GENERALIZED GREEN'S EQUIVALENCES ON THE SUBSEMIGROUPS OF THE BICYCLIC MONOID

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ABSTRACT

We study generalized Green's equivalences on all subsemigroups of the bicyclic monoid **B** and determine the abundant (and adequate) subsemigroups of **B**.

1 INTRODUCTION

The bicyclic monoid **B**, is one of the most fundamental semigroups, with many remarkable properties and generalizations; see [1, 2, 6, 7, 8, 9, 10, 11, 12].

A description of the subsemigroups of the bicyclic monoid was obtained in [3], and by using this description several properties about all subsemigroups of **B** have been proved in [4] . In this paper we use this description to study the generalized Green's relations $\mathcal{L}^*$ and $\mathcal{R}^*$ of the subsemigroups of **B**. This study is motivated by a J. Fountain's question, who asked if the description can be used to say which are the abundant and adequate subsemigroups of the bicyclic monoid.

Let S be a semigroup and $a, b \in S$. We say that $a\,\mathcal{L}^*\,b$, if there is an oversemigroup of S (a semigroup having S as a subsemigroup) where $a\,\mathcal{L}\,b$. It is known and it is easy to check that (see [5]) $a\,\mathcal{L}^*\,b$ if and only if,

$$\text{for all } x, y \in S^1 \text{ we have } ax = ay \Leftrightarrow bx = by. \tag{1}$$

The relation $\mathcal{R}^*$ is defined analogously as is the corresponding property. We say that a semigroup is *abundant* if every $\mathcal{L}^*$-class has an idempotent and every $\mathcal{R}^*$-class has an idempotent. An abundant semigroup is *adequate* if the set of its idempotents forms a semilattice.

The *bicyclic monoid* **B** is defined by the monoid presentation $\langle b, c \mid bc = 1\rangle$; a natural set of unique normal forms for **B** is $\{c^i b^j : i, j \geq 0\}$ and we shall identify **B** with this set. The normal forms multiply according to the following rule:

$$c^i b^j c^k b^l = \begin{cases} c^{i-j+k} b^l \text{ if } j \leq k \\ c^i b^{j-k+l} \text{ if } j > k. \end{cases}$$

We are going to study the $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes of all subsemigroups of the bicyclic monoid in order to determine the abundant subsemigroups. We note that every set of idempotents from the bicyclic monoid is a semilattice (indeed a chain) and so a subsemigroup of the bicyclic monoid is adequate if and only if it is abundant.

We start by noting that two idempotents in the bicyclic monoid are always in separated $\mathcal{L}^*$-classes ($\mathcal{R}^*$-classes). In fact, given two idempotents say, $c^i b^i, c^j b^j$ with $i < j$ we can use (1) choosing $x = c^i b^i$ and $y = c^j b^j$. We have $c^j b^j x = c^j b^j y = c^j b^j$ but $c^i b^i x = c^i b^i$ which is not equal to $c^i b^i y = c^j b^j$.

We will consider the different types of semigroups of **B** separately. Diagonal subsemigroups, one of the types, are formed by idempotents and so trivially are abundant. We begin by presenting some previous results giving the description of the subsemigroups of **B** in Section 2, then in Sections 3 and 4 we make some remarks that will be useful to study their $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes. Finally, in Sections 5 and 6 we consider the two relevant types of subsemigroups, the Upper and Two-sided subsemigroups, respectively.

2 PREVIOUS RESULTS

In this section we introduce the necessary notation and present the main result from [3] with the description of the subsemigroups of **B**.

In order to define subsets of the bicyclic monoid it is convenient to see **B** as an infinite square grid, as shown in Figure 1. We start by introducing some

	0	1	2	3	
0	1	b	b^2	b^3	$\cdots$
1	c	cb	cb^2	cb^3	$\cdots$
2	c^2	c^2b	c^2b^2	c^2b^3	$\cdots$
3	c^3	c^3b	c^3b^2	c^3b^3	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 1: The bicyclic monoid

basic subsets of $\mathbf{B}$:

$$\begin{aligned} D &= \{c^i b^i : i \geq 0\} - \text{ the \textit{diagonal}}, \\ L_p &= \{c^i b^j : 0 \leq j < p, i \geq 0\} - \text{ the \textit{left strip (determined by } p)}, \end{aligned}$$

for $p \geq 0$. For $0 \leq q \leq p \leq m$ we define the *triangle*

$$T_{q,p} = \{c^i b^j : q \leq i \leq j < p\}.$$

Note that for $q = p$ this set is empty. For $i, m \geq 0$ and $d > 0$ we define the *rows*

$$\Lambda_i = \{c^i b^j : j \geq 0\},\ \Lambda_{i,m,d} = \{c^i b^j : d \mid j - i,\ j \geq m\}$$

and in general for $I \subseteq \{0, \ldots, m-1\}$,

$$\Lambda_{I,m,d} = \bigcup_{i \in I} \Lambda_{i,m,d} = \{c^i b^j : i \in I, d \mid j - i, j \geq m\}.$$

For $p \geq 0$, $d > 0$, $r \in [d] = \{0, \ldots, d-1\}$ and $P \subseteq [d]$ we define the *squares*

$$\begin{aligned} \Sigma_p &= \{c^i b^j : i, j \geq p\},\ \Sigma_{p,d,r} = \{c^{p+r+ud} b^{p+r+vd} : u, v \geq 0\}, \\ \Sigma_{p,d,P} &= \bigcup_{r \in P} \Sigma_{p,d,r} = \{c^{p+r+ud} b^{p+r+vd} : r \in P; u, v \geq 0\}. \end{aligned}$$

Pictures illustrating some of these sets can be found in [3].

The function $\rho : \mathbf{B} \to \mathbf{B}$ defined by $c^i b^j \mapsto (c^i b^j)\rho = c^j b^i$ is an anti-isomorphism. Geometrically ρ is the reflection with respect to the main diagonal.

We can now present the main result from [3]:

Proposition 2.1 *Let S be a subsemigroup of the bicyclic monoid. Then one of the following conditions holds:*

1. *S is a subset of the diagonal;* $S \subseteq D$.

