Fibonacci type semigroups

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Abstract

We study "Fibonacci type" groups and semigroups. By establishing asphericity of their presentations we show that many of the groups are infinite. We combine this with Adjan graph techniques and the classification of the finite Fibonacci semigroups (in terms of the finite Fibonacci groups) to extend it to the Fibonacci type semigroups.

Keywords: Fibonacci group, Fibonacci semigroup, cyclic presentation, asphericity, Adjan graph.
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1 Introduction

In [6] the *Fibonacci semigroups* $S(r,n,k)$ defined by the cyclic presentations

$$\mathcal{F}(r,n,k) = \langle x_0,\ldots,x_{n-1} \mid x_i x_{i+1} \ldots x_{i+(r-1)} = x_{i+(r-1)+k} \quad (0 \leq i \leq n-1)\rangle$$

were studied and the finite semigroups $S(r,n,k)$ were classified in terms of the finite *Fibonacci groups* $F(r,n,k)$ (i.e. the groups defined by $\mathcal{F}(r,n,k)$). Here we prove the corresponding result for the more general *Fibonacci type* semigroups $T(r,n,k,h)$ and groups $R(r,n,k,h)$ defined by the cyclic presentations

$$\mathcal{R}(r,n,k,h) = \langle x_0,\ldots,x_{n-1} \mid x_i x_{i+h} \ldots x_{i+(r-1)h} = x_{i+(r-1)h+k} \quad (0 \leq i \leq n-1)\rangle$$

($k \geq 0$, $h \geq 1$, $n \geq 2$, $r \geq 2$, subscripts mod n), that were introduced in [5]. In that paper it was shown that for certain choices of the parameters the group $R(r,n,k,h)$ is finite and metacyclic, and that the family contains instances of finite, non-metacyclic cyclically presented groups (such groups are rare). More recently the groups $R(2,n,k,h)$ – the so-called *Cavicchioli-Hegenbarth-Repovš* groups $G_n(h,h+k)$ – have been of interest for their algebraic and topological properties (see [3],[9]). With the exception of two unresolved cases the finite groups $R(r,n,k,h)$ were classified in [17],[18],[10] and the present paper arose from a desire to classify the finite semigroups $T(2,n,k,h)$. In doing so we found that the asphericity methods used in [3],[17] are effective in the more general setting and can be combined with the Adjan graph and semigroup rewriting techniques of [6],[7] to classify the finite semigroups $T(r,n,k,h)$ in terms of the finite groups $R(r,n,k,h)$. Our main results are as follows.

Theorem A *Suppose $r \geq 3$, $((r-1)h+k,n) > 1$, $(k,n) > 1$, $(h,n) > 1$. Then $R(r,n,k,h)$ and $T(r,n,k,h)$ are infinite.*

Theorem B *The semigroup $T(r,n,k,h)$ is finite if and only if ($(n,(r-1)h+k) = 1$ or $(n,k) = 1$) and $R(r,n,k,h)$ is finite, in which case $T(r,n,k,h)$ is respectively the union of (n,k) disjoint left ideals or $(n,(r-1)h+k)$ disjoint right ideals, each isomorphic to $R(r,n,k,h)$.*

The classification of the finite semigroups $S(r,n,k)$ obtained in [6],[7] corresponds to the case $h=1$ in Theorem B. If the hypothesis $(h,n)>1$ is removed from Theorem A then the group $R(r,n,k,h)$ may be finite (for example $F(2,6,2)\cong \mathbb{Z}_7$) yet the semigroups $T(r,n,k,h)$ are all infinite. This was proved in [7, Theorem 3.5] as an application of the Reidemeister-Schreier style rewriting techniques for semigroups developed there. Our methods differ in that for $(h,n)>1$ we use asphericity of the presentations to prove both the semigroups and groups infinite.

