

Transfer Entropy and Directed Information in Gaussian Diffusion Processes

Nigel J. Newton *

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Abstract

Transfer Entropy and *Directed Information* are information-theoretic measures of the *directional dependency* between stochastic processes. Following the definitions of Schreiber and Massey in discrete time, we define and evaluate these measures for the components of multi-dimensional Gaussian diffusion processes. When the components are jointly Markov, the Transfer Entropy and Directed Information are both measures of *influence* according to a simple physical principle. More generally, the effect of other components has to be accounted for, and this can be achieved in more than one way. We propose two definitions, one of which preserves the properties of influence of the jointly Markov case. The Transfer Entropy and Directed Information are expressed in terms of the solutions of matrix Riccati equations, and so are easy to compute. The definition of continuous-time Directed Information we propose differs from that previously appearing in the literature. We argue that the latter is not strictly directional.

Keywords: Causality, Diffusions, Directed Information, Information Flow, Nonequilibrium Statistical Mechanics, Transfer Entropy.

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1 Introduction

Transfer Entropy and *Directed Information* are information-theoretic measures of the *directional dependency* between stochastic processes. They quan-

*School of Computer Science and Electronic Engineering, University of Essex, Wivenhoe Park, Colchester, CO4 3SQ, UK. (njn@essex.ac.uk)

tify the statistical dependency between the past of one process and the future of another, and are thus connected with notions of *causality* in physical systems. Directed information was developed in the context of Telecommunications, as a directional variant of the *mutual information* between the input and output data sequences in a communication channel [7]. Transfer entropy, on the other hand, was developed within the Physical Sciences community as a means of testing for directional influence in complex systems [19]. It has been applied in fields as diverse as *Neuroscience*, *Systems Biology*, *Climatology* and *Econometrics*. (See [1] for a review of both quantities in a discrete-time context, their connections with *Granger Causality*, and an extensive bibliography.)

This paper defines and evaluates the transfer entropy and directed information between components of continuous-time Gaussian diffusion processes, expressing both quantities in terms of the solutions of matrix Riccati equations. In this setting, both quantities lead to a single infinitesimal *rate* of information transfer. Many of the results herein carry over to diffusion processes with *nonlinear* dynamics. However, such extensions make use of the theory of nonlinear filtering [5], and so introduce considerable technicalities, which can all too easily obscure the meaning of the results.

Information theory has its origins in telecommunications [20], where it provides a fundamental limit on the rate at which data can be reliably communicated over error-prone communication channels. The central quantities in information theory are the *mutual information* between two random variables, and its conditional variant. *Conditional mutual information* is a measure of conditional dependency between a pair of random variables, $U : \Omega \rightarrow \mathbf{U}$ and $V : \Omega \rightarrow \mathbf{V}$, defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{P})$. If \mathbf{U} and \mathbf{V} are Polish spaces with Borel σ -algebras \mathcal{U} and \mathcal{V} , and \mathcal{H} is a sub- σ -algebra of \mathcal{G} then a regular \mathcal{H} -conditional probability distribution exists for (U, V) : $P_{UV|\mathcal{H}} : \mathcal{U} \times \mathcal{V} \times \Omega \rightarrow [0, 1]$. Let $P_{U|\mathcal{H}}$ and $P_{V|\mathcal{H}}$ be its marginals. The \mathcal{H} -conditional mutual information is then

$$I(U; V|\mathcal{H}) = \mathbb{E}\mathcal{D}(P_{UV|\mathcal{H}} | P_{U|\mathcal{H}} \otimes P_{V|\mathcal{H}}), \quad (1)$$

where $\mathcal{D}(P|Q)$ is the *Kullback-Leibler divergence* between two probability measures, P and Q , defined on a common measurable space:

$$\mathcal{D}(P|Q) = \begin{cases} \int \frac{dP}{dQ} \log \left(\frac{dP}{dQ} \right) dQ & \text{if } P \ll Q \\ +\infty & \text{otherwise.} \end{cases} \quad (2)$$

The integral in (2) is well defined since (with the convention $0 \log 0 = 0$) the integrand is bounded from below; the expectation in (1) is well defined since $\mathcal{D}(P|Q) \geq 0$. The (unconditional) mutual information $I(U; V)$ is obtained by setting $\mathcal{H} = \{\emptyset, \Omega\}$ in (1).

$I(U; V|\mathcal{H})$ is non-negative; it is zero if and only if U and V are \mathcal{H} -conditionally independent. An important property, of which we shall make frequent use, is the *chain rule*; this states that

$$I(U; (V, W)|\mathcal{H}) = I(U; V|\mathcal{H}) + I(U; W|\mathcal{H} \vee \sigma(V)), \quad (3)$$

where $W : \Omega \rightarrow \mathbf{W}$ is a third random variable, and $\sigma(V)$ is the σ -algebra generated by V . The reader is referred to [2] and [16] for a systematic development of such theorems of information theory; [16], in particular, develops the subject at an appropriate level of abstraction.

If \mathbf{U} and \mathbf{V} are the path spaces of continuous-time semimartingales, then Girsanov's theorem can be used in the computation of $I(U; V|\mathcal{H})$. The mutual information between a "signal process" X and an "observation process" Y , comprising the signal and independent additive Gaussian white noise, was first found in this way in [3]. Duncan's results were extended to the case of partial independence in [4], and to a more abstract result on Wiener space in [10]. The connection between the path mutual information and the instantaneous mutual information in this context was first investigated in [9]. These ideas were connected with information and energy flows in statistical mechanical systems in [11] and [14], and with time-reversal and notions of *dual* linear and nonlinear filters in [12] and [13]. These papers all address problems in which there is "one-way influence" between the components whose mutual information is sought, and so lead to notions of directional *information flow*.

Transfer entropy and directed information were both developed for problems in which there is "two-way influence" between processes. Their definitions are based on mutual information, and measure the dependency between the past of one process and the future of another. A notion of directed information between stationary discrete-time processes appeared in the context of *bidirectional communication* in [6]. It was refined in [7], and used to study communication channels with feedback. Massey's definition is as follows:

$$D_{X \rightarrow Y}^M(N) = \sum_{n=1}^N I((X_1, \dots, X_n); Y_n | Y_1, \dots, Y_{n-1}), \quad (4)$$

where (X_1, X_2, \dots) is the input sequence to a communication channel, and (Y_1, Y_2, \dots) is the corresponding output sequence. (By definition, an empty

sequence generates the trivial σ -algebra $\{\emptyset, \Omega\}$.) The term “transfer entropy” was first used in [19]. Schreiber’s definition is as follows:

$$T_{X \rightarrow Y}(k, l, n) = I((X_{n-l}, \dots, X_{n-1}); Y_n | Y_{n-k}, \dots, Y_{n-1}); \quad (5)$$

it is often described as a measure of “the disambiguation on the future of Y , over and above that provided by the past of Y , afforded by the past of X ”. It is frequently used with the common value $k = l = n - 1$ for the history lengths of X and Y , in which case its cumulative variant,

$$\begin{aligned} D_{X \rightarrow Y}(N) &= \sum_{n=2}^N T_{X \rightarrow Y}(n-1, n-1, n) \\ &= \sum_{n=2}^N I((X_1, \dots, X_{n-1}); Y_n | Y_1, \dots, Y_{n-1}), \end{aligned} \quad (6)$$

is similar to $D_{X \rightarrow Y}^M(N)$.

