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On the sign characteristics of Hermitian matrix polynomials

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The sign characteristics of Hermitian matrix polynomials are discussed, and in particular an appropriate definition of the sign characteristics associated with the eigenvalue infinity. The concept of sign characteristic arises in different forms in many scientific fields, and is essential for the stability analysis in Hamiltonian systems or the perturbation behavior of eigenvalues under structured perturbations. We extend classical results by Gohberg, Lancaster, and Rodman to the case of infinite eigenvalues. We derive a systematic approach, studying how sign characteristics behave after an analytic change of variables, including the important special case of Möbius transformations, and we prove a signature constraint theorem. We also show that the sign characteristic at infinity stays invariant in a neighborhood under perturbations for even

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degree Hermitian matrix polynomials, while it may change for odd degree matrix polynomials. We argue that the non-uniformity can be resolved by introducing an extra zero leading matrix coefficient.

1. Introduction

We study the sign characteristic of Hermitian matrix polynomials. The sign characteristic is an element of $\{+1, -1\}$ that is associated with particular eigenvalues of structured matrices, matrix pencils, matrix polynomials, or analytic matrix functions. Particular examples are Hamiltonian matrices, Hermitian, even/odd pencils, and their extensions to matrix polynomials [26]. We formulate our results in terms of Hermitian matrices, pencils, polynomials, or analytic functions and eigenvalues on the real line; however, at least in the complex case there are completely analogous results associated with Hamiltonian matrices, and also with even pencils, or polynomials, which are obtained by replacing λ with $\imath\lambda$, where $\imath = \sqrt{-1}$. The sign characteristic is very important for the understanding of several physical phenomena, such as bifurcation of solutions in dynamical systems or the perturbation behavior of eigenvalues under structured perturbations. This perturbation theory is essential in the stability analysis of Hamiltonian systems and in other applications in control theory, see [5]. The sign characteristic is also closely connected to inertias of bilinear forms as well as other invariants, and it comes in different forms and flavors in many scientific fields and applications.

Let us consider a few well known examples where the sign characteristic plays an important role, which are from [30] expressed in the framework of Hermitian pencils, see also the survey [5].

Example 1.1. In the optimal H_∞ control problem, see [3,4,39] one has (in the complex case) to deal with parameterized matrix pencils of the form

$$x \left[\begin{array}{cc|ccc} 0 & \imath E & 0 & 0 & 0 \\ -\imath E^* & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|ccc} 0 & -A & 0 & 0 & -B_1 \\ -A^* & 0 & C_1^* & C_2^* & 0 \\ \hline 0 & C_1 & \gamma^2 I_p & 0 & D_{11} \\ 0 & C_2 & 0 & 0 & D_{21} \\ -B_1^* & 0 & D_{11}^* & D_{21}^* & I_m \end{array}\right],$$

where $\gamma > 0$ is a real parameter. In the so called γ iteration one has to determine the smallest possible γ such that the pencil has no real eigenvalues, and it is essential that this γ is computed accurately. In the limiting situation when the optimal γ is achieved, the sign characteristic of the eigenvalue(s) on the real axis (and, if E is singular, the eigenvalue infinity) plays an essential role.

Example 1.2. Consider a control system

$$\begin{aligned} E\dot{v} &= Av + Bu,\ v(0) = v_0, \\ w &= Cv + Du, \end{aligned} \tag{1}$$

with real or complex matrices E, A, B, C, D of sizes $n \times n$, $n \times n$, $n \times m$, $p \times n$, $p \times m$, respectively. Suppose that all the finite eigenvalues of the pencil $xE - A$ are in the open left half complex plane. Then, the system is *passive*, i.e., it does not generate energy, if and only if the pencil

$$x \begin{bmatrix} 0 & \imath E & 0 \\ -\imath E^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & A & B \\ A^* & 0 & C^* \\ B^* & C & D + D^* \end{bmatrix}$$

has no real eigenvalues and the eigenvalue infinity has equal algebraic and geometric multiplicities. In industrial practice, these systems arise from the discretization of partial differential equations, model reduction, realization or system identification, and often they are non-passive even though the underlying physical problem is passive. In this case one is interested in constructing small perturbations to E, A, B, C, D such that the system becomes passive, see e.g., [2,5,6,12], and this requires explicit knowledge about the sign characteristic.

Example 1.3. The stability of linear second order gyroscopic systems, see [16,24,38], can be analyzed via the following quadratic eigenvalue problem

$$P(x)v = (-x^2 I + x\imath(2\delta G) - K)v = 0, \tag{2}$$

where $G, K \in \mathbb{C}^{n\times n}$, K is Hermitian positive definite, G is nonsingular skew-Hermitian, and $\delta > 0$ is a parameter. To stabilize the system one needs to find the smallest real δ such that all the eigenvalues of $P(x)$ are real, which means that the gyroscopic system is stable. For the system to be robustly stable it is essential that multiple real eigenvalues do not have mixed sign characteristic.

In all these applications, and many others, see [5] for a recent survey, the location of the real eigenvalues needs to be checked numerically at different values of parameters or perturbations. In this perturbation analysis, the sign characteristic is essential.

1.1. The sign characteristic for matrix polynomials

Historically, for matrices and matrix pencils, the theory of the sign characteristics goes back to Krein, see, e.g. [21,22] and the recent survey [19], which also motivates the term *Krein characteristic*. The first systematic treatment of the sign characteristic for Hermitian matrix polynomials is given by Gohberg, Lancaster and Rodman in [9], where they present three equivalent descriptions of the sign characteristics (see also [10,11]). However, their theory assumes matrix polynomials with nonsingular leading matrix coefficient, i.e., regular matrix polynomials with only finite eigenvalues, by requiring that the matrix polynomial is expressed in the monomial or other degree-graded bases.

In our opinion, a generalization to Hermitian matrix polynomials with singular leading coefficient should be independent of specific representations of the matrix polynomial (coefficient expansions in polynomial bases such as, e.g., monomials, Lagrange, Newton, Chebyshev, etc.) and should be constructed in such a way that it allows a perturbation, so the definition remains valid also in a small neighborhood. To achieve these goals, we discuss an extension of Gohberg, Lancaster and Rodman's third description of the sign characteristic, see [9–11], to general Hermitian matrix polynomials. We derive a systematic approach which allows to show that a signature constraint theorem still holds. We analyze in detail the consequences on the perturbation theory. We show that in the case of odd degree matrix polynomials this does not lead to a uniform treatment in the neighborhood of the eigenvalue infinity. This problem of non-uniformity can be resolved by adding higher powers with zero coefficients to the matrix polynomial. We also discuss the consequences of this procedure and present several examples.

Note that Gohberg, Lancaster and Rodman's first description of the sign characteristic in [9–11] relies on a special linearization of the matrix polynomial expressed in the monomial basis and does not easily extend to matrix polynomials with singular leading matrix coefficient or to matrix polynomials expressed in non-monomial bases.

Our approach to study the sign characteristic is analytic rather than algebraic, and hence, it is essentially basis-independent. However, for the sake of concreteness and simplicity, we have decided to present our results on matrix polynomials using the monomial basis. We note *en passant* that it would be straightforward to present the theory employing any other basis. The only potential exception is the notion of a leading coefficient, central in Section 4. This is only natural with a degree-graded basis. Yet, the problem is easily overcome via the concept of reversal, which is basis-independent: for the purposes of Section 4, in fact, the leading coefficient could be defined as the reversal polynomial evaluated at 0.

The paper is organized as follows. After some preliminaries, introducing the sign characteristics and the sign feature in Section 2, in Section 3 we discuss the effect of transformations. A signature constraint theorem and its applications are discussed in Section 4. In Section 5 we discuss the behavior of the sign characteristic under perturbations. A short summary concludes the paper. We note that, although our main focus is on matrix polynomials, many results in Sections 2, 3, and 5 apply to, and are stated for, the more general case of Hermitian analytic matrix functions, that include Hermitian matrix polynomials as a special case.

2. Notation and preliminaries

In the following by $\mathbb{R}, \mathbb{C}$ we denote the real and complex numbers and by $R^{m\times n}$, the set of $m \times n$ matrices with elements in a ring R. For an open interval $\Omega \subseteq \mathbb{R}$ we use the following sets: $C^{\omega}_{\mathbb{C}}(\Omega)$, the ring of complex valued functions that are analytic on Ω, $\mathcal{M}(\Omega)$, the field of fractions of $C^{\omega}_{\mathbb{C}}(\Omega)$, i.e., the field of functions that are meromorphic on Ω, and $\mathcal{A}_n(\Omega) := C^{\omega}_{\mathbb{C}}(\Omega)^{n\times n}$, the ring of $n \times n$ matrices with complex-valued Ω-analytic elements.

Furthermore, $\mathbb{F}[x]$ is the ring of univariate polynomials in x with coefficients in the field $\mathbb{F}$ and $\mathbb{F}(x)$ is the field of fractions of $\mathbb{F}[x]$, i.e., the field of rational functions in x. For $A(x) \in \mathcal{A}_n(\Omega)$, we denote by $A(x)^*$ the complex conjugate transpose of $A(x)$ and we write $A(x) = A(x)^*$ to express that $A(x)$ is Hermitian for all $x \in \Omega$. If a matrix function $A(x)$ is invertible over the field $\mathcal{M}(\Omega)$, then we say that it is *regular*; otherwise, it is said to be *singular*. The same binary classification applies to a *matrix polynomial* $P(x) \in \mathbb{F}[x]^{n\times n}$, according to whether it is invertible over the field $\mathbb{F}(x)$ or not.

One of the key ingredients of our approach to studying the sign characteristic is a theorem of Rellich [32,33], that is used in several classical monographs such as, e.g., [11,17].

Theorem 2.1 *(Rellich's Theorem). Let $H(x) \in \mathcal{A}_n(\Omega)$ be such that $H(x) = H(x)^*$. Then there exist $V(x), D(x) \in \mathcal{A}_n(\Omega)$ such that for all $x \in \Omega$, $H(x)$ has the decomposition*

$$H(x) = V(x)D(x)V(x)^{-1},$$

where $D(x) = \mathrm{diag}(d_{11}(x), \ldots, d_{nn}(x))$ is real and diagonal and $V(x)^{-1} = V(x)^*$. Moreover, the matrix $D(x)$ is uniquely determined by $H(x)$, up to a permutation of its diagonal elements.

Note that the nature of the proof in [11] for $\Omega = \mathbb{R}$ is completely local, and therefore, $\mathbb{R}$ can be safely replaced by any simply connected open subset of $\mathbb{R}$, i.e., any open interval Ω. In the following we call a decomposition as in Theorem 2.1 a *Rellich decomposition*. It is the analytic function analogue of the spectral theorem for complex Hermitian matrices.

The *normal rank of a polynomial matrix* $P(x) \in \mathbb{C}[x]^{n\times n}$, denoted by $\mathrm{rank}_{\mathbb{C}(x)} P(x)$, is the rank of $P(x)$ as a matrix over $\mathbb{C}(x)$. A finite eigenvalue of $P(x)$ is an $x_0 \in \mathbb{C}$ such that the rank over $\mathbb{C}$ of $P(x_0) \in \mathbb{C}^{n\times n}$ is strictly less than the normal rank of $P(x)$, i.e.,

$$\mathrm{rank}_{\mathbb{C}}\, P(x_0) < \mathrm{rank}_{\mathbb{C}(x)}\, P(x).$$

Similarly, the *normal rank of an analytic matrix function* $A(x) \in \mathcal{A}_n(\Omega)$ is defined as its rank over the field $\mathcal{M}(\Omega)$, and a finite eigenvalue of an analytic matrix function $A(x)$ is an $x_0 \in \mathbb{C}$ such that

$$\mathrm{rank}_{\mathbb{C}}\, A(x_0) < \mathrm{rank}_{\mathcal{M}(\Omega)}\, A(x).$$

Throughout the paper, we denote by $\Lambda(A)$ the set of the finite eigenvalues of an analytic matrix function $A(x)$. We change the definition slightly in the special case of a polynomial function $P(x)$, denoting by $\Lambda(P)$ the set of the eigenvalues (possibly including the eigenvalue infinity, whose definition will be given below). We will be particularly interested in real eigenvalues of analytic matrix functions and matrix polynomials, because, unlike nonreal eigenvalues, they are associated to a sign characteristic. If $A(x)$ is an analytic matrix function, we define $\Lambda_{\mathbb{R}}(A) =: \Lambda(A) \cap \mathbb{R}$.

The following proposition shows that the normal rank of a Hermitian matrix function is equal to the number of nonvanishing diagonal elements of $D(x)$ in the Rellich decomposition.

Proposition 2.2. *Let $H(x) = H(x)^* \in \mathcal{A}_n(\Omega)$ with normal rank r. Then $n-r$ is the number of diagonal elements of $D(x)$ that are identically zero in any Rellich decomposition $H(x) = V(x)D(x)V(x)^*$.*

Proof. Since $V(x)$ is invertible over the field $\mathcal{M}(\Omega)$, the normal ranks of $H(x)$ and $D(x)$ coincide, and are equal to the number of diagonal elements of $D(x)$ that are not identically zero. □

In the following, we will sometimes refer to the set of non-identically vanishing diagonal elements of $D(x)$ in a Rellich decomposition:

$$\mathcal{I} = \{i \mid 1 \leq i \leq n, \text{ there exists } x_0 \in \Omega \mid d_{ii}(x_0) \neq 0\}. \tag{3}$$

The Rellich decomposition naturally separates the invertible and non-invertible parts of a Hermitian matrix function by means of a congruence transformation with a unitary matrix function $V(x)$ in $\mathcal{A}_n(\Omega)$. The decomposition and the separation of the parts are actually numerically computable, see [23, Theorem 3.9], where an explicit differential equation for $V(x)$ is derived and there exist numerical methods that can be used to compute the decomposition [7,29]. It should be noted that, even if the matrix function is polynomial, usually the factors $V(x), D(x)$ are not polynomial.

We use the Rellich decomposition to define the sign characteristic and a related property, the sign feature, of a real eigenvalue of a (possibly singular) analytic Hermitian matrix function $H(x)$ of normal rank r, by just considering the nonzero elements of $D(x)$.

