Arithmeticity of orbifold generalised triangle groups[1]

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Abstract

Maclachlan and Martin have proved that only finitely many arithmetic Kleinian groups can be generated by 2 elliptic elements, and have classified these groups in the non-cocompact case.

Here we investigate the cocompact case, restricting to a class of generalised triangle groups considered by Jones and Reid which arise as the fundamental groups of hyperbolic 3-dimensional orbifolds. We obtain 21 arithmetic groups and provide a description of the corresponding orbifolds.

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1 Introduction

A *generalised triangle group* is a group with a presentation

$$GT(l,m,n;w) = \langle\ a,b \mid a^l = b^m = w^n = 1\ \rangle,$$

where $l,m,n \geq 2$ and w is a cyclically reduced word in the free product on $\{a,b\}$. We call two words $w, w' \in \langle\ a,b \mid a^l = b^m = 1\ \rangle$ *equivalent* if one can be transformed to the other by a sequence of the following moves:

1. cyclic permutation,
2. inversion,
3. automorphism of $\mathbb{Z}_l$ or $\mathbb{Z}_m$,
4. interchanging the two free factors (if $l = m$).

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If w and w' are equivalent we shall write $w \sim w'$. We shall call two generalised triangle group presentations $GT(l,m,n;w)$ and $GT(l',m',n';w')$ *equivalent* if $l=l'$, $m=m'$, $n=n'$, and $w \sim w'$.

In [15] Maclachlan and Martin initiated a programme to investigate the 2-generator arithmetic Kleinian groups. There are precisely four such groups generated by a pair of parabolic elements [5], precisely 12 when one of the generators is elliptic and the other parabolic [3], and only finitely many generated by 2 elliptic (i.e. finite order) elements [15]. A consequence of this last result is that only finitely many generalised triangle groups can have faithful representations as arithmetic Kleinian groups.

The non-cocompact arithmetic Kleinian groups generated by 2 elliptic elements were classified in [14], where it was shown that there are 21 non-conjugate groups, 15 of which are generalised triangle groups. Moreover the generalised triangle groups arise as the fundamental groups of 3-dimensional orbifolds whose singular sets in S^3 all admit a simple, uniform description.

In this paper we address the question as to which generalised triangle groups admit faithful representations as cocompact arithmetic Kleinian groups. The first problem encountered when tackling this question is in determining which generalised triangle groups have faithful representations as Kleinian groups (i.e. as discrete subgroups of $PSL(2,\mathbb{C})$). We simplify this issue by restricting to a class of generalised triangle groups considered by Jones and Reid [13].

They show that any group $GT(l,m,n;W(r,s,a,b))$ where $r<s$ are positive coprime integers and

$$W(r,s,a,b)=\prod_{i=0}^{s-1} a^{(-1)^{[(2i+1)r/s]}} b^{(-1)^{[(2i+2)r/s]}},$$

arises as the fundamental group of a 3-dimensional orbifold Q whose singular set in S^3 is a graph (with 3 edges and 2 vertices) formed by adding an extra edge to a 2-bridge knot or link L, and labelling the edges l,m,n to denote cone angles of $2\pi/l, 2\pi/m, 2\pi/n$ respectively. Moreover they show that the fundamental group of every such orbifold is of the form $GT(l,m,n;W(r,s,a,b))$ for some pair (r,s). We write $Q=Q_{r,s}(l,m,n)$.

By showing that Q satisfies the hypotheses of Thurston's orbifold conjecture [12, 21], Jones and Reid show that (subject to this conjecture) Q has a geometric structure, unless L is a link of two unknotted, unlinked components. Further, by referring to Dunbar's classification of non-hyperbolic orbifolds [4], they show that in most cases Q is hyperbolic. Indeed this is true for all cases we shall consider. Thus $G=GT(l,m,n;W(r,s,a,b))$ admits a faithful representation as a Kleinian group.

In [22] it was shown that if $G=GT(l,m,n;W(r,s,a,b))$ admits a faithful representation as a Kleinian group then the representation is cocompact if and only if

$$c_1(G)>0 \text{ and } c_2(G)>0,$$

where

$$c_1(G)=\begin{cases} 1/l+1/m+1/n-1 & \text{if } s \text{ is odd} \\ 2/l+1/n-1 & \text{if } s \text{ is even} \end{cases}$$

$$c_2(G)=\begin{cases} 1/l+1/m+1/n-1 & \text{if } s \text{ is odd} \\ 2/m+1/n-1 & \text{if } s \text{ is even} \end{cases}$$