$D_{X \rightarrow Y}$ seems a more natural definition of directed information since it does not involve the mutual information between the values of X and Y at the same time instant, and so is more faithful to the notion of directional dependency. However, it is important to remember that Massey’s definition was made in the context of a specific application. In [7], Y_n is the output of a communication channel corresponding to the input X_n , and so Y_n actually occurs *after* X_n in physical time (but before X_{n+1}), a notion that can be made mathematically explicit by switching to a finer time scale. Let $\tilde{Y}_1 = 0$ and, for any $m \in \mathbb{N}$, let $\tilde{X}_m = X_{\lceil m/2 \rceil}$ and $\tilde{Y}_{m+1} = Y_{\lfloor (m+1)/2 \rfloor}$, where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ are the ceiling and floor functions, respectively. Note that \tilde{X} changes only when m increments from an even value, whereas \tilde{Y} changes only when m increments from an odd value, reflecting the causal relations of communication channels with feedback. An easy calculation shows that

$$D_{X \rightarrow Y}^M(N) = D_{\tilde{X} \rightarrow \tilde{Y}}(2N). \quad (7)$$

(NB. alternate terms in the sum on the right-hand side here are zero.) The use of $D_{X \rightarrow Y}$ rather than $D_{X \rightarrow Y}^M$ also symmetrises Massey’s “law of conservation of directed information” [8], thus avoiding shifted sequences. For a general pair of sequences (X_1, X_2, \dots) and (Y_1, Y_2, \dots) :

$$\begin{aligned} I((X_1, \dots, X_N); (Y_1, \dots, Y_N)) &= D_{X \rightarrow Y}(N) + D_{Y \rightarrow X}(N) \\ &\quad + \sum_{n=1}^N I(X_n; Y_n | X_1, Y_1, \dots, X_{n-1}, Y_{n-1}). \end{aligned} \quad (8)$$

The third term on the right-hand side here (called the “instantaneous information exchange” in [1]) is zero if X_n and Y_n are independent, conditioned on their pasts, which is the case for the sequences \tilde{X} and \tilde{Y} defined above.

Since it is more faithful to the notion of directional dependency, we shall base our definition of continuous-time directed information on $D_{X \rightarrow Y}$ rather than $D_{X \rightarrow Y}^M$. A definition based on $D_{X \rightarrow Y}^M$ was developed in [21], in a very general context. However, the hypotheses used are somewhat unripe and, like $D_{X \rightarrow Y}^M$, the information quantity obtained is not strictly directional.

In applications, one often wants to identify *causal influence* between stochastic processes. In Neuroscience, for example, it is useful to know which group of neurons causes which to fire. Of course, the existence of statistical dependency between the past of one process and the future of another is no guarantee of causation; this can only be determined by examining the underlying physics (and usually requires physical intervention) [15]. This cannot be done in the abstract setting of this paper. However, in order to motivate some of the definitions we give, it is useful to have a *Principle of Influence* between processes, that corresponds to causal influence in many applications. It attributes a physical property to the time variable.

PI: *The future of one process cannot influence the past of another.*

According to this principle, any statistical dependency between the past of one process and the future of another arises from a combination of the influence that the former has on the latter, and the influence that other processes have on both. In the context of a closed system having a filtration $(\mathcal{F}(t), 0 \leq t < \infty)$ to which all processes are adapted, a process X that is Markov with respect to \mathcal{F} (meaning that $X(t)$ is $X(s)$ -conditionally independent of $\mathcal{F}(s)$ for all $s \leq t$) cannot be influenced by any other process. Section 2 of this paper defines and evaluates the transfer entropy and directed information between two processes that are jointly Markov in this sense; according to PI, they measure one-way influence. When processes are not jointly Markov, there is more than one plausible definition of transfer entropy and directed information. We explore marginal and conditional variants of two such definitions in section 3, only one of which preserves the properties of influence of the jointly Markov case.

The following notation will be used frequently in what follows:

- $\mathbb{M}_{m,n}$ is the set of real matrices of dimension $m \times n$;

- $\mathbb{S}_n^{++} \subset \mathbb{S}_n^+ \subset \mathbb{S}_n \subset \mathbb{M}_{n,n}$ are the subsets of positive-definite, positive-semi-definite, and symmetric $n \times n$ matrices, respectively;
- $I_n \in \mathbb{S}_n^{++}$ is the multiplicative identity matrix;
- For $\mu \in \mathbb{R}^n$ and $v \in \mathbb{S}_n^+$, $N(\mu, v)$ is the multivariate Gaussian distribution with mean vector μ and covariance matrix v .

2 Jointly Markov Processes

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space, on which is defined a filtration $(\mathcal{F}(t) \subset \mathcal{G}, 0 \leq t < \infty)$. (All filtrations will be assumed to satisfy the “usual conditions” [17].) For some $n \geq 2$, let $(X(t) \in \mathbb{R}^n, 0 \leq t < \infty)$ be a continuous, Gaussian, \mathcal{F} -diffusion process defined on Ω , with initial distribution $N(\mu, v)$, drift vector $b(t)X(t)$ and diffusion matrix $a(t)$, where $\mu \in \mathbb{R}^n$, $v \in \mathbb{S}_n^+$, and $b : [0, \infty) \rightarrow \mathbb{M}_{n,n}$ and $a : [0, \infty) \rightarrow \mathbb{S}_n^+$ are measurable functions satisfying

$$\int_0^t (\|b(s)\| + \|a(s)\|) ds < \infty \quad \text{for all } 0 \leq t < \infty. \quad (9)$$

By this we mean that the distribution of $X(0)$ is $N(\mu, v)$, X is adapted to \mathcal{F} , and the process (M, \mathcal{F}) , defined by

$$M(t) = X(t) - X(0) - \int_0^t b(s)X(s) ds, \quad (10)$$

is a non-standard \mathbb{R}^n -valued Brownian motion with quadratic covariation $[M](t) = \int_0^t a(s) ds$.

For some $n_1, n_2 \geq 1$ with $n_1 + n_2 = n$, let

$$e_1 = [I_{n_1} \ 0] \in \mathbb{M}_{n_1,n} \quad \text{and} \quad e_2 = [0 \ I_{n_2}] \in \mathbb{M}_{n_2,n}. \quad (11)$$

We shall use the notation $X_i := e_i X$, $\mu_i := e_i \mu$, $v_{ij} := e_i v e_j'$, $b_{ij} := e_i b e_j'$ and $a_{ij} := e_i a e_j'$ for the block components of X , μ , v , b and a . This section defines and evaluates the transfer entropy and the directed information from X_2 to X_1 . The values of these quantities between different components of X can be found by using the methods herein on the process πX for an appropriately chosen non-singular matrix $\pi \in \mathbb{M}_{n,n}$.

