The Cyclically Presented Groups with Relators $x_i x_{i+k} x_{i+l}$

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Abstract

Continuing Cavicchioli, Repovš, and Spaggiari's investigations into the cyclic presentations $\langle x_1, \ldots, x_n \mid x_i x_{i+k} x_{i+l} = 1\ (1 \leq i \leq n)\rangle$ we determine when they are aspherical and when they define finite groups; in these cases we describe the groups' structures. In many cases we show that if the group is infinite then it contains a non-abelian free subgroup.

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1 Introduction

In this paper we consider the cyclic presentations

$$\mathcal{P}_n(k,l) = \langle x_1,\ldots,x_n \mid x_i x_{i+k} x_{i+l}\ (1 \le i \le n)\rangle$$

and the groups $G_n(k,l)$ they define (where $1 \le k,l \le n-1$ and subscripts are taken $\bmod n$). We classify the finite groups $G_n(k,l)$ and determine when the presentations $\mathcal{P}_n(k,l)$ are aspherical (that is, when $\pi_2(K) = 0$ where K is the standard 2-dimensional CW-complex associated with $\mathcal{P}$). Similar investigations were carried out in [1],[14] for the cyclic presentations $\mathcal{Q}_n(m,k)$ with relators $x_i x_{i+m} x_{i+k}^{-1}$ and the groups $H_n(m,k)$ they define. (The groups $H_n(m,k)$ were introduced in [5] and generalize Conway's Fibonacci groups $F(2,n)$ and the Sieradski groups $S(2,n)$). It turns out that for $n \ge 10$ the finite groups $G_n(k,l)$ have a richer structure than the finite groups $H_n(m,k)$, which are cyclic.

The presentations $\mathcal{P}_n(k,l)$ and $\mathcal{Q}_n(m,k)$ fit into the more general class of cyclic presentations $\mathcal{G}_n^{(a,b,r,s)}(m,k,h)$ introduced by Cavicchioli, Repovš and Spaggiari in [6]. It is hoped that the results here, together with those in [14], will provide insight into Problem 4.4 of [6] which asks for necessary and sufficient conditions for asphericity of those presentations.

Our main results are the following.

Theorem A. *Suppose $(n,k,l)=1$ and let $\mathcal{P} = \mathcal{P}_n(k,l)$. Then $\mathcal{P}$ is aspherical if and only if $k \ne l$, $k+l \not\equiv 0 \pmod n$, $2l-k \not\equiv 0 \pmod n$, $2k-l \not\equiv 0 \pmod n$, $3l \not\equiv 0 \pmod n$, $3k \not\equiv 0 \pmod n$, $3(l-k) \not\equiv 0 \pmod n$ and either*

(i) $n \ne 18$; or

(ii) $n = 18$ and $k+l \not\equiv 0 \pmod 3$.

Theorem B. *The group $G = G_n(k,l)$ is finite if and only if $(n,k,l)=1$ and one of the following conditions holds:*

(i) $k = l$ in which case $G \cong \mathbb{Z}_s$ where $s = 2^n - (-1)^n$;

(ii) $k \ne l$, $n \not\equiv 0 \pmod 3$ and either $k+l \equiv 0 \pmod n$ or $2l-k \equiv 0 \pmod n$ or $2k-l \equiv 0 \pmod n$ in which case $G \cong \mathbb{Z}_3$;

(iii) $k \ne l$, $k+l \not\equiv 0 \pmod 3$ and either $3l \equiv 0 \pmod n$ or $3k \equiv 0 \pmod n$ or $3(l-k) \equiv 0 \pmod n$ in which case G is metacyclic of order $s = 2^n - (-1)^n$ and we have the metacyclic extension

$$\mathbb{Z}_{s/3} \hookrightarrow G \twoheadrightarrow \mathbb{Z}_3;$$

and the metacyclic extension

$$G' \cong \mathbb{Z}_\beta \hookrightarrow G \twoheadrightarrow \mathbb{Z}_\alpha;$$

where $\alpha = 3(2^{n/3} - (-1)^{n/3})$, $\beta = s/\alpha$.

(A)	(B)	(C)		Aspherical	Abelianization	Group
F	F	F		Yes	finite $\neq 1$	∞
F	F	T		No	$\mathbb{Z}_\alpha$	Metacyclic
F	T	F		No	$\mathbb{Z}_3$	$\mathbb{Z}_3$
T	F	F	$n \neq 18$	Yes	∞	Large
T	F	F	$n = 18$	No	$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_{19}$	$\mathbb{Z} * \mathbb{Z} * \mathbb{Z}_{19}$
T	F	T		No	$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_\gamma$	$\mathbb{Z} * \mathbb{Z} * \mathbb{Z}_\gamma$
T	T	F		No	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$
T	T	T		No	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} * \mathbb{Z}$

Table 1: Summary of results for $(n, k, l) = 1$, $k \neq l$.

Let $d = (n, k, l)$. Then $\mathcal{P}_n(k, l)$ is aspherical if and only if $\mathcal{P}_{n/d}(k/d, l/d)$ is aspherical. (This is why we assume $(n, k, l) = 1$ in Theorem A). Moreover, by Lemma 2.4 of [6] $G_n(k, l)$ is isomorphic to the free product of d copies of the non-trivial group $G_{n/d}(k/d, l/d)$ so $G_n(k, l)$ is infinite when $d > 1$. Furthermore, if $d = 1$ and $k = l$ then an elementary argument using Tietze transformations shows that $G_n(k, l) \cong \mathbb{Z}_s$ where $s = 2^n - (-1)^n$.

We shall state some of our results in terms of the following three conditions:

(A) $n \equiv 0 \pmod 3$ and $k + l \equiv 0 \pmod 3$;

(B) $k + l \equiv 0 \pmod n$ or $2l - k \equiv 0 \pmod n$ or $2k - l \equiv 0 \pmod n$;

(C) $3l \equiv 0 \pmod n$ or $3k \equiv 0 \pmod n$ or $3(l - k) \equiv 0 \pmod n$.

These conditions were derived in part from computational experiments using GAP [8] which was invaluable in formulating our results. Note that if (B) and (C) hold then (A) holds.

It follows that there are precisely seven (out of the possible eight) combinations of (A), (B), (C) being true or false. These are listed in Table 1 where we summarize our results (here $\alpha = 3(2^{n/3} - (-1)^{n/3}), \gamma = (2^{n/3} - (-1)^{n/3})/3)$. In this table ∞ denotes a group of infinite order whose structure is unknown, Metacyclic denotes metacyclic of order $s = 2^n - (-1)^n$, Large denotes a large group (that is, one that has a finite index subgroup that maps homomorphically onto the free group of rank 2). Note also in Table 1 that the 2nd line corresponds to Theorem B(iii); and the 3rd corresponds to Theorem B(ii). Further, the 8th line only occurs when $n = 3$ or 6.

In Section 2 we obtain information about the structure of $G_n(k, l)$ for various combinations of (A),(B),(C) being true or false; in Section 3 we study the metacyclic case (Theorem B(iii)); in Section 4 we prove Theorem A and make other remarks on asphericity; in

Section 5 we prove Theorem B and consider whether the Tits alternative holds. For basic concepts used in this paper we refer the reader to [12].

2 Preliminaries

Lemma 2.1. *In each of the following cases the standard 2-complexes associated with the presentations $\mathcal{P}_n(k,l)$ and $\mathcal{P}_n(k',l')$ are homotopy equivalent. Moreover the triple (n,k,l) satisfies condition (A),(B), or (C) if and only if (n,k',l') does.*

(i) Let $k' = l-k$, $l' = -k \pmod n$.

(ii) Let $k' = l$, $l' = k$.

(iii) Let $k' = k-l$, $l' = -l \pmod n$.

(iv) Let $k' = k$, $l' = k-l \pmod n$.

(v) If $(k,n) = 1$ let $k' = 1$, $l' = Kl \pmod n$, where $Kk \equiv 1 \pmod n$.

(vi) If n is even and $(l,n) = 1$ let $k' = 1$, $l' = Lk+1 \pmod n$, where $Ll \equiv -1 \pmod n$.

Proof. (i) Setting $j = i+k$ in the relators $x_i x_{i+k} x_{i+l}$ and cyclically permuting gives $x_j x_{j+(l-k)} x_{j-k}$.

