Debt Overhang and Non-Distressed Debt Restructuring*

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Abstract

In this paper, we analyse the restructuring of debt in the presence of debt overhang. The firm starts out with a debt liability and an investment opportunity. Then with unrestructured debt, the firm maintains the current borrowing payments until default or investment. If the creditors allow the parties to restructure the debt with exchange offers, then the borrowing payments change as well as the default and investment points. We find that there is a unique optimal restructuring path which maintains debt at positive levels but defers default indefinitely. This path is optimal regardless of whether the debt holders or the firm control the process through superior bargaining power. Moreover, a debt-for-equity exchange to remove all existing debt takes place just before investment that is followed by the issue of an optimal amount of new debt as part of the funding for the investment cost. The optimal investment trigger is higher along the optimal restructuring path than it is for an unlevered firm. We discuss the findings in the light of existing empirical evidence.

JEL numbers: G32; G33; G34

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1 Introduction

In this paper, we study the restructuring of debt for a firm with debt overhang. As pointed out by Myers (1977), a debt overhang leads to underinvestment. The firm only services the debt payments in the region where the potential earnings flow is sufficiently high. Therefore, the firm might default before making the investment. Thus, changing the debt burden through debt restructuring can change both the timing of the default and investment. Therefore, we create a scenario where one party makes debt-for-equity exchange offers to reduce the debt burden, or equity-for-debt offers to increase the debt burden, that is accepted or rejected by the other party. Our primary focus is to study such debt restructuring.

The model is substantially equivalent to Myers (1977) except set within a continuous time framework, and furthermore the firm pays corporate taxes. The firm owns an investment opportunity as its only asset and undertakes an obligation to pay a constant coupon flow indefinitely. The firm has deep pockets and continues to inject cash to enable payments of the coupon flow until it is optimal to default, or it is optimal to make the investment and harvest the earnings flow. The firm can borrow more at investment. However, the option to make exchange offers to change the debt burden is valuable. The parties hold bargaining power that is perfectly and unevenly distributed (either 100% to the firm and 0% to the debt holders or the other way around). The party with the bargaining power can make exchange offers to the other party in the form and at a time that is optimal. We ask two questions: How and when is the debt overhang restructured? Is the debt restructuring process efficient?

The answer to the first question is that firms actively restructure debt in all non-distressed states of nature through small debt-for-equity and equity-for-debt exchanges. The restructuring path is the same whether the debt holder or the firm holds the bargaining power. Surprisingly, the firm maintains an optimal positive level of debt in all states before investment, but replaces all existing debt with new debt as part of the investment process. Therefore, the firm defers the removal of debt in anticipation of investment until the investment happens. Leverage is valuable before investment because of the tax advantage of debt. However, to carry old debt over the investment threshold causes distortions to the timing of the investment decision. Therefore, the firm makes a massive debt restructuring just before the investment to remove these distortions. The firm immediately takes on new debt to acquire a new tax shield. The answer to the second question is that the debt restructuring is efficient. Regardless of which party controls the debt restructuring process through its bargaining power, the debt restructuring follows the same path where the value of the firm is always maximised.

There are additional features to note. First, with optimal debt overhang, the investment trigger along the optimal debt restructuring path is higher than the investment trigger for a corresponding unlevered firm. A levered firm takes advantage of the debt tax shield before investment that an unlevered firm cannot. Both firms choose the optimal leverage after investment. Increasing the investment trigger is optimal for the levered firm to take advantage of the debt tax shield before investment. Second, the debt restructuring path leads to a reduction in borrowing if the firm is close to default, which lowers the default trigger to the point that default never happens. In existing optimal capital structure models, the option to exercise limited liability tends to be more valuable than reducing the debt burden for the firm (see, e.g., Dangl and Zechner (2004)). In our model, the debt holder and the firm both have a stake in maximising the value of the investment opportunity. Debt restructuring achieves this objective by deferring default indefinitely. Post-investment, the situation changes as non-distressed debt restructuring is typically not feasible. Instead, the parties engage in debt restructuring only in distressed states of nature.

This paper falls into the study of Coasian renegotiation of debt contracts, but the focus on debt overhang situations makes the results appear different from existing models in this area. The unconditional promise of payment that firms imply by borrowing can lead to ex-post inefficient defaults. The literature predicts, therefore, the renegotiation of distressed debt. For instance, the firm might choose to default on their debt even if the liquidation value is less than the continuation value of the firm. In this case, debt restructuring generates a bargaining surplus for both parties. Such an ex-post inefficient default has been the primary focus of the debt restructuring theory such as in Hart and Moore (1998), Mella-Barral and Perraudin (1997), and Mella-Barral (1999). Debt overhang is, in contrast, a situation where the mix of debt and equity ex-ante distorts the investment decision. Therefore, the debt restructuring process is principally aimed at managing the borrowing policy in non-distressed states to avoid such distortions.

Our theoretical results are consistent with the empirical findings that models of distressed debt restructuring such as in Hart and Moore (1998) and Mella-Barral and Perraudin (1997) cannot easily explain. First, non-distressed debt restructuring is a common occurrence, see Roberts and Sufi (2009) and Nikolaev (2015). Second, the debt holders often receive equity as payment for debt in workouts, see Franks and Torous (1994). Moreover, equity as payment for debt is associated with the firm's growth opportunities, see James (1995). And, the model predicts that firms restructure the bulk of their old debt only upon reaching the investment point, followed by new debt issues to fund investment. The pattern of the retirement of old debt followed by sizeable new debt corresponds to the finding of large debt-for-debt exchanges for fast-growing firms, see Gilson and Warner (1998).

Industry competition, however, matters for our conclusions. In an extension of the model, we address the problem of renegotiating the debt overhang in firms that operate in a competitive industry. In contrast to our primary model, the firm optimally writes off its debt overhang immediately. The reason is that the firm needs to unburden itself from the debt liability as quickly as possible to be able to compete with its unlevered peers. Therefore, industry competition matters to the debt-restructuring path.

A paper close to ours is Manso (2008) who studies the problem of risk shifting. The problem in Manso (2008) is that existing debt distorts the choice of risk in new investments. This choice can be reversed by default when the debt holder forecloses on the firm's assets and find it optimal to switch back to less risky investments. In our debt overhang model, the timing of the firm's investment is distorted not the level of risk. When the debt holder takes control of the investment opportunity in default states, it restores optimal timing. Debt restructuring, however, fixes the problem before the default state. Manso (2008) does not consider debt restructuring in non-default states. Another related paper is Pawlina (2010). But, a fundamental difference is that Pawlina (2010) restricts debt restructuring to default states. Mella-Barral and Perraudin (1997) and Mella-Barral (1999) use a similar modelling technology but do not consider debt overhang, as is the case for Hart and Moore (1998). Hart and Moore (1998) use different modelling techniques.

In Section 2, we describe the model. In Section 3, we solve the model for when debt restructuring is not allowed to happen. Section 4 outlines the optimal solutions for debt restructuring in a non-competitive industry. In Section 5, we analyse a version of the model set in a competitive industry. Section 6 has a description of the empirical predictions of the model, and Section 7 is the conclusion.

2 Model

In broad terms, the following describes the model. A firm's only asset is an investment opportunity. A potential earnings process y_t represents the project's profitability. The process y_t is an observable geometric

Brownian motion with drift μ and diffusion σ:

$$dy_t = y_t\left(\mu dt + \sigma dB_t\right). \tag{1}$$

The net cost of the investment opportunity is I. The risk-free rate is $r > \mu$, and a risk-neutral probability measure governs the Brownian motion B_t. The firm has an exogenous debt liability which it continues to serve until default. The firm pays a corporate tax at the rate of τ on net earnings. We allow full loss offset provisions, so the firm always pays the coupon flow net of tax. This assumption is also made in related models such as Fischer et al. (1989), Goldstein et al. (2001) and Strebulaev (2007). Investors pay zero investor tax. The shareholders control the investment policy unless the firm defaults and the debt holder forecloses on the firm's asset. The key assumptions are as follows.

2.1 Leverage

- *The firm owns no other assets except the investment opportunity and has an existing exogenous debt liability with perpetual coupon flow c.*
- *The firm services the debt liability by injecting cash, and it has unlimited funds.*
- *The debt liability remains in place until the firm defaults, or it is renegotiated (along lines described below).*

The debt holder can thus expect to receive the coupon flow up to the point that the firm defaults. The assumption that the firm continuously injects cash to continue debt service is equivalent to Leland's (1994) assumption that the firm continually sells additional equity to fund the coupon flow. Unlike Hart and Moore (1998), we assume the firm has deep pockets and never becomes cash constrained. The firm injects cash, therefore, as long as this is in its interest to do so.

2.2 Default and Debt Recovery

• *If the firm defaults on the debt liability at a potential earnings level y, then the full value of the firm is given by the function $X(y)$.*

• *If the firm defaults on the debt liability at the realised earnings level y after the investment has been made, then the full value of the firm is given by the function $\bar{X}(y)$. In line with Mella-Barral and Perraudin (1997), we implement the assumption that $\bar{X}(y)$ is the unlevered value of the firm, that is, $\bar{X}(y) = \frac{y(1-\tau)}{r-\mu}$.*

• *The debt holder recovers in either case $0 \leq \xi \leq 1$ percent of the value of $X(y)$ or $\bar{X}(y)$ which depends on whether the default happens before or after the investment is made, respectively.*

Mella-Barral and Perraudin (1997) assume $\bar{X}(y)$ is the unlevered value of the firm, whereas Leland (1994) assumes $\bar{X}(y)$ is unaffected by the capital structure (i.e., the unlevered, untaxed value of the firm). A realistic value is likely to be somewhere in between these two where leverage is a trade-off between the tax benefits and the financial distress costs of borrowing. Our choice of $\bar{X}(y)$ underestimates the value of borrowing in default; but since the recovered value is $\xi\bar{X}(y)$ where the constant ξ is arbitrary, it can reflect the gains from leverage.[1]

Pawlina (2010) assumes $X(y)$ is zero, but this assumption might be unrealistic because the debt holder cannot recover any value from an investment opportunity. Tax benefits from leverage and costs linked to the debt overhang might be embedded in $X(y)$. Moreover, $X(y)$ should also reflect the value of further debt restructuring (the point in Mella-Barral and Perraudin (1997) is precisely that debt restructuring

[1]This argument is not entirely satisfactory. Both recovery values and the derivative of recovery values enter the analysis. Even if calibrating ξ such that the recovered value of the unlevered firm matches the optimally levered value of the firm, it does not follow that one matches the derivatives.

avoids deadweight costs in default). We merely make $X(y)$ and $\bar{X}(y)$, which represent exit points from the model, exogenous to focus on the debt restructuring that takes place within the context of the model.

2.3 Investment

- *The firm has control of the investment process until default.*
- *New debt issued at the investment stage is always junior to the firm's existing debt.*
- *All existing debt at the investment stage remains in place after investment.*
- *Further changes to the capital structure after the investment point are not permitted.*

The model prevents a transfer of the control of the investment process from the firm in any other way than through default. In practice, debt contracts contain covenants restricting the firm's choices concerning its assets (e.g., controlling risk). Since our model is about the investment in an asset that generates positive cash flow, the debt holder has no interest in reducing the firm's ability to invest; so in the context of our model, this assumption is not particularly restrictive. It could be restrictive in cases where the firm could increase the riskiness of the investment (see Manso (2008)). The firm is allowed to put in place a capital structure that maximises its wealth at the point of investment, but it cannot make further changes to the capital structure. We restrict the analysis to the case where the firm's existing debt is senior to all new debt, and where no changes to the capital structure happen after investment. A body of literature on debt overhang with varying degrees of seniority exists (see Sundaresan et al. (2015)) and on the use of secured debt (see Hennessy and Whited (2005)) exist. This assumption is relaxed, therefore, elsewhere. There is also a vast literature on dynamic capital structure choices under transaction costs, see, for example, Fischer et al. (1989), Dangl and Zechner (2004), Titman and Tsyplakov (2007) and Srebulaev (2007). If leverage changes are free of cost, then such changes would be continuous, see Leland (1994). Leland (1994), however, makes the point that increases in borrowing are likely to be resisted by the existing debt holder because of dilution effects, and reductions in borrowing are never optimal for the firm. These effects mean that continuous changes to leverage might never happen even if the firm has the option to carry them out. In our paper, we do not engage with the issue of optimal borrowing after investment and merely prevent changes from happening. But, this simplification has no bearing on our results beyond a potential underestimation of the actual value of the firm at the investment point.

