preference profile $u \in \mathcal{U}$ with the property that $\sum_i u_i(z_C) < \sum_i u_i(z_B)$ admitting an equilibrium with brokers with $\delta \to 1$. It can be verified that this happens for example with the preferences of Table 1, given uniform matching.

Party/Policy	z_A	z_B	z_C	Q
A	10	0	-485	0
B	10	20	10	0
C	505	990	1000	0

Table 1: A Preference Profile admitting an Inefficient Equilibrium with Brokers with uniform matching and $\delta \to 1$.

On the other hand, broker equilibria are always *welfare improving*: for any $\delta \in (0,1)$, conditions $(1b)$ and $(6b)$ imply that in any broker equilibrium with a dominant majority party A, $\sum_i [W^i(\mathbf{k}) - u_i(z_A)] > 0$.

The possible inefficiency of equilibria in our model contrasts with the result in Gul (1989), which establishes efficiency for $\delta \to 1$. However, it is aligned with similar results in the literature on non-cooperative coalitional bargaining games with externalities, in which inefficiency is a robust phenomenon (Ray and Vohra (2013)).

5.1.2 The Role of the Broker

Theorem 5.4 shows that under a broad set of conditions, there exists an equilibrium with brokers. The theorem, however, is silent about the preference profiles under which brokers can emerge in equilibrium. Thus, it is still possible that these preference profiles are in some sense exceptional, and not likely to arise in applications. In this section we show that this is not the case. We also establish properties of the broker and the environment under which brokers emerge in equilibrium.[13]

Does the existence of brokers relies in some way on *pathological* preference profiles? To address this question, we ask whether a broker equilibrium can be consistent with the standard notion of 'well-behaved' preferences: the class of single-peaked preference profiles, $\mathcal{U}^{SP}$. Our next Proposition states this result.

Proposition 5.7 *Suppose there exists a dominant majority party A. Let* $\Omega^{**} \equiv \{\omega \in \Omega : (1-\delta\rho_{BC})\upsilon < \delta^2 \min\{\theta, \mu\}\}$*. Then for any* $\omega \in \Omega^{**}$*, there exists a compact set* $U_\omega \subset \mathcal{U}^{SP}$ *such that all* $u \in U_\omega$*, the legislative bargaining game with parameters* (ω, u) *admits a broker equilibrium.*

In fact, linear and quadratic payoffs are also admissible. Figure 5.1.2 illustrates this in the quadratic payoff case. To maintain the normalization that $u_i(Q) = 0$, we write i's payoff function as $u_i(x) = -\beta_i(x - z_i)^2 + \beta_i(Q - z_i)^2$ in the case of the quadratic utility function, and similarly for the linear payoffs. Note that here the condition $i \gg j$ boils down to $\beta_i > \beta_j$. Thus, in these examples we must have $\beta_C > \beta_B$. This must be the case, because in a broker equilibrium C (the final buyer) has to dominate the broker so that there is a final transaction. In addition to this, in the example in the figure we have $u_B(z_C) \geq u_B(z_A)$ and $u_B(z_C) \geq u_B(Q)$, which in this context imply that the broker's ideal policy z_B must be closer to z_C than to both z_A and the status quo Q. Thus, if for example $z_C < z_A < Q$, as in the figure, the broker's preferred policy cannot be to the right of that of the majority party. This result extends beyond the example, and in fact beyond the class of single-peaked preference profiles, to all preference profiles that are consistent with a broker equilibrium.

Proposition 5.8 *Suppose there is a broker equilibrium with a dominant majority party A. Then the broker directly benefits from policy change; i.e., prefers the policy implemented*

[13] To obtain these results, we further exploit the duality results from convex analysis, transforming restrictions on preference profiles into a modified matrix Λ' of matching parameters, and obtaining conditions for existence of a solution to the underlying system of inequalities following the same steps as in Theorem 5.4. The algebraic derivations are relegated to Online Appendix A.

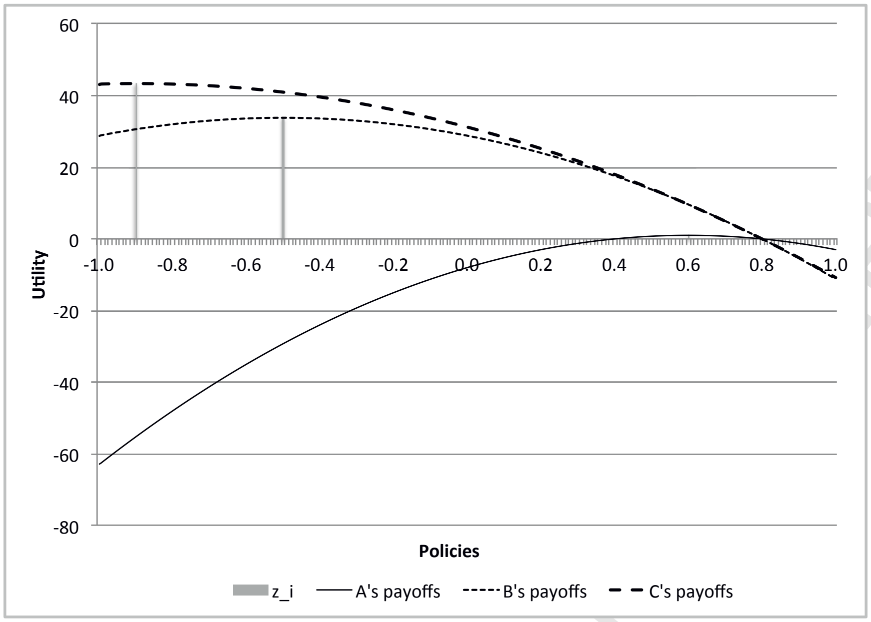

Figure 3: Quadratic utility functions admitting an equilibrium with brokers. Here $\rho_{23} = 0.2$, and p is uniform. $\delta = 0.95$. $\beta_A = 25, \beta_B = 15$ and $\beta_C = 20$.

in equilibrium to the status quo, $u_B(z_C) > u_B(Q)$, and to the ideal policy of the majority party, $u_B(z_C) > u_B(z_A)$.

Proposition 5.8 shows that in order to have an equilibrium with brokers, the party acting as broker *must* have a stake in the policy outcome.

Corollary 5.9 *Suppose that there is a dominant majority party A, and consider any matching parameters $\omega \in \Omega$ and preference profile $u \in \mathcal{U}$. If $u_B(x) = u_B \in \mathbb{R}$ for all $x \in X$, the game with parameters (ω, u) does not admit a broker equilibrium.*

The result of Proposition 5.8, however, goes well beyond this. Since the broker must prefer implementing z_C to the status quo, its presence increases the aggregate surplus of implementing z_C *relative to inaction*. In addition, Proposition 5.8 shows that the broker must also prefer implementing z_C to z_A. As a result, whenever there is a broker equilibrium, the broker must also increase the aggregate surplus of implementing z_C *relative to the majority policy*.

