A Finite Screw Approach to Type Synthesis of Three-DOF Translational Parallel Mechanisms

Shuofei Yang[1], Tao Sun [1,*], Tian Huang[1,3], Qinchuan Li[2], Dongbing Gu[4]

1 Key Laboratory of Mechanism Theory and Equipment Design of State Ministry of Education, Tianjin University, Tianjin 300072, China

2 Mechatronic Institute, Zhejiang Sci-Tech University, Hangzhou 310018, China

3 School of Engineering, University of Warwick, Coventry CV4 7AL, UK

4 School of Computer Science and Electronic Engineering, University of Essex, Colchester CO4 3SQ, UK

Abstract: This paper for the first time presents a finite screw approach to type synthesis of three-degree-of-freedom (DOF) translational parallel mechanisms (TPMs). Firstly, the finite motions of a rigid body, a TPM and its limbs are described by finite screws. Secondly, given the standard form of a limb with the specified DOF, the analytical expressions of the finite screw attributed to the limb are derived using the properties of screw triangle product, resulting in a full set of the 3-, 4- and 5-DOF limbs that can readily be used for determining all the potential topological structures of TPMs. Finally, the assembly conditions for type synthesis of TPMs are proposed by taking into account the inclusive relationship between the finite motions of a TPM and those of its limbs. The merit of this approach lies in that the limb structures can be formulated in a justifiable manner that naturally ensures the full cycle finite motion pattern specified to the moving platform.

Keywords: Three-DOF translational parallel mechanisms, Finite screw, Type synthesis

1. Introduction

Parallel mechanisms having three translational movement capabilities, also known as the three-DOF translational parallel mechanisms (TPMs), have been extensively studied in academia and widely applied in industry [1-5]. Type synthesis is one of important issues in the development of TPMs [5]. The approaches available to hand can roughly be divided into two categories, i.e. the finite motion based methods [6-10] and the instantaneous constraint based methods [11-13]. The basic idea of the first category is to find all the possible limb sub-manifolds or characteristic subsets so that their intersection leads to three-DOF finite translational motions of the moving platform. This is followed by seeking all the potential equivalences of the sub-manifolds or subsets attributed to a limb, a process that was conventionally done with the aid of the intuition-based rules or the properties of subgroup products, resulting in numerous possible limb structures that may generate the specified sub-manifolds or subsets. The proposed approaches along this track include the displacement sub-manifold method [6-8], the position and orientation characteristic method [9], and the differential geometry method [10]. However, developing a more vigorous approach that enables the rationality of the produced limb structures to be ensured remains an open issue to be tackled, relying upon analytical formulations of finite motions of limbs and joints. As the counterpart of the methods falling into the first category, the basic idea of the instantaneous constraint based methods is to seek all the possible limb wrench systems whose union spans the resultant wrench system that is reciprocal to three-DOF instantaneous translational motions of the moving platform. This is followed by enumerating the limb structures whose twists are reciprocal to the wrench system attributed to a limb. The proposed approaches along this track include the constraint synthesis method [11-12] and its variants [13]. Although the methods falling into this category are simple and intuitive, the process to verify the consistency between the finite and

Corresponding Author, Email: stao@tju.edu.cn

instantaneous motions must be considered.

By reviewing the characteristics of type synthesis approaches mentioned above, it is expected to develop a more vigorous approach that enables the rationality of the produced limb structures to be ensured in an explicit and analytical manner. In this sense, finite screw theory would have the potential to do so. This is because the composition of a number of successive finite screws can be explicitly represented and algebraically operated by screw triangle product which is the specific representation of Baker-Campbell-Hausdorff formula [14]. The idea of finite screw was first proposed by Dimentberg [15] and developed by many others over the last few decades [16-25]. For example, Parkin [16-17] proposed a specific finite screw in a quasi-vector form with elaborately designed magnitude that is particularly suitable for finite motion composition. Huang [18-19] developed the screw triangle product of two finite screws as a linear combination of five meaningful terms. Recently, intensive efforts have been made by Dai [20] to investigate the interrelationships among finite displacement screws, the point/line transformation matrix representations of SE(3) as well as dual quaternions [21-23]. Having firstly developed the eigen/differential map of finite/instantaneous screws [22] and rigorously proven its consistency with the exponential map of Lie group/algebra or quaternions and Euler-Rodrigues formula [22-24], Dai [25] established the interrelationship theory that enables algebraic properties of various mathematic descriptions of rigid body motions to be closely connected. However, use of finite screw for type synthesis of serial and parallel kinematic chains remains an open issue to be investigated.

Drawing mainly on the finite screw theory, this paper intends to develop an approach for type synthesis of TPMs with particular focus upon formulating analytical expressions of finite screws that can be used to represent the limb structures. The paper is organized as follows. Having had a brief review of state-of-the art in type synthesis of TPMs and finite screw theory in Section 1, Section 2 addresses the parametric representation of finite motions of a TPM and its limbs using finite screws. This is followed in Section 3 by the formulations of analytical expressions of finite screws of the limbs within a TPM, resulting in numerous and justified limb structures having 3-, 4- and 5-DOF. The conditions for assembling TPMs using these limbs are then presented in Section 4 before the conclusions are drawn in Section 5.

2. Parametric Representation of Finite Motions of a TPM

Finding analytical representation that enables to describe the finite motions of the limbs is a prerequisite for type synthesis of TPMs. This section will develop a finite screw representation to relate finite motions of the moving platform of a TPM to those of its limbs and joints.

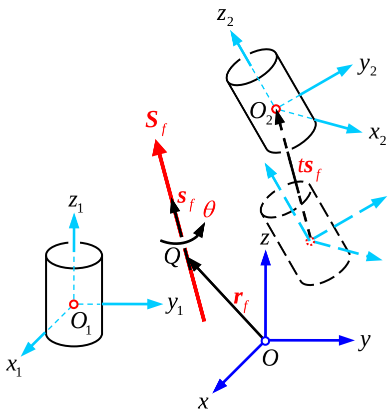


Fig. 1. Finite motion of a rigid body

As shown in Fig. 1, general finite motion of a rigid body moving from pose 1 to pose 2 can be represented by a finite screw $\boldsymbol{S}_f$ in terms of a rotation about an axis followed by a translation along the same axis [18, 22, 25]. The axis is referred to as the Chasles' axis or the finite screw axis [21, 24-27], and hereafter we refer the axis as screw axis for simplicity.

$$\boldsymbol{S}_f = 2\tan\frac{\theta}{2}\begin{pmatrix} \boldsymbol{s}_f \\ \boldsymbol{r}_f \times \boldsymbol{s}_f \end{pmatrix} + t\begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_f \end{pmatrix} \tag{1}$$

where $\boldsymbol{s}_f$ denotes the unit vector of the screw axis and $\boldsymbol{r}_f$ denotes the position vector pointing from the origin O to an arbitrary point Q on the screw axis, θ and t are known as screw parameters ,representing respectively the rotation angle about and the translation distance along the screw axis. In addition, the term $2\tan(\theta/2)$ will serve as a measure of rotation for the composition of finite motions [16].

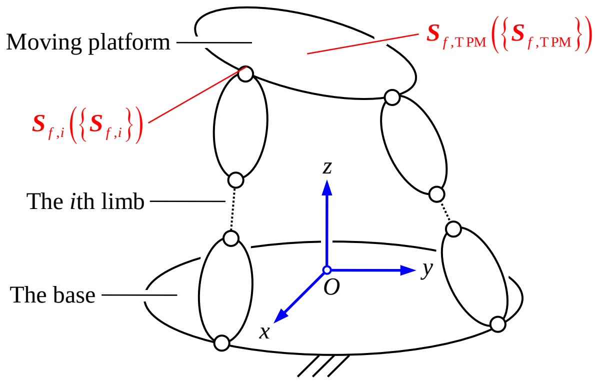


Fig. 2. Finite screw of a TPM

Now, let us consider a TPM composed of l ($l \geq 2$) limbs connecting the moving platform with the base as shown in Fig. 2. To ensure finite and continuous motion of the platform, we assume that the finite screw of the platform $\boldsymbol{S}_{f,\mathrm{TPM}}$ is continuous such that a 'continuous' set, denoted by $\{\boldsymbol{S}_{f,\mathrm{TPM}}\}$, can be formed. The same conventions will be applied to the finite screws of limbs and joints throughout this article. According to Eq.(1), $\{\boldsymbol{S}_{f,\mathrm{TPM}}\}$ can be expressed as

$$\{\boldsymbol{S}_{f,\mathrm{TPM}}\} = \left\{ t\begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s} \end{pmatrix} \middle| t\boldsymbol{s} \in \mathbb{R}^3 \right\} \tag{2}$$

Meanwhile, note that all limbs share the same platform, $\{\boldsymbol{S}_{f,\mathrm{TPM}}\}$ can also be expressed as the intersection of finite screws $\{\boldsymbol{S}_{f,i}\}$ ($i = 1, 2 \ldots, l$), each representing the finite screw of the ith limb.

$$\{\boldsymbol{S}_{f,\mathrm{TPM}}\} = \{\boldsymbol{S}_{f,1}\} \cap \{\boldsymbol{S}_{f,2}\} \cap \cdots \cap \{\boldsymbol{S}_{f,l}\} \tag{3}$$

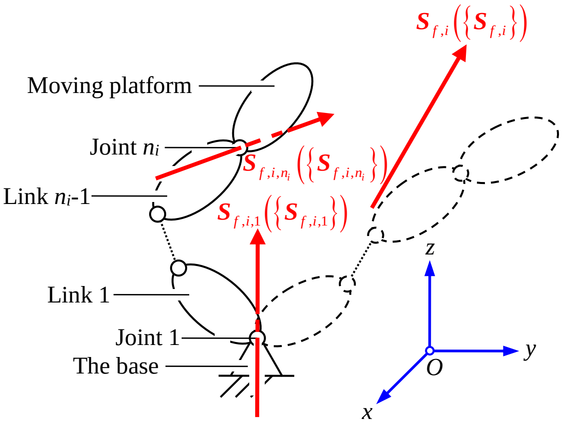
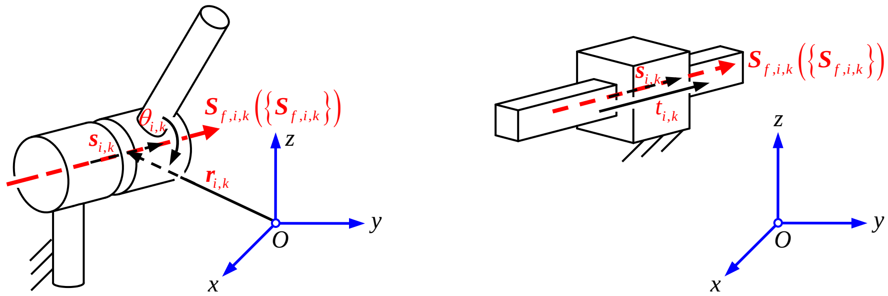


