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Efficiency of Lowest-Unmatched Price Auctions

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***Highlights (for review)**

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Highlights

In a LUPA, the winning bid is the lowest one among those submitted by only one player

All bidders pay a fee; the auctioneer retains the item if there is no winner.

Despite apparent unfairness, expected payoffs of bidders and organizers are zero.

LUPAs act as a price revealing mechanism in expected terms.

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Efficiency of Lowest-Unmatched Price Auctions

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December 2015

Abstract

In Lowest-Unmatched Price Auctions (LUPA) all participants pay a bidding fee and the lowest bid placed by only one participant wins. Many LUPAs do not specify what happens with the item on offer if there is no unmatched bid. The item may remain with the auctioneer which may appear unfair given that the auctioneer collects the bidding fees. We show that in a symmetric Nash equilibrium of a LUPA with known prize both players and the auctioneer will have an expected profit of zero. Moreover, LUPAs may be seen as a value-revealing mechanism.

JEL Classification: C71, D44

Keywords: unmatched bid auction, selling mechanism, efficiency

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Efficiency of Lowest-Unmatched Price Auctions

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Abstract

In Lowest-Unmatched Price Auctions (LUPA) all participants pay a bidding fee and the lowest bid placed by only one participant wins. Many LUPAs do not specify what happens with the item on offer if there is no unmatched bid. The item may remain with the auctioneer which may appear unfair given that the auctioneer collects the bidding fees. We show that in a symmetric Nash equilibrium of a LUPA with known prize both players and the auctioneer will have an expected profit of zero. Moreover, LUPAs may be seen as a value-revealing mechanism.

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1. Introduction

A Lowest-Unmatched Price Auction (LUPA) specifies that the winning bid is the lowest among all unmatched bids, i.e. those placed by only one player. Although participating in a LUPA requires strategic skills, they may be mistaken for gambles (see Raviv and Virag, 2009 and the clarifications by the Gambling Commission in 2008). Efficiency of LUPAs as trading mechanisms has not been investigated, except for a partial characterization by Scarsini et al. (2010) who employ the zero-sum property to derive that the organizer's expected payoff is non-positive (non-negative for participants); yet, according to them, LUPAs seem to generate more money than the value of the object auctioned. Besides, in most LUPAs bidders pay participation fees and with a strictly positive probability there is no winner, in which case

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Media	Item	Value, €	Total bids	Winner, €	Profit, €
Radio	monetary	10 000	47872	14.55	13 457.28
Radio	monetary	10 000	52847	14.65	15 895.03
Radio	monetary	1 000	1798	0.60	– 118.98
Radio	monetary	3 000	6732	5.82	298.68
Radio	monetary	5 000	6201	11.16	– 1 961.51
Newspaper	bike	1 099	1272	1.51	– 475.72
TV	car	20 000	266824	20.65	11 074.80
Radio	house	350 000	610104	99.82	– 51 049

Table 1: Summary of some LUPAs run in Germany in 2005-2006.

the organizer would retain the item.[1] This rule seems unfair, as the organizer also obtains the bidding fees. In this paper we explicitly show that the expected payoff of participants and organizers is exactly zero.

From a strategic perspective, LUPAs give bidders incentives to outguess bids of their rivals, unlike, say, first (or second) price private value auctions, where bidders have incentives to reveal their valuation of the auctioned item through bids placed. We show that despite this, the value of the auctioned item is reflected in the bidding behavior, and conclusions can be drawn about the bidders' valuation of the item, as well as about efficiency of LUPAs as trading mechanisms.

In some LUPAs, organizers specify a total number of bids that need to be placed in order for the sale to take place. This number is typically high enough to cover the cost of the auctioned item through bidding fees. In others, the number of required bids is not specified. In Table 1, 4 out of 8 LUPAs of the second type resulted in losses for the organizers, while the remaining auctions were profitable.[2] Games with high and relatively low stakes seem equally likely to be profitable or unprofitable, independent of the media (newspaper, radio or TV) used as the auction's platform.

Papers dealing with LUPAs often assume that players are only allowed

[1]This rule was used in Eichberger and Vinogradov (2008, 2015) and Östling et al. (2011).