2.4 Debt Restructuring and Bargaining Power

- *Debt restructuring can take place at zero cost in the earnings window between default and investment, that is, when the earnings are too large for a default to be optimal and too small for the investment to be optimal.*
- *The bargaining power in debt renegotiations is always perfectly and unevenly distributed with either the firm holding 100% of the bargaining power at all times or the debt holder holding 100% of the bargaining power at all times.*
- *The party with bargaining power chooses the timing and form of debt restructuring, which the party without bargaining power can accept or reject in a take-it-or-leave-it offer.*
- *The parties cannot use cash payments as part of the debt restructuring process. If the rejection of an offer takes place the restructuring game stops, and it is not possible to make further debt restructuring offers.*

Although the firm always control the investment process, they can not control the debt restructuring process. Debt restructuring is controlled by the party that holds the bargaining power. The assumptions regarding bargaining power are problematic for several reasons. First, in practice, some sharing of bargaining power should occur. In theory, however, it is convenient to focus on the extreme cases. Since the two parties disagree in general on the timing of the debt restructuring, the party that prefers delay can reject the offer and defer agreement. The profitability of suspending agreement is a function of the bargaining power. With our assumptions, the party without bargaining power can never expect to earn rent in the bargaining process, now or in the future. Therefore, the value of vetoing an agreement that meets the reservation level is zero, which simplifies the analysis. We leave out a rigorous study of shared bargaining power.

The restrictions on cash payments serve to rule out any promise by the firm or the debt holder to make cash payments to the other party as part of a debt restructuring game. The main issue is to prevent a debt holder and a firm to make untaxed cash payments to each other in exchange for an increase in after-tax coupon payments. Since the model assumes taxation only at the corporate level with full offset provisions, this strategy represents a "money machine" that generates a tax subsidy. The assumption of stopping at rejection is made to restrict the strategy options for the players. In continuous time games, the set of equilibria can be hard to analyse with necessary rigour. For instance, the concept of rejection and a follow-up offer "in the next instance" require technical modelling assumptions that lie outside our model. See Rosu (2006) for a discussion of continuous time games.

Existing debt restructuring games include the continuous auction model used in Mella-Barral and Perraudin (1997) and Mella-Barral (1999), and the continuous bargaining model used in Pawlina (2010). These models assume debt forgiveness or payment holidays where temporary changes to the payment schedule are made to avoid the deadweight costs of default, but ultimately the original contractual payment schedule resumes. Avoiding the deadweight cost of default generates a bargaining surplus for both parties.

Our model rules out such renegotiations and focuses instead on the debt restructuring that happens in the window between default and investment. Within this window, the existing contract is by definition acceptable for both parties, but it might not be the optimal contract because it distorts the timing of the investment decision. Therefore, we need to consider permanent changes to the contractual arrangements in the form of debt-for-equity or equity-for-debt exchanges as a means of debt restructuring. An agreement for a party to pay more (or less) in current states must be matched by a promise to pay less (or more) in future states. Payment holidays of the type mentioned above cannot achieve this. An example of a game using exchange offers is Christensen et al. (2014) in the context of adjustments to capital structures with callable debt.

3 Unrestructured Debt

In this section, we solve the model by using all the modelling assumptions presented in Section 2 except those outlined in subsection 2.4 which allow take-it-or-leave-it offers. Therefore, paragraphs 2.1-3 fully describe the model. The firm observes the potential earnings flow y_t and makes one of three decisions. First, they do nothing and continue paying the coupon flow c according to the original contractual arrangements. Second, they default on the coupon flow c which leads to the debt holder foreclosing on the firm's assets. Third, they invest at cost I which yields the (now realised) earnings flow y_t. The model continues as long as the firm chooses the first option. The model stops at the second option when the debt holder forecloses on the investment opportunity. The model also ends at the investment point, which creates a levered firm with a risky earnings flow (as in Leland (1994)).

3.1 Trigger Strategies

We study the use of trigger strategies to stop the model. Let y^* denote the default trigger and y_I the investment trigger, such that for $y^* < y_t < y_I$ the firm continues paying the coupon flow. The first time y_t equals y^* or y_I, represented by the stopping time T, the firm respectively defaults or invests. We use V (with no bar) to denote the value of the firm and V_B to denote the value of the firm's debt for $t < T$. We also use $\bar{V}$ (with bar) to denote the value of the firm and $\bar{V}_B$ to denote the value of the firm's debt for $t = T$. Since the debt is never restructured, the debt holder recovers $\xi X(y)$ from a default before the investment is made and $\xi \bar{X}(y^*)$ after. At investment, the firm's value is $\bar{V}(y_I, c)$, which is shared between the (old) debt, $\bar{V}_B(y_I, c)$, and the equity, $\bar{V}(y_I, c) - \bar{V}_B(y_I, c)$. The firm has the option to borrow new debt at the point of investment and incur a new coupon liability of $\delta \geq 0$. The value of the new debt is $\bar{V}_B(y_I, \delta)$. The firm funds the investment net of selling new debt. So, correcting for the contribution of the new debt, the net equity value is $\bar{V}(y_I, c) - \bar{V}_B(y_I, c) - I$. Building on the boundary conditions outlined above, we can express the debt values in terms of expected cash flows:

$$V_B(y_t, c) = \mathbb{E}\left(\int_t^T e^{-r(s-t)} c ds + e^{-r(T-t)} \left(\mathbb{I}\xi X(y^*) + (1 - \mathbb{I})\bar{V}_B(y_I, c)\right)\right), \tag{2}$$

where $\mathbb{I}$ is an indicator function which takes the value one for $y_T = y^*$ and zero for $y_T = y_I$. Similarly, the value of the equity is the residual value until default or investment happens:

$$V(y_t, c) - V_B(y_t, c) = \mathbb{E}\left(-\int_t^T e^{-r(s-t)} c(1-\tau) ds + e^{-r(T-t)}(1 - \mathbb{I})(\bar{V}(y_I, c) - \bar{V}_B(y_I, c) - I)\right). \tag{3}$$

Combining (2) and (3), we find the value of the firm for $t < T$:

$$V(y_t, c) = \mathbb{E}\left(\int_t^T e^{-r(s-t)} c\tau ds + e^{-r(T-t)}\left(\mathbb{I}\xi X(y^*) + (1-\mathbb{I})(\bar{V}(y_I, c) - I)\right)\right), \tag{4}$$

where we recognise that the firm benefits from the tax shield of borrowing until the stopping time T at the cost of deadweight losses in the event that $y_T = y^*$.

The essential problem in this section is to choose the optimal stopping time T. Consider first $y_T = y_I$. At investment, the firm can incur new junior debt, and a new default trigger is formed which depends on the total coupon flow $c+\delta$, which is denoted $\bar{y}^*$ (we impose the same notational convention outlined above and use a single bar to indicate that the trigger applies to the period after the investment is made). We can use Leland's (1994) formula for default directly which yields the default trigger $\bar{y}^* = \frac{c+\delta}{r}(r-\mu)\frac{\lambda_1}{\lambda_1 - 1}$, where λ_1 is given by $\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \left(\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right)^{1/2}$. The restructured value of the firm at the default trigger is $\bar{X}(y^*)$ from which the debt holder can recover a fraction ξ. The value of the firm after investment, $\bar{V}(y_t, c)$, can therefore be written in the following way:

$$\bar{V}(y_t, c) = \frac{y_t(1-\tau)}{r-\mu} + \frac{(c+\delta)\tau}{r}\left(1 - \left(\frac{y_t}{\bar{y}^*}\right)^{\lambda_1}\right) - (1-\xi)\bar{X}(y^*)\left(\frac{y_t}{\bar{y}^*}\right)^{\lambda_1}, \tag{5}$$

In equation (5) the firm's value is decomposed into three terms on the right-hand side. The first term is the discounted value of the unlevered earnings flow after tax. The second term is the value of the debt tax shield in non-default states. The probability $1 - \left(\frac{y}{\bar{y}^*}\right)^{\lambda_1}$ can be interpreted as the risk neutral probability of no default. Finally, the third term is the expected value of the deadweight costs of default, which happens with risk neutral probability $\left(\frac{y}{\bar{y}^*}\right)^{\lambda_1}$.

Using (5), we obtain expressions for the optimal level of new debt δ at $y_T = y_I$ provided we have an expression for $\bar{X}(y)$. Next, assume first that $c = 0$ so that all debt is issued at the point of investment. By differentiating $\bar{V}(y_I, 0)$ with respect to δ, we find the optimal debt $\hat{\delta}$ which is the first best level of debt,

$$\hat{\delta} = \left(\frac{r}{r-\mu}\frac{\lambda_1 - 1}{\lambda_1}\right)\left(\frac{\tau}{\tau - \lambda_1(1-\xi(1-\tau))}\right)^{-\frac{1}{\lambda_1}} y_I = \left(\frac{r}{r-\mu}\frac{\lambda_1 - 1}{\lambda_1}\right)\pi^{-\frac{1}{\lambda_1}} y_I, \tag{6}$$

where we have defined $\pi = \frac{\tau}{\tau - \lambda_1(1-\xi(1-\tau))}$. The expression in (6) is linear in y_I which means that the risk neutral probability of default, $\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1}$, is independent of y and given by $\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1} = \pi$ at the investment trigger y_I. Therefore, the optimal investment trigger y_I can therefore be identified by standard smooth pasting techniques (shown for instance in Dixit (1993)):

$$\begin{aligned} y_I &= \left[\frac{1-\tau}{r-\mu} + \frac{\tau}{r-\mu}\frac{\lambda_1 - 1}{\lambda_1}\left(\frac{y_I}{\bar{y}^*}\right)^{-1}\left(1 - \left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1}\right) - (1-\xi)\left(\frac{y_I}{\bar{y}^*}\right)^{-1}\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1}\right]^{-1} I\frac{\lambda_2}{\lambda_2 - 1} \\ &= \left[\frac{1-\tau}{r-\mu} + \frac{\tau}{r-\mu}\frac{\lambda_1 - 1}{\lambda_1}\pi^{-\frac{1}{\lambda_1}}(1-\pi) - (1-\xi)\pi^{1-\frac{1}{\lambda_1}}\right]^{-1} I\frac{\lambda_2}{\lambda_2 - 1}, \end{aligned} \tag{7}$$

where $\lambda_2 = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \left(\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right)^{1/2}$. Further, in this expression, y_I can be identified exactly since the ratio $\frac{y_I}{\bar{y}^*}$ is constant. If the tax rate is zero so that debt had no value, then the investment trigger is that of an unlevered firm, $\hat{\delta} = 0$ and $y_I = I(r-\mu)\frac{\lambda_2}{\lambda_2 - 1}$, which is identical to the investment trigger derived in Dixit (1993). Therefore, the expressions inside the large bracket are associated with the optimal debt tax shield and the cost of default.

Now consider that $c > 0$. In this case we cannot easily pin down the investment trigger point, and moreover the risk neutral probability of default is not necessarily independent of y or c at the investment trigger. The recovered assets in default at the default trigger $\bar{y}^*$ between the old debt holder (entitled to the coupon flow c) and the new one (entitled to the coupon flow δ) depends on seniority. Using the assumptions about seniority and debt recovery the value of the old debt after the investment is made is given by:

$$\bar{V}_B(y_t, c) = \frac{c}{r}\left(1 - \left(\frac{y_t}{\bar{y}^*}\right)^{\lambda_1}\right) + \min\left(\xi\frac{\bar{y}^*(1-\tau)}{r-\mu}, \frac{c}{r}\right)\left(\frac{y_t}{\bar{y}^*}\right)^{\lambda_1}, \tag{8}$$

and the value of the new debt is given by:

$$\bar{V}_B(y_t, \delta) = \frac{\delta}{r}\left(1 - \left(\frac{y_t}{\bar{y}^*}\right)^{\lambda_1}\right) + \left(\xi\frac{\bar{y}^*(1-\tau)}{r-\mu} - \min\left(\xi\frac{\bar{y}^*(1-\tau)}{r-\mu}, \frac{c}{r}\right)\right)\left(\frac{y_t}{\bar{y}^*}\right)^{\lambda_1}. \tag{9}$$

Both (8) and (9) have the same general structure. The first term is the contribution to the debt value from the coupon payments that are received in no-default states. This contribution is multiplied by the risk neutral probability of no default. The second term is the contribution to the debt value from the debt recovery that takes place in default states that is multiplied by the risk neutral probability of default. The firm can default in states where the recovered value of the firm exceeds the nominal claim of the old debt holder, which explains the use of the minimum operators in the second terms of (8) and (9). Equations (5), (8) and (9) provide the values of the firm and the debt after the investment is made, which can be used to work out the optimal timing of investment and default triggers prior to the investment decision.