2.1 A Directed Representation for X

We shall make use of the following hypotheses:

(H1) for any $0 \leq t < \infty$, $\text{rank}(a_{11}(t)) = k > 0$;

(H2) for any $0 \leq t < \infty$, $b_{12}(t) = a_{11}(t)\gamma(t)$ for some measurable function $\gamma : [0, \infty) \rightarrow \mathbb{M}_{n_1, n_2}$, satisfying

$$\int_0^t \|\gamma(s)'(I_{n_1} + a_{11}(s))\gamma(s)\| ds < \infty \quad \text{for all } 0 \leq t < \infty. \quad (12)$$

Let $a_{11}(t) = u(t)\lambda(t)u(t)'$ be the ‘‘reduced’’ eigen-decomposition of $a_{11}(t)$. By this, we mean that $\lambda(t) \in \mathbb{S}_k^{++}$ is a diagonal matrix containing the *non-zero* eigenvalues of $a_{11}(t)$, and the columns of $u(t) \in \mathbb{M}_{n_1, k}$ are the corresponding (orthonormal) eigenvectors. Let $\sigma(t) \in \mathbb{M}_{n, k+n_2}$ and $a^W(t) \in \mathbb{S}_{k+n_2}^+$ be defined as follows:

$$\sigma(t) = \begin{bmatrix} \sigma_{11}(t) & 0 \\ \sigma_{21}(t) & I_{n_2} \end{bmatrix} \quad \text{and} \quad a^W(t) = \begin{bmatrix} I_k & 0 \\ 0 & \alpha(t) \end{bmatrix}, \quad (13)$$

where $\sigma_{i1}(t) = a_{i1}(t)u(t)\lambda(t)^{-1/2}$ ($i = 1, 2$), and

$$\alpha(t) = a_{22}(t) - \sigma_{21}(t)\sigma_{21}(t)' \in \mathbb{S}_{n_2}^+; \quad (14)$$

then $a = \sigma a^W \sigma'$. Let W and \hat{M} be defined as follows:

$$\begin{aligned} W(t) &= \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix} = \int_0^t (\sigma(s)'\sigma(s))^{-1} \sigma(s)' dM(s) \\ \hat{M}(t) &= \int_0^t \sigma(s) dW(s), \end{aligned} \quad (15)$$

where $W_1(t) \in \mathbb{R}^k$ and $W_2(t) \in \mathbb{R}^{n_2}$. (W, \mathcal{F}) is a non-standard $(k + n_2)$ -vector Brownian motion with quadratic covariation $[W](t) = \int_0^t a^W(s) ds$ and, since $\sigma(\sigma'\sigma)^{-1}\sigma'a = a$, the process $(M - \hat{M}, \mathcal{F})$ is a non-standard n -vector Brownian motion with quadratic covariation zero. So M and \hat{M} are indistinguishable.

We factorise the initial covariance matrix, v , in a similar way. Let $l = \text{rank}(v_{11})$, and let $\psi \in \mathbb{M}_{n, l+n_2}$, $v^\Xi \in \mathbb{S}_{l+n_2}^+$ and $\phi \in \mathbb{S}_{n_2}^+$ be defined as follows:

$$\psi = \begin{bmatrix} \psi_{11} & 0 \\ \psi_{21} & I_{n_2} \end{bmatrix}, \quad v^\Xi = \begin{bmatrix} I_l & 0 \\ 0 & \phi \end{bmatrix} \quad \text{and} \quad \phi = v_{22} - \psi_{21}\psi_{21}' \in \mathbb{S}_{n_2}^+, \quad (16)$$

where $\psi_{i1} = v_{i1}u_0\lambda_0^{-1/2}$ ($i = 1, 2$), and $v_{11} = u_0\lambda_0u'_0$ is the reduced eigen-decomposition of v_{11} . (If $l = 0$ then ψ_{11} and ψ_{21} are void, and $v^\Xi = \phi$.) Let

$$\Xi = \begin{bmatrix} \Xi_1 \\ \Xi_2 \end{bmatrix} = (\psi'\psi)^{-1}\psi'(X(0) - \mu), \quad (17)$$

where $\Xi_1 \in \mathbb{R}^l$ and $\Xi_2 \in \mathbb{R}^{n_2}$. (Ξ_1 is void if $l = 0$.) Straightforward calculations show that $\mathbb{E}\|X(0) - \mu - \psi\Xi\|^2 = 0$, and that Ξ has the Gaussian distribution, $N(0, v^\Xi)$.

The foregoing arguments show that X satisfies the following Itô equation:

$$X(t) = \mu + \psi\Xi + \int_0^t b(s)X(s) ds + \int_0^t \sigma(s) dW(s). \quad (18)$$

Remark 2.1. *In numerical implementations of the eigen-decomposition of a_{11} , it may be difficult to distinguish between an eigenvalue that is zero and one that is merely small, resulting in uncertainty about the “noise” dimension k . Although not mathematically necessary, the inclusion of the identity matrix in (12) ensures that the transfer entropy and directed information of sections 2.2 and 2.3 are not sensitive to the choice of k in such cases. X is often defined in the form (18), “up-front”. (See, for example, [18].)*

In order to compute the disambiguation on the future of X_1 afforded by its past, it is convenient to bring the relevant part of that past into the present. This can be achieved by means of a Kalman-Bucy filter for X based on “observations” of X_1 . Let \mathcal{F}_1 be the filtration generated by X_1 , let $(q_2(t) \in \mathbb{S}_{n_2}^+, 0 \leq t < \infty)$ satisfy the matrix Riccati equation:

$$\begin{aligned} q_2(0) &= \phi \\ \dot{q}_2 &= (b_{22} - a_{21}\gamma)q_2 + q_2(b_{22} - a_{21}\gamma)' + \alpha - q_2\gamma'a_{11}\gamma q_2, \end{aligned} \quad (19)$$

where ϕ , γ and α are as defined in (16), (H2) and (14), respectively, and let $((\hat{X}, \bar{W}_1)(t) \in \mathbb{R}^{n+k}, 0 \leq t < \infty)$ satisfy the Itô equation:

$$\begin{aligned} \hat{X}(0) &= \begin{cases} \mu & \text{if } l = 0 \\ \mu + \psi [I_l \ 0]' \Xi_1 & \text{otherwise} \end{cases}, \quad \bar{W}_1(0) = 0 \\ d\hat{X} &= b\hat{X}dt + \left(\sigma [I_k \ 0]' + e'_2 q_2 \gamma' \sigma_{11} \right) d\bar{W}_1 \\ d\bar{W}_1 &= \sigma'_{11} \gamma e_2 (X - \hat{X}) dt + dW_1. \end{aligned} \quad (20)$$

Proposition 2.1. *If (H1) and (H2) hold, then:*

(i) *for any $0 \leq t < \infty$, the Gaussian distribution $N(\hat{X}(t), e_2' q_2(t) e_2)$ is a regular $\mathcal{F}_1(t)$ -conditional distribution for $X(t)$;*

(ii) *$(\bar{W}_1, \mathcal{F}_1)$ is an \mathbb{R}^k -valued standard Brownian motion;*

(iii) *for any $0 \leq s \leq t < \infty$, $\mathcal{F}_1(t) = \mathcal{F}_1(s) \vee \sigma(\bar{W}_1(r) - \bar{W}_1(s), s \leq r \leq t)$.*

Proof. We begin by proving part (i) in the special case that $t = 0$. If $l = 0$ then $X_1(0)$ is non-random, and the (unconditional) distribution, $N(\mu, e_2' \phi e_2)$, is also a regular $X_1(0)$ -conditional distribution. On the other hand, if $l > 0$ then $\Xi_1 = (\psi_{11}' \psi_{11})^{-1} \psi_{11}' (X_1(0) - \mu_1)$ is $\mathcal{F}_1(0)$ -measurable, and Ξ_2 is independent of $\mathcal{F}_1(0)$. That $N(\hat{X}_2(0), \phi)$ is a regular $X_1(0)$ -conditional distribution for $X_2(0)$ follows from the fact that $X_2(0) = \mu_2 + \psi_{21} \Xi_1 + \Xi_2$.

