ORIGAMI AS A TOOL FOR MATHEMATICAL INVESTIGATION AND ERROR MODELLING IN ORIGAMI CONSTRUCTION

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I owe my gratitude to the British Origami Society, BOS, who have supported me with my research with financial support to attend 6OSME in Japan and academic support through access to their extensive library.

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Abstract

Origami is the ancient Japanese art of paper folding [1]. It has inspired applications in industries ranging from Bio-Medical Engineering [2] to Architecture [3]. This thesis reviews ways in which Origami is used in a number of fields and investigates unexplored areas providing insight and new results which may lead to better understanding and new uses.

The OSME conference series arguably covers most of the research activities in the field of Origami and its links to Science and Mathematics. The thesis provides a comprehensive review of the work that has been presented at these conferences and published in their proceedings.

The mathematics of Origami has been explored before and much of the fundamental work in this field is presented in chapter 3. Here an attempt is made to

push the bounds of this field by suggesting ways in which Origami can be used as a mathematical tool for in-depth exploration of non trivial problems. A particular problem we consider is the 4-colour theorem and its proof.

Looking at some well known methods for producing angles and lengths mathematically the thesis also explores how accurate these might be. This leads to the surprisingly unstudied field of error modelling in Origami. Errors in folding processes have not previously been looked at from a mathematical point of view. The thesis develops a model for error estimation in crease patterns and a framework for error modelling in Origami applications. By introducing a standardised error into alignments, uniform error bounds for each of the one-fold constructions are generated. This defines a region in which a crease could lie in order to satisfy the alignments of a given fold within a specified tolerance. Analysis of this method on some examples provides insight into how this might be used in multi-fold constructions. An algorithm to that effect is introduced.

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, University of Essex. No part of this thesis has been submitted elsewhere for any other degree or qualification, and unless referenced to the contrary in the text it is all the work of the author.

Contents

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Nomenclature

A

Alignment : bringing two or more objects to the same location when making a fold

B

Base : a regular geometric shape that has a structure similar to that of the desired subject.

Blintz : folding all corners to the centre.

C

Crease : a mark left in the paper after a fold has been unfolded.

Crease assignment : determination of whether each crease is a mountain fold, valley fold, or flat (unfolded) crease. It is also called crease parity.

Crease pattern : the pattern of creases left behind on the paper after a model has been unfolded.

D

Dual crease pattern : the dual graph of a graph representing a crease pattern.

F

Fold : a bend produced in a material.

Fold angle : The angle of intersection between the two faces intersecting at a crease.

Flap : a region of paper in an Origami shape that is attached only along one edge so that it can be easily manipulated by itself.

J

Judgement fold : a fold created without alignments, often made to copy a diagram or made "to taste"

M

Measured fold : a fold which created a crease along a mark which has been measured using tools other than through the process of folding.

Mountain fold : a crease that looks like a mountain when viewed form above. Usually indicated by a dot-dot-dash line in sequential diagrams or a solid black line in crease patterns.

O

Origami : the art of folding paper into decorative shapes, usually from uncut squares.

U

Unfold : removing a valley fold or mountain fold (or a group of same), leaving behind one or more creases.

V

Valley fold : a crease that looks like a valley when viewed from above. Usually indicated by a dashed line in sequential folding diagrams and a dashed or colored line in crease patterns.

Acronyms

CB: Crease Boundary

COET: Conference on Origami in Education and Therapy

CP: Crease Pattern

CPG: Connected Planar Graph

DG: Dual Graph

EC: Example Crease

EPQ: Extended Project Qualification

HJA: Huzita Justin Axiom

TC: Target Crease

BOS: British Origami Society

JOAS: Japan Origami Academic Society

OSC: One Straight Cut

OSME: Origami, Science, Mathematics and Education

1 Introduction: Origami, Art, Engineering and Mathematics

Origami is the ancient Japanese art of paper folding [4]. In the last century it has become popular around the world. As well as creating beautiful artistic models, many have looked into the mathematics underpinning it and how it can be used in real world applications.

There is now a vast array of applications utilising these age-old techniques and the field is rapidly expanding. From collapsible stents [5] which provide a less invasive surgery to optimised folding processes for car air-bags [6] and even folding lenses and solar panels for use in the space industry, it is increasingly becoming a ubiquitous tool in all sectors of industry.

This thesis reviews ways in which the art is being utilised in a rapidly expanding number of fields and investigates several previously unexplored areas providing insight and new results which may lead to new research areas and applications.

The Origami, Science, Maths and Education conference series, OSME, arguably covers most of the research activities in the field of Origami and its links to Science and Mathematics. The thesis provides a comprehensive review of the work that has been presented at these conferences and published in their proceedings. It includes an overview of the framework which they developed starting in 1989 with the first international meeting of Origami Science and Technology and concluding with the 6th conference which was held in 2014. It reviews all of the papers individually which are then separated into categories. A summary of each paper is presented as well as an extensive list of references.