2. *S is a union of a subset of a triangle, a subset of the diagonal above the triangle, a square below the triangle and some rows belonging to a strip determined by the square and the triangle, or the reflection of such a union with respect to the diagonal. Formally there exist* $q, p \in \mathbb{N}_0$ *with* $q \leq p$, $d \in \mathbb{N}$, $I \subseteq \{q, \ldots, p-1\}$ *with* $q \in I$, $P \subseteq \{0, \ldots, d-1\}$ *with* $0 \in P$, $F_D \subseteq D \cap L_q$, $F \subseteq T_{q,p}$ *such that S is of one of the following forms:*

   (i) $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$; *or*

   (ii) $S = F_D \cup (F)\rho \cup (\Lambda_{I,p,d})\rho \cup \Sigma_{p,d,P}$.

3. *There exist* $d \in \mathbb{N}$, $I \subseteq \mathbb{N}_0$, $F_D \subseteq D \cap L_{\min(I)}$ *and sets* $S_i \subseteq \Lambda_{i,i,d}$ $(i \in I)$ *such that S is of one of the following forms:*

   (i) $S = F_D \cup \bigcup_{i \in I} S_i$; *or*

   (ii) $S = F_D \cup \bigcup_{i \in I} (S_i)\rho$;

   where each S_i *has the form*

   $$S_i = F_i \cup \Lambda_{i,m_i,d}$$

   for some $m_i \in \mathbb{N}_0$ *and some finite set* F_i, *and*

   $$I = I_0 \cup \{r + ud : r \in R, u \in \mathbb{N}_0, r + ud \geq N\}$$

   for some (possibly empty) $R \subseteq \{0, \ldots, d-1\}$, *some* $N \in \mathbb{N}_0$ *and some finite set* $I_0 \subseteq \{0, \ldots, N-1\}$.

We call *diagonal subsemigroups* those defined by *1.*, *two-sided subsemigroups* those defined by *2.*, *upper subsemigroups* those defined by *3.(i)* and *lower subsemigroups* those defined by *3.(ii)*. Pictures illustrating the several types of semigroups can be found in [3].

3 $\mathcal{L}^*$-CLASSES

In general, to study the $\mathcal{L}^*$-classes of a subsemigroup S of $\mathbf{B}$ we have to consider the equation $ax = ay$ appearing in (1), in our introductory section, and the following fact will be useful:

Lemma 3.1 *We have $ax = ay$ with $a = c^i b^j, x = c^r b^s, y = c^u b^v \in S$, $x \neq y$, i.e,*

$$c^i b^j \; c^r b^s = c^i b^j \; c^u b^v$$

if and only if

$$j \geq r, j \geq u \text{ and } s - r = v - u.$$

PROOF. If $j \geq r, j \geq u$ and $s - r = v - u$ then $c^i b^j c^r b^s = c^i b^{j+s-r} = c^i b^{j+v-u} = c^i b^j c^u b^v$. For the converse let's consider the four cases in the equation $c^i b^j \; c^r b^s = c^i b^j \; c^u b^v$. (i) $j \geq r, j \geq u$. In this case the equation becomes $c^i b^{j-r+s} = c^i b^{j-u+v}$ and so $s - r = v - u$ as stated. (ii) $j \geq r, j < u$. In this case we obtain $c^i b^{j-r+s} = c^{i-j+u} b^v$ and so we have $i = i - j + u$ (and $j - r + s = v$) which implies $j = u$, a contradiction. Analogously we cannot have (iii) $j < r, j \geq u$. (iv) Finally we show that is also not possible to have $j < r, j < u$. In this case the equation becomes $c^{i-j+r} b^s = c^{i-j+u} b^v$ which implies $r = u$, $s = v$ and so $x = y$, which contradicts the hypothesis. ∎

Lemma 3.2 *Let $c^i b^j, c^k b^l \in \mathbf{B}$, with $j \leq l$. If $c^i b^j x = c^i b^j y$ for some $x, y \in \mathbf{B}$ then $c^k b^l x = c^k b^l y$.*

PROOF. The statement holds trivially if $x = y$, so assume that $x \neq y$. Let $x = c^r b^s$ and $y = c^u b^v$. Since $c^i b^j \; c^r b^s = c^i b^j \; c^u b^v$ with $c^r b^s \neq c^u b^v$, using Lemma 3.1, we have $j \geq r, j \geq u$ and $s - r = v - u$. So, since $l \geq j \geq r$ and $l \geq j \geq u$, we have $c^k b^l \; c^r b^s = c^k b^{l-r+s} = c^k b^{l+v-u} = c^k b^l \; c^u b^v$. ∎

As an immediate consequence of this fact, we just have to check one of the equivalences in (1):

Corollary 3.3 *Two elements $c^i b^j, c^k b^l$ $(j \leq l)$ in a subsemigroup S of $\mathbf{B}$ are $\mathcal{L}^*$-related if and only if*

$$c^k b^l x = c^k b^l y \implies c^i b^j x = c^i b^j y, \forall x, y \in S^1.$$

Using this we can state a necessary and sufficient condition for two elements A and B in a subsemigroup of $\mathbf{B}$ to be $\mathcal{L}^*$-related, illustrated in Figure 2 (x is in the horizontal shaded strip determined by the columns of A and B and y in the shaded diagonal):

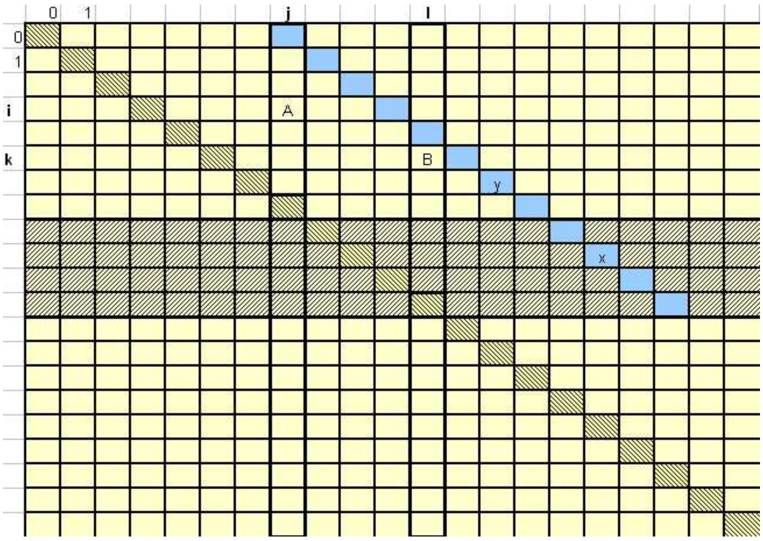

Figure 2: $\mathcal{L}^*$-relation in the subsemigroups of the bicyclic monoid

Lemma 3.4 *Two elements c^ib^j, c^kb^l $(j \leq l)$ in a subsemigroup S of $\mathbf{B}$ are not $\mathcal{L}^*$-related if and only if there exist two different elements $x = c^rb^s, y = c^ub^v \in S$ such that $j < r \leq l$, $u \leq l$ and $s - r = v - u$.*