[2]Examples are from Eichberger and Vinogradov (2015). In all cases, the bidding cost is 49c per bid. Profit is calculated as the total revenue from bids less the [advertized] value of the prize. On top of this, there was a fee paid by the organizers of the auction to service providers, typically 12c per bid.

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to place a single bid, see Rapoport et al. (2007), Östling, et al. (2011) and Houba et al. (2011). In contrast, here, as in Eichberger and Vinogradov (2008, 2015) and Scarsini et al. (2010), the game has no restrictions on the number of bids placed by each player.

2. The model

One can classify the LUPAs in Table 1 into three groups. Firstly, these are LUPAs with a monetary prize. In these auctions the value of the prize is identical for all bidders and for the organizer, justifying the common value assumption. Secondly, there are LUPAs which sell standard items with a well-defined market value, as the bicycle in Table 1; web-based LUPAs would sell iPhones, iPads, cameras, or camcorders. In this case, a common value assumption can also be justified for the bidders; for the organizers the valuation may be different. If the organizer manufactures the item or obtains a bulk purchase discount from the manufacturer, he may procure the item at a price below its market value. Thirdly, some LUPAs sell items that different participants are likely to value differently, e.g., a tuned car or a house.

We begin our analysis with the common value case, proceed with the case of a different valuation by the organizer, and finally apply results to the private value variant of a LUPA. The formulation of the game follows closely Eichberger and Vinogradov (2015), henceforth EV (2015).

2.1. The Game

Consider a finite set of identical potential bidders $I = \{1, .., N\}$ who value the item which is to be sold in the auction at A. A player faces bidding costs of c per bid. Players may become active bidders or may choose not to bid at all.

We allow for multiple bids and define a strategy s_i of player i by a vector of binaries $s_i = (1, 0, ..., 0, 1, ...)$. The position b in strategy s_i refers to the bid b and $s_i(b)$ indicates whether player i places bid b ($s_i(b) = 1$) or does not place this bid ($s_i(b) = 0$). Bidding above A is unprofitable, therefore bids above A are dominated by the non-participation option, hence the number of undominated bids is finite. We denote the highest undominated bid by $\overline{b}$. Formally, a strategy is a mapping $s_i : \mathbb{N}(\overline{b}) \to \{0, 1\}$, where $\mathbb{N}(\overline{b})$ denotes the set of integers up to $\overline{b}$. The strategy set S of each player is the set of these mappings.

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A strategy combination $\mathbf{s} = (s_1, s_2, ..s_N)$ is an element of S^N. A strategy combination $\mathbf{s}$ can be written $(s_i, \mathbf{s}_{-i})$, where $\mathbf{s}_{-i}$ is the strategy combination played by player i's rivals. Given a strategy combination $\mathbf{s}$, one can determine the lowest unmatched bid $\mu(\mathbf{s})$. If there is no unmatched bid, we assign $\mu(\mathbf{s}) = 0$.

Let $\pi_i \in \Delta(S)$ be a mixed strategy of player i and denote by π_{-i} the combination of mixed strategies of his rivals. For a pure strategy s_i of player i a profile π_{-i} of mixed strategies of his rivals determines the probability $w_b(s_i, \pi_{-i})$ of bid b winning: $w_b(s_i, \pi_{-i}) = \sum_{\{\mathbf{s}_{-i}:\mu(s_i,\mathbf{s}_{-i})=b\}} \pi(\mathbf{s}_{-i})$, where $\pi(\mathbf{s}_{-i}) = \prod_{i \neq j} \pi(s_j)$ denotes the probability of the respective strategy combination. With this notation, the expected payoff of player i from playing s_i is

$$P_i(s_i, \pi_{-i}) = \sum_{b=1}^{\bar{b}} s_i(b) \left[(A - b) w_b(s_i, \pi_{-i}) - c\right]. \tag{1}$$

Defining the expected payoff $P_i(\pi_i, \pi_{-i})$ of a mixed strategy π_i for player i in the usual way, one can apply the standard definition of a Nash equilibrium in mixed strategies. A combination of mixed strategies $\left(\pi_i^*, \pi_{-i}^*\right)$ is a Nash equilibrium if

$$P_i\left(\pi_i^*, \pi_{-i}^*\right) \geq P_i\left(\pi_i, \pi_{-i}^*\right) \quad \text{for all } \pi_i \in \Delta(S) \quad \text{and all } i \in I.$$

We will focus on symmetric equilibria and, therefore, omit the indices of players[3]. We are now ready to state our main results.