Table 1
Arithmetic Kleinian $GT(l,m,n;W(r,s,a,b))$

(l,m,n)	(r,s)	$m_\gamma(z)$	γ	Covolume
$(3,3,2)$	$(1,3)$	z^3+4z^2+6z+2	$-1.7718+1.1151i$	0.2646
$(2,3,4)$	$(1,3)$	z^3+5z^2+7z+1	$-2.4196+0.6063i$	0.1323
$(2,3,2)$	$(1,6)$	z^3+4z^2+4z+2	$-0.5804+0.6063i$	0.1323
$(3,3,2)$	$(1,4)$	z^3+2z^2+4z+2	$-0.6806+1.6332i$	0.6616
$(2,3,2)$	$(1,8)$	z^3+3z^2+z+1	$-0.1154+0.5897i$	0.3308
$(2,3,2)$	$(3,8)$	$z^3+6z^2+10z+2$	$-2.8846+0.5897i$	0.3308
$(3,3,2)$	$(1,5)$	$z^4+2z^3+2z^2+6z+1$	$0.1993+1.5895i$	?
$(2,3,4)$	$(2,5)$	$z^4+8z^3+21z^2+18z+1$	$-3.1385+0.4851i$	?
$(2,3,2)$	$(1,10)$	$z^4+4z^3+3z^2+1$	$0.1385+0.4851i$	?
$(3,3,2)$	$(2,5)$	$z^5+8z^4+24z^3+30z^2+14z+2$	$-2.8061+1.1564i$	?
$(2,3,4)$	$(1,5)$	$z^5+7z^4+18z^3+22z^2+13z+1$	$-0.8970+0.9590i$	?
$(2,3,2)$	$(3,10)$	$z^5+8z^4+24z^3+32z^2+16z+2$	$-2.1030+0.9590i$	?
$(3,3,2)$	$(1,6)$	$z^5+2z^4+4z^2+8z+2$	$0.8424+1.3553i$	?
$(2,3,2)$	$(1,12)$	$z^5+6z^4+10z^3+2z^2-2z+2$	$0.2984+0.3768i$	?
$(2,3,2)$	$(5,12)$	$z^5+9z^4+28z^3+34z^2+13z+1$	$-3.2984+0.3768i$	?
$(3,3,2)$	$(3,7)$	$z^6+10z^5+38z^4+66z^3+51z^2+14z+1$	$-3.1969+0.9018i$	?
$(2,3,4)$	$(2,7)$	$z^6+10z^5+40z^4+80z^3+79z^2+30z+1$	$-1.9247+1.0617i$	?
$(2,3,2)$	$(3,14)$	$z^6+8z^5+25z^4+40z^3+34z^2+12z+1$	$-1.0753+1.0617i$	?
$(3,3,2)$	$(2,11)$	$z^5+2z^4+2z^3+8z^2+10z+2$	$0.6809+1.7333i$	?
$(2,3,4)$	$(1,11)$	$z^5+5z^4+6z^3+z+1$	$0.2799+0.4869i$	?
$(2,3,2)$	$(9,22)$	$z^5+10z^4+36z^3+54z^2+28z+2$	$-3.2799+0.4869i$	?

If s is odd then there are infinitely many triples (l,m,n) for which these conditions are satisfied. If we restrict to cases where s is even then $(l,m,n)=(2,3,2)$ or $(3,3,2)$, and the problem is more manageable.

Whilst we do not have the machinery to investigate the problem for all groups $G=GT(l,m,n;W(r,s,a,b))$ with these triples, we can certainly make some progress when s is small. Our methods will necessarily include some groups $GT(l,m,n;W(r,s,a,b))$ where s is odd, but since these groups are integral to our calculations we shall include them in our results. Our main result is the following

Theorem 1.1 *Suppose* $\Gamma = GT(l,m,n;W(r,s,a,b))$ *where* $\gcd(r,s)=1$ *admits a faithful discrete representation in* $PSL(2,\mathbb{C})$*, and assume one of the following holds:*

1. $(l,m,n)=(3,3,2)$, $s\leq 15$,
2. $(l,m,n)=(2,3,2)$, $s\leq 30$ *and is even,*
3. $(l,m,n)=(2,3,4)$, $s\leq 15$.

Then (up to equivalence) Γ *admits a faithful representation as an arithmetic Kleinian group if and only if* $(l,m,n),(r,s)$ *are contained in Table 1.*

An approximation to the parameter $\gamma=\mathrm{tr}[\rho(a),\rho(b)]-2$*, its minimum polynomial over* $\mathbb{Q}$ *and the covolume (where available) are as given in Table 1. The corresponding orbifold* $Q_{r,s}(l,m,n)$ *is as given in figure 1.*