Proof. Using Corollary 3.3, c^ib^j and c^kb^l are not $\mathcal{L}^*$-related if and only if there exist two elements $x, y \in S$ such that $c^kb^lx = c^kb^ly$ and $c^ib^jx \neq c^ib^jy$. Let $x = c^rb^s$ and $y = c^ub^v$. Using Lemma 3.1, $c^kb^lx = c^kb^ly$ is equivalent to $l \geq r, l \geq u$ and $s - r = v - u$, and $c^ib^jx \neq c^ib^jy$ is equivalent to $j < r \vee j < u \vee s - r \neq v - u$. Since $s - r = v - u$ it must be that $j < r \vee j < u$. We can assume, without loss of generality, that $j < r$, whence we have $j < r \leq l, u \leq l, s - r = v - u$. ∎

As a trivial consequence we have the following useful sufficient condition for two elements to be $\mathcal{L}^*$-related:

Lemma 3.5 *Let S be a subsemigroup of* $\mathbf{B}$ *and let* $c^ib^j, c^kb^l \in S$ $(j \le l)$*. If* S *has no elements in rows* $j+1, \ldots, l$ *then* $c^ib^j \, \mathcal{L}^* \, c^kb^l$*.*

And we have the following corollary:

Corollary 3.6 *Two elements of a subsemigroup S of* $\mathbf{B}$ *in the same column are* $\mathcal{L}^*$*-related.*

This we knew already because two elements in the same column are $\mathcal{L}$-related in the bicyclic monoid.

Another consequence of Lemma 3.4 is the following:

Corollary 3.7 *An* $\mathcal{L}^*$*-class of* S *consists of a union of adjacent columns, i.e., there cannot exist two* $\mathcal{L}^*$*-related elements* A *and* B *and another element* C *not* $\mathcal{L}^*$*-related to* A *and* B *in a column between them.*

PROOF. Let $A = c^ib^j$, $B = c^kb^l$ and $C = c^mb^n$ $(j \le n \le l)$. If A and B are $\mathcal{L}^*$-related then, by Lemma 3.4, elements $x = c^rb^s, y = c^ub^v$ with $j < r \le l, u \le l$ and $s - r = v - u$ cannot exist. Hence such elements cannot exist with $j < r \le n \le l$ and $u \le n$, so $C \; \mathcal{L}^* \; A \; \mathcal{L}^* B$ and $C \; \mathcal{L}^* \; B$ by transitivity. ■

4 $\mathcal{R}^*$-CLASSES

To obtain the corresponding facts for $\mathcal{R}^*$-classes we will use the standard anti-isomorphism of an inverse semigroup T to itself, $\rho : T \to T;\; x \mapsto x^{-1}$. We note that $(xy)\rho = (xy)^{-1} = y^{-1}x^{-1}$. If S is a subsemigroup of T, we denote by S^{-1} the subsemigroup $S\rho$. If T is the bicyclic monoid $\mathbf{B}$ then $\rho : \mathrm{B} \to \mathrm{B};\; c^ib^j \mapsto c^jb^i$ and

$$(c^ib^jc^kb^l)^{-1} = c^lb^kc^jb^i. \tag{2}$$

The following fact will be useful:

Lemma 4.1 *If S is a subsemigroup of an inverse semigroup T and* $a, b \in S$ *then* $(a,b) \in \mathcal{L}^*_S$ *if and only if* $(a^{-1}, b^{-1}) \in \mathcal{R}^*_{S^{-1}}$*.*

PROOF. Let $a, b \in S$. We have $(a, b) \in \mathcal{L}^*_S$ if and only if $ax = ay \Leftrightarrow bx = by$ $(\forall x, y \in S^1)$. This happens if and only if $(ax)^{-1} = (ay)^{-1} \Leftrightarrow (bx)^{-1} = (by)^{-1}$ $(\forall x, y \in S^1)$. This is equivalent to $x^{-1}a^{-1} = y^{-1}a^{-1} \Leftrightarrow x^{-1}b^{-1} = y^{-1}b^{-1}$ $(\forall x, y \in S^1)$ what is the same as $ua^{-1} = va^{-1} \Leftrightarrow ub^{-1} = vb^{-1}$ $(\forall u, v \in (S^{-1})^1)$ and so $(a^{-1}, b^{-1}) \in \mathcal{R}^*_{S^{-1}}$. ■

In the case where $T = S$ we have $S = S^{-1}$ and so we can say:

Lemma 4.2 *If S is an inverse semigroup and $a, b \in S$ then $(a, b) \in \mathcal{L}^*$ if and only if $(a^{-1}, b^{-1}) \in \mathcal{R}^*$.*

Lemma 4.3 *If $xa = ya$ with $a = c^i b^j, x = c^r b^s, y = c^u b^v \in S$, $x \neq y$, i.e,*

$$c^r b^s \ c^i b^j = c^u b^v \ c^i b^j$$

then

$$i \geq s, i \geq v \text{ and } r - s = u - v.$$

PROOF.

We have $xa = ya$ if and only if $(xa)^{-1} = (ya)^{-1}$. By (2), we have $(xa)^{-1} = c^j b^i c^s b^r$ and $(ya)^{-1} = c^j b^i c^v b^u$ and so, by Lemma 3.1, we have $i \geq s, i \geq v$ and $r - s = u - v$.

■

Lemma 4.4 *Let $c^i b^j, c^k b^l \in \mathbf{B}$, $i \leq k$. If $x \ c^i b^j = y \ c^i b^j$ then $x \ c^k b^l = y \ c^k b^l$, for any $x, y \in \mathbf{B}$, $x \neq y$.*

PROOF. If $xc^i b^j = yc^i b^j$ then $(xc^i b^j)^{-1} = (yc^i b^j)^{-1}$. So $c^j b^i x^{-1} = c^j b^i y^{-1}$ and, by Lemma 3.2, $c^l b^k x^{-1} = c^l b^k y^{-1}$. Hence, $(xc^k b^l)^{-1} = (yc^k b^l)^{-1}$ and so $xc^k b^l = yc^k b^l$. ■

As an immediate consequence of this fact we have

Corollary 4.5 *Two elements $c^i b^j, c^k b^l$ $(i \leq k)$ in a subsemigroup S of $\mathbf{B}$ are $\mathcal{R}^*$-related if and only if*

$$x \ c^k b^l = y \ c^k b^l \implies x \ c^i b^j = y \ c^i b^j, \forall x, y \in S^1.$$

The following lemma gives a necessary and sufficient condition for two elements to be $\mathcal{R}^*$-related and it is illustrated by Figure 3.

