

# Isomorphism theorems for classes of cyclically presented groups

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## Abstract

We consider two multi-parameter classes of cyclically presented groups, introduced by Cavicchioli, Repovš, and Spaggiari, that contain many previously considered families of cyclically presented groups of interest both for their algebraic and for their topological properties. Building on results of Bardakov and Vesnin, O'Brien and the previously named authors, we prove theorems that establish isomorphisms of groups within these families.

Keywords: Cyclically presented group, isomorphism.

MSC: 20F05.

## 1 Introduction

The *cyclically presented group*  $G_n(w)$  is defined by the *cyclic presentation*

$$\{x_0, \dots, x_{n-1} \mid w, \theta(w), \dots, \theta^{n-1}(w)\}$$

where  $w = w(x_0, \dots, x_{n-1})$  is a word in the free group  $F_n$  with generators  $x_0, \dots, x_{n-1}$  and  $\theta : F_n \rightarrow F_n$  is the automorphism of  $F_n$  given by  $\theta(x_i) = x_{i+1}$  for each  $0 \leq i < n$  (subscripts mod  $n$ ).

The first family of cyclically presented groups that we will consider is the following eight parameter family, introduced in [3], where it is shown that this family contains many classes of cyclically presented groups, previously considered in the literature, of interest both for their algebraic and for their topological properties. Let  $n \geq 2$ ,  $r, s \geq 0$ ,  $0 \leq p, h < n$ , and  $l, k \in \mathbb{Z}$  (note that unlike in [3] we allow  $k, l < 0$  and  $r, s \in \{0, 1\}$ ) and define

$$G_n(h, k; p, q; r, s; l) = G_n \left( \left( \prod_{j=0}^{r-1} x_{jp} \right)^l \left( \prod_{j=0}^{s-1} x_{h+jq} \right)^{-k} \right) \quad (1)$$

(where an empty product corresponds to the empty word). The groups  $G_n(k, 1; m, 0; 2, 1; 1)$  are the *groups of Fibonacci type*  $G_n(m, k) = G_n(x_0 x_m x_k^{-1})$  introduced in [11], and independently in [4]. In particular, the groups  $H(n, m) = G_n(m, 1) = G_n(1, 1; m, 0; 2, 1; 1)$  are the *Gilbert-Howie groups* of [10] and the groups  $F(2, n) = G_n(2, 1; 1, 0; 2, 1; 1)$  are the *Fibonacci groups* of [7]. Different values of the parameters of these cyclically presented groups may yield isomorphic groups. Our starting point is the following theorem:

**Theorem 1** ([5, Theorem 2],[1, Theorem 1.1]). *Let  $n \geq 2$ ,  $1 \leq m, k, m', k' < n$ ,  $(n, m, k) = 1$ , and assume  $(n, k') = 1$ . If  $m'(m-k) \equiv mk' \pmod n$  then  $G_n(m, k) \cong G_n(m', k')$ .*

Setting  $k = k' = 1$  in Theorem 1 recovers [10, Lemma 2.1]. A less concise version of Theorem 1, and without the hypothesis  $(n, k') = 1$ , was asserted in [1, Theorem 1.1]; however, it was pointed out in [5] that the hypothesis  $(n, k') = 1$  is necessary (a counterexample of  $G_6(1, 3) \not\cong G_6(3, 4)$  being provided), which led to the formulation of [5, Theorem 2] given above.

In this note we extend Theorem 1 to the class of groups  $G_n(h, k; p, q; r, s; l)$ . It is convenient to express our main result in terms of parameters  $A, B, A', B'$  where

$$A = h, B = A - p(r-1) + q(s-1), A' = h', B' = A' - p'(r-1) + q'(s-1).$$

**Theorem A.** *If  $(n, h', p', q') = 1$ ,  $(n, B) = 1$ ,  $pA' + p'B \equiv 0 \pmod n$ , and  $qA' + q'B \equiv 0 \pmod n$ , then  $G_n(h, k; p, q; r, s; l) \cong G_n(h', k; p', q'; r, s; l)$ .*