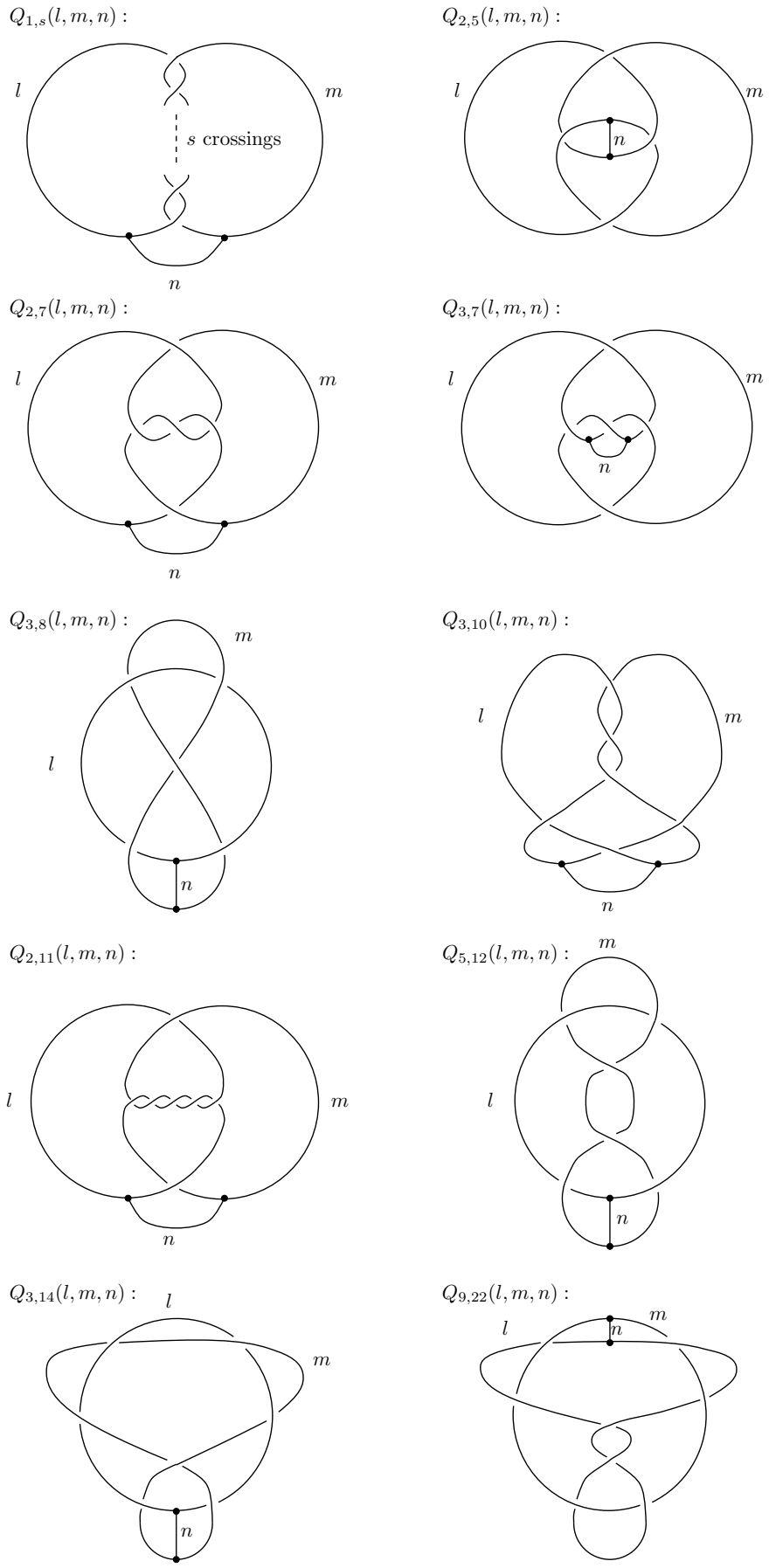


Figure 1: Table of orbifolds.

2 Arithmetic Kleinian groups

A Kleinian group Γ is a discrete subgroup of $PSL(2,\mathbb{C})$, the group of all orientation preserving isometries of H^3. The quotient H^3/Γ is a hyperbolic orbifold, or a hyperbolic manifold if Γ is torsion free. For isometries f, g the complex numbers

$$\beta(f) = (\mathrm{tr}(f))^2 - 4,\ \beta(g) = (\mathrm{tr}(g))^2 - 4,\ \gamma(f,g) = \mathrm{tr}[f,g] - 2 \tag{1}$$

where $[f,g] = fgf^{-1}g^{-1}$ are called the *parameters* of $\langle\ f, g\ \rangle$ and we write

$$par(\Gamma) = (\gamma(f,g), \beta(f), \beta(g)). \tag{2}$$

These parameters are independent of the choice of matrix representatives of f, g in $SL(2,\mathbb{C})$ and they determine Γ uniquely up to conjugacy whenever $\gamma(f,g) \neq 0$ [7]. If f, g are elliptic elements of orders l, m repectively then $\beta(f) = -4\sin^2(\pi/l), \beta(g) = -4\sin^2(\pi/m)$. Thus if $G = \langle\ f, g\ \rangle$ then

$$par(G) = (\gamma, -4\sin^2(\pi/l), -4\sin^2(\pi/m)).$$

For fixed l, m the space of all such discrete groups is determined by the single parameter $\gamma(f,g)$. Note that when $l = 2$ the subgroup $\langle\ g, fgf\ \rangle$ is of index 2 in $\langle\ f, g\ \rangle$ and has parameters

$$(\gamma(f,g)(\gamma(f,g) - \beta(g)), \beta(g), \beta(g)).$$

Conversely, each group generated by a pair of elements g_1, g_2 of the same order can be extended by elements of order 2 which conjugate g_1 to $g_2^{\pm 1}$ [8].

We now give a definition of arithmetic Kleinian groups in terms of quaternion algebras; for further details on this see [2]. Let k be a number field with one complex place, A be a quaternion algebra over k ramified at all real places, and let ρ be an embedding of A into $M(2,\mathbb{C})$, the 2×2 matrix algebra over $\mathbb{C}$. If $\mathcal{O}$ is an order in A, and $\mathcal{O}^1$ denotes the elements of norm 1 in $\mathcal{O}$ then $\rho(\mathcal{O}^1)$ is a discrete group of finite covolume in $SL(2,\mathbb{C})$. The set of *arithmetic Kleinian groups* consists of all subgroups of $PSL(2,\mathbb{C})$ which are commensurable with some such $\pi\rho(\mathcal{O}^1)$, where $\pi : SL(2,\mathbb{C}) \to PSL(2,\mathbb{C})$ is the natural map.

If Γ is a finite covolume Kleinian group then we can determine whether or not Γ is arithmetic by the following method [16]. Let $\Gamma^{(2)} = \{g^2 \mid g \in \Gamma\}$ and define the field $k\Gamma$ by

$$k\Gamma = \mathbb{Q}(\mathrm{tr}(\Gamma^{(2)})), \tag{3}$$

where $\mathrm{tr}(\Gamma) = \{\pm\mathrm{tr}(g) \mid g \in \Gamma\}$, and define the quaternion algebra $A\Gamma$ over $k\Gamma$ by

$$A\Gamma = \{\textstyle\sum a_i\gamma_i | a_i \in k\Gamma, \gamma_i \in \Gamma^{(2)}\}. \tag{4}$$

The field $k\Gamma$ and the quaternion algebra $A\Gamma$ are invariants of the commensurability class of the group Γ [18, 17]. In the cases where Γ is generated by two elements $\overline{A} = \pi(A), \overline{B} = \pi(B)$ (not of order 2) with $\alpha = \mathrm{tr}A, \beta = \mathrm{tr}B, \lambda = \mathrm{tr}(AB)$ then

$$k\Gamma = \mathbb{Q}(\alpha^2, \beta^2, \alpha\beta\lambda), \tag{5}$$

and $A\Gamma$ is given by the Hilbert symbol

$$A\Gamma = \left(\frac{\alpha^2(\alpha^2-4), \alpha^2\beta^2(\alpha^2+\beta^2+\lambda^2-\alpha\beta\lambda-4)}{k\Gamma} \right) \tag{6}$$

see [16, 19]. Among the finite covolume Kleinian groups we can identify which are arithmetic by the following theorem.