Definition 2.3. Let $H(x) = H(x)^* \in \mathcal{A}_n(\Omega)$ have a Rellich decomposition $H(x) = V(x)D(x)V(x)^*$ with $D(x) = \operatorname{diag}(d_{11}(x), \ldots, d_{nn}(x))$. Let $\lambda \in \Omega$ be a real root of some $d_{ii}(x)$ that is not identically zero on Ω, and consider a Taylor expansion

$$d_{ii}(x) = \epsilon_i^\lambda c_i^\lambda (x-\lambda)^{m_i^\lambda} + O((x-\lambda)^{m_i^\lambda+1}), \tag{4}$$

where $m_i^\lambda \in \mathbb{N}$, $c_i^\lambda \in \mathbb{R}$ is positive and $\epsilon_i^\lambda \in \{1, -1\}$. We say that m_i^λ is the *ith partial multiplicity* of the real eigenvalue λ of $H(x)$ and that ϵ_i^λ is its *ith sign characteristic*. The *algebraic multiplicity* of $\lambda \in \mathbb{R}$ is the number $\sum_i m_i^\lambda$, whereas its *geometric multiplicity* is the number of nonzero partial multiplicities m_i^λ. Furthermore,

$$\phi_i^\lambda = \frac{1 - (-1)^{m_i^\lambda}}{2} \epsilon_i^\lambda$$

is called the *ith sign feature* of λ.

Remark 2.4. It should be noted that the ordering of the indices i, implicit in Definition 2.3, is arbitrary. However, all such orderings are equivalent for our purposes, and therefore throughout the manuscript we implicitly suppose to have fixed one particular, but arbitrary, Rellich decomposition.

Remark 2.5. If we express (4) as

$$d_{ii}(x) = (x - \lambda)^{m_i^\lambda} \nu(x),$$

with $\nu(\lambda) \neq 0$, then the ith sign characteristic of λ is given by $\operatorname{sign}(\nu(\lambda))$.

Remark 2.6. Usually, for a real analytic $A(x)$, the partial multiplicities of an eigenvalue are defined via the *Smith normal form*, which is possible since $C_{\mathbb{C}}^{\omega}(\Omega)$ is an elementary divisor domain [14], and hence, the Smith form exists. When a Smith form exists, then also a local Smith form exists, a fact that can be shown in a similar way as for the polynomial case in [11, Theorem S1.10]. However, note that in the Rellich decomposition of a Hermitian $H(x) = V(x)D(x)V(x)^*$, we have that $V(\lambda)$ is nonsingular for any $\lambda \in \Omega$. Therefore, the local Smith forms at λ of $H(x)$ and $D(x)$ are the same, and hence, the partial multiplicities are also equal.

The sign characteristics have the property of being invariant under analytic congruence transformations. In [9] this is proved for a regular analytic matrix function. We present the result for the general case.

Theorem 2.7. *Let $H(x) = H(x)^* \in \mathcal{A}_n(\Omega)$, and assume that $\lambda \in \Omega$ is an eigenvalue of $H(x)$. Let $R(x) \in \mathcal{A}_n(\Omega)$ satisfy* $\det R(\lambda) \neq 0$*. Then $H(x)$ and $R(x)H(x)R(x)^*$ have the same sign characteristics at λ.*

Proof. The statement is obvious, albeit vacuous, for $H(x) = 0$, so we may assume that the normal rank of $H(x)$ is $r \geq 1$. Consider a Rellich decomposition $H(x) = V(x)D(x)V(x)^*$. Then $R(x)H(x)R(x)^* = [R(x)V(x)]D(x)[R(x)V(x)]^*$. By Proposition 2.2 we may assume

$$D(x) = \begin{bmatrix} D_r(x) & 0 \\ 0 & 0 \end{bmatrix},$$

with $D_r(x)$ diagonal and regular. Since $R(x)V(x)$ is real-analytic on Ω, it has an analytic QR factorization $R(x)V(x) = Q(x)T(x)$ [34], i.e., $T(x) = \begin{bmatrix} T_{11}(x) & T_{12}(x) \\ 0 & T_{22}(x) \end{bmatrix}$ is real-analytic on Ω and upper triangular with $T_{11}(x)$ of the same size as $D_r(x)$, and $Q(x)$ is real-analytic on Ω and unitary. Then $R(x)H(x)R(x)^* = Q(x)[T(x)D(x)T(x)^*]Q(x)^*$, where

$$T(x)D(x)T(x)^* = \begin{bmatrix} T_{11}(x)D_r(x)T_{11}(x)^* & 0 \\ 0 & 0 \end{bmatrix}.$$

Observe that $R(x)H(x)R(x)^*$ has the same sign characteristics at λ as $T(x)D(x)T(x)^*$ by Theorem 2.1, hence also $T_{11}(x)D_r(x)T_{11}(x)^*$. The statement now follows by applying Theorem 3.6 in [9] for the regular case to $T_{11}(x)D_r(x)T_{11}(x)^*$, observing that $\det R(\lambda) \neq 0$ implies $\det T_{11}(\lambda) \neq 0$. $\square$

One of the main goals of this paper is to extend the concept of sign characteristic to the eigenvalue at infinity and thus to extend the classical results of Gohberg, Lancaster and Rodman in [9] to matrix polynomials with singular leading matrix coefficient. To do this in a systematic way, we need the concept of the *grade of a matrix polynomial*. Consider a matrix polynomial

$$P(x) = \sum_{j=0}^{k} P_j x^j, \; P_j \in \mathbb{C}^{n\times n}, \; j = 0, 1, \ldots, k$$

of *degree* k, i.e., the leading matrix coefficient P_k is not the zero matrix. Then we can associate with $P(x)$ an integer $g \geq k$, called *grade of* $P(x)$ and express $P(x)$ as

$$P(x) = \sum_{j=0}^{g} P_j x^j, \; P_i \in \mathbb{C}^{n\times n}, \; j = 0, 1, \ldots, k, \qquad P_{k+1}, \ldots, P_g = 0.$$

At first sight this looks artificial, but under some circumstances, and especially for structured matrix polynomials, it is a very useful concept, see [27,28]. In particular, and this is the main reason for using the grade instead of the degree, it has been shown in [27,31] that Möbius transformations, which play an important role in our analysis, are grade-preserving, but in general not degree-preserving.

Once the grade g of a matrix polynomial $P(x) = \sum_{j=0}^{g} P_j x^j$ is fixed, the *reversal* of $P(x)$ is given by

$$\mathrm{rev}_g\, P(x) = x^g P(x^{-1}) = \sum_{j=0}^{g} P_j x^{g-j},$$

and the ith partial multiplicity of the eigenvalue 0 of $\operatorname{rev}_g P(x)$ is defined to be the ith *partial multiplicity of the eigenvalue* ∞ of $P(x)$.

Using the reversal we can then also introduce the ith *sign characteristic of infinity.*

Definition 2.8 *(Sign characteristic and sign feature of the eigenvalue infinity).* Let $P(x) \in \mathbb{C}[x]^{n\times n}$ be Hermitian, have grade g and let $S(v) = -\operatorname{rev}_g P(v)$. If $P(x)$ has an eigenvalue at infinity with ith partial multiplicity m_i^∞, then we say that the *ith sign characteristic of infinity*, ϵ_i^∞, is the sign characteristic of the eigenvalue 0 of $S(v)$ having corresponding ith partial multiplicity. Furthermore, we call

$$\phi_i^\infty = \frac{1-(-1)^{g+m_i^\infty}}{2}\epsilon_i^\infty$$

the *ith sign feature at* ∞ of $P(x)$.

To see that this definition is reasonable, we first show that is does not depend on the particular choice of the grade.

Proposition 2.9. *Let $P(x) \in \mathbb{C}[x]^{n\times n}$ be Hermitian, have grade g and degree k. The sign characteristics and the sign features at ∞ as defined in Definition 2.8 do not depend on the particular choice of the grade, i.e., the ith sign characteristic ϵ_i^∞ and the ith sign feature ϕ_i^∞ are the same for all $g > k$. For $g = k$, the definition remains consistent provided that the corresponding partial multiplicity does not become zero (since otherwise there is no corresponding eigenvalue infinity and therefore there is no corresponding sign feature).*

Proof. If $g = k + g_1$, then $S(v) = -\operatorname{rev}_g(P)(v) = -v^{g_1}\operatorname{rev}_k(P)(v) = v^{g_1}\widetilde{S}(v)$. Let $\widetilde{D}(v)$ be the diagonal factor in a Rellich decomposition of $\widetilde{S}(v)$, then we obtain that $D(v) = v^{g_1}\widetilde{D}(v)$ is the diagonal factor in a Rellich decomposition of $S(v)$. Then, by Definition 2.3, the sign characteristics at 0 of $S(v)$ and $\widetilde{S}(v)$ are the same. Moreover, since both the ith partial multiplicity of 0 and the degree k are increased by g_1, it is clear that the sign feature at infinity of $P(x)$ is independent of the choice of grade. □

Remark 2.10. From the relation $S(v) = v^{g_1}\widetilde{S}(v)$ in the proof of Proposition 2.9, the ith partial multiplicity m_i^∞ of ∞ of $P(x)$ with *grade* g corresponds to a partial multiplicity $m_i^\infty - (g-k)$ of $P(x)$ with *grade* k, i.e., grade equal to degree. The use of grade (when $g > k$) introduces additional eigenvalues at ∞, all with partial multiplicities exactly $g-k$ (from $m_i^\infty - (g-k) = 0$). This provides a simple way to distinguish the “original” infinite eigenvalues from the “artificial” ones, the former having partial multiplicities $> g-k$ with grade g.

The motivation for the minus sign in the definition of $S(v)$ and the presence of g in the definition of the sign feature is that we aim to obtain an elegant signature constraint theorem, as we will see in the next sections. This goal could have also been achieved via the definition of an antireversal, $x^g P(-x^{-1})$. It is not clear which choice is better, but we prefer our definition, since it has been used in the previous literature [1,36].

3. Transformations and their effect on the sign characteristics

In this section we study the effect of transformations of the form

$$H(x) \mapsto E(y) = w(y)H(f(y)) \tag{5}$$

on the sign characteristics and the sign features, where in (5) $f(y)$ is a diffeomorphism and $w(y)$ is a nonvanishing function. We restrict our attention to smooth real-analytic transformations, as we want to preserve analyticity.

Definition 3.1. Let $\Omega \subseteq \mathbb{R}$ be an open interval, and let $f : \Omega \to f(\Omega)$ be a real-valued real-analytic diffeomorphism. We say that f is *orientation-preserving* if $f' := \frac{df}{dy} > 0$ while f is *orientation-inverting* if $f' < 0$.

Observe that this definition makes sense, because any real diffeomorphism must have a derivative of constant sign. Note that this is a simpler version (on a one-dimensional Euclidean space) of the more general concept of an *orientation-preserving diffeomorphism* [35, Definition 4.1.3].

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}$ be an open interval, and let $f : \Omega \to f(\Omega)$ be a real-valued analytic diffeomorphism. Let $H(x) \in \mathcal{A}_n(f(\Omega))$ be an analytic Hermitian matrix function, and suppose that $x_0 \in f(\Omega)$ is an eigenvalue of $H(x)$. Consider the map defined via $H(x) \mapsto E(y) = H(f(y))$. Then the following assertions hold.*

1. *The partial multiplicities of $y_0 = f^{-1}(x_0)$ as an eigenvalue of $E(y)$ are equal to the partial multiplicities of x_0 as an eigenvalue of $H(x)$.*
2. *If f is orientation-preserving, then the sign characteristics of $y_0 = f^{-1}(x_0)$ as an eigenvalue of $E(y)$ are equal to the sign characteristics of x_0 as an eigenvalue of $H(x)$.*
3. *If f is orientation-inverting, then the sign characteristics of $y_0 = f^{-1}(x_0)$ as an eigenvalue of $E(y)$ are equal to the sign characteristics of x_0 as an eigenvalue of $H(x)$ for the even partial multiplicities, and are equal to the negatives of the sign characteristics of x_0 as an eigenvalue of $H(x)$ for the odd partial multiplicities.*

Proof. Since f is diffeomorphic, it is in particular an open map, and thus $f(\Omega)$ is open and (simply) connected, i.e., it is an open interval. Moreover, $H(x)$ is Hermitian and analytic for all $x \in f(\Omega)$ if and only if $E(y)$ is Hermitian and analytic for all $y \in \Omega$, using the fact that the composition of two analytic functions is analytic. Therefore, $H(x)$ has a

Rellich decomposition $H(x) = V(x)D(x)V(x)^*$ for any $x \in f(\Omega)$, and analogously, $E(y)$ has a Rellich decomposition for any $y \in \Omega$. It follows that $V(x)$ is analytic and unitary for all $x \in f(\Omega)$ if and only if $V(f(y))$ is for all $y \in \Omega$, as f is locally analytic and invertible. This implies that $V(f(y))D(f(y))V(f(y))^* = E(y)$ is again a Rellich decomposition. Suppose that $d_{ii}(x) = (x - x_0)^{m_i^{x_0}} \nu(x)$ for some $i \in \mathcal{I}$, where $\mathcal{I}$ is defined as in (3). Then $d_{ii}(f(y)) = (f(y) - f(y_0))^{m_i^{y_0}} \nu(f(y))$, with $m_i^{x_0} = m_i^{y_0}$, and $\nu(x_0) = \nu(f(y_0)) \neq 0$. Yet, using a Taylor expansion, for any $y \in \Omega$ we can write $f(y) = f(y_0) + (y - y_0)f'(\upsilon)$, for some $\upsilon \in [y, y_0] \subset \Omega$, or $\upsilon \in [y_0, y] \subset \Omega$, according to whether $y < y_0$ or $y > y_0$. Hence $(f(y) - f(y_0))^{m_i^{y_0}} \nu(f(y)) = (y - y_0)^{m_i^{y_0}} (f'(\upsilon))^{m_i^{y_0}} \nu(f(y))$. Therefore, the sign characteristic does not change if f is orientation-preserving, while it is multiplied by $(-1)^{m_i^{y_0}}$ if f is orientation-inverting. $\square$

Theorem 3.2 emphasizes the intuitive fact that the orientation plays an important role, and that one needs to keep track of whether a change of variable is orientation-preserving or orientation-inverting.

Remark 3.3. Note that the statement of Theorem 3.2 includes the special cases $\Omega = \mathbb{R}$ or $f(\Omega) = \mathbb{R}$, i.e., a diffeomorphism from the real line to an open interval, or vice versa. This could be exploited to define the signs at infinity using, for example, the map $P(x) \mapsto (\sin\theta)^g P(\cot\theta)$. Note in fact that $f(\theta) = \cot\theta$ is analytic and diffeomorphic in $(0, \pi)$, so that this approach is essentially equivalent to the one via reversals (it excludes one point). However, we will not follow this approach as we prefer to map polynomials to polynomials.

From Theorem 3.2, we can easily deduce as a corollary the effect of a re-parametrization on the sign feature.