Who appropriates these policy gains? To address this question, we decouple the equilibrium payoff W^i of each party i into two components: a *policy value* PV_i capturing the payoff attributable to policy, and an expected transfer T_i. In a broker equilibrium where B transfers votes from A to C, the policy value PV_i of player i is $\delta u_i(z_C)$ with probability $(p_{AB} + p_{BA})$ and δPV_i with probability $(1 - p_{AB} - p_{BA})$. Thus

$$PV_i = \frac{\delta(p_{AB} + p_{BA})}{\upsilon} u_i(z_C) \tag{7}$$

Given (7) and the total equilibrium payoffs from Lemma 5.2, we can then compute the expected transfer to player i in this equilibrium by $T_i = W_i - PV_i$. Doing this for the final buyer C, we obtain

$$T_C = -\frac{\delta\,(p_{AB} + p_{BA})}{\upsilon} \times [u_C^* - \rho_{CB}\, S_{BC}(z_C)]$$

Now, condition (2b), which assures that the broker has incentives to *carry* the trade after acquiring the votes from A, requires that $\delta\,\rho_{BC}\, S_{BC}(z_C) \geq u_B^*$. But then $\rho_{BC}\, S_{BC}(z_C) > u_B(z_C)$, or equivalently, $u_C^* - \rho_{CB}\, S_{BC}(z_C) > 0$. Thus, in equilibrium, the party implementing its preferred policy, C, makes a positive transfer to the broker B; i.e., $T_C < 0$. Similarly, as we show in the proof of the next result, (2b), (5b), and $A \gg C$ imply $T_B < 0$. Thus,

Proposition 5.10 *In any broker equilibrium with a dominant majority party A, both the broker B and the final buyer C make ex ante positive transfers to other parties, while the majority party A receives ex ante positive transfers from other parties.*

Proposition 5.10 says that A can extract rents from both B and C. Thus, while B and C benefit from implementing z_C relative to both the status quo and the preferred policy of the majority party, z_A, A is able to extract some of this policy benefit from both parties.

Together, Propositions 5.8 and 5.10 highlight a fundamental difference between intermediaries in politics and in exchange economies. Differently to intermediaries in exchange economies – who can only benefit by making a monetary profit – the legislative intermediary must care about policy outcomes. In fact, in equilibrium the broker is rewarded with the prospect of a policy gain when making the initial trade with the majority party A, *and* recovers some of its monetary loses in bilateral trading. Each of these parts is important. The first part says that the political broker is not a two-sided platform that can charge both sides for its services, but an agent who derives surplus from facilitating a beneficial policy change. The second part is important too, for A must anticipate that once the broker is in a position of power it will keep on negotiating, instead of using this power to implement its preferred policy.

Proposition 5.10 also shows that the broker will not be the sole source of compensation to the majority party. In fact, the strategic environment must be such that the broker can extract sufficient rents from the final buyer, putting in motion a *chain* of rent extraction. The point is that the broker equilibrium provides the incentives for the broker and the ultimate buyer to compensate the majority party. It is not the only way to extract rents from B, and in fact it is not the efficient way to extract rents from B (see section 5.1.1). However, the broker is an instrument to make this transfer of resources incentive compatible.

5.2 Arbitrary Dominances with a Majority Party

A key feature of our model is that parties bargain over public decisions, which affect the payoff of all players. Because of this feature, the trade-offs that parties face when negotiating with one another in decentralized bargaining depend on their equilibrium beliefs about the path of play following each possible trade, on and off the equilibrium path. These equilibrium conjectures were uniquely pinned down in the case of a dominant majority party, but will generally differ across dominance relations.

Consider for example meetings between A and B. In a broker equilibrium B must buy A's votes, and then support C's preferred policy in exchange for rents. Thus the joint continuation value for A and B after A sells its votes to B is $\Pi(AB, C)$. How does this compare to the prospect of A buying B's votes instead? In the case of a dominant majority party, $A \gg C$ by hypothesis, so when A buys votes from B it implements its preferred policy immediately. As a result, selling to B dominates buying from B if $\Pi(AB, C) \geq S_{AB}(z_A)$ (eq.3). However, if A is not a dominant majority party, it is possible that $C \gg A$. Indeed, if $C \gg A$ and $u_A^* \leq \delta\rho_{AC}S_{AC}(z_C)$, then after buying B's votes A would have incentives to sell her votes to C in exchange for rents. In this case the joint continuation value for A and B after A buys B's votes is $\Pi(BA, C) \neq S_{AB}(z_A)$.

The fact that equilibrium conjectures generally differ across dominance relations poses a natural question. Is the assumption that the majority party A dominates B and C necessary for the existence of a broker equilibrium? Is it possible that a broker equilibrium exists even if the initial seller is dominated by the broker, the ultimate buyer, or both? Theorem 5.11 establishes that with the exception of the *brokerage condition* (2), which requires that the broker is dominated by the ultimate buyer, the restrictions on dominance relations imposed in the dominant majority party case are not essential for the result. We show that there are matching parameters and preferences for which a broker equilibrium exists for *each* dominance relation.

Theorem 5.11 *Suppose $k_A \geq r$ and fix any dominance relation $\gg$. There is a set of matching parameters $\Omega^{\dagger}_{\gg} \subset \Omega$ such that if $\omega \in \Omega^{\dagger}_{\gg}$, then there exists a compact set of preference profiles $U_\omega \subset \mathcal{U}$ such that for all $u \in U_\omega$, (ω, u) admits a broker equilibrium.*

The proof of this result consists of several steps. First, we generalize the analysis of incentives in decentralized bargaining; i.e., we characterize which trades a party would want to carry out in each bilateral meeting, having anticipated the consequences of alternative trades on rent and policy outcomes (Propositions B.2 and B.3).[14] The resulting conditions characterize all broker equilibria, for given continuation values (Lemma B.4). The values, $W^i(\mathbf{k})$, in turn, are still determined by Lemma 5.2. Substituting, and adding the dominance conditions and strict maxima conditions, we obtain, for each dominance relation $\gg$, $L_{\gg}$ systems of the form $\Lambda^\ell u \leq \alpha^\ell$, $\ell = 1, \ldots, L_{\gg}$, characterizing the conditions on the primitives of the model under which there is a broker equilibrium given $\gg$.[15]

To prove the theorem, we first show that if there is a preference profile that admits brokers when $(A \gg C, C \gg B, A \gg B)$, there is one that admits brokers with $(A \gg C, C \gg B, B \gg A)$ (Proposition B.5). Thus, the sufficient condition for brokers in Theorem

[14]A key result in this direction is the efficiency of bilateral meetings for the parties involved in the trade. In any meeting between two players i and j, *given continuation values* – and thus given the strategies of all players, including their own future play – the outcome of the meeting between i and j is efficient for i and j (Remark B.1).