Fig. 3. Finite screw of a TPM limb

We model each limb of a TPM by assuming that it contains n_i 1-DOF joints. Note that the finite motion of the end-link of the ith ($i=1,2\ldots,l$) limb shown in Fig. 3 can be virtualized as the composition of those produced by all joints in the limb. Thus, $\left\{\boldsymbol{S}_{f,i}\right\}$ can be represented as the screw triangle product [19, 28] of n_i successive finite screws

$$\left\{\boldsymbol{S}_{f,i}\right\}=\left\{\boldsymbol{S}_{f,i,n_i}\vartriangle\boldsymbol{S}_{f,i,n_i-1}\vartriangle\cdots\vartriangle\boldsymbol{S}_{f,i,1}\middle|\theta_{i,k},\ t_{i,k}\in\mathbb{R},\ k=1,2,\cdots,n_i\right\},\quad i=1,2\ldots,l \tag{4}$$

where $\left\{\boldsymbol{S}_{f,i,k}\right\}$ denotes the finite screw produced by the kth ($k=1,2,\cdots,n_i$) joint with its screw axis being the joint axis.



(a) R joint (b) P joint

Fig. 4. Finite screws produced by 1-DOF joints

If the kth 1-DOF joint is considered as either a revolute (R) joint or a prismatic (P) joint as shown in Fig. 4, $\left\{\boldsymbol{S}_{f,i,k}\right\}$ can be expressed from Eq. (1) as

$$\left\{\boldsymbol{S}_{f,i,k}\right\}=\begin{cases}\left\{2\tan\dfrac{\theta_{i,k}}{2}\begin{pmatrix}\boldsymbol{s}_{i,k}\\ \boldsymbol{r}_{i,k}\times\boldsymbol{s}_{i,k}\end{pmatrix}\right\} & \text{Revolute joint}\\ \left\{t_{i,k}\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_{i,k}\end{pmatrix}\right\} & \text{Prismatic joint}\end{cases},\quad k=1,2,\cdots,n_i \tag{5}$$

where $\boldsymbol{s}_{i,k}$, $\boldsymbol{r}_{i,k}$, $\theta_{i,k}$ and $t_{i,k}$ have the similar meanings given in Eq. (1). As for the screw triangle product of two successive finite screws produced by either R or P joint, please refer to Appendix A.

3. Parametric Generation of TPM Limbs

Having the parametric representation of finite motions established in Section 2 to hand, we will develop an analytical and hierarchical approach to synthesizing all possible simple limb structures of TPMs. Given the standard form of the finite screw of a limb having the specified DOF, this can be done by analytically deriving the full set of finite screws whose expressions either are equivalent to the standard form or contain three translational motions. This implementation allows each derived analytical expression physically corresponds to a specific limb structure. The 3-, 4- and 5-DOF TPM limbs will be considered because 6-DOF limbs do not affect the finite motions. For simplicity, we omit the limb's identifier since the finite screws generated by joints in the limb are the only concerns. Therefore, symbols $\boldsymbol{s}$, $\boldsymbol{r}$, θ and t in a finite screw should be understood as those associated with either a R joint or a P joint.

3.1 3-DOF and 4-DOF TPM limb structures

As the simplest case of Eq. (4), the finite screw of a 3-DOF TPM limb can be formulated as the screw triangle product of three translational factors according to Eq. (5) and the Case 4 in Appendix A, i.e.

$$\left\{\boldsymbol{S}_{f,l_{\mathrm{I}}}\right\}=\left\{t_1\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_1\end{pmatrix}\triangle t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\} \tag{6}$$

This leads to the limb structure denoted by PPP where the directions of three P joints are assumed to be non-coplanar.

By adding a rotational factor at the right end of Eq. (6), we define

$$\left\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\right\}=\left\{t_1\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_1\end{pmatrix}\triangle t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\triangle 2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\right\} \tag{7a}$$

as the standard form of the finite screw of a 4-DOF TPM limb. According to Theorem 2 in Appendix B, three equivalent expressions of Eq. (7a) can be formulated by placing the rotational factor in all possible positions in the screw triangle product whilst keeping $\left\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\right\}$ unchanged.

$$\left\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\right\}=\left\{t_1\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_1\end{pmatrix}\triangle t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle 2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\} \tag{7b}$$

$$\left\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\right\}=\left\{t_1\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_1\end{pmatrix}\triangle 2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\} \tag{7c}$$

$$\left\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\right\}=\left\{2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle t_1\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_1\end{pmatrix}\triangle t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\} \tag{7d}$$

Since each factor in the screw triangle product corresponds to a R or a P joint, four 4-DOF TPM limb structures can be synthesized and denoted respectively by RPPP, PRPP, PPRP, and PPPR. Note that we sequence the joints in a reverse order to that of the finite screws in the corresponding screw triangle product. This conversion will be applied throughout the article. With the above 4-DOF TPM limbs containing three P joints to hand, the others with different joint types will be synthesized using the properties of screw triangle.

Firstly, it can be proved by Theorem 1 in Appendix B that three translational factors in Eq. (7d) can be rewritten as

$$\left\{t_1\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_1\end{pmatrix}\triangle t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\}=\left\{\begin{pmatrix}\mathbf{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_4-\boldsymbol{r}_4'\right)\end{pmatrix}\triangle t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\} \tag{8}$$

This means that the finite motion generated by three translations with fixed directions is equivalent to that generated by one translation along a circle and two translations with fixed directions though the first factor on the right side dose not equal to that on the left side of Eq. (8). Geometrically, the first factor on the right side represents a finite screw of translation with screw parameter θ_4' where θ_4' represents the angle which the translational arc length corresponds to, $\boldsymbol{r}_4-\boldsymbol{r}_4'$ denotes the radius of the translational circle. $[\boldsymbol{s}_4\times]$ is the skew-symmetric matrix of $\boldsymbol{s}_4$, and $\boldsymbol{I}$ denotes an identity matrix of order 3.

Then, substituting Eq. (8) into Eq. (7d) and implementing the finite motion analysis of the dyad composed of two R joints whose axes are parallel to each other (see Appendix C), leads to

$$\left\{\boldsymbol{S}_{f,l_{\text{II}}}\right\}=\left\{2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle\begin{pmatrix}\mathbf{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_4-\boldsymbol{r}_4'\right)\end{pmatrix}\triangle t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\} \tag{9a}$$

$$=\left\{2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\triangle t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\}$$

(9b) (9c) (9d) (9e) (9f)

Eq. (9a) shows that the screw triangle product composed by a rotational factor and three translational factors is equivalent to that composed of two rotational factors and two translational factors. This property enables five equivalent expressions of Eq. (9a) to be obtained by placing two translational factors in all possible positions whilst keeping $\left\{\boldsymbol{S}_{f,l_{\text{II}}}\right\}$ unchanged.

$$\begin{aligned}\left\{\boldsymbol{S}_{f,l_{\text{II}}}\right\}&=\left\{2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle\begin{pmatrix}\mathbf{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_4-\boldsymbol{r}_4'\right)\end{pmatrix}\triangle t_2\begin{pmatrix}\mathbf{0}\\ e^{\theta_4'[\boldsymbol{s}_4\times]}\boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ e^{\theta_4'[\boldsymbol{s}_4\times]}\boldsymbol{s}_3\end{pmatrix}\right\}\\&=\left\{2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\triangle 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\right\}\end{aligned} \tag{9b}$$

$$\begin{aligned}\left\{\boldsymbol{S}_{f,l_{\text{II}}}\right\}&=\left\{t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\triangle 2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle\begin{pmatrix}\mathbf{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_4-\boldsymbol{r}_4'\right)\end{pmatrix}\right\}\\&=\left\{t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\triangle 2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\right\}\end{aligned} \tag{9c}$$

$$\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\} = \left\{2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle\begin{pmatrix}\boldsymbol{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)(\boldsymbol{r}_4-\boldsymbol{r}_4')\end{pmatrix}\triangle t_2\begin{pmatrix}\boldsymbol{0}\\ e^{\theta_4'[\boldsymbol{s}_4\times]}\boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\}$$
$$= \left\{2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\triangle t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\} \quad (9d)$$

$$\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\} = \left\{t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle 2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle\begin{pmatrix}\boldsymbol{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)(\boldsymbol{r}_4-\boldsymbol{r}_4')\end{pmatrix}\triangle t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\}$$
$$= \left\{t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle 2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\triangle t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\} \quad (9e)$$

$$\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\} = \left\{t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle 2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle\begin{pmatrix}\boldsymbol{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)(\boldsymbol{r}_4-\boldsymbol{r}_4')\end{pmatrix}\triangle t_3\begin{pmatrix}\boldsymbol{0}\\ e^{\theta_4'[\boldsymbol{s}_4\times]}\boldsymbol{s}_3\end{pmatrix}\right\}$$
$$= \left\{t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle 2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\triangle 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\right\} \quad (9f)$$

Thus, denoted by PPRR, RPPR, RRPP, PRPR, PRRP and RPRP, six 4-DOF TPM limbs can analytically be synthesized. It should be noted that since Eq. (9) can also be derived from any of four equivalent expressions in Eq. (7), we only take into account the limb structures generated from one of them.