2.2. Bidders' equilibrium payoffs

In a symmetric Nash equilibrium π, all pure strategies s in the support of the equilibrium mixed strategy $\operatorname{supp}\pi$ must yield an equal expected payoff:

$$P(s, \pi) = P(s', \pi), \quad \text{for all} \quad s, s' \in \operatorname{supp}\pi. \tag{2}$$

The following proposition establishes that, in equilibrium, players in a LUPA will face an expected payoff of zero. The proof rests on showing that the strategy s^0 of not bidding and, hence, obtaining a payoff of zero has positive probability in any symmetric Nash equilibrium. Hence, s^0 belongs

[3] Formal definitions of $\mu(\mathbf{s})$ and $w_b(s_i, \pi)$ and a more detailed discussion are in EV (2015).

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to the support of the equilibrium mixed strategy and the expected payoff of all strategies which are played with positive probability in equilibrium must be zero.

Proposition 1. *For an equilibrium mixed strategy π, the expected payoff $P(s, \pi)$ is zero for all $s \in \operatorname{supp} \pi$.*

Proposition 1 establishes that participating in a LUPA is not profitable in expected terms. For an item auctioned in a LUPA which the bidders value at the market price A, risk-neutral players are indifferent between obtaining the good in the market and participating in the LUPA. In this sense, LUPA is a "*fair lottery*". This interpretation of LUPAs as trading mechanisms rests upon the option not to participate and instead to obtain the item in the market.

2.3. The equilibrium payoff of the auctioneer

The auctioneer's valuation of the item on sale may or may not differ from that of the bidders. If the object is a good which the auctioneer manufactures himself or obtains at a discount, its value for the auctioneer may differ from the market price at which bidders may acquire it. Denote the value of the item for the auctioneer by C which is not necessarily equal to the value of the object for the bidders A. Moreover, although the bidders'valuation A determines the expected payoff of the auctioneer, it may be unobservable for him.

Given a strategy combination $\mathbf{s} = (s_1, .., s_i, .., s_N)$, denote the total amount of bidding fees collected by $F(\mathbf{s}) = \sum_{i=1}^{N} c \cdot \sum_{b=1}^{\bar{b}} s_i(b)$, and define the realized revenue of the auctioneer as

$$R(\mathbf{s}) = \begin{cases} \mu(\mathbf{s}) + F(\mathbf{s}) & \text{if} \quad \mu(\mathbf{s}) > 0, \\ C + F(\mathbf{s}) & \text{if} \quad \mu(\mathbf{s}) = 0. \end{cases} \tag{3}$$

This formulation assumes that the auctioneer always keeps the fees collected and, on top of this, either obtains the winning bid in exchange for the item if there is a winner ($\mu(\mathbf{s}) > 0$), or keeps the item if there is a tie ($\mu(\mathbf{s}) = 0$) and thus saves C. For a profile of mixed strategies π, the expected revenue of the auctioneer is

$$R(\pi) = \sum_{\mathbf{s} \in S^N} \pi(\mathbf{s}) \cdot R(\mathbf{s}). \tag{4}$$

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Denote by $\tau(\pi) = \Pr\{\mathbf{s} : \mu(\mathbf{s}) = 0\} = \sum_{\mathbf{s}\in S^N : \mu(\mathbf{s})=0} \pi(\mathbf{s})$ the probability of a tie.

Proposition 2. *If π is an equilibrium mixed strategy then*

$$R(\pi) = (1 - \tau(\pi)) A + \tau(\pi) C. \tag{5}$$

The valuation of the item by participants, A, enters the expected revenue of the auctioneer through the strategic behavior of players in equilibrium. To see this, recall that Proposition 1 ensures that, in equilibrium, bidders' expected costs (bidding fees plus the winning bid) equal their expected gains (the value of the prize). For this reason, in expected terms, all bidding fees and the winning bid sum up to $(1 - \tau(\pi)) A$ in Equation (5). The remaining term of the expression is the expected value of not allocating the item if there is no winner.