[15]Note that there is one way in which the dominant majority party case is special. For *this* dominance relation, all equilibrium continuations are uniquely determined. As a result, substituting the values from Lemma 5.2, the system (1)-(6) completely characterizes the set of parameters for which a broker equilibrium exists. In general, however, off-path continuations can vary for different parameters, even fixing the dominance relation. As a result, generally there are multiple systems of the form $\Lambda' u \leq \alpha$ for each possible dominance relation.

5.4 is also sufficient for brokers whenever $A \gg C$ and $C \gg B$. We then extend the existence result to the remaining dominance relations, and provide alternative conditions under which a similar result holds when $(C \gg B, B \gg A, C \gg A)$ (Proposition B.6) and $(C \gg A, A \gg B, C \gg B)$ (Proposition B.7). Due to space limitations, these proofs are included in Online Appendix B.

5.3 A Minority Legislature

In this section we extend our analysis of political intermediation to legislatures in which no party has a majority of the votes. Fractionalized power is a relatively common occurrence in legislatures across the world: in 45% of the seat distributions in presidential democracies and 57% of seat distributions in parliamentary democracies, no party controlled a majority of seats in the legislature (Cheibub, Przeworski, and Saiegh (2004)). In these cases, either minority parties form relatively stable policy coalitions, or policy compromises are attained on a case-by-case basis, suggesting that political intermediation can be particularly important in these settings. In fact, Cheibub, Przeworski, and Saiegh (2004) show that the absence of a majority party doesn't affect legislative success, as measured by the proportion of government bills turned into law: single-party minority governments are at least as successful as majority coalitions.

From a theoretical standpoint, minority legislatures introduce two new considerations. First, since any two parties form a majority, any vote share is strategically equivalent to one in which each party has one vote. In this context, there is no natural assignment of parties to roles (seller, broker, ultimate buyer), and it is possible that more than one strategy profile is supported in equilibrium for any given dominance relation, given appropriate parameters. Second, for a *given strategy profile*, parties face different incentives in decentralized bargaining. When A has a majority, any trade involving A resolves in the buyer having a majority of the votes, but any trade between the two minority parties leaves the majority unchanged. When no party has a majority, on the other hand, any trade between two parties resolves in the buyer having a majority of the votes, independently of the initial distribution of voting rights.

Do these changes in the structure of the game result in any constraint on the dominance relations $\gg$ under which political intermediaries can be supported in equilibrium? Our next result establishes that – as in the case of a majority party – a particular configuration of dominance relations is not necessary for the existence of a broker equilibrium when power is fractionalized. However, it also establishes restrictions on the roles that parties can play for a given dominance relation.[16]

Theorem 5.12 *Suppose $k_i < r$ for all $i = A, B, C$, and fix any dominance relation $\gg$.*

1. *There is a $\Omega^{\ddagger}_{\gg} \subset \Omega$ such that if $\omega \in \Omega^{\ddagger}_{\gg}$, then there exists a compact set of preference profiles $U_\omega \subset \mathcal{U}$ such that for all $u \in U_\omega$, (ω, u) admits a broker equilibrium.*
2. *If $\gg$ is cyclic ($A \gg C, C \gg B, B \gg A$), a broker equilibrium can occur if the initial seller dominates or is dominated by the broker or the ultimate buyer.*
3. *If $\gg$ is transitive, neither the broker nor the ultimate buyer can dominate the initial seller. Thus, in a broker equilibrium the seller is the dominant party, and the broker is a dominated party.*

It follows that if we fix the role parties play in a broker equilibrium, a broker equilibrium can only exist for some configurations of the dominance relation $\gg$. In particular, if the dominance relation is transitive, only the dominant party can be the seller, and only the dominated party can be the broker.

To see the intuition for this result, consider a broker equilibrium in which B buys votes from A in decentralized bargaining, and then sells its votes to C. Why doesn't B sell its votes to C directly in decentralized bargaining? This is surprising, because we know that both B and C give money away in this broker equilibrium. The reason why B does not sell to C directly is that if it were to do that, C would then sell its votes to A, who would end up implementing a policy B dislikes. Because of this, B is willing to give away rents to avoid the anticipated policy loss. The same logic explains why it must be the case that $A \gg C$: otherwise C would implement its preferred policy right away after buying from B in decentralized bargaining, and B would prefer to sell its votes to C. This would upset

[16]Since in this case any trade between any two parties resolves in the buyer having a majority of the votes, Proposition B.2 applies to all pairwise meetings. Thus, for any candidate equilibrium σ, the conditions that result from applying Proposition B.2 to the trades prescribed by σ, together with the brokerage condition (eq.2), characterize the conditions on parameters for σ to be an equilibrium. We then prove our next result proceeding as in Section 5.2 for any given strategy profile.

the broker equilibrium.[17] Now, $C \gg B$ is fixed by assumption. And we have just argued that $A \gg C$. What remains to complete is either $A \gg B$ or $B \gg A$. But $B \gg A$ implies that $\gg$ is cyclic. Thus, if $\gg$ is transitive, it must be that $A \gg C$ and $C \gg B$.

5.4 Transfers

In Proposition 5.10 we showed for the case of a dominant majority party that in any broker equilibrium the final buyer transfers rents to the broker. Because the proof of this result relies only on the brokerage condition (eq. 2), it applies generically, for any dominance relation $\gg$ and any initial distribution of voting rights $\mathbf{k}$. The same proposition also established – for the case of a dominant majority party – that in a broker equilibrium the broker must be a net loser of rents in expectation. Thus, in equilibrium, the majority party can extract some of the policy gains from the ultimate buyer.

While the proof of this result does not extend immediately to all cases, the result does hold generically. In fact, since we know from our previous results that for each $(\mathbf{k}, \gg)$ the set of matching parameters and preference profiles (ω, u) that admit a broker equilibrium, say $\mathcal{M}$, is closed and has a nonempty interior, we can just write this problem as that of choosing $(\omega, u) \in \mathcal{M}$ to maximize the net expected transfers to the broker, T_B, and check whether the solution, T_B^*, is such that $T_B^* < 0$. We can then simply solve this problem numerically, for each $(\mathbf{k}, \gg)$. We do this in matlab, using the Global Optimization Toolbox.[18] Our numerical results extend the result of Proposition 5.10 to all $(\mathbf{k}, \gg)$, and confirm that in a broker equilibrium the broker must also be a net loser of rents in expectation.[19]

[17] This does not happen when A has a majority of the votes because any trade between B and C leads A to implement her preferred policy outright, which prevents them from trading.

[18] All codes are available upon request.