Lemma 4.6 *Two elements c^ib^j, c^kb^l $(i \leq k)$ in a subsemigroup S of $\mathbf{B}$ are not $\mathcal{R}^*$-related if and only if there exist two different elements $x = c^rb^s, y = c^ub^v \in S$ such that $i < s \leq k$, $v \leq k$ and $r - s = u - v$.*

Proof. The elements c^ib^j, c^kb^l are not $\mathcal{R}^*_S$-related if and only if the elements c^jb^i, c^lb^k are not $\mathcal{L}^*_{S^{-1}}$-related. By Lemma 3.4 this happens if and only if there exists $x^{-1} = c^sb^r \neq c^vb^u = y^{-1}$ in S^{-1} such that $i < s \leq k$, $v \leq k$ and $r - s = u - v$. And so, if and only if, there exists $x = c^rb^s \neq c^ub^v = y$ in S such that $i < s \leq k$, $v \leq k$ and $r - s = u - v$. ■

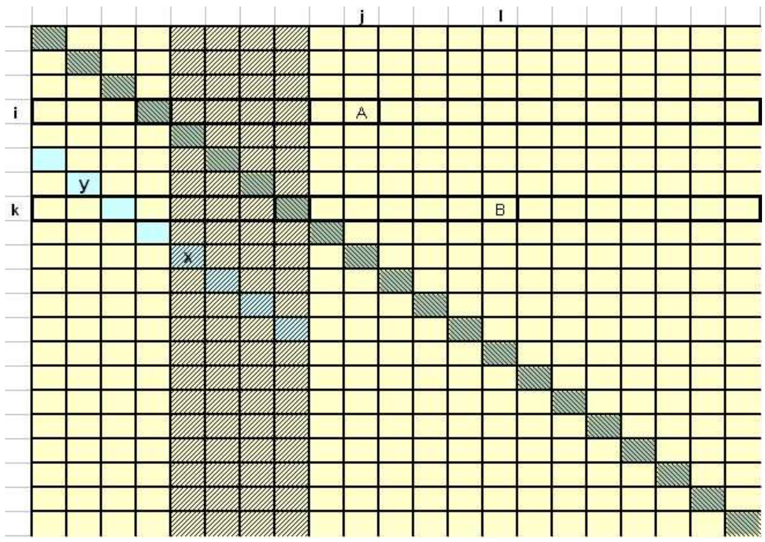

Figure 3: $\mathcal{R}^*$-relation in subsemigroups of the bicyclic monoid

Lemma 4.7 *Let S be a subsemigroup of $\mathbf{B}$ and let $c^ib^j, c^kb^l \in S$ $(i \leq k)$. If S has no elements in columns $i+1, \ldots, k$ then $c^ib^j \, \mathcal{R}^* \, c^kb^l$.*

Corollary 4.8 *Two elements of a subsemigroup S of $\mathbf{B}$ in the same row are $\mathcal{R}^*$-related.*

Corollary 4.9 *An $\mathcal{R}^*$-class of S consists of adjacent rows, i.e., there cannot exist two $\mathcal{R}^*$-related elements A and B in S and another element C in S not related with A and B in a row between them.*

PROOF. If $(c^ib^j, c^kb^l) \in \mathcal{R}_S^*$ then, by Lemma 4.1, $(c^jb^i, c^lb^k) \in \mathcal{L}_{S^{-1}}^*$. By Corollary 3.7, c^jb^i and c^lb^k are in union of adjacent rows in S^{-1}, which means that c^ib^j and c^kb^l are in a union of adjacent columns in S. ■

5 UPPER SUBSEMIGROUPS

Upper semigroups may be abundant or not. A simple example is the free monogenic semigroup, generated by b, which is a non abundant upper semigroup, since it has no idempotents. We note that, since this semigroup is cancellative, it has a unique $\mathcal{L}^*$-class and a unique $\mathcal{R}^*$-class. If we adjoin the identity to it, we obtain the free monogenic monoid, which is an abundant upper subsemigroup of the bicyclic monoid, having one $\mathcal{L}^*$-class and one $\mathcal{R}^*$-class; both contain an idempotent, the identity of the monoid.

We start by considering finitely generated upper subsemigroups. They have the form $S = F_D \cup F \cup \Lambda_{I,m,d}$ where $I \subseteq \mathbb{N}_0$, $q = \min(I) \leq p = \max(I) \leq m$, $d \in \mathbb{N}$, $F_D \subseteq \{c^ib^i : i < q\}$, $F \subseteq \{c^ib^j : q \leq i \leq p, i \leq j < m\}$ are finite sets, and $\Lambda_{I,m,d} = \{c^ib^j : i \in I, d \mid j - i, j \geq m\}$ (see [4]). This semigroup is illustrated by Figure 4. In this section we assume that S is a semigroup of this kind.

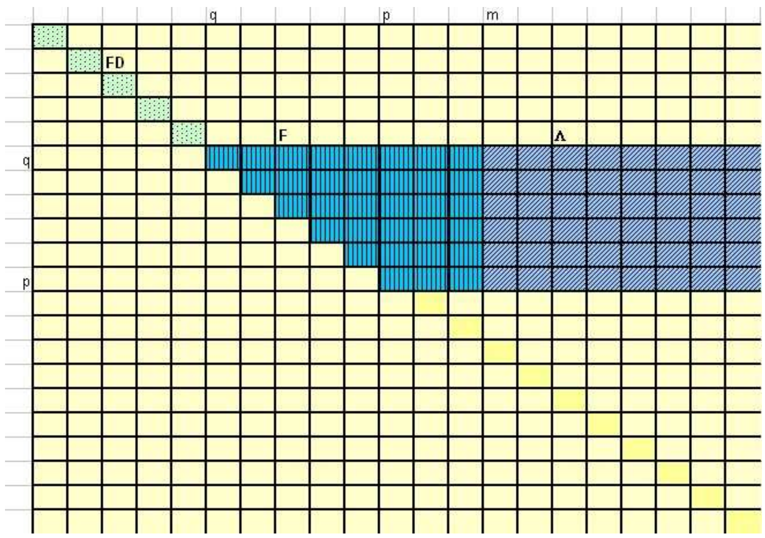

Figure 4: The region containing a semigroup $S = F_D \cup F \cup \Lambda_{I,m,d}$