2 The groups

Let F_n be the free group with generators $x_0,\ldots,x_{n-1}$ and let $\theta : F_n \to F_n$ be the automorphism of F_n given by $\theta(x_i)=x_{i+1}$ for each $0\le i\le n-1$ (subscripts mod n). This induces an action of the cyclic group $T=\langle t \mid t^n\rangle$ of order n on the presentation $G_n(w)$. Specifically, $tx_it^{-1}=x_{i+1}$ $(0\le i\le n-1)$ and therefore $t^ix_0t^{-i}=x_i$. Writing $x=x_0$ we see that the relator $x_0x_h\ldots x_{(r-1)h}x_{(r-1)h+k}^{-1}$ of $R(r,n,k,h)$ rewrites to $(xt^h)^r t^{k-h}x^{-1}t^{-(r-1)h-k}$. Setting $y=t^{-h}x^{-1}$ and eliminating x this becomes $y^{-r}t^kyt^{-(r-1)h-k}$ and so the split extension $N(r,n,k,h)$ of $R(r,n,k,h)$ by T has a presentation

$$\mathcal{N}(r,n,k,h)=\langle y,t \mid t^n, y^rt^\alpha y^{-1}t^\beta\rangle$$

where

$$\alpha=(r-1)h+k, \beta=-k.$$

The presentation $\mathcal{N}(r,n,k,h)$ may be regarded as a *relative presentation* $\langle T,y \mid y^rt^\alpha y^{-1}t^\beta\rangle$, in the sense of Bogley and Pride [4], and we will require their concept of *relative asphericity.* We will call an (ordinary) presentation *aspherical* if the standard 2-complex associated with the presentation is topologically aspherical. The following result is a special case of [15, Lemma 5] and a routine generalization of [13, Lemma 3.1].

Theorem 1 ([13],[15]) *If the relative presentation $\mathcal{N}(r,n,k,h)$ is (relatively) aspherical then the presentation $\mathcal{R}(r,n,k,h)$ is aspherical.*

Our method of proof is to establish asphericity of the presentations $\mathcal{R}(r,n,k,h)$ by combining the above result with theorems concerning relative asphericity of relative presentations of the form $\langle H,y \mid y^ray^{-1}b=1\rangle$. For the case $r\ge 4$, the appropriate theorem is provided by Davidson [11], and for $r=3$ it is provided by Ahmad [2].

Theorem 2 ([11, Theorem 1.2]) *Let $\mathcal{P}=\langle H,y \mid y^ray^{-1}b=1\rangle$ be a relative presentation, where $r\ge 4$ and a,b are non-trivial elements of H with $b\neq a^{\pm 1}$. Suppose $o(a)\neq 2$, $o(b)\neq 2$, and that $\mathcal{P}$ is not one of the following exceptional cases:*

(E1a) $a=b^2$ and $3<o(b)<\infty$; (E2a) $a=b^{-2}$ and $3<o(b)<\infty$; (E3a) $a=b^3$ and $o(b)=9$;
(E1b) $b=a^2$ and $3<o(a)<\infty$; (E2b) $b=a^{-2}$ and $3<o(a)<\infty$; (E3b) $b=a^3$ and $o(a)=9$.

Then $\mathcal{P}$ is relatively aspherical if and only if $1/o(a)+1/o(b)+1/o(ab^{-1})\le 1$.

Theorem 3 ([2, Theorem 5.2.1]) *Let $\mathcal{P} = \langle H, y \mid y^r a y^{-1} b = 1\rangle$ be a relative presentation, where $r = 3$ and a, b are non-trivial elements of H with $b \neq a^{\pm 1}$. Suppose that $\mathcal{P}$ is not one of the following exceptional cases:*

(F1a) $ab = ba$ and $o(a) = 2$, $o(b) \geq 4$; *(F2a) $a = b^2$ and $3 \leq o(a) < \infty$;*
(F1b) $ba = ab$ and $o(b) = 2$, $o(a) \geq 4$; *(F2b) $b = a^2$ and $3 \leq o(b) < \infty$.*

Then $\mathcal{P}$ is relatively aspherical if and only if $1/o(a) + 1/o(b) + 1/o(ab^{-1}) \leq 1$ and none of the following hold:

(S1a) $a^2 = 1$, $b^3 = 1$ and $\langle a, b\rangle \cong \mathbb{Z}_6$; *(S2a) $a^2 = 1$ and $a = b^2$;* *(S3a) $a^2 = 1$ and $a = b^3$;*
(S1b) $b^2 = 1$, $a^3 = 1$ and $\langle b, a\rangle \cong \mathbb{Z}_6$; *(S2b) $b^2 = 1$ and $b = a^2$;* *(S3b) $b^2 = 1$ and $b = a^3$.*

(Where $\langle a, b\rangle$ denotes the subgroup of H generated by a and b.)