Let $p = e_2' q_2 e_2$; since $b_{12} = a_{11} \gamma$,

$$\begin{aligned} \dot{p} &= (b - \rho \sigma_{11}' \gamma e_2) p + p (b - \rho \sigma_{11}' \gamma e_2)' + a - \rho \rho' - p e_2' \gamma' a_{11} \gamma e_2 p \\ &= b p + p b' + a - (\rho + p e_2' \gamma' \sigma_{11}) (\rho + p e_2' \gamma' \sigma_{11})', \end{aligned}$$

where $\rho = \sigma [I_k \ 0]'$. Let Y be the following k -vector ‘‘observations’’ process,

$$Y(t) = \int_0^t \sigma_{11}(s)' \gamma(s) e_2 X(s) ds + W_1(t),$$

and let \mathcal{F}^Y be the filtration it generates. Now $e_2' q_2 = p e_2'$, and so the diffusion coefficient in the second equation in (20) is $\rho + p e_2' \gamma' \sigma_{11}$. It is a standard result of Kalman-Bucy filtering with correlated noise processes (see, for example, Theorem 10.3 in [5]) that $N(\hat{X}(t), p(t))$ is a regular $\mathcal{F}_1(0) \vee \mathcal{F}^Y(t)$ -conditional distribution for $X(t)$. Since $\hat{X}_1 = X_1$, $\mathcal{F}_1(t) \subset \mathcal{F}_1(0) \vee \mathcal{F}^Y(t)$ for all t . It follows from (H2) that

$$Y(t) = \int_0^t (\sigma_{11}(s)' \sigma_{11}(s))^{-1} \sigma_{11}(s)' (dX_1(s) - b_{11}(s) X_1(s) ds),$$

and since (X_1, \mathcal{F}_1) is a semimartingale (Stricker’s theorem), $\mathcal{F}^Y(t) \subset \mathcal{F}_1(t)$. This completes the proof of part (i).

The fact that $(\bar{W}_1, \mathcal{F}_1)$ is a martingale with quadratic covariation $[\bar{W}_1](t) = I_k t$ is easily established, and this proves part (ii). Part (iii) follows since the second equation in (20) has a strong solution. \square

2.2 The Transfer Entropy

For any continuous, vector-valued process $(\Theta(t) \in \mathbb{R}^m, 0 \leq t < \infty)$ and any $0 \leq s < t < \infty$, we use the notation $\Theta(s, t)$ for the $C([s, t]; \mathbb{R}^m)$ -valued random variable $(\Theta(r), s \leq r \leq t)$. (The space $C([s, t]; \mathbb{R}^m)$ is assumed to be metrised by the maximum norm.) We define the transfer entropy from X_2 to X_1 as follows:

$$T_{2 \rightarrow 1}(s, t) = I(X_2(0, s); X_1(s, t) | \mathcal{F}_1(s)) \quad \text{for } 0 \leq s \leq t < \infty. \quad (21)$$

Proposition 2.2. *If (H1) and (H2) hold then, for any $0 \leq s \leq t < \infty$,*

$$T_{2 \rightarrow 1}(s, t) = \frac{1}{2} \int_s^t \text{tr}(\gamma(r)' a_{11}(r) \gamma(r) (q_2(r) - \tilde{q}_2(r))) dr, \quad (22)$$

where γ and q_2 are as defined in (H2) and (19), respectively, $(\tilde{q}_2(r), s \leq r < \infty)$ satisfies (19) over the time interval $[s, \infty)$ and $\tilde{q}_2(s) = 0$.

Proof. Fix $0 \leq s < \infty$, let $(Z(t) \in \mathbb{R}^{n+n_2}, s \leq t < \infty)$ be the process with components,

$$Z_1(t) = \begin{bmatrix} X_1(t) \\ X_2(s) \end{bmatrix} \quad \text{and} \quad Z_2(t) = X_2(t),$$

and let $(\mathcal{F}_1^Z(t), s \leq t < \infty)$ be the filtration generated by Z_1 . Then Z satisfies the Itô equation,

$$Z(t) = Z(s) + \int_s^t e' b(r) e Z(r) dr + \int_s^t e' \sigma(r) dW(r),$$

where $e = \begin{bmatrix} e'_1 & 0 & e'_2 \end{bmatrix} \in \mathbb{M}_{n, n+n_2}$. Let $((\hat{Z}, \bar{W}_1^Z)(t) \in \mathbb{R}^{n+n_2+k}, s \leq t < \infty)$ satisfy the Itô equation

$$\begin{aligned} \hat{Z}(s) &= Z(s), & \bar{W}_1^Z(s) &= 0 \\ d\hat{Z} &= e' b e \hat{Z} dt + e' (\rho + e'_2 \tilde{q}_2 \gamma' \sigma_{11}) d\bar{W}_1^Z, \\ d\bar{W}_1^Z &= \sigma'_{11} \gamma e_2 e (Z - \hat{Z}) dt + dW_1. \end{aligned}$$

It is easily verified that Z satisfies the hypotheses of Proposition 2.1, with s playing the role of time 0, n playing the role of n_1 , and $n + n_2$ playing the role of n . Proposition 2.1 shows that $N(\hat{Z}(t), e' e'_2 \tilde{q}_2(t) e_2 e)$ is a regular $\mathcal{F}_1^Z(t)$ -conditional distribution for $Z(t)$, and that $(\bar{W}_1^Z, \mathcal{F}_1^Z)$ is a k -dimensional standard Brownian motion. (In particular, since X is Markov with respect to \mathcal{F} , \bar{W}_1^Z is independent of $\mathcal{F}(s)$.)