3.2 Optimal Investment and Default

Before we proceed with the analysis, we define the following matrix which we use for the results that follow. The matrix provides the solution to smooth pasting problems in a compact way.

$$\mathbb{M}(y) := \begin{pmatrix} y^{\lambda_1} & y^{\lambda_2} \\ \lambda_1 y^{\lambda_1 - 1} & \lambda_2 y^{\lambda_2 - 1} \end{pmatrix}. \tag{10}$$

Appendix A explains the parameters $\lambda_1 = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$ and $\lambda_2 = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$. Also define $\Gamma(y_I, c, \delta) = \bar{V}(y_I, c) - \bar{V}_B(y_I, c) - I$, which is a measure of the equity value at the investment trigger point y_I. Γ depends on the old debt c but also on the new debt δ because the firm can borrow fresh debt as part of funding the investment cost. The objective is to derive the stopping times for the default and investment associated with a debt overhang c. The following result helps set out the conditions for optimal new debt at the investment point.

Lemma 1: *Assume $\xi \bar{X}(\bar{y}^*) < \frac{c}{r}$. The first order condition to the problem $\max_\delta \Gamma(y_I, c, \delta)$ is $\bar{\delta}$ which is implicitly given by the following equation:*

$$\ln y_I = \ln\left(\frac{r-\mu}{r}\frac{\lambda_1}{\lambda_1 - 1}c\right) + \frac{1}{\lambda_1}\ln\left(\frac{\tau}{\tau - 2\lambda_1}\right) + \frac{\lambda_1 + 1}{\lambda_1}\ln\left(1 + \frac{\bar{\delta}}{c}\right) + \ln\left(1 + \frac{\tau - \lambda_1}{\tau - 2\lambda_1}\frac{\bar{\delta}}{c}\right). \tag{11}$$

The solution $\bar{\delta}$ can be positive or negative depending on c.

The condition $\xi \bar{X}(\bar{y}^*) < \frac{c}{r}$ states that the recovered restructured value in default, the left-hand side, is less than the risk-free value of the senior debt holders, the right-hand side. Therefore, this restriction ensures that the old debt is risky post-investment. If the old debt becomes risk-free post-investment, the new debt depends directly on the unlevered solution given in (6). From Lemma 1, we have the following proposition.

Proposition 1 (Optimal New Junior Debt and Investment/Default Triggers): *There are three cases, listed in the following table:*

Case	Condition	Optimal δ	$\mathbb{P}(\text{Default}) = \left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1}$
1	$\bar{\delta} < 0$	0	$\left(\frac{r}{r-\mu}\frac{\lambda_1 - 1}{\lambda_1}\right)^{\lambda_1}\left(\frac{y_I}{c}\right)^{\lambda_1}$
2	$\bar{\delta} \geq 0,\ \xi \bar{X}(\bar{y}^*) < \frac{c}{r}$	$\bar{\delta}$	$\frac{\tau - \lambda_1(1-\xi(1-\tau))}{\tau - \lambda_1(2c+\delta)/(c+\delta)}\pi$
3	$\bar{\delta} \geq 0,\ \xi \bar{X}(\bar{y}^*) \geq \frac{c}{r}$	$\hat{\delta} - c$	π

In all cases, the optimal default trigger y^* and the investment trigger y_I are determined by the following system:

$$\mathbb{M}^{-1}(y^*)\begin{pmatrix} \frac{c}{r}(1-\tau) \\ 0 \end{pmatrix} = \mathbb{M}^{-1}(y_I)\begin{pmatrix} \Gamma(y_I, c, \delta) + \frac{c}{r}(1-\tau) \\ \Gamma'(y_I, c, \delta) \end{pmatrix} \tag{12}$$

The three cases differ mainly in the firm's ability to make use of new junior debt. In Case 1 the firm has already exceeded the optimal threshold of debt through its debt overhang, so no new debt is issued. In Case 3 the debt overhang is so small that the amount of new debt is merely the difference between the desired level of borrowing for an unlevered firm, that is, $\hat{\delta}$, and the old debt, c. Case 2 is somewhere in between. The firm borrows new debt, but the debt overhang distorts the total debt burden.

We identify three sources of inefficiency. First, the fact that the firm can choose default before investment means a loss due to restructuring in default. Second, the debt overhang distorts the timing of the investment decision. Third, the debt overhang prevents the firm from obtaining optimal leverage at the investment trigger. These three sources of inefficiency generate incentives for the firm to restructure its debt overhang.

4 Debt Restructuring

In this section, we analyse the full model.[2] The procedure consists of two steps. First, we give the parties the option to renegotiate the debt away in one lump sum and investigate the optimal timing of conversion to unlevered status. Second, we allow smaller debt-for-equity or equity-for-debt exchanges before the firm carries out a full conversion to unlevered status.

4.1 Optimal Timing of Conversion to Unlevered Status

In this subsection, we restrict the set of actions to debt-for-equity exchange offers where equity replaces all existing debt. Therefore, the outcome is a conversion to unlevered status. As in Section 3, we use trigger strategies to investigate this issue. We define a stopping time T where the firm makes a debt-for-equity offer to remove the entire debt burden. There may be multiple triggers for such offers. Stopping time T is the first opportunity to one of these trigger values. Let $y_T = \hat{y}$. If $\hat{y} < y^*$ and the firm holds the bargaining power, then the debt holder needs to receive at least $\xi X(\hat{y})$ worth of equity and the firm retains the residual equity $(1-\xi)X(\hat{y})$. If the debt holder holds the bargaining power, then the firm needs to receive at least

[2] In a previous draft, we included the possibility of distressed debt restructuring, but that approach was not fruitful. Since the firm never invests in default states, debt forgiveness does not improve investment efficiency.

zero with the debt holder retaining all equity $X(\hat{y})$. Similarly if $\hat{y} \geq y^*$ and the firm holds the bargaining power, then the debt holder needs to receive at least $V_B(\hat{y}, c)$ worth of equity with the firm retaining the residual equity $V(\hat{y}, 0) - V_B(\hat{y}, c)$. If the debt holder holds the bargaining power, the firm needs to receive at least $V(\hat{y}, c) - V_B(\hat{y}, c)$ with the debt holder retaining the residual equity $V_B(\hat{y}, c) + V(\hat{y}, 0) - V(\hat{y}, c)$. These constraints form the following result:

Proposition 2 (Optimal Conversion to Unlevered Status): *Consider a debt overhang c and associated unrestructured default and investment trigger y^* and y_I respectively, that satisfy (12). There are two restructuring point $\hat{y}_L \leq \hat{y}_H$. The lower trigger point $\hat{y}_L = y^*$ and the upper trigger point $\hat{y}_H$ solve the following system:*

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} \Gamma(\hat{y}_H, 0, \hat{\delta}) - \frac{c\tau}{r} \\ \\ \Gamma'(\hat{y}_H, 0, \hat{\delta}) \end{pmatrix} = V(\hat{y}_L, 0) - \frac{c\tau}{r}, \tag{13}$$

where $\Gamma(\hat{y}, 0, \hat{\delta}) = \bar{V}(\hat{y}, 0) - I$, and the new debt $\hat{\delta}$ is determined by (6). For $y_t \in (\hat{y}_L, \hat{y}_H)$, the debt holder or the firm, no matter which one holds the bargaining power, holds out for a restructuring until $y_t = \hat{y}_L$ or $y_t = \hat{y}_H$, whichever happens first, where a conversion to unlevered status occurs. For $y_t \notin (\hat{y}_L, \hat{y}_H)$ there is immediate restructuring. If the debt holder holds the bargaining power, then it offers the firm new equity worth $V(\hat{y}, c) - V_B(\hat{y}, c)$ at the restructuring point $\hat{y}$, and if the firm holds the bargaining power, then it offers the debt holder new equity worth $V_B(\hat{y}, c)$.

For the debt holder, the problem is to find the optimal time to switch a coupon flow c for a share in the potential earnings flow y_t. Any restructuring in the region below y^* cannot happen because the debt holder cannot make cash payments to the firm, so the firm instead exercises its right to default on the

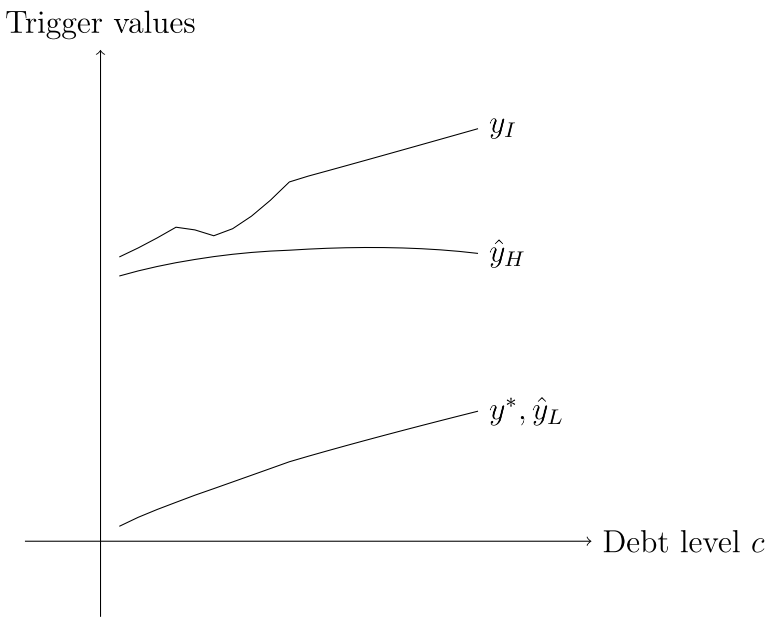

Figure 1: The figure shows the unrestructured default and investment triggers (y^* and y_I, resp.) and the lower and upper restructuring triggers ($\hat{y}_L$ and $\hat{y}_H$, resp.).

coupon flow. Moreover, any promise of an increased equity stake in the firm to continue debt service for $y_t < y^*$ is not credible. Therefore, the lower restructuring point is given by $\hat{y}_L = y^*$. When the firm holds the bargaining power, restructuring the debt is always optimal is always optimal at the default trigger y^* rather than below because any buyout of debt below the default trigger must compensate the debt holder for ξ percent of the restructured value of the firm. Therefore, any restructuring below y^* yields the same sharing of equity between the debt holder and the firm but costs the firm additional coupon payments. Therefore, the firm also prefers that the lower restructuring trigger is at y^*. What may seem surprising is that the optimisation problem that determines the upper restructuring point is identical for the two parties. This is so because the efficiency gain is captured entirely by the party holding the bargaining power. So, the optimal timing of the conversion to equity, which leads to investment, is the same for either party.

In Figure 1, we show the numerical values for the default trigger and the lower restructuring point (y^* and $\hat{y}_L$) as well as the investment trigger with unrestructured debt (y_I) and the upper restructuring point ($\hat{y}_H$). We note that the investment trigger curve shows erratic behaviour near the left-hand starting point. The irregularity is because, with unrestructured debt, the ability to issue new debt at investment (δ) depends on the current debt levels. The transition from Case 1 to Case 2 in Proposition 1 leads to a benefit for the firm as it can now issue new debt at the investment point. The transition from Case 2 to Case 3 leads to a disadvantage for the firm as the old debt now becomes risk-free. There is no noticeable irregularity in the default trigger curve nor the upper restructuring point curve. Near default, the amount of new debt issued at the investment trigger has a negligible impact; for the upper restructuring point, δ does not affect the investment decision. We also note that while the lower restructuring trigger (identical to the unrestructured default trigger) increases monotonically with the debt liability, the upper restructuring point, $\hat{y}_H$, is non-monotonic. As $c \to 0$ the default trigger goes to zero and the investment trigger goes to the unlevered investment trigger. At the limiting point, there is no debt liability to restructure, but the limiting values of the upper restructuring point do go towards the unlevered investment trigger as the existing debt liability becomes small. At first, the upper restructuring trigger increases with the debt liability, but beyond a certain threshold, it starts to decrease. We use this feature to generate a full restructuring equilibrium in the next subsection. Before that, we address the question of whether the restructuring policy maximises the firm's value as well as the value to the individual party with the bargaining power. The fact that the firm and the debt holder agree on the restructuring policy suggests it is, and we confirm this with the following result:

Proposition 3 (Restructuring Efficiency): *The restructuring policy described in Proposition 3 max-*

imises the value of the restructured firm.