Theorem 2.1 (Maclachlan and Reid [16]). *Let Γ be a finite covolume Kleinian group. Then Γ is arithmetic if and only if*

1. *$k\Gamma$ has exactly one complex place, and*
2. *$\mathrm{tr}(\Gamma^{(2)})$ consists of algebraic integers, and*
3. *the quaternion algebra $A\Gamma$ is ramified at all real places.*

3 The case $GT(3,3,2;W(r,s,a,b))$

We shall see in Section 4 that the groups

$$\begin{array}{ll} L_1 = GT(2,3,2;W(r,2s,x,y)) & \text{where } \gcd(r,2s)=1, \\ L_2 = GT(2,3,4;W(r,s,x,y)) & \text{where } s \text{ is odd and } \gcd(r,s)=1 \end{array}$$

are $\mathbb{Z}_2$-extensions of $GT(3,3,2;W(r,s,xyx,y))$ or $GT(3,3,2;W(2r,s,xyx,y)$ respectively. Since arithmeticity and discreteness are preserved by taking $\mathbb{Z}_2$-extensions, in classifying the arithmetic groups $G = GT(3,3,2;W(r,s,a,b))$ with s up to some given value, we are doing the same for the groups L_1 and L_2.

Let $\rho : G \to PSL(2,\mathbb{C})$ be a representation satisfying $\mathrm{tr}\rho(a) = \mathrm{tr}\rho(b) = 1$, then $\mathrm{tr}\rho(w)$ is a polynomial with integer coefficients in $\lambda = \mathrm{tr}\rho(ab)$ [1]. We call this the *trace polynomial* and write $\mathrm{tr}\rho(w) = \tau_w(\lambda)$.

We will re-express theorem 2.1 for the groups $GT(3,3,2;W(r,s,a,b))$ in terms of irreducible factors of τ_w. For this we will need some preliminary results.

Proposition 3.1 *For every root μ of $m_\lambda(z)$, there is an embedding $\sigma : \mathbb{Q}(\lambda) \to \mathbb{C}$ given by $\sigma(\lambda) = \mu$. Conversely if $\sigma : \mathbb{Q}(\lambda) \to \mathbb{C}$ is an embedding then $\sigma(\lambda)$ is a root of $m_\lambda(z)$.*

Lemma 3.2 (Gehring et al. [6]). *Let ν be a real place corresponding to a real embedding $\sigma : k \to \mathbb{R}$ and let $A = \left(\frac{p,q}{k}\right)$. Then A is ramified at the real place ν if and only if $\sigma(p)$ and $\sigma(q)$ are both negative.*

Theorem 3.3 *Let $G = GT(3,3,2;W(r,s,a,b))$ and suppose $\rho : G \to PSL(2,\mathbb{C})$ is a faithful discrete representation of G as a Kleinian group with $\mathrm{tr}\rho(a) = \mathrm{tr}\rho(b) = 1$. Let $\lambda = \mathrm{tr}\rho(ab)$, and let $m_\lambda(z)$ be the minimum polynomial of λ over $\mathbb{Q}$. Then $\Gamma = \rho(G)$ is an arithmetic Kleinian group if and only if*

1. *$m_\lambda(z)$ has one pair of complex conjugate roots, and*
2. *the real roots of $m_\lambda(z)$ lie in the interval $(-1,2)$.*

Proof. By (5) and (6) the invariant trace field and quaternion algebras are given by

$$k\Gamma = \mathbb{Q}(\lambda),\ \ A\Gamma = \left(\frac{-3, \lambda^2 - \lambda - 2}{\mathbb{Q}(\lambda)}\right)$$

Since Γ is of finite covolume it is an arithmetic Kleinian group if and only if conditions (1)–(3) of Theorem 2.1 are true.

Condition (2) holds because $\Gamma^{(2)} \subseteq \Gamma$, and $\mathrm{tr}(\Gamma)$ consists of algebraic integers by [1]. By Proposition 3.1, condition (1) of Theorem 2.1 is equivalent to condition (1) above. By Lemma 3.2, condition (3) of Theorem 2.1 holds if and only if for every real place corresponding to a real embedding $\sigma : k \to \mathbb{R}$,

$$\sigma(-3) < 0,\ \ \sigma(\lambda^2 - \lambda - 2) < 0.$$

The first part is clearly true, and by Proposition 3.1 the second part is equivalent to saying that for every real root x of $m_\lambda(z)$,

$$x^2 - x - 2 < 0$$

and this is the same as our condition (2) above. □

Under the hypothesis that G admits a faithful representation as a Kleinian group, we will use Theorem 3.3 to classify (up to equivalence of G) the pairs (r, s) with $s \leq 15$ for which G admits a faithful discrete representation as an arithmetic Kleinian group.

If ρ is any faithful representation of G, then $\mathrm{tr}\rho(ab)$ is a root of the trace polynomial τ_w [1]. Thus if there is to be a faithful representation of G as an arithmetic Kleinian group then for some root λ of τ_w the minimum polynomial $m_\lambda(z)$ of λ over $\mathbb{Q}$ must satisfy conditions (1) and (2) of Theorem 3.3. Now the minimum polynomial of any root of τ_w is an irreducible factor of τ_w, so this is equivalent to saying that for some irreducible factor $m(z)$ of $\tau_w(z)$, conditions (1) and (2) must hold.

