Free subgroups in certain generalized triangle groups of type $(2, m, 2)$

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Abstract

A generalized triangle group is a group that can be presented in the form $G = \langle\ x, y \mid x^p = y^q = w(x, y)^r = 1\ \rangle$ where $p, q, r \geq 2$ and $w(x, y)$ is a cyclically reduced word of length at least 2 in the free product $\mathbb{Z}_p * \mathbb{Z}_q = \langle\ x, y \mid x^p = y^q = 1\ \rangle$. Rosenberger has conjectured that every generalized triangle group G satisfies the Tits alternative. It is known that the conjecture holds except possibly when the triple (p, q, r) is one of $(3, 3, 2)$, $(3, 4, 2)$, $(3, 5, 2)$, or $(2, m, 2)$ where $m = 3, 4, 5, 6, 10, 12, 15, 20, 30, 60$. In this paper we show that the Tits alternative holds in the cases $(p, q, r) = (2, m, 2)$ where $m = 6, 10, 12, 15, 20, 30, 60$.

1 Introduction

A *generalized triangle group* is a group that can be presented in the form

$$G = \langle\ x, y \mid x^p = y^q = w(x, y)^r = 1\ \rangle$$

where $p, q, r \geq 2$ and $w(x, y)$ is a cyclically reduced word of length at least 2 in the free product $\mathbb{Z}_p * \mathbb{Z}_q = \langle\ x, y \mid x^p = y^q = 1\ \rangle$ that is not a proper power. It was conjectured by Rosenberger [15] that every generalized triangle group G satisfies the Tits alternative. That is, G either contains a non-abelian free subgroup or has a soluble subgroup of finite index.

It is now known that the Tits alternative holds for a generalized triangle group G except possibly when the triple (p, q, r) is one of $(3, 3, 2)$, $(3, 4, 2)$, $(3, 5, 2)$, or $(2, m, 2)$ where $m \geq 3$. (See [9] for a survey of these results.) In recent work Benyash-Krivets [3, 4] considers the case $(2, m, 2)$. He has shown that if $m \geq 7$, $m \neq 10, 12, 15, 20, 30, 60$ then the Tits alternative holds for G. In this paper we augment that result to prove the following:

Main Theorem. *Let $G = \langle\ x, y \mid x^2 = y^m = w(x, y)^2 = 1\ \rangle$ where $w(x, y) = xy^{\alpha_1} \ldots xy^{\alpha_k}$, $1 \leq \alpha_i < m$, $m \geq 6$. Then the Tits alternative holds for G.*

If $k = 1$ then the Tits alternative holds for G by [8]. If $m = 6$ and $k = 2$ or 3 then the Tits alternative holds for G by [15, 14] respectively. The Main Theorem then follows from Theorems 1, 2 and 3:

Theorem 1 *Let G be as defined in the Main Theorem. If $m = 6$ and $k > 3$, then G contains a non-abelian free subgroup.*

Theorem 2 *Let G be as defined in the Main Theorem. If $m = 5p$ where $p \neq 5$ is prime and $k > 1$, then G contains a non-abelian free subgroup.*

Theorem 3 *Let G be as defined in the Main Theorem. If $k > 1$ and $m = 12, 20, 30,$ or 60 then G contains a non-abelian free subgroup.*

Theorem 1 has independently been obtained by Barkovich and Benyash-Krivets [1, 5], and for this reason we do not give a complete proof. However, we require Theorem 1 in an essential way in the proofs of the other results, so in order to make our paper self-contained we have included a sketch proof in an Appendix.

2 Preliminaries

We first recall some definitions and well-known facts concerning generalized triangle groups; further details are available in (for example) [9]. Let G be as defined in the Main Theorem, but with $m \geq 3$. A homomorphism $\rho : G \to H$ (for some group H) is said to be *essential* if $\rho(x), \rho(y), \rho(w)$ are of orders $2, m, 2$ respectively. By [2] G admits an essential representation into $PSL(2, \mathbb{C})$.

A projective matrix $A \in PSL(2, \mathbb{C})$ is of order n if and only if $\mathrm{tr}(A) = 2\cos(q\pi/n)$ for some $(q, n) = 1$. Note that in $PSL(2, \mathbb{C})$ traces are only defined up to sign. A subgroup of $PSL(2, \mathbb{C})$ is said to be *elementary* if it has a soluble subgroup of finite index, and is said to be *non-elementary* otherwise.

Let $\rho : \langle\, x, y \mid x^2 = y^m = 1 \,\rangle \to PSL(2, \mathbb{C})$ be given by $x \mapsto X,\ y \mapsto Y$ where X, Y have orders $2, m$, respectively. Then $w(x, y) \mapsto w(X, Y)$. By Horowitz [12] $\mathrm{tr}w(X, Y)$ is a polynomial with rational coefficients in $\mathrm{tr}X, \mathrm{tr}Y, \lambda := \mathrm{tr}XY$, of degree k in λ. Since X, Y have orders $2, m$, respectively, we may assume (by composing ρ with an automorphism of $\langle\, x, y \mid x^2 = y^m = 1 \,\rangle$ if necessary), that $\mathrm{tr}X = 0,\ \mathrm{tr}Y = 2\cos(\pi/m)$. Moreover (again by [12]) X and Y can be any elements of $PSL(2, \mathbb{C})$ with these traces. Suppressing $\mathrm{tr}X, \mathrm{tr}Y$ in the notation we define the *trace polynomial* of G to be $\tau(\lambda) := \mathrm{tr}w(X, Y)$.

The representation ρ induces an essential representation $G \to PSL(2, \mathbb{C})$ if and only if $\mathrm{tr}\rho(w) = 0$; that is, if and only if λ is a root of τ. Note that $\tau(\lambda) = \pm\tau(-\lambda)$ so the roots $\lambda, -\lambda$ occur with equal multiplicity.

By [12] the leading coefficient of τ is given by

$$c = \frac{1}{(\sin(\pi/m))^k} \prod_{i=1}^{k} \sin\left(\frac{\pi\alpha_i}{m}\right).$$

(This expression can also be obtained from Lemma 12 in the Appendix, where we obtain a formula for each of the coefficients of τ.) For each $1 \leq j \leq m/2$ we shall let $t_j = \sin(j\pi/m)$ and let k_j denote the number of times $\alpha_i = j$ or $\alpha_i = (m - j)$ in the word $w(x,y)$ (so that $k = k_1 + \ldots + k_{\lfloor m/2 \rfloor}$). The above formula then becomes $c = (t_1^{k_1} \ldots t_{\lfloor m/2 \rfloor}^{k_{\lfloor m/2 \rfloor}})/(\sin(\pi/m)^k)$.

Now if X, Y generate a non-elementary subgroup of $PSL(2,\mathbb{C})$ then $\rho(G)$ (and hence G) contains a non-abelian free subgroup. Thus in proving that G contains a non-abelian free subgroup we may assume that X, Y generate an elementary subgroup of $PSL(2,\mathbb{C})$. By Corollary 2.4 of [15] there are then three possibilities: (i) X, Y generate a finite subgroup of $PSL(2,\mathbb{C})$; (ii) $\mathrm{tr}[X,Y] = 2$; or (iii) $\mathrm{tr}XY = 0$.

The finite subgroups of $PSL(2,\mathbb{C})$ are the alternating groups A_4 and A_5, the symmetric group S_4, cyclic and dihedral groups (see for example [7]). Manipulation using trace identities shows that (ii) is equivalent to $\mathrm{tr}XY = \pm\sin(\pi/m)$. These values occur as roots of τ if and only if G admits an essential cyclic representation. Such a representation can be realized as $x \mapsto A, y \mapsto B$ where

$$A = \begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & e^{-i\pi/2} \end{pmatrix}, \quad B = \begin{pmatrix} e^{i\pi/m} & 0 \\ 0 & e^{-i\pi/m} \end{pmatrix}.$$

In case (iii) X and Y generate the finite dihedral group D_{2m}. We summarize the above as

Lemma 4 *Let G be as defined in the Main Theorem, with $m \geq 3$. Suppose $G \to PSL(2,\mathbb{C})$ is an essential representation given by $x \mapsto X, y \mapsto Y$, where $\mathrm{tr}X = 0$, $\mathrm{tr}Y = 2\cos(\pi/m)$. If G does not contain a non-abelian free subgroup then one of the following occurs:*

1. *X, Y generate A_4, S_4, or A_5;*
2. *$\mathrm{tr}XY = \pm 2\sin(\pi/m)$;*
3. *$\mathrm{tr}XY = 0$ and $\langle X, Y \rangle \cong D_{2m}$.*

Case (2) occurs if and only if G admits an essential cyclic representation.