Why is the investment trigger with the restructuring policy in Proposition 2 not the same as the investment trigger for the unlevered firm? The answer lies in the treatment of taxes. With full offset provisions the firm pays only the after-tax coupon $(1-\tau)c$ whereas the debt holder receives the full coupon flow c. Therefore, the levered firm earns a tax credit flow $c-(1-\tau)c = \tau c$ between the default trigger and the investment trigger. The tax credit increases the value of the levered firm relative to the unlevered firm. In practice, firms in the US can to some extent carry a loss to a period where they make offsets. Therefore, our model overstates the tax benefit from borrowing during a period where the firm does not earn income. An alternative tractable modelling specification is that corporate taxes are paid only when the firm has positive earnings (Titman and Tsyplakov (2007) and Pawlina (2010)). However, this modelling specification understates the tax effects. Most tax systems allow losses to be carried forward for some time. The reality lies somewhere in between these two modelling extremes. Therefore, the efficiency improvements arise in the context of allowing the levered firm to obtain some tax credits in the period before restructuring the debt. A debt overhang can, therefore, be valuable. A full debt restructuring avoids the distortions to the investment trigger just before investment.

In this subsection, we have assumed repurchases of debt with equity in a single transaction. In Mao and Tserlukevich (2015), firms use cash to repurchase debt. US firms that operate under Chapter 11 can accumulate cash reserves, which makes this a plausible proposition. Our model does not allow for Chapter 11 creditor protection; and Franks and Torous (1994) show that the use of equity, preferred stock or new debt is more common for debt restructuring by firms that operate outside Chapter 11. The management of debt in between the two restructuring points motivates the search for an overall restructuring equilibrium which we carry out next.

4.2 Full Restructuring Equilibrium

In this section, we characterise the full restructuring equilibrium. Before we state the problem, we define the following expressions:

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \mathbb{M}^{-1}(\hat{y}_L) \begin{pmatrix} \frac{c}{r}(1-\tau) \\ 0 \end{pmatrix} - \mathbb{M}^{-1}(y_I) \begin{pmatrix} \Gamma(y_I, c, \delta) + \frac{c}{r}(1-\tau) \\ \Gamma'(y_I, c, \delta) \end{pmatrix}, \tag{14}$$

which is the condition described in Proposition 1, and

$$K_3 = \begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} \Gamma(\hat{y}_H, 0, \hat{\delta}) - \frac{c\tau}{r} \\ \Gamma'(\hat{y}_H, 0, \hat{\delta}) \end{pmatrix} - \left(V(\hat{y}_L, 0) - \frac{c\tau}{r} \right), \tag{15}$$

which is the condition described in Proposition 3. The expression in (14) determines the unrestructured default trigger $\hat{y}_L$ and the unrestructured investment trigger y_I when K_1 and K_2 are equal to zero and the expression in (15) determines the upper restructuring trigger $\hat{y}_H$ when K_3 is equal to zero. Consider the following Lagrange program:

$$\max_{c, \hat{y}_L, \hat{y}_H} L(c, \hat{y}_L, \hat{y}_H | y) = \begin{pmatrix} y^{\lambda_1} & y^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} \Gamma(\hat{y}_H, 0, \hat{\delta}) - \frac{c\tau}{r} \\ \Gamma'(\hat{y}_H, 0, \hat{\delta}) \end{pmatrix} + \frac{c\tau}{r} \tag{16.a}$$

$$\text{subject to} \quad K_i = 0, \quad i = 1, 2, 3 \tag{16.b}$$

The program in (16.a-b) maximises the value of the firm that is constrained by its default and investment policy if no restructuring takes place (constraints K_1 and K_2) and a complete of all debt in debt-for-equity exchanges at the optimal times (constraint K_3). We demonstrate that the solution for this program, which essentially creates a map between the current state y and the value maximising debt burden c, also defines the optimal debt restructuring solutions for the firm because if the firm does not choose the optimal borrowing level for the current state both parties are made better off by choosing a different borrowing policy. The Lagrangian is (using Lagrange multipliers φ_i, $i = 1, 2, 3$):

$$\max_{c,\hat{y}_L,\hat{y}_H,\varphi_i,i=1,2,3} \mathcal{L}(c, \hat{y}_L, \hat{y}_H, \varphi_i, i = 1, 2, 3|y) = L(c, \hat{y}_L, \hat{y}_H|y) - \sum_{i=1}^{3} \varphi_i K_i \tag{17}$$

The following result outlines the solution to this program.

Lemma 2 *The solution to (17) is given by the following condition:*

$$\begin{aligned} \partial L/\partial c = &\begin{pmatrix} \partial K_1/\partial c & \partial K_2/\partial c \end{pmatrix} \begin{pmatrix} \partial K_1/\partial c & \partial K_2/\partial c \\ \partial K_1/\partial \hat{y}_L & \partial K_2/\partial \hat{y}_L \end{pmatrix}^{-1} \begin{pmatrix} \partial L/\partial c - \partial L/\partial \hat{y}_H \frac{\partial K_3/\partial c}{\partial K_3/\partial \hat{y}_H} \\ \partial L/\partial \hat{y}_L - \partial L/\partial \hat{y}_H \frac{\partial K_3/\partial \hat{y}_L}{\partial K_3/\partial \hat{y}_H} \end{pmatrix} \\ &+ \partial L/\partial \hat{y}_H \frac{\partial K_3/\partial \hat{y}_L}{\partial K_3/\partial \hat{y}_H} \end{aligned} \tag{18}$$

This problem has a unique solution $c^*(y_t)$ for all $0 < y_t \leq \bar{y}$, with $\bar{y}$ the maximal value for $\hat{y}_H$ satisfying (13).

The point $\bar{y}$ marks the ultimate conversion point where the firm eventually retires all debt and issues new debt to invest. Before reaching this investment point, the optimisation program chooses the lower and upper restructuring points to maximise the value of the firm given the current state y. When y gets sufficiently close to the lower restructuring point, it is optimal to lower the debt burden to push the lower restructuring point even further down. When y gets sufficiently close to the upper restructuring point, it is optimal to increase the debt burden to push the upper restructuring point even higher up. This process ultimately stops when y goes to zero where existing debt also goes to zero, and when y gets close to $\bar{y}$, where the investment takes place. A continuous sequence of small debt-for-equity or equity-for-debt exchanges achieves the implementation of the borrowing policy in Lemma 2. We now address the issue of whether this sequence represents an equilibrium:

Proposition 4: *If at time 0 the state variable is $y_0 \leq \bar{y}$ and the firm carries a debt burden $c^*(y_0)$, then the continuous sequence of debt-for-equity and equity-for-debt exchanges that implement the optimal debt burden $c^*(y_t)$, $t \geq 0$ are each a solution to Lemma 3. Thus, for $y_t < \bar{y}$, the maintenance of the debt burden $c^*(y_t)$ and the full retirement of all debt in a debt-for-equity exchange at $y_t = \bar{y}$ represent an equilibrium. If the firm carries a debt burden $c \neq c^*(y_0)$, then in this case the firm carries out a large debt-for-equity or equity-for-debt exchange such that the debt burden is equal to $c^*(y_0)$. There is no other equilibrium.*

Proposition 4 sets out the optimal path of borrowing up to the point where the investment takes place. The policy maximises the value of the firm by picking the appropriate lower and upper restructuring points at all times. The continuous sequence of debt-for-equity and equity-for-debt exchanges implements the equilibrium. Equilibrium arises by exploring whether alternative paths generate profits for the parties. However, any restructuring path must be tied down to a full retirement of all debt at the investment point. Thus, a search for deviations before reaching the investment trigger point $\bar{y}$ suffices. A deviation is not possible unless the parties can make cash payments directly to each other; so the assumption preventing such payments is crucial to obtaining a unique equilibrium. If the firm makes a one-dollar coupon payment to the debt holders, then the cost to the firm is net of tax (this is a direct consequence of the assumption of full offset provisions), and the debt holder benefits in full. Therefore, if the debt holder pays cash payments to the firm as compensation, both parties are always better off. Therefore, the identifation of the optimal restructuring path relies on the restriction of the use of cash payments. The only way the firm has an incentive to pay an extra one-dollar coupon payment in some state of nature is either if it increases the value of the firm or if the firm could pay less to the debt holders in other states. The first is not feasible along the optimal path. The second is never incentive compatible for the debt holder. Therefore, the only restructuring equilibrium follows precisely the path laid out by Lemma 2.

4.3 Links to Existing Dynamic Capital Structure Theory

Our model outlines a debt restructuring equilibrium with actively managed debt in all non-distressed states before investing. The existing theory on a dynamic capital structure also studies similar optimal borrowing problems. There is a relatively clear divide between capital structure theory which deals with the management of the debt burden in non-distressed states and the debt restructuring theory which typically describes the renegotiation between the firm and the debt holder in a distressed state. In our debt overhang model, this distinction breaks down. Therefore, we discuss the similarities and differences between the theory on dynamic capital structure and the restructuring of the debt overhang.

At a superficial level, the research often specifies the terms of redemption of old debt differently from the debt restructuring games we play in our model: for instance, redemption at par values (Dangl and

Zechner (2004)) with call provisions (Christensen et al. (2002)), or at market prices (Leland (1994)). Prepayment covenants where repurchases are at prices above market values typically prevent leverage reductions. Therefore, debt becomes expensive to retire near distress states. The policies for optimal dynamic borrowing can involve a stepwise increasing debt burden up to the point where the firm eventually defaults (Dangl and Zechner (2004)). In debt restructuring models, either the debt is restructured in default (Mella-Barral and Perraudin (1997)) or to avoid default such as in our model. There is a softening of the debt burden near default states in our model which does not occur in optimal capital structure models. Therefore, the relaxation of the prepayment covenants of debt partially explains the different prediction in debt management.

However, there are more profound differences. Dynamic capital structure models are essentially trade-off models where borrowing aims to protect the firm from losses due to taxes and bankruptcy costs (Fischer et al. (1989), Dangl and Zechner (2004), Strebulaev (2007)). Leland (1994) considers small changes to leverage that are carried out at market prices. Such changes create a dilution effect for the debt holder with increases in borrowing and losses for the firm with reductions in borrowing. In Leland's (1994) model, changes in leverage have no real effects other than changing the timing of default. In contrast, in our model leverage has a real direct impact since it influences the timing of the investment. The essential problem in a debt restructuring model with debt overhang is to induce the correct investment incentives for the firm. But this is of similar concern for the debt holder because it also stand to gain from investment efficiency. Therefore, we are more likely to see that the debt holder and the firm agree on the management of debt, as predicted by our model.

5 An Extension of the Model to a Competitive Industry

As an extension, we consider a firm with an existing debt overhang which seeks entry into a competitive industry. There are several existing industry equilibrium models that show financing in continuous time, and our extension builds on the model by Leahy (1993), and its extensions by Fries et al. (1997) and Zhdanov (2007). An alternative model is Miao (2005); but in contrast to the previous models cited, Miao's (2005) model does not accommodate the uncertainty on the product's price in equilibrium which makes it inconsistent with our primary model. We build mainly on Fries et al. (1997), where firms make investments and enter an industry when product prices are high, and exit when they are low. The endogenous entry decisions create downward pressure on product prices, and the endogenous exit decisions create a corresponding upward pressure; so, the competitive forces in the industry affect the product's price process. In between the entry and exit points, however, the price uncertainty is driven by the same exogenous consumption shocks that drive equation (1), which makes this model a suitable extension of the primary model outlined in the previous sections.