[19] In the same way, we also showed that the results of Proposition 5.8 also extend to all $(\mathbf{k}, \gg)$. Thus, if there is a broker equilibrium, the broker prefers the policy of the ultimate buyer to both the policy of the seller and the status quo.

6 Conclusion

Enacting new policies in collective bodies requires compromises and political exchanges among legislators with different political views. This process of legislative bargaining has two readily observable characteristics. First, compromises among members of a legislative coalition are typically made in a decentralized bargaining process (i.e., backroom deals). Second, whenever there are more than two legislative blocks, this process of decentralized bargaining leads naturally to the emergence of legislative intermediaries. In this paper, we proposed a model of legislative bargaining that captures this decentralized sequential bargaining process, and focused on the role that political intermediaries can have in this setting.

We showed that the emergence of legislative intermediaries is a robust equilibrium phenomenon. The existence of a broker equilibrium (i) is generic, and does not depend (ii) on special frictions in the opportunities that parties have to trade with one another, (iii) on the initial allocation of voting rights, (iv) on the existence of cycles in the majority preference or (v) on particular constraints in the dominance relations.

We showed that the triangulation of political agreements implemented by the broker equilibrium can impact outcomes and welfare. In fact, we established conditions for the existence of a broker equilibrium even when there is a *dominant* majority party; i.e., a party that has both a majority of the votes, and a higher willingness to pay than all others in binary comparisons. In this case, the broker equilibrium implements a different policy outcome than the one that would result in the absence of the broker, or in any equilibrium maintaining the composition of the legislature.[20] In particular, whenever a broker equilibrium exists, it is the unique equilibrium that implements C's preferred outcome, and there is no equilibrium in which B implements its preferred outcome. In addition, under some additional conditions the broker equilibrium is the unique Markov Perfect equilibrium.

The existence of a broker equilibrium has direct empirical implications regarding the nature of political trades, including which agents will, and which agents will not negotiate

[20] This equilibrium outcome is inefficient whenever there are bargaining frictions, and can be inefficient even as frictions vanish. The inefficiency of vote trading in this setting is consistent with results in the literature of noncooperative dynamic coalition formation in the presence of externalities, and provides further evidence against the ability of markets for votes to attain efficient outcomes.

with each other, or will do so only after observing or negotiating other trades. In particular, when one party has a majority of the votes or there is no majority party but the dominance relation cycles, the only constraint is that the ultimate buyer has a higher preference intensity than (i.e., *dominates*) the broker; i.e., that C's willingness to pay to retain its preferred policy instead of that of B be larger than B's willingness to pay to retain its preferred policy instead of C's. This is necessary of course, because otherwise the broker would simply implement its preferred policy when it attains a majority of the votes. When instead there is no majority party and the dominance relation is transitive, neither the broker nor the ultimate buyer can dominate the seller. Thus, there can be a broker equilibrium in which B brokers a deal transferring decision power from A to C only if A would *buy* C's votes in a two party committee.

We also establish the following additional empirical implications:

First, we show that in order to be able to fulfill this role, the broker *must* have a stake in the policy outcome. In particular, the legislative intermediary must prefer the final policy outcome to both the status quo and the preferred policy of the party whom he initially transacts with. In the case of euclidean preferences in a unidimensional policy space, for instance, the broker's ideal point must be closer to C's ideal policy than to A's ideal policy. Second, we establish precise implications regarding the direction of transfers. As one could anticipate, the agent implementing its preferred policy will be a net contributor in expectation, and the original seller will be a net recipient of transfers in expectation. What is less immediate, perhaps, is that for all allocations of voting rights and dominance relations, the broker must also be a net contributor.

Third, we also show that reducing bargaining frictions (or increasing the discount factor δ) makes political intermediation more likely – in the sense that the set of parameters for which a broker equilibrium exists is increasing in δ – and increases all agents' equilibrium payoffs in a broker equilibrium.

More broadly, our results also suggest that in the context of decentralized bargaining, the power of a legislative actor can be unrelated to its vote share or bargaining power, instead being determined by the role they can attain in bringing a coalition together.[21] This introduces a new mechanism to what has been outlined in the literature through

[21] This is different, for example, that in Baron-Ferejohn models, where agents' value is monotonic in their proposal power; see Eraslan (2002).

which small parties can have a disproportionate effect on policy outcomes.

Throughout the paper, we have maintained the assumption that all actors in the model are members of the legislature, and thus endowed with voting rights. However, nothing in the model prevents the possibility that an outside party (say an interest group) plays the role of the broker, if allowed to participate in backroom deals. In fact, given access, an interest group is strategically equivalent to internal members, except that it cannot sell votes in the initial round of decentralized bargaining. The conditions for existence of a broker equilibrium in the paper are therefore also sufficient for existence of a broker equilibrium with an outside broker.

The three party model that we studied in this paper has the minimal structure required to study the emergence of *an* intermediary in legislative bargaining. The general set-up of the model of decentralized bargaining with an arbitrary committee size n can be useful to study issues that go beyond the analysis in this paper. A natural and interesting direction for future research is to allow competition between middlemen. As we discussed in the paper, a key consideration for the existence of a broker equilibrium is establishing a chain of rent extraction from the party implementing its preferred policy to the initial seller. This suggests the question: can competition among middlemen make equilibria with intermediation more elusive? Does this answer depend on the position of these agents in a network?[22] We leave these questions for future research.

[22]With more than three players, the particular definition of a broker equilibrium that we used in this paper would need to be amended slightly to reflect the fact that more than one player could act as a broker (either in competition with one another, or as part of a chain of trades). Thus we would not necessarily require that any particular broker trades on the equilibrium path with probability one, but that *some* broker does.

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7 Appendix

Lemma 7.1 *Suppose that the initial vote allocation at the beginning of a trading round is* $\mathbf{k} = (k_A, k_B, k_C)$ *with at least* $k_i > 0$ *and* $k_j > 0$ *for* $i, j \in \{A, B, C\}$, *let* $\mathbf{k}'_{-j}$ *denote the vote allocation that would result after* i *buys from* $j \neq i$ *in that round. Then (0.a)* $W^i_{ij}(\mathbf{k}; buy) = w^i(\mathbf{k}'_{-j}) + w^j(\mathbf{k}'_{-j}) - w^j(\mathbf{k})$, *(0.b)* $W^i_{ij}(\mathbf{k}; sell) = w^i(\mathbf{k}'_{-i}) + w^j(\mathbf{k}'_{-i}) - w^j(\mathbf{k})$, *and (0.c)* $W^i_{ij}(\mathbf{k}; wait) = w^i(\mathbf{k})$, *and thus:*

1. $W^j_{ji}(\mathbf{k}, buy) + w^i(\mathbf{k}) = W^i_{ij}(\mathbf{k}, sell) + w^j(\mathbf{k})$.
2. $W^i_{ij}(\mathbf{k}; sell) \geq W^i_{ij}(\mathbf{k}; buy)$ *if and only if* $w^i(\mathbf{k}'_{-i}) + w^j(\mathbf{k}'_{-i}) \geq w^i(\mathbf{k}'_{-j}) + w^j(\mathbf{k}'_{-j})$.
3. $W^i_{ij}(\mathbf{k}; sell) \geq W^i_{ij}(\mathbf{k}; buy)$ *iff* $W^j_{ji}(\mathbf{k}; buy) \geq W^j_{ji}(\mathbf{k}; sell)$.
4. $W^i_{ij}(\mathbf{k}; sell) \geq W^i_{ij}(\mathbf{k}; wait)$ *iff* $W^j_{ji}(\mathbf{k}; buy) \geq W^j_{ji}(\mathbf{k}; wait)$.