Remark 5 If X, Y generate A_4 then $m = 3$ and XY has order 3, so $\mathrm{tr}XY = \pm 1$. If X, Y generate S_4 then either (a) $m = 3$ and XY has order 4, so $\mathrm{tr}XY = \pm\sqrt{2}$; or (b) $m = 4$ and XY has order 3, so $\mathrm{tr}XY = \pm 1$. If X, Y generate A_5 then either (a) $m = 3$ and XY has order 5; or (b) $m = 5$ and XY has order 3, so $\mathrm{tr}XY = \pm 1$; or (c) $m = 5$ and XY has order 5, in which case XY is conjugate to Y^2 so $\mathrm{tr}XY = \pm\mathrm{tr}Y^2 = \pm((\mathrm{tr}Y)^2 - 2)$.

3 The case $m = 4$

Lemma 6 *Let $G = \langle\, x, y \mid x^2 = y^4 = (xy^{\alpha_1} \ldots xy^{\alpha_k})^2 = 1 \,\rangle$ and let k_2 denote the number of values of i for which $\alpha_i = 2$. Then G contains a non-abelian free subgroup unless one of the following holds:*

1. *k is odd and one of the following holds:*

(a) $\sum_{i=1}^{k} \alpha_i = 0 \bmod 4$;

(b) $\sum_{i=1}^{k} \alpha_i = 2 \bmod 4$ and $k_2 = 1$;

(c) $\sum_{i=1}^{k} \alpha_i = 1, 3 \bmod 4$ and $k_2 = 0$;

2. *k is even and one of the following holds:*

(a) $\sum_{i=1}^{k} \alpha_i = 2 \bmod 4$;

(b) $\sum_{i=1}^{k} \alpha_i = 0 \bmod 4$ and either

(i). $k_2 = 0$ and $k = 2 \bmod 4$; or

(ii). $k_2 = 2$;

(c) $\sum_{i=1}^{k} \alpha_i = 1, 3 \bmod 4$ and $k_2 = 1$.

Proof
By Lemma 4 and Remark 5 we may assume that the roots of the trace polynomial τ are among $\pm\sqrt{2}, 0, \pm 1$. Thus

$$\tau(\lambda) = c\lambda^s(\lambda^2 - 1)^t(\lambda^2 - 2)^u$$

where $s + 2t + 2u = k$ and

$$c = \frac{1}{(\sin(\pi/4))^k}(\sin(\pi/4))^{k_1}(\sin(2\pi/4))^{k_2} = \sqrt{2}^{k_2},$$

where k_1, k_2 denote the number of times α_i takes the values $\pm 1, 2$ respectively. (Note that k and s are of the same parity.)

Let

$$A = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}, \quad B = \begin{pmatrix} (1+i)/\sqrt{2} & z \\ 0 & (1-i)/\sqrt{2} \end{pmatrix}$$

be elements of $PSL(2, \mathbb{C})$ so that $\mathrm{tr}A = 0$, $\mathrm{tr}B = \sqrt{2}$, $\mathrm{tr}AB = z - \sqrt{2}$. Consider the representation $\rho : \langle\, x, y \mid x^2 = y^4 = 1 \,\rangle \to PSL(2, \mathbb{C})$ given by $x \mapsto A$, $y \mapsto B$ then

$$\begin{aligned} \mathrm{tr}\rho(xy^{\alpha_1} \dots xy^{\alpha_k}) &= \tau(z - \sqrt{2}) \\ &= \pm(\sqrt{2})^{k_2}(z - \sqrt{2})^s(z^2 - 2\sqrt{2}z + 1)^t(z - 2\sqrt{2})^u z^u \end{aligned}$$

whose constant term is 0 if $u > 0$, and $\pm(\sqrt{2})^{k_2+s}$ if $u = 0$. Now the constant term in $\mathrm{tr}(AB^{\alpha_1} \dots AB^{\alpha_k})$ is given by $2\cos((2k + \sum_{i=1}^{k} \alpha_i)\pi/4) \in \{\pm 2, \pm\sqrt{2}\}$. If $u > 0$ we have that $2k + \sum_{i=1}^{k} \alpha_i = 2 \bmod 4$, and one of the conclusions 1(a) or 2(a) holds. Thus we may assume $u = 0$, and therefore $k_2 + s = 1$ or 2.

Suppose k is odd. Then s is odd. Since $2k + \sum_{i=1}^{k} \alpha_i \neq 2 \bmod 4$ we have $\sum_{i=1}^{k} \alpha_i = 1, 2$, or $3 \bmod 4$. If $\sum_{i=1}^{k} \alpha_i = 2 \bmod 4$ then k_2 is odd so $k_2 = 1$, $s = 1$ and we are in case 1(b). If $\sum_{i=1}^{k} \alpha_i = 1, 3 \bmod 4$ then k_2 is even so $k_2 = 0$, $s = 1$ and we are in case 1(c).

Suppose k is even. Then s is even. Since $2k + \sum_{i=1}^{k} \alpha_i \neq 2 \bmod 4$ we have $\sum_{i=1}^{k} \alpha_i = 0, 1$, or $3 \bmod 4$. If $\sum_{i=1}^{k} \alpha_i = 1$ or $3 \bmod 4$ then k_2 is odd so $k_2 = 1, s = 0$

and we are in case 2(c). If $\sum_{i=1}^{k} \alpha_i = 0$ mod 4 then k_2 is even so either $k_2 = 0, s = 2$ or $k_2 = 2, s = 0$. In the latter option we are in case 2(b)(ii). In the former 0 is a root of $\tau(\lambda)$ so G admits an essential dihedral representation. Thus $\sum_{i=1}^{k}(-1)^i \alpha_i = 2$ mod 4. Combining this with $\sum_{i=1}^{k} \alpha_i = 0$ mod 4 and the fact that each α_i is odd, we obtain $k = 2$ mod 4 and we are in case 2(b)(i). $\square$

4 The cases $m = 10, 15$

In this section we consider the following situation. Let G be as defined in the Main Theorem where $m = 5p$ for some prime p. We first consider the case where k is even.

Lemma 7 *Let G be as defined in the Main Theorem, where $m = 5p$ for some prime p and where k is even. Then G contains a non-abelian free subgroup.*

Proof
If $p = 2$ then G contains a non-abelian free subgroup by [16, Theorem A]. Suppose then that p is odd.

Consider a homomorphism $\theta : G \to \mathbb{Z}_{10p} \cong \mathbb{Z}_2 \times \mathbb{Z}_{5p}$ such that $\theta(x), \theta(y)$ have orders $2, 5p$ respectively. Then, up to an automorphism of $\mathbb{Z}_{10p}$ we may assume that $\theta(x) = 5p,\ \theta(y) = 2$. Then $\theta(w) = 5pk + 2\sum_{i=1}^{k} \alpha_i$, which is not of order 2, since k is even and p is odd. Hence we must have $\theta(w) = 0$, so θ is not essential.

In a similar way, consider a homomorphism $\theta : G \to \langle\, a, b \mid a^2 = b^{5p} = (ab)^2 = 1 \,\rangle \cong D_{10p}$ such that $\theta(x), \theta(y)$ have orders $2, 5p$ respectively. Then, up to an automorphism of D_{10p} we may assume that $\theta(x) = a,\ \theta(y) = b$. Then $\theta(w) = b^{\sum_{i=1}^{k}(-1)^i \alpha_i}$, which is not of order 2, since p is odd. Hence we must have $\theta(w) = 1$, so θ is not essential.

Thus G admits no essential cyclic or dihedral representation, so (since we also have $m > 5$) Lemma 4 implies that G contains a non-abelian free subgroup. $\square$

By Lemma 7 we may restrict attention to the case where k is odd. We do so throughout the remainder of this section without further comment.

Now G maps homomorphically onto the group

$$\overline{G} = \langle\, x, y \mid x^2 = y^5 = \overline{w}(x, y)^2 = 1 \,\rangle \tag{1}$$

where $\overline{w} \in \langle\, x, y \mid x^2 = y^5 = 1 \,\rangle$ is given by $\overline{w} = xy^{\beta_1} \dots xy^{\beta_k}$ where $\beta_i = \alpha_i$ mod 5 $(1 \le i \le k)$. Now $\overline{w} \ne y^\beta$ for any β, since k is odd. If $\overline{w} = x$ then $\overline{G} \cong \mathbb{Z}_2 * \mathbb{Z}_5$ and so $\overline{G}$, and hence G, contains a non-abelian free subgroup. If $\overline{w}$ is a proper power then $\overline{G}$, and hence G, contains a non-abelian free subgroup by [2].

Thus we will assume that $\overline{w}$ can be freely reduced to a word of the form $\overline{w} = xy^{\gamma_1} \dots xy^{\gamma_\ell}$ that is not a proper power, where $1 \le \gamma_i \le 4$ $(1 \le i \le \ell)$, $\ell \ge 1$. Hence the corresponding presentation (1) is a presentation of $\overline{G}$ as a generalized triangle group. We let $\tau(\lambda)$, $\sigma(\mu)$ denote the trace polynomials of G and $\overline{G}$ respectively.