Using a Maple program (available from the author's homepage) to check these conditions for the irreducible factors of trace polynomials of groups $G = GT(3, 3, 2; W(r, s, a, b))$ where $s \leq 15$ we rule out all but the following 7 pairs (r, s):

$$(1, 3),\ (1, 4),\ (1, 5),\ (2, 5),\ (1, 6),\ (3, 7),\ (2, 11).$$

We must now determine whether the groups $GT(3, 3, 2; W(r, s, a, b))$ with the above pairs (r, s) do in fact have faithful representations as arithmetic Kleinian groups. We note the following

Proposition 3.4 *Let $G = GT(3, 3, 2; W(r, s, a, b))$ and suppose $\rho : G \to PSL(2, \mathbb{C})$ is a faithful discrete representation with $\mathrm{tr}\rho(a) = \mathrm{tr}\rho(b) = 1$, $\mathrm{tr}\rho(ab) = \lambda$. If the real roots of $m_\lambda(z)$ lie in the interval $(-1, 2)$ then $\lambda \notin \mathbb{R}$.*

Proof. Suppose for contradiction that $\lambda \in \mathbb{R}$. By [11, Appendix 3] the axes $\rho(a), \rho(b)$ intersect if and only if the matrix

$$T = \begin{pmatrix} 2 & \mathrm{tr}\rho(a) & \mathrm{tr}\rho(b) \\ \mathrm{tr}\rho(a) & 2 & \mathrm{tr}\rho(ab) \\ \mathrm{tr}\rho(b) & \mathrm{tr}\rho(ab) & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & \lambda \\ 1 & \lambda & 2 \end{pmatrix}$$

is positive definite. The eigenvalues of T are given by

$$2-\lambda,\ \frac{\lambda+4\pm\sqrt{\lambda^2+8}}{2}$$

and since $\lambda \in (-1,2)$, these are all positive. Thus $\rho(a), \rho(b)$ intersect at a point $P \in H^3$ (say) and the set

$$\{g \in \rho(G) \mid P \cap gP \neq \emptyset\} = \rho(G)$$

which is infinite, so $\rho(G)$ does not act properly discontinuously. But by [20, Exercise 3.5.10], $\rho(G)$ acts properly discontinuously on H^3 and we have a contradiction. □

If $(r,s) = (1,3),\ (1,4),\ (1,5),\ (2,5),\ (1,6)$, or $(3,7)$ then the Maple program shows that the trace polynomial factors as

$$\tau_w(z) = p(z)q(z)$$

where $p(z) = 1, z$, or $(z-1)$, and $q(z)$ is irreducible over $\mathbb{Q}$ with one pair of complex roots, and with all real roots in the interval $(-1,2)$.

If $\rho : G \to PSL(2,\mathbb{C})$ is a faithful discrete representation with $\mathrm{tr}\rho(a) = \mathrm{tr}\rho(b) = 1$ then by Proposition 3.4 $\mathrm{tr}\rho(ab) \notin \mathbb{R}$ and so is one of the two complex (conjugate) roots of $q(z)$, and hence $\rho(G)$ is an arithmetic Kleinian group.

In the case $GT(3,3,2;W(2,11,a,b))$ then the Maple program shows that the trace polynomial $\tau_w(z) = \mathrm{tr}\rho(w)$ factors as

$$\tau_w(z) = p(z)q(z)$$

where

$$p(z) = z^5 - 4z^3 - 2z^2 + 4z + 2$$

is irreducible with two complex roots,

$$\lambda, \overline{\lambda} = -1.2798796 \pm 0.4869155i,$$

and

$$q(z) = z^6 - 2z^5 - 2z^4 + 4z^3 + z^2 - 2z + 1$$

is irreducible with 4 complex roots,

$$\begin{aligned} \mu_1, \overline{\mu_1} &= -1.0709625 \pm 0.2505454i, \\ \mu_2, \overline{\mu_2} &= 0.3770064 \pm 0.3894352i. \end{aligned}$$

The following proposition shows that if $\Gamma = \rho(G)$ is a faithful representation of G in $PSL(2,\mathbb{C})$ then $\mathrm{tr}\rho(ab) = \lambda$. Since $p(z)$ has only one pair of complex roots, and all real roots of $p(z)$ lie in the interval $(-1,2)$ we can deduce that Γ is an arithmetic Kleinian group.

Proposition 3.5 *Let $G = GT(3,3,2;W(2,11,a,b))$ and suppose $\rho : G \to PSL(2,\mathbb{C})$ is a faithful representation with $\mathrm{tr}\rho(a) = \mathrm{tr}\rho(b) = 1$. If $\mathrm{tr}\rho(ab) = \mu_1$ or μ_2 then $\rho(G)$ is not discrete.*

Proof. We will refer to results of Gehring et al. [6], obtained using the disc covering method of Gehring and Martin [8, 9], concerning the discreteness of 2-generator subgroups of $PSL(2,\mathbb{C})$. More precisely, the result we will use states that if g is an element of order 3, and $\langle\, f, g\,\rangle$ is a discrete 2-generator subgroup of $PSL(2,\mathbb{C})$ and if $\gamma(f,g)$ lies within a certain subset of $\mathbb{C}$, then $\gamma(f,g)$ must take one of only a handful of exceptional values.

Suppose $\rho(a) = A,\ \rho(b) = B$, then $\Gamma = \rho(G)$ has parameters

$$(\gamma(A,B), -3, -3)$$

where, by the Fricke identity,

$$\begin{aligned}\gamma(A,B) &= (\mathrm{tr}A)^2 + (\mathrm{tr}B)^2 + (\mathrm{tr}AB)^2 - (\mathrm{tr}A)(\mathrm{tr}B)(\mathrm{tr}AB) - 4\\ &= (\mathrm{tr}AB)^2 - (\mathrm{tr}AB) - 2.\end{aligned}$$

Thus

$$\gamma(A,B) = 0.1551500 \pm 0.7871949i \qquad \text{if } \mathrm{tr}AB = \mu_1, \overline{\mu_1}$$

and

$$\gamma(A,B) = -2.3865324 \pm 0.0957961i \qquad \text{if } \mathrm{tr}AB = \mu_2, \overline{\mu_2}.$$

Then by [6] if $\mathrm{tr}AB = \mu_2, \overline{\mu_2}$, Γ is not discrete.