The mathematics of Origami has been explored before. This thesis attempts to push the boundaries of this field by suggesting ways in which Origami can be used as a mathematical tool for in depth exploration of problems such as the 4-colour problem.

The thesis looks at some well known methods for producing angles and lengths mathematically and how accurate these might be. This leads to the surprisingly unstudied field of error modelling in Origami. Errors in folding processes have not been looked at from a mathematical point of view. The thesis develops a model for error estimation in crease patterns and a framework for error modelling in Origami applications. By introducing a standardised error into alignments, uniform error bounds for each of the one-fold constructions are generated. This defines a region in which a crease could lie in order to satisfy the alignments of a given fold within a specified tolerance. Analysis of this method on some examples provides insight into how this can be extended to cater for multi-fold constructions. An algorithm that minimises the error state of compound creases is introduced for this purpose.

1.1 Motivation

The author has been an Origami enthusiast since childhood when he was given a copy of Robert Harbin's book "Origami 1: The art of Paper-Folding" [7]. At secondary school he worked to convince mathematics and art teachers that Origami was both artistic and mathematical and need not be confined to one category. Building on this he incorporated Origami in GCSE and AS level Art projects as well as an Extended Project Qualification, EPQ, which focused on the applications of Origami. It was this project which lead him to become a member of the British Origami Society, BOS, and attend their conventions. Shortly after starting a BSc in mathematics he founded the University of Essex Origami society and began an Origami project with the Students Union's volunteering team which runs weekly Origami clubs teaching basic geometry through fun kinaesthetic activities in several nearby schools. Interested in how things work from the most fundamental levels, in 2013 as a final year undergraduate project he researched the mathematics of Origami which lead him to discover the OSME conference series [8]. This convinced him that investigating the mathematical concepts of Origami was something he would attempt at the first opportunity. When some funding became available in the Department of Mathematical Sciences of the University of Essex he applied for funding to do a PhD on the mathematics of Origami.

In 2014 he attended the 6th OSME conference and the Japan Origami Academic Society, JOAS, convention in Japan funded as a representative of the BOS. In September 2015 he organised a BOS convention held at the University of Essex where he was elected as the society librarian and trustee of the charity tasked with leading an archiving project on the worlds largest library of Origami resources. This library is now based near the university and has been an invaluable resource for his work. He is now treasurer of the BOS and continues to help organising conventions and leads with the library archive project.

The author became fascinated as to how a single uncut sheet of paper could be made into incredibly complex and elegant shapes through only a finite number of operations. At the back of his mind there has always been much more to Origami than folding paper. To explore this further as a PhD student was the main motivation behind this thesis.

The starting point was to see how Origami could solve difficult problems such as trisecting angles proving some trigonometric identities and basic algebra. Later it was to become clear that modern complex results such as the 4-colour theorem could be examined again from the perspective of Origami. The 4-colour theorem has been proved but the proof is not to the taste of many serious mathematicians mainly because it relies too much on computers and codes [9]. Finding an easier more succinct proof using Origami has been one of the driving questions of this thesis.

Another idea which motivated this work is the idea that Origami constructions of the same type are realised looking different. This is often due to the expertise of the folder but there is more behind that: Errors. We know that all physical manipulation cannot be done with perfect accuracy. Since Origami is now commonly used in manufacture it is essential that an estimate of errors likely to be introduced in the final product be calculated. Such an error may, for example, help the scheduling of the maintenance of machinery where the folds are machine made. To illustrate, consider the boxes used in packaging Easter eggs. Errors accumulated over many creases may be so great that the boxes would not stack beyond a certain height. This is no good for transportation and displays in retail surfaces. The investigation has lead to a model of errors incurred when making creases.

1.2 Thesis organisation

Chapter 2 is an extensive review of the literature on Origami with emphasis on the proceedings of the OSME Conferences. Chapter 3 reviews the fundamental concepts of folding. It presents the work that has been done to solve mathematical problems using Origami constructions, comparing the scope of Origami construction with that of straight edge and compass. Chapter 4 aims to show how origami can be used as a mathematical tool and is not just limited to solving ancient problems which can be simply calculated using algebraic methods. It looks at complex problems such as the 4-colour problem, to see whether Origami could offer a viable solution approach. Through exploration of the problem it suggests the possibility of a potential Origami inspired proof. Chapter 5 looks at error modelling in Origami and provides novel methods to minimise errors in any application of folding. Chapter 6 presents the conclusions and suggestions for possible further research. The Thesis includes an extensive bibliography. Note that the expressions 'to crease' and 'to fold' are often used loosely. This document will refer to a fold as the action of bending a piece of paper which produces a crease; thus a crease is the mark left in the material when a fold is pressed flat.