Lemma 8 *If 1 is a repeated root of $\sigma(\mu)$ then G contains a non-abelian free subgroup.*

Proof

Let $q : G \to \overline{G}$ denote the canonical epimorphism. By hypothesis, there is an essential representation $\rho : \overline{G} \to PSL_2(\mathbb{C}[\mu]/(\mu-1)^2)$. Indeed, we can construct ρ explicitly via:

$$\rho(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} e^{i\pi/5} & \mu \\ 0 & e^{-i\pi/5} \end{pmatrix}.$$

Composing this with the canonical epimorphism

$$\psi : PSL_2(\mathbb{C}[\mu]/(\mu-1)^2) \to PSL_2(\mathbb{C}[\mu]/(\mu-1)) \cong PSL_2(\mathbb{C})$$

gives an essential representation $\tilde{\rho} = \psi \circ \rho : \overline{G} \to PSL_2(\mathbb{C})$ with image A_5, corresponding to the root 1 of the trace polynomial.

Let $\overline{K}$ denote the kernel of $\tilde{\rho}$, V the kernel of ψ, and K the kernel of the composite map $\tilde{\rho} \circ q : G \to PSL_2(\mathbb{C})$. Then V is a complex vector space, since its elements have the form $\pm(I + (\mu-1)A)$ for various 2×2 matrices A, with multiplication

$$[\pm(I + (\mu-1)A)][\pm(I + (\mu-1)B)] = \pm(I + (\mu-1)(A+B)).$$

Our strategy is to apply the techniques of [13] to K to obtain the existence of a non-abelian free subgroup. To this end we will first analyse the structure of $V \supset \rho(\overline{K}) = \rho(q(K))$ to obtain a large free abelian quotient K/N of K with suitable properties. We will then exhibit K as the fundamental group of a certain CW-complex X, and show that the second homology group of the covering complex of X corresponding to N has a free $\mathbb{Z}(K/N)$-submodule of large rank.

Now $\overline{K}$ is generated by conjugates of $(xy)^3$. Consider four such conjugates: $c_1 = (xy)^3$, $c_2 = x(xy)^3x$, $c_3 = yxy^3(xy)^3y^2xy^4$, and $c_4 = yxy^4(xy)^3yxy^4$. A calculation shows that $\rho(c_i) = \pm(I + (\mu-1)M_i)$ where

$$M_1 = \begin{pmatrix} -1 & z_1 \\ -\overline{z}_1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & \overline{z}_1 \\ -z_1 & -1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} z_2 & -z_3 \\ -z_3 & -z_2 \end{pmatrix}, \quad M_4 = \begin{pmatrix} \overline{z}_2 & \overline{z}_3 \\ \overline{z}_3 & -\overline{z}_2 \end{pmatrix},$$

where

$$z_1 = \frac{-(1+\sqrt{5})}{2} + i\frac{\sqrt{10-2\sqrt{5}}}{2},$$
$$z_2 = \frac{3+\sqrt{5}}{2} + i\frac{\sqrt{10-2\sqrt{5}}}{2},$$
$$z_3 = -1 + i\frac{(3+\sqrt{5})\sqrt{10-2\sqrt{5}}}{4}.$$

By considering (for example) the upper right hand entries, it is easy to verify that M_1, M_2, M_3, M_4 are linearly independent over $\mathbb{Q}$. The group A_5 acts on V via conjugation and since $\tilde{\rho}(x)$ is of order 2, the action of $\tilde{\rho}(x)$ on V is diagonalizable. Moreover, the only possible eigenvalues are ± 1. Thus V splits as a $\mathbb{Q}$-direct sum $V_+ \oplus V_-$, where $\tilde{\rho}(x)$ acts as the identity on V_+ and as the antipodal map $v \mapsto -v$ on V_-. The canonical projection $V \to V_-$ with kernel V_+ is $\tilde{\rho}(x)$-equivariant.

For $j = 3, 4$, the off-diagonal entries of M_j are equal. It follows easily that $\rho(xc_j)$ has trace 0, so is of order 2, and hence $\rho(xc_jx) = \rho(c_j^{-1})$. Note also that $xc_1x = c_2$ and $xc_2x = c_1$. Thus $\rho(c_1c_2^{-1}), \rho(c_3), \rho(c_4) \in V_-$ and $\rho(c_1c_2) \in V_+$. Let N be the pre-image of V_+ in K. Then N is normal in K and is invariant under conjugation by x. It follows that K/N is free abelian of rank at least 3 and that $\tilde{\rho}(x)$ acts on K/N as the antipodal map.

Note that K is the fundamental group of a 2-dimensional CW-complex X arising from the given presentation of G. This complex X has 60 cells of dimension 0, 120 cells of dimension 1, and $60(\frac{1}{2}+\frac{1}{5}+\frac{1}{2}) = 72$ cells of dimension 2. Here, $60/5 = 12$ of the 2-cells (call them $\alpha_1, \ldots, \alpha_{12}$, say) arise from the relator y^{5p}, $60/2 = 30$ ($\alpha_{13}, \ldots, \alpha_{42}$, say) arise from the relator x^2, and $60/2 = 30$ ($\alpha_{43}, \ldots, \alpha_{72}$, say) arise from the relator $w(x, y)^2$. Moreover, $\alpha_1, \ldots, \alpha_{12}$ are attached by maps which are pth powers. Let $\widehat{X}$ be the regular covering complex of X corresponding to the normal subgroup N of K and let $\widehat{\alpha}_i$ denote a lift of the 2-cell α_i. Then each of $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{12}$ is a 2-cell attached by a map which is a pth power.

Let GF_p denote the field with p elements. Now $H_2(\widehat{X}, GF_p)$ is a subgroup of the 2-chain group $C_2(\widehat{X}, GF_p)$ and since K/N freely permutes the cells of $\widehat{X}$, $C_2(\widehat{X}, GF_p)$ is a free $GF_p(K/N)$-module on the basis $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{72}$. Let Q be the free $GF_p(K/N)$-submodule of $C_2(\widehat{X}, GF_p)$ of rank 12 generated by $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{12}$. Since these 2-cells are attached by maps which are pth powers, their boundaries in the 1-chain group $C_1(\widehat{X}, GF_p)$ are zero. Thus Q is a subgroup of $H_2(\widehat{X}, GF_p)$.

Suppose $Q \neq H_2(\widehat{X}, GF_p)$, and let $\widehat{\beta} \in H_2(\widehat{X}, GF_p) \backslash Q$. Then $\widehat{\beta} = \sum_{i=1}^{72} \mu_i \widehat{\alpha}_i$ where $\mu_i \in GF_p(K/N)$ ($1 \leq i \leq 72$) and $\mu_q \neq 0$ for some $q > 12$. Let L be the submodule of $H_2(\widehat{X}, GF_p)$ generated by $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{12}, \widehat{\beta}$. Let $\pi_q : C_2(\widehat{X}, GF_p) \to GF_p(K/N)$ denote the projection map on the basis element $\widehat{\alpha}_q$ and suppose $\lambda, \lambda_1, \ldots, \lambda_{12} \in GF_p(K/N)$ satisfy

$$v := \lambda\widehat{\beta} + \lambda_1\widehat{\alpha}_1 + \ldots + \lambda_{12}\widehat{\alpha}_{12} = 0$$

in $C_2(\widehat{X}, GF_p)$. Then $0 = \pi_q(v) = \lambda\mu_q$, and since $GF_p(K/N)$ is an integral domain we have that $\lambda = 0$ so $\lambda_1\widehat{\alpha}_1 + \ldots + \lambda_{12}\widehat{\alpha}_{12} = 0$ in Q. But $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{12}$ form a $GF_p(K/N)$-basis for Q so $\lambda_1 = \cdots = \lambda_{12} = 0$ and hence L is free on $\{\widehat{\alpha}_1, \ldots, \widehat{\alpha}_{12}, \widehat{\beta}\}$. Thus $H_2(\widehat{X}, GF_p)$ contains a free $GF_p(K/N)$-submodule of rank $13 = 1 + \chi(X)$ so by [13, Proposition 2.1 and Theorem 2.2], $K = \pi_1(X)$ contains a non-abelian free subgroup.