(ii) Taking the inverse of the relators $x_i x_{i+k} x_{i+l}$ and replacing each generator by its inverse gives $x_{i+l} x_{i+k} x_i$; cyclically permuting yields $x_i x_{i+l} x_{i+k}$.

(iii) Setting $j = i+l$ in the relators $x_i x_{i+k} x_{i+l}$ and cyclically permuting gives $x_j x_{j-l} x_{j+(k-l)}$. Then apply part (ii).

(iv) Negating each subscript of the relators $x_i x_{i+l} x_{i+k}$ of $\mathcal{P}_n(l,k)$ and letting $j = -i-k$, then cyclically permuting yields the relators $x_j x_{j+l} x_{j+(k-l)}$.

(v) Applying the subscript shift $i \to iK$ to the relators $x_i x_{i+k} x_{i+l}$ yields the relators $x_i x_{i+1} x_{i+Kl}$.

(vi) Applying the subscript shift $i \to iL$ to the relators $x_i x_{i+k} x_{i+l}$ yields $x_i x_{i+kL} x_{i-1}$. Writing $j = i-1$ and cyclically permuting gives $x_j x_{j+1} x_{j+kL+1}$.

□

Thus if $(k,n) = 1$ or $(l,n) = 1$ or $(k-l,n) = 1$ then $G_n(k,l) \cong G_n(1,l')$ for some l'. Parts (iii),(v) of Lemma 2.1 are contained in Lemmas 2.1 and 2.2 of [6]. More equivalences amongst the presentations $\mathcal{P}_n(k,l)$ can be established using the other results in Section 2 of [6].

Since the exponent sum of $x_i x_{i+k} x_{i+l}$ is not equal to ± 1 the abelianization $G_n(k,l)^{\mathrm{ab}}$ is non-trivial. Moreover, as a corollary to Theorem 5.1 of [4] we know precisely when the abelianization is infinite. (Strictly, all parameters in that theorem are positive whereas we require one of them to be negative; this does not affect the proof, however.)

Lemma 2.2 ([4])**.** *Suppose $(n,k,l)=1$, $k\neq l$. The abelianization $G_n(k,l)^{\mathrm{ab}}$ is infinite if and only if (A) holds.*

Lemma 2.3. *Suppose $(n,k,l)=1$, $k\neq l$. If (A) holds then $G_n(k,l)$ is large.*

Proof. The standard split extension of $G_n(k,l)$ by the cyclic group of order n has presentation $E_n(k,l)=\langle x,t \mid t^n, xt^{-k}xt^{k-l}xt^l\rangle$. We have that $l\equiv -k\equiv k-l \bmod 3$ so adjoining the relator t^3 gives that $\langle x,t \mid t^3,(xt^l)^3\rangle \cong \mathbb{Z}_3*\mathbb{Z}_3$ is a homomorphic image of $E_n(k,l)$. Thus $E_n(k,l)$, and hence $G_n(k,l)$, is large. $\square$

Lemma 2.4. *Suppose $(n,k,l)=1$, $k\neq l$. If (B) holds then $\mathcal{P}_n(k,l)$ is not aspherical. If, in addition, (A) holds then $G_n(k,l)\cong \mathbb{Z}*\mathbb{Z}$ otherwise $G_n(k,l)\cong\mathbb{Z}_3$.*

Proof. If $k+l\equiv 0 \pmod n$ set $k'=-l$, $l'=k-l \pmod n$; if $2l-k\equiv 0 \pmod n$ set $k'=l-k$, $l'=-k \pmod n$. This gives that $l'=2k'$ and $G_n(k',l')\cong G_n(k,l)$ by Lemma 2.1. Thus we may assume that $l\equiv 2k \pmod n$.

Since $(n,k,l)=1$ we have $(n,k)=1$ so $G_n(k,l)\cong G_n(1,2)$, by Lemma 2.1. The relators $x_ix_{i+1}x_{i+2}$, $x_{i+1}x_{i+2}x_{i+3}$ together imply that $x_i=x_{i+3}$ for all i. Suppose (A) holds, so that $n\equiv 0 \pmod 3$; then the generating set $\{x_i \mid 1\leq i\leq n\}=\{x_1,x_2,x_3\}$ and thus $G_n(1,2)=\langle x_1,x_2,x_3 \mid x_1x_2x_3\rangle\cong\mathbb{Z}*\mathbb{Z}$. An aspherical presentation of any given group has the maximal possible deficiency of all presentations of that group [13, page 478]. The group $\mathbb{Z}*\mathbb{Z}$ has a presentation of deficiency 2 so $\mathcal{P}_n(k,l)$ is not aspherical. Suppose then that (A) does not hold. Then $x_i=x_1$ for all i so $G_n(1,2)=\langle x_1 \mid x_1^3\rangle\cong\mathbb{Z}_3$, a finite non-trivial group, so $\mathcal{P}_n(k,l)$ is not aspherical. $\square$

Thus we may have $G_n(k,l)\cong G_{n'}(k',l')$ with $n\neq n'$, for finite and for infinite groups. In connection with this and with Question 5 of [1] we note that this behaviour cannot occur for the groups $H_n(m,k)$ (of the introduction) when they are finite (by [14],[15]), and that there are no recorded examples of it when they are infinite.

Lemma 2.5. *Suppose $(n,k,l)=1$, $k\neq l$. If (B) does not hold and (A),(C) both hold then $G_n(k,l)\cong\mathbb{Z}*\mathbb{Z}*\mathbb{Z}_\gamma$ where $\gamma=(2^{n/3}-(-1)^{n/3})/3$ and thus $\mathcal{P}_n(k,l)$ is not aspherical.*

Proof. It follows from the hypotheses that either $(n,k)=1$ or $(n,l)=1$ so by Lemma 2.1 we may assume $k=1$. The conditions imply also that $n=3m$ where $m\geq 4$ and so $l\in\{m,m+1,2m,2m+1\}$, $1+l\equiv 0 \pmod 3$. Lemma 2.1 also implies that $G_n(1,m)\cong G_n(1,2m+1)$ and $G_n(1,m+1)\cong G_n(1,2m)$ so it is enough to consider $l\in\{m,m+1\}$. We give only the proof for $l=m$, the case $l=m+1$ being similar.

Let $n=3m$ where $1+m\equiv 0 \pmod 3$ and $m\geq 5$. Then $m=3\hat{m}+2$ and $n=9\hat{m}+6$ where $\hat{m}\geq 1$. We prove that $G=\langle x_1,x_2,x_3 \mid (x_1x_2x_3)^\gamma\rangle$ from which the result follows. Our first step is to re-order the relators (for convenience we will write i for x_i and $\bar{i}$ for x_i^{-1}):

$$\begin{array}{ccc}
1 & 2 & m+1 \\
2 & 3 & m+2 \\
m+1 & m+2 & 2m+1 \\
\\
2m+1+3j & 2m+2+3j & 1+3j \\
2m+2+3j & 2m+3+3j & 2+3j \\
2m+3+3j & 2m+4+3j & 3+3j \\
m+2+3j & m+3+3j & 2m+2+3j \\
m+3+3j & m+4+3j & 2m+3+3j \\
m+4+3j & m+5+3j & 2m+4+3j \\
3+3j & 4+3j & m+3+3j \\
4+3j & 5+3j & m+4+3j \\
5+3j & 6+3j & m+5+3j \\
\\
2m & 2m+1 & 3m \\
3m-1 & 3m & m-1 \\
3m & 1 & m
\end{array}$$

where $0 \le j \le \hat{m}-1$.

The first three relators yield $m+1 = \bar{2}\,\bar{1}$; $m+2 = \bar{3}\,\bar{2}$ and $2m+1 = 2\,3\,1\,2$. Then there follows $9\hat{m} = n-6$ relators in blocks of 9 the first $9\hat{m}-1$ of which together with $2m\;2m{+}1\;3m$ show that $G = \langle 1,2,3 \rangle$ subject to the three relators $5+3(\hat{m}-1)\;6+3(\hat{m}-1)\;m+5+3(\hat{m}-1)$; $3m-1\;3m\;m-1$ and $3m\;1\;m$. In fact $3(\hat{m}-1) = m-5$ so the first of these relators is $m\;m+1\;2m$.