Applying this result to the case where the auctioneer has to procure the item at the market price (*common value* assumption), $C = A$, one obtains immediately that the expected return of the LUPA, $R(\pi) = A$, equals the market value at which the item can be purchased. Hence, an auctioneer who values the item the same as the participants cannot expect a strictly positive profit by selling the item through this mechanism.

Corollary 3. *For $C = A$ in equilibrium $R(\pi) = A$ holds.*

If the organizer and the bidders have *different valuations* of the item, proposition 1 also shows that LUPAs are *fair mechanisms*. For $A \geq C$, the auctioneer's expected revenue is limited above by the market valuation of the item. For $A < C$, the auctioneer would better not auction the item at all since expected revenue falls below the auctioneer's procurement cost C. These findings are in stark contrast with the first-glance impression that collecting bidding fees from a large number of participants would generate unlimited profits for the auctioneer.

Profitability of the LUPA for the auctioneer depends on the equilibrium probability of a tie $\tau(\pi)$, which depends on the number of players N, bidding fee c and bidders' valuation A. The latter may be unknown to the organizer (*private value*). The auctioneer starts the auction only if he believes his rent is positive, $A - C > 0$; this gives incentives to minimize the probability of a tie. This can be done, for example, by appropriately choosing

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the bidding fee, c. If the probability of a tie is close to zero, LUPAs act as a *value-revealing mechanism* in expected terms: the expected revenue of the auctioneer converges to the valuation of the good by the participants.

Re-auctioning has a similar effect. If a LUPA results in a tie, the auctioneer with $C < A$ has incentives to re-auction the item. If re-auctioning is always done through a LUPA, the expected value of the auctioneer is again equal to the market valuation of the item.

Corollary 4. *If the auctioneer re-auctions the item through a LUPA, whenever there is no winner, then* $R(\pi) = A$.

To see the last result, in (5) replace value C by the expected revenue from re-auctioning the item: $R(\pi) = (1 - \tau) A + \tau R(\pi)$, which yields $R(\pi) = A$. Although the item may be re-auctioned in a LUPA with a different bidding fee, by Proposition 2, the expected revenue of the auctioneer does not depend on the choice of fees.

3. Conclusion

LUPAs still are unusual selling mechanisms, yet they possess attractive features: the expected value for participants is the same as if they would obtain the auctioned item in the market and the seller's expected revenue cannot exceed the market price. Moreover, LUPAs are capable of revealing the market valuation of the auctioned item if the probability of a tie (no winner) is close to zero or if the seller keeps re-auctioning the item if ties occur. All these properties are in expected terms. For applications, this means that LUPAs should be repeated many times in order to deliver these results *on average*.

These considerations may explain why some LUPAs ended with losses for the organizers whereas others were beneficial for them. Most of auctions referred to in this paper were linked to a marketing campaign by TV or radio broadcasters without the intention of frequent repetitions. In these cases, potential losses were covered by marketing expenses. In contrast, the LUPAs conducted on the internet sell homogenous goods in a series of repeated auctions, which provide a better setting for testing our zero-expected profit result. However, this type of LUPAs often imposes an upper limit on the number of bids in order to guarantee a total amount of bidding fees which covers the cost of the auctioned item. According to our theory, provisions of this type are unnecessary.