Secondly, according to Theorem 1 in Appendix B and the finite motion analysis of the triad composed of three R joints whose axes are parallel to one another (see Appendix C), Eq. (7d) can also be expressed as

$$\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\} = \left\{2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle t_1\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_1\end{pmatrix}\triangle t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\triangle t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\}$$
$$= \left\{2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle\begin{pmatrix}\boldsymbol{0}\\ \left(e^{(\theta_4'+\theta_4'')[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)(\boldsymbol{r}_4-\boldsymbol{r}_4'')\end{pmatrix}\triangle\begin{pmatrix}\boldsymbol{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)(\boldsymbol{r}_4''-\boldsymbol{r}_4')\end{pmatrix}\triangle t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\} \quad (10a)$$
$$= \left\{2\tan\frac{\theta_4-\theta_4'-\theta_4''}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle 2\tan\frac{\theta_4''}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4''\times\boldsymbol{s}_4\end{pmatrix}\triangle 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\triangle t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\}$$

(10d) (10c) (10b)

This means that the screw triangle product formed by a rotational factor and three translational factors is equivalent to that formed by three rotational factors and a translational factor. This property allows three equivalent expressions to be achieved by placing the remained one translational factor in all possible positions in the screw triangle product whilst keeping $\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\}$ unchanged.

$$\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\} = \left\{2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle\begin{pmatrix}\boldsymbol{0}\\ \left(e^{(\theta_4'+\theta_4'')[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)(\boldsymbol{r}_4-\boldsymbol{r}_4'')\end{pmatrix}\triangle\begin{pmatrix}\boldsymbol{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)(\boldsymbol{r}_4''-\boldsymbol{r}_4')\end{pmatrix}\triangle t_3\begin{pmatrix}\boldsymbol{0}\\ e^{\theta_4'[\boldsymbol{s}_4\times]}\boldsymbol{s}_3\end{pmatrix}\right\}$$
$$= \left\{2\tan\frac{\theta_4-\theta_4'-\theta_4''}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\triangle 2\tan\frac{\theta_4''}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4''\times\boldsymbol{s}_4\end{pmatrix}\triangle t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\triangle 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\right\} \quad (10b)$$

$$\left\{ \boldsymbol{S}_{f,l_{\mathrm{II}}} \right\} = \left\{ 2\tan\frac{\theta_4}{2}\begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \square \begin{pmatrix} \boldsymbol{0} \\ \left(e^{(\theta_4'+\theta_4'')[\boldsymbol{s}_4\times]} - \boldsymbol{I}\right)\left(\boldsymbol{r}_4 - \boldsymbol{r}_4''\right) \end{pmatrix} \square \begin{pmatrix} \boldsymbol{0} \\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]} - \boldsymbol{I}\right)\left(\boldsymbol{r}_4'' - \boldsymbol{r}_4'\right) \end{pmatrix} \square t_3 \begin{pmatrix} \boldsymbol{0} \\ e^{(\theta_4'+\theta_4'')[\boldsymbol{s}_4\times]}\boldsymbol{s}_3 \end{pmatrix} \right\}$$
$$= \left\{ 2\tan\frac{\theta_4 - \theta_4' - \theta_4''}{2}\begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \square t_3 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_3 \end{pmatrix} \square 2\tan\frac{\theta_4''}{2}\begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4'' \times \boldsymbol{s}_4 \end{pmatrix} \square 2\tan\frac{\theta_4'}{2}\begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4' \times \boldsymbol{s}_4 \end{pmatrix} \right\} \tag{10c}$$

$$\left\{ \boldsymbol{S}_{f,l_{\mathrm{II}}} \right\} = \left\{ t_3 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_3 \end{pmatrix} \square 2\tan\frac{\theta_4}{2}\begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \square \begin{pmatrix} \boldsymbol{0} \\ \left(e^{(\theta_4'+\theta_4'')[\boldsymbol{s}_4\times]} - \boldsymbol{I}\right)\left(\boldsymbol{r}_4 - \boldsymbol{r}_4''\right) \end{pmatrix} \square \begin{pmatrix} \boldsymbol{0} \\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]} - \boldsymbol{I}\right)\left(\boldsymbol{r}_4'' - \boldsymbol{r}_4'\right) \end{pmatrix} \right\}$$
$$= \left\{ t_3 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_3 \end{pmatrix} \square 2\tan\frac{\theta_4 - \theta_4' - \theta_4''}{2}\begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \square 2\tan\frac{\theta_4''}{2}\begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4'' \times \boldsymbol{s}_4 \end{pmatrix} \square 2\tan\frac{\theta_4'}{2}\begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4' \times \boldsymbol{s}_4 \end{pmatrix} \right\} \tag{10d}$$

Eq. (10) physically corresponds to four 4-DOF TPM limb structures denoted respectively by PRRR, RPRR, RRPR and RRRP. For the reason addressed previously, we only take into account the limb structures derived from Eq. (7d).

As a result of the foregoing analysis, fourteen 4-DOF TPM limb structures in total can be synthesized in a hierarchical and analytical manner using the process shown in Fig. 5. Note that the geometry of the joint axes in a 4-DOF TPM limb should satisfy the following conditions: (1) the axes of all R joints should be parallel to one another; (2) the directions of any two P joints are non-colinear, the directions of any three P joints are non-coplanar, and (3) at least one P joint should not be normal to the axis of any R joint.

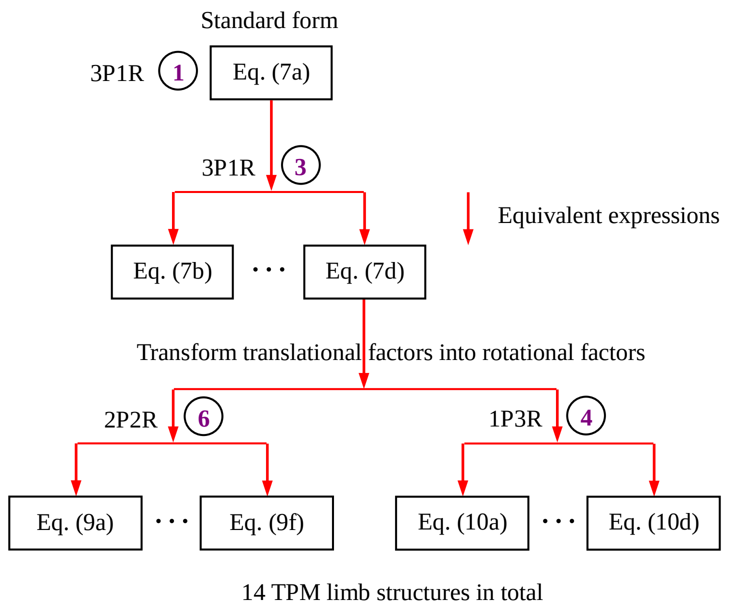


Fig. 5. The process to generate the 4-DOF TPM limb structures

3.2 5-DOF TPM limb structures

Following the process developed in Section 3.1, we will synthesize 5-DOF TPM limb structures. By adding two rotational factors having non-colinear axis directions at the right end of Eq. (6), we define the standard form of the finite screw of a 5-DOF TPM limb

$$\left\{ \boldsymbol{S}_{f,l_{\mathrm{III}}} \right\} = \left\{ t_1 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_1 \end{pmatrix} \square t_2 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_2 \end{pmatrix} \square t_3 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_3 \end{pmatrix} \square 2\tan\frac{\theta_4}{2}\begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \square 2\tan\frac{\theta_5}{2}\begin{pmatrix} \boldsymbol{s}_5 \\ \boldsymbol{r}_5 \times \boldsymbol{s}_5 \end{pmatrix} \right\} \tag{11}$$

Compared with 4-DOF TPM limbs, the process to synthesize 5-DOF TPM limbs is more complicated because some finite screws are not equivalent to the standard form given in Eq. (11) though they contain three translational motions. In order for the readers to have a clearer picture of the whole process, we divide 5-DOF TPM limbs into two categories, depending upon whether they contain inactive R joints, i.e. the R joints that do not get involved in generating translational motions.

3.2.1 5-DOF TPM limb structures with inactive R joints

Firstly, we assume that 5-DOF TPM limbs contain two inactive R joints. If so, three P joints must be required to generate three translations. Thus, similar to the process to derive Eq. (7), nine equivalent expressions of Eq. (11) can be formulated using Theorem 3 in Appendix B by placing two rotational factors (without changing the sequence between them) in all possible positions in the screw triangle product while keeping $\left\{ \boldsymbol{S}_{f,l_{\mathrm{III}}} \right\}$ unchanged. Only one of them is presented here due to space limitation.

$$\left\{ \boldsymbol{S}_{f,l_{\mathrm{III}}} \right\} = \left\{ 2\tan\frac{\theta_4}{2} \begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \Delta t_1 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_1 \end{pmatrix} \Delta t_2 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_2 \end{pmatrix} \Delta t_3 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_3 \end{pmatrix} \Delta 2\tan\frac{\theta_5}{2} \begin{pmatrix} \boldsymbol{s}_5 \\ \boldsymbol{r}_5 \times \boldsymbol{s}_5 \end{pmatrix} \right\} \tag{12}$$

Combined with the standard form given in Eq. (11), these equivalent expressions physically correspond to ten 5-DOF TPM limb structures denoted respectively by R$\underline{\text{R}}$PPP (Eq. (11)), PR$\underline{\text{R}}$PP, PPR$\underline{\text{R}}$P, PPPR$\underline{\text{R}}$, RP$\underline{\text{R}}$PP, RPP$\underline{\text{R}}$P, RPPP$\underline{\text{R}}$ (Eq. (12)), PRP$\underline{\text{R}}$P, PRPP$\underline{\text{R}}$ and PPRP$\underline{\text{R}}$. Here, the axis of the underlined R joint ($\underline{\text{R}}$) is not parallel to that of the R joint without underline.