Put $T_v = (2^v - (-1)^v)/3$ where $v \ge 1$. A calculation shows that for $j = 2\hat{j} \ge 0$ the corresponding block of 9 relators yield

$$\begin{array}{rclrclrcl}
2m+2+3j &=& \bar{2}\,\bar{1}(123)^{-u(j,1)} & 2m+3+3j &=& (123)^{u(j,2)}1 & 2m+4+3j &=& \bar{1}(123)^{-u(j,3)}\bar{3} \\
m+3+3j &=& 2\,3(123)^{v(j,1)}1\,2 & m+4+3j &=& \bar{2}\,\bar{1}(123)^{-v(j,2)} & m+5+3j &=& (123)^{v(j,3)}1 \\
4+3j &=& (123)^{-w(j,1)}\bar{3}\,\bar{2} & 5+3j &=& 2\,3(123)^{w(j,2)} & 6+3j &=& \bar{2}\,\bar{1}(123)^{-w(j,3)}
\end{array}$$

where

$$\begin{array}{lll}
u(j,1) = T_{1+6\hat{j}} & u(j,2) = T_{2+6\hat{j}} & u(j,3) = T_{3+6\hat{j}} - 2 \\
v(j,1) = T_{2+6\hat{j}} & v(j,2) = T_{3+6\hat{j}} & v(j,3) = T_{4+6\hat{j}} \\
w(j,1) = T_{3+6\hat{j}} - 1 & w(j,2) = T_{4+6\hat{j}} & w(j,3) = T_{5+6\hat{j}}
\end{array}$$

and if $j = 2\hat{j}+1 \ge 1$ the corresponding block yields

$$\begin{array}{rclrclrcl}
2m+2+3j &=& 3(123)^{u(j,1)} & 2m+3+3j &=& (123)^{-u(j,2)}\bar{3}\,\bar{2} & 2m+4+3j &=& 2\,3(123)^{u(j,3)}1\,2 \\
m+3+3j &=& \bar{1}(123)^{v(j,1)}\bar{3} & m+4+3j &=& 3(123)^{v(j,2)} & m+5+3j &=& (123)^{-v(j,3)}\bar{3}\,\bar{2} \\
4+3j &=& (123)^{w(j,1)}1 & 5+3j &=& \bar{1}(123)^{-w(j,2)}\bar{3} & 6+3j &=& 3(123)^{w(j,3)}
\end{array}$$

where

$$\begin{array}{rclrclrcl}
u(j,1) &=& T_{4+6\hat{j}} - 1 & u(j,2) &=& T_{5+6\hat{j}} - 1 & u(j,3) &=& T_{6+6\hat{j}} \\
v(j,1) &=& T_{5+6\hat{j}} - 2 & v(j,2) &=& T_{6+6\hat{j}} - 1 & v(j,3) &=& T_{7+6\hat{j}} - 1 \\
w(j,1) &=& T_{6+6\hat{j}} & w(j,2) &=& T_{7+6\hat{j}} - 2 & w(j,3) &=& T_{8+6\hat{j}} - 1
\end{array}$$

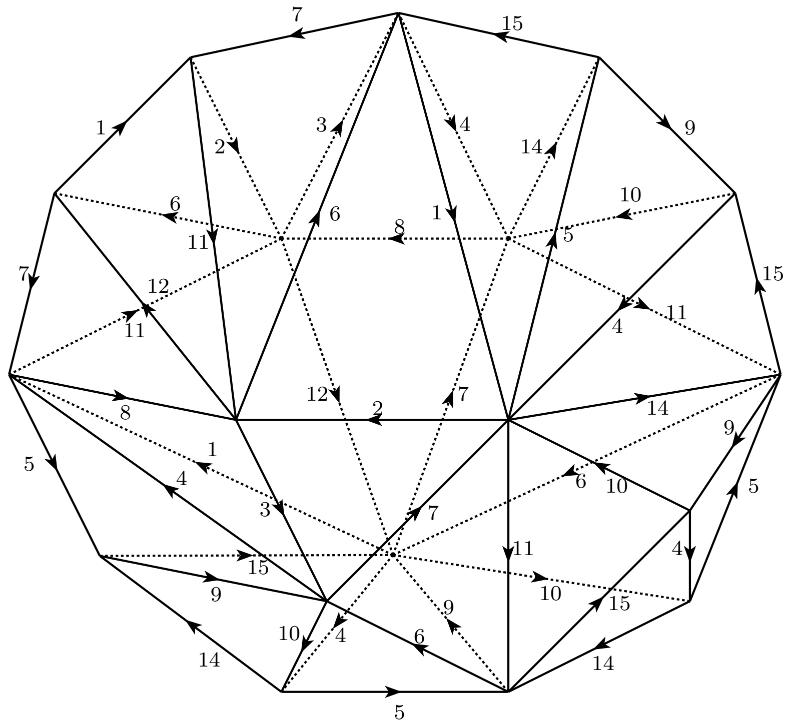

Figure 1: Sphere for $\mathcal{P}_{15}(1,5)$

It follows that when m is even the relator $m\ m+1\ 2m$ rewrites to

$$\bar{1}(123)^{-w(\hat{m}-1,2)}\bar{3}\,\bar{2}\,\bar{1}(123)^{-v(\hat{m}-1,3)} = (123)^{-T_m};$$

using $2m\ 2m+1\ 3m$ the relator $3m-1\ 3m\ m-1$ rewrites to

$$2\,3(123)^{u(\hat{m}-1,3)}1\,2\overline{(2312)}2\,3(123)^{v(\hat{m}-1,3)}(123)^{2(\hat{m}-1,1)}1 = (123)^{T_m};$$

and the relator $3m\ 1\ m$ rewrites to

$$\overline{(2312)}2\,3(123)^{v(\hat{m}-1,3)}1\bar{1}(123)^{-w(\hat{m}-1,2)}\bar{3} = (123)^0$$

from which we obtain the result. The consequences when m is odd are similar and we omit the details. $\square$

In certain cases of Lemma 2.5 we can explicitly obtain spheres. An application of Lemma 2.1 shows that when $n = 15$ there is (up to homotopy) only one presentation to be considered, namely $\mathcal{P}_{15}(1,5)$. We give a sphere for this case in Figure 1.

3 The metacyclic cases

In this section we deal with the cases where $(n,k,l) = 1$, $k \neq l$, (C) holds and (A) does not. It follows that (B) does not hold. These conditions imply that either $(n,k) = 1$ or

$(n,l)=1$ or $(n,k-l)=1$ so by Lemma 2.1 we may assume that $k=1$. Thus it is enough to consider $G=G_n(1,l)$ where $1+l\not\equiv 0$, $2l-1\not\equiv 0$, $2-l\not\equiv 0$ and either $3l\equiv 0$ or $3(l-1)\equiv 0$ all modulo n; and where $n\equiv 0 \pmod 3$ and $1+l\not\equiv 0 \pmod 3$.

Lemma 3.1. *Suppose $(n,k,l)=1$, $k\neq l$. If (C) holds and (A) does not then $|G_n(k,l)^{\mathrm{ab}}|=\alpha$ where $\alpha=3(2^{n/3}-(-1)^{n/3})$.*

Proof. As explained above it is enough to consider $G_n(1,l)$ together with the conditions on l and n listed there. Let $n=3m$. Then there are four cases: (i) $l=m$ and $m\equiv 0$ or $1 \pmod 3$; (ii) $l=m+1$ and $m\equiv 0$ or $2 \pmod 3$; (iii) $l=2m$ and $m\equiv 0$ or $2 \pmod 3$; and (iv) $l=2m+1$ and $m\equiv 0$ or $1 \pmod 3$. But the substitution $M=m+1$, $M=2m$, $M=2m+1$ (respectively) transforms case (ii), (iii), (iv) (respectively) to case (i) so it is enough to consider (i) only. Now the relation matrix of a cyclic presentation is a circulant matrix and it follows (see, for example, [7, page 77]) that $|G_n(k,l)^{\mathrm{ab}}|=P$ where

$$P=\prod_{j=0}^{n-1} f(\zeta)$$

where $f(x)=1+x+x^l$ and $\zeta=\mathrm{e}^{2\pi i/3m}$.