We briefly outline the Fries et al. (1997) setup. The aggregate earnings flow in the industry is written as $y_t = x_t D(q_t)$ where x_t represents consumption shocks modelled as a geometric Brownian motion with drift μ and diffusion σ, and $D(q_t)$ is an inverse demand function (product price) of the aggregate capacity q_t in the industry. Thus, applying Ito's Lemma, we find:

$$\begin{aligned} dy_t &= D(q_t)dx_t + x_t D'(q_t)dq_t = x_t D(q_t)\left(\mu dt + \sigma dB_t + \frac{D'(q_t)}{D(q_t)}dq_t\right) \\ &= y_t\left(\mu dt + \sigma dB_t + \frac{D'(q_t)}{D(q_t)}dq_t\right) \end{aligned} \tag{20}$$

This expression is identical to (1) except for the third term which responds to capacity changes in the industry. This term becomes non-zero only at the entry and exit triggers. As long as the capacity q_t is constant the third term is zero and the earnings flow is described by (1) as in the non-competitive case. If entry into the industry takes place and the term $dq_t > 0$ and product prices fall, then the influx of new firms negatively influences the earnings flow y_t. Further, if exit takes place and the term $dq_t < 0$ and product prices increase, then the earnings flow is affected but in the opposite direction. The Fries et al. (1997) model assumes instantaneous entry and exit so that there is a cap on total earnings y_E where the entry takes place (and on the boundary $dy_t = 0$), and a floor on total earnings y_B where the exit takes place (and on this boundary $dy_t = 0$ also), and an inverse demand function which is iso-elastic $D(q_t) = q_t^{-\frac{1}{\kappa}}$ where $\kappa > 0$ is constant. Therefore, $D'(q_t) = -\frac{1}{q_t \kappa} D(q_t)$. At the exit point, the probability of exit of any arbitrary firm is approximated by $\frac{dq_t}{q_t}$, such that $\frac{dq_t}{q_t} = -\kappa \frac{D'(q_t)}{D(q_t)} dq_t$. Therefore, the inverse demand function for the industry has a direct impact on the likelihood of default. This is the essential point made in Fries et al. (1997). The model itself is a version of a stochastic flow model as outlined in Harrison (1985).

Consider an unlevered firm. Perfect competition leads to entry at the entry trigger y_E such that the firm makes zero profits, which means its value minus the investment cost is exactly zero. The firm can issue debt to partially finance the investment cost as the debt leads to tax advantages. As demonstrated in Fries et al. (1997) this type of financing means that a competitive firm has value equal to the investment cost at the entry point. The valuation of the debt issued at the point of investment, which has a coupon flow δ if we adopt the notation used in the previous sections, is as follows: The debt value is constant at the entry trigger y_E but its behaviour is a bit more complicated at the exit trigger y_B. The relative exit of capacity at this trigger is $\frac{dq_t}{q_t}$, and assuming the firm is as likely to default as any other firm, the probability of default is also $\frac{dq_t}{q_t}$. The value of the firm's assets is at this stage zero, so the expected loss to the debt holder is $\frac{dq_t}{q_t}\bar{V}_B(y_B)$. Therefore, the drift in the debt value $\bar{V}_B(y_B)$ if the firm does not default must be exactly equal to the expected loss $\frac{dq_t}{q_t}\bar{V}_B(y_B)$. The two boundary conditions describe the valuation formula for debt (as demonstrated in Fries et al. (1997)):

$$\bar{V}_B(y, \delta) = \frac{c}{r} + \begin{pmatrix} y^{\lambda_1} & y^{\lambda_2} \end{pmatrix} \begin{pmatrix} \lambda_1 y_E^{\lambda_1} & \lambda_2 y_E^{\lambda_2} \\ (\lambda_1 - \kappa) y_B^{\lambda_1} & (\lambda_2 - \kappa) y_B^{\lambda_2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{\kappa c}{r} \end{pmatrix} \tag{21}$$

The valuation of equity accounts for the debt tax shield.

We extend this model into a model of debt overhang by assuming the firm carries a debt liability before entering the industry. The entry point marks an investment decision which can become distorted by such debt overhang, and therefore creates incentives for debt restructuring. Consider a competitive firm that is planning to enter the industry at the entry trigger point y_E. The value of that firm just after entry is exactly equal to the investment cost I (this is true even if the firm issues debt as a partial funding of the investment cost), and since the industry is competitive the value of the investment opportunity must be constant at this point. Therefore, we can solve for the unknown parameters A and B in the general form of the value of the investment opportunity $Ay^{\lambda_1} + By^{\lambda_2}$ by evaluating these coefficients at the entry trigger y_E, where both the value of the investment opportunity and its derivative are zero:

$$\mathbb{M}(y_E) \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which means $A = B = 0$ since the matrix $\mathbb{M}$ is non-singular. The value of a competitive firm with an identical investment plan but which carries a debt overhang with a coupon flow of c is, therefore, negative.

Hence, in the absence of debt restructuring the firm defaults immediately.

Hence some debt restructuring is necessary to allow a competitive firm with debt overhang to enter the industry. The only possible equilibrium is one in which there is a complete debt write-off (a debt-for-nothing exchange) which takes place the moment the debt overhang arises. Therefore, industry competition makes the debt restructuring process a lot simpler than what is the case in the primary model. The firm cannot expect to make any tax-related gains from debt in the period before entry into the industry, so the debt tax shield falls to the debt issued after entry. In our primary model, the firm can earn a debt tax shield before the investment point because its investment trigger can change to accommodate various debt levels. In non-competitive industries, the investment point is the optimal time to redeem old debt, whereas in competitive industries the optimal time is when the debt overhang arises. In an oligopoly model, we conjecture that the retirement of debt overhang happens at some intermediate stage, because an optimal trade-off should exist between keeping the tax benefits of debt alive at the same time as not giving up too much on the firm's competitive advantage. We leave a rigorous study of this point to future work.

6 Conclusions

We study the restructuring process of a pure debt overhang and find that the firm actively manages its borrowing by a sequence of continuous debt-for-equity or equity-for-debt exchanges to maintain positive leverage before reaching the investment trigger point. At the investment trigger, the firm retires all old debt and borrows new fresh debt to partially finance the investment cost, which is consistent with the existing empirical evidence. The existing debt restructuring models do not predict these findings.

Industry competition matters to our conclusions. Whereas in non-competitive industries the firm maintains an optimal positive debt overhang and retires all existing debt only when it reaches the investment trigger point, in competitive industries the optimal debt overhang is always zero. There is a tension between keeping a debt overhang alive to capture the tax benefits of debt and eliminating the debt overhang to strengthen the firm's competitive position. In a non-competitive industry, the first effect dominates whereas in a perfectly competitive industry the second does.

References

[1] Christensen, P.O., C.R. Flor, and D. Lando, 2014, Dynamic Capital Structure with Callable Debt and Debt Renegotiations, *Journal of Corporate Finance* **29**, 644 - 661.

[2] Dangl, T. and J. Zechner, 2004, Credit Risk and Dynamic Capital Structure Choice, *Journal of Financial Intermediation* **2**, 183 - 204.

[3] Dixit, A., 1993, *The Art of Smooth Pasting*, Harwood Fundamentals of Pure & Applied Economics.

[4] Fischer, E.O., R. Heinkel, and J. Zechner, 1989, Dynamic Capital Structure Choice: Theory and Tests, *Journal of Finance* **44**, 19 - 40.

[5] Franks, J.R. and W.N. Torous, 1994, A Comparison of Financial Recontracting in Distressed Exchanges and Chapter 11 Reorganizations, *Journal of Financial Economics* **35**, 349 - 370.

[6] Fries, S., M. Miller, and W. Perraudin, 1997, Debt in Industry Equilibrium, *Review of Financial Studies* **10**, 39 – 67.

[7] Gilson, S.C. and J.B. Warner, 1998, Private versus Public Debt: Evidence from Firms That Replace Bank Loans with Junk Bonds, ssrn.com/abstract=140093.

[8] Goldstein, R., N. Ju, and H. Leland, 2001, An EBIT-Based Model of Dynamic Capital Structure, *Journal of Business* **74**, 483 - 512.

[9] Harrison, J.M., 1985, *Brownian Motion and Stochastic Flow Systems*, Krieger Publishing Company, Malabar, Florida.

[10] Hart, O. and J. Moore, 1998, Default and Renegotiation: A Dynamic Model of Debt, *Quarterly Journal of Economics* **113**, 1 - 41.

[11] Hennessy, C.A. and T.M. Whited, 2005, Debt Dynamics, *Journal of Finance* **60**, 1129 - 1165.

[12] James, C., 1995, When Do Banks Take Equity in Debt Restructurings? *Review of Financial Studies* **8**, 1209 – 1234.

[13] Leahy, J.V., 1993, Investment in Competitive Equilibrium: The Optimality of Myopic Behavior, *Quarterly Journal of Economics* **108**, 1105 – 1133.

[14] Leland, H., 1994, Corporate Debt Value, Bond Covenants, and Optimal Capital Structure, *Journal of Finance*, **49**, 1213 – 1252.

[15] Manso, G., 2008, Investment Reversibility and Agency Cost of Debt, *Econometrica* **76**, 437 – 442.

[16] Mao, L. adn Y. Tserlukevich, 2015, Repurchasing Debt, *Management Science* **61**, 1648 - 1662.

[17] Mella-Barral, P., 1999, The Dynamics of Default and Debt Reorganization, *Review of Financial Studies* **12**, 535 – 578.

[18] Mella-Barral, P. and W. Perraudin, 1997, Strategic Debt Service, *Journal of Finance* **52**, 531 – 556.

[19] Miao, J., 2005, Optimal Capital Structure and Industry Dynamics, *Journal of Finance* **60**, 2621 – 2659.

[20] Myers, S.C., 1977, The Determinants of Corporate Borrowing, *Journal of Financial Economics* **5**, 147 - 175.

[21] Nikolaev, V.V., 2015, Scope for Renegotiation in Private Debt Contracts, Working Paper No. 15-15, The University of Chicago Booth School of Business and http://ssrn.com/abstract=20125226.

[22] Pawlina, G., 2010, Underinvestment, Capital Structure and Strategic Debt Restructuring, *Journal of Corporate Finance* **16**, 679 - 702.

[23] Roberts, M.R. and A, Sufi, 2009, Renegotiation of Financial Contracts: Evidence from Private Credit Agreements, *Journal of Financial Economics* **93**, 159 – 184.

[24] Rosu, I., 2006, Multi-Stage Game Theory in Continuous Time, available at https://hal-hec.archives-ouvertes.fr/hal-00515909.

[25] Sundaresan, S., N. Wang, and J. Yang, 2015, Dynamic Investment, Capital Structure, and Debt Overhang, *Review of Corporate Finance Studies*, **4**, 1 – 42.

[26] Strebulaev, I.A., 2007, Do Tests of Capital Structure Theory Mean What They Say? *Journal of Finance* **62**, 1747 - 1787.

[27] Titman, S. and S. Tsyplakov, 2007, A Dynamic Model of Optimal Capital Structure, *Review of Finance* **11**, 401 - 451.

[28] Zhdanov, A., 2007, Competitive Equilibrium with Debt, *Journal of Financial and Quantitative Analysis* **42**, 709 - 734.

Appendix A: Valuation

Dixit (1993) describes the valuation framework. The value of a claim F on an earnings flow y_t which generates a cash flow $f(y_t)$ satisfies the ODE $\mathbb{L}(F(y_t)) + f(y_t) = 0$ where the infinitesimal operator is:

$$\mathbb{L} = \frac{\sigma^2}{2} y_t^2 \frac{d^2}{dy^2} + \mu y_t \frac{d}{dy} - r. \tag{A.1}$$

The homogeneous part of this ODE has the general solution $Ay_t^{\lambda_1} + By_t^{\lambda_2}$ for arbitrary constants A and B which span the entire solution set. The non-homogeneous term determines the particular solution. For instance, if $f(y_t) = c$, then the particular solution is $\frac{c}{r}$ since $\mathbb{L}\left(\frac{c}{r}\right) = -c$; and if $f(y_t) = y_t$, then the particular solution is $\frac{y_t}{r-\mu}$ since $\mathbb{L}\left(\frac{y_t}{r-\mu}\right) = -y_t$. The coefficients λ_1 and λ_2 are the negative and positive roots, respectively, of the characteristic equation $\frac{1}{2}\sigma^2\lambda(\lambda-1) + \mu\lambda - r = 0$, and are given by $\lambda_1 = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$, and $\lambda_2 = \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$.