Suppose then that $H_2(\widehat{X}, GF_p) = Q$. We argue as in the proof of [13, Corollary 3.2]. The element $c_1c_2 \in N$ is mapped to the element $\pm(I + (\mu - 1)(M_1 + M_2))$ of infinite order in V_+ so N^{ab} has torsion-free rank at least 1. Thus $H_1(\widehat{X}, GF_p) \cong N^{ab}/pN^{ab} \neq 0$. We also have that $H_2(\widehat{X}, GF_p)$ is a free $GF_p(K/N)$-module and K/N is a free abelian group of rank at least 3, so by [13, Theorem D] there is a subgroup J/N of K/N such that $(K/N)/(J/N) \cong K/J \cong \mathbb{Z}^2$ and $H_1(\widehat{X}, GF_p)$ contains a non-zero free $GF_p(J/N)$-submodule. Moreover, J/N is infinite so this module is of infinite GF_p-dimension.

Thus, by definition, the Bieri-Strebel invariant ([6]) Σ of the $GF_p(K/N)$-module $H_1(\widehat{X}, GF_p)$ is a proper subset of the sphere S^{d-1} (where d is the rank of the free abelian group K/N). But $\Sigma = -\Sigma$, since $\tilde{\rho}(x)$ acts as the antipodal map on K/N. Hence $\Sigma \cup -\Sigma \neq S^{d-1}$, and so N has a non-abelian free subgroup by [6, Theorem 4.1]. $\square$

Lemma 9 *If $\overline{G}$ has an essential cyclic representation then G contains a non-abelian free subgroup.*

Proof
Let $q : G \to \overline{G}$ denote the canonical epimorphism. Since $\overline{G}$ admits an essential cyclic representation, $\pm 2\sin(\pi/5)$ are roots of its trace polynomial, so there also exists an essential representation $\rho : \overline{G} \to PSL(2,\mathbb{C})$ given by $x \mapsto X,\ y \mapsto Y$, where

$$X = \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}, \quad Y = \begin{pmatrix} e^{i\pi/5} & 0 \\ 0 & e^{-i\pi/5} \end{pmatrix}.$$

Let $\psi : \rho(\overline{G}) \to PSL(2,\mathbb{C})$ be given by

$$X \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Y \mapsto Y$$

then $\tilde{\rho} := \psi \circ \rho : \overline{G} \to PSL(2,\mathbb{C})$ is an essential representation with image $\mathbb{Z}_{10}$. Let $K, \overline{K}, \overline{N}$ denote the kernels of the maps $\tilde{\rho} \circ q, \tilde{\rho}, \rho$, respectively. Then $\overline{K}$ is generated by $c_t := y^t x y^{-t} x$ $(t = 1,2,3,4)$. Now for each t

$$\rho(c_t) = \begin{pmatrix} 1 & i(e^{2\pi t i/5} + 1) \\ 0 & 1 \end{pmatrix}$$

so $\rho(c_1), \rho(c_2), \rho(c_3), \rho(c_4)$ are linearly independent over $\mathbb{Q}$ and hence $\rho(\overline{K}) \cong \mathbb{Z}^4$. Thus $\overline{G}/\overline{K} \cong \mathbb{Z}_{10}$ and $\overline{K}/\overline{N} \cong \mathbb{Z}^4$, so if N denotes the preimage of $\overline{N}$ in G then $N \lhd K \lhd G$ and $G/K \cong \mathbb{Z}_{10}$, $K/N \cong \mathbb{Z}^4$. Moreover, $x c_t x = c_t^{-1}$ for each t so $\tilde{\rho}(x)$ acts as the antipodal map on K/N.

Now K is the fundamental group of a 2-dimensional CW-complex with 10 0-cells, 20 1-cells and 12 2-cells, 2 of which correspond to the relator y^{5p}, and so are attached by pth powers. The argument given in the proof of Lemma 8 then shows that K has a non-abelian free subgroup. $\square$

For the following lemma, recall that 2ℓ is the (free product) length of $\overline{w}(x,y)$ and that $\sigma(\mu)$ denotes the trace polynomial of $\overline{G}$.

Lemma 10 *Suppose that ℓ is odd and that $\overline{G}$ admits no essential cyclic representation. If 0 is a repeated root of $\sigma(\mu)$ then $\overline{G}$ (and hence G) contains a non-abelian free subgroup.*

Proof
Let $\eta = 2\cos(\pi/5) = (1+\sqrt{5})/2$ and note that $\eta^4 - 3\eta^2 + 1 = 0$. By Lemma 4 and

Remark 5 we may assume that the roots of σ are among $\pm(\eta^2 - 2) = \pm\eta^{\pm 1}$, ± 1, $\pm 2\sin(\pi/5) = \pm\sqrt{4-\eta^2}$, 0. The leading coefficient of $\sigma(\mu)$ is given by $c = \eta^{k_2}$. Thus $\sigma(\mu)$ takes the form

$$\sigma(\mu) = \eta^{k_2}\mu^s(\mu^2 - 1)^t(\mu^2 - \eta^{-2})^u(\mu^2 - (4 - \eta^2))^v$$

where $s + 2t + 2u + 2v = \ell$. Let $A, B \in PSL(2, \mathbb{C})$ be defined as follows:

$$A = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}, \quad B = \begin{pmatrix} e^{i\pi/5} & z \\ 0 & e^{-i\pi/5} \end{pmatrix}.$$

Then $\mathrm{tr}A = 0$, $\mathrm{tr}B = \eta$, $\mathrm{tr}AB = z - \sqrt{4-\eta^2}$.

Consider the representation $\rho : \langle\ x, y \mid x^2 = y^5 = 1\ \rangle \to PSL(2, \mathbb{C})$ given by $x \mapsto A$, $y \mapsto B$, then

$$\begin{aligned} \mathrm{tr}\rho(xy^{\gamma_1}\dots xy^{\gamma_\ell}) &= \sigma(z - \sqrt{4-\eta^2}) \\ &= \eta^{k_2}(z - \sqrt{4-\eta^2})^s(z^2 - 2z\sqrt{4-\eta^2} + \eta^{-2})^t \\ &\quad \cdot (z^2 - 2z\sqrt{4-\eta^2} + 1)^u(z - 2\sqrt{4-\eta^2})^v z^v \end{aligned}$$

whose constant term is 0 if $v > 0$ and is $\eta^{k_2 - 2t}(\sqrt{4-\eta^2})^s$ if $v = 0$. Now the constant term in $\mathrm{tr}(AB^{\gamma_1}\dots AB^{\gamma_\ell})$ is $2\cos((5\ell + 2\sum_{i=1}^{\ell}\gamma_i)\pi/10)$. Since ℓ is odd and $\overline{G}$ admits no essential cyclic representation, this constant term is either $\pm 2\cos(\pi/10) = \pm\eta\sqrt{4-\eta^2}$ or $\pm 2\cos(3\pi/10) = \pm\sqrt{4-\eta^2}$. Thus we can conclude that $v = 0$, that

$$\eta^{k_2 - 2t}(\sqrt{4-\eta^2})^s = \eta\sqrt{4-\eta^2} \quad \text{or} \quad \sqrt{4-\eta^2},$$

and therefore that $s = 1$ and $t = k_2/2$ or $t = (k_2 - 1)/2$. Hence 0 is not a repeated root of $\sigma(\mu)$, contrary to hypothesis. □

For the proof of Theorem 2 we shall require the following proposition.

Proposition 11 *Let $p \neq q$ be prime numbers, and let $1 \leq t \leq pq - 1$. Then*

$$\prod_{\psi \in \mathrm{Aut}(\mathbb{Z}_{pq})} 2\sin\left(\frac{\psi(t)\pi}{pq}\right) = \begin{cases} q^{p-1} & \text{if } p|t \\ p^{q-1} & \text{if } q|t \\ 1 & \text{otherwise} \end{cases}$$

Proof

By identity 1.392(1) of [11] we have that for all real numbers x and $n \geq 2$

$$\sin(x) \prod_{1 \leq r < n} 2\sin(x + r\pi/n) = \sin(nx).$$

Differentiating and substituting $x = 0$ we obtain

$$\prod_{1 \leq r < n} 2\sin\left(\frac{r\pi}{n}\right) = n. \tag{2}$$

We now claim that the identity

$$\prod_{\substack{1\le r<n\\(r,n)=1}} 2\sin\left(\frac{r\pi}{n}\right) = \begin{cases} u & \text{if } n \text{ is a power of a prime } u\\ 1 & \text{otherwise}\end{cases} \tag{3}$$

holds for all $n \geq 2$. This clearly holds when $n = 2$. Let $N \geq 3$ and suppose inductively that it holds for all $n < N$. Now

$$\prod_{1\le r<N} 2\sin\left(\frac{r\pi}{N}\right) = \prod_{\substack{1\le r<N\\(r,N)=1}} 2\sin\left(\frac{r\pi}{N}\right)\cdot \prod_{\substack{d|N\\d>1}}\prod_{\substack{1\le r<N\\(r,N)=d}} 2\sin\left(\frac{r\pi}{N}\right). \tag{4}$$