Secondly, we assume that the 5-DOF TPM limbs contain one R inactive joint and two R joints whose axes are parallel to each other. By grouping the first four factors in Eq. (12) as Eq. (7d) and the last four factors in Eq. (12) as Eq. (7a), respectively, twelve equivalent expressions of Eq. (12) can be formulated by implementing the process to derive Eq. (9). One of six equivalent expressions corresponding to Eq. (7d) and that to Eq. (7a) are presented as follows.

$$\left\{ \boldsymbol{S}_{f,l_{\mathrm{III}}} \right\} = \left\{ 2\tan\frac{\theta_4 - \theta_4'}{2} \begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \Delta 2\tan\frac{\theta_4'}{2} \begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4' \times \boldsymbol{s}_4 \end{pmatrix} \Delta t_2 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_2 \end{pmatrix} \Delta t_3 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_3 \end{pmatrix} \Delta 2\tan\frac{\theta_5}{2} \begin{pmatrix} \boldsymbol{s}_5 \\ \boldsymbol{r}_5 \times \boldsymbol{s}_5 \end{pmatrix} \right\} \tag{13}$$

$$\left\{ \boldsymbol{S}_{f,l_{\mathrm{III}}} \right\} = \left\{ 2\tan\frac{\theta_4}{2} \begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \Delta t_2 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_2 \end{pmatrix} \Delta t_3 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_3 \end{pmatrix} \Delta 2\tan\frac{\theta_5 - \theta_5'}{2} \begin{pmatrix} \boldsymbol{s}_5 \\ \boldsymbol{r}_5 \times \boldsymbol{s}_5 \end{pmatrix} \Delta 2\tan\frac{\theta_5'}{2} \begin{pmatrix} \boldsymbol{s}_5 \\ \boldsymbol{r}_5' \times \boldsymbol{s}_5 \end{pmatrix} \right\} \tag{14}$$

These equivalent expressions physically correspond to twelve 5-DOF TPM limb structures denoted by RPP$\underline{\text{RR}}$ (Eq. (13)), R$\underline{\text{R}}$PP$\underline{\text{R}}$, R$\underline{\text{RR}}$PP, RP$\underline{\text{R}}$P$\underline{\text{R}}$, RP$\underline{\text{RR}}$P, R$\underline{\text{R}}$P$\underline{\text{R}}$P, RRPP$\underline{\text{R}}$ (Eq. (14)), RPPR$\underline{\text{R}}$, PPRR$\underline{\text{R}}$, RPRP$\underline{\text{R}}$, PRRP$\underline{\text{R}}$ and PRPR$\underline{\text{R}}$. Here, it is assumed that (1) the axes of the R joints without underline are parallel to each other; (2) the axes of the underlined R joints ($\underline{\text{R}}$) are parallel to each other, but they are not parallel to the axes of the R joints without underline.

The same routine is applicable to the 5-DOF TPM limbs containing one R inactive joint and three R joints whose axes are parallel to one another. Implementing the process to derive Eq. (10) results in eight equivalent expressions of Eq. (12). One of four equivalent expressions corresponding to Eq. (7d) and that to Eq. (7a) are presented here.

$$\left\{ \boldsymbol{S}_{f,l_{\mathrm{III}}} \right\} = \left\{ 2\tan\frac{\theta_4 - \theta_4' - \theta_4''}{2} \begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \Delta 2\tan\frac{\theta_4''}{2} \begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4'' \times \boldsymbol{s}_4 \end{pmatrix} \Delta 2\tan\frac{\theta_4'}{2} \begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4' \times \boldsymbol{s}_4 \end{pmatrix} \Delta t_3 \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_3 \end{pmatrix} \Delta 2\tan\frac{\theta_5}{2} \begin{pmatrix} \boldsymbol{s}_5 \\ \boldsymbol{r}_5 \times \boldsymbol{s}_5 \end{pmatrix} \right\} \tag{15}$$

$$\left\{\boldsymbol{S}_{f,l_{\mathrm{III}}}\right\}=\left\{2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\Delta 2\tan\frac{\theta_5-\theta_5'-\theta_5''}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\Delta 2\tan\frac{\theta_5''}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5''\times\boldsymbol{s}_5\end{pmatrix}\Delta 2\tan\frac{\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5'\times\boldsymbol{s}_5\end{pmatrix}\right\} \tag{16}$$

Thus, eight 5-DOF TPM limb structures can be synthesized and denoted by RPRRR (Eq. (15)), RRPRR, RRRPR, RRRRP, RRRPR (Eq. (16)), RRPRR, RPRRR, PRRRR using the same conventions defined previously.

By commuting the inactive R joint with its adjacent P joint, another ten equivalent expressions of Eq. (12) can be obtained because this operation does not affect the finite motion of a 5-DOF TPM limb (see Theorem 3 in Appendix B), leading to ten 5-DOF limb structures denoted by PRPRR (Eq. (17)), PPRRR, PRRPR, PRRRP, PRRRR, RRPRP, RRRPP, RPRRP, PRRRP and RRRRP. For example, the derivation to obtain PRPRR can be detailed as follows.

$$\begin{aligned}\left\{\boldsymbol{S}_{f,l_{\mathrm{III}}}\right\}&=\left\{2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\Delta t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\Delta t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\Delta 2\tan\frac{\theta_5}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\right\}\\ &=\left\{2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta\begin{pmatrix}\boldsymbol{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_4-\boldsymbol{r}_4'\right)\end{pmatrix}\Delta t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\Delta t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\Delta 2\tan\frac{\theta_5}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\right\}\\ &=\left\{2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\Delta t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\Delta 2\tan\frac{\theta_5}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\Delta t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\}\end{aligned} \tag{17}$$

It should be pointed out that if we commute the inactive R joint with its adjacent R joint, the produced finite screws still contain three translations though they are no longer equivalent to Eq. (11). For example, the finite screw of RRPPR does not equal to Eq. (11) but equals to Eq. (9b) if $\theta_5=0$.

$$\left\{2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\Delta t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\Delta 2\tan\frac{\theta_5}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\Delta 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\right\}\neq\left\{\boldsymbol{S}_{f,l_{\mathrm{III}}}\right\} \tag{18}$$

$$\begin{aligned}&\left\{2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\Delta t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\Delta 2\tan\frac{\theta_5}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\Delta 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\middle|\theta_5=0\right\}\\ &=\left\{2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta t_2\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_2\end{pmatrix}\Delta t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_3\end{pmatrix}\Delta 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\right\}=\left\{\boldsymbol{S}_{f,l_{\mathrm{II}}}\right\}\end{aligned} \tag{19}$$

These expressions lead to twenty 5-DOF TPM limb structures denoted respectively by RRPPR (Eq. (18)), RPRPR, RPPRR, RRRPP, RRPRP, RPRRP, RRPRR, RPRRR, RPRRR, RRRPR, RRRPR, RRPRR, RRRRP, RRRRP, PPRRR, PRRPR, PRPRR, PRRRP, PRRRR, and PRRRR.

At this stage, sixty 5-DOF TPM limb structures with inactive R joints can be created. Among them, ten are composed of three P joints and two inactive R joints, and the rest are composed of two (three) R joints whose axes are parallel to one another, two (one) P joints, and an inactive R joint.

3.2.2 5-DOF TPM limb structures without inactive R joint

For the 5-DOF TPM limbs without inactive R joint, two cases should be considered. The first case is that the limbs are composed of one P joint and two groups of R joints, each consisting of two R joints whose axes are parallel to each other. According to Theorems 1 and 3 in Appendix B and the finite motion analysis of the dyad in Appendix C, one more equivalent expression of Eq. (11) can be derived.

$$\begin{aligned}\left\{\boldsymbol{S}_{f,l_{\mathrm{III}}}\right\}&=\left\{\begin{pmatrix}\mathbf{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_4-\boldsymbol{r}_4'\right)\end{pmatrix}\Delta\begin{pmatrix}\mathbf{0}\\ \left(e^{\theta_5'[\boldsymbol{s}_5\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_5-\boldsymbol{r}_5'\right)\end{pmatrix}\Delta t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\Delta 2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta 2\tan\frac{\theta_5}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\right\}\\ &=\left\{2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta\begin{pmatrix}\mathbf{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_4-\boldsymbol{r}_4'\right)\end{pmatrix}\Delta 2\tan\frac{\theta_5}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\Delta\begin{pmatrix}\mathbf{0}\\ \left(e^{\theta_5'[\boldsymbol{s}_5\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_5-\boldsymbol{r}_5'\right)\end{pmatrix}\Delta t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\}\\ &=\left\{2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\Delta 2\tan\frac{\theta_5-\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\Delta 2\tan\frac{\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5'\times\boldsymbol{s}_5\end{pmatrix}\Delta t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\right\}\end{aligned}\tag{20}$$

Similar to the process to derive Eqs. (9) and (10), four equivalent expressions of Eq. (20) can be formulated by placing the translational factor in all possible positions, leading to five 5-DOF TPM limb structures denoted by PRR$\underline{\text{RR}}$ (Eq. (20)), RPR$\underline{\text{RR}}$, RRP$\underline{\text{RR}}$, RR$\underline{\text{R}}$P$\underline{\text{R}}$ and RR$\underline{\text{RR}}$P.

If we place the two R joints belonging to one group plus the P joint between two R joints belonging to another group, it can be proved that the resultant finite screw contains three translational motions though it is no longer equivalent to Eq. (11). For example, the finite screw of $\underline{\text{R}}$RPR$\underline{\text{R}}$ dose not equal to Eq. (11) but equals to Eq. (6) provided that $\theta_4=0$ and $\theta_5=0$, i.e.

$$\left\{2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta 2\tan\frac{\theta_5-\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\Delta t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\Delta 2\tan\frac{\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5'\times\boldsymbol{s}_5\end{pmatrix}\Delta 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\right\}\neq\left\{\boldsymbol{S}_{f,l_{\mathrm{III}}}\right\}\tag{21}$$

$$\begin{aligned}&\left\{2\tan\frac{\theta_4-\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta 2\tan\frac{\theta_5-\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\Delta t_3\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_3\end{pmatrix}\Delta 2\tan\frac{\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5'\times\boldsymbol{s}_5\end{pmatrix}\Delta 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\middle|\begin{matrix}\theta_4=0,\\ \theta_5=0\end{matrix}\right\}\\ &=\begin{pmatrix}\mathbf{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_4-\boldsymbol{r}_4'\right)\end{pmatrix}\Delta\begin{pmatrix}\mathbf{0}\\ e^{\theta_4'[\boldsymbol{s}_4\times]}\left(e^{\theta_5'[\boldsymbol{s}_5\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_5-\boldsymbol{r}_5'\right)\end{pmatrix}\Delta t_3\begin{pmatrix}\mathbf{0}\\ e^{\theta_4'[\boldsymbol{s}_4\times]}e^{\theta_5'[\boldsymbol{s}_5\times]}\boldsymbol{s}_3\end{pmatrix}=\left\{\boldsymbol{S}_{f,l_{\mathrm{I}}}\right\}\end{aligned}\tag{22}$$

In this way, five 5-DOF TPM limb structures can be generated and denoted by P$\underline{\text{R}}$RR$\underline{\text{R}}$, $\underline{\text{R}}$PRR$\underline{\text{R}}$, $\underline{\text{R}}$RPR$\underline{\text{R}}$ (Eq. (21), $\underline{\text{R}}$RRP$\underline{\text{R}}$ and $\underline{\text{R}}$RR$\underline{\text{R}}$P, respectively.

The second case is that the limbs are composed of five R joints. For the similar reason mentioned above, we divide the R joints into two groups. One group contains two R joints and another contains three R joints. Again, we assume that the axes of R joints belonging to the same group are parallel to one another, and those belonging to different groups are not parallel to one another.