Corollary 4 *Suppose $r \geq 3$, $\alpha \not\equiv 0$, $\beta \not\equiv 0$, $\beta \not\equiv \pm\alpha$, $2\alpha \not\equiv 0$, $2\beta \not\equiv 0$, $\alpha \not\equiv \pm 2\beta$, $\beta \not\equiv \pm 2\alpha$, $\alpha \not\equiv 3\beta$, $\beta \not\equiv 3\alpha$ (all mod n). Then $\mathcal{R}(r, n, k, h)$ is aspherical.*

Proof
The presentation $\mathcal{N}(r, n, k, h)$ may be regarded as a relative presentation $\langle H, y \mid y^r a y^{-1} b = 1\rangle$, where $H = \langle t \mid t^n\rangle$, $a = t^\alpha$, $b = t^\beta$. The hypotheses imply that a, b are non-trivial elements of H, $b \neq a^{\pm 1}$ and none of the exceptional cases (E1),(E2),(E3) of Theorem 2 and none of the exceptional cases (F1),(F2) or the cases (S1),(S2),(S3) of Theorem 3 hold. If $1/o(a) + 1/o(b) + 1/o(ab^{-1}) > 1$ then (since $o(a), o(b) \neq 2$) we have that $o(ab^{-1}) = 2$ and $o(a) = 3$ or $o(b) = 3$. That is, $2\alpha \equiv 2\beta$ mod n and $3\alpha \equiv 0$ mod n or $3\beta \equiv 0$ mod n, which contradicts $\alpha \not\equiv -2\beta$ and $\beta \not\equiv -2\alpha$. Thus Theorems 2 and 3 imply that $\mathcal{N}(r, n, k, h)$ is relatively aspherical, and so $\mathcal{R}(r, n, k, h)$ is aspherical, by Theorem 1. $\square$

Further technical conditions that ensure the asphericity of $\mathcal{R}(r, n, k, h)$ may be extracted from Theorems 2 and 3. In investigating the (S1),(S2),(S3) conditions of Theorem 3 we found that the corresponding groups $R(r, n, k, h)$ are free products of copies of $F(3, 12, 4)$ (for (S1)), of $F(3, 8, 2)$ (for (S2)), and of either $F(3, 6, 1)$ or $R(3, 6, 5, 2)$ (for (S3)). Simplifying the presentations in GAP [12] reveals that $F(3, 12, 4) = \langle x_2, x_5 \mid (x_2x_5)^{37}\rangle \cong \mathbb{Z}_{37} * \mathbb{Z}$ and that (writing $a = x_1x_0$, $b = x_1x_2$, $c = x_0$) $F(3, 8, 2) = \langle a, b, c \mid a^3, b^3, aba^{-1}b^{-1}\rangle \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) * \mathbb{Z}$. The groups $F(3, 6, 1)$ and $R(3, 6, 5, 2)$ were identified in [5] as distinct, non-metacyclic groups of order 1512. Since these all have torsion it follows that none of the corresponding presentations are aspherical.

We can now prove Theorem A.

Proof of Theorem A
If the semigroup $T(r, n, k, h)$ is finite then the group $R(r, n, k, h)$ is a homomorphic image of $T(r, n, k, h)$, under the natural homomorphism, so it suffices to show that $R(r, n, k, h)$ is infinite.

The group $N(r, n, k, h)$ maps onto $\langle y, t \mid t^{(\alpha,\beta,n)}, y^{r-1}\rangle \cong \mathbb{Z}_{(\alpha,\beta,n)} * \mathbb{Z}_{r-1}$. If $(\alpha, \beta, n) > 1$ then this is infinite, and hence so is $R(r, n, k, h)$, so we may assume $(\alpha, \beta, n) = 1$. Together with the hypotheses $(\alpha, n) > 1$, $(\beta, n) > 1$ this implies $\alpha \not\equiv 0$, $\beta \not\equiv 0$, $\beta \not\equiv \pm\alpha$, $\alpha \not\equiv \pm 2\beta$, $\beta \not\equiv \pm 2\alpha$, $\alpha \not\equiv 3\beta$, $\beta \not\equiv 3\alpha$ (all mod n). Suppose $2\alpha \equiv 0$ mod n. If β is even then α is odd and hence $(r-1)h$ is odd so $1 < (h, n) | ((r-1)h, n) = ((r-1)h, n/2, n) = (\alpha + \beta, \alpha, n) = (\alpha, \beta, n) = 1$, a contradiction. If β is odd then $1 < (\beta, n) = (\beta, n/2) = (\alpha, \beta, n) = 1$, again a contradiction. Thus $2\alpha \not\equiv 0$ mod n;