Now

$$\prod_{\substack{d|N\\d>1}}\prod_{\substack{1\le r<N\\(r,N)=d}} 2\sin\left(\frac{r\pi}{N}\right) = \prod_{\substack{d|N\\d>1}}\prod_{\substack{1\le s<N/d\\(s,N/d)=1}} 2\sin\left(\frac{s\pi}{N/d}\right). \tag{5}$$

Applying the inductive hypothesis, the right hand side of (5) is equal to the product of all primes u such that N/d is a power of u, where $d > 1$ ranges over all divisors of N. Thus

$$\prod_{\substack{d|N\\d>1}}\prod_{\substack{1\le r<N\\(r,N)=d}} 2\sin\left(\frac{r\pi}{N}\right) = \begin{cases} u^{\alpha-1} & \text{if } N=u^{\alpha}, \text{ where } \alpha\ge 1 \text{ and } u \text{ is prime}\\ N & \text{otherwise}\end{cases}$$

Substituting this into (4) and applying (2) to the left hand side we get that the identity (3) holds for $n = N$ and hence for all $n \geq 2$. Finally,

$$\prod_{\psi\in \mathrm{Aut}(\mathbb{Z}_{pq})} 2\sin\left(\frac{\psi(t)\pi}{pq}\right) = \prod_{\substack{1\le\alpha<pq\\(\alpha,pq)=1}} 2\sin\left(\frac{\alpha t\pi}{pq}\right)$$
$$= \begin{cases} \prod_{\substack{1\le\alpha<pq\\(\alpha,pq)=1}} 2\sin(\alpha\pi/q) = \left(\prod_{\substack{1\le\alpha<q\\(\alpha,q)=1}} 2\sin(\alpha\pi/q)\right)^{p-1} & \text{if } p|t\\ \prod_{\substack{1\le\alpha<pq\\(\alpha,pq)=1}} 2\sin(\alpha\pi/p) = \left(\prod_{\substack{1\le\alpha<p\\(\alpha,p)=1}} 2\sin(\alpha\pi/p)\right)^{q-1} & \text{if } q|t\\ \prod_{\substack{1\le\alpha<pq\\(\alpha,pq)=1}} 2\sin(\alpha\pi/pq) & \text{otherwise}\end{cases}$$

and an application of (3) completes the proof. □

Proof of Theorem 2

We will consider the homomorphic image $\overline{G}$ of G defined by the presentation (1). As explained at the start of this section we will assume that $\overline{w}(x, y)$ is not a proper power and can be freely reduced to the form $\overline{w}(x, y) = xy^{\gamma_1} \ldots xy^{\gamma_\ell}$ where $1 \leq \gamma_i \leq 4$ $(1 \leq i \leq \ell - 1)$, $\ell \geq 1$.

By [13, Theorem E] we may assume that G admits no essential cyclic representation, and since $m > 5$ Lemma 4 implies that the trace polynomial for G has the form $\tau(\lambda) = c\lambda^k$, where

$$c = \frac{1}{(\sin(\pi/5p))^k}\prod_{i=1}^{k}\sin\left(\frac{\pi\alpha_i}{5p}\right).$$

Let $X, Y \in PSL(2,\mathbb{C})$ be elements of orders $2, 5p$ that generate a cyclic subgroup of $PSL(2,\mathbb{C})$. We may assume that

$$X = \begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & e^{-i\pi/2} \end{pmatrix}, \quad Y = \begin{pmatrix} e^{i\pi/5p} & 0 \\ 0 & e^{-i\pi/5p} \end{pmatrix}$$

so that $\mathrm{tr}XY = 2\sin(\pi/5p)$. Let $\rho : \langle\, x, y \mid x^2 = y^{5p} = 1 \,\rangle \to PSL(2,\mathbb{C})$ be given by $x \mapsto X,\ y \mapsto Y$. Then $\mathrm{tr}\rho(w) = \mathrm{tr}(X^k Y^a) = \pm 2\sin(a\pi/5p)$, where $a = \sum_{i=1}^{k} \alpha_i$. On the other hand $\mathrm{tr}\rho(w) = \tau(2\sin(\pi/5p)) = \prod_{i=1}^{k} 2\sin(\alpha_i\pi/5p)$. Thus

$$2\sin(a\pi/5p) = \pm \prod_{i=1}^{k} 2\sin(\alpha_i\pi/5p)$$

and hence

$$\prod_{\psi\in \mathrm{Aut}(\mathbb{Z}_{5p})} 2\sin(\psi(a)\pi/5p) = \pm \prod_{i=1}^{k} \prod_{\psi\in \mathrm{Aut}(\mathbb{Z}_{5p})} 2\sin(\psi(\alpha_i)\pi/5p). \qquad (6)$$

Suppose $5|\alpha_i$ for some $1 \le i \le k$. Then by Proposition 11 p^4 divides the right hand side of (6). If $5|a$ then $\overline{G}$ admits an essential cyclic representation and so $\overline{G}$ (and hence G) contains a non-abelian free subgroup, by Lemma 9. Thus we may assume $5 \nmid a$. Proposition 11 then implies that the left hand side of (6) is either equal to 1 or 5^{p-1} and we have a contradiction. Thus $5 \nmid \alpha_i$ for any $1 \le i \le k$ so the (free product) length of $w(x,y)$ is equal to the (free product) length of $\overline{w}(x,y)$. Hence $\ell = k$, and thus the trace polynomial $\sigma(\mu)$ of $\overline{G}$ is of degree $k \ge 3$.

As explained in the proof of Lemma 10 we may assume that $\sigma(\mu)$ is of the form $\sigma(\mu) = c'\mu^s(\mu^2-1)^t(\mu^2-\eta^{-2})^u$ where $\eta = 2\cos(\pi/5)$ and s is odd. By Lemma 10 we may assume $s = 1$, and by Lemma 8 we may assume $t \le 1$. The automorphism θ of $\mathbb{Z}_5$ generated by the map $1 \mapsto 2$ yields the alternative presentation $\overline{G} = \langle\, x, y \mid x^2 = y^5 = (xy^{\theta(\beta_1)} \dots xy^{\theta(\beta_k)})^2 = 1 \,\rangle$. The potential roots ± 1 and $\pm\eta^{-1}$ for σ correspond to essential representations $\overline{G} \to A_5$ that map xy to elements of order 3 or 5 respectively (cf. Remark 5). The automorphism θ has the effect of interchanging these two possibilities. Thus the trace polynomial corresponding to this new presentation has the form $\sigma'(\mu) = c''\mu^s(\mu^2-\eta^{-2})^t(\mu^2-1)^u$, for some c''. By another application of Lemma 8 we may assume $u \le 1$. Since $k = s + 2t + 2u > 1$ we are reduced to the cases $k = 3, 5$.

If $k = 3$ then G contains a non-abelian free subgroup by [14, Theorem 1]. If $k = 5$ then $s = t = 1$ so $\sigma(\mu) = c'\mu(\mu^2-1)(\mu^2-\eta^{-2})$. A computer search reveals that the only words $w(x,y)$ (up to cyclic permutation, inversion, and automorphisms of $\langle\, y \mid y^5 = 1 \,\rangle$) with trace polynomial of that form are $xyxy^3xy^2xy^4xy^t$ with $t \in \{1,2\}$. In each case, a GAP [10] calculation shows that $\overline{G}$ has a subgroup of index 11 admitting the free group of rank 2 as a homomorphic image, and hence G contains a non-abelian free subgroup. □

5 The cases $m = 12, 20, 30, 60$

Proof of Theorem 3
We shall consider alternative presentations for G:

$$G = \langle\, x, y \mid x^2 = y^m = (xy^{\psi(\alpha_1)} \dots xy^{\psi(\alpha_k)})^2 = 1 \,\rangle$$

where ψ is an automorphism of $\mathbb{Z}_m$. By [14, Theorem 5] we may assume that k is odd. By [13, Theorem E] we may assume that G admits no essential cyclic representation. Since $m > 5$, Lemma 4 implies that the trace polynomial for G takes the form $\tau(\lambda) = c\lambda^k$ where $c = (t_1^{k_1} \dots t_{m/2}^{k_{m/2}})/(\sin(\pi/m))^k$. Let $X, Y \in PSL(2, \mathbb{C})$ have orders 2 and m respectively that generate a cyclic group of order m. We may assume $\mathrm{tr}(XY) = 2\sin(\pi/m)$. Fix ρ to be the representation $\rho : \langle\, x, y \mid x^2 = y^m = 1 \,\rangle \to PSL(2, \mathbb{C})$ given by $x \mapsto X,\ y \mapsto Y$. Then

$$\mathrm{tr}\rho(xy^{\psi(\alpha_1)} \dots xy^{\psi(\alpha_k)}) = \pm 2\cos(q\pi/m) \quad \text{for some} \quad 1 \leq q < m/2. \tag{7}$$

(Note that if $q = m/2$ then ρ induces an essential cyclic representation of G, contrary to our earlier assumption.) In particular,

$$-1 \leq \prod_{\psi \in A} \frac{\mathrm{tr}\rho(xy^{\psi(\alpha_1)} \dots xy^{\psi(\alpha_k)})}{2} \leq 1 \tag{8}$$

for any group A of automorphisms of $\mathbb{Z}_m$.