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4. Proofs

The following result from EV(2015) is used in the proofs. Consider a LUPA, denoted as Γ_k, in which players are restricted to place bids only up to $k \leq A$, i.e. they can only play strategies from the constrained set $S_k = \{s \in S : s(b) = 0, \forall b > k\}$. Let π_k be the mixed strategy played symmetrically by all players in Γ_k and π_{k+1} the mixed strategy played in a similarly defined "larger"game Γ_{k+1}. If π_{k+1} is an equilibrium in Γ_{k+1}, the following proposition links it to an equilibrium in Γ_k:

Proposition 5 (EV, 2014). *Consider game Γ_{k+1}. For any strategy s such that $s(k+1) = 0$ denote with $s'(s)$ its counterpart such that $s'(k+1) = 1$ and $s'(l) = s(l), \forall l \leq k$. With this notation, if π_{k+1} is an equilibrium mixed strategy in Γ_{k+1}, then a mixed strategy $\widetilde{\pi}_k$ with elements $\widetilde{\pi}_k(s)$ as defined below is an equilibrium mixed strategy in Γ_k:*

$$\widetilde{\pi}_k(s|\pi_{k+1}) := \begin{cases} \pi_{k+1}(s) + \pi_{k+1}(s') & \textit{if} \quad s(k+1) = 0, \\ 0 & \textit{if} \quad s(k+1) = 1. \end{cases} \tag{6}$$

Moreover, the two equilibria generate equal expected payoffs $P_i(s, \pi_{k+1}) = P_i(s, \widetilde{\pi}_k)$.

This proposition combines Proposition 2 and Lemma A.4 from EV (2015). Note that in the above proposition, any s has exactly one counterpart $s'(s)$.

Proof of Proposition 1

Proof. It suffices to show that if $s \in \operatorname{supp} \pi_{k+1}$ then $P_i(s, \pi_{k+1}) = 0$.

First, we will show that this holds in Γ_1 (LUPA with only one bid $b = 1$ allowed). The strategy set S_1 is $\{s^0, s^1\} = \{0, 1\}$. Let $\pi_1 = (\pi_1^0, \pi_1^1)$ be symmetric mixed equilibrium in Γ: $\pi_1^0, \pi_1^1 > 0 \Rightarrow P_i(s^1, \pi_1) = (A-1)\, w_1(s^1, \pi_1) - c = P_i(s^0, \pi_1) = 0$. This implies $w_1(s^1, \pi_1) = (\pi_1^0)^{N-1} = \frac{c}{A-1}$, which results in $\pi_1^0 = \sqrt[N-1]{\frac{c}{A-1}}$. Since $\pi_1^0 + \pi_1^1 = 1$, we obtain $\pi_1^1 = 1 - \sqrt[N-1]{\frac{c}{A-1}}$. By computation, this is the only equilibrium. Since s^0 is in the support of it, the expected payoff of players from playing any $s \in \operatorname{supp} \pi_1$ is $P_i(s, \pi_1) = 0$, $\forall i \in I$, hence also the expected payoff $P_i(\pi_1) = 0$.

Now assume that the proposition holds for Γ_k and consider game Γ_{k+1}. Let π_{k+1} be equilibrium in Γ_{k+1}. Construct $\widetilde{\pi}_k$ as in (6): $\widetilde{\pi}_k(s) = \pi_{k+1}(s) +$

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$\pi_{k+1}(s'(s)), \forall s \in S_k$. By Proposition 5, $\widetilde{\pi}_k$ is equilibrium in Γ_k, and $P_i(s, \pi_{k+1}) = P_i(s, \widetilde{\pi}_k)$ for any player i and any strategy $s \in S_k$. By inductive hypothesis, $P_i(s, \widetilde{\pi}_k) = 0 = P_i(s, \pi_{k+1})$. Since all strategies in the support of the mixed equilibrium deliver equal expected payoffs, it follows that if $s \in \operatorname{supp} \pi_{k+1}$ then $P_i(s, \pi_{k+1}) = 0$. ■

Proof of Proposition 2

Proof. From (3) and (4)

$$R(\pi) = \sum_{\mathbf{s} \in S^N} \pi(\mathbf{s}) \cdot R(\mathbf{s}) = \sum_{\mathbf{s} \in S^N : \mu(\mathbf{s}) > 0} \pi(\mathbf{s}) [\mu(\mathbf{s}) + F(\mathbf{s})] + \sum_{\mathbf{s} \in S^N : \mu(\mathbf{s}) = 0} \pi(\mathbf{s}) [C + F(\mathbf{s})].$$