According to Theorems 1 and 3 in Appendix B and the finite motion analysis of the dyad and the triad in Appendix C, one more equivalent expression of Eq. (11) can be derived

$$\begin{aligned}\left\{\boldsymbol{S}_{f,l_{\mathrm{III}}}\right\}&=\left\{2\tan\frac{\theta_4}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta\begin{pmatrix}\mathbf{0}\\ \left(e^{(\theta_4'+\theta_4'')[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_4-\boldsymbol{r}_4''\right)\end{pmatrix}\Delta\begin{pmatrix}\mathbf{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_4''-\boldsymbol{r}_4'\right)\end{pmatrix}\Delta 2\tan\frac{\theta_5}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\Delta\begin{pmatrix}\mathbf{0}\\ \left(e^{\theta_5'[\boldsymbol{s}_5\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_5-\boldsymbol{r}_5'\right)\end{pmatrix}\right\}\\ &=\left\{2\tan\frac{\theta_4-\theta_4'-\theta_4''}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix}\Delta 2\tan\frac{\theta_4''}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4''\times\boldsymbol{s}_4\end{pmatrix}\Delta 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix}\Delta 2\tan\frac{\theta_5-\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix}\Delta 2\tan\frac{\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5'\times\boldsymbol{s}_5\end{pmatrix}\right\}\end{aligned}\tag{23}$$

Eq. (23) corresponds to the limb structure denoted by RR$\underline{\text{RRR}}$. And RRR$\underline{\text{RR}}$ can be generated in similar way.

If we place two (three) R joints belonging to one group between three (two) R joints in another group, it can be proved that the resultant finite screw contains three translational motions though it is no longer equivalent to Eq. (11). For example, the finite screw of $\underline{\text{R}}$RR$\underline{\text{RR}}$ does not equal to Eq. (11) but equals Eq. (6) if $\theta_4=0$ and $\theta_5=0$, i.e.

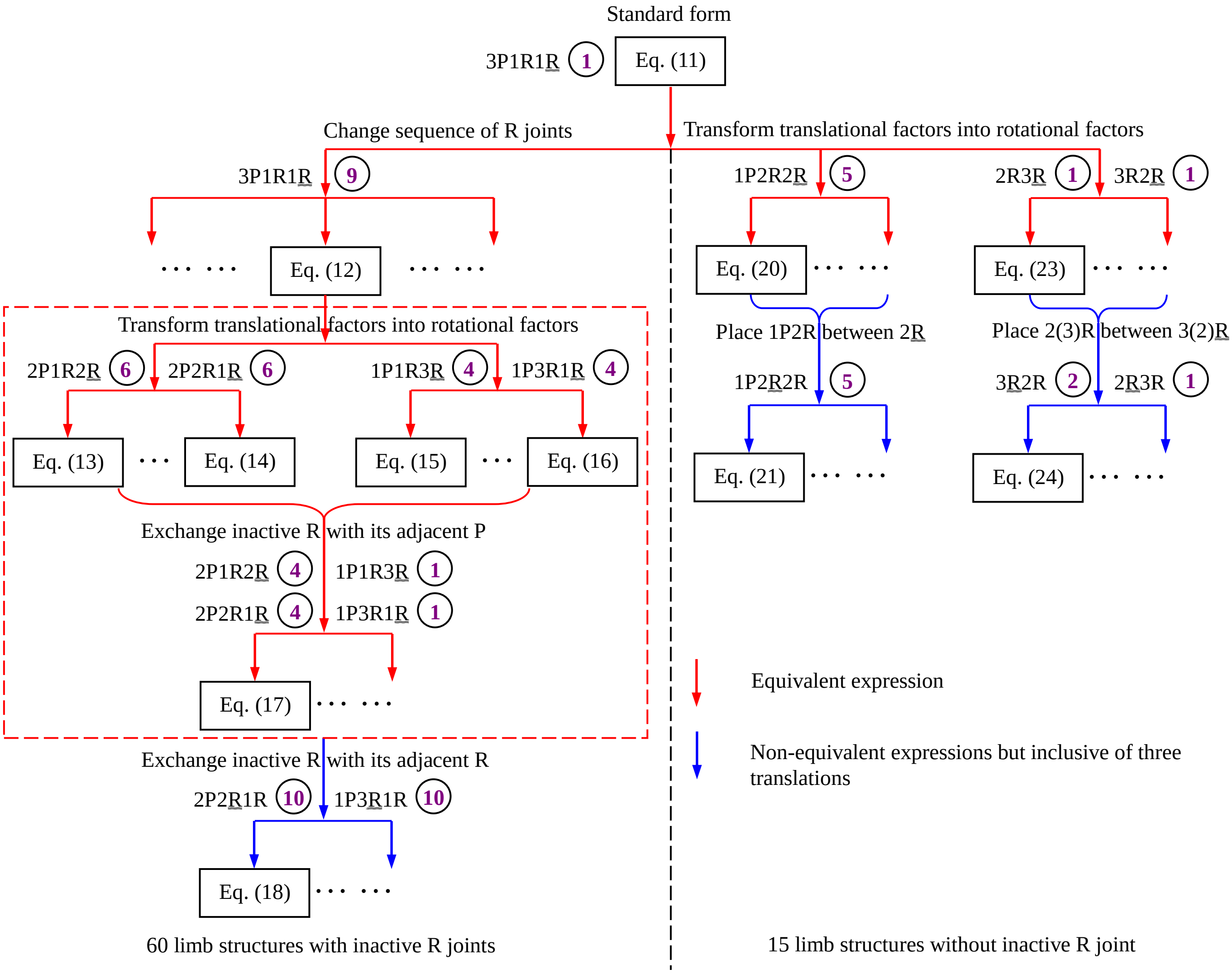


Fig. 6. The process to synthesize 5-DOF TPM limb structures with and without inactive R joints

$$\left\{ 2\tan\frac{\theta_4-\theta_4'-\theta_4''}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix} \square 2\tan\frac{\theta_4''}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4''\times\boldsymbol{s}_4\end{pmatrix} \square 2\tan\frac{\theta_5-\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix} \square 2\tan\frac{\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5'\times\boldsymbol{s}_5\end{pmatrix} \square 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix} \right\} \neq \left\{ \boldsymbol{S}_{f,l_{\mathrm{III}}} \right\} \quad (24)$$

$$\left\{ 2\tan\frac{\theta_4-\theta_4'-\theta_4''}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4\times\boldsymbol{s}_4\end{pmatrix} \square 2\tan\frac{\theta_4''}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4''\times\boldsymbol{s}_4\end{pmatrix} \square 2\tan\frac{\theta_5-\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5\times\boldsymbol{s}_5\end{pmatrix} \square 2\tan\frac{\theta_5'}{2}\begin{pmatrix}\boldsymbol{s}_5\\ \boldsymbol{r}_5'\times\boldsymbol{s}_5\end{pmatrix} \square 2\tan\frac{\theta_4'}{2}\begin{pmatrix}\boldsymbol{s}_4\\ \boldsymbol{r}_4'\times\boldsymbol{s}_4\end{pmatrix} \middle| \theta_4=0,\ \theta_5=0 \right\}$$
$$= \begin{pmatrix}\boldsymbol{0}\\ \left(e^{(\theta_4'+\theta_4'')[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)(\boldsymbol{r}_4-\boldsymbol{r}_4'')\end{pmatrix} \square \begin{pmatrix}\boldsymbol{0}\\ \left(e^{\theta_4'[\boldsymbol{s}_4\times]}-\boldsymbol{I}\right)(\boldsymbol{r}_4''-\boldsymbol{r}_4')\end{pmatrix} \square \begin{pmatrix}\boldsymbol{0}\\ e^{\theta_4'[\boldsymbol{s}_4\times]}\left(e^{\theta_5'[\boldsymbol{s}_5\times]}-\boldsymbol{I}\right)(\boldsymbol{r}_5-\boldsymbol{r}_5')\end{pmatrix} = \left\{ \boldsymbol{S}_{f,l_{\mathrm{I}}} \right\}$$

(25)

In this way, another three 5-DOF TPM limb structures, $\underline{R}RR\underline{RR}$ (Eq. (24)), $\underline{RR}RR\underline{R}$ and $\underline{R}RRR\underline{R}$, can be created.

At this stage, fifteen 5-DOF TPM limb structures without inactive R joints can be synthesized. Among them, ten are composed of one P joint and two groups of R joints, each consisting of two R joints. The rest five are composed of two groups of R joints, one consisting of two R joints, and another consisting of three R joints.

Consequently, seventy-five 5-DOF limb structures in total can be synthesized in a hierarchical and analytical manner using the processes shown in Fig. 6. Note that the geometry of the joint axes in a 5-DOF TPM limb should satisfy the following conditions: (1) the axes of the R joints without underline are parallel to each other; (2) the axes of the underlined R joints (R) are parallel to each other, but they are not parallel to the axes of the R joints without underline; (3) the directions of any two P joints are non-colinear, the directions of any three P joints are non-coplanar; (4) any P joint is not normal to the axes of three parallel R joints and any two P joints are not both normal to the axes of two parallel R joints.

4. Assembly Conditions

Having the limb structures developed in Section 3, the assembly conditions to synthesize TPMs using these limbs will be addressed. We know from Eq. (3) that the finite screw of a TPM should be the intersection of those of all limbs. Here, we do not consider the TPMs having at least a 3-DOF limb since the platform has naturally three translational finite motions. So, we only consider the assembly conditions for the TPMs composed of 4 and/or 5-DOF limbs by identifying the independent equations amongst the following $6\times(l-1)$ constraint equations.

$$\boldsymbol{S}_{f,1}=\boldsymbol{S}_{f,2}=\cdots=\boldsymbol{S}_{f,l} \tag{26}$$

In order to do so, assume that a TPM consists of l_{II} 4-DOF limbs and l_{III} 5-DOF limbs such that $l_{\mathrm{II}}+l_{\mathrm{III}}=l$. Then, $6\times(l_{\mathrm{II}}+l_{\mathrm{III}}-1)$ scalar equations containing $4l_{\mathrm{II}}+5l_{\mathrm{III}}$ screw parameters can be formulated using Eq. (26). Note that each TPM limb can realize three translations but less than three rotations. So, out of $3\times(l_{\mathrm{II}}+l_{\mathrm{III}}-1)$ scalar equations associated with the rotations, at most $l_{\mathrm{II}}+2l_{\mathrm{III}}$ of them are independent and they thereby can be used to solve $l_{\mathrm{II}}+2l_{\mathrm{III}}$ screw parameters in terms of rotation. Consequently, the total number of the independent equations and the relevant screw parameters can be determined by

$$m_e=6\times(l_{\mathrm{II}}+l_{\mathrm{III}}-1)-m_e' \tag{27}$$

$$m_e'=2l_{\mathrm{II}}+l_{\mathrm{III}}-3+f$$

$$m_v=4l_{\mathrm{II}}+5l_{\mathrm{III}} \tag{28}$$

where m_e and m_v represent the number of the independent equations and the screw parameters in Eq. (26); m_e' denotes the number of the dependent equations in Eq. (26); f denotes the coefficient for the two special cases as follows.