Theorem 3.4. *Under the assumptions of Theorem 3.2, the following assertions hold.*

1. *If f is orientation-preserving, then the sign features of $y_0 = f^{-1}(x_0)$ as an eigenvalue of $E(y)$ are equal to the sign features of x_0 as an eigenvalue of $H(x)$.*
2. *If f is orientation-inverting, then the sign features of $y_0 = f^{-1}(x_0)$ as an eigenvalue of $E(y)$ are the negatives of the sign features of x_0 as an eigenvalue of $H(x)$.*

As a second step we analyze the effect on the sign characteristic of multiplications by non-vanishing functions.

Theorem 3.5. *Let $H(x) = H(x)^* \in \mathcal{A}_n(\Omega)$ and let $E(x) = w(x)H(x)$, with an analytic non-vanishing function $w : \Omega \to \mathbb{R}$. Then the sign characteristics (resp., features) of an eigenvalue $x_0 \in \Omega$ of $E(x)$ are equal to the sign characteristics (resp., features) of x_0 as eigenvalue of $H(x)$ multiplied by* $\operatorname{sign}(w(x_0))$.

Proof. With a Rellich decomposition $H(x) = V(x)D(x)V(x)^*$, we obtain a Rellich decomposition $E(x) = V(x)[w(x)D(x)]V(x)^*$ and $d_{ii}(x) = (x - x_0)^{m_i^{x_0}} \nu(x)$ if and only if $[w(x)D(x)]_{ii} = (x - x_0)^{m_i^{x_0}} w(x)\nu(x)$, from which the claim follows. □

Example 3.6 *(Effect of a Möbius transformation on the sign characteristics and on the sign features).* As an application of the discussed transformations, we study the effect of a real Möbius transformation on the sign characteristics of a Hermitian matrix polynomial. Suppose that $P(x) \in \mathbb{C}[x]^{n\times n}$ is Hermitian and has grade g, and for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ let $\Delta := \det\left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right] \neq 0$. Then with the Möbius transformation $f(y) = \frac{\alpha y+\beta}{\gamma y+\delta}$ we have that $f'(y) = \frac{\Delta}{(\gamma y+\delta)^2}$, and hence, f is a diffeomorphism on $(-\infty, -\delta/\gamma)$ and $(-\delta/\gamma, +\infty)$. It is either orientation inverting or orientation preserving according to the sign of Δ.

Now consider the mapping

$$P(x) \mapsto Q(y) = (\gamma y + \delta)^g P\Big(\frac{\alpha y + \beta}{\gamma y + \delta}\Big).$$

Applying Theorems 3.2, 3.4, and 3.5, as well as Definitions 2.3 and 2.8, we obtain the following results.

- A finite eigenvalue $\lambda \neq \alpha/\gamma$ of $P(x)$ is mapped to a finite eigenvalue $\mu = \frac{\delta\lambda-\beta}{\alpha-\gamma\lambda}$ of $Q(y)$.
- If λ has ith partial multiplicity m_i^λ, sign characteristic ϵ_i^λ and sign feature ϕ_i^λ, then one has the following cases:
  - if m_i^λ is even, then by definition both λ and μ must have sign feature 0;
  - if m_i^λ is even and g is even, then λ and μ must have the same sign characteristic;
  - if m_i^λ is even and g is odd, then λ has sign characteristic ϵ_i^λ if and only if μ has sign characteristic $\operatorname{sign}(\gamma\mu + \delta)\epsilon_i^\lambda$;
  - if m_i^λ is odd and g is even, then λ has sign characteristic ϵ_i^λ (resp. sign feature ϕ_i^λ) if and only if μ has sign characteristic $\operatorname{sign}(\Delta)\epsilon_i^\lambda$ (resp. sign feature $\operatorname{sign}(\Delta)\phi_i^\lambda$);
  - if m_i^λ is odd and g is odd, then λ has sign characteristic ϵ_i^λ (resp. sign feature ϕ_i^λ) if and only if μ has sign characteristic $\operatorname{sign}(\gamma\mu + \delta)\operatorname{sign}(\Delta)\epsilon_i^\lambda$ (resp. sign feature $\operatorname{sign}(\gamma\mu + \delta)\operatorname{sign}(\Delta)\phi_i^\lambda$).

Let us now first assume that $\gamma \neq 0$. In this case, one has the following.

- The finite eigenvalue $\widetilde{\lambda} = \alpha/\gamma$ of $P(x)$ is mapped to the eigenvalue $\widetilde{\mu} = \infty$ of $Q(y)$.
- If $\widetilde{\lambda}$ has ith partial multiplicity $m_i^{\widetilde{\lambda}}$, sign characteristic $\epsilon_i^{\widetilde{\lambda}}$ and sign feature $\phi_i^{\widetilde{\lambda}}$, then one has the following cases:
  - if $m_i^{\widetilde{\lambda}}$ is even and g is even, then $\widetilde{\lambda}$ and $\widetilde{\mu}$ must have opposite sign characteristic, and the sign feature of $\widetilde{\mu}$ is by definition equal to 0;
  - if $m_i^{\widetilde{\lambda}}$ is even and g is odd, then $\widetilde{\lambda}$ has sign characteristic $\epsilon_i^{\widetilde{\lambda}}$ if and only if $\widetilde{\mu}$ has sign characteristic $-\operatorname{sign}(\gamma)\epsilon_i^{\widetilde{\lambda}}$. Moreover, $\widetilde{\mu}$ has sign feature $-\operatorname{sign}(\gamma)\epsilon_i^{\widetilde{\lambda}}$;

– if $m_i^{\widetilde{\lambda}}$ is odd and g is even, then $\widetilde{\lambda}$ has sign characteristic $\epsilon_i^{\widetilde{\lambda}}$ (resp. sign feature $\phi_i^{\widetilde{\lambda}}$) if and only if $\widetilde{\mu}$ has sign characteristic $\operatorname{sign}(\Delta)\epsilon_i^{\widetilde{\lambda}}$ (resp. sign feature $\operatorname{sign}(\Delta)\phi_i^{\widetilde{\lambda}}$);
– if $m_i^{\widetilde{\lambda}}$ is odd and g is odd, then $\widetilde{\lambda}$ has sign characteristic $\epsilon_i^{\widetilde{\lambda}}$ if and only if $\widetilde{\mu}$ has sign characteristic $\operatorname{sign}(\gamma)\operatorname{sign}(\Delta)\epsilon_i^{\widetilde{\lambda}}$. Moreover, $\widetilde{\mu}$ has sign feature 0.

- The eigenvalue $\hat{\lambda} = \infty$ is mapped to the finite eigenvalue $\hat{\mu} = -\delta/\gamma$.
- If $\hat{\lambda}$ has ith partial multiplicity $m_i^{\hat{\lambda}}$, sign characteristic $\epsilon_i^{\hat{\lambda}}$ and sign feature $\phi_i^{\hat{\lambda}}$, then one has the following cases:
  – if $m_i^{\hat{\lambda}}$ is even, then by definition $\hat{\mu}$ must have sign feature 0;
  – if $m_i^{\hat{\lambda}}$ is even and g is even, then $\hat{\lambda}$ and $\hat{\mu}$ must have opposite sign characteristic;
  – if $m_i^{\hat{\lambda}}$ is even and g is odd, then $\hat{\lambda}$ has sign characteristic $\epsilon_i^{\hat{\lambda}}$ if and only if $\hat{\mu}$ has sign characteristic $\operatorname{sign}(\gamma)\operatorname{sign}(\Delta)\epsilon_i^{\hat{\lambda}}$;
  – if $m_i^{\hat{\lambda}}$ is odd and g is even, then $\hat{\lambda}$ has sign characteristic $\epsilon_i^{\hat{\lambda}}$ (resp. sign feature $\phi_i^{\hat{\lambda}}$) if and only if $\hat{\mu}$ has sign characteristic $\operatorname{sign}(\Delta)\epsilon_i^{\hat{\lambda}}$ (resp. sign feature $\operatorname{sign}(\Delta)\phi_i^{\hat{\lambda}}$);
  – if $m_i^{\hat{\lambda}}$ is odd and g is odd, then $\hat{\lambda}$ has sign characteristic $\epsilon_i^{\hat{\lambda}}$ if and only if $\hat{\mu}$ has sign characteristic $-\operatorname{sign}(\gamma)\epsilon_i^{\hat{\lambda}}$. Moreover, $\hat{\mu}$ has sign feature $-\operatorname{sign}(\gamma)\epsilon_i^{\hat{\lambda}}$.

Conversely, if $\gamma = 0$, then the eigenvalue infinity stays at infinity. Assuming that ∞, as an eigenvalue of $P(x)$, has partial multiplicity m_i^{∞}, sign characteristic ϵ_i^{∞}, and sign feature ϕ_i^{∞}, then one has the following (note that $\alpha \neq 0 \neq \delta$ since otherwise $\Delta = \alpha\delta = 0$):

– if m_i^{∞} and g are either both even or both odd, then ∞ has sign feature 0 both as an eigenvalue of $P(x)$ and as an eigenvalue of $Q(y)$;
– if m_i^{∞} is even and g is even, then ∞ must have the same sign characteristic when seen as an eigenvalue of $P(x)$ and when seen as an eigenvalue of $Q(y)$;
– if m_i^{∞} is even and g is odd, then ∞ has sign characteristic ϵ_i^{∞} (resp. sign feature ϕ_i^{∞}) as an eigenvalue of $P(x)$ if and only if it has sign characteristic $\operatorname{sign}(\alpha)\epsilon_i^{\infty}$ (resp. sign feature $\operatorname{sign}(\alpha)\phi_i^{\infty}$) as an eigenvalue of $Q(y)$;
– if m_i^{∞} is odd and g is even, then ∞ has sign characteristic ϵ_i^{∞} (resp. sign feature ϕ_i^{∞}) as an eigenvalue of $P(x)$ if and only if it has sign characteristic $\operatorname{sign}(\Delta)\epsilon_i^{\infty}$ (resp. sign feature $\operatorname{sign}(\Delta)\phi_i^{\infty}$) as an eigenvalue of $Q(y)$;
– if m_i^{∞} is odd and g is odd, then ∞ has sign characteristic ϵ_i^{∞} as an eigenvalue of $P(x)$ if and only if it has sign characteristic $\operatorname{sign}(\delta)\epsilon_i^{\infty}$ as an eigenvalue of $Q(y)$.

In this section we have studied the effect of transformations on the sign feature and sign characteristic. These results will be used in the following section to derive a global constraint for these quantities.

4. A signature constraint theorem

In this section we discuss a constraint law for the sign feature and sign characteristic, extending to possibly singular matrix polynomials the signature constraint theorem [11], that we state below.

Theorem 4.1 *(Proposition 10.12 in [11]). Let* $P(x) = I_n x^g + \sum_{j=0}^{g-1} P_j x^j$ *be a monic Hermitian matrix polynomial. For* $\lambda \in \Lambda_{\mathbb{R}}(P)$ *and* $1 \leq i \leq n$*, let* $\phi_i^\lambda(P)$ *be the corresponding sign features. Then*

$$\sum_{\lambda \in \Lambda_{\mathbb{R}}(P),\ 1 \leq i \leq n} \phi_i^\lambda(P) = \begin{cases} 0 \text{ if } 2 \mid g \\ n \text{ if } 2 \nmid g \end{cases}$$

Consider now a Hermitian matrix polynomial $P(x)$, possibly not monic, of grade g. Then $P(x)$ is holomorphic on the whole complex plane $\mathbb{C}$, so in particular its restriction to the real line is real analytic, and the results of the previous section apply with $\Omega = \mathbb{R}$. Recall that the *Sylvester inertia index*, or simply inertia, of a Hermitian matrix H is the triple (n_+, n_0, n_-), where n_+ (resp. n_0, n_-) is the number of positive (resp. zero, negative) eigenvalues of H. Furthermore, the *signature* of H is defined as $\mathrm{sig}(H) = n_+ - n_-$.

To derive a signature constraint law, it is convenient to first discuss the case where $P(x)$ has no infinite eigenvalues. A sufficient condition for this is that $P(x)$ has nonsingular leading matrix coefficient P_g. In this case, a proof can be easily obtained by generalizing Theorem 4.1 to any nonsingular leading matrix coefficient using [10, Eqn. 12.2.12]: in this case, in the statement n is replaced by the signature of the leading matrix coefficient P_g (note that $\mathrm{sig}(I_n) = n$). Our signature constraint result, Theorem 4.5, is stronger than those in [10,11], because it allows for a general Hermitian matrix polynomial $P(x)$, including the case that the leading matrix coefficient P_g is singular. In the following we denote by $\Lambda_{\mathbb{R}^*}(P)$ the set of all real eigenvalues of the Hermitian matrix polynomial $P(x)$ including ∞, and we use again the set $\mathcal{I}$ as defined in (3). For $i \in \mathcal{I}$, $\lambda \in \Lambda_{\mathbb{R}^*}(P)$ we denote by m_i^λ, $\phi_i^\lambda(P)$, respectively, the ith partial multiplicity and sign feature associated with λ and $P(x)$.

Theorem 4.2. *Let* $P(x) = \sum_{j=0}^{g} P_j x^j$ *be a Hermitian matrix polynomial of grade* g *with no infinite eigenvalues. For* $\lambda \in \Lambda_{\mathbb{R}^*}(P)$ *and* $i \in \mathcal{I}$*, let* $\phi_i^\lambda(P)$ *be the corresponding sign features. Then*

$$\sum_{\lambda \in \Lambda_{\mathbb{R}^*}(P),\ i \in \mathcal{I}} \phi_i^\lambda(P) = \begin{cases} 0 \text{ if } 2 \mid g \\ \mathrm{sig}(P_g) \text{ if } 2 \nmid g \end{cases}$$

Proof. Since there are no eigenvalues at infinity, it follows that $\operatorname{rank} P_g = r = \operatorname{rank}_{\mathbb{C}(x)} P(x)$. Observe that this implies that either $P(x) \equiv 0$ or that g is equal to the degree k of P. If $P(x) \equiv 0$, then the assertion holds trivially, so we consider the case $k = g$ and let (n_+, n_-, n_0) be the inertia of P_g. Note that $n_+ + n_- + n_0 = n$ and that $n_0 = n - r$. Then the proof follows by a counting argument on the number of zeros with odd multiplicity of $d_{ii}(x)$, $i \in \mathcal{I}$.