Proof of Lemma 7.1. To establish this result, note that $W^i_{ij}(\mathbf{k}; buy) = w^i(\mathbf{k}'_{-j}) - t^{buy}_{ij}(\mathbf{k})$. For j to accept, $w^j(\mathbf{k}'_{-j}) + t^{buy}_{ij}(\mathbf{k}) \geq w^j(\mathbf{k})$. Then in equilibrium $t^{buy}_{ij}(\mathbf{k}) = w^j(\mathbf{k}) - w^j(\mathbf{k}'_{-j})$. Substituting, $W^i_{ij}(\mathbf{k}; buy) = w^i(\mathbf{k}'_{-j}) + w^j(\mathbf{k}'_{-j}) - w^j(\mathbf{k})$. Similarly, $W^i_{ij}(\mathbf{k}; sell) = w^i(\mathbf{k}'_{-i}) - t^{sell}_{ij}(\mathbf{k})$, and for j to accept, $w^j(\mathbf{k}'_{-i}) + t^{sell}_{ij}(\mathbf{k}) \geq w^j(\mathbf{k})$, so in equilibrium $t^{sell}_{ij}(\mathbf{k}) = w^j(\mathbf{k}) - w^j(\mathbf{k}'_{-i})$. Substituting, $W^i_{ij}(\mathbf{k}; sell) = w^i(\mathbf{k}'_{-i}) + w^j(\mathbf{k}'_{-i}) - w^j(\mathbf{k})$. This establishes part 0. Parts 1 and 2 follow immediately from 0. Part 3 follows from 2, and part 4 follows from 1. ■

Proposition 7.2 *Suppose at time* τ_0 *two parties,* B *and* C, *have voting rights, where* $k_B > k_C$ *but* $C \gg B$. *Then there exists a MPE in which, independently of who has the opportunity to propose,* C *buys* B*'s votes and implements its preferred policy; i.e.,* $y_\tau = z_C$ *for all* $\tau \geq \tau_0$. *Moreover,*

1. *If* $u^*_B \leq \delta\, \rho_{BC}\, S_{BC}(z_C)$, *the majority party extends negotiations after disagreement. Here* $W^B(\mathbf{k}) = \rho_{BC}\, S_{BC}(z_C)$, $W^C(\mathbf{k}) = \rho_{CB}\, S_{BC}(z_C)$, *and* $u^*_B \leq \delta W^B(\mathbf{k})$.
2. *If* $u^*_B \geq \delta\, \rho_{BC}\, S_{BC}(z_C)$ *and* $(1-\delta)u^*_B \geq \delta\rho_{BC}\,[S_{BC}(z_C) - S_{BC}(z_B)]$, B *implements* z_B *after disagreement. Here* $W^B(\mathbf{k}) = u^*_B + \rho_{BC}(S_{BC}(z_C) - S_{BC}(z_B))$, $W^C(\mathbf{k}) = u_C(z_B) + \rho_{CB}(S_{BC}(z_C) - S_{BC}(z_B))$, *and* $u^*_B \geq \delta W^B(\mathbf{k})$.

3. *If neither of these conditions hold, there is no MPE in pure strategies. In equilibrium, the majority party implements its preferred policy after disagreement with probability*

$$\alpha^* = \frac{(1-\delta)}{\delta\rho_{BC}} \left(\frac{u_B^* - \delta\rho_{BC} S_{BC}(z_C)}{\delta S_{BC}(z_C) - S_{BC}(z_B)} \right) \quad (8)$$

Here $\delta W^B(\mathbf{k}) = u_B^*$ *and* $\delta W^C(\mathbf{k}) = \delta S_{BC}(z_C) - u_B^*$.

Proof of Proposition 7.2. As in the statement of the proposition, suppose that two parties, B and C, have voting rights, where $k_B > k_C$, and $C \gg B$. Let $\mathbf{k} = (k_B, k_C, 0)$. Consider the problem of the majority party when it has an opportunity to propose. B can, first of all, choose not to make a relevant offer (wait), guaranteeing its post trade continuation value $W_{BC}^B(\mathbf{k}, wait) = w^B(\mathbf{k})$. The conjectures for the post-trade continuation values $w^B(\mathbf{k})$ and $w^C(\mathbf{k})$ depend on whether B prefers to implement its preferred policy or extend negotiations after disagreement: if $u_B^* \geq \delta W^B(\mathbf{k})$, B prefers to implement z_B and $w^B(\mathbf{k}) = u_B^*$, $w^C(\mathbf{k}) = u_C(z_B)$, and if $u_B^* < \delta W^B(\mathbf{k})$, then B prefers to extend negotiations after disagreement, so $w^B(\mathbf{k}) = \delta W^B(\mathbf{k})$ and $w^C(\mathbf{k}) = \delta W^C(\mathbf{k})$.

But B can also exchange policy for rents by trading with C. If B makes a relevant sell offer to C, C will then implement z_C, so B gets a payoff $W_{BC}^B(\mathbf{k}, sell) = u_B(z_C) - t_{BC}^{sell}(\mathbf{k})$. For the minority party to accept the offer, $W_{BC}^C(\mathbf{k}, sell) = u_C^* + t_{BC}^{sell}(\mathbf{k}) \geq w^C(\mathbf{k})$. Thus in equilibrium a relevant sell offer has a transfer $-t_{BC}^{sell}(\mathbf{k}) = u_C^* - w^C(\mathbf{k})$, and

$$W_{BC}^B(\mathbf{k}; sell) = S_{BC}(z_C) - w^C(\mathbf{k})$$

Similarly, if B makes a relevant buy offer,

$$W_{BC}^B(\mathbf{k}, buy) = S_{BC}(z_B) - w^C(\mathbf{k})$$

Note that selling dominates buying and implementing z_B if and only if $C \gg B$, and implementing z_B dominates extending negotiations after disagreement if and only if $u_B^* \geq \delta W^B(\mathbf{k})$. Thus, if $u_B^* \geq \delta W^B(\mathbf{k})$, B makes C a relevant sell offer. When instead $u_B^* \leq \delta W^B(\mathbf{k})$, B either waits or makes C a relevant sell offer. Since in this case the majority party extends negotiations in the event that C rejects an offer, $w^B(\mathbf{k}) = \delta W^B(\mathbf{k})$ and $w^C(\mathbf{k}) = \delta W^C(\mathbf{k})$, and B sells to C if and only if $S_{BC}(z_C) \geq \delta \left[W^B(\mathbf{k}) + W^C(\mathbf{k})\right]$. To summarize, when B has the opportunity to propose and $C \gg B$, it sells if either (i) $u_B^* \geq \delta W^B(\mathbf{k})$ or (ii) $u_B^* \leq \delta W^B(\mathbf{k})$ and $S_{BC}(z_C) \geq \delta\left[W^B(\mathbf{k}) + W^C(\mathbf{k})\right]$, and otherwise waits and extends negotiations.