We will first consider the case where $F_D = F = \emptyset$. In this case, $S = \Lambda_{I,m,d}$ is a finite union of special subsemigroups of $\mathbb{N}_0$ (numerical semigroups of the form $\{kd : k \in \mathbb{N}_0,\ kd \geq N\}$ with $d, N \in \mathbb{N}_0, d > 0$). We will show that this subsemigroup has only one $\mathcal{L}^*$-class and only one $\mathcal{R}^*$-class. In fact, given two elements $c^ib^j, c^kb^l \in S$, with $j \leq l$, there are no elements of S in rows $j+1, \ldots, l$ because $j+1 > m$ and all elements of S are in rows $q, \ldots, p$ with $p \leq m$. So using Lemma 3.5 we see that c^ib^j, c^kb^l are $\mathcal{L}^*$-related. To see that there is also only one $\mathcal{R}^*$-class, we can take two arbitrary elements $c^ib^j, c^kb^l \in S$ with $i \leq k$ $(\leq p \leq m)$. Since S has no elements in columns $i+1, \ldots, k-1$, we cannot find two different elements x, y in the conditions of Lemma 4.6, not even in the case where $p = m$. Hence c^ib^j, c^kb^l are $\mathcal{R}^*$-related. Having only a $\mathcal{L}^*$-class and only a $\mathcal{R}^*$-class, for the subsemigroup to be abundant it just needs to contain one idempotent. This is only possible if $m = p$ and the idempotent c^pb^p belongs to S. We summarize this in the following

Proposition 5.1 *An upper subsemigroup S of* $\mathbf{B}$ *of the form $S = \Lambda_{I,m,d}$ has a unique $\mathcal{R}^*$-class and a unique $\mathcal{L}^*$-class. It is abundant if and only if $m = p$ and $c^pb^p \in S$.*

We consider now subsemigroups of the form $S = F_D \cup \Lambda_{I,m,d}$ where $F_D = \{e_1, \ldots, e_n\}$ $(n \geq 1)$. If I has only one element, say $I = \{p\}$, then S is in fact obtained starting from the numerical semigroup $\Lambda_{I,m,d}$ and adding successively the identities $e_n, e_{n-1}, \ldots e_1$. The elements of $\Lambda_{p,m,d}$ together with the idempotent of S that is lower in the diagonal (which may be c^pb^p) form a cancellative monoid, hence having a unique $\mathcal{L}^*$-class and $\mathcal{R}^*$-class. Each other idempotent in F_D is by itself an $\mathcal{L}^*$-class and an $\mathcal{R}^*$-class. So, if $c^pb^p \in S$ the classes are $\{e_1\}, \{e_2\}, \ldots, \{e_n\}, \Lambda_{p,m,d}$. Otherwise the classes are $\{e_1\}, \{e_2\}, \ldots, \{e_n\} \cup \Lambda_{p,m,d}$. In any case the subsemigroup is abundant.

Proposition 5.2 *An upper semigroup of the form $S = F_D \cup \Lambda_{p,m,d}$ with $F_D = \{e_1, \ldots, e_n\}$ $(n \geq 1)$ is abundant. If $\Lambda_{p,m,d}$ has an idempotent the $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes are $\{e_1\}, \{e_2\}, \ldots, \{e_n\}, \Lambda_{p,m,d}$. Otherwise they are $\{e_1\}, \{e_2\}, \ldots, \{e_n\} \cup \Lambda_{p,m,d}$.*

We continue with $S = F_D \cup \Lambda_{I,m,d}$ where $F_D = \{e_1, \ldots, e_n\}$ $(n \geq 1)$ but assuming now that I has more than one element. Let's first consider the case where $c^pb^p \in \Lambda_{I,m,d}$ $(m = p)$. In this case, two elements $c^ib^j, c^pb^k \in \Lambda_{I,m,d}$ with $i < p$ are not $\mathcal{R}^*$-related since, letting x be the lower idempotent in F_D and $y = c^pb^p$, the elements x and y are in the conditions of Lemma 4.6. But any two elements $c^ib^j, c^kb^l \in \Lambda_{I,m,d}$ with $i, j < p$ are $\mathcal{R}^*$-related since S has no elements in columns $q, \ldots, p-1$ and we can use Lemma 4.7. The lower idempotent in F_D, say $e = c^ib^i$, is $\mathcal{R}^*$-related to any element c^kb^l in rows $I \backslash \{p\}$ because $k < p$ and S has no elements in columns $i+1, \ldots, p-1$. The other idempotents in F_D are $\mathcal{R}^*$-classes by themselves. So every $\mathcal{R}^*$-class has an idempotent. The $\mathcal{R}^*$-classes of these subsemigroups are illustrated by the example in Figure 5.

Two elements $c^ib^j, c^kb^l \in \Lambda_{I,m,d}$ are $\mathcal{L}^*$-related because $j, l \geq p$, there are no elements in S below row p and so we can use Lemma 3.5. These $\mathcal{L}^*$-class contains already one idempotent, c^pb^p, and so the other idempotents in F_D are $\mathcal{L}^*$-classes by themselves. Hence, also every $\mathcal{L}^*$-class has an idempotent and the semigroup is abundant.

The case where $c^pb^p \notin S$ can also be illustrated by Figure 5, removing the last row. We show that, in this case, the set $\Lambda_{I,m,d}$ is an $\mathcal{L}^*$-class of S without an idempotent, and so the semigroup is not abundant. The elements in $\Lambda_{I,m,d}$ are still $\mathcal{L}^*$-related. But the lower idempotent $c^ib^i \in F_D$ is not related to them, since, given $c^kb^l \in \Lambda_{I,m,d}$, we can find two elements $x = c^jb^{j+ud}, y = c^kb^{k+vd} \in \Lambda_{I,m,d}$ in the same diagonal and in different rows, in the conditions of Lemma 3.4.

Proposition 5.3 *An upper semigroup of the form* $S = F_D \cup \Lambda_{I,m,d}$ *with* $F_D = \{e_1, \ldots, e_n\}$ $(n \geq 1)$ *and* $|I| > 1$ *is abundant if and only if* $m = p$ *and* $c^pb^p \in \Lambda_{I,m,d}$. *In this case the* $\mathcal{L}^*$*-classes and* $\mathcal{R}^*$*-classes are* $\{e_1\}, \{e_2\}, \ldots, \{e_n\}, \Lambda_{I,m,d}$.