Indeed, for $i \in \mathcal{I}$ a root $\lambda \in \Lambda_{\mathbb{R}^*}(P)$ of $d_{ii}(x)$ has odd multiplicity m_i^λ if and only if it is associated with an eigenvalue of nonzero sign feature. In other words, the sign feature is -1 if $d_{ii}(x)$ is positive to the left of the root and negative to the right, and it is $+1$ if it is negative to the left and positive to the right. Now let $\beta > 0$ be larger than the largest (in absolute value) real eigenvalue of $P(x)$. Then

$$\frac{\operatorname{sign}(d_{ii}(\beta)) - \operatorname{sign}(d_{ii}(-\beta))}{2}$$

counts the sum of total sign features associated to that value of i. Summing over all $i \in \mathcal{I}$ we get that the sum of all the sign features is

$$\sum_{i\in\mathcal{I}} \frac{\operatorname{sign}(d_{ii}(\beta)) - \operatorname{sign}(d_{ii}(-\beta))}{2} = \frac{\sum_{i\in\mathcal{I}} \operatorname{sign}(d_{ii}(\beta)) - \sum_{i\in\mathcal{I}} \operatorname{sign}(d_{ii}(-\beta))}{2}. \tag{6}$$

Note now that

$$\sum_{i\in\mathcal{I}} \operatorname{sign}(d_{ii}(\beta)) = \operatorname{sig}(P(\beta)), \qquad \sum_{i\in\mathcal{I}} \operatorname{sign}(d_{ii}(-\beta)) = \operatorname{sig}(P(-\beta))$$

Suppose first that g is even. Then $P(\beta)$ and $P(-\beta)$ both have the same inertia as P_g, as can be checked by a continuity argument and the fact that $\operatorname{rank} P(x)$ is constant for all x on $(-\infty, -\beta] \cup [\beta, \infty)$, and by using

$$\lim_{\beta\to+\infty} \beta^{-g} P(\beta) = \lim_{\beta\to+\infty} \beta^{-g} P(-\beta) = P_g.$$

Therefore, the right hand side of (6) is equal to

$$\frac{1}{2}(n_+ - n_- - n_+ + n_-) = 0.$$

If g is odd, then $P(\beta)$ has the same inertia as P_g and $P(-\beta)$ has the same inertia as $-P_g$. This can be shown again by a continuity argument, observing

$$\lim_{\beta\to+\infty} \beta^{-g} P(\beta) = -\lim_{\beta\to+\infty} \beta^{-g} P(-\beta) = P_g.$$

Thus, the right hand side of (6) becomes

$$\frac{1}{2}(n_+ - n_- + n_+ - n_-) = \operatorname{sig}(P_g). \quad \square$$

To extend the result to the case where $P(x)$ has infinite eigenvalues, it is convenient to consider three auxiliary matrix polynomials. Let $\beta > |\lambda_{\max}|$, where $\lambda_{\max}$ is the finite real eigenvalue of $P(x)$ of maximal absolute value. Then introduce

$$Q(y) := (-y)^g P\Big(\frac{\beta y + 1}{-y}\Big), \qquad R(z) := z^g P\Big(\frac{\beta z - 1}{z}\Big). \tag{7}$$

Observe that neither $Q(y)$ nor $R(z)$ has an infinite eigenvalue, so that we can apply Theorem 4.2 to them. We have the following lemma.

Lemma 4.3. *Let $P(x) = \sum_{j=0}^{g} P_j x^j$ be a Hermitian matrix polynomial of grade g. Let $\mathcal{I}$ be defined as in (3). If λ is a finite real eigenvalue of $P(x)$ with partial multiplicities m_i^λ and sign features $\phi_i^\lambda(P)$, $i \in \mathcal{I}$, then $\frac{-1}{\beta+\lambda}$ is a finite eigenvalue of $Q(y)$ with partial multiplicities m_i^λ and sign features $\phi_i^\lambda(P)$, $i \in \mathcal{I}$, and in the same way, $\frac{1}{\beta-\lambda}$ is a finite eigenvalue of $R(z)$ with partial multiplicities m_i^λ and sign features $\phi_i^\lambda(P)$, $i \in \mathcal{I}$.*

If $\lambda = \infty$ is an eigenvalue of $P(x)$ with partial multiplicities $m_i^\infty(P)$, $i \in \mathcal{I}$, then 0 is an eigenvalue of both $Q(y)$ and $R(z)$ each with multiplicities $m_i^0(Q) = m_i^0(R) = m_i^\infty(P)$, $i \in \mathcal{I}$, and furthermore, if g is even, then the sign features of 0 as an eigenvalue of $Q(y)$ and $R(z)$ are the same, while if g is odd, then the sign features of 0 as an eigenvalue of $Q(y)$ and $R(z)$ are opposite in sign.

Proof. The conservation of the partial multiplicities follows immediately from [27, Theorem 5.3] or [31, Theorem 4.1]. Thus, it suffices to prove the statements on the sign features for which we apply Theorems 3.4 and 3.5, or equivalently Example 3.6. We observe that both Möbius re-parameterizations $y = \frac{-1}{\beta+x}$ and $z = \frac{1}{\beta-x}$ are orientation preserving (on the open intervals where they are a diffeomorphism), because they have determinant 1. Therefore the sign features of a finite nonzero real eigenvalue of $Q(y)$ (resp. $R(z)$) can only differ from those of the corresponding finite real eigenvalue λ of $P(x)$ if g is odd and $\frac{1}{\beta+\lambda}$ (resp. $\frac{1}{\beta-\lambda}$) is negative. But this happens if and only if $\lambda < -\beta$ (resp. if and only if $\lambda > \beta$), which is impossible by the definition of β.

Finally, by comparing the two Möbius transformations in (7), we see that $R(z) = (2\beta z - 1)^g Q(\frac{-z}{2\beta z-1})$. Using Theorem 3.4 we see that the reparametrization has no effect because it is orientation preserving. However, by Theorem 3.5, the global factor $(-1)^g$ comes into play, thus proving the assertions on the sign features associated with the 0 eigenvalue of $Q(y)$ and $R(z)$. □

A third matrix polynomial with eigenvalues at 0 when $P(x)$ has eigenvalues at infinity is $S(v)$ as constructed in Definition 2.8. Comparing $S(v)$ with $Q(y)$ and $R(z)$ we have the following Lemma.

Lemma 4.4. *Let $P(x) = \sum_{j=0}^{g} P_j x^j$ be a Hermitian matrix polynomial of grade g with eigenvalue $\lambda = \infty$ associated with sign characteristics ϵ_i^∞ and partial multiplicities m_i^∞, $i \in \mathcal{I}$. Then the eigenvalue 0 of $Q(y), R(z)$ and $S(v)$ has partial multiplicities $m_i^0(Q)$, $m_i^0(R)$, $m_i^0(S)$, respectively, equal to m_i^∞, for all $i \in \mathcal{I}$. Moreover, for the sign characteristics $\epsilon_i^0(Q)$, $\epsilon_i^0(R)$, $\epsilon_i^0(S)$, of the eigenvalue 0 of $Q(y)$, $R(z)$, and $S(v)$, respectively, the following statements hold.*

- *If m_i^∞ is odd and*
  - *if g is odd then $\epsilon_i^0(Q) = \epsilon_i^0(S) = -\epsilon_i^0(R)$;*

   – *if g is even then $\epsilon_i^0(Q) = \epsilon_i^0(S) = \epsilon_i^0(R)$;*
- *If m_i^∞ is even and*
   – *if g is odd then $\epsilon_i^0(R) = \epsilon_i^0(S) = -\epsilon_i^0(Q)$;*
   – *if g is even then $\epsilon_i^0(R) = \epsilon_i^0(Q) = -\epsilon_i^0(S)$.*

Proof. The proof follows from the relations $Q(y) = -(\beta y + 1)^g S(\frac{-y}{\beta y+1})$ and $R(z) = -(\beta z - 1)^g S(\frac{z}{\beta z-1})$, and by repeated application of Theorems 3.2 and 3.5 (and of the details in the proof of Theorem 3.2) analogous to the proof of Lemma 4.3. □

Combining these results we have the following theorem.

Theorem 4.5 *(Signature Constraint Theorem). Let $P(x) = \sum_{j=0}^{g} P_j x^j$ be a Hermitian matrix polynomial of grade g. For $\lambda \in \Lambda_{\mathbb{R}^*}(P)$ and $i \in \mathcal{I}$, let $\phi_i^\lambda(P)$ be the corresponding sign features. Then*

$$\sum_{\lambda \in \Lambda_{\mathbb{R}^*}(P),\ i \in \mathcal{I}} \phi_i^\lambda(P) = \begin{cases} 0 \text{ if } 2 \mid g, \\ \operatorname{sig}(P_g) \text{ if } 2 \nmid g. \end{cases}$$

Proof. Suppose first that g is even. Applying Theorem 4.2 to $Q(y)$ with real eigenvalue set $\Lambda_{\mathbb{R}}(Q) = \Lambda(Q) \cap \mathbb{R}$, we get that

$$0 = \sum_{0 \neq \lambda \in \Lambda_{\mathbb{R}}(Q),\ i \in \mathcal{I}} \phi_i^\lambda(Q) + \sum_{i \in \mathcal{I}} \phi_i^0(Q).$$

By Lemma 4.3 we have

$$\sum_{0 \neq \lambda \in \Lambda_{\mathbb{R}}(Q),\ i \in \mathcal{I}} \phi_i^\lambda(Q) = \sum_{\infty \neq \lambda \in \Lambda_{\mathbb{R}^*}(P),\ i \in \mathcal{I}} \phi_i^\lambda(P),$$

whereas by Lemma 4.4

$$\sum_{i \in \mathcal{I}} \phi_i^0(Q) = \sum_{i \in \mathcal{I}} \phi_i^0(S),$$

using the fact that g is even and that only the values of i such that $m_i^0(Q)$ is odd contribute to the summation, because $m_i^0(Q)$ being even implies $\phi_i^0(Q) = 0$. The assertion follows, since by definition $\phi_i^0(S) = \phi_i^\infty(P)$ as g is even.

The case of odd g requires some further discussion. Consider β as a parameter varying in $(|\lambda_{\max}|, +\infty)$. Let $A(\beta)$ (resp. $B(\beta)$) be the leading matrix coefficient of $Q(y)$ (resp. $R(z)$). From the formula in [31, Proof of Proposition 3.2, second bullet] we get $A(\beta) = (-1)^g P(-\beta)$ and $B(\beta) = P(\beta)$. Moreover, both A and B are Hermitian matrices that depend analytically on the real parameter β, and hence, by Theorem 2.1 and Proposition 2.2 we have that their eigenvalues are analytic functions of β, of which $n - r$

are constantly zero, where r is the normal rank of $P(x)$ and n is its size. In particular, since there is no eigenvalue of $P(x)$ in the interval $(|\lambda_{\max}|, +\infty)$, the number of positive and negative eigenvalues of $A(\beta)$ and $B(\beta)$ must be independent of β. As a consequence, their signatures are constant, and we may simply write $\operatorname{sig}(A)$ and $\operatorname{sig}(B)$, omitting β. Let $b = \beta^{-1}$. It is easy to check that $\beta^{-g}A(\beta) = -S(-b)$, while $\beta^{-g}B(\beta) = -S(b)$, and that $P_g = -\lim_{b\to 0^+} S(-b) = -\lim_{b\to 0^+} S(b)$.

Being polynomial, $S(b)$ is analytic at 0, and hence, it admits a Rellich decomposition. Setting $\gamma = \dim\ker P_g + r - n$, such a decomposition is given by

$$S(b) = V(b)\Big(0_{n-r} \oplus \bigoplus_{j=1}^{r-\gamma} \big(\alpha_j^0 + O(b)\big) \oplus \bigoplus_{j=1}^{\gamma} \big(\epsilon_j^0 c_j^0 b^{m_j^0} + O(b^{m_j^0+1})\big)\Big)V(b)^*, \tag{8}$$

where 0_k is the $k \times k$ zero matrix, $\oplus$ denotes the direct sum, α_j^0 are some nonzero constants, c_j^0 are *positive* constants, and ϵ_j^0 (resp. m_j^0) are the sign characteristics (resp. partial multiplicities) at 0 of $S(b)$, which are, by definition, the sign characteristics (resp. partial multiplicities) at ∞ of $P(x)$. Clearly, the signature of $S(b)$ is the same as the signature of the diagonal matrix in (8).

When $|b| > 0$ is small enough, then only the lowest order terms in b matter. Thus, there exists $b_0 > 0$ such that for $0 < b < b_0$ we have that

$$\operatorname{sig}(S(b)) = \sum_{i=1}^{r-\gamma} \operatorname{sign}(\alpha_i^0) + \sum_{i=1}^{\gamma} \epsilon_i^0.$$

Similarly there exists $b_1 > 0$ such that for $-b_1 < b < 0$ it holds that

$$\operatorname{sig}(S(-b)) = \sum_{i=1}^{r-\gamma} \operatorname{sign}(\alpha_i^0) + \sum_{i=1}^{\gamma} (-1)^{m_i^0} \epsilon_i^0.$$

On the other hand

$$\operatorname{sig}(P_g) = -\operatorname{sig} \lim_{b\to 0^+} S(b) = -\operatorname{sig}(S(0)) = -\sum_{i=1}^{r-\gamma} \operatorname{sign}(\alpha_i^0).$$

Using that $-\operatorname{sig}(A) = -\operatorname{sig}(\beta^{-g}A) = \operatorname{sig}(S(-b))$ and $-\operatorname{sig}(B) = -\operatorname{sig}(\beta^{-g}B) = \operatorname{sig}(S(b))$, we obtain

$$2\operatorname{sig}(P_g) - \operatorname{sig}(A) - \operatorname{sig}(B) = 2 \sum_{m_i^0 \text{ even}} \epsilon_i^0.$$

On the other hand, applying Theorem 4.2 twice, we get

$$\operatorname{sig}(A) + \operatorname{sig}(B) = \sum_{0 \neq \lambda \in \Lambda_{\mathbb{R}}(Q),\ i \in \mathcal{I}} \phi_i^\lambda(Q) + \sum_{i\in\mathcal{I}} \phi_i^0(Q) + \sum_{0 \neq \lambda \in \Lambda_{\mathbb{R}}(R),\ i \in \mathcal{I}} \phi_i^\lambda(R) + \sum_{i\in\mathcal{I}} \phi_i^0(R).$$

Using Theorem 4.2, Lemma 4.3 and Lemma 4.4 (with g odd), this is in turn equal to

$$\mathrm{sig}(A) + \mathrm{sig}(B) = 2 \sum_{\infty \neq \lambda \in \Lambda_{\mathbb{R}^*}(P),\ i \in \mathcal{I}} \phi_i^\lambda(P).$$

The result follows by observing that, when m_i^0 is even, ϵ_i^0 is, by definition, the sign feature at infinity of $P(x)$. □

Remark 4.6. In this remark, we sketch an alternative proof of Theorem 4.5. The complete proof that we give above emphasizes the role of Möbius transformations to extend the theorem to cover the point at infinity, while the approach that we follow below gives some further intuition to explain why for odd grade it is the even partial multiplicities at infinity that matter for the sign constraint theorem.