We prove Theorem A in Section 2. The hypothesis  $(n, h', p', q') = 1$ , while necessary, is not strong: writing  $d' = (n, h', p', q')$  we see that  $G_n(h', k; p', q'; r, s; l)$  is isomorphic to the free product of  $d'$  copies of  $G_{n/d'}(h'/d', k; p'/d', q'/d'; r, s; l)$  (see, for example, [8] or [15, Theorem 1]). As a corollary to Theorem A we obtain:

**Corollary 2.** *Let  $n \geq 2$ ,  $1 \leq m, k, m', k' < n$ ,  $(n, m', k') = 1$ , and assume  $(n, m-k) = 1$ . If  $m'(m-k) \equiv mk' \pmod n$  then  $G_n(m, k) \cong G_n(m', k')$ .*

Noting that if the congruence  $m'(m-k) \equiv mk' \pmod n$  holds then  $(n, m', k') = 1$  and  $(n, m-k) = 1$  both hold if and only if  $(n, m, k) = 1$  and  $(n, k') = 1$  both hold, we obtain Theorem 1.

We now record some further consequences of Theorem A concerning other cyclically presented groups that have arisen in the literature. The *cyclically presented groups with length three positive relators* are the groups  $\Gamma_n(k, l) = G_n(x_0 x_k x_l)$  ( $0 \leq k, l < n$ ) considered in [6, Section 4],[9],[12],[2]. These are both the groups  $G_n(k, -1; 0, l-k; 1, 2; 1)$  and the groups  $G_n(l, -1; k, 0; 2, 1; 1)$ . Applying Theorem A to these expressions gives:

**Corollary 3.** (a) If  $(n, k', l') = 1$ ,  $(n, l) = 1$  and  $k'k \equiv l'l \pmod n$  then  $\Gamma_n(k, l) \cong \Gamma_n(k', l')$ .

(b) If  $(n, k', l') = 1$ ,  $(n, l-k) = 1$  and  $k'l \equiv k(k'-l') \pmod n$  then  $\Gamma_n(k, l) \cong \Gamma_n(k', l')$ .

The Prishchepov groups  $P(r, n, k, s, q) =$

$$G_n((x_0 x_q \dots x_{(r-1)q})(x_{(k-1)} x_{(k-1)+q} \dots x_{(k-1)+(s-1)q})^{-1})$$

( $n \geq 2, r, s \geq 1, 0 \leq k, q < n$ ), introduced in [13] and studied further in [15], are the groups  $G_n(k-1, 1; q, q; r, s; 1)$ . Applying Theorem A to these we obtain:

**Corollary 4.** If  $n \geq 2$ ,  $(n, k'-1, q') = 1$ ,  $(n, (k-1) - q(r-s)) = 1$ , and  $q'(k-1) + q(k'-1) \equiv qq'(r-s) \pmod n$ , then  $P(r, n, k, s, q) \cong P(r, n, k', s, q')$ .

The generalized Prishchepov groups  $\tilde{P}(r, n, k, s, p, q) =$

$$G_n((x_0 x_q \dots x_{(r-1)q})(x_k x_{k+p} \dots x_{k+(s-1)p})^{-1})$$

( $n \geq 2, r, s \geq 1, 0 \leq k, p, q < n$ ), studied in [14], are the groups  $G_n(k, 1; q, p; r, s; 1)$ .

(An unfortunate conflict of notation arises:  $P(r, n, k, s, q) = \tilde{P}(r, n, k-1, s, q, q)$ , rather than the more desirable  $P(r, n, k, s, q) = \tilde{P}(r, n, k, s, q, q)$ .)

Applying Theorem A we obtain:

**Corollary 5.** Suppose  $n \geq 2$ ,  $(n, k', p', q') = 1$ ,  $(n, k+p(s-1) - q(r-1)) = 1$ ,  $qk' + q'k \equiv q'(q(r-1) - p(s-1)) \pmod n$  and  $pk' + p'k \equiv p'(q(r-1) - p(s-1)) \pmod n$ . Then  $\tilde{P}(r, n, k, s, p, q) \cong \tilde{P}(r, n, k', s, p', q')$ .