Put $w=\mathrm{e}^{2\pi i/3}$, $\theta=\mathrm{e}^{2\pi i/m}$. Then $j=3t$ yields

$$P_1=\prod_{t=0}^{m-1}(1+\theta^t+1)=\prod_{t=0}^{m-1}(-1)((-2)-\theta^t)=2^m-(-1)^m;$$

$j=3t+1$ yields

$$P_2=\prod_{t=0}^{m-1}(1+\zeta\theta^t+w)=\prod_{t=0}^{m-1}(-\zeta)\left[-\left(\frac{1+w}{\zeta}\right)-\theta^t\right]=(1+w)^m-(-\zeta)^m;$$

and $j=3t+2$ yields

$$P_3=\prod_{t=0}^{m-1}(1+\zeta^2\theta^t+w^2)=(1+w^2)^m-(-\zeta^2)^m.$$

Then $P=P_1P_2P_3=3(2^m-(-1)^m)$. □

Lemma 3.2. *Suppose $(n,k,l)=1$, $k\neq l$. If (C) holds and (A) does not then $G=G_n(k,l)$ is metacyclic of order $s=2^n-(-1)^n$ and we have the metacyclic extension*

$$\mathbb{Z}_{s/3}\hookrightarrow G\twoheadrightarrow \mathbb{Z}_3$$

and the metacyclic extension

$$G'\cong\mathbb{Z}_\beta\hookrightarrow G\twoheadrightarrow\mathbb{Z}_\alpha$$

where $\alpha=3(2^{n/3}-(-1)^{n/3})$, $\beta=s/\alpha$.

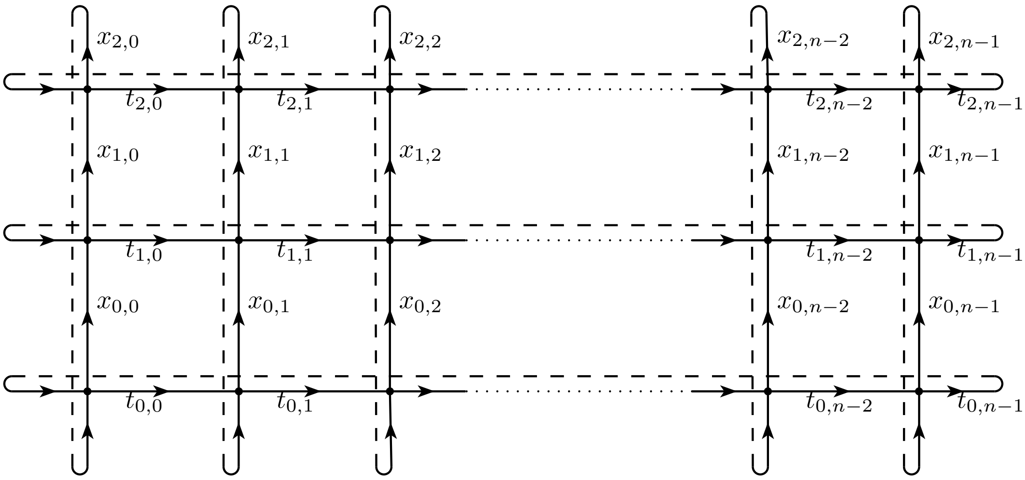

Figure 2: Covering complex

Proof. Again it is enough to consider $G_n(1,l)$ (with the conditions on l and n listed above).

Let E be the standard split extension of G by the cyclic group of order n. Then $|E:G| = n$ and E has the presentation

$$E = \langle x, t \mid t^n, xt^{-1}xt^{-(l-1)}xt^l \rangle.$$

We claim that E', the derived subgroup of index $3n$ in E, is cyclic of order $s/3$ where $s = 2^n - (-1)^n$ and it follows that G is metacyclic of order s. Since E' is a subgroup of index 3 in (the isomorphic copy in E of) G we obtain the first metacyclic extension in the statement of the lemma. Moreover, it follows that G' and G^{ab} are cyclic so by Lemma 3.1 $G^{\mathrm{ab}} \cong \mathbb{Z}_\alpha$ and we obtain the second metacyclic extension.

To prove our claim first observe that $E^{\mathrm{ab}} \cong \mathbb{Z}_3 \times \mathbb{Z}_n$ and so the covering complex corresponding to E' has 1-skeleton as given by Figure 2. The 2-cells are obtained from the lifts of t^n and $xt^{-1}xt^{-(l-1)}xt^l$ at each vertex and these are (up to cyclic permutation):

$$t_{j,0}\, t_{j,1}\, \ldots\, t_{j,n-1} \tag{3.1}$$

and

$$x_{j,i}t^{-1}_{j+1,i-1}x_{j+1,i-1}t^{-1}_{j+2,i-2}\ldots t^{-1}_{j+2,i-l}x_{j+2,i-l}t_{j,i-l}\ldots t_{j,i-1}, \tag{3.2}$$

where $0 \le j \le 2$, $0 \le i \le n-1$ and the subscripts are taken modulo 3, modulo n respectively.

A presentation for E' is obtained by collapsing a maximal tree. We first collapse the edges labelled $t_{j,i}$ apart from $t_{0,n-1}$, $t_{1,n-1}$ and $t_{2,n-1}$. Note however that the t-lifts (3.1) now yield $t_{0,n-1} = t_{1,n-1} = t_{2,n-1} = 1$ in E'. Thus the lifts (3.2) become

$$x_{j,i}x_{j+1,i-1}x_{j+2,i-l} \quad (0 \le j \le 2,\ 0 \le i \le n-1). \tag{3.3}$$

Before choosing which two x-edges to collapse we first rearrange the $3n$ words in (3.3) into n rows each having a triple of words.

Assume that $3l \equiv 0 \pmod{n}$. The first row of the new arrangement is

$$x_{0,0}x_{1,n-1}x_{2,n-l} \quad x_{2,n-l}x_{0,n-l-1}x_{1,n-2l} \quad x_{1,l}x_{2,l-1}x_{0,0} \tag{3.4}$$

and since $3l \equiv 0 \pmod{n}$ these words are

$$x_{0,0}x_{1,n-1}x_{2,2l} \quad x_{2,2l}x_{0,2l-1}x_{1,l} \quad x_{1,l}x_{2,l-1}x_{0,0}. \tag{3.5}$$

To obtain the next $n-1$ rows we repeatedly make the shift $x_{j,i} \to x_{j,i+2l-1}$ starting at (3.5). The point being that the gcd $(2l-1, 3l) = 1$ since if $q > 1$ divides $3l$ and $2l-1$ then q divides $l+1$. Since $l+1 \not\equiv 0 \pmod{3}$ it follows that q divides l, a contradiction. Therefore the shift induces a permutation of our set.

The n rows are

$$\begin{array}{ccccccccc}
x_{0,0} & x_{1,n-1} & x_{2,2l} & x_{2,2l} & x_{0,2l-1} & x_{1,l} & x_{1,l} & x_{2,l-1} & x_{0,0} \\
x_{0,2l-1} & x_{1,2l-2} & x_{2,l-1} & x_{2,l-1} & x_{0,l-2} & x_{1,n-1} & x_{1,n-1} & x_{2,n-2} & x_{0,2l-1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
x_{0,2l+2} & x_{1,2l+1} & x_{2,l+2} & x_{2,l+2} & x_{0,l+1} & x_{1,2} & x_{1,2} & x_{2,1} & x_{0,2l+2} \\
x_{0,l+1} & x_{1,l} & x_{2,1} & x_{2,1} & x_{0,0} & x_{1,2l+1} & x_{1,2l+1} & x_{2,2l} & x_{0,l+1}
\end{array} \tag{3.6}$$

Observe that (3.6) is also arranged into three columns each of n words. We label the words in the first column r_i $(0 \le i \le n-1)$; the second column s_i $(0 \le i \le n-1)$; and the third column u_i $(0 \le i \le n-1)$. To obtain a presentation for E' collapse the edges labelled by $x_{2,2l}$ and $x_{1,l}$ giving

$$E' = \langle x_{j,i} \mid r_i, s_i, u_i \rangle$$

where $0 \le j \le 2$, $0 \le i \le n-1$ and $(j,i) \ne (2,2l), (1,l)$.