Partition $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ so that $\lim_{x\to\infty} x^{-g} d_{ii}(x)$ is nonzero for any $i \in \mathcal{I}_1$ and it is 0 for any $i \in \mathcal{I}_2$. We will argue that Definition 2.8 guarantees that for each $i \in \mathcal{I}_2$, the sum of all sign features associated with $d_{ii}(x)$ is zero, and that this fact yields Theorem 4.5.

Following the same proof as for Theorem 4.2, one has that

$$\sum_{\lambda \in \Lambda_{\mathbb{R}}(P),\ i \in \mathcal{I}_1} \phi_i^\lambda(P) = \begin{cases} 0 \text{ if } 2 \mid g \\ \mathrm{sig}(P_g) \text{ if } 2 \nmid g \end{cases}$$

Letting $v = 1/x$,

$$S(v) = -v^g P(1/v) = -x^{-g} P(x) = V(x)(-x^{-g} D(x)) V(x)^*.$$

Since $x^{-g} d_{ii}(x)$ will not converge to zero as $x \to \pm\infty$ (or $v \to 0$) for all $i \in \mathcal{I}_1$, the eigenvalue 0 of $S(v)$, or equivalently the eigenvalue ∞ of $P(x)$, is associated with $-v^g d_{ii}(1/v) = -x^{-g} d_{ii}(x)$ if and only if $i \in \mathcal{I}_2$. Now, still following the proof of Theorem 4.2, for each fixed $i \in \mathcal{I}_2$,

$$\sum_{\lambda \in \Lambda_{\mathbb{R}}(P)} \phi_i^\lambda(P) = \frac{\mathrm{sign}(d_{ii}(\beta)) - \mathrm{sign}(d_{ii}(-\beta))}{2},$$

where $\beta > 0$ is larger than the maximal absolute value of all the finite real eigenvalues of $P(x)$. It can be shown using Definition 2.8 that, regardless of the grade of $P(x)$, the sign feature $\phi_i^\infty(P)$ of $P(x)$ is precisely

$$\phi_i^\infty(P) = \frac{\mathrm{sign}(d_{ii}(-\beta)) - \mathrm{sign}(d_{ii}(\beta))}{2}.$$

For the latter equation to hold, it is crucial that in Definition 2.8 the sign features at infinity can be nonzero only for odd grade and even partial multiplicities or for even grade and odd partial multiplicities. As a consequence, for each $i \in \mathcal{I}_2$,

$$\sum_{\lambda \in \Lambda_{\mathbb{R}^*}(P)} \phi_i^\lambda(P) = \sum_{\lambda \in \Lambda_{\mathbb{R}}(P)} \phi_i^\lambda(P) + \phi_i^\infty(P) = 0.$$

Hence

$$\sum_{\lambda \in \Lambda_{\mathbb{R}^*}(P), i \in \mathcal{I}_2} \phi_i^\lambda(P) = 0$$

and then the result is established from

$$\sum_{\lambda \in \Lambda_{\mathbb{R}^*}(P), i \in \mathcal{I}} \phi_i^\lambda(P) = \sum_{\lambda \in \Lambda_{\mathbb{R}}(P), i \in \mathcal{I}_1} \phi_i^\lambda(P) + \sum_{\lambda \in \Lambda_{\mathbb{R}^*}(P), i \in \mathcal{I}_2} \phi_i^\lambda(P).$$

Remark 4.7. Observe that, when $g > \deg(P)$, the sum of the sign feature is always zero for any g because $P_g = 0$. The difference occurs only when $g = \deg(P)$. If $\deg(P)$ is even the sum is still zero, but when $\deg(P)$ is odd, the sum is $\mathrm{sig}(P_{\deg(P)})$.

However, the proof of Theorem 4.5 shows that the sign characteristics associated with partial multiplicities $g - \deg(P)$ (that are, by Remark 2.10, associated with those infinite eigenvalues that are "artificial") are the inertia indices of $-P_{\deg(P)}$. Moreover, their sign features are all zero if $\deg(P)$ is even and are their sign characteristics if $\deg(P)$ is odd. Hence, the sum of the "extra" sign features at infinity is zero when $\deg(P)$ is even and is $-\mathrm{sig}(P_{\deg(P)})$ if $\deg(P)$ is odd, making the whole picture coherent.

Remark 4.8. Observe that Theorem 4.5 can also be obtained by defining the sign features at infinity as the sign features of the antireversal $T(z) = z^g P(-z^{-1})$. Indeed, it is immediate that $T(z) = (-1)^{g+1} S(-z)$, and hence, the sign characteristic of a zero eigenvalue of partial multiplicity m_i^0 of $T(z)$ is $(-1)^{g+1+m_i^0}$ times the sign characteristic of a zero eigenvalue of $S(v)$, of the corresponding partial multiplicity. In particular, when $g + m_i^0$ is odd, then these signs are unchanged. But given Definition 2.8, the case of $g + m_i^0$ odd is precisely the one that is relevant in Theorem 4.5.

4.1. Connection with the canonical form of Hermitian pencils

In this section we discuss the connection of our results to the canonical form for Hermitian pencils under congruence, see [25,37] and the references therein.

Theorem 4.9 *(Theorem 6.1 in [25]). Every Hermitian pencil $A + xB$ is congruent to a pencil of the form*

$$0_{u\times u} \oplus \bigoplus_{i=1}^{p} x \begin{bmatrix} 0 & 0 & F_{\rho_i} \\ 0 & 0 & 0 \\ F_{\rho_i} & 0 & 0 \end{bmatrix} + G_{2\rho_i+1} \oplus \bigoplus_{i=1}^{r} \delta_i [F_{k_i} + x G_{k_i}] \oplus \bigoplus_{i=1}^{q} \eta_i [(x + \alpha_i) F_{\ell_i} + G_{\ell_i}]$$
$$\oplus \bigoplus_{i=1}^{s} \begin{bmatrix} 0 & (x+\beta_i) F_{m_i} \\ (x + \overline{\beta_i}) F_{m_i} & 0 \end{bmatrix} + \begin{bmatrix} 0 & G_{m_i} \\ G_{m_i} & 0 \end{bmatrix},$$

where u, $\rho_1 \le \cdots \le \rho_p$, $k_1 \le \cdots \le k_r$, $l_1 \le \cdots \le l_q$, and $m_1 \le \cdots \le m_s$ are positive integers, α_j are real numbers, β_j are complex nonreal numbers, $\delta_1, \dots, \delta_r, \eta_1, \dots, \eta_q$ are equal to $+1$ or -1, and

$$F_n = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad G_n = \begin{bmatrix} F_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times n}. \tag{9}$$

This canonical form is uniquely determined up to permutations of the blocks, and up to replacing β_j by $\overline{\beta_j}$ inside the corresponding blocks.

We first give a technical lemma that is useful to compute the signature of the leading matrix coefficient of a Hermitian pencil.

Lemma 4.10. *The signatures of the coefficients of x in each diagonal block in the canonical form of Theorem 4.9 are:*

$$\operatorname{sig}(0_{u\times u}) = \operatorname{sig}\left(\begin{bmatrix} 0 & 0 & F_{\rho_i} \\ 0 & 0 & 0 \\ F_{\rho_i} & 0 & 0 \end{bmatrix}\right) = \operatorname{sig}\left(\begin{bmatrix} 0 & F_{m_i} \\ F_{m_i} & 0 \end{bmatrix}\right) = 0,$$
$$\operatorname{sig}(G_{k_i}) = \frac{1+(-1)^{k_i}}{2}, \quad \operatorname{sig}(F_{\ell_i}) = \frac{1-(-1)^{\ell_i}}{2}.$$

Proof. We just need to prove that $\operatorname{sig}(F_{\ell_i}) = \frac{1-(-1)^{\ell_i}}{2}$, as all the other claims follow immediately (recalling that $\left[\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right]$ is similar to $A \oplus -A$.) Suppose first that $\ell_i = 2\mu_i$ is even. Then, block-diagonalizing $xI_{2\mu_i} - F_{2\mu_i}$ by an appropriate permutation similarity, it is readily seen that $\det(xI_{2\mu_i} - F_{2\mu_i}) = (x^2-1)^{\mu_i}$, yielding $\operatorname{sig}(F_{2\mu_i}) = 0$. The case of odd $\ell_i = 2\mu_i + 1$ can be reduced to the previous one, as by a Laplace expansion by the central row, we have $\det(xI_{2\mu_i+1} - F_{2\mu_i+1}) = (x-1)\det(xI_{2\mu_i} - F_{2\mu_i})$, and hence, $\operatorname{sig}(F_{2\mu_i+1}) = 1$. □

It turns out that the signs $\delta_1, \dots, \delta_r, \eta_1, \dots, \eta_q$ in Theorem 4.9 determine the sign characteristics associated with real and infinite eigenvalues, as the next results show. Note that in the literature there is a minor incoherence in the description of the exact relation between these signs and the sign characteristic, see e.g., [25].

Theorem 4.11. *The analytic Hermitian matrix pencil $(x+\alpha)F_\ell + G_\ell$, where F_ℓ and G_ℓ are as in (9), has a unique real eigenvalue at $-\alpha$ of partial multiplicity ℓ and sign characteristic $(-1)^{\ell+1}$.*

The analytic Hermitian matrix pencil $F_k + xG_k$ has a unique eigenvalue at infinity of partial multiplicity k and sign characteristic $(-1)^k$.

Proof. It suffices to prove the first statement, as together with Definition 2.8 it immediately implies the second. Observe that by a simple change of variable we may assume that $\alpha = 0$. It is clear by direct inspection that $A(x) = xF_\ell + G_\ell$ has an eigenvalue at 0 of partial multiplicity ℓ and geometric multiplicity 1. It remains to compute its sign characteristic.

By the definition of G_ℓ, $A(0)$ has precisely one zero eigenvalue. Therefore, using the Rellich decomposition (Theorem 2.1) of $A(x)$ and Definition 2.3, it is clear that the sign characteristic at 0 of $A(x)$ is just

$$\lim_{x \to 0^+} \mathrm{sig}(A(x)) - \mathrm{sig}(A(0)).$$

By Lemma 4.10, $\mathrm{sig}(A(0)) = \frac{1+(-1)^\ell}{2}$. On the other hand, for any $x > 0$, $A(x)$ is congruent to xF_ℓ. Indeed, first one can take $A(x)$ to $x(F_\ell + G_\ell)$ by either the simple diagonal congruence $\mathrm{diag}(\dots, x, 1, x^{-1}, \dots)$, for odd values of ℓ, or the simple diagonal congruence $\mathrm{diag}(\dots, x, , x^{\frac{1}{2}}, x^{-\frac{1}{2}}, x^{-1}, \dots)$, for even values of ℓ. To show that $F_\ell + G_\ell$ and F_ℓ are congruent, let N_ℓ be the nilpotent Jordan block of size ℓ and observe that $G_\ell = N_\ell F_\ell = F_\ell N_\ell^T$. It easily follows that for any *real* polynomial p, $p(I_\ell + N_\ell)F_\ell = F_\ell p(I_\ell + N_\ell)^T = F_\ell p(I_\ell + N_\ell)^*$. Let S_ℓ be the principal square root of $I_\ell + N_\ell$, see [15], then S_ℓ is a real polynomial in $I_\ell + N_\ell$, and $S_\ell F_\ell S_\ell^* = S_\ell^2 F_\ell = F_\ell + G_\ell$, displaying the desired congruence.

Thus, again by Lemma 4.10, $\mathrm{sig}(A(x)) = \frac{1-(-1)^\ell}{2}$, and hence, the sign characteristic of $A(x)$ at 0 is $(-1)^{\ell+1}$. □

Hence, we may easily obtain an alternative proof of Theorem 4.5 for the special case of pencils, i.e., $g = 1$. Indeed, observe that there is no loss of generality in assuming that a pencil $A + Bx$ is in the canonical form described in Theorem 4.9, for if it is not, we may just apply Theorem 2.7 (specialized to the case where $A(x)$ is a pencil and $R(x)$ is constant and nonsingular). Then, since B is block diagonal, its signature is the sum of the signatures of each block, i.e., by Lemma 4.10,

$$\mathrm{sig}(B) = \sum_{i\ :\ \ell_i \text{ odd}} \eta_i + \sum_{i\ :\ k_i \text{ even}} \delta_i.$$

But on the other hand, by Theorem 4.11, the sign feature of any finite real eigenvalue α_i is precisely 0 if ℓ_i is even and η_i if ℓ_i is odd, whereas the sign feature of any infinite eigenvalue is 0 if k_i is odd and δ_i if k_i is even. Therefore, we have verified that Theorem 4.5 is coherent with Theorem 4.9.

5. Perturbation theory and sign features: a local conservation rule

Theorem 4.5 can be interpreted as a *global* conservation law. If the Hermitian matrix polynomial $P(x)$ is perturbed, then the sum of its sign features (for even g) or the sum of its sign features minus the signature of its leading matrix coefficient (for odd g) is preserved.

However, as we will discuss in this section, a stronger result can be proved, that the sign features of a regular Hermitian matrix function are *locally* preserved. Related results are obtained in [9, Section 3.2] in the case of a polynomial with nonsingular leading matrix coefficient. Here we give a more general statement with our own proof. We will also explain why the result is false for singular analytic matrix functions. Then, we will see some application to the perturbation theory of regular Hermitian matrix polynomials, discussing the nontrivial role of the grade.

5.1. Classical results on the smoothness of eigenvalues

Before considering the local conservation results, it is convenient to recall some basic results about the smoothness of the eigenvalues of a matrix. It is known that, for analytic perturbations, non-analyticity can only occur when eigenvalues coalesce [17, Ch. II]. Clearly, the analysis can be reduced to the problem of determining the smoothness of the roots of a polynomial for which we have the following well-known result.