Finally we consider arbitrary finitely generated upper semigroups $S = F_D \cup F \cup \Lambda_{I,m,d}$ where $F \neq \emptyset$. We note that, if I has only one element, then S is again obtained from a numerical semigroup adding finitely many idempotents and so we have:

Proposition 5.4 *A subsemigroup of the form* $F_D \cup F \cup \Lambda_{p,m,d}$, $F_D = \{e_1, \ldots, e_n\}$ $(n \geq 0)$ *is abundant if and only if it contains at least one idempotent. If* $c^pb^p \in S$ *the* $\mathcal{L}^*$*-classes and* $\mathcal{R}^*$*-classes are* $\{e_1\}, \{e_2\}, \ldots, \{e_n\}, \Lambda_{I,m,d}$. *Otherwise the classes are* $\{e_1\}, \{e_2\}, \ldots, \{e_n\} \cup \Lambda_{I,m,d}$.

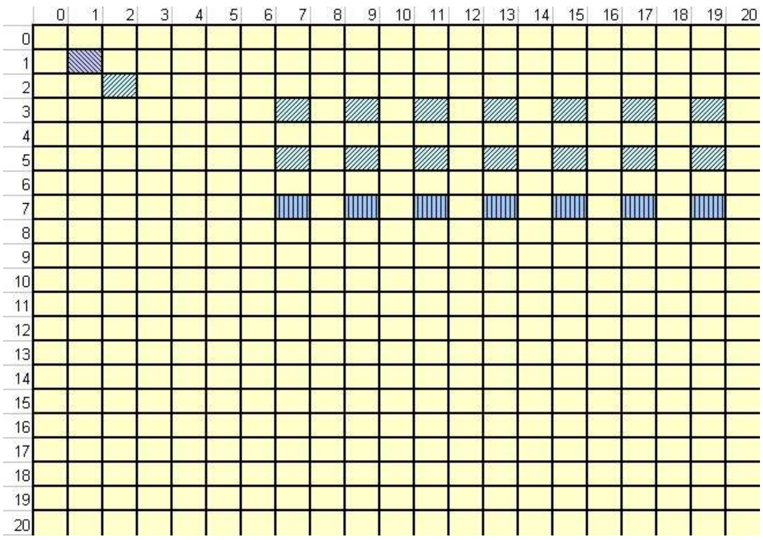

Figure 5: $\mathcal{R}$-classes of $S = F_D \cup \Lambda_{I,m,d}$

So we assume now that I has a at least two elements. To identify the $\mathcal{L}^*$-classes of S, it is convenient to write $S = F_D \cup F' \cup S'$, where $F' = S \cap \{c^i b^j : q \leq i \leq j < p\}$ and $S' = S \cap \{c^i b^j : j \geq p\}$, as illustrated in Figure 6.

The elements in S' are $\mathcal{L}^*$-related because they are on columns $p, p+1, \ldots$ and, since there are no elements in S below row p, we can use Lemma 3.5. An element $c^i b^j \in F'$ is not $\mathcal{L}^*$-related to an element $c^k b^l \in S'$. In fact, since S has elements in row p and $i, j < p$, we can choose two different elements $x = c^p b^{p+ud}$, $y = c^i b^{i+vd}$ in the same diagonal, in the conditions of Lemma 3.4. Hence, if $F' \neq \emptyset$ then S' is an $\mathcal{L}^*$-classe of S. Also in the case where $F' = \emptyset$ the set S' is an $\mathcal{L}^*$-class of S. This is shown if $F_D = \emptyset$. And if $F_D \neq \emptyset$, we can see that the lower idempotent in F_D is not $\mathcal{L}^*$-related say, with an element $c^p b^k \in S'$ because we can choose two different elements $x = c^p b^{p+ud}$ and $y = c^i b^{i+vd}$ in S with $i < p$ in the same diagonal, and use Lemma 3.4. So, in any case, S' is an $\mathcal{L}^*$-class of S and for S to be abundant it must contain the idempotent $c^p b^p$.

We continue the study of $\mathcal{L}^*$-classes considering now the elements in F', which are in finitely many columns. For the semigroup to be abundant, each $\mathcal{L}^*$-classe, that may be formed by the elements in one or more columns, must contain an idempotent. In Figure 7, we find an example of an abundant

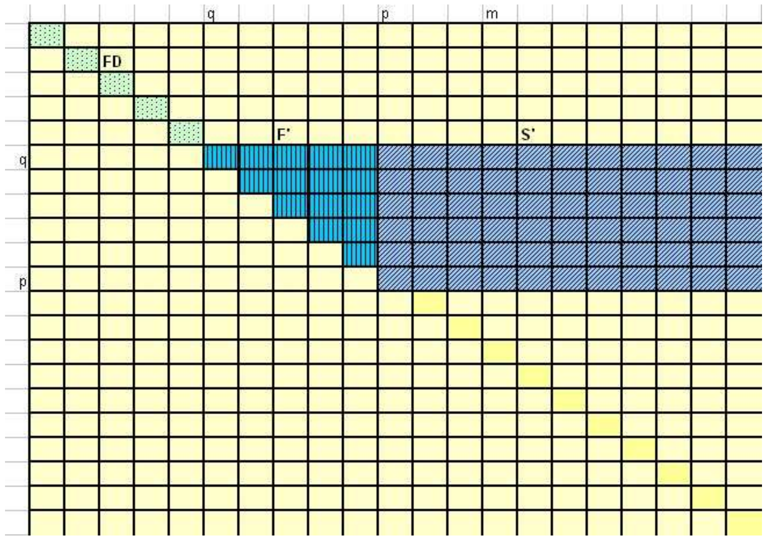

Figure 6: Determining the $\mathcal{L}^*$-classes of $S = F_D \cup F \cup \Lambda_{I,m,d}$

subsemigroup where some columns with elements do not have idempotents.