Another class of groups fitting within the family  $G_n(h, k; p, q; r, s; l)$  was considered in [2]. Given  $f \in \mathbb{Z}$  and  $r \geq 0$  let  $\Lambda(r, f) = \prod_{i=0}^{r-1} x_{if}$  be the positive word of length  $r$  whose first letter is  $x_0$  and whose step size is  $f$ . We call any shift of  $\Lambda(r, f)$  or its inverse an  $f$ -block (of block length  $r$ ) and we say that a cyclically presented group  $G$  is of type  $\mathfrak{F}$  if there is some  $f$  such that  $G \cong G_n(w)$  where  $w$  is the product of two  $f$ -blocks. Thus a group is of type  $\mathfrak{F}$  if it is isomorphic to

$$G_n \left( \left( \prod_{i=0}^{r_1-1} x_{if} \right) \left( \prod_{i=0}^{r_2-1} x_{a+if} \right)^\epsilon \right) \quad (2)$$

for some  $r_1, r_2 \geq 0, 0 \leq f, a < n, \epsilon = \pm 1$ . (See the proof of [2, Lemma 1].) Of course the case  $\epsilon = -1$  gives the Prishchepov groups (which contain the groups of Fibonacci type) and the case  $\{r, s\} = \{2, 1\}$ ,  $\epsilon = 1$  gives the cyclically presented groups with length three positive relators. The group of type  $\mathfrak{F}$  given at (2) is the group  $G_n(a, -\epsilon; f, f; r_1, r_2; 1)$ . Applying Theorem A to this we obtain:

**Corollary 6.** *Suppose  $(n, a', f') = 1$ ,  $(n, a - f(r_1 - r_2)) = 1$ ,  $f'a + fa' \equiv ff'(r_1 - r_2) \pmod n$ . Then*

$$G_n \left( \left( \prod_{i=0}^{r_1-1} x_{if} \right) \left( \prod_{i=0}^{r_2-1} x_{a+if} \right)^\epsilon \right) \cong G_n \left( \left( \prod_{i=0}^{r_1-1} x_{if'} \right) \left( \prod_{i=0}^{r_2-1} x_{a'+if'} \right)^\epsilon \right).$$

In [2] groups of type  $\mathfrak{F}$  that satisfy a certain condition relating the step size, block lengths, and sign, were called *groups of type  $\mathfrak{M}$* . Specifically, the *cyclically presented group in the class  $\mathfrak{M}$  defined by the parameters  $(r, n, s, f, A)$  where  $r \geq 0$ ,  $s \in \mathbb{Z}$ ,  $0 \leq f, A < n$ , and  $f(r - s) \equiv 0 \pmod n$  is the group  $G_n(w)$  where*

$$w = \begin{cases} \left( \prod_{i=0}^{r-1} x_{if} \right) \left( \prod_{i=0}^{s-1} x_{A+if} \right)^{-1} & \text{if } s \geq 0, \\ \left( \prod_{i=0}^{r-1} x_{if} \right) \left( \prod_{i=0}^{|s|-1} x_{A+(r+i)f} \right) & \text{if } s \leq 0 \end{cases}$$

(see equation (6) of [2]). Thus the group in the class  $\mathfrak{M}$ , defined by the parameters  $(r, n, s, f, A)$ , where  $r \geq 0$ ,  $s \in \mathbb{Z}$  and  $f(r - s) \equiv 0 \pmod n$  is the group

$$G = \begin{cases} G_n(A, 1; f, f; r, s; 1) & \text{if } s \geq 0, \\ G_n(A + rf, -1; f, f; r, |s|; 1) & \text{if } s \leq 0. \end{cases}$$

Applying Theorem A to these groups we have the following:

**Corollary 7.** (a) *Suppose  $r, s \geq 0$ ,  $f(r - s) \equiv 0 \pmod n$ ,  $f'(r - s) \equiv 0 \pmod n$ . If  $(n, A', f') = 1$ ,  $(n, A) = 1$  and  $f'A + fA' \equiv 0 \pmod n$  then the groups in the class  $\mathfrak{M}$ , defined by the parameters  $(r, n, s, f, A)$ , and by the parameters  $(r, n, s, f', A')$  are isomorphic.*

(b) *Suppose  $r \geq 0$ ,  $s \leq 0$ ,  $f(r - s) \equiv 0 \pmod n$ ,  $f'(r - s) \equiv 0 \pmod n$ . If  $(n, A', f') = 1$ ,  $(n, A + |s|f) = 1$  and  $f'A + fA' \equiv 0 \pmod n$  then the groups in the class  $\mathfrak{M}$ , defined by the parameters  $(r, n, s, f, A)$ , and by the parameters  $(r, n, s, f', A')$  are isomorphic.*

We now turn to a second family of cyclically presented groups introduced in [6]. For  $(a, b, r, s) \in \mathbb{Z}^4$ ,  $n \geq 2$ ,  $0 \leq m, k, h < n$  let

$$G_n^{(a,b,r,s)}(m, k, h) = G_n(x_0^a x_k^b x_{h+m}^a (x_h^r x_m^r)^{-s}). \quad (3)$$

It was shown in [6] that this family also contains many classes of cyclically presented groups previously considered in the literature. In particular,  $G_n^{(1,1,2,1)}(m, k, 0)$  are the groups of Fibonacci type  $G_n(m, k)$  and  $G_n^{(1,1,0,0)}(0, k, l)$  are the cyclically presented groups with length three positive relators  $\Gamma_n(k, l)$ . We have the following theorem, a generalization of [1, Theorem 1.1], which establishes isomorphisms of groups within this class:

**Theorem 8** ([6, Theorem 2.6]). *Suppose  $0 \leq m, k, h < n$ ,  $(n, k - h - m) = 1$ ,  $(n, m', k', h') = 1$  and that  $\rho = (n, k - h - m)$  divides  $k'$  and there exist positive integers  $\alpha, \beta, \gamma, \delta$  such that*

$$\alpha + \beta(k - h - m) \equiv 1 - m \pmod{n}, \quad (4)$$

$$\gamma + \delta(k - h - m) \equiv 1 - h \pmod{n}, \quad (5)$$

$$\alpha + \beta k' \equiv 1 + m' \pmod{n}, \quad (6)$$

$$\gamma + \delta k' \equiv 1 + h' \pmod{n}, \quad (7)$$

where  $1 \leq \alpha, \gamma \leq \rho$  and  $1 \leq \beta, \delta \leq n/\rho$ . Then for all  $(a, b, r, s) \in \mathbb{Z}^4$   $G_n^{(a,b,r,s)}(m, k, h) \cong G_n^{(a,b,r,s)}(m', k', h')$ .

Note that the hypothesis  $(n, k - h - m) = 1$  was omitted from the original statement [6, Theorem 2.6] and is necessary: just as the pair of groups  $G_6(1, 3)$ ,  $G_6(3, 4)$  provided a counterexample to [1, Theorem 1.1], the same pair of groups  $G_6^{(1,1,2,1)}(1, 3, 0) = G_6(1, 3)$ ,  $G_6^{(1,1,2,1)}(3, 4, 0) = G_6(3, 4)$  provide a counterexample to the original statement [6, Theorem 2.6]. In the same way that the (necessary) imposition of the hypothesis  $(n, k') = 1$  leads to the more concise formulation of [1, Theorem 1.1] given in Theorem 1, the (necessary) imposition of the hypothesis  $(n, k - h - m) = 1$  in Theorem 8 leads to the following formulation:

**Corollary B.** *Suppose  $0 \leq m, k, h < n$ ,  $(n, k - h - m) = 1$ ,  $(n, m', k', h') = 1$ ,  $m'(h + m - k) \equiv mk'$ ,  $h'(h + m - k) \equiv hk'$ . Then for all  $(a, b, r, s) \in \mathbb{Z}^4$   $G_n^{(a,b,r,s)}(m, k, h) \cong G_n^{(a,b,r,s)}(m', k', h')$ .*

*Proof.* Suppose  $d = (n, k') > 1$ . Since  $d$  divides  $k'$ , it must also divide  $mk'$  and  $hk'$  and hence  $m'(h + m - k)$  and  $h'(h + m - k)$ . But since  $(n, k - h - m) = 1$ ,  $d$  divides  $m'$  and  $h'$ , and so  $d$  divides  $(n, m', k', h') = 1$ , a contradiction. Thus  $(n, k') = 1$ . Writing  $(k')^{-1}$  for the multiplicative inverse of  $k' \pmod{n}$ , let  $\beta = m'(k')^{-1} \pmod{n}$ ,  $\delta = h'(k')^{-1} \pmod{n}$ . In the notation of Theorem 8,  $\rho = 1$  and hence  $\alpha = \gamma = 1$ . The hypotheses imply that the congruences (4)–(7) hold, so the result follows.  $\square$

Putting  $(a, b, r, s) = (1, 1, 2, 1)$  and  $h = h' = 0$  in Corollary B yields Corollary 2, and hence Theorem 1. Since there was a missing hypothesis in the original statement of Theorem 8 we include a direct proof of Corollary B (based on the proof of [6, Theorem 2.6]) in Section 3.

## 2 Proof of Theorem A

Suppose  $d = (n, A') > 1$ . Since  $d$  divides  $A'$ , it must also divide  $pA'$  and  $qA'$ , and hence  $p'B$  and  $q'B$ . But, since  $(n, B) = 1$ ,  $d$  divides  $p'$  and  $q'$  so  $d$  divides  $(n, A', p', q') = (n, h', p', q') = 1$ , a contradiction. Thus  $(n, A') = 1$ .

Writing  $(A')^{-1}$  for the multiplicative inverse of  $A' \pmod n$ , let  $\sigma = p'(A')^{-1}$ ,  $\tau = q'(A')^{-1}$ , so

$$\sigma A' \equiv p' \pmod n, \quad (8)$$

$$\tau A' \equiv q' \pmod n. \quad (9)$$

Further  $\sigma B \equiv p'(A')^{-1}B \equiv (A')^{-1}(p'B) \equiv (A')^{-1}(-pA') \equiv -p \pmod n$ ; similarly  $\tau B \equiv -q \pmod n$ , i.e.

$$\sigma B \equiv -p \pmod n, \quad (10)$$

$$\tau B \equiv -q \pmod n. \quad (11)$$

Let  $G = G_n(h, k; p, q; r, s; l)$  as defined at (1). Inverting the relators gives

$$G = G_n \left( \left( \prod_{j=0}^{r-1} x_{(r-1)p-jp}^{-1} \right)^l \left( \prod_{j=0}^{s-1} x_{(s-1)q+h-jq}^{-1} \right)^{-k} \right).$$

Replacing each generator by its inverse and then subtracting  $(r-1)p$  from all subscripts gives

$$\begin{aligned} G &= G_n \left( \left( \prod_{j=0}^{r-1} x_{-jp} \right)^l \left( \prod_{j=0}^{s-1} x_{-(r-1)p+(s-1)q+h-jq} \right)^{-k} \right) \\ &= G_n \left( \left( \prod_{j=0}^{r-1} x_{-jp} \right)^l \left( \prod_{j=0}^{s-1} x_{B-jq} \right)^{-k} \right) \\ &= \langle x_0, \dots, x_{n-1} \mid \left( \prod_{j=0}^{r-1} x_{i-jp} \right)^l \left( \prod_{j=0}^{s-1} x_{i+B-jq} \right)^{-k} \quad (0 \leq i < n) \rangle. \end{aligned}$$