Case 1: $f=1$ when $l_{\mathrm{II}}\geq 1$ and the finite screws of all 4-DOF limbs are identical and are the subsets of those of all 5-DOF limbs, or $l_{\mathrm{II}}=0$ and the finite screws of all 5-DOF limbs are not equal but have common finite screw of rotation;

Case 2: $f=2$ when $l_{\mathrm{II}}=0$ and the finite screws of all 5-DOF limbs are identical.

Combining Eq. (27) with (28) yields the number of free screw parameters in Eq. (26)

$$m_{fv}=m_v-m_e=3+f \tag{29}$$

As mentioned above, TPM has three DOFs, the constraint equations (Eq. (26)) should contain only three free screw parameters. Therefore, $f=0$, i.e. the case 1 ($f=1$) and case 2 ($f=2$) should be avoided during assembling a TPM. This consideration finally yields the assembly conditions of a TPM composed of 4 and/or 5-DOF limbs: (1) the axis of a R joint in a limb should not be parallel to that in another limb; (2) a TPM cannot be synthesized by only two 5-DOF limbs and their motion-equivalent limbs.

5. Conclusions

This paper for the first time uses finite screw theory for type synthesis of TPMs. The following conclusions are drawn.

1) The continuous finite motion of a TPM limb can be represented as the screw triangle product of the finite screws of joints.

2) The 3-, 4-DOF TPM limbs can be synthesized by simply deriving the equivalent expressions of the standard forms of finite screws of the limbs using properties of screw triangle product. This can be extended to the 5-DOF TPM limbs by additionally considering the finite screws containing three translational motions though they are not equivalent to the corresponding standard form. The processes to synthesize the full set of 3-, 4-, and 5-DOF TPM limbs have been developed in a parametric and hierarchical manner.

3) Compared with the other methods developed previously, the merit of the proposed approach lies in that the limb structures can be synthesized in an analytical manner that naturally ensures the full cycle finite motion pattern specified to the platform.

4) On the basis of the finite screw approach proposed in this paper, our future work will focus upon type synthesis problem of parallel mechanisms having other motion patterns, particularly for those having parasitic motions. These issues, however, deserve to be addressed in separate reports.

Acknowledgements

This research work is partially supported by the National Natural Science Foundation of China (NSFC) under Grant 51135008 and 51205278.

References

[1] K. H. Hunt, Kinematic Geometry of Mechanisms, Oxford University Press, 1978.

[2] M. Carricato, V. Parenti-Castelli, A family of 3-DOF translational parallel manipulators, ASME J. Mech. Des. 125 (2) (2003) 302-307.

[3] Q. C. Li, Z. Huang, Mobility analysis of a novel 3-5R parallel mechanism family, ASME J. Mech. Des. 126 (1) (2004) 79-82.

[4] X. W. Kong, C. M. Gosselin, Kinematics and singularity analysis of a novel type of 3-CRR 3-DOF translational parallel manipulator, Int. J. Robot. Res. 21 (9) (2002) 791-798.

[5] J. P. Merlet, C. M. Gosselin, T. Huang, Handbook of Robotics, Chapter 18: Parallel Mechanisms, Springer, 2014.

[6] J. M. Hervé, F. Sparacino, Structural synthesis of “parallel” robots generating spatial translation, Proc. of the 1991 5th International Conference on Advanced Robotics, Pisa, Italy, 1991, pp. 808-813.

[7] Q. C. Li, Type synthesis theory of lower-mobility parallel mechanisms and synthesis of new architectures (Ph.D. thesis) Yanshan University, China, 2003.

[8] W. Wei, Type synthesis and kinematic analysis of translational parallel manipulators (Master thesis) McGill University, Canada, 2005.

[9] Q. Jin, T. L. Yang, Theory for topology synthesis of parallel manipulators and its application to three-dimension-translation parallel manipulators, ASME J. Mech. Des. 126 (4) (2004) 625-639.

[10] J. Meng, G. F. Liu, Z. X. Li, A geometric theory for analysis and synthesis of sub-6 DOF parallel manipulators, IEEE Trans. Robot. 23 (4) (2007) 625-649.

[11] Z. Huang, Q. C. Li, General methodology for type synthesis of symmetrical lower-mobility parallel manipulators and several novel manipulators, Int. J. Robot. Res. 21 (2) (2002) 131-145.

[12] Z. Huang, Q. C. Li, Type synthesis of symmetrical lower-mobility parallel mechanisms using the constraint synthesis method, Int. J. Robot. Res. 22 (1) (2003) 59-79.

[13] X. W. Kong, C. M. Gosselin, Type synthesis of 3-DOF translational parallel manipulators based on screw theory, ASME J. Mech. Des. 126 (1) (2004) 83-92.

[14] B. C. Hall, Lie Groups, Lie Algebras, and Representations: An Elementary Introduction, Springer-Verlag, 2003.

[15] F. M. Dimentberg, The Screw Calculus and Its Application to Mechanics, Moscow, 1965, English Translation, U.S. Department of Commerce, (N.T.I.S), No. AD, 1969, pp. 680–993.

[16] I. A. Parkin, A third conformation with the screw systems: finite twist displacements of a directed line and point, Mech. Mach. Theory 27 (2) (1992) 177-188.

[17] K. H. Hunt, I. A. Parkin, Finite displacements of points, planes, and lines via screw theory, Mech. Mach. Theory 30 (2) (1995) 177-192.

[18] C. T. Huang. The finite screw systems associated with a prismatic-revolute dyad and the screw displacement of a point, Mech. Mach. Theory 29 (8) (1994) 1131-1142.

[19] C. T. Huang, C. M. Chen, The linear representation of the screw triangle—a unification of finite and infinitesimal kinematics, ASME J. Mech. Des. 117 (4) (1995) 554-560.

[20] J. S. Dai, N. Holland, D. R. Kerr, Finite twist mapping and its application to planar serial manipulators with revolute joints, Proc. IMechE, Journal of Mechanical Engineering Science 209 (4) (1995) 263-271.

[21] J. S. Dai, An historical review of the theoretical development of rigid body displacements from Rodrigues parameters to the finite twist, Mech. Mach. Theory 41 (1) (2006) 41-52.

[22] J. S. Dai, Finite displacement screw operators with embedded Chasles' motion, ASME J. Mech. Robot. 4 (4) (2012) 041002-1-041002-9.

[23] J. S. Dai, Euler-Rodrigues formula variations, quaternion conjugation and intrinsic connections, Mech. Mach. Theory 92 (2015) 144-152.

[24] J. S. Dai, Historical relation between mechanisms and screw theory and the development of finite displacement screws, Journal of Mechanical Engineering 51 (13) (2015) 13-26.

[25] J. S. Dai, Geometrical Foundations and Screw Algebra for Mechanisms and Robotics, Higher Education Press, Beijing, 2014, ISBN: 9787040334838, (translated from J. S. Dai, Screw Algebra and Kinematic Approaches for Mechanisms and Robotics, to be published by Springer, London, 2016).

[26] F. Sticher, On the finite screw axis cylindroid, Mech. Mach. Theory 24 (3) (1989) 143-155.

[27] M. F. Chasles, Note sur le propriétés générales du système de deux corps semblables entr'eux et places d'une maniére quelconque dans l'espace; et sur le déplacement fini ou infiniment petis d'un corps solide libre, Férussac, Bulletin Des Sciences Mathematiques 14 (1830) 321-326.

[28] D. Zarrouk, M. Shoham, A note on the screw triangle, ASME J. Mech. Robot. 3 (1) (2011) 014502-1-014502-4.

[29] W. Rudin, Principles of Mathematical Analysis, 3rd ed., McGraw-Hill, Inc., 1976.

Appendix A

The resultant finite screw $\boldsymbol{S}_{f,ab}$ generated by two successive finite screws $\boldsymbol{S}_{f,a}$ and $\boldsymbol{S}_{f,b}$ of two 1-DOF joints can be represented as a screw triangle product [19, 28]

$$\boldsymbol{S}_{f,ab} = \boldsymbol{S}_{f,a} \triangle \boldsymbol{S}_{f,b} = \frac{1}{1 - \tan\frac{\theta_a}{2}\tan\frac{\theta_b}{2}\boldsymbol{s}_{f,a}^{\mathrm{T}}\boldsymbol{s}_{f,a}}\left(\boldsymbol{S}_{f,a} + \boldsymbol{S}_{f,b} + \frac{1}{2}\boldsymbol{S}_{f,b} \times \boldsymbol{S}_{f,a} - \tan\frac{\theta_a}{2}\tan\frac{\theta_b}{2}\left(t_b\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_{f,a}\end{pmatrix} + t_a\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_{f,a}\end{pmatrix}\right)\right) \quad \text{(A-1)}$$

with

$$\boldsymbol{S}_{f,i} = \begin{cases} 2\tan\frac{\theta_i}{2}\begin{pmatrix}\boldsymbol{s}_{f,i}\\ \boldsymbol{r}_{f,i} \times \boldsymbol{s}_{f,i}\end{pmatrix} & \text{Revolute joint } (t_i = 0) \\ t_i\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_{f,i}\end{pmatrix} & \text{Prismatic joint } (\theta_i = 0)\end{cases} \qquad i=a,\ b \quad \text{(A-2)}$$

$\boldsymbol{S}_{f,ab}$ can be normalized into the finite screw form shown in Eq. (1)

$$\boldsymbol{S}_{f,ab} = \boldsymbol{S}_{f,a} \triangle \boldsymbol{S}_{f,b} = 2\tan\frac{\theta_{ab}}{2}\begin{pmatrix}\boldsymbol{s}_{f,ab}\\ \boldsymbol{r}_{f,ab} \times \boldsymbol{s}_{f,ab}\end{pmatrix} + t_{ab}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_{f,ab}\end{pmatrix} \quad \text{(A-3)}$$

Then, θ_{ab}, t_{ab}, $\boldsymbol{s}_{f,ab}$ and $\boldsymbol{r}_{f,ab}$ in $\boldsymbol{S}_{f,ab}$ in the following cases can be determined by those in $\boldsymbol{S}_{f,a}$ and $\boldsymbol{S}_{f,b}$.