To deal with the case when $\mathrm{tr}AB = \mu_1$ or its conjugate, we must consider $\mathbb{Z}_2$-extensions of Γ. Suppose Λ is a $\mathbb{Z}_2$-extension of Γ by the order 2 element X. Then Λ has parameters

$$(\gamma(X,Y), \beta(X), \beta(Y)) = (\gamma(X,Y), -4, -3)$$

where $\gamma(X,Y)$ satisfies

$$\gamma(X,Y)^2 - \beta(Y)\gamma(X,Y) - \gamma(A,B) = 0$$

(see [15]), i.e.

$$\begin{aligned}\gamma(X,Y) &= \frac{-3 \pm \sqrt{9 + 4\gamma(A,B)}}{2}\\ &= 0.0709625 \pm 0.2505454i \text{ or } -3.0709625 \mp 0.2505454i.\end{aligned}$$

By [6] Λ cannot be discrete, and hence nor can Γ. □

Hence in each case G has a faithful discrete representation in $PSL(2,\mathbb{C})$ satisfying the conditions of Theorem 3.3, and we have proved:

Lemma 3.6 *Suppose $G = GT(3,3,2;W(r,s,a,b))$ where $\gcd(r,s) = 1$, $s \le 15$ admits a faithful discrete representation in $PSL(2,\mathbb{C})$. Then (up to equivalence) G has a faithful representation as an arithmetic Kleinian group if and only if*

$$(r,s) = (1,3),\ (1,4),\ (1,5),\ (2,5),\ (1,6),\ (3,7) \text{ or } (2,11). \tag{7}$$

An interesting problem would be to classify all groups $GT(3,3,2;W(r,s,a,b))$ which have faithful representations as arithmetic Kleinian groups. The trace polynomial is, on the whole, very hard to predict, so it is difficult to make progress in the general case. However, if we restrict the problem to pairs $(r,s) = (1,s)$ then

$$W(1,s,a,b) \sim (ab)^{s/2}(a^{-1}b^{-1})^{s/2} \qquad \text{if } s \text{ is even,}$$
$$W(1,s,a,b) \sim (ab)^{s/2}ab^{-1}(a^{-1}b^{-1})^{s/2} \qquad \text{if } s \text{ is odd.}$$

The regularity of these words suggests that the trace polynomial may be of a predictable nature. Calculations using a Maple program (available from the author's homepage) indicate that this is the case, and we make the following conjecture which we have verified for $s \leq 70$.

Conjecture 3.7 *Let $G = GT(3,3,2;W(1,s,a,b))$, then the trace polynomial $\tau_w(z)$ is as follows:*

- *Suppose s is even*
  1. *If $s/2 = 0 \mod 6$ then $\tau_w(z) = q(z)$ where $q(z)$ is irreducible with $s/2$ complex roots.*
  2. *If $s/2 = 1,5 \mod 6$ then $\tau_w(z) = z(z-1)q(z)$ where $q(z)$ is irreducible with $s/2-1$ complex roots.*
  3. *If $s/2 = 2,4 \mod 6$ then $\tau_w(z) = (z-1)q(z)$ where $q(z)$ is irreducible with $s/2$ complex roots.*
  4. *If $s/2 = 3 \mod 6$ then $\tau_w(z) = zq(z)$ where $q(z)$ is irreducible with $s/2-1$ complex roots.*
- *Suppose s is odd*
  1. *If $(s-1)/2 = 0,2 \mod 3$ then $\tau_w(z) = (z-1)q(z)$ where $q(z)$ is irreducible with $(s+(-1)^{(s+1)/2})/2$ complex roots.*
  2. *If $(s-1)/2 = 1 \mod 3$ then $\tau_w(z) = q(z)$ where $q(z)$ is irreducible with $(s+(-1)^{(s+1)/2})/2$ complex roots.*

From this it would follow that $GT(3,3,2;W(1,s,a,b))$ is an arithmetic Kleinian group if and only if $3 \leq s \leq 6$.

Note that we could obtain the same result by proving only that $\tau_w(z) = p(z)q(z)$ where $p(z) = 1, z, (z-1)$ or $z(z-1)$ and that $q(z)$ is irreducible. For then, by [15] if G is an arithmetic Kleinian group then $deg(q) \leq 44$, and by our previous calculations we must have that $3 \leq s \leq 6$.

4 $\mathbb{Z}_2$-extensions

In this section we obtain the remaining groups in Table 1. We first show that if $\gcd(r,s) = 1$ then the group $GT(2,3,2;W(r,2s,x,y))$ is a $\mathbb{Z}_2$-extension of $GT(3,3,2;$

$W(r,s,xyx,y)$). To see this, consider the orbifold corresponding to the first group, $Q = Q_{r,2s}(2,3,2)$.

As described in [13], Q can be realised as a genus-two handlebody (with fundamental group generated by x and y) with singular 2-handles (singularities of order $2,3,2$, respectively) attached along closed curves in the boundary, representing x, y, and $W(r,2s,x,y)$.

The 2-fold branched covering $\overline{Q}$ of Q, whose fundamental group is the index 2 subgroup of H generated by x^2, xyx, y, corresponds to a genus-three handlebody with singular 2-handles (singularities of orders $3,3,2$ respectively) attached to the lifts of $a = xyx, b = y$ and $W(r,2s,x,y)$, and a non-singular 2-handle attached to the lift of x^2.