Establishing C's best response in the trading node $((C,B),\mathbf{k})$ for fixed continuation values follows from Lemma 7.1. The minority party B can buy, sell or wait. If it waits, it gets $W^C_{CB}(\mathbf{k}, wait) = w^C(\mathbf{k})$, and by Lemma 7.1, $W^C_{CB}(\mathbf{k}, buy) = S_{BC}(z_C) - w^B(\mathbf{k})$ and $W^C_{CB}(\mathbf{k}, sell) = S_{BC}(z_B) - w^B(\mathbf{k})$. Thus, given $C \gg B$, C either waits or makes B a relevant buy offer. If $u_B^* \geq \delta W^B(\mathbf{k})$, the majority party implements z_B after disagreement, and buying dominates waiting for C since $W^C_{CB}(\mathbf{k}; buy) \geq W^C_{CB}(\mathbf{k}; wait) \Leftrightarrow C \gg B$. If instead $u_B^* \leq \delta W^B(\mathbf{k})$, C prefers to trade if and only if $S_{BC}(z_C) \geq \delta\left[W^B(\mathbf{k}) + W^C(\mathbf{k})\right]$.

Proving the statements in the proposition now only requires to check the consistency of these best responses when values are determined endogenously.

Part 1. Suppose that in equilibrium (i) $u_B^* \leq \delta W^B(\mathbf{k})$ and (ii) $S_{BC}(z_C) \geq \delta[W^B(\mathbf{k}) + W^C(\mathbf{k})]$. Then C buys from B in both trading nodes and implements z_C, while the majority party extends negotiations after disagreement. Then

$$W^j(\mathbf{k}) = \delta W^j(\mathbf{k}) + \rho_{ji}\left\{S_{BC}(z_C) - \delta[W^B(\mathbf{k}) + W^C(\mathbf{k})]\right\} \quad \text{for } j = B, C,$$

and therefore $W^B(\mathbf{k}) = \rho_{BC} S_{BC}(z_C)$ and $W^C(\mathbf{k}) = \rho_{CB} S_{BC}(z_C)$. Substituting in $u_B^* < \delta W^B(\mathbf{k})$ gives $u_B^* \leq \delta \rho_{BC} S_{BC}(z_C)$. Substituting in $S_{BC}(z_C) \geq \delta[W^B(\mathbf{k}) + W^C(\mathbf{k})]$ gives $S_{BC}(z_C) \geq 0$, which is implied by $u_B^* \leq \delta \rho_{BC} S_{BC}(z_C)$.

Part 2. Suppose that in equilibrium $u_B^* \geq \delta W^B(\mathbf{k})$. Then C buys from B in both trading nodes and implements z_C, while B implements its preferred policy after disagreement, so

$$W^\ell(\mathbf{k}) = u_\ell(z_B) + \rho_{\ell j}(S_{BC}(z_C) - S_{BC}(z_B)) \text{ for } \ell = B, C$$

Substituting back in $u_B^* \geq \delta W^B(\mathbf{k})$ gives $(1-\delta)u_B^* \geq \delta \rho_{BC}\left[S_{BC}(z_C) - S_{BC}(z_B)\right]$.

Part 3. Finally, suppose that in equilibrium (i) $u_B^* < \delta W^B(\mathbf{k})$ and (ii) $S_{BC}(z_C) < \delta[W^B(\mathbf{k}) + W^C(\mathbf{k})]$. Then B does not make a relevant offer in $((B,C),\mathbf{k})$ and C does not make a relevant offer in $((C,B),\mathbf{k})$, after which B extends negotiations. As a result, agreement is never reached, and therefore $W^B(\mathbf{k}) = 0$ and $W^C(\mathbf{k}) = 0$. Substituting in (i), we get $u_B^* < 0$, which is impossible as long as $z_B \neq Q$. This shows that (i) there does not exist a MPE in which B and C do not trade in $((B,C),\mathbf{k})$ or $((C,B),\mathbf{k})$ and B extends negotiations. It remains to show that if (i) $u_B^* > \delta\rho_{BC}S_{BC}(z_C)$ and (ii) $(1-\delta)u_B^* < \delta\rho_{BC}\left[S_{BC}(z_C) - S_{BC}(z_B)\right]$, there exists a MPE in which B sells to C when they meet and B implements z_B after disagreement with probability $\alpha^* \in (0,1)$, and extends negotiations with probability $1-\alpha^*$, with α^* as defined in (8). To show this, we compute the values implied by this strategy profile and use the fact that B has to be indifferent between implementing z_B and extending negotiations after disagreement to compute α^* (details are available from the authors upon request). ■

Proof of Lemma 5.2. Consider $W^C(\mathbf{k})$. Note that in all trading nodes $((i,j),\mathbf{k})$ other than $((A,B),\mathbf{k})$ or $((B,A),\mathbf{k})$, $W_{ij}^C(\mathbf{k}) = \delta W^C(\mathbf{k})$, and $W_{AB}^C(\mathbf{k}) = W_{BA}^C(\mathbf{k}) = \delta W^C(\mathbf{k}'_{BC}) = \delta\rho_{CB}S_{CB}(z_C)$. Then

$$W^C(\mathbf{k}) = \frac{\delta(p_{AB}+p_{BA})\rho_{CB}S_{CB}(z_C)}{(1-\delta)+\delta(p_{AB}+p_{BA})}$$

Now consider $W^\ell(\mathbf{k})$ for $\ell \in \{A,B\}$. Note that in all trading nodes $((i,j),\mathbf{k})$ other than $((A,B),\mathbf{k})$ and $((B,A),\mathbf{k})$, $W_{ij}^\ell(\mathbf{k}) = \delta W^\ell(\mathbf{k})$, while $W_{AB}^\ell(\mathbf{k}) = W_{BA}^\ell(\mathbf{k}) = \Pi(AB,C) - \delta W^j(\mathbf{k})$, for $j \in \{A,B\}\setminus \ell$. Then

$$W^\ell(\mathbf{k}) = \frac{p_{\ell j}[\Pi(AB,C) - \delta W^j(\mathbf{k})]}{1-\delta(1-p_{\ell j})} \quad (9)$$