Let $\bar{b} = b_{22} - (\sigma_{21} + \tilde{q}_2 \gamma' \sigma_{11}) \sigma'_{11} \gamma$ and $\bar{\sigma} = (\tilde{q}_2 - q_2) \gamma' \sigma_{11}$. It follows from (9) and (12), and the continuity of q_2 and \tilde{q}_2 , that $\int_0^t (\|\bar{b}(s)\| + \|\bar{\sigma}(s)\|^2) ds < \infty$ for all $0 \leq t < \infty$. Let $\Theta(t) = \hat{Z}_2(t) - \hat{X}_2(t)$ and $\bar{W}_1^+(t) = \bar{W}_1(t) - \bar{W}_1(s)$; then (Θ, \bar{W}_1^+) satisfies the equations

$$\begin{aligned} \Theta(s) &= X_2(s) - \hat{X}_2(s), & \bar{W}_1^+(s) &= 0 \\ d\Theta &= \bar{b}\Theta dt + \bar{\sigma} d\bar{W}_1^+, \\ d\bar{W}_1^+ &= \sigma'_{11} \gamma \Theta dt + d\bar{W}_1^Z, \end{aligned} \quad (23)$$

It is easily verified that $\mathbb{E}(\Theta(t) | \mathcal{F}_1(t)) = 0$, and $\mathbb{E}(\Theta(t)\Theta(t)' | \mathcal{F}_1(t)) = q_2(t) - \tilde{q}_2(t)$. Furthermore, (23) has a strong solution, and so Theorem 3.1 in [10] shows that

$$I(X(0, s); \bar{W}_1^+(s, t)) = \frac{1}{2} \int_s^t \text{tr}(\gamma(r)' a_{11}(r) \gamma(r) (q_2(r) - \tilde{q}_2(r))) dr. \quad (24)$$

(See also [4] for the case in which $k = 1$.) Now

$$\begin{aligned} I(X(0, s); \bar{W}_1^+(s, t)) &= I(X(0, s); \bar{W}_1^+(s, t)) - I(X_1(0, s); \bar{W}_1^+(s, t)) \\ &= I(X_2(0, s); \bar{W}_1^+(s, t) | \mathcal{F}_1(s)) \\ &= I(X_2(0, s); (X_1(s), \bar{W}_1^+(s, t)) | \mathcal{F}_1(s)) \\ &= T_{2 \rightarrow 1}(s, t), \end{aligned} \quad (25)$$

where we have used Proposition 2.1(ii) and the fact that the mutual information between independent random variables is zero in the first step, the chain rule in the second step, and Proposition 2.1(iii) and the invariance of conditional mutual information under measurable isomorphisms in the final step. The statement of the proposition now follows from (24) and (25). \square

Remark 2.2. *Since X is Markov, $T_{2 \rightarrow 1}(s, t) = I(X_2(s); X_1(s, t) | \mathcal{F}_1(s))$, and so the history of X_2 is unimportant in (21). The transfer entropy for different histories of X_1 can be obtained by re-defining the time origin at which the Riccati equation (19) is initialised.*

Substituting W_1 from the third equation of (20) into (18), we can express X_2 in the form $X_2 = X_2^1 + X_2^2$, where

$$\begin{aligned} X_2^1(0) &= \mu_2 + \psi_{21} \Xi_1, & X_2^2(0) &= \Xi_2 \\ dX_2^1 &= (b_{22} - a_{21} \gamma) X_2^1 dt + b_{21} X_1 dt + a_{21} \gamma \hat{X}_2 dt + \sigma_{21} d\bar{W}_1 \\ dX_2^2 &= (b_{22} - a_{21} \gamma) X_2^2 dt + dW_2, \end{aligned} \quad (26)$$

and $\psi_{21}\Xi_1$ is void if $l = 0$. Since these equations have strong solutions, the components X_2^1 and X_2^2 are adapted to the filtrations generated by X_1 and (Ξ_2, W_2) , respectively. (This is perhaps surprising since, according to (18), W_1 “drives” X_2 and is typically not adapted to \mathcal{F}_1 .) It follows that

$$T_{2 \rightarrow 1}(s, t) = I((\Xi_2, W_2(0, s)); X_1(s, t) | \mathcal{F}_1(s)); \quad (27)$$

in particular, no part of $T_{2 \rightarrow 1}$ has its origins in the fact that X_2 shares the noise component W_1 with X_1 if $a_{12} \neq 0$.

2.3 The Directed Information

Let $D_{2 \rightarrow 1}(t)$ be the following *directed information*:

$$D_{2 \rightarrow 1}(t) = \int_0^t R_{2 \rightarrow 1}(s) ds, \quad \text{where} \quad R_{2 \rightarrow 1}(t) = \lim_{\delta \downarrow 0} \delta^{-1} T_{2 \rightarrow 1}(t, t + \delta). \quad (28)$$

It is natural to think of $R_{2 \rightarrow 1}$ as representing a *flow* of Shannon information from X_2 to X_1 . The following proposition develops this idea, showing that it is the rate at which X_1 learns about the process (Ξ_2, W_2) .

Proposition 2.3. *If (H1) and (H2) hold then, for any $0 \leq t < \infty$,*

$$\begin{aligned} D_{2 \rightarrow 1}(t) &= I((\Xi_2, W_2(0, t)); X_1(0, t)) \\ &= \frac{1}{2} \int_0^t \text{tr}(\gamma(s)' a_{11}(s) \gamma(s) q_2(s)) ds. \end{aligned} \quad (29)$$

Proof. Let $\bar{b} = b_{22} - a_{21}\gamma$ and $\bar{\sigma} = -q_2\gamma'\sigma_{11}$. It follows from (9) and (12), and the continuity of q_2 , that $\int_0^t (\|\bar{b}(s)\| + \|\bar{\sigma}(s)\|^2) ds < \infty$ for all $0 \leq t < \infty$. Let $\Phi(t) = X_2(t) - \hat{X}_2(t)$; then (Φ, \bar{W}_1) satisfies the equations

$$\begin{aligned} \Phi(0) &= \Xi_2, \quad \bar{W}_1(0) = 0 \\ d\Phi &= \bar{b}\Phi dt + \bar{\sigma}d\bar{W}_1 + dW_2 \\ d\bar{W}_1 &= \sigma'_{11}\gamma\Phi dt + dW_1. \end{aligned} \quad (30)$$

Now (30) has a strong solution, and so $(\Xi_2, W_2(0, t), \bar{W}_1(0, t))$ is independent of Ξ_1 . It thus follows from Proposition 2.1(iii) and the chain rule (3) that

$$\begin{aligned} I((\Xi_2, W_2(0, t)); X_1(0, t)) &= I((\Xi_2, W_2(0, t)); \bar{W}_1(0, t)) \\ &= \frac{1}{2} \int_0^t \text{tr}(\gamma(s)' a_{11}(s) \gamma(s) q_2(s)) ds, \end{aligned}$$

where we have used Theorem 3.1 of [10] in the second step. \square

There is an important difference between $D_{2 \rightarrow 1}$ and the obvious continuous-time extension of Massey's definition (4). From the latter we might propose the definition $D_{2 \rightarrow 1}^M(t) = \int_0^t R_{2 \rightarrow 1}^M(s) ds$, where

$$R_{2 \rightarrow 1}^M(t) = \lim_{\delta \downarrow 0} \delta^{-1} I(X_2(0, t + \delta); X_1(t, t + \delta) | \mathcal{F}_1(t)); \quad (31)$$

cf. (32) in [21]. However, as pointed out in the introduction, this is truly directional only if there is a physical time delay between $X_2(t)$ and $X_1(t)$. This is not so here since $X_1(t)$ can influence $X_2(t + \delta)$, through the coefficient b_{21} , for arbitrarily small $\delta > 0$. The mutual information in (31) can be infinite since $X_1(t, t + \delta)$ and $X_2(t, t + \delta)$ typically have non-zero quadratic covariation, $[X_1, X_2](t) = \int_0^t a_{12}(s) ds$, which results in the singularity of their joint distribution with respect to its product of marginals. If X_1 and X_2 share no noise ($a_{12} = 0$) then $R_{2 \rightarrow 1}^M$ coincides with $R_{2 \rightarrow 1}$.