Note that $\sum_{\mathbf{s} \in S^N : \mu(\mathbf{s}) > 0} \pi(\mathbf{s}) + \sum_{\mathbf{s} \in S^N : \mu(\mathbf{s}) = 0} \pi(\mathbf{s}) = \sum_{\mathbf{s} \in S^N} \pi(\mathbf{s}) = 1$, and that the probability of a tie is defined as $\tau = \sum_{\mathbf{s} \in S^N : \mu(\mathbf{s}) = 0} \pi(\mathbf{s})$, therefore

$$R(\pi) = \sum_{\mathbf{s} \in S^N : \mu(\mathbf{s}) > 0} \pi(\mathbf{s}) \mu(\mathbf{s}) + \sum_{\mathbf{s} \in S^N} \pi(\mathbf{s}) F(\mathbf{s}) + \tau C.$$

It remains show that the first two terms on the right-hand side add to $(1 - \tau) A$. For the first of them we need to show

$$\sum_{\mathbf{s} \in S^N : \mu(\mathbf{s}) > 0} \pi(\mathbf{s}) \mu(\mathbf{s}) = \sum_{i=1}^{N} \sum_{s_i \in S} \pi(s_i) \sum_{b=1}^{\bar{b}} s_i(b) w_b(s_i, \pi) b. \tag{7}$$

To prove this, introduce the indicator function

$$\delta_b(s_i, \mathbf{s}_{-i}) = \begin{cases} 1 & \text{if } \mu(s_i, \mathbf{s}_{-i}) = b \\ 0 & \text{otherwise,} \end{cases}$$

and represent the probability of winning $w_b(s_i, \pi)$ as

$$w_b(s_i, \pi) = \sum_{\mathbf{s}_{-i} \in S^{N-1}} \pi(\mathbf{s}_{-i}) \cdot \delta_b(s_i, \mathbf{s}_{-i}).$$

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Substitute this to the right-hand side of (7) to obtain

$$\begin{aligned}
& \sum_{i=1}^{N} \sum_{s_i \in S} \pi(s_i) \sum_{b=1}^{\bar{b}} s_i(b) \sum_{\mathbf{s}_{-i} \in S^{N-1}} \pi(\mathbf{s}_{-i}) \cdot \delta_b(s_i, \mathbf{s}_{-i})\, b \\
= & \sum_{i=1}^{N} \sum_{s_i \in S} \pi(s_i) \sum_{\mathbf{s}_{-i} \in S^{N-1}} \pi(\mathbf{s}_{-i}) \sum_{b=1}^{\bar{b}} s_i(b) \cdot \delta_b(s_i, \mathbf{s}_{-i})\, b = \\
& \sum_{i=1}^{N} \sum_{\mathbf{s} \in S^N} \pi(\mathbf{s}) \sum_{b=1}^{\bar{b}} s_i(b) \cdot \delta_b(s_i, \mathbf{s}_{-i})\, b = \sum_{\mathbf{s} \in S^N} \pi(\mathbf{s}) \sum_{i=1}^{N} \sum_{b=1}^{\bar{b}} s_i(b) \cdot \delta_b(s_i, \mathbf{s}_{-i})\, b.
\end{aligned}$$

Here on each strategy combination $(s_i, \mathbf{s}_{-i})$ the indicator $\delta_b(s_i, \mathbf{s}_{-i})$ will take value 1 either for exactly one player and one bid b, in which case $\delta_b(s_i, \mathbf{s}_{-i})\, b = \mu(\mathbf{s}) > 0$, or for nobody, in which case $\mu(\mathbf{s}) = 0$. With this in mind, we obtain

$$\sum_{\mathbf{s} \in S^N} \pi(\mathbf{s}) \sum_{i=1}^{N} \sum_{b=1}^{\bar{b}} s_i(b) \cdot \delta_b(s_i, \mathbf{s}_{-i})\, b = \sum_{\mathbf{s} \in S^N : \mu(\mathbf{s}) > 0} \pi(\mathbf{s})\, \mu(\mathbf{s}).$$