To see that $E' = \langle x_{0,0} \rangle$ we consider each of the $n-1$ triples r_i, s_i, u_i $(0 \le i \le n-2)$ in turn. The triple r_0, s_0, u_0 yields $x_{1,n-1} = x_{0,0}^{-1}$, $x_{0,2l-1} = x_{0,0}^{0}$ and $x_{2,l-1} = x_{0,0}^{-1}$. The next triple r_1, s_1, u_1 now yields $x_{1,2l-2} = x_{0,0}$, $x_{0,l-2} = x_{0,0}^{2}$ and $x_{2,n-2} = x_{0,0}$. More generally, the triple r_i, s_i, u_i will yield $x_{1,i(2j-1)-1}$, $x_{0,(2l-1)+i(2j-1)}$, $x_{2,(l-1)+i(2j-1)}$ are each in $\langle x_{0,0} \rangle$ and so E' is indeed cyclic generated by $x_{0,0}$. Observe also that the sequence powers of $x_{0,0}$ obtained by r_i, s_i, u_i is as follows: $-1, 0, -1$; $1, 2, 1$; $-3, -2, -3$; $5, 6, 5$, and so on. Solving the recurrence relation shows that we obtain from $r_{n-2}, s_{n-2}, u_{n-2}$ the following identities:

$$x_{1,2l+1} = x_{0,0}^{p_1}; \quad x_{0,l+1} = x_{0,0}^{p_2}; \quad x_{2,1} = x_{0,0}^{p_3}, \tag{3.7}$$

where p_1, p_2, p_3 (respectively) equals $(2^{n-1}-1)/3$, $(2^{n-1}+2)/3$, $(2^{n-1}-1)/3$ (respectively) (n odd), or equals $-(2^{n-1}+1)/3$, $-(2^{n-1}-2)/3$, $-(2^{n-1}+1)/3$ (respectively) (n even).

It follows from all this that $E' = \langle x_{0,0} \rangle$ subject to the relators $r_{n-1}, s_{n-1}, u_{n-1}$. But an easy check using (3.7) shows that each of these yields the relator $x_{0,0}^{s/3}$ and this proves our claim.

If we assume that $3(l-1) \equiv 0 \pmod{n}$ the argument is similar. This time our first triple r_0, s_0, u_0 is:

$$x_{0,2l-1}x_{1,2l-2}x_{2,l-1} \quad x_{1,l}x_{2,l-1}x_{0,0} \quad x_{2,1}x_{0,0}x_{1,2l-2}.$$

The shift is again $x_{j,i} \rightarrow x_{j,i+2l-1}$ and the x-edges collapsed to produce the presentation for E' are $x_{1,2l-2}$ and $x_{2,l-1}$. We omit the details. $\square$

It is well-known (see, for example, Chapter 3 in [12]) that any finite metacyclic group L, with metacyclic extension $\mathbb{Z}_M \hookrightarrow L \twoheadrightarrow \mathbb{Z}_N$ has a presentation of the form

$$B(M,N,r,\lambda) = \langle a,b \mid a^M = 1,\, bab^{-1} = a^r,\, b^N = a^{\lambda M/(M,r-1)} \rangle$$

for some r, λ where $r^N \equiv 1 \pmod{M}$. Moreover, by [2] if L has a balanced presentation then $\lambda = 1$, $H_2(L,\mathbb{Z}) = 0$ and L has a 2-generator, 2-relator presentation. Thus we have:

Corollary 3.3. *Let $G = G_n(k,l)$ and suppose $(n,k,l) = 1$, $k \neq l$. If (C) holds and (A) does not hold then*

(i) $H_2(G,\mathbb{Z}) = 0$;

(ii) G has a presentation $B((2^n - (-1)^n)/3, 3, r, 1)$ for some r where $r^3 \equiv 1 \pmod{(2^n - (-1)^n)/3}$;

(iii) G has a presentation with 2 generators and 2 relators.

Computer experiments in GAP [8] in the cases $n = 9, 12, 15$ suggest a value for r for the presentation in part (ii).

Conjecture 3.4. *Suppose $(n,k,l) = 1$, $k \neq l$. If (C) holds and (A) does not hold then $G_n(k,l) \cong \Gamma$ where $\Gamma = B((2^n - (-1)^n)/3, 3, 2^{2n/3}, 1)$.*

An analysis of the presentation for Γ yields the following result which, in particular, shows that Γ has the desired abelianisation.

Lemma 3.5. *Let $n = 3m$. Then*

(i) $\Gamma = \langle a,b \mid b^3 = a^{(2^{2m}+(-2)^m+1)/3}, ba^{2^m} = a^{(-1)^m}b \rangle$;

(ii) $\Gamma^{\mathrm{ab}} \cong \mathbb{Z}_\alpha$ *where* $\alpha = 3(2^m - (-1)^m)$.

Proof. It follows from [2] that Γ has a presentation

$$\Gamma = \langle a,b \mid b^3 = a^V,\, ba^s b^{-1} a^{-s} = a^{(M,r-1)} \rangle$$

where $V = M/(M,r-1)$, $M = (2^n - (-1)^n)/3$, $r = 2^{2n/3}$ and where s is defined as follows. If s_1 and k_1 are integers such that $(M,r-1) = s_1(r-1) + k_1 M$ and d (taken mod M) is the greatest factor of M that is prime to s_1 then put $s = s_1 + dV$.

To prove (i) observe that $M = (2^m - (-1)^m)(2^{2m} + (-2)^m + 1)/3$ and $r - 1 = (2^m - (-1)^m)(2^m + (-1)^m)$ so $3M - 2^m(r-1) = 2^m - (-1)^m$. From this it follows that $(M, r-1) = 2^m - (-1)^m$, that $V = (2^{2m} + (-2)^m + 1)/3$ and that we can take $s_1 = -2^m$ and $k_1 = 3$. Since M is odd and s_1 is a power of 2 we have $d = 0$ hence $s = -2^m$ yielding the desired presentation for Γ.

For (ii) observe that $V - \frac{2}{3}(2^{m-1} - (-1)^{m-1})(M, r-1) = 1$ so $(V, (M, r-1)) = 1$ and there exists v with $(v, (M, r-1)) = 1$ such that $vV \equiv 1 \pmod{(M, r-1)}$. It now follows that $\Gamma^{ab} = \langle a, b \mid b^3 = a^V,\ a^{(M,r-1)} = 1,\ ab = ba\rangle = \langle a, b, c \mid b^3 = a^V,\ a^{(M,r-1)} = 1,\ ab = ba,\ c = a^V\rangle = \langle b, c \mid b^3 = c,\ c^{(M,r-1)} = 1,\ bc = cb\rangle = \langle b \mid b^\alpha = 1\rangle$ as required. $\square$

4 Asphericity

The standard split extension of $G_n(k,l)$ by the cyclic group of order n has presentation $E_n(k,l) = \langle x, t \mid t^n, xt^{-k}xt^{k-l}xt^l\rangle$. If we put $T = \langle t \mid t^n\rangle$ then $E_n(k,l)$ has a so-called relative presentation $\mathcal{R}_n(k,l) = \langle T, x \mid xt^{-k}xt^{k-l}xt^l\rangle$. Lemma 4.1 of [6] gives that if $\mathcal{R}_n(k,l)$ is aspherical (in the sense that any non-empty spherical picture over $\mathcal{R}$ contains a dipole) then the presentation $\mathcal{P}_n(k,l)$ is aspherical (more precisely, it is diagrammatically reducible in the sense of Gersten [10] which implies that $\pi_2(K) = 0$, where K is the standard CW-complex associated with $\mathcal{P}$). Theorem 4.1 of [3] gives necessary and sufficient conditions for $\mathcal{R}_n(k,l)$ to be aspherical. Following [1],[9], this approach was used in Theorem 4.3 of [6] to obtain sufficient conditions for $\mathcal{P}_n(k,l)$ to be aspherical. Unfortunately that theorem is incorrect, implying (for example) that $\mathcal{P}_9(2,3)$ is an aspherical presentation whereas in fact it defines a metacyclic group of order 513. We correct and improve that result by including the missing condition and strengthening the other conditions:

Theorem 4.1. *The presentation $\mathcal{R} = \mathcal{R}_n(k,l)$ is aspherical if and only if none of the following conditions (a)–(e) is satisfied and $\mathcal{P} = \mathcal{P}_n(k,l)$ is aspherical if none of them is satisfied.*

- *(a)* $(n, 2k-l) + (n, 2l-k) + (n, k+l) > n$;
- *(b)* $n = 6(n, 2k-l)$ *and* $(k-2l) \equiv \alpha(l-2k) \pmod n$ *where* $\alpha = 2$ *or* 3;
- *(c)* $n = 6(n, 2l-k)$ *and* $(k+l) \equiv \alpha(k-2l) \pmod n$ *where* $\alpha = 2$ *or* 3;
- *(d)* $n = 6(n, k+l)$ *and* $(l-2k) \equiv \alpha(k+l) \pmod n$ *where* $\alpha = 2$ *or* 3;
- *(e)* n *divides* $3l$ *or* n *divides* $3k$ *or* n *divides* $3(l-k)$.