A routine calculation shows $W(r,2s,x,y) = W(r,s,xyx,y)$ so the group $\pi_1^{orb}(\overline{Q}) = GT(3,3,2;W(r,s,a,b))$ is an index 2 subgroup of $GT(2,3,2;W(r,2s,x,y))$, as required. Applying a similar argument to the group $GT(2,3,4;W(r,s,x,y))$ and noting that $W(r,s,x,y)^2 = W(2r,s,xyx,y)$ shows that the group $GT(3,3,2;W(2r,s,a,b))$ is an index 2 subgroup of $GT(2,3,4;W(r,s,x,y))$.

Thus $GT(2,3,2;W(r,2s,x,y))$ where $\gcd(r,s) = 1$ and $s \leq 15$ is an arithmetic Kleinian group if and only if $GT(3,3,2;W(r,s,a,b))$ is equivalent to one of the groups in Lemma 3.6. Similarly $GT(2,3,4;W(r,s,x,y))$ where $\gcd(r,s) = 1$ and $s \leq 15$ is odd is an arithmetic Kleinian group if and only if $GT(3,3,2;W(2r,s,a,b))$ is equivalent to a group in Lemma 3.6.

For each of the pairs (r,s) at (7) it is easy to show that $W(r,s,a,b) \sim W(s-r,s,a,b)$. Using the equivalent presentations $GT(3,3,2;W(r,s,a,b))$ and $GT(3,3,2;W(s-r,s,a,b))$ of G in turn, we can obtain the two $\mathbb{Z}_2$-extensions of G and thus complete Table 1.

5 γ-values and covolumes

Using (1) and the fact that for matrices $A, B \in SL(2,\mathbb{C})$

$$\mathrm{tr}[A,B] = (\mathrm{tr}A)^2 + (\mathrm{tr}B)^2 + (\mathrm{tr}AB)^2 - (\mathrm{tr}A)(\mathrm{tr}B)(\mathrm{tr}AB) - 2,$$

the minimum polynomial $m_\gamma(z)$ over $\mathbb{Q}$ of the parameter $\gamma = \gamma(a,b)$ of any of our groups $GT(l,m,n;W(r,s,a,b))$ in Table 1 (and hence an approximation to γ) can be calculated directly from the minimum polynomial $m_\lambda(z)$ of λ.

Jones and Reid have developed a computer program to study explicitly how the geometry and topology of certain arithmetic Kleinian groups varies as the number theoretic data is changed. Amongst other things, this program calculates the covolumes of these groups.

In [6] arithmetic Kleinian groups $G_{n,i}$, generated by a pair of elements of orders 2 and n where $3 \leq n \leq 7$ are considered. By applying the program of Jones and Reid to these groups their covolumes were obtained. Since the parameters of a Kleinian group determine the group up to conjugacy, comparing the minimum polynomial $m_\gamma(z)$ of our arithmetic Kleinian groups $GT(2,3,q;W(\alpha,\beta,a,b))$ with the groups $G_{3,i}$ allows us to make the following identifications:

$$GT(2,3,2;W(1,6,a,b)) \cong G_{3,5},$$

and

$$GT(2,3,2;W(1,8,a,b)) \cong G_{3,14}.$$

Thus we can provide the covolumes of these groups, of their index 2 subgroups, and of the second $\mathbb{Z}_2$-extension of these subgroups.

6 The orbifolds

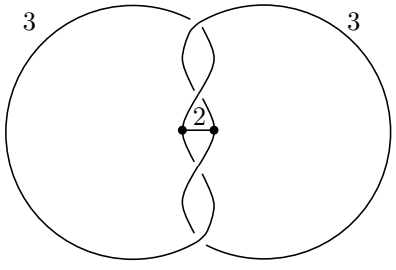
It remains to prove that the orbifolds in figure 1 have the required fundamental groups. Using the Wirtinger algorithm, the fundamental groups of the orbifolds $Q_{1,2s}(l,m,n)$ and $Q_{2,5}(l,m,n)$ were calculated in [13, 10] respectively. The remaining fundamental groups can be calculated in a similar manner. We leave the detailed calculations as an exercise.

We will now show how we arrived at these orbifolds and describe the symmetries some of them exhibit. In finding the orbifolds corresponding to the arithmetic Kleinian groups $GT(3,3,2;W(r,s,a,b))$ obtained in Section 3, we need to consider the pairs (r,s) at (7).

The orbifolds $Q_{1,s}(l,m,n)$ were obtained in [14] and the orbifold $Q_{2,5}(l,m,n)$ was obtained in [10]. To find the orbifolds $Q_{3,7}(l,m,n)$, $Q_{2,11}(l,m,n)$ we start with the 2-bridge knots B_3^7, B_2^{11} and then experiment with the positioning of the unknotting tunnels so as to produce graphs Gr which, when labelled l,m,n to indicate branching indices, describe orbifolds with the required fundamental groups.

We can obtain the orbifolds corresponding to the $\mathbb{Z}_2$-extensions of the groups $GT(3,3,2;W(r,s,a,b))$ by considering the symmetries of the orbifolds $Q_{r,s}(3,3,2)$ we have just described. The singular sets of each of the orbifolds $Q_{r,s}(3,3,2)$ can each be redrawn so as to admit two distinct automorphisms X,Y (say) which interchange the edges labelled 3, and which can be realised as rotations by an angle π about some axes x,y.

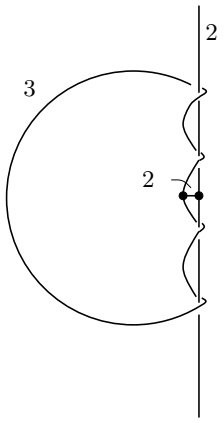
The quotient orbifolds $Q_{r,s}(3,3,2)/\langle\ X\ \rangle$ and $Q_{r,s}(3,3,2)/\langle\ Y\ \rangle$ correspond to the $\mathbb{Z}_2$-extensions of $GT(3,3,2;W(r,s,a,b))$. We illustrate this with the orbifold $Q_{1,4}(3,3,2)$, which we redraw as



Let X,Y be rotations by π about axes x,y respectively where the direction of x is from the top to the bottom of the page, and y is perpendicular to the page.