Solving the system (9) for $\ell = A, B$, we get

$$W^A(\mathbf{k}) = \frac{p_{AB}\Pi(AB,C)}{(1-\delta)+\delta(p_{AB}+p_{BA})} \quad \text{and} \quad W^B(\mathbf{k}) = \frac{p_{BA}\Pi(AB,C)}{(1-\delta)+\delta(p_{AB}+p_{BA})}$$

■

Lemma 7.3 *Assume the dominance relation $A \gg C \gg B, A \gg B$. There exists a broker equilibrium with a dominant majority party A if and only if there are payoffs $u_i(z_j) \in \mathbb{R}$ for $i,j \in N$ such that and the following system of linear inequalities is satisfied:*

$$\upsilon u_A^* - \delta^2\, p_{AB}\, u_A(z_C) - \delta^2\, p_{AB}\,\rho_{BC}\, u_C^* - \delta^2\, p_{AB}\,\rho_{BC}\, u_B(z_C) \leq 0 \quad \text{(1b)}$$

$$-\delta\,\rho_{BC}\, u_C^* - \delta\,\rho_{BC}\, u_B(z_C) + u_B^* \leq 0 \quad \text{(2b)}$$

$$u_A^* + u_B(z_A) - \delta u_A(z_C) - \delta\, \rho_{BC}\, u_C^* - \delta\, \rho_{BC}\, u_B(z_C) \leq 0 \quad \text{(3b)}$$

$$-u_A(z_C) - \rho_{BC}\, u_C^* - \rho_{BC}\, u_B(z_C) \leq 0 \quad \text{(4b)}$$

$$\upsilon u_A^* + \upsilon u_C(z_A) - \delta^2\, p_{AB}\, u_A(z_C) - \delta^2\, \theta\, u_C^* - \delta^2\, \theta\, u_B(z_C) \leq 0 \quad \text{(5b)}$$

$$\upsilon u_C(z_A) + \upsilon u_B(z_A) - \delta^2\, p_{BA}\, u_A(z_C) - \delta^2\, \mu\, u_C^* - \delta^2\, \mu\, u_B(z_C) \leq 0 \quad \text{(6b)}$$

Proof of Theorem 5.4. The equilibrium conditions (1)-(6) together with the requirement that $u \in \mathcal{U}$, and the dominance relations $A \gg C$, $C \gg B$, and $A \gg B$ form a system of linear inequalities in the unknowns $u_i(z_j)$, which can be written as $\Lambda u \leq \alpha$, where $\alpha^T \equiv (\mathbf{0}_9, -\mathbf{b}_9)$,

$$u^T = \begin{pmatrix} u_A^* & u_C(z_A) & u_C(z_A) & u_A(z_C) & u_C^* & u_B(z_C) & u_A(z_B) & u_C(z_B) & u_C^* \end{pmatrix},$$

and

$$\Lambda = \begin{pmatrix}
\upsilon & \upsilon & 0 & -\delta^2 p_{AB} & -\delta^2\theta & -\delta^2\theta & 0 & 0 & 0 \\
0 & \upsilon & \upsilon & -\delta^2 p_{BA} & -\delta^2\mu & -\delta^2\mu & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\delta\rho_{BC} & -\delta\rho_{BC} & 0 & 0 & 1 \\
1 & 0 & 1 & -\delta & -\delta\rho_{BC} & -\delta\rho_{BC} & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -\rho_{BC} & -\rho_{BC} & 0 & 0 & 0 \\
\upsilon & 0 & 0 & -\delta^2 p_{AB} & -\delta^2 p_{AB}\rho_{BC} & -\delta^2 p_{AB}\rho_{BC} & 0 & 0 & 0 \\
-1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 \\
-1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1
\end{pmatrix}$$

(The rows in the matrix correspond to the inequalities in the text in the following order. The first six rows are inequalities (6b), (7b), (3b), (4b), (5b), (2b). The next three rows are the dominance order, and the last nine rows guarantee that for all $i, j \in N$, $u_i^* > u_i(z_j)$

and $u_i^* > 0 = u_i(Q)$.)

It follows from Lemma 5.3 that our original system of inequalities does not have a solution if there exists a $\lambda \geq 0$ such that:

$$\upsilon\lambda_1 + \lambda_4 + \upsilon\lambda_6 - \lambda_7 - \lambda_9 - \lambda_{10} - \lambda_{11} - \lambda_{12} = 0$$

$$\upsilon\lambda_1 + \upsilon\lambda_2 - \lambda_7 + \lambda_{14} = 0$$

$$\upsilon\lambda_2 + \lambda_4 - \lambda_9 + \lambda_{17} = 0 \tag{10}$$

$$-\delta^2 p_{AB}\lambda_1 - \delta^2 p_{BA}\lambda_2 - \delta\lambda_4 - \lambda_5 - \delta^2 p_{AB}\lambda_6 + \lambda_7 + \lambda_{11} = 0$$

$$-\delta^2\theta\lambda_1 - \delta^2\mu\lambda_2 - \delta\rho_{BC}\lambda_3 - \delta\rho_{BC}\lambda_4 - \rho_{BC}\lambda_5 - \delta^2 p_{AB}\rho_{BC}\lambda_6 + \lambda_7 - \lambda_8 - \lambda_{13} - \lambda_{14} - \lambda_{15} = 0$$

$$-\delta^2\theta\lambda_1 - \delta^2\mu\lambda_2 - \delta\rho_{BC}\lambda_3 - \delta\rho_{BC}\lambda_4 - \rho_{BC}\lambda_5 - \delta^2 p_{AB}\rho_{BC}\lambda_6 - \lambda_8 + \lambda_{18} = 0 \tag{11}$$

$$\lambda_9 + \lambda_{12} = 0 \tag{12}$$

$$\lambda_8 + \lambda_{15} = 0 \tag{13}$$

$$\lambda_3 + \lambda_8 + \lambda_9 - \lambda_{16} - \lambda_{17} - \lambda_{18} = 0$$

and

$$\sum_{i=10}^{18} \lambda_i > 0 \tag{14}$$

From (12), $\lambda_9 = \lambda_{12} = 0$, from (13), $\lambda_8 = \lambda_{15} = 0$, and from (10) and $\lambda_9 = 0$, $\lambda_2 = \lambda_4 = \lambda_{17} = 0$. After substituting, we can further obtain $\lambda_3 = \lambda_{16} + \lambda_{18} \geq 0$, and $\lambda_7 = \upsilon\lambda_1 + \lambda_{14} \geq 0$. Substituting, the dual system becomes