If $(n,k,l) = 1$ we have a simpler formulation:

Corollary 4.2. *Suppose $(n,k,l) = 1$, $k \neq l$, that (C) does not hold, $(n, 2k-l) + (n, 2l-k) + (n, k+l) \leq n$ and that if $n = 18$ then (A) does not hold. Then $\mathcal{P}_n(k,l)$ is aspherical.*

Proof. Clearly (a) and (e) do not hold. Suppose for contradiction that (b) holds. Then $(n, 2k - l) = n/6$ so $n/6$ divides $2k - l$ and since $(k - 2l) = \alpha(l - 2k)$ we have that $n/6$ divides $k-2l$. Thus $n/6$ divides $3k$ and $3l$ so $n/18$ divides $(n, k, l) = 1$ and hence $n = 6, 12$, or 18, and in these cases the hypotheses are never satisfied. Thus (b) does not hold; similar arguments show that (c),(d) do not hold and hence $\mathcal{P}_n(k, l)$ is aspherical. □

We can now prove Theorem A.

Proof of Theorem A. If $k = l$ then $G_n(k, l) \cong \mathbb{Z}_s$ so $\mathcal{P}_n(k, l)$ is not aspherical, so assume $k \neq l$. Note that the remaining conditions are that neither (B) nor (C) holds and either $n \neq 18$ or ($n = 18$ and (A) does not hold). By Lemma 2.4 we may assume that (B) does not hold.

Suppose that (C) holds. If (A) holds then $\mathcal{P}_n(k, l)$ is not aspherical by Lemma 2.5. If (A) does not hold then $G_n(k, l)$ is a finite non-trivial group by Lemma 3.2, so $\mathcal{P}_n(k, l)$ is not aspherical. Thus we may assume that neither (B) nor (C) holds.

If $n = 18$ and (A) holds then either $(k, n) = 1$ or $(l, n) = 1$ so by Lemma 2.1 we may assume that $k = 1$. The conditions then imply that $l = 5, 8, 11$, or 14. Another application of Lemma 2.1 yields that $G_{18}(1, 5) \cong G_{18}(1, 8) \cong G_{18}(1, 11) \cong G_{18}(1, 14)$. By eliminating generators using a routine application of Tietze transformations we may show that $G_{18}(1, 5) \cong \langle x_3, x_5, x_{14} \mid (x_5 x_{14}^{-1})^{19} \rangle \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}_{19}$ which has torsion so $\mathcal{P}$ is not aspherical. So we may assume that neither (B) nor (C) hold and that if $n = 18$ then (A) does not hold. By Corollary 4.2 it suffices to show that $(n, 2k - l) + (n, 2l - k) + (n, k + l) \leq n$.

Let $p = n/(n, 2k - l)$, $q = n/(n, 2l - k)$, $r = n/(n, k + l)$, then this fails to hold if and only if $\{p, q, r\} \in S$ where $S = \{\{2, 3, 3\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 2, N\}\ (N \geq 2)\}$. (Note that $p, q, r \neq 1$ since (A) does not hold.) We shall assume, for contradiction, that $\{p, q, r\} \in S$. From the definition of p, $(n, l - 2k) = n/p$ so n/p divides $l - 2k$ and therefore for some $1 \leq \alpha \leq p - 1$

$$l - 2k \equiv \alpha n/p \bmod n. \tag{4.1}$$

Similarly, for some $1 \leq \beta \leq q - 1$ and some $1 \leq \gamma \leq r - 1$

$$k - 2l \equiv \beta n/q \bmod n, \tag{4.2}$$
$$k + l \equiv \gamma n/r \bmod n. \tag{4.3}$$

Summing (4.1)–(4.3) we obtain $\alpha n/p + \beta n/q + \gamma n/r \equiv 0 \bmod n$, so setting $\kappa = \alpha/p + \beta/q + \gamma/r$ we have $\kappa \in \mathbb{Z}$. For each triple $\{p, q, r\} \in S$ it is easy to check that $\kappa \notin \mathbb{Z}$ for any choice of α, β, γ, and the proof is complete. □

As a corollary to Theorem A we have that the converse of Lemma 4.1 of [6] holds.

Corollary 4.3. *The relative presentation $\mathcal{R}_n(k, l)$ is aspherical if and only if the absolute presentation $\mathcal{P}_n(k, l)$ is aspherical.*

Proof. If $\mathcal{R}_n(k,l)$ is aspherical then $\mathcal{P}_n(k,l)$ is aspherical, by Lemma 4.1 of [6]. Let $d = (n,k,l)$, $N = n/d$, $K = k/d$, $L = l/d$. If $\mathcal{P}_n(k,l)$ is aspherical then $\mathcal{P}_N(K,L)$ is aspherical. Then none of (a)–(e) of Theorem 4.1 hold for the numbers N, K, L and hence none of them do for n, k, l, so $\mathcal{R}_n(k,l)$ is aspherical. $\square$

The standard split extension of $H_n(m,k)$ (from the introduction) has a relative presentation $\mathcal{S}_n(m,k) = \langle T, x \mid xt^m xt^{-k}x^{-1}t^{k-m}\rangle$. Lemma 2.2 of [1] (a generalization of Lemma 3.1 of [9]) gives that if $\mathcal{S}_n(m,k)$ is aspherical then $\mathcal{Q}_n(m,k)$ is aspherical. Using Theorem 2 of [14] and Theorem 3.2 of [9] we can obtain the (analogous result to Corollary 4.3) that the converse holds in many cases. For simplicity we only state a result for the 'strongly irreducible' cases (see [14]).

Theorem 4.4. *Suppose that $(n,m,k) = 1$ and $(n,k) > 1$, $(k-m,n) > 1$. Then the relative presentation $\mathcal{S}_n(m,k)$ is aspherical if and only if the absolute presentation $\mathcal{Q}_n(m,k)$ is aspherical.*

5 Finiteness and the Tits alternative

Proof of Theorem B. If $d = (n,k,l) > 1$ then $G_n(k,l)$ is isomorphic to the free product of d copies of $G_{n/d}(k/d, l/d)$, which has non-trivial abelianization, so $G_n(k,l)$ is infinite. Thus we may assume $(n,k,l) = 1$. If $k = l$ then $G \cong \mathbb{Z}_s$, and this is condition (i), so assume $k \neq l$.

If (B) holds and (A) does not hold then $G \cong \mathbb{Z}_3$ by Lemma 2.4 and this is condition (ii) of the theorem; and if (A) does not hold and (C) holds then this is condition (iii) and the result follows from Lemma 3.2.