Appendix B: Proofs

Derivation of Equation (6): The first order condition obtained by differentiating $\bar{V}(y_I, 0)$ with respect to δ and set equal to zero is:

$$\begin{aligned} \frac{d\bar{V}(y_I,0)}{d\delta} &= \frac{\tau}{r}\left(1 - \left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1}\right) - \frac{\delta\tau}{r}\frac{d}{d\delta}\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1} - (1-\xi)\frac{1-\tau}{r-\mu}\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1}\frac{d\bar{y}^*}{d\delta} \\ &\quad - (1-\xi)\frac{\bar{y}^*(1-\tau)}{r-\mu}\frac{d}{d\delta}\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1} \end{aligned} \tag{B.1}$$

We use the definition of $\bar{y}^* = \frac{\delta}{r}(r-\mu)\frac{\lambda_1}{\lambda_1 - 1}$ to obtain

$$\frac{d\bar{y}^*}{d\delta} = \frac{r-\mu}{r}\frac{\lambda_1}{\lambda_1 - 1} \tag{B.2.a}$$

$$\frac{d}{d\delta}\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1} = \lambda_1 \left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1 - 1}\left(-\frac{y_I}{\bar{y}^{*2}}\right)\frac{d\bar{y}^*}{d\delta} = \frac{\lambda_1}{\bar{y}^*}\frac{r-\mu}{r}\frac{\lambda_1}{\lambda_1 - 1}\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1} \tag{B.2.b}$$

Incorporating these expression into the first order condition above we find

$$\begin{aligned}0 = {} & \frac{\tau}{r}\left(1 - \left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1}\right) + \frac{\delta\tau}{r}\frac{\lambda_1}{\bar{y}^*}\frac{r-\mu}{r}\frac{\lambda_1}{\lambda_1 - 1}\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1} \\ & - (1-\xi)\frac{1-\tau}{r-\mu}\frac{r-\mu}{r}\frac{\lambda_1}{\lambda_1 - 1}\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1} + (1-\xi)\frac{\bar{y}^*(1-\tau)}{r-\mu}\frac{\lambda_1}{\bar{y}^*}\frac{r-\mu}{r}\frac{\lambda_1}{\lambda_1 - 1}\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1}\end{aligned} \tag{B.3}$$

Rearranging and eliminating terms we eliminate all terms containing δ except where δ appears as part of the risk neutral probability of default:

$$\left(\frac{y_I}{\bar{y}^*}\right)^{\lambda_1} = \frac{\tau}{-(\lambda_1 - 1)\tau - \lambda_1(1-\tau)(1-\xi)}. \tag{B.4}$$

Taking both sides to the power of $\frac{1}{\lambda_1}$, rearranging and using the definition of $\bar{y}^*$, Equation (6) follows. $\square$

Proof of Lemma 1: At the investment trigger point y_I the firm makes the investment and retain its value, $\bar{V}(y_I, c)$, receives the proceeds from new borrowing $\bar{V}_B(y_I, \delta)$, but must pay the investment cost I and retain the total debt liability $\bar{V}_B(y_I, c) + \bar{V}_B(y_I, \delta)$. Then, the firm sets δ such as to maximise the net value, $\bar{V}(y_I, c) - I - \bar{V}_B(y_I, c)$. This program can be expressed as:

$$\max_{\delta \geq 0} \left(\frac{1-\tau}{r-\mu} y_I + \frac{c+\delta}{r} \tau \left(1 - \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1} \right) - (1-\xi) \frac{1-\tau}{r-\mu} \bar{y}^* \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1} \right.$$
$$\left. - I - \frac{c}{r} \left(1 - \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1} \right) - \min \left(\xi \frac{1-\tau}{r-\mu} \bar{y}^*, \frac{c}{r} \right) \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1} \right), \tag{B.5}$$

where $\bar{y}^* = \frac{r-\mu}{r} \frac{\lambda_1}{\lambda_1 - 1}(c+\delta)$. The assumption made in the result implies that $\min \left(\xi \frac{1-\tau}{r-\mu} \bar{y}^*, \frac{c}{r} \right) = \xi \frac{1-\tau}{r-\mu} \bar{y}^*$. The first order condition for the program above then becomes

$$0 = \frac{\tau}{r} \left(1 - \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1} \right) - \frac{c+\delta}{r} \tau \lambda_1 \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1 - 1} \left(-\frac{y_I}{\bar{y}^{*2}} \right) \frac{r-\mu}{r} \frac{\lambda_1}{\lambda_1 - 1} - \frac{1-\tau}{r-\mu} \frac{r}{r-\mu} \frac{\lambda_1}{\lambda_1 - 1} \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1}$$
$$- \frac{1-\tau}{r-\mu} \bar{y}^* \lambda_1 \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1 - 1} \left(-\frac{y_I}{\bar{y}^{*2}} \right) \frac{r-\mu}{r} \frac{\lambda_1}{\lambda_1 - 1} + \frac{c}{r} \lambda_1 \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1 - 1} \left(-\frac{y_I}{\bar{y}^{*2}} \right) \frac{r-\mu}{r} \frac{\lambda_1}{\lambda_1 - 1}$$
$$= \frac{\tau}{r} + \frac{\tau}{r}(\lambda_1 - 1) \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1} + \frac{1-\tau}{r} \lambda_1 \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1} - \frac{c}{c+\delta} \frac{\lambda_1}{r} \left(\frac{y_I}{\bar{y}^*} \right)^{\lambda_1}. \tag{B.6}$$

Define $F(y_I)$ and $G(\delta)$ as:

$$F(y_I) = \left(\frac{r}{r-\mu} \right)^{\lambda_1} \left(\frac{\lambda_1 - 1}{\lambda_1} \right)^{\lambda_1} y_I^{\lambda_1}, \tag{B.7.a}$$

$$G(\delta) = \frac{\tau}{(\tau - \lambda_1)(c+\delta) - c\lambda_1} (c+\delta)^{1+\lambda_1}. \tag{B.7.b}$$

Then, by manipulating the first order condition above, we find that it means that $F(y_I) = G(\delta)$. Therefore, we can identify $F(y_I^0) = G(0)$ as one point on this curve, which yields

$$y_I^0 = \left(\frac{r-\mu}{r} \frac{\lambda_1}{\lambda_1 - 1} c \right) \left(\frac{\tau}{\tau - 2\lambda_1} \right)^{\frac{1}{\lambda_1}}. \tag{B.8}$$

The point y_I^0 is the point at which the first order condition is solved for $\delta = 0$. Next we work out the total differential $dF(y_I)$ and $dG(\delta)$:

$$dF(y_I) = \frac{\lambda_1}{y_I} F(y_I) dy_I, \tag{B.9.a}$$

$$dG(\delta) = \left(\frac{1+\lambda_1}{c+\delta} - \frac{1}{(c+\delta) + \frac{\lambda_1}{\lambda_1 - \tau c}} \right) G(\delta) d\delta. \tag{B.9.b}$$

Along the path where $F(y_I) = G(\delta)$ the relationship $dF(y_I) = dG(\delta)$ means that

$$\frac{\lambda_1}{y_I} dy_I = \left(\frac{1+\lambda_1}{c+\delta} - \frac{1}{(c+\delta) + \frac{\lambda_1}{\lambda_1 - \tau c}} \right) d\delta$$

and we find therefore the following:

$$F(y_I) = F(y_I^0) + \int_{y_I^0}^{y_I} \frac{\lambda_1}{y} dy = G(0) + \int_0^{\delta} \left(\frac{1+\lambda_1}{c+\delta'} - \frac{1}{(c+\delta') + \frac{\lambda_1}{\lambda_1 - \tau c}} \right) d\delta' = G(\delta). \tag{B.10}$$

By eliminating $F(y_I^0)$ and $G(0)$ and integrating out, we find:

$$\int_{y_I^0}^{y_I} \frac{-\lambda_1}{y} dy = \int_0^{\delta} \left(\frac{(\lambda_1+1)}{c+\delta'} - \frac{1}{(c+\delta') + \frac{\lambda_1}{\lambda_1 - \tau} c} \right) d\delta', \tag{B.11}$$

which means

$$\ln \frac{y_I}{y_I^0} = \frac{\lambda_1 + 1}{\lambda_1} \ln \left(1 + \frac{\delta}{c} \right) - \frac{1}{\lambda_1} \ln \left(1 + \frac{\lambda_1 - \tau}{2\lambda_1 - \tau} \frac{\delta}{c} \right), \tag{B.12}$$

and Lemma 1 follows. □

Proof of Proposition 1: Step 1 (Optimal borrowing): From Lemma 1 we have

$$\ln(y_I) = \ln(y_I^0) + \frac{\lambda_1 + 1}{\lambda_1} \ln\left(1 + \frac{\bar{\delta}}{c}\right) - \frac{1}{\lambda_1} \ln\left(1 + \frac{\lambda_1 - \tau}{2\lambda_1 - \tau} \frac{\bar{\delta}}{c}\right). \tag{B.13}$$

If $\bar{\delta} < 0$ then the optimal borrowing policy is $\delta = 0$ (Case 1). If $\bar{\delta} \geq 0$ and the assumption that $\min\left(\xi \frac{1-\tau}{r-\mu} \bar{y}^*, \frac{c}{r}\right) = \xi \frac{1-\tau}{r-\mu} \bar{y}^*$ is true, then $\delta = \bar{\delta}$ (Case 2). If the assumption is not true, the value of the old debt is $V_B(y_I) = \frac{c}{r}$ which is independent of δ, therefore the firm seeks to set δ such as to maximise its value $\bar{V}(y_I)$, which is also the objective when there is no debt overhang. Therefore, the optimal delta satisfies the condition that $c + \delta = \left(\frac{r}{r-\mu} \frac{\lambda_1 - 1}{\lambda_1}\right) \left(\frac{\tau}{-(\lambda_1 - 1)\tau - \lambda_1(1-\xi)(1-\tau)}\right)^{-\frac{1}{\lambda_1}} y_I$.

Step 2 (Smooth pasting condition firm): The firm defaults at y^*, makes the investment at y_I, and collects the cash flow $-c(1-\tau)$ in the intermediate region where debt service is upheld. The general form of the equity value function is $Ay_t^{\lambda_1} + By_t^{\lambda_2} - \frac{c}{r}(1-\tau)$ for arbitrary constants A and B. This function means that we can write the smooth pasting problem at y^* as:

$$\mathbb{M}(y^*) \begin{pmatrix} A \\ B \end{pmatrix} - \begin{pmatrix} \frac{c}{r}(1-\tau) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{B.14}$$

and the smooth pasting problem at y_I as:

$$\mathbb{M}(y_I) \begin{pmatrix} A \\ B \end{pmatrix} - \begin{pmatrix} \frac{c}{r}(1-\tau) \\ 0 \end{pmatrix} = \begin{pmatrix} \bar{V}(y_I, c) - I - \bar{V}_B(y_I, c) \\ \bar{V}'(y_I, c) - \bar{V}_B'(y_I, c) \end{pmatrix} = \begin{pmatrix} \Gamma(y_I, c, \delta) \\ \Gamma'(y_I, c, \delta) \end{pmatrix}. \tag{B.15}$$

By solving for the vector with the coefficients A and B for both problems, and setting the right-hand sides equal, we achieve the result. $\square$

Proof of Proposition 2: To make the notation more compact, we use $V_E(y,c) = V(y,c) - V_B(y,c)$ and $V_E(y,0) = V(y,0) - V_B(y,c)$. We first investigate the optimal restructuring point from the point of view of the debt holder.

Step 1 (Sharing): Consider restructuring at $y^* \leq \hat{y} \leq y_I$ such that destructing takes place below $\hat{y}_L$ and above $\hat{y}_H$ but not in between (see step 2). Since the debt holder keeps whatever is left of the equity once the original equity holders receive their outside option, any restructuring solution at $\hat{y}$ must involve an offer of $V_E(\hat{y},c)$ worth of equity to the original equity holders and the debt holder retaining $V(\hat{y},0) - V_E(\hat{y},c) = V_B(\hat{y},c) + (V(\hat{y},0) - V(\hat{y},c))$ worth of equity.

Step 2 (Restructuring region): Any restructuring point below y^* involves the restructuring of an already restructured firm in default which is worth ξ percent of the original firm and with no coupon flow, so it is never desirable for the debt holder that has the option to receive more if it restructures in a non-distressed state. Any restructuring point above y_I involves the restructuring of a firm that has already made the investment where the restructuring gains are zero. Therefore, all optimal restructuring points $\hat{y}$ must be in the interval $[y^*, y_I]$.