To check if all $\mathcal{L}^*$-classes of elements in $F_D \cup F'$ have idempotents we just have to form unions with the $p-q$ columns. To do that, we start by observing that if two columns i, j with elements with $q \leq i < j < p$, are in the same $\mathcal{L}^*$-class, then $c^j b^j$ cannot be in the class. In fact, if $c^j b^j$ were in the class then, given two elements $c^k b^i, c^l b^j$ we could obtain $x = c^j b^{j+ud}, y = c^k b^{k+vd}$ in the conditions of Lemma 3.4 and $c^k b^i, c^l b^j$ would not be related. So, the idempotent in an $\mathcal{L}^*$-class with elements from F' is either in the leftmost column or in F_D. Hence, to check if all classes have idempotents we can proceed the following way. We begin by forming a union of rows with elements, L, starting from first column $i \leq p-1$ with elements and going left. If we have already an idempotent we start forming next class. If not we add next column $j < i$ with elements to L, if there are no elements in rows $j+1, \dots, i$. We proceed adding columns until no more columns can be added. After that, if the last column does not have an idempotent in S (and there are still other columns with elements in F' on the left) we have found an $\mathcal{L}^*$-class without an idempotent. Otherwise L is an $\mathcal{L}^*$-class with idempotent and we start forming the next class. After going through all columns of F' two things may happen. If the final union of rows L has an idempotent then all $\mathcal{L}^*$-classes have idempotents.

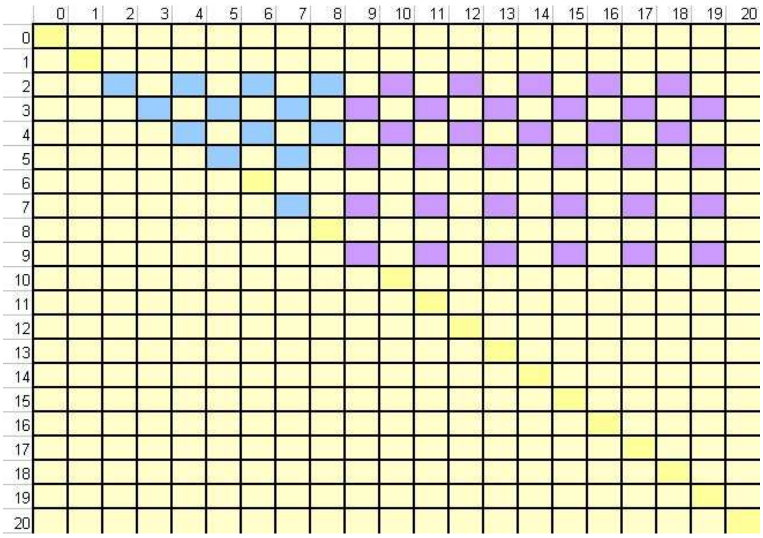

Figure 7: An abundant upper subsemigroup

If not, it may be possible that all $\mathcal{L}^*$-classes have idempotents if $F_D \neq \emptyset$. This can only occur if there is only one row in S with elements in rows $k+1, \ldots, l$, where k is the minimum of column indices in L and l is the maximum. In fact, we see using Lemma 3.4 that only in this case the lower idempotent in F_D is $\mathcal{L}^*$-related to the elements in L.

To check if the $\mathcal{R}^*$-classes have idempotents we can proceed in a similar way. There are finitely many rows with elements and $\mathcal{R}^*$-classes are unions of adjacent rows. We start from row $i = p$ and go up adding rows and forming $\mathcal{R}^*$-classes. We consider the next row j $(= \max(I \backslash \{i\})$ in I. Using Lemma 4.6 we see that, elements in rows i and j are $\mathcal{R}^*$-related if and only if S has no elements in columns $i+1, \ldots, j$ or, for each element A in columns $i+1, \ldots, j$ we cannot find another element $\mathbf{B}$ in columns $0, \ldots, j$ in the same diagonal as A. Proceeding this way we can form the $\mathcal{R}^*$-classes, which are at most $|I| + |F_D|$, and check if they all have idempotents.

These algorithms allows us to check if a general upper subsemigroup of the form $F_D \cup F \cup \Lambda_{I,m,d}$ is abundant. Hence we can say the following

Proposition 5.5 *Let $S = F_D \cup F \cup \Lambda_{I,m,d}$ be an upper subsemigroup of $\mathbf{B}$. Writing $S = F_D \cup F' \cup S'$, where $F' = S \cap \{c^i b^j : q \leq i \leq j < p\}$ and $S' = S \cap \{c^i b^j : j \geq p\}$, the set S' is an $\mathcal{L}^*$-class of S. For S to be abundant* *it is necessary that $c^p b^p \in S$. There exist algorithms to construct the other finitely many $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes of S from the finitely many elements of S in columns $0, \ldots, p$ and the finite set I. The semigroup is abundant if and only if these finitely many $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes have idempotents.*

The algorithms to check if a upper subsemigroup is abundant follows. We have seen it is necessary that $c^p b^p \in S$ for S to be abundant. Then we can check if all $\mathcal{L}^*$-classes have idempotents with the algorithm in Figure 8, where C is the set of indices of columns having elements in $F' \cap S$.

```
isabundant ← true
L ← ∅
while C ≠ ∅ and isabundant
do
    i ← max(C); C ← C\{i}; L ← L ∪ {i}
    if c^i b^i ∈ S then L ← ∅
    else
        if C = ∅ then
            if F_D = ∅ or {min(C) + 1, ..., max(C)} ∩ I > 1
            then isabundant ← false
        else
            j ← max(C)
            if {j + 1, ..., i} ∩ I = ∅ then isabundant ← false fi
        fi
    fi
od
```

Figure 8: Algorithm to check if all $\mathcal{L}^*$-classes have indempotents

To check if all $\mathcal{R}^*$-classes have idempotents we can the use the algorithm in Figure 9.

Finitely generated lower subsemigroups are similar, just replacing rows by columns.

If S is a non finitely generated upper subsemigroup, so with elements in an infinite number of rows, then there is no algorithm to check if S is abun-

```
isabundant ← true
R ← ∅
while I ≠ ∅ and isabundant
do
    i ← max(I); I ← I\{i}; R ← R ∪ {i}
    if I = ∅ then
        if ⋃_{k∈R} {c^k b^k} = ∅ and F_D = ∅ then isabundant ← false fi
    else
        j ← max(I)
        if ¬R_related(j, i) then
            if ⋃_{k∈R} {c^k b^k} = ∅ then isabundant ← false fi
        fi
    fi
od
where
    R_related(j, i) = (∃u : −p ≤ u ≤ p : c^{j+u} b^j, c^{i+u} b^i ∈ S)
```

Figure 9: Algorithm to check if all $\mathcal{R}^*$-classes have indempotents

dant. In fact, we cannot decide if S is abundant, looking to finitely many rows, because we can always add a row without idempotent to an abundant semigroup obtaining a non abundant subsemigroup. For example, the semigroup $S = \{c^i b^j : 0 \leq i < p, j \geq i\}$ $(p > 0)$ is abundant and the semigroup $S \cup \{c^p b^j : j > p\}$ is not.