Theorem 5.1 *(Theorem A in [13]). Let $p(z) = z^n + \sum_{i=0}^{n-1} a_i z^i$ be a monic polynomial with complex coefficients and with roots $r_1, \ldots, r_n$. Moreover, denote by $\sim$ the equivalence class on $\mathbb{C}^n$ defined by $v_1 \sim v_2$ if and only if v_2 is a permutation of v_1. Then the function that maps the coefficients of $p(z)$ to its roots is a homeomorphism, when seen as a function from $\mathbb{C}^n$ to $\mathbb{C}^n/\sim$.*

In [13, Theorem A] the Euclidian topology on $\mathbb{C}^n$ is used, whereas on $\mathbb{C}^n/\sim$ the quotient topology is employed [18, pp. 94–99]. An entirely different question is whether one can obtain an *inverse function theorem*, i.e., whether one can label n continuous functions $r_i(a_0, \ldots, a_{n-1})$, $i = 1, \ldots, n$, such that $p(z) = \prod_{i=1}^n (z - r_i(a_0, \ldots, a_{n-1}))$. In general, the answer to this question is negative, as shown by the example $p(z) = z^2 - x$ for a complex parameter x. Two important exceptions are discussed in [17, Section II.5.2]. First, if all the coefficients of $p(z)$ depend continuously on a single *real* parameter t, then one can pick n continuous functions of t to represent the roots [17, Theorem 5.2]. Furthermore, if the coefficients of the polynomial depend analytically on t, then the n functions are analytic as well. The second important exception is when all the roots are real, or more generally, as our presentation will illustrate, when they lie on any set where the topology induced by the Euclidean topology on $\mathbb{C}$ becomes an order topology, e.g., a simple and open curve. Essentially, the key property is the ability to continuously reorder an n-tuple. For this we introduce the *reordering map*:

$$v = \{v_1, \ldots, v_n\} \in \mathbb{R}^n/\sim \;\mapsto\; \chi(v) = \begin{bmatrix} v_{\sigma(1)} & \cdots & v_{\sigma(n)} \end{bmatrix}^T \in \mathbb{R}^n,$$

where σ is any permutation of $\{1, \ldots, n\}$ such that $v_{\sigma(1)} \geq \cdots \geq v_{\sigma(n)}$.

Then we have the following theorem which is implicit in [17].

Lemma 5.2. *The reordering map is continuous.*

Proof. Let $(v_m)_m \subset \mathbb{R}^n/\sim$ be any sequence satisfying $\lim_{m\to\infty} v_m = v \in \mathbb{R}^n/\sim$. Denote by ℓ the number of distinct entries in v, i.e., suppose that there exist $w_1 > \cdots > w_\ell \in \mathbb{R}$ such that μ_k entries of v are equal to w_k, with $\sum_{k=1}^{\ell} \mu_k = n$. Let $\delta = \min_{i\neq j} |w_i - w_j|$. Then, since $(v_m)_m$ is a convergent sequence in the quotient topology, given any $0 < \varepsilon < \delta/2$, for m large enough and for any $k = 1, \ldots, \ell$, v_m has exactly μ_k components in the open interval $\mathcal{J}_k = (w_k - \varepsilon, w_k + \varepsilon)$. Then, for any $x_i \in \mathcal{J}_i$, $x_j \in \mathcal{J}_j$, we have $x_i > x_j$ if and only if $i < j$. This holds because the intervals $\mathcal{J}_k$ are disjoint by construction, and because the Euclidean topology on $\mathbb{R}$ is the order topology induced by $<$. (Note that this is not true, e.g., for $\mathbb{C}$.) Therefore, for m large enough, $\chi(v_m)$ is such that its first μ_1 components lie in $\mathcal{J}_1$, the second μ_2 components lie in $\mathcal{J}_2$, and so on. Hence, $\lim_{m\to\infty} \chi(v_m) = \chi(v)$, implying that χ is continuous. □

By the above results, we have the following theorem, stated (without proof) in [17, Section II.5.7].

Theorem 5.3. *Let $A(x, \zeta)$ be an $n \times n$, possibly complex, matrix whose elements depend (jointly) continuously on the real parameters (x, ζ), and such that for any (x, ζ) in a certain domain $\Omega \subset \mathbb{R}^2$ all the eigenvalues of $A(x, \zeta)$ are real. Then there exist n jointly continuous functions $f_j(x, \zeta)$, $j = 1, \ldots, n$, that are the eigenvalues of $A(x, \zeta)$ for all $(x, \zeta) \in \Omega$.*

Proof. For the proof it suffices to compose two continuous functions: the map from (x, ζ) to the real coefficients of the characteristic polynomial of A, and the map from those coefficients to the (ordered) n-tuple $\chi(f_1(x, \zeta), \ldots, f_n(x, \zeta)) \in \mathbb{R}^n$ of the eigenvalues of $A(x, \zeta)$, which is continuous by [13, Theorem A] and Lemma 5.2. □

Remark 5.4. Another interesting question is whether in the case that the coefficients of a monic polynomial are jointly analytic functions of two real parameters (x, ζ), we can find n jointly *analytic* functions $f_1(x, \zeta), \ldots, f_n(x, \zeta)$, that are the roots of the polynomial at each point. The answer is again negative as the example $p(z) = z^2 - 3xz + 2x^2 - \zeta^2(x-1)^2$ demonstrates, see [17, Section II.5.7] for further remarks and examples.

Note that, by Rellich's Theorem 2.1, for any *fixed* ζ and for any polynomial whose coefficients depend jointly analytically on x and ζ, e.g., the characteristic polynomial of an Hermitian matrix function, we can find two eigenvalue functions that are analytic *in* x, and vice versa for any fixed x we obtain analytic eigenvalue functions in ζ. Unfortunately, unlike for complex holomorphic functions, in the real case this condition does *not* imply that we have n jointly analytic functions, as the standard counterexample [20] $f(x, \zeta) = \dfrac{2\, x\, \zeta}{x^2 + \zeta^2}$, $f(0, 0) = 0$ shows. Indeed, the latter function is separately analytic on $\mathbb{R}^2$, but not even jointly continuous at $(0, 0)$.

In the next subsection we will expand on this discussion and derive some perturbation results for regular Hermitian functions.

5.2. Perturbation theory for regular Hermitian matrix functions

To derive our perturbation analysis for regular Hermitian matrix functions, it is convenient to introduce some further notation. Let $\lambda \in \Omega \subseteq \mathbb{R}$, and $\delta > 0$ be such that $\mathcal{J} := [\lambda - \delta, \lambda + \delta] \subset \Omega$. For any nonzero $f(x)$, that is analytic in Ω and such that $f(\lambda - \delta)f(\lambda + \delta) \neq 0$, we define the *local type of f in the interval $\mathcal{J}$* to be the ordered pair

$$\big(\operatorname{sign} f(\lambda - \delta), \operatorname{sign} f(\lambda + \delta)\big).$$

Note that since $\mathcal{J}$ is compact, the function $f(x)$ can only have finitely many roots in $\mathcal{J}$. Observe furthermore that, by continuity, the local type of a function determines the parity of the number of roots of odd multiplicity that $f(x)$ has in $\mathcal{J}$. It also determines the associated sign characteristics at such roots, i.e., the sign of the first nonzero derivative evaluated at the roots of odd multiplicities. More specifically we have the following result.

Proposition 5.5. *Consider a function $f(x)$ that is analytic in the interval $\mathcal{J}$. Then, the following statements on the sign characteristics of the roots of f in $\mathcal{J}$ with odd multiplicity hold.*

1. *If the local type of f in $\mathcal{J}$ is $(+,+)$, then f has an even number of roots of odd multiplicity in $\mathcal{J}$. Moreover, the sign characteristics at such roots (if any) alternate in sign starting with -1, i.e., they are $-1, 1, -1, \ldots, 1$.*
2. *If the local type of f in $\mathcal{J}$ is $(+,-)$, then f has an odd number (in particular, at least one) of roots of odd multiplicity in $\mathcal{J}$. Moreover, the sign characteristics at such roots alternate in sign starting with -1, i.e., they are $-1, 1, -1, \ldots, -1$.*
3. *If the local type of f in $\mathcal{J}$ is $(-,+)$, then f has an odd number (in particular, at least one) of roots of odd multiplicity in $\mathcal{J}$. Moreover, the sign characteristics at such roots alternate in sign starting with 1, i.e., they are $1, -1, 1, \ldots, 1$.*
4. *If the local type of f in $\mathcal{J}$ is $(-,-)$, then f has an even number of roots of odd multiplicity in $\mathcal{J}$. Moreover, the sign characteristics at such roots (if any) alternate in sign starting with 1, i.e., they are $1, -1, 1, \ldots, -1$.*

Proof. We only give a proof of item 1., as the other cases are analogous. The argument can be best followed by considering Fig. 1 below.

Since f is analytic, it is in particular continuous. Thus, each time that f has a root of odd multiplicity at a point, say, $x_0 \in J$, then it must have opposite signs in an interval containing real numbers strictly smaller than x_0 and in an interval containing real numbers strictly larger than x_0. Conversely, for any root of even multiplicity, say, x_1, there exists a neighborhood of x_1 such that f is constant in sign. Now suppose that $f(x)$ has some roots of odd multiplicity in $\mathcal{J}$, as otherwise there is nothing to prove. Let r be the smallest one. Since $f(x) > 0$ at the left endpoint of $\mathcal{J}$, and since there are no roots of

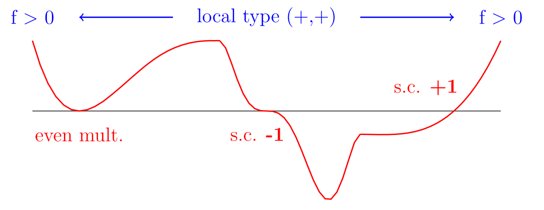

Fig. 1. An analytic function $f(x)$ having local type $(+,+)$ in the interval $\mathcal{J}$.

Table 1
Rules for the local sum of sign features according to the local type.

Local type of f in $\mathcal{J}$	Sum of sign features of the roots of f in $\mathcal{J}$
$(+,+)$	0
$(+,-)$	-1
$(-,+)$	1
$(-,-)$	0

odd multiplicity smaller than r, we have $f(x) > 0$ in a left neighborhood of r, and hence, $f(x) < 0$ in a right neighborhood of r. Therefore, expanding $f(x) = \sum_{k=m}^{\infty} c_k(x-r)^k$ for some odd m, we see that necessarily $c_m < 0$, proving that the sign characteristic at r is -1. Repeating the argument yields the fact the sign characteristics at the roots of odd multiplicity must alternate in sign, whereas the fact that $f(x) > 0$ at the right endpoint of $\mathcal{J}$ guarantees that the largest such root must have sign characteristic $+1$, and hence, there are an even number of roots of odd multiplicity. $\square$

Note that, generally, from the local type nothing can be inferred about the roots with even multiplicities. Nonetheless, using Proposition 5.5, we can associate any local type with a specific value of the sum of the sign features over all the roots of f that lie in the interval $\mathcal{J}$. The cases are summarized in Table 1.

The following results illustrate why the local types are a useful tool for studying the local sum of sign features on a given interval.

Proposition 5.6. *Let $H(x) \in \mathcal{A}_n(\Omega)$ be a regular Hermitian matrix function, and let $\eta_j(x)$, for $j = 1, 2, \dots, n$, be the zeros of the polynomial $p(z) = \det(H(x) - zI)$ considered as functions of x. Let $\mathcal{J} = [a, b] \subset \Omega$ be an interval with the property that $\det H(a) \det H(b) \neq 0$. Let q_i, $i = 1, \dots, 4$, be the number of $\eta_j(x)$ that are, resp., positive at a, negative at a, positive at b, negative at b. Then the local sum of all the sign features of $H(x)$ in $\mathcal{J}$ is equal to $q_3 - q_1$ and to $q_2 - q_4$.*

Proof. Clearly, the $\eta_j(x)$ are the diagonal elements $d_{jj}(x)$ in the Rellich decomposition of $H(x)$. Observe that by Rellich's Theorem 2.1 the condition $\det H(a) \det H(b) \neq 0$ implies $\eta_j(a)\eta_j(b) \neq 0$, for all $j = 1, \dots, n$. Denote by $\nu_{(\cdot,\cdot)}$ the number of $\eta_j(x)$, $j = 1, \dots, n$ that are of type $(\cdot, \cdot)$ in $\mathcal{J}$. Observe that, by definition, the q_i are simply related to the local types by the following formulae, subject to the constraints $q_1 + q_2 = q_3 + q_4 = n$:

$$q_1 = \nu_{(+,+)} + \nu_{(+,-)}, \qquad q_2 = \nu_{(-,+)} + \nu_{(-,-)},$$

$$q_3 = \nu_{(-,+)} + \nu_{(+,+)}, \qquad q_4 = \nu_{(+,-)} + \nu_{(-,-)}.$$

By Table 1, the local sum of the sign features of $H(x)$ in $\mathcal{J}$ is equal to the number of $\eta_j(x)$ that are of type $(-,+)$ in $\mathcal{J}$ minus the number of $\eta_j(x)$ that are of type $(+,-)$ in $\mathcal{J}$. The statement follows immediately. □

We then obtain the following perturbation result.

Theorem 5.7. *Let $H(x) \in \mathcal{A}_n(\Omega)$ be regular and Hermitian, let $E(x) \in \mathcal{A}_n(\Omega)$ be Hermitian, and suppose that $\lambda \in \Omega \subseteq \mathbb{R}$ is a real eigenvalue of $H(x)$ with geometric multiplicity $\ell \leq n$, partial multiplicities $m_1^\lambda, \ldots, m_\ell^\lambda$, and associated sign features $\phi_1^\lambda, \ldots, \phi_\ell^\lambda$. For any $\zeta > 0$ consider the function $\hat{H}(x) := H(x) + \zeta E(x) \in \mathcal{A}_n(\Omega)$. Then, for a sufficiently small but fixed ζ, there exists an interval $\mathcal{J} \subset \Omega$ such that*

(i) $\lambda \in \mathcal{J}$;
(ii) the following conservation law holds:

$$\sum_{j=1}^{p} \sum_{i=1}^{\ell_j} \phi_{i,j}^{\hat{\lambda}_j} = \sum_{j=1}^{\ell} \phi_j^\lambda,$$

where $\hat{\lambda}_1, \ldots, \hat{\lambda}_p$ are the eigenvalues of $\hat{H}(x)$ lying in $\mathcal{J}$, each with geometric multiplicity ℓ_j, partial multiplicities $m_{i,j}^{\hat{\lambda}_j}$ and sign features $\phi_{i,j}^{\hat{\lambda}_j}$, for $i = 1, \ldots, \ell_j$.