Now

$$\begin{aligned} \mathrm{tr}\rho(xy^{\psi(\alpha_1)} \dots xy^{\psi(\alpha_k)}) &= \tau(2\sin(\pi/m)) \\ &= 2^k \prod_{i=1}^{k} \sin\left(\frac{\pi\psi(\alpha_i)}{m}\right) \end{aligned}$$

so

$$\frac{\mathrm{tr}\rho(xy^{\psi(\alpha_1)} \dots xy^{\psi(\alpha_k)})}{2} = 2^{k-1} \cdot t_1^{k_{\psi(1)}} \cdots t_{m/2}^{k_{\psi(m/2)}}. \tag{9}$$

We now consider each value of m separately.

The case $m = 12$.
Let ψ be the automorphism of $\mathbb{Z}_{12}$ generated by the map $1 \mapsto 5$ and let $A = \langle \psi \rangle$. Then using (8) and (9) we obtain

$$2^{2(k-1)}(t_1 t_5)^{k_1+k_5} \cdot (t_2)^{2k_2} \cdot (t_3)^{2k_3} \cdot (t_4)^{2k_4} \cdot (t_6)^{2k_6} \leq 1$$

which (using (3)) simplifies to

$$2^{k_3+2k_6-2} \cdot 3^{k_4} \leq 1.$$

We shall consider the following homomorphic images of G:

$$\begin{aligned} H &= \langle\, x, y \mid x^2 = y^6 = (xy^{\beta_1} \ldots xy^{\beta_k})^2 = 1 \,\rangle, \\ L &= \langle\, x, y \mid x^2 = y^4 = (xy^{\gamma_1} \ldots xy^{\gamma_k})^2 = 1 \,\rangle, \end{aligned}$$

where $\beta_i = \alpha_i \bmod 6$ and $\gamma_i = \alpha_i \bmod 4$ for each $1 \le i \le k$. Suppose $k_6 = 0$. Then each β_i is non-zero. If $k > 3$ then by Theorem 1 H, and hence G, contains a non-abelian free subgroup. If $k = 3$ then by [14, Theorem 1] G contains a non-abelian free subgroup. Thus we may assume $k_6 \ge 1$ and hence $k_6 = 1, k_3 = k_4 = 0$. Moreover we may assume

$$\mathrm{tr}\rho(xy^{\alpha_1} \ldots xy^{\alpha_k}) = \pm 2 \tag{10}$$

for otherwise one of $\rho(xy^{\alpha_1} \ldots xy^{\alpha_k})$ or $\rho(xy^{\psi(\alpha_1)} \ldots xy^{\psi(\alpha_k)})$ provides a contradiction to (7). Using (9) equation (10) simplifies to

$$\begin{aligned} 2 &= 2^{k_1+k_2+k_5+1} \cdot t_1^{k_1} t_2^{k_2} t_5^{k_5} t_6^1 \\ &= 2\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^{k_1-k_5} \end{aligned}$$

so $k_1 = k_5$. Since the image of ρ is isomorphic to $\mathbb{Z}_{12}$ and by equation (10) $\rho(w)$ is the zero of this group we have that $6k + \sum_{i=1}^{k} \alpha_i = 0 \bmod 12$, and k is odd so

$$\sum_{i=1}^{k} \alpha_i = 6 \bmod 12, \tag{11}$$

which implies $\sum_{i=1}^{k} \gamma_i = 2 \bmod 4$. By Lemma 6 L (and hence G) contains a non-abelian free subgroup unless precisely one $\gamma_i = 2$. This implies that $k_2 + k_6 = 1$, but $k_6 = 1$ so $k_2 = 0$.

Let $\overline{w}(x, y) = xy^{\beta_1} \ldots xy^{\beta_k}$. Using the relations $x^2 = 1, y^6 = 1$ of H we can cyclically reduce $\overline{w}(x, y)$ to x (in which case $H \cong \mathbb{Z}_2 * \mathbb{Z}_6$, so G contains a non-abelian free subgroup) or to the form $\overline{w}(x, y) = xy^{\delta_1} \ldots xy^{\delta_\ell}$ where ℓ is odd and $1 \le \delta_i \le 5$ for each $1 \le i \le \ell$. If $\ell > 3$ then by Theorem 1 H, and hence G, contains a non-abelian free subgroup. Thus we may assume $\ell = 1$ or 3. The words $w, \overline{w}$ then take the following forms:

$$\begin{aligned} &\ell = 1: \quad w = xy^{\xi_1} xy^{\xi_2} u(x, y) xy^6 v(x, y) && \overline{w} = xy^{\xi_1+\xi_2}, \\ &\ell = 3: \quad w = xy^{\xi_1} xy^{\xi_2} xy^{\xi_3} xy^{\xi_4} u(x, y) xy^6 v(x, y) && \overline{w} = xy^{\xi_1+\xi_4} xy^{\xi_2} xy^{\xi_3}, \end{aligned}$$

where $\xi_1, \xi_2, \xi_3, \xi_4 \in \{1, 5\}$ and

$$\begin{aligned} u(x, y) &= xy^{a_1} \ldots xy^{a_n}, \\ v(x, y) &= xy^{b_n} \ldots xy^{b_1}, \end{aligned}$$

with $a_i + b_i = 0 \bmod 6$ for each $1 \le i \le n$.

In the case $\ell = 1$ equation (11) implies $\sum_{i=1}^{k} \alpha_i = 0 \bmod 6$ so

$$\xi_1 + \xi_2 + (a_1 + \cdots + a_n) + 6 + (b_n + \cdots + b_1) = 0 \bmod 6$$

which implies $\xi_1 + \xi_2 = 0 \bmod 6$ contradicting our assumption that the exponents of y in $\overline{w}$ are non-zero. In the case $\ell = 3$, since $\xi_1 + \xi_2 + \xi_3 + \xi_4$ is even, Theorem 1 of [14] implies that H, and hence G, contains a non-abelian free subgroup.

The case $m = 20$.
We shall consider the following homomorphic image of G:

$$H = \langle\ x, y \mid x^2 = y^{10} = (xy^{\beta_1} \ldots xy^{\beta_k})^2 = 1\ \rangle$$

where $\beta_i = \alpha_i \bmod 10$ for each $1 \leq i \leq k$.

Let ψ be the automorphism of $\mathbb{Z}_{20}$ generated by the map $1 \mapsto 3$ and let $A = \langle \psi \rangle$. Then using (8) and (9) we obtain

$$2^{4(k-1)}(t_1t_3t_7t_9)^{k_1+k_3+k_7+k_9}(t_2t_6)^{2(k_2+k_6)}(t_4t_8)^{2(k_4+k_8)}t_5^{4k_5}t_{10}^{4k_{10}} \leq 1$$

which (using (3)) simplifies to

$$2^{2k_5+4k_{10}-4} \cdot 5^{k_4+k_8} \leq 1.$$

If $k_{10} = 0$ then each β_i is non-zero so H contains a non-abelian free subgroup by Theorem 2. Thus we may assume that $k_{10} \geq 1$ and hence $k_{10} = 1$, $k_5 = k_4 = k_8 = 0$. Moreover we may assume

$$\mathrm{tr}\rho(xy^{\alpha_1} \ldots xy^{\alpha_k}) = \pm 2 \qquad (12)$$

for otherwise for some $\phi \in A$ the element $\rho(xy^{\phi(\alpha_1)} \ldots xy^{\phi(\alpha_k)})$ provides a contradiction to (7). The image of ρ is isomorphic to $\mathbb{Z}_{20}$ and by equation (12) $\rho(w)$ is the zero of this group so we have that $\sum_{i=1}^{k} \alpha_i = 10 \bmod 20$ (since k is odd). Thus $\sum_{i=1}^{k} \beta_i = 0 \bmod 10$ so H admits an essential cyclic representation, and the result follows from [13, Theorem E].

The case $m = 30$.
We shall consider the following homomorphic images of G:

$$\begin{aligned} H &= \langle\ x, y \mid x^2 = y^{10} = (xy^{\beta_1} \ldots xy^{\beta_k})^2 = 1\ \rangle, \\ L &= \langle\ x, y \mid x^2 = y^{15} = (xy^{\gamma_1} \ldots xy^{\gamma_k})^2 = 1\ \rangle, \end{aligned}$$

where $\beta_i = \alpha_i \bmod 10$, $\gamma_i = \alpha_i \bmod 15$ for each $1 \leq i \leq k$.