Now suppose that conditions (i),(ii),(iii) do not hold. If (A) holds then G is infinite by Lemma 2.2 so assume otherwise. This in particular forces both (B) and (C) not to hold. It follows from Theorem A that the presentation $\mathcal{P}_n(k,l)$ is aspherical so G is torsion-free and since G is non-trivial it is infinite. $\square$

Recall that a group satisfies the *Tits alternative* if it either contains a non-abelian free subgroup or is virtually soluble. As noted in the introduction, $G_n(k,l)$ is isomorphic to the free product of $d = (n,k,l)$ copies of $G_{n/d}(k/d, l/d)$. Since $|G_{n/d}(k/d, l/d)| \geq 3$, $G_n(k,l)$ is large when $d > 1$ so we may assume that $(n,k,l) = 1$. Our results, as summarized in Table 1, show that the Tits alternative holds except possibly when none of (A),(B),(C) hold. We now show that it often holds in these cases as well. We introduce a fourth condition:

(D) $2(k+l) \equiv 0 \pmod n$ or $2(2l-k) \equiv 0 \pmod n$ or $2(2k-l) \equiv 0 \pmod n$

Lemma 5.1. *Suppose $(n,k,l) = 1$, $k \neq l$.*

(i) $\mathcal{P}_n(k,l)$ satisfies the small cancellation condition C(3) if and only if (B) does not hold;

(ii) $\mathcal{P}_n(k,l)$ satisfies the small cancellation condition T(6) if and only if none of (B),(C),(D) hold.

Proof. Let $\mathcal{P}_n(k,l)$ have associated star graph Γ. Then Γ is a bipartite graph with vertices x_i, x_i^{-1} $(1 \leq i \leq n)$ which we shall denote by $i, \bar{i}$ (respectively). The undirected edge with vertices i and j will be denoted by $\{i,j\}$.

(i) Clearly C(3) does not hold if and only if there is a piece of length 2 and this occurs if and only if Γ contains a closed path, γ say, of length 2. Using symmetry it can be assumed that γ contains one of the edges $\{\bar{1}, 1+k\}$, $\{\overline{1+k}, 1+l\}$, $\{\overline{1+l}, 1\}$ obtained from the relator $x_1x_{1+k}x_{1+l}$. Suppose γ contains $\{\bar{1}, 1+k\}$. Since the other two edges involving $\bar{1}$ are $\{\bar{1}, 1+l-k\}$ and $\{\bar{1}, 1-l\}$ obtained from the relators $x_{1-k}x_1x_{1+l-k}$, $x_{1-l}x_{1+k-l}x_1$ (respectively) it follows that γ is closed of length 2 if and only if either $1+k \equiv 1+l-k$ or $1+k \equiv 1-l$ (mod n) and this occurs if and only if either $2k-l \equiv 0$ or $k+l \equiv 0$ (mod n). Similarly if γ contains $\{\overline{1+k}, 1+l\}$ then γ is closed of length 2 if and only if either $2k-l \equiv 0$ or $2l-k \equiv 0$ (mod n); and if γ contains $\{\overline{1+l}, 1\}$ then γ is closed of length 2 if and only if either $2l-k \equiv 0$ or $k+l \equiv 0$ (mod n) and the result follows.

(ii) Since Γ is bipartite it follows from [11] that $\mathcal{P}_n(k,l)$ fails to satisfy T(6) if and only if Γ contains a closed path γ of length 2 or 4. The case of length 2 is dealt with in (i) so assume length 4. Without loss of generality it can be assumed that γ contains the vertex $\bar{1}$. The three edges involving $\bar{1}$ are $\{\bar{1}, 1+k\}, \{\bar{1}, 1+l-k\}, \{\bar{1}, 1-l\}$ obtained from the relators $x_1x_{1+k}x_{1+l}, x_{1-k}x_1x_{1+l-k}, x_{1-l}x_{1+k-l}x_1$ (respectively). Suppose that γ contains $\{\bar{1}, 1+k\}$. The other two edges involving $1+k$ are $\{1+k, \overline{1+k+l}\}$ and $\{1+k, \overline{1+2k-l}\}$. The two further edges involving $\overline{1+k+l}$ are $\{\overline{1+k+l}, 1+2k+l\}$ and $\{\overline{1+k+l}, 1+2l\}$; and involving $\overline{1+2k-l}$ are $\{\overline{1+2k-l}, 1+3k-l\}$ and $\{\overline{1+2k-l}, 1+2k-2l\}$. It then follows that γ is closed of length 4 if and only if either $1+2k+l$ or $1+2l$ or $1+3k-l$ or $1+2k-2l$ coincides with one of $1+l-k$ or $1-l$; this occurs if and only if either (C) holds or $l+k \equiv 0$ or $2k-l \equiv 0$ or $2(k+l) \equiv 0$ or $2(2k-l) \equiv 0$ (mod n). Similarly if γ contains $\{\bar{1}, 1+l-k\}$ then γ is closed of length 4 if and only if either (C) holds or $2k-l \equiv 0$ or $2l-k \equiv 0$ or $2(2k-l) \equiv 0$ or $2(2l-k) \equiv 0$ (mod n); or if γ contains $\{\bar{1}, 1-l\}$ then γ is closed of length 4 if and only if either (C) holds or $2l-k \equiv 0$ or $l+k \equiv 0$ or $2(2l-k) \equiv 0$ or $2(k+l) \equiv 0$ (mod n). The result now follows. □

Corollary 5.2. *Suppose $(n,k,l) = 1$, $k \neq l$. If none of (B),(C),(D) hold then $G_n(k,l)$ contains a non-abelian free subgroup.*

Proof. Since $\mathcal{P}_n(k,l)$ satisfies C(3)+T(6) this follows from Theorem 8.1 of [7]. □

It remains to consider the cases where $(n,k,l) = 1$, $k \neq l$, (A),(B),(C) do not hold and (D) does hold. In these cases either $(k,n) = 1$ or $(l,n) = 1$ so by using Lemma 2.1 we may assume that $k = 1$. The conditions then imply that $n = 2$ or 4 mod 6, $k = 1$, and $l = n/2 - 1$. When $n = 8$ a calculation in GAP [8] shows that the subgroup H of $G = G_n(1, n/2-1)$ generated by $\{x_1x_5, x_2x_6, x_3x_7, x_4x_8, x_5x_1, x_6x_2, x_7x_3, x_8x_4\}$ is normal, isomorphic to $\mathbb{Z}^8$, and that $G/H \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. That is, G is infinite and metabelian. We have not been able to determine the situation for $n > 8$.

A Appendix

As the proofs of Lemma 2.5 and Lemma 3.2 are somewhat technical we include here some examples to aid their understanding. We write i, j for $x_{i,j}$.

A.1 Proof of Lemma 2.5 in the case $(n, k, l) = (42, 1, 14)$

relators			**re-order**			**consequences**
1	2	15	1	2	15	$15 = \bar{2}\bar{1}$
2	3	16	2	3	16	$16 = \bar{3}\bar{2}$
3	4	17	15	16	29	$29 = 2312$
4	5	18	29	30	1	$30 = \bar{2}\bar{1}(123)^{-1}$
5	6	19	30	31	2	$31 = (123)1$
6	7	20	31	32	3	$32 = \bar{1}(123)^{-1}\bar{3}$
7	8	21	16	17	30	$17 = 23(123)12$
8	9	22	17	18	31	$18 = \bar{2}\bar{1}(123)^{-3}$
9	10	23	18	19	32	$19 = (123)^{5}1$
10	11	24	3	4	17	$4 = (123)^{-2}\bar{3}\bar{2}$
11	12	25	4	5	18	$5 = 23(123)^{5}12$
12	13	26	5	6	19	$6 = \bar{2}\bar{1}(123)^{-11}$
13	14	17	32	33	4	$33 = 3(123)^{4}$
14	15	28	33	34	5	$34 = (123)^{-10}\bar{3}\bar{2}$
15	16	29	34	35	6	$35 = 23(123)^{21}12$
16	17	30	19	20	33	$20 = \bar{1}(123)^{-9}\bar{3}$
17	18	31	20	21	34	$21 = 3(123)^{20}$
18	19	32	21	22	35	$22 = (123)^{-42}\bar{3}\bar{2}$
19	20	33	6	7	20	$7 = (123)^{21}1$
20	21	34	7	8	21	$8 = \bar{1}(123)^{-41}\bar{3}$
21	22	35	8	9	22	$9 = 3(123)^{84}$