Step 3 (Optimal timing): Suppose the debt holder defers the restructuring from above to a point $\hat{y}_L$ such that if $y_t > \hat{y}_L$, then the debt holder collects the coupon flow c. Then, at $y_t = \hat{y}_L$ the debt holder's claim is converted into equity where the debt holder collects a fraction, specifically $\frac{V_B(\hat{y}_L,c)+(V(\hat{y}_L,0)-V(\hat{y}_L,c))}{\bar{V}(\hat{y}_L,c)}$, of the firm's earnings flow y_t. This fraction decreases with $\hat{y}_L$, so if the debt holder finds that deferment of the restructuring is optimal at $\hat{y}_L$ it also finds that deferment of the restructuring to be optimal at a point below $\hat{y}_L$ in order to receive a greater fraction of the firm's earnings flow and to keep receiving the coupon payments of c in the mean time. Therefore, ultimately $\hat{y}_L$ is lowered to the point where $\hat{y}_L = y*$.

Similarly, the debt holder defers the restructuring from below to a point $\hat{y}_H$ such that if $y_t < \hat{y}_H$, then the debt holder collects the coupon flow c and at $y_t = \hat{y}_H$ the debt holder's claim is converted into equity. In this case deferral beyond y_I might be desirable but not feasible (Step 2) Therefore, the only problem is to determine the optimal timing of $\hat{y}_H$. If a smooth pasting solution determines the timing, then it can be written as:

$$\mathbb{M}(\hat{y}_H)\begin{pmatrix} A \\ B \end{pmatrix} + \begin{pmatrix} \frac{c}{r} \\ 0 \end{pmatrix} = \begin{pmatrix} V_B(\hat{y}_H, c) + V(\hat{y}, 0) - V(\hat{y}_H, c) \\ V'_B(\hat{y}, c) + V'(\hat{y}, 0) - V'(\hat{y}, c) \end{pmatrix}, \tag{B.17}$$

where A and B are arbitrary constants. We can solve for these constants:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \mathbb{M}^{-1}\begin{pmatrix} V_B(\hat{y}_H, c) + V(\hat{y}, 0) - V(\hat{y}_H, c) - \frac{c}{r} \\ V'_B(\hat{y}, c) + V'(\hat{y}, 0) - V'(\hat{y}, c) \end{pmatrix}. \tag{B.18}$$

Also, we know that the debt value at $\hat{y}_L$ is given by $A\hat{y}_L^{\lambda_1} + B\hat{y}_L^{\lambda_2} + \frac{c}{r} = V(\hat{y}_L, 0)$, so

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix}\mathbb{M}^{-1}(\hat{y}_H)\begin{pmatrix} V_B(\hat{y}_H, c) + V(\hat{y}_H, 0) - V(\hat{y}_H, c) - \frac{c}{r} \\ V'_B(\hat{y}_H, c) + V'(\hat{y}_H, 0) - V'(\hat{y}_H, c) \end{pmatrix} + \frac{c}{r} = V(\hat{y}_L, 0), \tag{B.19}$$

which determines the optimal timing of $\hat{y}_H$. This solution never exceeds y_I as beyond this point there are no efficiency gains from the point of view of the debt holder. What remains is to show that (B.19) yields (14). To do this we rewrite (B.19) in the following way:

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix}\mathbb{M}^{-1}(\hat{y}_H)\begin{pmatrix} V(\hat{y}_H, 0) - \frac{c}{r} \\ V'(\hat{y}_H, c) \end{pmatrix} - \begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix}\mathbb{M}^{-1}(\hat{y}_H)\begin{pmatrix} V_E(\hat{y}_H, c) \\ V'_E(\hat{y}_H, c) \end{pmatrix} + \frac{c}{r} = V(\hat{y}_L, 0). \tag{B.20}$$

Since the equity claim can be written as $A'y^{\lambda_1} + B'y^{\lambda_2} - \frac{c}{r}(1-\tau)$ for some constants A' and B', we find

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} V(\hat{y}_H, 0) - \frac{c}{r} \\ V'(\hat{y}_H, 0) \end{pmatrix} - \begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} A'\hat{y}_H^{\lambda_1} + B'\hat{y}_H^{\lambda_2} - \frac{c}{r}(1-\tau) \\ \lambda_1 A'\hat{y}_H^{\lambda_1 - 1} + \lambda_2 B'\hat{y}_H^{\lambda_2 - 1} \end{pmatrix} + \frac{c}{r}$$

$$= V(\hat{y}_L, c). \tag{B.21}$$

By combining the terms involving c on the left-hand side, and by working out explicitly the second term on the left-hand side, we find:

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} V(\hat{y}_H, 0) - \frac{c\tau}{r} \\ V'(\hat{y}_H, 0) \end{pmatrix} - \begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \begin{pmatrix} A' \\ B' \end{pmatrix} + \frac{c}{r} = V(\hat{y}_L, 0). \tag{B.22}$$

Finally, using the fact that the second term on the left hand side reduces to $A'\hat{y}_L^{\lambda_1} + B'\hat{y}_L^{\lambda_2} = V_E(\hat{y}_L, c) + \frac{c}{r}(1-\tau) = \frac{c}{r}(1-\tau)$ (since the equity value is zero at $\hat{y}_L = y^*$) we find

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} V(\hat{y}_H, 0) - \frac{c\tau}{r} \\ V'(\hat{y}_H, 0) \end{pmatrix} = V(\hat{y}_L, 0) - \frac{c\tau}{r}. \tag{B.23}$$

Recognising that $V(\hat{y}_H, 0) = \Gamma(\hat{y}_H, 0, \hat{\delta})$ we obtain equation (14).

Now consider the shareholders' problem:

Step 1 (Sharing): Consider $\hat{y}_L \leq \hat{y}_H \leq y_I$ such that restructuring takes place at $\hat{y}_L$ and $\hat{y}_H$ but not in between. Since the firm keeps whatever is left of the equity once the debt holder receives its outside option, any restructuring solution at $\hat{y} \geq y^*$ must involve an offer of $V_B(\hat{y}, c)$ worth of equity to the original debt holder and the remaining $V(\hat{y},0) - V_B(\hat{y},c)$ worth of equity is retained by the firm. Any restructuring solution at $\hat{y} < y^*$ must involve an offer of $\xi V(\hat{y},0)$ worth of equity to the original debt holder and the remaining $(1-\xi)V(\hat{y},0)$ worth of equity is retained by the firm.

Step 2 (Restructuring region): Any restructuring point strictly below y^* yields the same sharing of equity between the debt holder and the firm but will cost the firm coupon payments to achieve. Therefore there are no gains achievable by deferring the restructuring at $\hat{y} < y^*$. Thus, $y^* \leq \hat{y}_L$. Any restructuring point strictly above y_I is incentive compatible only if investment is deferred. This is never optimal unless efficiency gains are achieved, which cannot be true since the only source of inefficiency in this setup is associated with underinvestment. Therefore $\hat{y}_H \leq y_I$.

Step 3 (Optimal timing): Consider $\hat{y}_L = y^*$ and the smooth pasting problem to determine $\hat{y}_H$. The optimality condition is:

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} V(\hat{y}_H,0) - V_B(\hat{y}_H,c) + \frac{c}{r}(1-\tau) \\ V'(\hat{y}_H,0) - V'_B(\hat{y}_H,c) \end{pmatrix} - \frac{c}{r}(1-\tau) = V(\hat{y}_L,0) - V_B(\hat{y}_L,c). \quad \text{(B.24)}$$

We show that this problem has the same solution as (14) and therefore solves the timing problem for the firm. Using the same procedure as above, we isolate the debt values:

$$\begin{aligned} &\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} V(\hat{y}_H,0) + \frac{c}{r}(1-\tau) \\ V'(\hat{y}_H,0) \end{pmatrix} - \begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} V_B(\hat{y}_H,c) \\ V'_B(\hat{y}_H,c) \end{pmatrix} - \frac{c}{r}(1-\tau) \\ &= V(\hat{y}_L,0) - V_B(\hat{y}_L,c). \qquad \text{(B.25)} \end{aligned}$$

The debt value can be written as $A''y^{\lambda_1} + B''y^{\lambda_2} + \frac{c}{r}$ for constants A'' and B'', so we can reduce further the left-hand side to the following:

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} V(\hat{y}_H, 0) - \frac{c\tau}{r} \\ \\ V'(\hat{y}_H, 0) \end{pmatrix} - A''\hat{y}_L^{\lambda_1} - B''\hat{y}_L^{\lambda_2} - \frac{c}{r}(1-\tau) = V(\hat{y}_L, 0) - V_B(\hat{y}_L, c). \quad \text{(B.26)}$$

Since $A''\hat{y}_L^{\lambda_1} + B''\hat{y}_L^{\lambda_2} = V_B(\hat{y}_L, c) - \frac{c}{r}$, we ultimately find

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} V(\hat{y}_H, c) - \frac{c\tau}{r} \\ \\ V'(\hat{y}_H, c) \end{pmatrix} = V(\hat{y}_L, c) - \frac{c\tau}{r}, \quad \text{(B.27)}$$

which is identical to (B.23). □

Proof of Proposition 3: It suffices to show that a restructuring process that maximises the value of the firm, that is, the joint value of the firm's debt and equity, is the same as the restructuring process that maximises the value of the firm's debt or equity individually, as outlined in Proposition 3. Therefore, using the same argument as outlined in the proof of Proposition 3, a restructuring at $\hat{y}_L = y^*$ and $\hat{y}_H$ determined by a smooth pasting program satisfies the condition:

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} V(\hat{y}_H, 0) - V(\hat{y}_H, c) \\ \\ V'(\hat{y}_H, 0) - V'(\hat{y}_H, c) \end{pmatrix} = V(\hat{y}_L, 0) - V(\hat{y}_L, c). \quad \text{(B.28)}$$

We can write $V(y,c) = A'''y^{\lambda_1} + B'''y^{\lambda_2} + \frac{c\tau}{r}$ for some constants A''' and B''', and rewriting (B.28) we find:

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} V(\hat{y}_H,0) - \frac{c\tau}{r} \\ \\ V'(\hat{y}_H,0) \end{pmatrix} - \begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} A'''\hat{y}_L^{\lambda_1} + B'''\hat{y}_L^{\lambda_2} \\ \\ \lambda_1 A'''\hat{y}_L^{\lambda_1 - 1} + \lambda_2 B'''\hat{y}_L^{\lambda_2 - 1} \end{pmatrix}$$

$$= V(\hat{y}_L,0) - V(\hat{y}_L,c). \tag{B.29}$$

Using the fact that the second term on the left-hand side equals $A'''\hat{y}_L^{\lambda_1} + B'''\hat{y}_L^{\lambda_2} = V(\hat{y}_L,c) - \frac{c\tau}{r}$ we find:

$$\begin{pmatrix} \hat{y}_L^{\lambda_1} & \hat{y}_L^{\lambda_2} \end{pmatrix} \mathbb{M}^{-1}(\hat{y}_H) \begin{pmatrix} V(\hat{y}_H,0) - \frac{c\tau}{r} \\ \\ V'(\hat{y}_H,0) \end{pmatrix} - V(\hat{y}_L,c) + \frac{c\tau}{r} = V(\hat{y}_L,0) - V(\hat{y}_L,c). \tag{B.30}$$

and this reduces further to equation (B.23). $\square$

Proof of Lemma 2: The first order conditions of the Lagrangian are as follows:

$$0 = \frac{\partial L}{\partial c} - \varphi_1 \frac{\partial K_1}{\partial c} - \varphi_2 \frac{\partial K_2}{\partial c} - \varphi_3 \frac{\partial K_3}{\partial c} \tag{B.31.a}$$

$$0 = \frac{\partial L}{\partial \hat{y}_L} - \varphi_1 \frac{\partial K_1}{\partial \hat{y}_L} - \varphi_2 \frac{\partial K_2}{\partial \hat{y}_L} - \varphi_3 \frac{\partial K_3}{\partial \hat{y}_L} \tag{B.31.b}$$

$$0 = \frac{\partial L}{\partial \hat{y}_H} - \varphi_3 \frac{\partial K_3}{\partial \hat{y}_H} \tag{B.31.c}$$

The third condition determines φ_3:

$$\varphi_3 = \frac{\partial L}{\partial \hat{y}_H} \frac{1}{\partial K_3 / \partial \hat{y}_H} \tag{B.32}$$

This can be substituted into the two first conditions, and we find the system:

$$\begin{pmatrix} \partial K_1/\partial c & \partial K_2/\partial c \\ \partial K_1/\partial \hat{y}_L & \partial K_2/\partial \hat{y}_L \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \partial L/\partial c - \partial L/\partial \hat{y}_H \frac{\partial K_3/\partial c}{\partial K_3/\partial \hat{y}_H} \\ \partial L/\partial \hat{y}_L - \partial L/\partial \hat{y}_H \frac{\partial K_3/\partial \hat{y}_L}{\partial K_3/\partial \hat{y}_H} \end{pmatrix} \tag{B.33}$$

Pre-multiplying the equation by the inverse of the matrix on the left-hand side isolates φ_1 and φ_2. Substituting these into the first of the first order conditions above yields (18).