Let ψ be the automorphism of $\mathbb{Z}_{30}$ generated by the map $1 \mapsto 7$ and let $A = \langle \psi \rangle$. Then using (8) and (9) we obtain

$$\begin{aligned} 2^{4(k-1)}(t_1t_7t_{11}t_{13})^{k_1+k_7+k_{11}+k_{13}}(t_2t_{14}t_8t_4)^{k_2+k_{14}+k_8+k_4} \\ \cdot(t_3t_9)^{2(k_3+k_9)}(t_5)^{4k_5}(t_6t_{12})^{2(k_6+k_{12})}t_{10}^{4k_{10}}t_{15}^{4k_{15}} \\ \leq 1 \end{aligned}$$

which (using (3)) simplifies to

$$2^{4k_{15}-4} \cdot 5^{k_6+k_{12}} \cdot 9^{k_{10}} \leq 1.$$

If $k_{15} = 0$ then each γ_i is non-zero which implies that L, and hence G, contains a non-abelian free subgroup by Theorem 2. If $k_{15} > 0$ then $k_{10} = 0$, so H, and hence G, contains a non-abelian free subgroup by Theorem 2.

The case $m = 60$.
We shall consider the following homomorphic images of G:

$$\begin{aligned} H &= \langle\, x, y \mid x^2 = y^{20} = (xy^{\beta_1} \ldots xy^{\beta_k})^2 = 1 \,\rangle, \\ L &= \langle\, x, y \mid x^2 = y^{30} = (xy^{\gamma_1} \ldots xy^{\gamma_k})^2 = 1 \,\rangle, \end{aligned}$$

where $\beta_i = \alpha_i$ mod 20, $\gamma_i = \alpha_i$ mod 30 for each $1 \leq i \leq k$.

Consider the group $A \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ of automorphisms of $\mathbb{Z}_{60}$ generated by $\psi : 1 \mapsto 7$ and $\phi : 1 \mapsto 29$. Using (8) and (9) we obtain

$$\begin{aligned} 1 \geq\ & 2^{8(k-1)} \\ & \cdot (t_1 t_7 t_{11} t_{13} t_{17} t_{19} t_{23} t_{29})^{k_1+k_7+k_{11}+k_{13}+k_{17}+k_{19}+k_{23}+k_{29}} \\ & \cdot (t_2 t_{14} t_{22} t_{26})^{2(k_2+k_{14}+k_{22}+k_{26})} \cdot (t_3 t_{21} t_{27} t_9)^{2(k_3+k_{21}+k_{27}+k_9)} \\ & \cdot (t_4 t_{28} t_{16} t_8)^{2(k_4+k_{28}+k_{16}+k_8)} \cdot (t_5 t_{25})^{4(k_5+k_{25})} \cdot (t_6 t_{18})^{4(k_6+k_{18})} \cdot (t_{12} t_{24})^{4(k_{12}+k_{24})} \\ & \cdot (t_{10})^{8k_{10}} \cdot (t_{15})^{8k_{15}} \cdot (t_{20})^{8k_{20}} \cdot (t_{30})^{8k_{30}} \end{aligned}$$

which (using (3)) simplifies to

$$1 \geq 2^{4k_{15}+8k_{30}-8} \cdot 5^{2(k_{12}+k_{24})} \cdot 3^{4k_{20}}$$

In particular one of k_{20}, k_{30} is zero so either all β_i's are non-zero or all γ_i's are non-zero. Hence, by the above, one of H or L (and hence G) contains a non-abelian free subgroup. $\square$

A Appendix: The case $m = 6$

This appendix gives a sketch proof of Theorem 1. We begin by giving a complete calculation of *all* the coefficients of the trace polynomial.

Let $\mathcal{A}(k)$ denote the set of subsets $S \subset \{1, \ldots, k\}$ such that $s_1 - s_2 \not\equiv 1 \pmod{k}$ for $s_1, s_2 \in S$. The maximum cardinality of $S \in \mathcal{A}(k)$ is the integer part $\lfloor k/2 \rfloor$ of $k/2$. For $0 \leq j \leq \lfloor k/2 \rfloor$, let $\mathcal{A}(k, j)$ denote the set of sets $S \in \mathcal{A}(k)$ of cardinality j.

Lemma 12 *Let $X, Y \in SL(2, \mathbb{C})$ be matrices with* $\mathrm{tr}(X) = 0$, $\mathrm{tr}(Y) = 2\cos(\pi/m)$, $\mathrm{tr}(XY) = \lambda$, *for some integer $m \geq 2$. Let $W = XY^{\alpha_1} \ldots XY^{\alpha_k}$, where $1 \leq \alpha_i < m$ for each $1 \leq i \leq k$. Then the trace of W is given by the polynomial*

$$\mathrm{tr}(W) = c \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j B_j \lambda^{k-2j},$$

where

$$c = \prod_{j=1}^{k} \frac{\sin(\alpha_j \pi/m)}{\sin(\pi/m)},$$

$$B_j = \sum_{\{t_1,\ldots,t_j\} \in \mathcal{A}(k,j)} \left(\prod_{s=1}^{j} b(t_s) \right),$$

$$b(j) = \frac{\sin^2(\pi/m) e^{i\pi(\alpha_{j+1}-\alpha_j)/m}}{\sin(\alpha_j \pi/m)\sin(\alpha_{j+1}\pi/m)}.$$

Proof
By [12] the trace of $W(X,Y)$ is determined by the traces of X, Y and XY, so it is sufficient to work with fixed matrices with the given traces. We define

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} e^{i\pi/m} & \lambda \\ 0 & e^{-i\pi/m} \end{pmatrix},$$

Then, for $1 \leq \alpha \leq m-1$,

$$XY^{\alpha} = \begin{pmatrix} 0 & -e^{-i\alpha\pi/m} \\ e^{i\alpha\pi/m} & p(\alpha)\lambda \end{pmatrix}$$

with $p(\alpha) = \sin(\alpha\pi/m)/\sin(\pi/m)$. Now each entry in $W(X,Y)$ is a sum of terms, each of which is a product of an entry from each of XY^{α_j} $(1 \leq j \leq k)$. The leading monomial of $\mathrm{tr}(W(X,Y))$ necessarily consists of the product of the lower right entries of the XY^{α_j}, so is $c\lambda^k = \prod_{j=1}^{k} p(\alpha_j)\lambda^k$, as claimed. Each term contributing to the λ^{k-2j} monomial can be obtained from c by replacing each of j (non-overlapping) pairs of (cyclically) consecutive lower right entries by the upper right entry of the first member of the pair, followed by the lower left entry of the second member. Such a term is thus equal to $cb(s_1)\cdots b(s_j)$ for some $\{s_1,\ldots,s_j\} \in \mathcal{A}(k,j)$, and the result follows. $\square$

Sketch proof of Theorem 1
Let

$$G = \langle\ x, y \mid x^2 = y^6 = w(x,y)^2 = 1\ \rangle,$$
$$\overline{G} = \langle\ x, y \mid x^2 = y^3 = \overline{w}(x,y)^2 = 1\ \rangle,$$

where $w(x,y) = xy^{\alpha_1}\ldots xy^{\alpha_k}$, $\overline{w}(x,y) = xy^{\beta_1}\ldots xy^{\beta_k}$ where for $1 \leq i \leq k$, $\beta_i = \alpha_i \bmod 3$, and $k > 3$. Let $\tau(\lambda), \sigma(\mu)$ denote the trace polynomials of $G, \overline{G}$ respectively. By Lemma 4 if G contains no non-abelian free subgroup then the roots of τ are among 0, corresponding to an essential representation onto the dihedral group D_{12}, or ± 1, which occur if and only if G admits an essential cyclic representation.

Suppose first that G admits an essential cyclic representation, with kernel K. Then ± 1 are roots of $\tau(\lambda)$. By [13, Theorem 4.8] if 1 or -1 is a repeated root of

$\tau(\lambda)$ then G has a non-abelian free subgroup. Thus we may assume that $\tau(\lambda) = c\lambda^{k-2}(\lambda^2-1)$ and in particular that G has an essential representation ρ onto D_{12}. Now K has a deficiency 0 presentation, its abelianization K/K' is free abelian of rank 3, and conjugation by x induces the antipodal automorphism on K/K'. Moreover, a calculation shows that $\rho(K')$ is a non-trivial abelian subgroup of D_{12}, so K'/K'' is non-trivial. By [13, Corollary 3.2], K' (and hence G) contains a non-abelian free subgroup.