$$\upsilon\lambda_6 - \lambda_{14} - \lambda_{10} - \lambda_{11} = 0 \tag{15}$$

$$[\upsilon - \delta^2 p_{AB}]\lambda_1 - \lambda_5 - \delta^2 p_{AB}\lambda_6 + \lambda_{14} + \lambda_{11} = 0$$

$$[\upsilon - \delta^2\theta]\lambda_1 - \delta\rho_{BC}\lambda_{16} - \delta\rho_{BC}\lambda_{18} - \rho_{BC}\lambda_5 - \delta^2 p_{AB}\rho_{BC}\lambda_6 - \lambda_{13} = 0$$

$$-\delta^2\theta\lambda_1 - \rho_{BC}\lambda_5 - \delta^2 p_{AB}\rho_{BC}\lambda_6 - \delta\rho_{BC}\lambda_{16} + [1 - \delta\rho_{BC}]\lambda_{18} = 0 \tag{16}$$

and

$$\lambda_{10} + \lambda_{11} + \lambda_{13} + \lambda_{14} + \lambda_{16} + \lambda_{18} > 0$$

From (15), $\upsilon\lambda_6 = \lambda_{10} + \lambda_{11} + \lambda_{14} \geq 0$, and from (16), $[1 - \delta\rho_{BC}]\lambda_{18} = \delta^2\theta\lambda_1 + \rho_{BC}\lambda_5 + \delta^2 p_{AB}\rho_{BC}\lambda_6 + \delta\rho_{BC}\lambda_{16} \geq 0$. Substituting, and simplifying, the dual system is

$$[\upsilon - \delta^2 p_{AB}]\lambda_1 - \lambda_5 - \frac{\delta^2 p_{AB}}{\upsilon}\lambda_{10} + \left[\frac{\upsilon - \delta^2 p_{AB}}{\upsilon}\right]\lambda_{11} + \left[\frac{\upsilon - \delta^2 p_{AB}}{\upsilon}\right]\lambda_{14} = 0 \tag{17}$$

$$\left[(1 - \delta\rho_{BC})\upsilon - \delta^2\theta\right]\lambda_1 = \rho_{BC}\lambda_5 + (1 - \delta\rho_{BC})\lambda_{13} + \delta\rho_{BC}\lambda_{16} + \frac{\delta^2 p_{AB}\rho_{BC}}{\upsilon}(\lambda_{10} + \lambda_{11} + \lambda_{14}) \tag{18}$$

and

$$\delta^2\theta\lambda_1 + \rho_{BC}\lambda_5 + \delta^2 p_{AB}\rho_{BC}\lambda_6 + (1 - \delta\rho_{BC})[\lambda_{10} + \lambda_{11} + \lambda_{13} + \lambda_{14}] + \lambda_{16} > 0 \tag{19}$$

Since all the coefficients on the RHS of (18) are positive, a necessary and sufficient condition for the solution of the dual system is that the coefficient of λ_1 is positive as well, i.e., $(1 - \delta\rho_{BC})\upsilon - \delta^2\theta > 0$. Therefore $(1 - \delta\rho_{BC})\upsilon - \delta^2\theta \leq 0$ is a necessary and sufficient condition for the existence of a solution to the primal. ■

The proof of Proposition 5.5 uses the following lemma.

Lemma 7.4 *If $\omega \equiv (p, \rho, \delta) \in \Omega^*$, there is an open subset $U \subset \mathcal{U}$ such that for every $u \in U$, the legislative bargaining game with parameters (ω, u) admits a broker equilibrium.*

Proof of Lemma 7.4. Denote the dominance relation under preference profile u by $\gg_u$. We say that $i \gg_u j$ is *stronger* than $i \gg_{u'} j$ if $u_i^* + u_j(z_i) - (u_i(z_j) + u_j^*) > u_i^{*\prime} + u_j'(z_i) - (u_i'(z_j) + u_j^{*\prime})$.

Take any pair $(\omega, u_\omega) \in \Omega^* \times \mathcal{U}$ such that (ω, u_ω) admits a broker equilibrium. First note that reducing $u_A(z_B)$ makes the dominance relation $A \gg B$ stronger and does not affect any of the conditions in $(1b) - (6b)$. Note that by increasing $u_B(z_C)$ all conditions in $(1b) - (6b)$ hold with strict inequality and the dominance relation $C \gg B$ becomes stronger. Moreover, since $u_\omega \in \mathcal{U}$, $u_B(z_C) < u_B^*$, the increment in $u_B(z_C)$ can be small enough to remain in $\mathcal{U}$. Note now that reducing $u_A(z_C)$ makes the dominance relation $A \gg C$ stronger but makes all conditions $(1b) - (6b)$ but $(2b)$, tighter. Therefore, choosing $u_A(z_C)$ and $u_A(z_B)$ appropriately we have that there is some u' close to u_ω such that

(ω, u') verifies all conditions in $(1b) - (6b)$ with strict inequality, the dominance relations are stronger under u' than under u_ω, and $u' \in int\mathcal{U}$. Because all inequalities are not strict there is an open ball $U_{u_\omega} \subset int\mathcal{U}$ around u' such that for every $u'' \in U_{u_\omega}$ the legislative bargaining (ω, u'') admits a broker equilibrium. ■

Proof of Proposition 5.5. By Lemma 7.4, for every $\omega \in int(\Omega^*)$ there is an open subset $U_\omega \subset \mathcal{U}$ such that for every $u \in U_\omega$, the legislative bargaining game with parameters (ω, u) admits a broker equilibrium. Since every (ω, u) verifies $(1b) - (6b)$ with strict inequality, the dominance relations are also strict, and $u \in int(\mathcal{U})$, it is easy to construct an open ball P_ω around ω such that any pair $(\omega, u) \in (P_\omega \times U_\omega)$ verifies $(1b) - (6b)$ with strict inequality, the dominance relations are also strict, and $u \in int(\mathcal{U})$ (and therefore admits a broker equilibrium). ■

Proof of Proposition 5.10. The result for C was proved in the text. Now consider the broker, party B. As before $T_B = W_B - PV_B$. Then substituting from Lemma 5.2, and (7), we have

$$\begin{aligned} T_B &= \frac{\delta}{\upsilon} \{p_{BA}[u_A(z_C) + \rho_{BC} S_{BC}(z_C)] - (p_{AB} + p_{BA}) u_B(z_C)\} \\ &\leq \frac{1}{\upsilon\delta} \left[\delta^2 (p_{BA} + p_{AB}) - \upsilon\right] S_{AC}(z_C) < 0, \end{aligned}$$

where we used (5b), $A \gg C$, and the fact that $S_{AC}(z_C) > 0$, by (2b). The result for A follows because $\sum_i W^i = \sum_i PV_i = \frac{\delta}{\upsilon}(p_{AB} + p_{BA}) \sum u_i(z_C)$. ■