similarly $2\beta \not\equiv 0 \bmod n$. Corollary 4 then implies that $\mathcal{R}(r,n,k,h)$ is aspherical and so $R(r,n,k,h)$ is torsion-free. Since the exponent sum of the relators is $r-1 \geq 2$, the abelianization $R(r,n,k,h)^{\mathrm{ab}}$ is non-trivial (see [14, Section 9]), so $R(r,n,k,h)$ is non-trivial and hence is infinite. $\square$

Although not necessary for the proof of Theorem B we take the opportunity to deal with the cases $\alpha \equiv 0$ or $\beta \equiv 0 \bmod n$. In these cases, by partitioning the generators and relators, it is easy to see that $R(r,n,k,h)$ is isomorphic to the free product of (n,h) copies of $G(r-1,n)$, where

$$G(s,n) = \langle x_0, \ldots, x_{n-1} \mid x_i x_{i+1} \ldots x_{i+s-1} = 1 \ (i = 0, \ldots, n-1) \rangle.$$

In [16, Theorems 2 and 3] Umar proved that $G(s,n)$ is finite if and only if $(n,s) = 1$, in which case $G(s,n) \cong \mathbb{Z}_s$. We generalize this by identifying the group.

Theorem C *The group* $G(s,n) \cong \mathbb{Z}_{s/(n,s)} * \underbrace{\mathbb{Z} * \ldots * \mathbb{Z}}_{(n,s)-1}$.

Proof
Let $\delta = (n,s)$. Then there exist $p, q \in \mathbb{Z}$ such that $\delta = ps + qn$ so $\delta \equiv ps \bmod n$. The relation $x_i x_{i+1} \ldots x_{i+s-1} = 1$ implies $x_i(x_{i+1} \ldots x_{i+s-1} x_{i+s}) = x_{i+s}$ so $x_i = x_{i+s}$ and hence $x_i = x_{i+s} = x_{i+2s} = \ldots = x_{i+ps}$. But $x_{i+ps} = x_{i+\delta}$ so we have $x_i = x_{i+\delta}$ for each $1 \leq i \leq n-1$. Eliminating generators $x_\delta, \ldots, x_{n-1}$ gives

$$\begin{aligned} G(s,n) &= \langle x_0, \ldots, x_{\delta-1} \mid (x_i x_{i+1} \ldots x_{i+\delta-1})^{s/\delta} = 1 \ (i = 0, \ldots, \delta-1) \rangle \\ &= \langle x_0, \ldots, x_{\delta-1} \mid (x_0 x_1 \ldots x_{\delta-1})^{s/\delta} = 1 \rangle \end{aligned}$$

and the result follows. $\square$

3 The semigroups

We recall the concept of Adjan graphs [1]. The *left (resp. right) Adjan graph* of the positive presentation

$$\langle x_0, \ldots, x_{n-1} \mid u_i(x_0, \ldots, x_{n-1}) = v_i(x_0, \ldots, x_{n-1}) \ (0 \leq i \leq m-1) \rangle$$

is defined to be the graph with vertices $x_0, \ldots, x_{n-1}$ and where vertices x_i, x_j are joined by an edge if and only if there is a relation $u_k = v_k$ $(0 \leq k \leq m-1)$ such that x_i, x_j are the initial (resp. final) letters of u_k, v_k. Thus, for the left Adjan graph of $\mathcal{R}(r,n,k,h)$ the edges are $\{x_i, x_{i+\alpha}\}$ and for the right Adjan graph the edges are $\{x_i, x_{i+\beta}\}$. Therefore the left graph has $d_L = (\alpha, n)$ components (each a cycle) and the right graph has $d_R = (\beta, n)$ components (each a cycle).