Finally, consider

$$\begin{aligned}
& \sum_{\mathbf{s} \in S^N : \mu(\mathbf{s}) > 0} \pi(\mathbf{s})\, \mu(\mathbf{s}) + \sum_{\mathbf{s} \in S^N} \pi(\mathbf{s})\, F(\mathbf{s}) = \\
& \sum_{i=1}^{N} \sum_{s_i \in S} \pi(s_i) \sum_{b=1}^{\bar{b}} s_i(b)\, w_b(s_i, \pi)\, b + \sum_{\mathbf{s} \in S^N} \pi(\mathbf{s}) \sum_{i=1}^{N} c \cdot \sum_{b=1}^{\bar{b}} s_i(b) = \\
& \sum_{i=1}^{N} \sum_{s_i \in S} \pi(s_i) \sum_{b=1}^{\bar{b}} s_i(b)\, w_b(s_i, \pi)\, b + \sum_{i=1}^{N} \sum_{s_i \in S} \pi(s_i) \sum_{\mathbf{s}_{-i} \in S^{N-1}} \pi(\mathbf{s}_{-i}) \cdot c \cdot \sum_{b=1}^{\bar{b}} s_i(b).
\end{aligned}$$

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Subtract $\sum_{i=1}^{N}\sum_{s_i\in S}\pi(s_i)\sum_{b=1}^{\bar{b}} s_i(b)\, w_b(s_i,\pi)\, A$ from both sides to obtain

$$\begin{aligned}
\sum_{\mathbf{s}\in S^N:\mu(\mathbf{s})>0}\pi(\mathbf{s})\,\mu(\mathbf{s}) + \sum_{\mathbf{s}\in S^N}\pi(\mathbf{s})\,F(\mathbf{s}) - \sum_{i=1}^{N}\sum_{s_i\in S}\pi(s_i)\sum_{b=1}^{\bar{b}} s_i(b)\,w_b(s_i,\pi)\,A &= \\
-\sum_{i=1}^{N}\sum_{s_i\in S}\pi(s_i)\left(\sum_{b=1}^{\bar{b}} s_i(b)\,w_b(s_i,\pi)\,(A-b) - \underbrace{\sum_{\mathbf{s}_{-i}\in S^{N-1}}\pi(\mathbf{s}_{-i})}_{=1}\cdot c\cdot\sum_{b=1}^{\bar{b}} s_i(b)\right) &= \\
-\sum_{i=1}^{N}\sum_{s_i\in S}\pi(s_i)\left(\sum_{b=1}^{\bar{b}} s_i(b)\,(w_b(s_i,\pi)\,(A-b)-c)\right) &= \\
-\sum_{i=1}^{N}\sum_{s_i\in S}\pi(s_i)\,P_i(s_i,\pi)\,.
\end{aligned}$$

By Proposition 1, in equilibrium $P_i(s_i,\pi)=0$ for any s_i such that $\pi(s_i)>0$, therefore

$$\sum_{\mathbf{s}\in S^N:\mu(\mathbf{s})>0}\pi(\mathbf{s})\,\mu(\mathbf{s}) + \sum_{\mathbf{s}\in S^N}\pi(\mathbf{s})\,F(\mathbf{s}) = \sum_{i=1}^{N}\sum_{s_i\in S}\pi(s_i)\sum_{b=1}^{\bar{b}} s_i(b)\,w_b(s_i,\pi)\,A.$$

It only remains to note that similarly to (7) one has

$$\sum_{\mathbf{s}\in S^N:\mu(\mathbf{s})>0}\pi(\mathbf{s})\,A = \sum_{i=1}^{N}\sum_{s_i\in S}\pi(s_i)\sum_{b=1}^{\bar{b}} s_i(b)\,w_b(s_i,\pi)\,A,$$

and hence the previous equality turns to

$$\sum_{\mathbf{s}\in S^N:\mu(\mathbf{s})>0}\pi(\mathbf{s})\,\mu(\mathbf{s}) + \sum_{\mathbf{s}\in S^N}\pi(\mathbf{s})\,F(\mathbf{s}) = \sum_{\mathbf{s}\in S^N:\mu(\mathbf{s})>0}\pi(\mathbf{s})\,A = (1-\tau)\,A.$$

This proves that

$$R(\pi) = (1-\tau)\,A + \tau C.$$

■

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