22	23	36	35	36	7	$36 = \bar{2}\bar{1}(123)^{-43}$
23	24	37	36	37	8	$37 = (123)^{85}1$
24	25	38	37	38	9	$38 = \bar{1}(123)^{-169}\bar{3}$
25	26	39	22	23	36	$23 = 23(123)^{85}12$
26	27	40	23	24	37	$24 = \bar{2}\bar{1}(123)^{-171}$
27	28	41	24	25	38	$25 = (123)^{341}1$
28	29	42	9	10	23	$10 = (123)^{-170}\bar{3}\bar{2}$
29	30	1	10	11	24	$11 = 23(123)^{341}12$
30	31	2	11	12	25	$12 = \bar{2}\bar{1}(123)^{-683}$
31	32	3	38	39	10	$39 = 3(123)^{340}$
32	33	4	39	40	11	$40 = (123)^{-682}\bar{3}\bar{2}$
33	34	5	40	41	12	$41 = 23(123)^{1365}12$
34	35	6	25	26	39	$26 = \bar{1}(123)^{-681}\bar{3}$
35	36	7	26	27	40	$27 = 3(123)^{1364}$
36	37	8	27	28	41	$28 = (123)^{-2730}\bar{3}\bar{2}$
37	38	9	12	13	26	$13 = (123)^{1365}1$
38	39	10	13	14	27	$14 = \bar{1}(123)^{-2729}\bar{3}$
39	40	11	14	15	28	$(\bar{1}(123)^{-2729}\bar{3})(\bar{2}\bar{1})(123)^{-2730}\bar{3}\bar{2} \sim (123)^{-5461}$
40	41	12	28	29	42	$42 = \bar{2}\bar{1}(123)^{2730}$
41	42	13	41	42	13	$(23(123)^{1365}12)(\bar{2}\bar{1}(123)^{2730})(123)^{1365}1) \sim (123)^{5461}$
42	1	14	42	1	14	$(\bar{2}\bar{1}(123)^{2730})(1)(\bar{1}(123)^{-2729}\bar{3}) \sim (123)^{0}$

Note Let $T_u = \frac{1}{3}(2^u - (-1)^u)$.

u	1	2	3	4	5	6	7	8	9	10	11	12	13	14
T_n	1	1	3	5	11	21	43	85	171	341	683	1365	2731	5461

A.2 Proof of Lemma 3.2 in the case $(n, k, l) = (15, 1, 10)$

We have $n = 15$, $l = 10$, so shift $= 2l - 1 = 4$.

Relators

0,0	1,14	2,5	1,0	2,14	0,5	2,0	0,14	1,5
0,1	1,0	2,6	1,1	2,0	0,6	2,1	0,0	1,6
0,2	1,1	2,7	1,2	2,1	0,7	2,2	0,1	1,7
0,3	1,2	2,8	1,3	2,2	0,8	2,3	0,2	1,8
0,4	1,3	2,9	1,4	2,3	0,9	2,4	0,3	1,9
0,5	1,4	2,10	1,5	2,4	0,10	2,5	0,4	1,10
0,6	1,5	2,11	1,6	2,5	0,11	2,6	0,5	1,11
0,7	1,6	2,12	1,7	2,6	0,12	2,7	0,6	1,12
0,8	1,7	2,13	1,8	2,7	0,13	2,8	0,7	1,13
0,9	1,8	2,14	1,9	2,8	0,14	2,9	0,8	1,14
0,10	1,9	2,0	1,10	2,9	0,0	2,10	0,9	1,0
0,11	1,10	2,1	1,11	2,10	0,1	2,11	0,10	1,1
0,12	1,11	2,2	1,12	2,11	0,2	2,12	0,11	1,2
0,13	1,12	2,3	1,13	2,12	0,3	2,13	0,12	1,3
0,14	1,13	2,4	1,14	2,13	0,4	2,14	0,13	1,4

Reordered relators

r_0:	0,0	1,14	2,5	s_0:	2,5	0,4	1,10	u_0:	1,10	2,9	0,0
r_1:	0,4	1,3	2,9	s_1:	2,9	0,8	1,14	u_1:	1,14	2,13	0,4
	0,8	1,7	2,13		2,13	0,12	1,3		1,3	2,2	0,8
	0,12	1,11	2,2		2,2	0,1	1,7		1,7	2,6	0,12
	0,1	1,0	2,6		2,6	0,5	1,11		1,11	2,10	0,1
	0,5	1,4	2,10		2,10	0,9	1,0		1,0	2,14	0,5
	0,9	1,8	2,14		2,14	0,13	1,4		1,4	2,3	0,9
	0,13	1,12	2,3		2,3	0,2	1,8		1,8	2,7	0,13
	0,2	1,1	2,7		2,7	0,6	1,12		1,12	2,11	0,2
	0,6	1,5	2,11		2,11	0,10	1,1		1,1	2,0	0,6
	0,10	1,9	2,0		2,0	0,14	1,5		1,5	2,4	0,10
	0,14	1,13	2,4		2,4	0,3	1,9		1,9	2,8	0,14
	0,3	1,2	2,8		2,8	0,7	1.13		1,13	2.12	0,3
	0,7	1,6	2,12		2,12	0,11	1,2		1,2	2,1	0,7
r_{14}:	0,11	1,10	2,1	s_{14}:	2,1	0,0	1,6	u_{14}:	1,6	2,5	0,11

Kill $1,10$ and $2,5$ to obtain the following consequences.

$1,14 = (0,0)^{-1}$	$0,4 = (0,0)^{0}$	$2,9 = (0,0)^{-1}$
$1,3 = (0,0)^{0}(0,0)$	$0,8 = (0,0)(0,0)$	$2,13 = (0,0)(0,0)^{0}$
$1,7 = (0,0)^{-2}(0,0)^{-1}$	$0,12 = (0,0)^{-1}(0,0)^{-1}$	$2,2 = (0,0)^{-1}(0,0)^{-2}$
$1,11 = (0,0)^{2}(0,0)^{3}$	$0,1 = (0,0)^{3}(0,0)^{3}$	$2,6 = (0,0)^{3}(0,0)^{2}$
$1,0 = (0,0)^{-6}(0,0)^{-5}$	$0,5 = (0,0)^{-5}(0,0)^{-5}$	$2,10 = (0,0)^{-5}(0,0)^{-6}$
$1,4 = (0,0)^{10}(0,0)^{11}$	$0,9 = (0,0)^{11}(0,0)^{11}$	$2,14 = (0,0)^{11}(0,0)^{10}$
$1,8 = (0,0)^{-22}(0,0)^{-21}$	$0,13 = (0,0)^{-21}(0,0)^{-21}$	$2,3 = (0,0)^{-21}(0,0)^{-22}$
$1,12 = (0,0)^{42}(0,0)^{43}$	$0,2 = (0,0)^{43}(0,0)^{43}$	$2,7 = (0,0)^{43}(0,0)^{42}$
$1,1 = (0,0)^{-86}(0,0)^{-85}$	$0,6 = (0,0)^{-85}(0,0)^{-85}$	$2,11 = (0,0)^{-85}(0,0)^{-86}$
$1,5 = (0,0)^{170}(0,0)^{171}$	$0,10 = (0,0)^{171}(0,0)^{171}$	$2,0 = (0,0)^{171}(0,0)^{170}$
$1,9 = (0,0)^{-342}(0,0)^{-341}$	$0,14 = (0,0)^{-341}(0,0)^{-341}$	$2,4 = (0,0)^{-341}(0,0)^{-342}$
$1,13 = (0,0)^{682}(0,0)^{683}$	$0,3 = (0,0)^{683}(0,0)^{683}$	$2,8 = (0,0)^{683}(0,0)^{682}$
$1,2 = (0,0)^{-1366}(0,0)^{-1365}$	$0,7 = (0,0)^{-1365}(0,0)^{-1365}$	$2,12 = (0,0)^{-1365}(0,0)^{-1366}$
$1,6 = (0,0)^{2730}(0,0)^{2731}$	$0,11 = (0,0)^{2731}(0,0)^{2731}$	$2,1 = (0,0)^{2731}(0,0)^{2730}$

$$\begin{aligned} H' &= \langle 0,0 \mid r_{14}, s_{14}, u_{14} \rangle \\ &= \langle 0,0 \mid (0,0)^{5462}(0,0)^{0}(0,0)^{5461}, (0,0)^{5461}(0,0)^{1}(0,0)^{5461}, (0,0)^{5461}(0,0)^{0}(0,0)^{5462} \rangle \\ &= \mathbb{Z}_{10923} \end{aligned}$$

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