Now  $(n, B) = 1$  implies that  $\{i \mid 0 \leq i < n\} = \{\alpha B \mid 0 \leq \alpha < n\}$  so

$$\begin{aligned} G &= \langle x_0, \dots, x_{n-1} \mid \left( \prod_{j=0}^{r-1} x_{\alpha B-jp} \right)^l \left( \prod_{j=0}^{s-1} x_{\alpha B+B-jq} \right)^{-k} \quad (0 \leq \alpha < n) \rangle \\ &= \langle x_0, \dots, x_{n-1} \mid \left( \prod_{j=0}^{r-1} x_{\alpha B+j\sigma B} \right)^l \left( \prod_{j=0}^{s-1} x_{\alpha B+B+j\tau B} \right)^{-k} \quad (0 \leq \alpha < n) \rangle \end{aligned}$$

by (10),(11), so

$$G = \langle x_0, \dots, x_{n-1} \mid \left( \prod_{j=0}^{r-1} x_{(\alpha+j\sigma)B} \right)^l \left( \prod_{j=0}^{s-1} x_{(\alpha+1+j\tau)B} \right)^{-k} \quad (0 \leq \alpha < n) \rangle.$$

Noting that  $(n, A') = 1$  and  $(n, B) = 1$  we may adjoin generators  $\{y_0, \dots, y_{n-1}\}$  as follows:

$$G = \left\langle \begin{array}{l} x_0, \dots, x_{n-1}, \\ y_0, \dots, y_{n-1} \end{array} \left| \begin{array}{l} \left( \prod_{j=0}^{r-1} x_{(\alpha+j\sigma)B} \right)^l \left( \prod_{j=0}^{s-1} x_{(\alpha+1+j\tau)B} \right)^{-k} \\ y_{\alpha A'} = x_{\alpha B} \quad (0 \leq \alpha < n) \end{array} \right. \right\rangle.$$

Eliminating generators  $x_0, \dots, x_{n-1}$  then gives

$$\begin{aligned} G &= \left\langle y_0, \dots, y_{n-1} \mid \left( \prod_{j=0}^{r-1} y_{(\alpha+j\sigma)A'} \right)^l \left( \prod_{j=0}^{s-1} y_{(\alpha+1+j\tau)A'} \right)^{-k} \quad (0 \leq \alpha < n) \right\rangle \\ &= \left\langle y_0, \dots, y_{n-1} \mid \left( \prod_{j=0}^{r-1} y_{\alpha A' + j(\sigma A')} \right)^l \left( \prod_{j=0}^{s-1} y_{\alpha A' + A' + j(\tau A')} \right)^{-k} \quad (0 \leq \alpha < n) \right\rangle \\ &= \left\langle y_0, \dots, y_{n-1} \mid \left( \prod_{j=0}^{r-1} y_{\alpha A' + jp'} \right)^l \left( \prod_{j=0}^{s-1} y_{\alpha A' + A' + jq'} \right)^{-k} \quad (0 \leq \alpha < n) \right\rangle \end{aligned}$$

using (8),(9). Now  $(n, A') = 1$  implies that  $\{\alpha A' \mid 0 \leq \alpha < n\} = \{i \mid 0 \leq i < n\}$  so

$$\begin{aligned} G &= \left\langle y_0, \dots, y_{n-1} \mid \left( \prod_{j=0}^{r-1} y_{i+jp'} \right)^l \left( \prod_{j=0}^{s-1} y_{i+A'+jq'} \right)^{-k} \quad (0 \leq i < n) \right\rangle \\ &= \left\langle y_0, \dots, y_{n-1} \mid \left( \prod_{j=0}^{r-1} y_{i+jp'} \right)^l \left( \prod_{j=0}^{s-1} y_{i+h'+jq'} \right)^{-k} \quad (0 \leq i < n) \right\rangle \\ &= G_n(h', k; p', q'; r, s; l) \end{aligned}$$

as required.