3 Processes that are not Jointly Markov

Let X be as defined in section 2, and satisfy (H1) and (H2). In this section we suppose that $n_2 \geq 2$, and sub-divide X_2 into two components. For some $\tilde{n}_2, \tilde{n}_3 \geq 1$ with $\tilde{n}_2 + \tilde{n}_3 = n_2$, let

$$\tilde{e}_1 = e_1, \quad \tilde{e}_2 = [I_{\tilde{n}_2} \quad 0] e_2 \in \mathbb{M}_{\tilde{n}_2, n} \quad \text{and} \quad \tilde{e}_3 = [0 \quad I_{\tilde{n}_3}] e_2 \in \mathbb{M}_{\tilde{n}_3, n}, \quad (32)$$

where e_i is as defined in (11). We shall use the notation $\tilde{X}_i := \tilde{e}_i X$, $\tilde{b}_{ij} := \tilde{e}_i b \tilde{e}_j'$ and $\tilde{a}_{ij} := \tilde{e}_i a \tilde{e}_j'$ ($i, j = 1, 2, 3$) for the block components of X , b and a . Proposition 2.2 evaluates the transfer entropy from $(\tilde{X}_2, \tilde{X}_3)$ to \tilde{X}_1 ($\tilde{T}_{(2,3) \rightarrow 1} = T_{2 \rightarrow 1}$). In what follows, we split $\tilde{T}_{(2,3) \rightarrow 1}$ into separate components from \tilde{X}_2 and \tilde{X}_3 , in two different ways.

3.1 Splitting by components of X_2

According to the chain rule (3):

$$\tilde{T}_{(2,3) \rightarrow 1}(s, t) = T_{2 \rightarrow 1}(s, t) = \tilde{T}_{2 \rightarrow 1}^X(s, t) + \tilde{T}_{3 \rightarrow 1|2}^X(s, t), \quad (33)$$

where

$$\begin{aligned} \tilde{T}_{2 \rightarrow 1}^X(s, t) &= I(\tilde{X}_2(0, s); \tilde{X}_1(s, t) | \mathcal{F}_1(s)), \\ \tilde{T}_{3 \rightarrow 1|2}^X(s, t) &= I(\tilde{X}_3(0, s); \tilde{X}_1(s, t) | \mathcal{F}_{12}(s)), \end{aligned} \quad (34)$$

and \mathcal{F}_{12} is the filtration generated by $(\tilde{X}_1, \tilde{X}_2)$.

Let $\alpha_{ij} := \tilde{e}_i e'_2 \alpha e_2 \tilde{e}'_j$ ($i, j = 2, 3$), where α is as defined in (14). In order to compute $\tilde{T}_{2 \rightarrow 1}^X$ and $\tilde{T}_{3 \rightarrow 1|2}^X$, we first factorise α as we did a in section 2.1, making use of the following hypotheses:

(H3) for any $0 \leq t < \infty$, $\text{rank}(\alpha_{22}(t)) = \tilde{k} > 0$;

(H4) for any $0 \leq t < \infty$, $\tilde{b}_{23}(t) = \tilde{a}_{21}(t)\gamma(t)e_2\tilde{e}'_3 + \alpha_{22}(t)c(t)$, where γ is as defined in (H2), and $c : [0, \infty) \rightarrow \mathbb{M}_{\tilde{n}_2, \tilde{n}_3}$, is a measurable function satisfying

$$\int_0^t \|c(s)'(I_{\tilde{n}_2} + \alpha_{22}(s))c(s)\| ds < \infty \quad \text{for all } 0 \leq t < \infty. \quad (35)$$

Let $\alpha_{22}(t) = \tilde{u}(t)\tilde{\lambda}(t)\tilde{u}(t)'$ be the reduced eigen-decomposition of $\alpha_{22}(t)$; then $a(t) = \tau(t)a^V(t)\tau(t)'$, where

$$\tau(t) = \begin{bmatrix} \tau_{11}(t) & 0 & 0 \\ \tau_{21}(t) & \tau_{22}(t) & 0 \\ \tau_{31}(t) & \tau_{32}(t) & I_{n_3} \end{bmatrix}, \quad a^V(t) = \begin{bmatrix} I_{k+\tilde{k}} & 0 \\ 0 & \beta(t) \end{bmatrix}, \quad (36)$$

$\tau_{11}(t) = \sigma_{11}(t)$, $\tau_{i1}(t) = \tilde{e}_i e'_2 \sigma_{21}(t)$ and $\tau_{i2}(t) = \alpha_{i2}(t)\tilde{u}(t)\tilde{\lambda}(t)^{-1/2}$ ($i = 2, 3$), and

$$\beta(t) = \alpha_{33}(t) - \tau_{32}(t)\tau_{32}(t)'. \quad (37)$$

Similarly, let $\phi_{ij} := \tilde{e}_i e'_2 \phi e_2 \tilde{e}'_j$ ($i, j = 2, 3$), where ϕ is as defined in (16), and let $\tilde{l} = \text{rank}(\phi_{22})$. If $\tilde{l} > 0$ then ϕ_{22} admits a reduced eigen-decomposition $\phi_{22} = \tilde{u}_0 \tilde{\lambda}_0 \tilde{u}'_0$, and $v = \eta v^\Theta \eta'$, where

$$\eta = \begin{bmatrix} \eta_{11} & 0 & 0 \\ \eta_{21} & \eta_{22} & 0 \\ \eta_{31} & \eta_{32} & I_{n_3} \end{bmatrix}, \quad v^\Theta = \begin{bmatrix} I_{l+\tilde{l}} & 0 \\ 0 & \theta \end{bmatrix}, \quad \theta = \phi_{33} - \eta_{32}\eta'_{32}, \quad (38)$$

$\eta_{11} = \psi_{11}$, and $\eta_{i1} = \tilde{e}_i e'_2 \psi_{21}$ and $\eta_{i2} = \phi_{i2} \tilde{u}_0 \tilde{\lambda}_0^{-1/2}$ ($i = 2, 3$). (η_{i1} is void if $l = 0$, η_{i2} is void if $\tilde{l} = 0$, and $v^\Theta = \theta$ if $l + \tilde{l} = 0$.)

The foregoing arguments show that X satisfies the Itô equation:

$$X(t) = \mu + \eta\Theta + \int_0^t b(s)X(s) ds + \int_0^t \tau(s) dV(s), \quad (39)$$

where

$$\Theta = (\eta' \eta)^{-1} \eta' (X(0) - \mu), \quad V(t) = \int_0^t (\tau(s)' \tau(s))^{-1} \tau(s)' dM(s), \quad (40)$$

and M is as defined in (10). Θ is an $\mathcal{F}(0)$ -measurable $(l + \tilde{l} + \tilde{n}_3)$ -vector Gaussian random variable with mean zero and covariance matrix v^Θ , and (V, \mathcal{F}) is a $(k + \tilde{k} + \tilde{n}_3)$ -vector, non-standard Brownian motion with quadratic covariation $[V](t) = \int_0^t a^V(s) ds$. Let

$$\tilde{e}_{(12)} = [I_{n_1 + \tilde{n}_2} \quad 0] \in \mathbb{M}_{n_1 + \tilde{n}_2, n}. \quad (41)$$

We shall use the notation $\tilde{X}_i := \tilde{e}_i X$, $\tilde{b}_{ij} := \tilde{e}_i b \tilde{e}_j'$ and $\tilde{a}_{ij} := \tilde{e}_i a \tilde{e}_j'$ ($i, j = 1, 2, 3, (12)$) for the block components of X , b and a .