As part of their investigations into the semigroups $S(r,n,k)$ Campbell, Robertson, Ruškuc, Thomas proved the following result:

Theorem 5 ([6, Theorem 1.3],[8, Theorem 4])
Let $\mathcal{Q} = \langle x_0, \ldots, x_{n-1} \mid x_j = v_j(x_0, \ldots, x_{n-1}) \ (0 \leq j \leq n-1) \rangle$ *where each* v_j *is a positive word of length at least* 2 *and each* x_i *occurs as the first (resp. last), the second (resp. penultimate), and the last (resp. first) letter of three of the* v_j*. Suppose that the right (resp. left) Adjan graph of* $\mathcal{Q}$ *is connected and that the left (resp. right) Adjan graph has* d *components. Then the semigroup defined by* $\mathcal{Q}$ *is a union of* d *disjoint right (resp. left) ideals each isomorphic to the group defined by* $\mathcal{Q}$*. In particular the semigroup is finite if and only if the group is finite.*

Applying this to the generalized Fibonacci groups $F(r,n,k)$ and semigroups $S(r,n,k)$ their main theorem is obtained:

Theorem 6 ([6, Theorem 1.2]) *If $(n, r-1+k) = 1$ (resp. $(n,k) = 1$) then $S(r,n,k)$ is the union of (n,k) (resp. $(n, r-1+k)$) disjoint left (resp. right) ideals each isomorphic to $F(r,n,k)$.*

It is also shown in [6, Section 3] that if $(n, k, r-1+k) > 1$ then $S(r,n,k)$ is infinite. The classification of the finite semigroups $S(r,n,k)$ in terms of the finite groups $F(r,n,k)$ (Theorem B with $h = 1$) is completed by the following theorem.

Theorem 7 ([7, Theorem 3.5]) *Suppose $(n, k, r-1+k) = 1$, $d_L = (n, r-1+k) > 1$ and $d_R = (n,k) > 1$. Then $S(r,n,k)$ is the disjoint union of $d_L d_R$ copies of a group that can be presented by $n + (d_L - 1)(d_R - 1)$ generators and n relations. In particular, $S(r,n,k)$ is infinite.*

We can now prove Theorem B.

Proof of Theorem B
If $((r-1)h + k, n) = 1$ or $(k,n) = 1$ then the result follows from Theorem 5 so we may assume that $((r-1)h + k, n) > 1$ and $(k,n) > 1$. If $(h,n) = 1$ then, by applying an automorphism of $\mathbb{Z}_n$ to the subscripts of the generators $x_0, \ldots, x_{n-1}$ we may assume $h = 1$ so the relations are those of the generalized Fibonacci semigroup $S(r,n,k)$, and this is infinite by Theorem 7 (together with the comments above it).

Suppose then that $(h,n) > 1$. If $r \geq 3$ then $R(r,n,k,h)$ and $T(r,n,k,h)$ are infinite by Theorem A, so we may assume $r = 2$. If $\alpha \not\equiv 0$ and $\beta \not\equiv 0$ mod n then $R(2,n,k,h)$, and hence $T(2,n,k,h)$, is infinite by [17],[18]. If $\alpha \equiv 0$ or $\beta \equiv 0$ mod n then $T(r,n,k,h)$ is the semigroup free product of (n,h) copies of $T(2,n,k,1) = S(2,n,k)$ which, by [16, Theorem 4], is the union of n trivial ideals, and hence $T(r,n,k,h)$ is infinite. $\square$

Remark 8 Question 1 of [3] asks for the classification of the finite groups $R(2,n,k,h)$. Except for two groups, this was provided in [13],[17],[18],[10]. The unresolved groups are the *Gilbert-Howie groups* ([13]) $H(9,4) = R(2,9,6,4)$ and $H(9,7) = R(2,9,3,7)$. Theorem B therefore classifies the finite semigroups $T(2,n,k,h)$ except for the two unresolved semigroups $T(2,9,6,4)$ and $T(2,9,3,7)$ (up to isomorphism and anti-isomorphism). By Theorem 5 the semigroups $T(2,9,6,4)$ and $T(2,9,3,7)$ are each the unions of 3 disjoint left ideals which are isomorphic to $R(2,9,6,4)$ and to $R(2,9,3,7)$, respectively. Therefore completing the classification of the finite groups $R(2,n,k,h)$ is equivalent to completing the classification of the finite semigroups $T(2,n,k,h)$. It seems unlikely, however, that semigroup techniques will shed more light on the problem.

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