We know that we can associate with each c a unique $\hat{y}_L(c)$ and $\hat{y}_H(c)$ as the solution to (13). Existence is established by the fact that for a given c and a $\hat{y}_L \leq y \leq \hat{y}_H$, the objective function characterises the firm's value for that borrowing level at the current state variable y. Since the smooth pasting problems that derive the boundary points have solutions, so must the objective function as the value function is bounded and continuous in y for a given c. Therefore, a solution exists for every c such that $\hat{y}_L(c) \leq \hat{y}_H(c)$. Maximising over all c must produce a solution within the values of c where $\hat{y}_L \leq \hat{y}_H(c)$, since otherwise it would not be the solution to the smooth pasting problem.

We claim there is a debt level c at which $\hat{y}_H(c)$ is maximal, which defines $\bar{y}$. For $c \to 0$, $\hat{y}_H(c)$ must approach the investment trigger for an unlevered firm. For $c \to \infty$ the firm always restructures immediately. Therefore, there is some $\bar{c} < \infty$ for which $\hat{y}_L(\bar{c}) = \hat{y}_H(\bar{c})$. Since $\hat{y}_H(c)$ is finite and continuous between zero and $\bar{c}$, a maximum point exists for $0 < c < \bar{c}$.

Uniqueness remains. Suppose for a given y, there are two solutions $c_1 < c_2$ to the maximisation problem. This means that $L(c_1, \hat{y}_L(c_1), \hat{y}_H(c_1)|y) = L(c_2, \hat{y}_L(c_2), \hat{y}_H(c_2)|y)$ for a given y. This means at this particular value of y:

$$\begin{pmatrix} y^{\lambda_1} & y^{\lambda_2} \end{pmatrix} \left(\mathbb{M}^{-1}(\hat{y}_H(c_1)) \begin{pmatrix} \Gamma(\hat{y}_H(c_1), 0, \hat{\delta}) - \frac{c_1 \tau}{r} \\ \Gamma'(\hat{y}_H(c_1), 0, \hat{\delta}) \end{pmatrix} - \mathbb{M}^{-1}(\hat{y}_H(c_2)) \begin{pmatrix} \Gamma(\hat{y}_H(c_2), 0, \hat{\delta}) - \frac{c_2 \tau}{r} \\ \Gamma'(\hat{y}_H(c_2), 0, \hat{\delta}) \end{pmatrix} \right) + \frac{(c_1 - c_2)\tau}{r} = 0 \tag{B.34}$$

However, by the assumption that c_1 and c_2 are optimal the transition from c_1 to c_2 at the threshold y must be smooth (by the smooth-pasting principle the firm can jump "smoothly" from one level of borrowing to the other). Therefore, differentiating (B.34) with respect to y, we should find:

$$\begin{pmatrix} \lambda_1 y^{\lambda_1 - 1} & \lambda_2 y^{\lambda_2 - 1} \end{pmatrix} \left(\mathbb{M}^{-1}(\hat{y}_H(c_1)) \begin{pmatrix} \Gamma(\hat{y}_H(c_1), 0, \hat{\delta}) - \frac{c_1 \tau}{r} \\ \Gamma'(\hat{y}_H(c_1), 0, \hat{\delta}) \end{pmatrix} - \mathbb{M}^{-1}(\hat{y}_H(c_2)) \begin{pmatrix} \Gamma(\hat{y}_H(c_2), 0, \hat{\delta}) - \frac{c_2 \tau}{r} \\ \Gamma'(\hat{y}_H(c_2), 0, \hat{\delta}) \end{pmatrix} \right) = 0 \tag{B.35}$$

Thus, the only way (B.34) and (B.35) can hold is that the difference between the coefficients inside the large matrices are both strictly positive, that is:

$$\left(\mathbb{M}^{-1}(\hat{y}_H(c_1)) \begin{pmatrix} \Gamma(\hat{y}_H(c_1), 0, \hat{\delta}) - \frac{c_1 \tau}{r} \\ \Gamma'(\hat{y}_H(c_1), 0, \hat{\delta}) \end{pmatrix} - \mathbb{M}^{-1}(\hat{y}_H(c_2)) \begin{pmatrix} \Gamma(\hat{y}_H(c_2), 0, \hat{\delta}) - \frac{c_2 \tau}{r} \\ \Gamma'(\hat{y}_H(c_2), 0, \hat{\delta}) \end{pmatrix} \right) > \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{B.36}$$

But if the difference is positive, then the value function associated with c_1 must be greater than the value function associated with c_2 for all $y' < y$ and $y' > y$, whereas at most one of these can be true. This is a contradiction. Therefore, there must be a kink in the optimal value function at y which contradicts the assumption that two solutions $c_1 \neq c_2$ exist. $\square$

Proof of Proposition 4: Consider the change $y + \Delta y$ along the path where the constraints in (17.a-b) hold, which means new coupon flow $c + \Delta c$ and associated triggers $\hat{y}_L + \Delta \hat{y}_L$ and $\hat{y}_H + \Delta \hat{y}_H$. The value of the equity and debt is then

$$\text{Equity} = L(c + \Delta c, \hat{y}_L + \Delta \hat{y}_L, \hat{y}_H + \Delta \hat{y}_H | y + \Delta y) - V_B(y + \Delta y, c + \Delta c) \tag{B.36.a}$$

$$\text{Debt} = V_B(y + \Delta y, c + \Delta c) \tag{B.36.b}$$

If the firm holds the bargaining power, it offers a fraction κ worth of $V_B(y + \Delta y, c + \Delta c)$ in exchange for its old claim $V_B(y + \Delta y, c)$. Since the debt holder must be made equally well off after the debt-for-debt exchange, it must be the case that $\kappa V_B(y + \Delta y, c + \Delta c) = V_B(y + \Delta y, c)$. Therefore, the value of the firm after the exchange is $L(c + \Delta c, \hat{y}_L + \Delta \hat{y}_L, \hat{y}_H + \Delta \hat{y}_H | y + \Delta y) - V_B(y + \Delta y, c + \Delta c) + (1 - \kappa) V_B(y + \Delta y, c + \Delta c) = L(c + \Delta c, \hat{y}_L + \Delta \hat{y}_L, \hat{y}_H + \Delta \hat{y}_H | y + \Delta y) - V_B(y + \Delta y, c)$. Therefore, the debt holder is equally well off and the firm increases its wealth if and only if the firm value is increased. The Lagrange programme ensures that this increase happens along the path where the constraints K_i, $i = 1, 2, 3$, are all equal to zero. If the debt holder holds the bargaining power, it offers the firm κ worth of $L(c + \Delta c, \hat{y}_L + \Delta \hat{y}_L, \hat{y}_H + \Delta \hat{y}_H | y + \Delta y) - V_B(y + \Delta y, c + \Delta c)$ in exchange for their old claim, and since the firm must be made equally well off after the exchange it must be the case that $\kappa(L(c + \Delta c, \hat{y}_L + \Delta \hat{y}_L, \hat{y}_H + \Delta \hat{y}_H | y + \Delta y) - V_B(y + \Delta y, c + \Delta c)) = L(c, \hat{y}_L, \hat{y}_H | y + \Delta y) - V_B(y + \Delta y, c)$. The debt holder retains $V_B(y + \Delta y, c + \Delta c) + (1 - \kappa)(L(c + \Delta c, \hat{y}_L + \Delta \hat{y}_L, \hat{y}_H + \Delta \hat{y}_H | y + \Delta y) - V_B(y + \Delta y, c + \Delta c)) = V_B(y + \Delta y, c) + L(c + \Delta c, \hat{y}_L + \Delta \hat{y}_L, \hat{y}_H + \Delta \hat{y}_H | y + \Delta y) - L(c, \hat{y}_L, \hat{y}_H | y + \Delta y)$. Therefore, the firm is equally well off and the debt holder increases its wealth if and only if the firm's value is increased. Therefore, a continuous sequence of exchanges can implement value maximisation along the path where the constraints $K_i = 0$, $i = 1, 2, 3$, are satisfied.

Any profitable deviation from the path along which the constraints $K_i = 0$, $i = 1, 2, 3$ must therefore involve violations of one or more of these constraints. Any deviation of this kind will not influence the ultimate investment trigger point $\bar{y}$ as long as it returns to the path where $K_i = 0$ before the firm chooses to invest. If it does not, the firm's value is not maximised in the neighbourhood of the new investment trigger point and therefore not incentive compatible. Thus, we can reduce the analysis of deviations to those that ultimately return to the path $K_i = 0$, $i = 1, 2, 3$ before the investment trigger point is influenced. Consider such a deviation where for $y_1 \leq y \leq y_2$ the coupon flow is maintained at $c^*(y) + \Delta c$, where $y_1 > 0$ and $y_2 < \hat{y}_H(c^*(y_2) + \Delta c)$ (the latter condition ensures that the deviation reverts to the optimal path before the investment is triggered). Suppose the debt holder holds the bargaining power, and consider that the deviation is followed by side payments of $(1 - \tau)\Delta c$ from the debt holders to the firm. The net cost to the firm is then zero: it pays Δc extra coupon payments to the debt holders but can claim τ percent in tax relief, and it receives $(1 - \tau)\Delta c$ from the debt holders in side payments. At the same time, the debt holder receives Δc from the firm in extra coupon payments but pays $(1 - \tau)\Delta c$ in side payments, which adds $\tau\Delta c$ in net coupon payments to the debt holder. If $\Delta c > 0$, therefore, then the debt holder is better off whereas the firm is equally well off. Consider the case where the firm holds the bargaining power and that the debt holder pays side payments of Δc to the firm. The net cash flow benefit to the firm is the difference between Δc and $\Delta c(1 - \tau)$, which is $\tau\Delta c$. For $\Delta c > 0$, the firm is therefore strictly better off whereas the debt holder is equally well off.

By the assumptions of the model cash payments are not allowed. Compensation equivalent to the side payments needs to be generated within the game itself. Any period of over payments $\Delta c > 0$ must be matched by periods of under payments, but such under payments are never incentive compatible. Therefore, a deviation involving $\Delta c > 0$ always is harmful to the firm and beneficial to the debt holder and therefore never incentive compatible for the firm. $\square$

Derivation of Equation (21): The valuation of debt takes the form $Ay^{\lambda_1} + By^{\lambda_2} + \frac{c}{r}$ where A and B are constants. The parameters λ_1 and λ_2 are given in Appendix A. Since the debt holder must have an incentive to hold the debt at the entry trigger point y_E the debt value must be constant along the boundary. Therefore $\lambda_1 A y_E^{\lambda_1 - 1} + \lambda_2 B y_E^{\lambda_2 - 1} = 0$. Next, since the debt holder must have an incentive to also hold the debt also at the exit trigger y_B, the drift in the debt value must exactly offset the expected cost of default, so $\bar{V}_B'(y_B, \delta) = \lambda_1 A y_B^{\lambda_1 - 1} + \lambda_2 y_B^{\lambda_2 - 1} = \frac{dq_t}{q_t} \bar{V}_B(y_B, \delta)$. Fries et al. (1997) show that $\frac{dq_t}{q_t} = \frac{\kappa}{y_B}$, and therefore the coefficients A and B are defined by the system:

$$\begin{pmatrix} \lambda_1 y_E^{\lambda_1 - 1} & \lambda_2 y_E^{\lambda_2 - 1} \\ \lambda_1 y_B^{\lambda_1 - 1} & \lambda_2 y_B^{\lambda_2 - 1} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\kappa}{y_B}(A y_B^{\lambda_1} + B y_B^{\lambda_2} + \frac{c}{r}) \end{pmatrix} \tag{B.37}$$

Multiplying the first row by y_E and the second row by y_B, and rearranging, we find:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \lambda_1 y_E^{\lambda_1} & \lambda_2 y_E^{\lambda_2} \\ (\lambda_1 - \kappa) y_B^{\lambda_1} & (\lambda_2 - \kappa) y_B^{\lambda_2} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \frac{\kappa c}{r} \end{pmatrix} \tag{B.38}$$

Substituting the coefficients into the valuation formula for $\bar{V}_B(y, \delta)$ equation (21) follows. $\square$