Let $(q_3(t) \in \mathbb{S}_{\tilde{n}_3}^+, 0 \leq t < \infty)$ satisfy the following matrix Riccati equation:

$$\begin{aligned} q_3(0) &= \theta \\ \dot{q}_3 &= (\tilde{b}_{33} - \tilde{a}_{3(12)} \tilde{\gamma}) q_3 + q_3 (\tilde{b}_{33} - \tilde{a}_{3(12)} \tilde{\gamma})' + \beta - q_3 \tilde{\gamma}' \tilde{a}_{(12)(12)} \tilde{\gamma} q_3, \end{aligned} \quad (42)$$

where θ and β are as defined in (38) and (37), and

$$\tilde{\gamma}(t) = \begin{bmatrix} \gamma(t) e_2 \tilde{e}_3' - \tau_{11}(t) (\tau_{11}(t)' \tau_{11}(t))^{-1} \tau_{21}(t)' c(t) \\ c(t) \end{bmatrix} \in \mathbb{M}_{n_1 + \tilde{n}_2, \tilde{n}_3}. \quad (43)$$

Proposition 3.1. *If (H1)–(H4) hold, then, for any $0 \leq s \leq t < \infty$,*

$$\tilde{T}_{3 \rightarrow 1|2}^X(s, t) = \frac{1}{2} \int_s^t \text{tr}(\tilde{e}_3 \tilde{e}_2' \gamma(r)' a_{11}(r) \gamma(r) e_2 \tilde{e}_3' (q_3(r) - \tilde{q}_3(r))) dr, \quad (44)$$

where $(\tilde{q}_3(t), s \leq t < \infty)$ satisfies (42) over the time interval $[s, \infty)$, and $\tilde{q}_3(s) = 0$. (NB. $\tilde{T}_{2 \rightarrow 1}^X$ can be found from (22), (33) and (44).)

Proof. It follows from (H2), (H4) and (43) that $\tilde{b}_{(12)3} = \tilde{a}_{(12)(12)} \tilde{\gamma}$, and so the process X , regarded as comprising the two components $\tilde{X}_{(12)}$ and \tilde{X}_3 , satisfies hypotheses (H1) and (H2), with $\tilde{\gamma}$ playing the role of γ . Proposition 2.1 thus shows that the Gaussian distribution $N(\bar{X}(t), \tilde{e}_3' q_3(t) \tilde{e}_3)$ is a regular $\mathcal{F}_{12}(t)$ -conditional distribution for $X(t)$, where $((\bar{X}, \bar{V}_{(12)})(t) \in \mathbb{R}^{n+k+\tilde{k}}, 0 \leq t < \infty)$ satisfies the Itô equation

$$\begin{aligned} \bar{X}(0) &= \begin{cases} \mu & \text{if } l + \tilde{l} = 0 \\ \mu + \eta [I_{l+\tilde{l}} \quad 0]' \Theta_{(12)} & \text{otherwise} \end{cases}, \quad \bar{V}_{(12)}(0) = 0 \\ d\bar{X} &= b\bar{X} dt + (I_n + \tilde{e}_3' q_3 \tilde{\gamma}' \tilde{e}_{(12)}) \tilde{\rho} d\bar{V}_{(12)} \\ d\bar{V}_{(12)} &= \tilde{\rho}' \tilde{e}_{(12)}' \tilde{\gamma} e_3 (X - \hat{X}) dt + dV_{(12)}. \end{aligned} \quad (45)$$

Here $\Theta_{(12)} = [I_{l+\tilde{l}} \ 0] \Theta$, $V_{(12)} = [I_{k+\tilde{k}} \ 0] V$ and $\tilde{\rho} = \tau [I_{k+\tilde{k}} \ 0]'$. ($\Theta_{(12)}$ is void if $l + \tilde{l} = 0$.)

The process \tilde{X} , regarded as comprising the two components \tilde{X}_1 and $(\tilde{X}_2, \tilde{X}_3)$, itself satisfies hypotheses (H1) and (H2), and so we may apply Propositions 2.1 and 2.2 to find the transfer entropy from $(\tilde{X}_2, \tilde{X}_3)$ to \tilde{X}_1 . Straightforward calculations show that the $\mathcal{F}_1(t)$ -conditional covariance matrix of $(\tilde{X}_2(t), \tilde{X}_3(t))$ is $q_2(t) - e_2 \tilde{e}'_3 q_3(t) \tilde{e}_3 e'_2$, where q_2 is as defined in (19), and so, according to (22), the transfer entropy from $(\tilde{X}_2, \tilde{X}_3)$ to \tilde{X}_1 is

$$T_{2 \rightarrow 1}(s, t) - \frac{1}{2} \int_s^t \text{tr}(\gamma(r)' a_{11}(r) \gamma(r) e_2 \tilde{e}'_3 (q_3(r) - \tilde{q}_3(r)) \tilde{e}_3 e'_2) dr.$$

Since \tilde{X}_3 is adapted to \mathcal{F}_{12} , the chain rule shows that

$$I((\tilde{X}_2, \tilde{X}_3)(0, s); \tilde{X}_1(s, t) | \mathcal{F}_1(s)) = I(\tilde{X}_2(0, s); \tilde{X}_1(s, t) | \mathcal{F}_1(s)) = \tilde{T}_{2 \rightarrow 1}^X(s, t).$$

Together with (33), this proves (44). \square

$\tilde{T}_{2 \rightarrow 1}^X$ and $\tilde{T}_{3 \rightarrow 1|2}^X$ are natural candidates for the transfer entropy between components of the three-component process X . They both quantify the disambiguation on the future of \tilde{X}_1 afforded by the past of another component. However, unlike $T_{2 \rightarrow 1}$, $\tilde{T}_{2 \rightarrow 1}^X$ is typically not a measure of the *influence* that \tilde{X}_2 has on \tilde{X}_1 , since it can be strictly positive even if $\tilde{b}_{12} = 0$ and $\tilde{b}_{32} = 0$. This is so if \tilde{X}_1 and \tilde{X}_2 inherit a common history from \tilde{X}_3 . Correspondingly, $\tilde{T}_{3 \rightarrow 1|2}^X$ measures only the *direct* influence that \tilde{X}_3 has on \tilde{X}_1 ; it does not include any influence that it may have on \tilde{X}_1 *via* \tilde{X}_2 .