Case 1: If $\boldsymbol{S}_{f,a} = 2\tan\frac{\theta_a}{2}\begin{pmatrix}\boldsymbol{s}_{f,a}\\ \boldsymbol{r}_{f,a} \times \boldsymbol{s}_{f,a}\end{pmatrix}$, $\boldsymbol{S}_{f,b} = 2\tan\frac{\theta_b}{2}\begin{pmatrix}\boldsymbol{s}_{f,b}\\ \boldsymbol{r}_{f,b} \times \boldsymbol{s}_{f,b}\end{pmatrix}$ and $\boldsymbol{s}_{f,a} = \boldsymbol{s}_{f,b}$, then

$$\theta_{ab} = \theta_a + \theta_b, \quad t_{ab} = 0, \quad \boldsymbol{s}_{f,ab} = \boldsymbol{s}_{f,a} = \boldsymbol{s}_{f,b}$$

$$\boldsymbol{r}_{f,ab} = \frac{\tan\frac{\theta_a}{2}}{\tan\frac{\theta_a}{2} + \tan\frac{\theta_b}{2}}\boldsymbol{r}_{f,a} + \frac{\tan\frac{\theta_b}{2}}{\tan\frac{\theta_a}{2} + \tan\frac{\theta_b}{2}}\boldsymbol{r}_{f,b} + \frac{\tan\frac{\theta_a}{2}\tan\frac{\theta_b}{2}}{\tan\frac{\theta_a}{2} + \tan\frac{\theta_b}{2}}\left(\left(\boldsymbol{r}_{f,b} - \boldsymbol{r}_{f,a}\right) \times \boldsymbol{s}_{f,ab}\right) \quad \text{(A-4)}$$

Case 2: If $\boldsymbol{S}_{f,a} = 2\tan\frac{\theta_a}{2}\begin{pmatrix}\boldsymbol{s}_{f,a}\\ \boldsymbol{r}_{f,a} \times \boldsymbol{s}_{f,a}\end{pmatrix}$ and $\boldsymbol{S}_{f,b} = t_b\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_{f,b}\end{pmatrix}$, then

$$\theta_{ab} = \theta_a, \quad t_{ab} = t_b\boldsymbol{s}_{f,b}^{\mathrm{T}}\boldsymbol{s}_{f,a}, \quad \boldsymbol{s}_{f,ab} = \boldsymbol{s}_{f,a}, \quad \boldsymbol{r}_{f,ab} = \boldsymbol{r}_{f,a} - \frac{t_b}{2\tan\frac{\theta_a}{2}}\left(\boldsymbol{s}_{f,b} \times \boldsymbol{s}_{f,a}\right) + \frac{t_b}{2}\boldsymbol{s}_{f,b} \quad \text{(A-5)}$$

Case 3: If $\boldsymbol{S}_{f,a} = t_a\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_{f,a}\end{pmatrix}$ and $\boldsymbol{S}_{f,b} = 2\tan\frac{\theta_b}{2}\begin{pmatrix}\boldsymbol{s}_{f,b}\\ \boldsymbol{r}_{f,b} \times \boldsymbol{s}_{f,b}\end{pmatrix}$, then

$$\theta_{ab} = \theta_b, \quad t_{ab} = t_a\boldsymbol{s}_{f,a}^{\mathrm{T}}\boldsymbol{s}_{f,b}, \quad \boldsymbol{s}_{f,ab} = \boldsymbol{s}_{f,b}, \quad \boldsymbol{r}_{f,ab} = \boldsymbol{r}_{f,b} - \frac{t_a}{2\tan\frac{\theta_b}{2}}\left(\boldsymbol{s}_{f,a} \times \boldsymbol{s}_{f,b}\right) - \frac{t_a}{2}\boldsymbol{s}_{f,a} \quad \text{(A-6)}$$

Case 4: if $\boldsymbol{S}_{f,a} = t_a \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_{f,a} \end{pmatrix}$ and $\boldsymbol{S}_{f,b} = t_b \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_{f,b} \end{pmatrix}$, then

$$\theta_{ab} = 0, \quad t_{ab} = \left| t_a \boldsymbol{s}_{f,a} + t_b \boldsymbol{s}_{f,b} \right|, \quad \boldsymbol{s}_{f,ab} = \frac{t_a \boldsymbol{s}_{f,a} + t_b \boldsymbol{s}_{f,b}}{\left| t_a \boldsymbol{s}_{f,a} + t_b \boldsymbol{s}_{f,b} \right|}, \quad \boldsymbol{r}_{f,ab} = \mathbf{0} \tag{A-7}$$

Appendix B

Theorem 1: The screw triangle product formed by three translational factors having decoupled screw parameters is equivalent to that formed by three translation factors having coupled screw parameters, i.e.

$$\left\{ t_1 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_1 \end{pmatrix} \triangle t_2 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_2 \end{pmatrix} \triangle t_3 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_3 \end{pmatrix} \middle| t_i \in \mathbb{R},\ i = 1,2,3 \right\} = \left\{ \begin{pmatrix} \mathbf{0} \\ \boldsymbol{t}_1 \end{pmatrix} \triangle \begin{pmatrix} \mathbf{0} \\ \boldsymbol{t}_2 \end{pmatrix} \triangle \begin{pmatrix} \mathbf{0} \\ \boldsymbol{t}_3 \end{pmatrix} \middle| \begin{array}{l} \boldsymbol{t}_i = f_i\left((x_1, x_2, x_3)^{\mathrm{T}} \right) \in \mathbb{R}^3, \\ x_i \in \mathbb{R},\ i = 1,2,3 \end{array} \right\} \tag{B-1}$$

Proof: On one hand, according to Case 4 in Appendix A, Eq. (B-1) can be rewritten as

$$\left\{ t_1 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_1 \end{pmatrix} \triangle t_2 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_2 \end{pmatrix} \triangle t_3 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \mathbf{0} \\ t_1 \boldsymbol{s}_1 + t_2 \boldsymbol{s}_2 + t_3 \boldsymbol{s}_3 \end{pmatrix} \right\} \tag{B-2}$$

Here, $t_1 \boldsymbol{s}_1 + t_2 \boldsymbol{s}_2 + t_3 \boldsymbol{s}_3$ can be understood as a function of three decoupled parameters t_i ($i = 1,2,3$) over $\mathbb{R}^3$, each associated with a translational factor. On the other hand, it is obvious that

$$\left\{ \begin{pmatrix} \mathbf{0} \\ \boldsymbol{t}_1 \end{pmatrix} \triangle \begin{pmatrix} \mathbf{0} \\ \boldsymbol{t}_2 \end{pmatrix} \triangle \begin{pmatrix} \mathbf{0} \\ \boldsymbol{t}_3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \mathbf{0} \\ \boldsymbol{t}_1 + \boldsymbol{t}_2 + \boldsymbol{t}_3 \end{pmatrix} \right\} \tag{B-3}$$

where $\boldsymbol{t}_i$ ($i = 1,2,3$) represents a function having three parameters x_i ($i = 1,2,3$), and thus their sum is also a function of these parameters over $\mathbb{R}^3$ while some constants in this function can be infinite. Hence, Eq. (B-3) is equivalent to Eq. (B-2) in a sense of the range of the function value [29]. In other words, Eqs. (B-3) and (B-2) are two different expressions of the same set. Once the values of the parameters t_i ($i = 1,2,3$) are given, the values of x_i ($i = 1,2,3$) can be determined.

Theorem 2: The rotational factor in the following screw triangle product can be placed in any position while keeping the resultant finite screw unchanged.

$$\left\{ t_1 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_1 \end{pmatrix} \triangle t_2 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_2 \end{pmatrix} \triangle t_3 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_3 \end{pmatrix} \triangle 2\tan\frac{\theta_4}{2} \begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \middle| t_1, t_2, t_3, \theta_4 \in \mathbb{R} \right\} \tag{B-4}$$

Proof: According to Case 3 of Appendix A, Eq. (B-4) can be rewritten as

$$\left\{ t_1 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_1 \end{pmatrix} \triangle t_2 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_2 \end{pmatrix} \triangle t_3 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_3 \end{pmatrix} \triangle 2\tan\frac{\theta_4}{2} \begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \right\} = \left\{ 2\tan\frac{\theta_4}{2} \begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r} \times \boldsymbol{s}_4 \end{pmatrix} + t \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_4 \end{pmatrix} \right\} \tag{B-5}$$

where

$$\boldsymbol{r} = \boldsymbol{r}_4 - \frac{1}{2\tan\frac{\theta_4}{2}} \left(\left(t_1 \boldsymbol{s}_1 + t_2 \boldsymbol{s}_2 + t_3 \boldsymbol{s}_3 \right) \times \boldsymbol{s}_4 \right) - \frac{1}{2} \left(t_1 \boldsymbol{s}_1 + t_2 \boldsymbol{s}_2 + t_3 \boldsymbol{s}_3 \right) + k \boldsymbol{s}_4,\quad k \in \mathbb{R}\ ;\quad t = \left(t_1 \boldsymbol{s}_1 + t_2 \boldsymbol{s}_2 + t_3 \boldsymbol{s}_3 \right)^{\mathrm{T}} \boldsymbol{s}_4$$

Comparing Case 2 and Case 3 in Appendix B shows that placing the rotational factor in different positions only change the signs of $t_1 \boldsymbol{s}_1$, $t_2 \boldsymbol{s}_2$, $t_3 \boldsymbol{s}_3$ but does not change the ranges of $\boldsymbol{r}$ and t. Therefore, Theorem 2 can be proved, and leads to the following theorem.