### 3 Proof of Corollary B

As in our earlier proof of Corollary B,  $(n, k') = 1$ . Again, let  $\beta = m'(k')^{-1}$ ,  $\delta = h'(k')^{-1} \pmod n$ . Then  $\beta(k-h-m) \equiv m'(k')^{-1}(k-h-m) \equiv (k')^{-1}m'(k-h-m) \equiv -(k')^{-1}mk' \equiv -m \pmod n$ ; similarly  $\delta(k-h-m) \equiv -h$ , i.e.

$$\beta(k-h-m) \equiv -m \pmod n, \quad (12)$$

$$\delta(k-h-m) \equiv -h \pmod n. \quad (13)$$

Let  $G = G_n^{(a,b,r,s)}(m, k, h)$  as defined at (3). Replacing each generator by its inverse, inverting the relators and subtracting  $(h+m)$  from the subscripts gives

$$G = \left\langle x_i \mid x_i^a x_{i+k-h-m}^b x_{i-h-m}^a = (x_{i-h}^r x_{i-m}^r)^s \quad (0 \leq i < n) \right\rangle.$$

Since  $(k-h-m, n) = 1$  the set  $\{i \mid 0 \leq i < n\} = \{t(k-h-m) \mid 0 \leq t < n\}$  so

$$G = \left\langle x_{t(k-h-m)} \mid \begin{array}{l} x_{t(k-h-m)}^a x_{t(k-h-m)+k-h-m}^b x_{t(k-h-m)-h-m}^a = \\ (x_{t(k-h-m)-h}^r x_{t(k-h-m)-m}^r)^s \quad (0 \leq t < n) \end{array} \right\rangle$$

and using (12),(13) we get

$$G = \left\langle x_{t(k-h-m)} \mid \begin{array}{l} x_{t(k-h-m)}^a x_{(t+1)(k-h-m)}^b x_{(t+\beta+\delta)(k-h-m)}^a = \\ (x_{(t+\delta)(k-h-m)}^r x_{(t+\beta)(k-h-m)}^r)^s \quad (0 \leq t < n) \end{array} \right\rangle.$$

Noting that  $(n, k') = 1$  and  $(n, k-h-m) = 1$  we may adjoin generators  $\{y_0, \dots, y_{n-1}\} = \{y_{tk'} \mid (0 \leq t < n)\}$  as follows:

$$G = \left\langle \begin{array}{l} x_{t(k-h-m)}, \\ y_{tk'} \end{array} \mid \begin{array}{l} x_{t(k-h-m)}^a x_{(t+1)(k-h-m)}^b x_{(t+\beta+\delta)(k-h-m)}^a = \\ (x_{(t+\delta)(k-h-m)}^r x_{(t+\beta)(k-h-m)}^r)^s, \\ y_{tk'} = x_{t(k-h-m)} \quad (0 \leq t < n) \end{array} \right\rangle.$$

Eliminating generators  $x_{t(k-h-m)}$  ( $0 \leq t < n$ ) gives

$$\begin{aligned} G &= \langle y_{tk'} \mid y_{tk'}^a y_{(t+1)k'}^b y_{(t+\beta+\delta)k'}^a = (y_{(t+\delta)k'}^r y_{(t+\beta)k'}^r)^s \quad (0 \leq t < n) \rangle \\ &= \langle y_{tk'} \mid y_{tk'}^a y_{tk'+k'}^b y_{tk'+\beta k'+\delta k'}^a = (y_{tk'+\delta k'}^r y_{tk'+\beta k'}^r)^s \quad (0 \leq t < n) \rangle \\ &= \langle y_{tk'} \mid y_{tk'}^a y_{tk'+k'}^b y_{tk'+m'+h'}^a = (y_{tk'+h'}^r y_{tk'+m'}^r)^s \quad (0 \leq t < n) \rangle \\ &= \langle y_i \mid y_i^a y_{i+k'}^b y_{i+m'+h'}^a = (y_{i+h'}^r y_{i+m'}^r)^s \quad (0 \leq i < n) \rangle \\ &= G_n^{(a,b,r,s)}(m', k', h') \end{aligned}$$

as required.

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