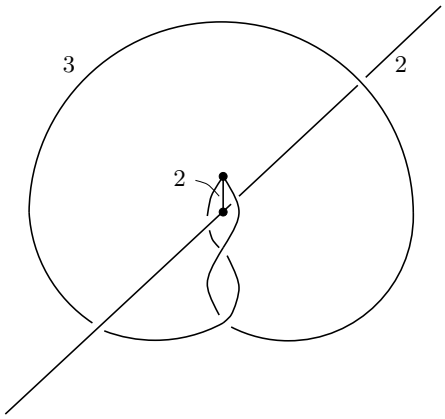
Then

$Q_{1,4}(3,3,2)/\langle\ X\ \rangle =$



where the open ends are joined at infinity, and this can easily be deformed to $Q_{1,8}(2,3,2)$. The second quotient

$Q_{1,4}(3,3,2)/\langle\ X\ \rangle =$



can be deformed to $Q_{3,8}(2,3,2)$. Thus $Q_{1,4}(3,3,2)$ double covers $Q_{1,8}(2,3,2)$ and $Q_{3,8}(2,3,2)$. In the same way we can show that for the remaining pairs (r,s) the orbifold $Q_{r,s}(3,3,2)$ double covers $\mathcal{Q}$ and $\mathcal{Q}'$ where

$$\begin{aligned} \mathcal{Q} &= Q_{s-p,2s}(2,3,2), \\ \mathcal{Q}' &= \begin{cases} Q_{s+r,2s}(2,3,2) & \text{if } s \text{ is even} \\ Q_{p/2,s}(2,3,4) & \text{if } s \text{ is odd} \end{cases} \end{aligned}$$

and $p = r$ if r is even and $p = s - r$ if r is odd.

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References

[1] G. Baumslag, J.W. Morgan, and P.B. Shalen. Generalized triangle groups. *Math. Proc. Cambridge Philos. Soc.*, 102(1):25–31, 1987.

[2] A. Borel. Commensurability classes and volumes of hyperbolic 3-manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 8(1):1–33, 1981.

[3] M.D.E. Condor, C. Maclachlan, G.J. Martin, and E.A. O'Brien. 2-generator arithmetic Kleinian groups III. Preprint.

[4] W.D. Dunbar. Geometric orbifolds. *Rev. Mat. Univ. Comput. Madrid 1*, 1-3:67–69, 1988.

[5] F. W. Gehring, C. Maclachlan, and G. J. Martin. Two-generator arithmetic Kleinian groups. II. *Bull. London Math. Soc.*, 30(3):258–266, 1998.

[6] F. W. Gehring, C. Maclachlan, G. J. Martin, and A. W. Reid. Arithmeticity, discreteness and volume. *Trans. Amer. Math. Soc.*, 349(9):3611–3643, 1997.

[7] F. W. Gehring and G. J. Martin. Stability and extremality in Jørgensen's inequality. *Complex Variables Theory Appl.*, 12(1-4):277–282, 1989.

[8] F. W. Gehring and G. J. Martin. Commutators, collars and the geometry of Möbius groups. *J. Anal. Math.*, 63:175–219, 1994.

[9] F. W. Gehring and G. J. Martin. On the Margulis constant for Kleinian groups. I. *Ann. Acad. Sci. Fenn. Math.*, 21(2):439–462, 1996.

[10] M.A. Hagelberg. Generalized triangle groups and 3-dimensional orbifolds. In *Second International Conference on Algebra (Barnaul, 1991)*, pages 185–192. Amer. Math. Soc., Providence, RI, 1995.

[11] H. B. Helling, Ĭ. Mennike, and È. Vinberg. On some generalized triangular groups and three-dimensional orbifolds. *Trudy Moskov. Mat. Obshch.*, 56:5–32, 350, 1995. Translation in *Trans. Moscow Math. Soc.* pages 1-21, 1995.

[12] C.D. Hodgson. Geometric structures on 3-dimensional orbifolds: Notes on Thurston's proof. Preprint.

[13] K.N. Jones and A.W. Reid. Minimal index torsion-free subgroups of Kleinian groups. *Math. Ann.*, 310(2):235–250, 1998.

[14] C. Maclachlan and G. J. Martin. The non-compact arithmetic generalised triangle groups. Preprint.

[15] C. Maclachlan and G. J. Martin. 2-generator arithmetic Kleinian groups. *J. Reine Angew. Math.*, 511:95–117, 1999.

[16] C. Maclachlan and A. W. Reid. Commensurability classes of arithmetic Kleinian groups and their Fuchsian subgroups. *Math. Proc. Cambridge Philos. Soc.*, 102(2):251–257, 1987.

[17] W.D. Neumann and A.W. Reid. Arithmetic of hyperbolic manifolds. In *Topology '90 (Columbus, OH, 1990)*, pages 273–310. de Gruyter, Berlin, 1992.

[18] A.W. Reid. A note on trace-fields of Kleinian groups. *Bull. London Math. Soc.*, 22(4):349–352, 1990.

[19] K Takeuchi. A characterization of arithmetic Fuchsian groups. *J. Math. Soc. Japan*, 27(4):600–612, 1975.

[20] W. P. Thurston. *Three-dimensional geometry and topology. Vol. 1.* Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[21] W.P. Thurston. Three manifolds with symmetry. Preprint.

[22] A.G.T. Williams. Euler characteristics of orbifold generalised triangle groups. *Math. Proc. Cambridge Philos. Soc.*, 132:435–438, 2002.