Hence we may assume that G has no essential cyclic representations, and thus $\tau(\lambda) = c\lambda^k$. Then as in the proof of Theorem 3 equations (8), (9) yield $(k_2, k_3) = (0,0),(1,0),(0,1)$ and thus $c = 1, \sqrt{3}, 2$, respectively. When k is even the existence of an essential dihedral representation implies that the alternating sum $\sum_{i=1}^{k}(-1)^i\alpha_i$ is congruent to 3 modulo 6 and thus $k_2 = 1$, $c = \sqrt{3}$.

We proceed by calculating the coefficients in $\tau(\lambda), \sigma(\mu)$ and split the proof into three cases, depending on the value of c. Consider first the form of $\sigma(\mu)$ in the cases $c = 1, \sqrt{3}$. By Lemma 4 and Remark 5 we may assume that the roots of σ are among $\pm1, \pm\sqrt{2}, (\pm1\pm\sqrt{5})/2, \pm\sqrt{3}, 0$. If ±1 or $\pm\sqrt{3}$ occurs as a root of σ then $\overline{G}$ admits an essential representation to A_4 or $\mathbb{Z}_6$. In either case $\sum_{i=1}^{k}\beta_i = 0 \bmod 3$, and we can define a representation $\rho : G \to \mathbb{Z}_6$ by $\rho(x) = 3 \bmod 6$ and $\rho(y) = 1 \bmod 6$. By assumption, ρ is not essential, so $\rho(w) = 0 \bmod 6$ and $c = \tau(1) = \pm2$, a contradiction. Since σ has rational coefficients we thus have

$$\sigma(\mu) = \mu^r(\mu^2-2)^s(\mu^4-3\mu^2+1)^t \tag{13}$$

where $r, s, t \geq 0$ satisfy $r + 2s + 4t = k$. Since $\sigma(\sqrt{3}) \in \{\pm1, \pm\sqrt{3}, \pm2\}$ we have $r = 0, 1$. If k is even then $r = 0$, and (since $\sum_{i=1}^{k}(-1)^i\alpha_i$ is congruent to 0 modulo 3) we also have $\sigma(0) = \pm2$ so $s = 1$.

Case 1: $c = 1$.
In this case k is odd and $\alpha_i \in \{1, 5\}$ for each $1 \leq i \leq k$. By Lemma 12, the coefficient $-B_1$ of λ^{k-2} in $\tau(\lambda)$ is given by $B_1 = \sum_{i=1}^{k} b(i)$, where for each $1 \leq i \leq k$

$$b(i) := \begin{cases} 1 & \text{if } \alpha_i = \alpha_{i+1} \\ \frac{-1+\sqrt{-3}}{2} & \text{if } \alpha_i = 1,\ \alpha_{i+1} = 5 \\ \frac{-1-\sqrt{-3}}{2} & \text{if } \alpha_i = 5,\ \alpha_{i+1} = 1 \end{cases}$$

(where α_{k+1} is defined equal to α_1). A similar analysis for $\sigma(\mu)$ shows that the coefficient $-B_1'$ of μ^{k-2} is given by $B_1' = \sum_{i=1}^{k} b'(i)$ where

$$b'(i) := \begin{cases} 1 & \text{if } \beta_i = \beta_{i+1} \\ \frac{1+\sqrt{-3}}{2} & \text{if } \beta_i = 1,\ \beta_{i+1} = 2 \\ \frac{1-\sqrt{-3}}{2} & \text{if } \beta_i = 2,\ \beta_{i+1} = 1 \end{cases}$$

Since the coefficient of λ^{k-2} in $\tau(\lambda)$ is zero, we have that k is a multiple of 3 – say $k = 3\ell$ where $\ell > 1$ – and each possible value of $b(i)$ occurs precisely ℓ times. It follows that $B_1' = 2\ell$. On the other hand we can compute the coefficient of μ^{k-2}

in $\sigma(\mu) = \mu(\mu^2 - 2)^s(\mu^4 - 3\mu^2 + 1)^t$ as $-2s - 3t$. We thus obtain the simultaneous diophantine equations

$$1 + 2s + 4t = 3\ell, \quad 2s + 3t = 2\ell, \quad s, t, \geq 0, \ell > 1$$

with the unique solution $s = 0, t = 2, \ell = 3$, and so $k = 9$.

Now consider the coefficient B_2 of λ^5 in $\tau(\lambda)$ and the coefficient B_2' of μ^5 in $\sigma(\mu)$. Using Lemma 12 we can deduce

$$2B_2 = B_1^2 - \sum_{i=1}^{9} b(i)^2 - 2\sum_{i=1}^{9} b(i)b(i+1)$$

where $b(10)$ is defined equal to $b(1)$. Since $B_1 = B_2 = 0$ and the $b(i)$'s are equally distributed amongst the three possible values it follows that $\sum_{i=1}^{9} b(i)b(i+1) = 0$.

A similar analysis shows that $\sum_{i=1}^{9} b'(i)^2 = 0$, $\sum_{i=1}^{9} b'(i)b'(i+1) = 6$, from which we can deduce $B_2' = 12$. But the coefficient of μ^5 in $\sigma(\mu) = \mu(\mu^4 - 3\mu^2 + 1)^2$ is 11. This contradiction completes Case 1.

Case 2: $c = \sqrt{3}$**.**
Then $\alpha_i \in \{1, 5\}$ for all but one value of i, for which $\alpha_i \in \{2, 4\}$. Without loss of generality we may assume that $\alpha_k = 2$ and $\alpha_i \in \{1, 5\}$ for $1 \leq i < k$. As in Case 1, consideration of the coefficient of λ^{k-2} in $\tau(\lambda)$ and of μ^{k-2} in $\sigma(\mu)$ yield diophantine equations in s, t, k. We find that the only solutions with $k > 3$ are (i) $s = 2$, $t = 0$, $k = 5$; (ii) $s = 0$, $t = 2$, $k = 9$; (iii) $s = 1$, $t = 2$, $k = 11$; (iv) $s = 0$, $t = 4$, $k = 17$; (v) $s = 0$, $t = 2$, $k = 8$. We can rule out solution (v) since k is even and $s \neq 1$.

For the remaining solutions, consideration of the coefficient of λ^{k-4} in $\tau(\lambda)$ and the coefficient of μ^{k-4} in $\sigma(\mu)$ yield additional diophantine equations which reduce us to solution (i). A computer search reveals that the only word $w(x, y)$ (up to cyclic permutation, inversion, and automorphisms of $\langle\, y \mid y^6 = 1 \,\rangle$) such that $\tau(\lambda), \sigma(\mu)$ are of the required form is $w(x, y) = xy^5xyxyxy^5xy^2$. A calculation in GAP [10] shows that in this case G has a subgroup of index 6 admitting a free homomorphic image of rank 2.

Case 3: $c = 2$**.**
In this case k is odd, the α_i are all odd, and $\alpha_i = 3$ for precisely one value of i. Without loss of generality we may assume that $\alpha_k = 3$ and $\alpha_i \in \{1, 5\}$ for $1 \leq i < k$. Again, the coefficient $-B_1$ of λ^{k-2} is given by $B_1 = \sum_{i=1}^{k} b(i)$ where $b(i)$ is as in Case 1 for $i < k - 1$,

$$b(k-1) := \begin{cases} \frac{1+\sqrt{-3}}{4} & \text{if } \alpha_{k-1} = 1 \\ \frac{1-\sqrt{-3}}{4} & \text{if } \alpha_{k-1} = 5 \end{cases}$$

and

$$b(k) := \begin{cases} \frac{1-\sqrt{-3}}{4} & \text{if } \alpha_1 = 1 \\ \frac{1+\sqrt{-3}}{4} & \text{if } \alpha_1 = 5 \end{cases}$$

Note that $b(1), \dots, b(k-2)$ are algebraic integers. From the equation $B_1 = 0$ it follows that $b(k-1) + b(k)$ is also an algebraic integer, and this can only happen if $\alpha_1 + \alpha_{k-1} = 6$. Assume inductively that $\alpha_t + \alpha_{k-t} = 6$ (and hence $b(k-t) = b(t-1)$, where $b(0)$ is defined equal to $b(k)$) for $1 \leq t < u$, for some $u \leq (k-1)/2$. Then from the equation $B_u = 0$ it turns out that $b(k-u) + b(u-1)$ is an algebraic integer, and this can only happen if $\alpha_u + \alpha_{k-u} = 6$.

Thus $\alpha_t + \alpha_{k-t} = 6$ for all $1 \leq t \leq (k-1)/2$, so the third relator of G has the form $(U(x,y)xU(x,y)^{-1}y^3)^2$ for some word U. In passing to $\overline{G}$, we kill y^3, so the relator collapses to x^2, and $\overline{G} \cong \mathbb{Z}_2 * \mathbb{Z}_3$. Hence $\overline{G}$, and so also G, contains a non-abelian free subgroup, as claimed. □

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