Applying Proposition 2.3 to the process \tilde{X} of (45), we obtain the directed information associated with $\tilde{T}_{2 \rightarrow 1}^X$:

$$\tilde{D}_{2 \rightarrow 1}^X(t) = \int_0^t \tilde{R}_{2 \rightarrow 1}^X(s) ds = I((\Theta_2, \bar{V}_2(0, s)); \tilde{X}_1(0, t)), \quad (46)$$

where

$$\tilde{R}_{2 \rightarrow 1}^X(t) = \lim_{\delta \downarrow 0} \delta^{-1} \tilde{T}_{2 \rightarrow 1}^X(t, t + \delta),$$

$\Theta_2 = [0 \ I_{\tilde{l}}] \Theta_{(12)}$ and $\bar{V}_2 = [0 \ I_{\tilde{k}}] \bar{V}$. $\tilde{T}_{2 \rightarrow 1}^X$ and $\tilde{D}_{2 \rightarrow 1}^X$ are measures of the influence that \tilde{X}_2 has on \tilde{X}_1 in the system (45), where the non- \mathcal{F}_{12} -adapted component \tilde{X}_3 is replaced by additional dynamics that preserve the joint distribution of \tilde{X}_1 and \tilde{X}_2 . (Θ_2, \bar{V}_2) is Markov with respect to \mathcal{F}_{12} , but not with respect to \mathcal{F} , illustrating the importance of the filtration in determining influence according to PI. We can clearly also define $\tilde{D}_{3 \rightarrow 1|2}^X = D_{2 \rightarrow 1} - \tilde{D}_{2 \rightarrow 1}^X$.

3.2 Splitting by components of (Ξ_2, W_2)

Unlike X_1 and X_2 , all components of (Ξ, W) are Markov with respect to \mathcal{F} , and so, according to PI, any dependency existing between the past of a component of (Ξ, W) and the future of X can be attributed to the influence that the former has on the latter. In this section we apply the chain rule to the right-hand side of (27) in order to identify the influence that sub-components of (Ξ_2, W_2) have on \tilde{X}_1 . Although this can be done without further assumptions, the split obtained is more directly connected with the components \tilde{X}_2 and \tilde{X}_3 under the following hypothesis:

(H5) $\phi_{23} = 0$, and for any $0 \leq t < \infty$, $\alpha_{23}(t) = 0$.

According to (27) and the chain rule

$$\tilde{T}_{(2,3) \rightarrow 1}(s, t) = T_{2 \rightarrow 1}(s, t) = \tilde{T}_{2 \rightarrow 1}^W(s, t) + \tilde{T}_{3 \rightarrow 1|2}^W(s, t), \quad (47)$$

where

$$\begin{aligned} \tilde{T}_{2 \rightarrow 1}^W(s, t) &= I((\tilde{\Xi}_2, \tilde{W}_2(0, s)); \tilde{X}_1(s, t) | \mathcal{F}_1(s)), \\ \tilde{T}_{3 \rightarrow 1|2}^W(s, t) &= I((\tilde{\Xi}_3, \tilde{W}_3(0, s)); \tilde{X}_1(s, t) | \mathcal{F}_1(s) \vee \mathcal{F}_2^W(s)), \end{aligned} \quad (48)$$

$\tilde{\Xi}_i = \tilde{e}_i e'_2 \Xi_2$ and $\tilde{W}_i = \tilde{e}_i e'_2 W_2$ ($i = 2, 3$), and \mathcal{F}_2^W is the filtration generated by $(\tilde{\Xi}_2, \tilde{W}_2)$. If (H5) holds then $(\tilde{\Xi}_2, \tilde{W}_2)$ and $(\tilde{\Xi}_3, \tilde{W}_3)$ influence X only through the components \tilde{X}_2 and \tilde{X}_3 , respectively, and so it is reasonable to think of $\tilde{T}_{2 \rightarrow 1}^W$ and $\tilde{T}_{3 \rightarrow 1|2}^W$ as being transfer entropies from \tilde{X}_2 and \tilde{X}_3 .

Let $(q_2^c(t) \in \mathbb{S}_{n_2}^+, 0 \leq t < \infty)$ satisfy the matrix Riccati equation,

$$\begin{aligned} q_2^c(0) &= e_2 \tilde{e}'_3 \phi_{33} \tilde{e}_3 e'_2 \\ \dot{q}_2^c &= (b_{22} - a_{21} \gamma) q_2^c + q_2^c (b_{22} - a_{21} \gamma)' + g - q_2^c \gamma' a_{11} \gamma q_2^c, \end{aligned} \quad (49)$$

where

$$g(t) = \begin{cases} e_2 \tilde{e}'_3 \alpha_{33}(t) \tilde{e}_3 e'_2 & \text{if } 0 \leq t \leq s \\ \alpha(t) & \text{otherwise.} \end{cases} \quad (50)$$

Proposition 3.2. *If (H1), (H2) and (H5) hold, then, for any $0 \leq s \leq t < \infty$,*

$$\tilde{T}_{3 \rightarrow 1|2}^W(s, t) = \frac{1}{2} \int_s^t \text{tr}(\gamma(r)' a_{11}(r) \tilde{\gamma}(r) (q_2^c(r) - \tilde{q}_2^c(r))) dr, \quad (51)$$

where $(\tilde{q}_2^c(t), s \leq t < \infty)$ satisfies (49) over the time interval $[s, \infty)$ and $\tilde{q}_2^c(s) = 0$. (NB. $\tilde{T}_{2 \rightarrow 1}^W$ can be found from (22), (47) and (51).)

Proof. Fix $0 \leq s < \infty$, let $(\Phi(t) \in \mathbb{R}^n, 0 \leq t < \infty)$ satisfy

$$\Phi(t) = \tilde{e}'_2 \tilde{\Xi}_2 + \int_0^t b(r) \Phi(r) dr + \tilde{e}'_2 \tilde{W}_2(s \wedge t),$$

and let $Z = X - \Phi$. Then $\sigma(\Phi(0, s)) = \mathcal{F}_2^W(s)$, $\Phi(s, t)$ is $\mathcal{F}_2^W(s)$ -measurable for all $t \geq s$, and Z is independent of Φ . Now

$$\begin{aligned} \tilde{T}_{3 \rightarrow 1|2}^W &= I((\tilde{\Xi}_3, \tilde{W}_3(0, s)); \tilde{X}_1(s, t) | \tilde{X}_1(0, s), \tilde{\Xi}_2, \tilde{W}_2(0, s)) \\ &= I((\tilde{\Xi}_3, \tilde{W}_3(0, s)); \Phi_1(s, t) + Z_1(s, t) | Z_1(0, s), \Phi(0, s)) \\ &= I((\tilde{\Xi}_3, \tilde{W}_3(0, s)); Z_1(s, t) | Z_1(0, s), \Phi(0, s)) \\ &= I((\Xi_2, W_2(0, s)); Z_1(s, t) | Z_1(0, s)) - I(\Phi(0, s); Z_1(s, t) | Z_1(0, s)) \\ &= I((\Xi_2, W_2(0, s)); Z_1(s, t) | Z_1(0, s)), \end{aligned}$$

where we have used a change of variables argument in the integral of (2) in the third step, the chain rule in the fourth step, and the independence of Φ and Z in the final step. The special case of Proposition 2.2, in which $\phi_{22} = 0$ and $\alpha_{22}(t) = 0$ for $0 \leq t \leq s$, shows that the final mutual information here is equal to the right-hand side of (51). \square

We can clearly also use $\tilde{T}_{2 \rightarrow 1}^W$ and $\tilde{T}_{3 \rightarrow 1|2}^W$ to define a pair of directed information quantities, $\tilde{D}_{2 \rightarrow 1}^W$ and $\tilde{D}_{3 \rightarrow 1|2}^W$.

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