Of course there is a procedure to check if S is not abundant. It suffices to construct the $\mathcal{L}^*$-classes ($\mathcal{R}^*$-classes), which are unions of columns (rows), until a class without idempotent is found, using Lemma 3.4 (Lemma 4.6).

6 TWO SIDED SUBSEMIGROUPS

In general, a two-sided semigroup has the form $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ (or the corresponding anti-isomorphic image) where $q, p \in \mathbb{N}_0$ with $q \leq p$, $d \in \mathbb{N}$, $I \subseteq \{q, \ldots, p-1\}$ with $q \in I$, $P \subseteq \{0, \ldots, d-1\}$ with $0 \in P$, $F_D \subseteq \{c^i b^i : i = 0, \ldots, q-1\}$, $F \subseteq \{c^i b^j : q \leq i < p, i \leq j < p\}$, $\Sigma_{p,d,P} = \{c^{p+r+ud} b^{p+r+vd} : r \in P; u, v \geq 0\}$ (see [3]). Figure 10 shows an example of one of this subsemigroups.

We note that a two-sided semigroup of the form $F_D \cup \Sigma_{p,d,P}$ is regular (see [3]) and so abundant. Each of its $\mathcal{L}^*$-classes and $\mathcal{R}^*$-classes is contained in single row or column and all have an idempotent. We start by showing the following:

Proposition 6.1 *Subsemigroups of the form $S = \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$ (with $F_D = F = \emptyset$) are abundant.*

Proof. If $I = \emptyset$ we have seen that S is abundant, so we assume $I \neq \emptyset$. We begin by showing that two columns i, j $(i < j)$ such that, the set $\{c^k b^k : i \leq k \leq j\} \cap S$ is either empty or equal to $\{c^i b^i\}$, are $\mathcal{L}^*$-related. In fact, since $i, j \geq p$, the rows $i+1, \ldots, j$ are in Σ_p. A row k in Σ_p has elements from S if and only if $c^k b^k \in S$. Hence S has no elements in rows $i+1, \ldots, j$ and Lemma 3.5 can be applied.

Using this, and observing thet $0 \in P$ and so $c^p b^p \in S$, we see that every $\mathcal{L}^*$-class is a union of columns starting from a column with an idempotent together with all columns on its right hand side not having idempotents. Hence, every $\mathcal{L}^*$-class has an idempotent.

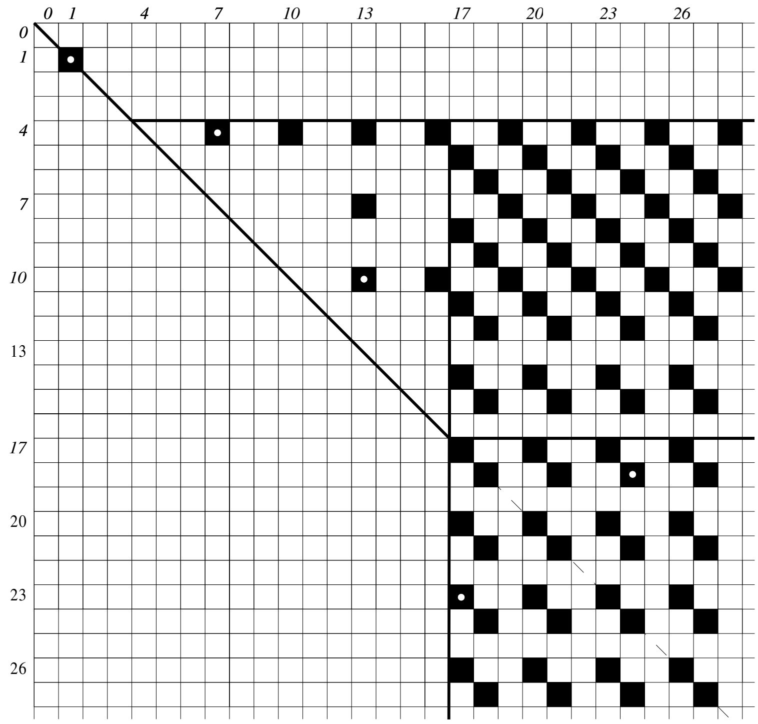

Figure 10: Two-sided subsemigroup generated by $\{cb, c^4b^7, c^{10}b^{13}, c^{18}b^{24}, c^{23}b^{17}\}$.

Each row in $p+1, p+2, \ldots$ with elements in S is an $\mathcal{R}^*$-class with idempotent. The elements in $S \cap \{c^ib^j : 0 \leq i \leq p, j \geq p\}$ are $\mathcal{R}^*$-related because S has no elements in columns $q, \ldots, p-1$ and Lemma 4.6 can be applied. Hence every $\mathcal{R}^*$-class has an idempotent. ■

Corollary 6.2 *Every simple subsemigroup of the bicyclic monoid is abundant.*

Proof. As shown in [3] these are the simple subsemigroups of the bicyclic monoid. ■

We consider now a general two sided semigroup $S = F_D \cup F \cup \Lambda_{I,p,d} \cup \Sigma_{p,d,P}$. For columns in $\Lambda_{I,p,d} \cup \Sigma_{p,d,P}$, the argument in the proof of Proposition 6.1 can be applied and so the classes with these elements have idempotents. Elements in columns $p-1$ and p are note related because we can take $x = c^pb^{p+ud}, y = c^ib^{i+vd}$ $(i < p)$ in the conditions of Lemma 3.4. To form the $\mathcal{L}^*$-classes with the elements in columns $p-1, p-2, \ldots$ we just look to rows $i < p$ and so to $F_D \cup F \cup \Lambda_{I,p,d}$. Hence we just have to apply the algorithm in Figure 8, where C is the set of indices of columns $i < p$ with elements in S.

Each row in $\Sigma_{p,d,P}$ is in a separate $\mathcal{R}^*$-class with idempotent. We note that rows p and $p-1$ are not related if $F_D \cup F \neq \emptyset$. In fact, if $F_D \neq \emptyset$ then the elements $x = c^p b^p$, $y \in F_D$ are in the conditions of Lemma 4.6. And if $F \neq \emptyset$ we can also find two elements $x = c^{p-ud} b^p$, $y \in F$ in the conditions of Lemma 4.6. Hence, to check if all $\mathcal{R}^*$-classes have idempotents, we just have to look to rows in $F_D \cup F \cup \Sigma_{p,d,P}$ and columns $0, \ldots, p-1$ and we can use the algorithm in Figure 9.

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