Proof. Denote by $\eta_j(x)$ the zeros of the polynomial $p(z) = \det(H(x) - zI)$ considered as functions of x. Clearly these are the functions $d_{jj}(x)$ in the Rellich decomposition of $H(x)$ and thus, the $\eta_j(x)$ are analytic functions of x, and the sign characteristics at λ are the signs of $a_{m_j^\lambda}$ in the series $\eta_j(x) = \sum_{i=m_j^\lambda}^{\infty} a_i(x-\lambda)^i$, whenever $m_j^\lambda > 0$, i.e., λ is an eigenvalue of $H(x)$ of partial multiplicity m_j^λ. We denote these signs by ϵ_j^λ. Now consider the perturbed Hermitian matrix function $\hat{H}(x) = H(x) + \zeta E(x)$, and let $q(z) = \det(H(x) + \zeta E(x) - zI)$.

By Theorem 5.3, we know that we can label n jointly continuous functions $f_j(x, \zeta)$ such that for any (x, ζ) in $\Omega \times \mathbb{R}$ they are the roots of $q(z)$, i.e., the eigenvalues of $\hat{H}(x, \zeta)$. Rellich's Theorem 2.1 and the uniqueness of the set of the eigenvalues of a square matrix guarantee the following fact.

Remark 5.8. For any fixed ζ, there are n functions $\hat{\eta}_j(x; \zeta)$, analytic in x, with $\hat{\eta}_j(x; 0) := \eta_j(x)$; and for any $x \in \Omega$, there exists a permutation σ (possibly depending on x) such that $f_{\sigma(j)}(x, \zeta) = \hat{\eta}_j(x; \zeta)$.

Now suppose without loss of generality that $\eta_j(\lambda) = 0$ if and only if $j \le \ell$, (this can be achieved via a relabeling of the n analytic functions η_j). Observe now that there exist δ, δ' such that, defining $\mathcal{J} := [\lambda - \delta, \lambda + \delta] \subset \Omega$, the following conditions are satisfied.

1. For any $j \le \ell$,
   - if m_j^λ is odd and $\epsilon_j^\lambda = 1$, then $\eta_j(x)$ is of local type $(-,+)$ in $\mathcal{J}$;
   - if m_j^λ is even and $\epsilon_j^\lambda = 1$, then $\eta_j(x)$ is of local type $(+,+)$ in $\mathcal{J}$;
   - if m_j^λ is odd and $\epsilon_j^\lambda = -1$, then $\eta_j(x)$ is of local type $(+,-)$ in $\mathcal{J}$;
   - if m_j^λ is even and $\epsilon_j^\lambda = -1$, then $\eta_j(x)$ is of local type $(-,-)$ in $\mathcal{J}$.
2. For any $j > \ell$,
   - if $\eta_j(\lambda) > 0$, then $\eta_j(x)$ is of local type $(+,+)$ in $\mathcal{J}$;
   - if $\eta_j(\lambda) < 0$, then $\eta_j(x)$ is of local type $(-,-)$ in $\mathcal{J}$.
3. For any $\zeta < \delta'$, there are two permutations σ_- and σ_+ such that for any $j = 1, \ldots, n$ the following conditions hold:
   - $\operatorname{sign} f_{\sigma_-(j)}(\lambda - \delta, \zeta) = \operatorname{sign} \eta_j(\lambda - \delta)$, and
   - $\operatorname{sign} f_{\sigma_+(j)}(\lambda + \delta, \zeta) = \operatorname{sign} \eta_j(\lambda + \delta)$.

That condition 2. can be satisfied follows by continuity in x: since for any $j > \ell$ $\eta_j(\lambda) \neq 0$, there exists a δ_j such that $|x - \lambda| < \delta_j \Rightarrow \eta_j(x)\eta_j(\lambda) > 0$. Similarly, that for a fixed $j \le \ell$ there exists a δ_j such that condition 1. can be satisfied follows from the analyticity of $\eta_j(x)$ and Definition 2.3. Note that we assume that, for all j, λ is a root of finite multiplicity for $\eta_j(x)$, i.e., $\eta_j(x) \not\equiv 0$. If this is not the case, then all the coefficients in the Taylor series are zero, and clearly no δ can be found such that $\eta_j(x)$ is of any local type. Thus, we can set $\delta := \min_j \delta_j$.

Finally, the existence of δ' follows by continuity in ζ of the $f_j(x, \zeta)$, by Remark 5.8, that implies that there exist permutations σ_- and σ_+ such that $\eta_j(\lambda - \delta) = f_{\sigma_-(j)}(\lambda - \delta, 0)$ and $\eta_j(\lambda + \delta) = f_{\sigma_+(j)}(\lambda + \delta, 0)$.

Let us now fix $\zeta_0 \in [0, \delta')$. Let $\hat{\eta}_j(x; \zeta_0)$ denote the zeros of $\det(H(x) + \zeta_0 E(x) - zI)$. By Remark 5.8, we know that for any fixed $x \in \mathcal{J}$ we can find a permutation σ' such that $\hat{\eta}_j(x; \zeta_0) = f_{\sigma'(j)}(x, \zeta_0)$.

Consider now the quantities q_i, defined as in the statement of Proposition 5.6 on $\hat{\eta}_j(x; \zeta)$ with $\mathcal{J} = [\lambda - \delta, \lambda + \delta]$. Clearly, q_i are integer-valued functions of ζ. Observe that

$$q_1(\zeta) = \sum_{j:\ \hat{\eta}_j(\lambda - \delta; \zeta) > 0} 1, \quad q_2(\zeta) = \sum_{j:\ \hat{\eta}_j(\lambda - \delta; \zeta) < 0} 1,$$

$$q_3(\zeta) = \sum_{j:\ \hat{\eta}_j(\lambda + \delta; \zeta) > 0} 1, \quad q_4(\zeta) = \sum_{j:\ \hat{\eta}_j(\lambda + \delta; \zeta) < 0} 1.$$

Hence, by the argument above and by conditions 1., 2. and 3., $q_i(0) = q_i(\zeta_0)$ for any $0 \le \zeta_0 < \delta'$. Invoking Proposition 5.6 concludes the proof. $\square$

Remark 5.9. We stress once more that it is crucial here that any $\eta_j(x)$ with $\eta_j(\lambda) = 0$ has a zero of finite multiplicity at λ, i.e., $\eta_j(x)$ is not identically zero. This is equivalent to the assumption that $H(x)$ is regular and is used to prove condition 2. in the proof of Theorem 5.7. If this does not hold, i.e., suppose without loss of generality $\eta_1(x) \equiv 0$, then $\eta_1(x, \zeta)$ may assume an arbitrary sign for all $x \in \Omega$, suggesting that no local conservation laws are possible. Indeed, this is illustrated by the following example which is a Hermitian variant of an unstructured example that appeared in [8, Equation 2].

Example 5.10. Let $H(x) = \left[\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then $\eta_1(x) = x, \eta_2(x) = 0$. The only eigenvalue is $\lambda = 0$, with geometric multiplicity 1, partial multiplicity 1 and sign feature 1. Consider the perturbation $E(x) = \left[\begin{smallmatrix} 0 & x-1 \\ x-1 & 0 \end{smallmatrix}\right]$. Then one can check that $2f_1(x, \zeta) = x + \sqrt{x^2 + 4\zeta^2(x-1)^2}$ and $2f_2(x, \zeta) = x - \sqrt{x^2 + 4\zeta^2(x-1)^2}$. For any arbitrarily small $\zeta > 0$ we see that neither $f_1(x, \zeta)$ nor $f_2(x, \zeta)$ have a root in a neighborhood of $x = 0$. Moreover, $f_2(x, \zeta)$ has a root of multiplicity 2 at $x = 1$. Therefore, the sum of the sign features is not locally preserved at 0.

Remark 5.11. One may wonder if the sum of sign features associated with *each* partial multiplicity is locally preserved. The answer is clearly negative, as is illustrated by the Hermitian matrix function $\left[\begin{smallmatrix} 0 & x \\ x & 0 \end{smallmatrix}\right]$, which has partial multiplicities $1, 1$ at the eigenvalue 0, with sign features 1 and -1. We can perturb it to $\left[\begin{smallmatrix} \zeta & x \\ x & 0 \end{smallmatrix}\right]$ which for any $\zeta > 0$ has partial multiplicity 2 at the eigenvalue 0, with sign feature 0.

Finally, the subtleties described in Section 5.1 are key in arguing that, in a sense made precise by Theorem 5.12, the geometric multiplicity of an eigenvalue cannot locally increase by a small perturbation. Note that Theorem 5.12 is based on Theorem 5.3, and hence, holds more generally for matrices (not necessarily Hermitian) depending continuously on a parameter. However, for simplicity we state it only for the special case that we need.

Theorem 5.12. *With the notation of Theorem 5.7, denote by $\hat{\eta}_j(x; \zeta)$ the eigenvalue functions of $H(x) + \zeta E(x)$. Then for small enough ζ there exists an interval $\mathcal{J}(\zeta)$ containing λ such that the number of $\hat{\eta}_j(x; \zeta)$ that have roots in $\mathcal{J}(\zeta)$ is not larger than ℓ, where ℓ is the geometric multiplicity of λ as an eigenvalue of the regular analytic matrix function $H(x)$.*

Proof. Using the notation of the proof of Theorem 5.7, suppose without loss of generality that $\eta_j(\lambda) = 0$ if and only if $j \leq \ell$. Let $\rho := \min_{j>\ell} |\eta_j(\lambda)| > 0$. By Theorem 5.3, the eigenvalues of $H(x) + \zeta E(x)$ are jointly continuous in ζ and x. Hence, by Remark 5.8, we deduce that there exists $\delta > 0$ such that, for all x, ζ satisfying $(x - \lambda)^2 + \zeta^2 < \delta^2$, and for all $j > \ell$, there is a permutation σ yielding $|f_{\sigma(j)}(x, \zeta)| > \rho/2$.

It follows that for any $\zeta < \delta$ there is an interval $\mathcal{J}(\zeta)$ containing λ with the property that at most ℓ eigenvalue functions of $H(x) + \zeta E(x)$ can have roots in the interval $\mathcal{J}(\zeta)$. $\square$

We stress that the results in this section imply that, at a finite real point, a set of real eigenvalues can be removed from the real line by a Hermitian perturbation if and only if the sum of their sign features is 0. This observation will be important in Subsection 5.4, and we give a proof in the next Theorem.

Theorem 5.13. *Let $H(x) \in \mathcal{A}_n(\Omega)$ be regular and Hermitian, let $E(x) \in \mathcal{A}_n(\Omega)$ be Hermitian, and suppose that $\lambda \in \Omega \subseteq \mathbb{R}$ is a real eigenvalue of $H(x)$ with geometric multiplicity $\ell \leq n$, and associated sign features $\phi_1^\lambda, \dots, \phi_\ell^\lambda$. Suppose that for a sufficiently small but fixed $\zeta > 0$, the Hermitian matrix function $\hat{H}(x) := H(x) + \zeta E(x) \in \mathcal{A}_n(\Omega)$ does not have any real eigenvalue in any small enough neighborhood of λ. Then,*

$$\sum_{i=1}^{\ell} \phi_i^\lambda = 0.$$

Conversely, suppose that λ is a real eigenvalue of $H(x)$ satisfying $\sum_{i=1}^{\ell} \phi_i^\lambda = 0$. Then, there exist a Hermitian $E(x) \in \mathcal{A}_n(\Omega)$ and an interval $\mathcal{J} \ni \lambda$, $\mathcal{J} \subset \Omega$, such that for sufficiently small values of $\zeta > 0$ the Hermitian $\hat{H}(x) := H(x) + \zeta E(x) \in \mathcal{A}_n(\Omega)$ does not have any real eigenvalue in $\mathcal{J}$.

Proof. By Theorem 5.7, there exists an interval $\mathcal{J} \subset \Omega$ such that $\lambda \in \mathcal{J}$ and the sum of the sign features of $H(x)$ at λ is equal to the sum of the sign features of $\hat{H}(x)$ over all its eigenvalues in $\mathcal{J}$; moreover, a careful look at the proof of Theorem 5.7 reveals that the same statement holds for any subinterval of $\mathcal{J}$ still containing λ. In particular, by assumption, we can take $\mathcal{J}$ to be small enough not to contain any real eigenvalue of $\hat{H}(x)$. Therefore,

$$\sum_{i=1}^{\ell} \phi_i^\lambda = 0.$$

To prove the converse implication, in view of Theorem 2.1 we may, without loss of generality, take $H(x) = D(x)$, where $D(x)$ is diagonal. We will explicitly construct the sought perturbation $E(x)$, showing that it actually can be chosen to be a constant matrix. Suppose that λ is a real eigenvalue of $H(x)$ and that the sum of its sign features is 0. Then, for each $d_{ii}(x)$ having λ as a zero, either the partial multiplicity m_i^λ is even, or it is odd but paired with another $d_{jj}(x)$ with λ as a zero also of odd partial multiplicity, and with opposite sign characteristic (feature). In the case of even partial multiplicity, suppose $d_{ii}(x) = (x - \lambda)^{m_i^\lambda} h_i(x)$, where $h_i(x)$ is analytic and satisfies $h_i(\lambda) \neq 0$. Perturb $d_{ii}(x)$ to $\hat{d}_{ii}(x, \zeta) = d_{ii}(x) + \zeta h_i(\lambda)$ with ζ real and positive. For a sufficiently small ζ, $\hat{d}_{ii}(x, \zeta)$ does not have a real zero in a small enough neighborhood of λ. In the case of odd partial multiplicity, we pair $d_{ii}(x)$ with $d_{jj}(x)$ described above. Let $d_{ii}(x) = (x - \lambda)^{m_i^\lambda} h_i(x)$ and $d_{jj}(x) = (x - \lambda)^{m_j^\lambda} h_j(x)$ with both m_i^λ and m_j^λ odd and $h_i(\lambda) h_j(\lambda) < 0$. Perturb $\operatorname{diag}(d_{ii}(x), d_{jj}(x))$ to

$$\begin{bmatrix} d_{ii}(x) & \zeta \\ \zeta & d_{jj}(x) \end{bmatrix}.$$

The eigenvalues of the perturbed 2×2 matrix are the roots of

$$(x - \lambda)^{m_i^\lambda + m_j^\lambda} h_i(x) h_j(x) = \zeta^2$$

Since $m_i^\lambda + m_j^\lambda$ is even and $h_i(\lambda)h_j(\lambda) < 0$, then for a real and sufficiently small ζ all the eigenvalues in a small enough complex neighborhood of λ are nonreal. □

Remark 5.14. Note that, in the notation of Theorems 5.7, 5.12 and 5.13, for sufficiently small values of ζ, $\hat{H}(x)$ is also regular, since $H(x)$ is regular.