Theorem 3: The two rotational factors in the following screw triangle product can be placed in any positions (without changing the sequence between them) while keeping the resultant finite screw unchanged.

$$\left\{ t_1 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_1 \end{pmatrix} \triangle t_2 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_2 \end{pmatrix} \triangle t_3 \begin{pmatrix} \mathbf{0} \\ \boldsymbol{s}_3 \end{pmatrix} \triangle 2\tan\frac{\theta_4}{2} \begin{pmatrix} \boldsymbol{s}_4 \\ \boldsymbol{r}_4 \times \boldsymbol{s}_4 \end{pmatrix} \triangle 2\tan\frac{\theta_5}{2} \begin{pmatrix} \boldsymbol{s}_5 \\ \boldsymbol{r}_5 \times \boldsymbol{s}_5 \end{pmatrix} \middle| t_1, t_2, t_3, \theta_4, \theta_5 \in \mathbb{R} \right\} \tag{B-6}$$

Appendix C

(1) Finite motion analysis of a dyad composed of two R joints whose axes are parallel to each other.

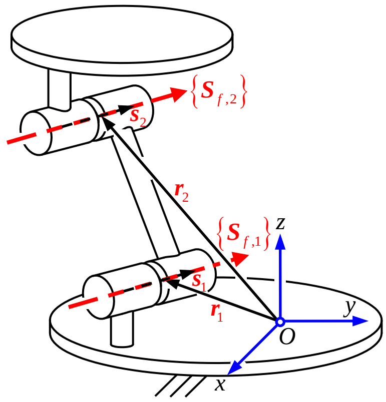


Fig. C-1. RR limb

The finite screw of the dyad shown in Fig. C-1 can be formulated as

$$\left\{\boldsymbol{S}_{f,\mathrm{RR}}\right\} = \left\{2\tan\frac{\theta_2}{2}\begin{pmatrix}\boldsymbol{s}_2\\ \boldsymbol{r}_2\times\boldsymbol{s}_2\end{pmatrix}\square\, 2\tan\frac{\theta_1}{2}\begin{pmatrix}\boldsymbol{s}_1\\ \boldsymbol{r}_1\times\boldsymbol{s}_1\end{pmatrix}\right\},\quad \boldsymbol{s}_2=\boldsymbol{s}_1=\boldsymbol{s} \tag{C-1}$$

Treating Eq. (C-1) as Case 1 in Appendix A, yields

$$\left\{\boldsymbol{S}_{f,\mathrm{RR}}\right\} = \left\{2\tan\frac{\theta_{\mathrm{RR}}}{2}\begin{pmatrix}\boldsymbol{s}_{\mathrm{RR}}\\ \boldsymbol{r}_{\mathrm{RR}}\times\boldsymbol{s}_{\mathrm{RR}}\end{pmatrix}+t_{\mathrm{RR}}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_{\mathrm{RR}}\end{pmatrix}\right\} \tag{C-2}$$

where

$$\theta_{\mathrm{RR}}=\theta_1+\theta_2,\quad t_{\mathrm{RR}}=0,\quad \boldsymbol{s}_{\mathrm{RR}}=\boldsymbol{s}$$

$$\boldsymbol{r}_{\mathrm{RR}}=\frac{\tan\frac{\theta_1}{2}}{\tan\frac{\theta_1}{2}+\tan\frac{\theta_2}{2}}\boldsymbol{r}_1+\frac{\tan\frac{\theta_2}{2}}{\tan\frac{\theta_1}{2}+\tan\frac{\theta_2}{2}}\boldsymbol{r}_2+\frac{\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}}{\tan\frac{\theta_1}{2}+\tan\frac{\theta_2}{2}}\left(\left(\boldsymbol{r}_2-\boldsymbol{r}_1\right)\times\boldsymbol{s}\right)$$

This allows to treat Eq. (C-2) as Case 2 in Appendix A such that

$$\left\{\boldsymbol{S}_{f,\mathrm{RR}}\right\} = \left\{2\tan\frac{\theta_1+\theta_2}{2}\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{r}_2\times\boldsymbol{s}\end{pmatrix}\square\begin{pmatrix}\boldsymbol{0}\\ \left(e^{\theta_1[\boldsymbol{s}\times]}-\boldsymbol{I}\right)\left(\boldsymbol{r}_2-\boldsymbol{r}_1\right)\end{pmatrix}\right\} \tag{C-3}$$

Examination of the algebraic structure of Eq. (C-3) shows that if $\theta_1+\theta_2$ and θ_1 are taken as two independent parameters, $\left\{\boldsymbol{S}_{f,\mathrm{RR}}\right\}$ can be visualized as the composition of a rotation and a translation, and it thereby can also be rewritten as

$$\left\{\boldsymbol{S}_{f,\mathrm{RR}}\right\} = \left\{2\tan\frac{\theta_{\mathrm{RR}}}{2}\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{r}_2\times\boldsymbol{s}\end{pmatrix}\square\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{t}_1\end{pmatrix}\Bigg|_{\boldsymbol{t}_1=\boldsymbol{f}_1(\theta_1)}\right\} \tag{C-4}$$

(2) Finite motion analysis of a triad composed of three R joints whose axes are parallel to one another.

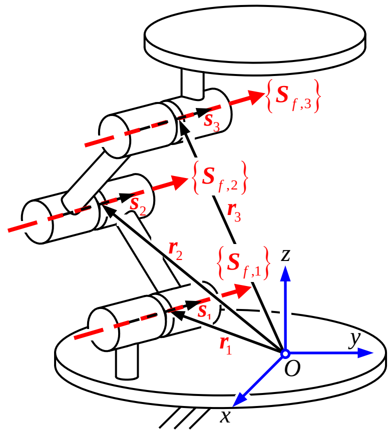


Fig. C-2. RRR limb

As a slightly more complicated case of the dyad, the finite screw of the triad shown in Fig. C-2 can be expressed by

$$\left\{ \boldsymbol{S}_{f,\mathrm{RRR}} \right\} = \left\{ 2\tan\frac{\theta_3}{2}\begin{pmatrix} \boldsymbol{s}_3 \\ \boldsymbol{r}_3 \times \boldsymbol{s}_3 \end{pmatrix} \Delta\, 2\tan\frac{\theta_2}{2}\begin{pmatrix} \boldsymbol{s}_2 \\ \boldsymbol{r}_2 \times \boldsymbol{s}_2 \end{pmatrix} \Delta\, 2\tan\frac{\theta_1}{2}\begin{pmatrix} \boldsymbol{s}_1 \\ \boldsymbol{r}_1 \times \boldsymbol{s}_1 \end{pmatrix} \right\}, \quad \boldsymbol{s}_3 = \boldsymbol{s}_2 = \boldsymbol{s}_1 = \boldsymbol{s} \tag{C-5}$$

Treating Eq. (C-5) as Case 1 in Appendix A, gives

$$\left\{ \boldsymbol{S}_{f,\mathrm{RRR}} \right\} = \left\{ 2\tan\frac{\theta_{\mathrm{RRR}}}{2}\begin{pmatrix} \boldsymbol{s}_{\mathrm{RRR}} \\ \boldsymbol{r}_{\mathrm{RRR}} \times \boldsymbol{s}_{\mathrm{RRR}} \end{pmatrix} + t_{\mathrm{RRR}}\begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_{\mathrm{RRR}} \end{pmatrix} \right\} \tag{C-6}$$

where

$$\theta_{\mathrm{RRR}} = \theta_1 + \theta_2 + \theta_3, \quad t_{\mathrm{RRR}} = 0, \quad \boldsymbol{s}_{\mathrm{RRR}} = \boldsymbol{s}$$

$$\boldsymbol{r}_{\mathrm{RRR}} = \frac{\tan\frac{\theta_1}{2}\left(1 - \tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\right)\boldsymbol{r}_1 + \tan\frac{\theta_2}{2}\left(1 + \tan\frac{\theta_1}{2}\tan\frac{\theta_3}{2}\right)\boldsymbol{r}_2 + \tan\frac{\theta_3}{2}\left(1 - \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\right)\boldsymbol{r}_3}{\tan\frac{\theta_1}{2} + \tan\frac{\theta_2}{2} + \tan\frac{\theta_3}{2} - \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}}$$
$$- \frac{\tan\frac{\theta_1}{2}\left(\tan\frac{\theta_2}{2} + \tan\frac{\theta_3}{2}\right)(\boldsymbol{r}_1 \times \boldsymbol{s}) - \tan\frac{\theta_2}{2}\left(\tan\frac{\theta_1}{2} - \tan\frac{\theta_3}{2}\right)(\boldsymbol{r}_2 \times \boldsymbol{s}) - \tan\frac{\theta_3}{2}\left(\tan\frac{\theta_1}{2} + \tan\frac{\theta_2}{2}\right)(\boldsymbol{r}_3 \times \boldsymbol{s})}{\tan\frac{\theta_1}{2} + \tan\frac{\theta_2}{2} + \tan\frac{\theta_3}{2} - \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}}$$

This allows to treat Eq. (C-6) as Cases 2 and 4 in Appendix A such that

$$\left\{ \boldsymbol{S}_{f,\mathrm{RRR}} \right\} = \left\{ 2\tan\frac{\theta_1 + \theta_2 + \theta_3}{2}\begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{r}_3 \times \boldsymbol{s} \end{pmatrix} \Delta \begin{pmatrix} \boldsymbol{0} \\ \left(e^{(\theta_1+\theta_2)[\boldsymbol{s}\times]} - \boldsymbol{I}\right)(\boldsymbol{r}_3 - \boldsymbol{r}_2) \end{pmatrix} \Delta \begin{pmatrix} \boldsymbol{0} \\ \left(e^{\theta_1[\boldsymbol{s}\times]} - \boldsymbol{I}\right)(\boldsymbol{r}_2 - \boldsymbol{r}_1) \end{pmatrix} \right\} \tag{C-7}$$

So, if we take $\theta_1 + \theta_2 + \theta_3$, $\theta_1 + \theta_2$ and θ_1 as three independent parameters, $\left\{ \boldsymbol{S}_{f,\mathrm{RRR}} \right\}$ can be visualized as the composition of a rotation and two translations, and it thereby can be rewritten as

$$\left\{ \boldsymbol{S}_{f,\mathrm{RRR}} \right\} = \left\{ 2\tan\frac{\theta_{\mathrm{RRR}}}{2}\begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{r}_3 \times \boldsymbol{s} \end{pmatrix} \Delta \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{t}_1 \end{pmatrix} \Delta \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{t}_2 \end{pmatrix} \middle| \boldsymbol{t}_1 = f_1(\theta_1 + \theta_2), \boldsymbol{t}_2 = f_2(\theta_1) \right\} \tag{C-8}$$