5.3. Perturbation theory of infinite eigenvalues for regular Hermitian matrix polynomials

In this section we discuss the local invariants at infinity for a regular Hermitian matrix polynomial $P(x)$. Assume that the perturbation $E(x)$ is also polynomial. Note that in this situation the most natural choice for the grade might *not* be $\deg P(x)$, but $\max(\deg P(x), \deg E(x))$, see also [28].

By Theorem 5.7, we know that, for any $\lambda \in \mathbb{R}$, there exists an interval $\mathcal{J}$ containing λ such that, for ζ small enough, the sum of the sign features for all eigenvalues of the perturbed polynomial $P(x) + \zeta E(x)$ that lie in $\mathcal{J}$ is equal to the sum of the sign features over all the partial multiplicities of λ seen as an eigenvalue of $P(x)$.

To simplify expressions, we rephrase this property as the statement: *the sum of the sign features is locally preserved on* $\mathbb{R}$. The question is whether we can extend this statement to a neighborhood of ∞, or at least whether we can find another local invariant at infinity.

If the grade is even, then this is straightforward. By Theorem 5.7 applied to $S(x) = -\operatorname{rev}_g P(x)$, the sum of the sign features at 0 of $S(x)$ is locally preserved. But by Theorem 3.4 and Theorem 3.5, the sign features of small eigenvalues of a perturbed $S(x) - \zeta \operatorname{rev}_g E(x)$ are precisely the same of those of large eigenvalues of $P(x) + \zeta E(x)$. Hence, the sum of the sign features of $P(x)$ is preserved in a neighborhood of infinity, i.e., in $(-\infty, -M) \cup (M, \infty)$ for sufficiently large $M > 0$, and thus we have the following theorem.

Theorem 5.15. *If $P(x)$ is a regular Hermitian matrix polynomial of even grade, then the sum of the sign features is locally preserved on $\mathbb{R} \cup \{\infty\}$, i.e., it is locally preserved at any $\lambda \in \mathbb{R}$* and *at $\lambda = \infty$ as well.*

We note that a reverse implication also holds, by applying Theorem 5.13 to $S(x)$ around the eigenvalue 0.

On the other hand, if the grade is odd, it is hopeless to have a local conservation of the sign features near ∞. Indeed, going to the reversal, what must be locally preserved is the sum of sign characteristics (or sign features) associated with the odd multiplicities corresponding to $S(x)$. In particular, for the eigenvalue zero of $S(x)$ and the eigenvalue ∞ of $P(x)$, the sign features corresponding to the former are associated with odd partial multiplicities whereas those corresponding to the latter are associated with even partial multiplicities. The mapping laws of the sign characteristics prescribed in Theorem 3.4 and Theorem 3.5 depend on which neighborhood of infinity (left or right) one considers. The only way to express a local conservation rule in a neighborhood of infinity is to go back to the sign features of the reversal $S(x)$. Unfortunately this does not yield a statement as nice as in the case of even grade.

Theorem 5.16. *Let $P(x)$ be a regular Hermitian matrix polynomial of odd grade and let the sign characteristic at infinity be defined as in Definition 2.8. Then in a neighborhood of ∞ the sum*

$$\sum_{\lambda\in\Lambda_{\mathbb{R}}(P),\lambda>0}\ \sum_{m_i^\lambda \text{ odd}} \epsilon_i^\lambda - \sum_{\lambda\in\Lambda_{\mathbb{R}}(P),\lambda<0}\ \sum_{m_i^\lambda \text{ odd}} \epsilon_i^\lambda + \sum_{m_i^\infty \text{ odd}} \epsilon_i^\infty$$

is locally preserved.

In other words, given any Hermitian polynomial $E(x)$ of the same grade as $P(x)$, there exists $M>0$ such that, for any small enough ζ, if $\epsilon_i^{\hat\lambda}$ are the sign characteristics associated with the eigenvalue $\hat\lambda$ arising from a slightly perturbed Hermitian polynomial $\hat P := P(x)+\zeta E(x)$ and ϵ_i^∞ are the sign characteristics of $P(x)$ associated with the eigenvalue ∞, then we have

$$\sum_{\hat\lambda\in\Lambda(\hat P),\hat\lambda>M}\ \sum_{m_i^{\hat\lambda} \text{ odd}} \epsilon_i^{\hat\lambda} - \sum_{\hat\lambda\in\Lambda(\hat P),\hat\lambda<-M}\ \sum_{m_i^{\hat\lambda} \text{ odd}} \epsilon_i^{\hat\lambda} + \sum_{\hat m_i^\infty \text{ odd}} \hat\epsilon_i^\infty = \sum_{m_i^\infty \text{ odd}} \epsilon_i^\infty,$$

where in abuse of notation $\hat\epsilon_i^\infty$ denote the sign characteristics of the eigenvalues of the perturbed polynomial that stay at ∞ and $\hat m_i^\infty$ are the corresponding partial multiplicities.

Proof. By definition, the sign characteristics at infinity of $P(x)$ are those of $S(v) = -v^g P(1/v)$ at 0. Applying Theorems 3.4 and 3.5, we see that for an odd partial multiplicity m the sign characteristics of $P(x)$ at a large λ are equal to (resp. opposite to) those of $S(v)$ at a small λ^{-1} if and only if $\lambda>0$ (resp. $\lambda<0$). Applying Theorem 5.7 to $S(v)$ and the appropriate neighborhood of 0 (that is mapped to a neighborhood of infinity for $P(x)$), and recalling that the sign features correspond to the sign characteristics for the odd partial multiplicities (and are 0 for the even partial multiplicities), the assertion follows. $\square$

Similarly, a reverse implication holds as a consequence of Theorem 5.13 applied to the antireversal polynomial.

The presented analysis shows that our definition of sign features, that lead to the global constraint of Theorem 4.5, fits well with the local conservation rule at infinity if and only if the grade is even. When the grade is odd, things are more complicated. This is not a defect of our definition, but a necessary consequence of the fact that, for odd grade, the signature of the leading matrix coefficient is involved in the signature constraint theorem. This makes it impossible to obtain a definition that works well both globally and locally.

There are two possible ways out of this global/local dichotomy for odd grade Hermitian matrix polynomials. Either one always forces the grade to be even by adding another zero coefficient, at the price of allowing a larger set of perturbations (including perturbation to the zero leading matrix coefficient), or one uses Theorem 5.16, at the price of having a much less elegant and more complicated rule. We give a few examples to illustrate these facts.

Example 5.17. Consider

$$P(x) = \begin{bmatrix} x^3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{bmatrix}$$

of grade 3. Note that $P(x)$ has an eigenvalue 0 of multiplicity 3 with sign feature 1, an eigenvalue 0 of multiplicity 1 with sign feature 1, an eigenvalue ∞ of multiplicity 3 with sign feature 0 and sign characteristic -1, and an eigenvalue ∞ of multiplicity 2 with sign feature -1. As shown in Theorem 4.5, the global sum of the sign features is 1. However, *any perturbation, however small, can change the signature of the leading matrix coefficient*. Suppose that there is a finite open cover of the compactification of the real line such that in each open subset of the cover there is a local conservation rule for the sum of the sign features. This would violate Theorem 4.5: to see this, take a perturbation that changes the signature of the leading matrix coefficient. Hence, there cannot be such an open cover. On the other hand, by Theorem 5.7, the sum of sign features is locally preserved on all $\mathbb{R}$. Therefore, there must be a possible exception at infinity, i.e., there cannot be any open subset of infinity that allows for a local conservation law of sign features. This is illustrated by

$$\hat{P}(x) = \begin{bmatrix} x^3 & 0 & 0 \\ 0 & 1-\zeta^3 x^3 & 0 \\ 0 & 0 & x-\zeta^2 x^3 \end{bmatrix}.$$

Note that neither the partial multiplicities nor the sign features of the zero eigenvalue are changed by this particular perturbation. However, for any $\zeta > 0$, there exist two real eigenvalues $\frac{1}{\zeta}$ each of partial multiplicity 1 with sign feature -1, and a real eigenvalue $(-\frac{1}{\zeta})$ of partial multiplicity 1 with sign feature -1. The global sum of sign features is now -1, as expected, since the signature of the leading matrix coefficient has changed. Yet, no matter how small $\zeta > 0$, the sum of sign features in a neighborhood of infinity is $-3 \neq -1$. Note that this example is coherent with Theorem 5.16, since $-\frac{1}{\zeta} < 0$ and hence we must multiply its sign characteristics by -1 in the summation of the statement of Theorem 5.16.

If we had picked even grade, say, 4, for $P(x)$ and $\hat{P}(x)$, then we would have a local conservation law at infinity of sign features, as predicted by Theorem 5.15. Indeed with this choice of the grade, the sum of sign features at infinity for P is -2, whereas $\hat{P}(x)$ has three extra simple eigenvalues at infinity, with sign features -1, 1, and 1, so that in a neighborhood of infinity the sum is still -2.

Example 5.18. Let $p(x) = 1$ have grade 1, i.e., it has a simple infinite eigenvalue with sign feature 0, and sign characteristic -1. Then any perturbation $\hat{p}(x) = 1 + \zeta_0 + \zeta_1 x$, $\zeta_1 \neq 0$, must have a real eigenvalue. (If $\zeta_1 = 0$, the eigenvalue stays at infinity and its sign feature and sign characteristic cannot change for small enough ζ_0.) Note that the product of the sign of the perturbed eigenvalue and its sign characteristic must be -1, coherently with Theorem 5.16.

Suppose now that we take the grade to be 2, then P has a double infinite eigenvalue with sign feature 0. It can be removed from the compactification of the real line by a degree 2 perturbation such as $\hat{p}(x) = 1 + \zeta x^2$, $\zeta > 0$. The reason why a degree 1, but grade 2, perturbation cannot remove it is that such a perturbation must still have a simple infinite eigenvalue, and hence, a complex conjugate pair of eigenvalues cannot be produced, i.e., it must also have another large real eigenvalue, of opposite sign feature.

Example 5.19. Let $p(x) = x$ of grade 3, then it has a double infinite eigenvalue with sign feature -1 and a simple zero eigenvalue with sign feature 1. However, the perturbation $\hat{p}(x) = x + \zeta x^3$ ($\zeta > 0$) has only one real eigenvalue at 0 and the double infinite eigenvalue has been removed from the compactification of the real line, in spite of having nonzero sign feature, but coherently with Theorem 5.16. Considering the grade to be 4, then originally there was a triple infinite eigenvalue, with sign feature -1. In this case it is impossible to remove all the three eigenvalues (counting multiplicity), although of course we may remove two of them while still locally preserving the sum of sign features: this is precisely what happens with $\hat{p}(x)$.

5.4. Coalescence of simple real eigenvalues

An application of the discussed theory is the analysis of what happens when a regular Hermitian matrix polynomial $P(x)$ has two nearby simple eigenvalues $\lambda_1 \simeq \lambda_2$, both lying on the compactification of the real line $\mathbb{R}^* := \mathbb{R} \cup \{\infty\}$. Clearly, a small Hermitian perturbation can make the two eigenvalues collide at a point $\lambda \in \mathbb{R}^*$. A subtler question is whether they can be removed from the compactification of the real line with a Hermitian perturbation, i.e., does there exist a Hermitian matrix polynomial $\hat{P}(x) = P(x) + \zeta E(x)$ that does not have real eigenvalues in any small enough interval containing λ_1 and λ_2? To clarify, we first consider the case that the point of collision $\lambda \in \mathbb{R}$ is finite. In this case we can apply a special case of Theorem 5.13. Then a pair of colliding finite real eigenvalues can be removed from the real line by a Hermitian perturbation if and only if they have opposite sign characteristics. Indeed, by Theorem 5.7, when we perturb a matrix polynomial then, for small enough perturbations, there is a real neighborhood of an eigenvalue such that the sum of the sign features stays constant. Suppose that $P(x)$ has two nearby simple real eigenvalues, say, $\lambda + \epsilon$ and $\lambda - \epsilon$. Assume moreover that there exists a matrix polynomial $\hat{P}(x)$, a small perturbation of $P(x)$, such that $\hat{P}(x)$ has no real eigenvalue near λ (because the perturbations moved the eigenvalues to some complex conjugate pair). It follows that in any small enough real neighborhood of λ the sum of the sign features of $\hat{P}(x)$ is zero, since there are no real eigenvalues. If we continuously move from $\hat{P}(x)$ to $P(x)$, the eigenvalues will also move continuously in the complex plane, collide at λ, and then separate along the real line. Along this process, the sum of the sign features must remain zero, and hence the two real eigenvalues of $P(x)$ must have *opposite* sign features, and therefore opposite sign characteristics as well, since we are assuming they are simple. Similarly, we can proceed as in the proof of Theorem 5.13 to show that the converse implication also holds.

Let us now analyze the situation when the point is infinity. When the grade is even, infinity is not special at all, so the rule is as usual, that they can be removed if and only if the sum of their sign features, or, equivalently in this case, their sign characteristics is 0, as prescribed by Theorem 5.15. For odd grade, we can apply the more complicated Theorem 5.16 to obtain the following cases.

- If both eigenvalues are finite, large and having the same sign, then they can be removed if and only if the sum of their sign characteristics is 0;
- If both eigenvalues are finite and large, one being positive and the other being negative, then they can be removed if and only if the sum of their sign characteristics is nonzero, i.e., either 2 or -2;
- If one eigenvalue is infinite and the other is finite, large and positive, then they can be removed if and only if the sum of their sign characteristics is 0;
- If one eigenvalue is infinite and the other is finite, large, and negative, then they can be removed if and only if the sum of their sign characteristic is nonzero, i.e., ± 2.

Note that in this case it is the sign characteristics at infinity, and not the sign features, that determine what happens. This is because with odd grade there is no local conservation of the sign features at infinity, and hence, one is forced to go to the reversal, where the sign features at zero correspond to the sign characteristics at zero.

Once again, the conclusion is that giving a simple local conservation law at infinity is not possible. One must either always see things as even grade, or alternatively, rely heavily on Theorems 3.4 and 3.5.

6. Conclusions

We have studied a systematic extension of the definition of sign characteristic for Hermitian matrix polynomials to the eigenvalue ∞. The goal was to achieve a concept that is uniform with the one for finite eigenvalues and that stays valid under small perturbations. For matrix polynomials of even grade (degree) we have realized this goal, while for odd grade we have argued that the task seems to be not possible, except if one resorts to increasing the grade to an even number. We have studied the change of sign characteristics under analytic re-parameterizations and multiplication by scalar functions, and we have shown a sign constraint theorem and studied the invariance of this result under perturbations.

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