2 A literature survey

There is a lot of written work on Origami as an Art form, more recently as an industrial process, and a medium for education and therapy. Since our interests are mainly on the mathematical side of it we shall concentrate on the OSME conference proceedings and anything else that is closely relevant.

2.1 The OSME Conference Series

Origami is a word of Japanese origin [10]; however, the early history of paper folding and where it began is somewhat unclear. There are records of paper folding in both Japan and Egypt from the 1600's and possibly earlier but it is also possible that paper folding began in China where paper was invented before this date [1]. Western and Eastern paper folding developed independently until 1945 when Gershon Legman began his studies of paper folding and brought together paper folders from across the world. By 1958 the modern Origami movement had begun and in 1967 the first members society, the British Origami Society, was founded. There are now many Origami societies around the world from OUSA (Origami USA) and BOS (British Origami Society) in the west to NOA (Nippon Origami Association) and JOAS (Japanese Origami Academic Society) in the East [11].

These organisations, in particular BOS, set up archives and libraries to preserve and share knowledge of Origami and for many years their publications were some of the only places where the mathematics of Origami was studied. Although their publications are neither journals nor peer reviewed, many of the articles presented in them laid the foundations for several fields of study within the mathematics of Origami.

One of the earliest references to Origami applications in an academic journal can be found in a New Scientist issue of 1981 with an article on the Miura Origami map fold [12]. Some of the articles presented in the first conference have also been published in the BOS magazine in the years following the conference [13–15]

There is now a large range of scientific books on Origami, some of their content is similar to that presented at the OSME conferences. There are several notable publications which in fact go further than the work in the OSME proceedings. Some of particular note are; 'Geometric Folding Algorithms' By Demaine and

O'Rourke which provides a comprehensive look at folding beginning with Linkages and working through Origami to folding polyhedra [16], this book goes further in terms of Computational Origami than much of the work in the OSME series; 'Origami Design Secrets' by Lang which was one of the first books on Origami design that builds from basic principles to complex design and optimisation algorithms [17] and provides a considerable amount of additional work on designing Origami; 'Origamics' by Haga which provides an in-depth exploration of Haga's theorems among others [18];'Geometric Origami' by Geretschlager which looks at algorithms and computational Origami [19]; and 'Roses, Origami and Maths' by Kawasaki which looks at much of the author's work on generalising Orizuru [20].

Much of the work described above looks at mathematics and Origami. The existing graduate level literature [16] is not directly looking at 'The Mathematics of Origami' as it is in the field of 'Computational Origami', and thus is primarily concerned with computational complexity and if a computer can efficiently solve those problems that arise from folding processes [21]. This work is not inherently looking at the mathematics behind paper folding. There is no pre-established graduate level literature providing definitions, notation or methodology and many variations of these exist in the literature looked at in this survey.

The BOS, Origami USA and other organisations published magazines and articles regarding paper folding which would eventually attract the attention of academics from mathematicians and architects to biologists and teachers. In 1989 Humiaki

Huzita (Humi) organised the first international meeting of Origami Science and Technology. This was the first meeting of its kind and Humi later published the proceedings from this conference himself [22]. Over the following years interest from the scientific community grew significantly and there have since been a further five conferences, seen in Table 2.1, with the proceedings from the latest conference, 6OSME, being published by the American Mathematical Society in 2016 [23]. Table 2.1 provides details of the dates and locations of the OSME conferences as well as the date the proceedings were published, the publisher and the editors.

Conference	**Date**	**Publisher, Publication Date, Editors**	**Location**
First International Meeting of Origami, Science and Technology	6th - 7th December 1989	Self published, 1990, H. Huzita [17]	University of Padova, Ferrara, Italy
Second International Meeting of Origami, Science and Scientific Origami	29th November - 2nd December 1994	The Organising Committee, 1997, K. Miura (Chief), T. Fuse, T. Kawasaki, J. Maekawa [18]	Seian University of Art and Design, Otsu, Japan
The Third International Meeting of Origami, Science, Mathematics, and Education (3OSME)	9th - 11th March 2001	A K Peters, 2002, Thomas Hull [19]	Asilomar, Pacific Grove, California, USA
The Fourth International Conference on Origami, Science, Mathematics, and Education (4OSME)	8th - 10th September 2006	A K Peters, 2009, R. Lang [20]	California Institute of Technology, Pasadena, California, USA
The Fifth International Conference on Origami, Science, Mathematics, and Education (5OSME)	13th 15th July 2010	CRC press, 2011, P. Wang-Iverson, R. Lang, M. Yim [21]	Management University, Singapore
The Sixth International Conference on Origami, Science, Mathematics, and Education (6OSME)	10th - 13th August 2014	The American Mathematical Society, 2016, K. Miura, T. Kawasaki, T. Tachi, R. Uehara, R. Lang, P. Wang-Iverson [24]	University of Tokyo, Tokyo Japan
The Seventh International Conference on Origami, Science, Mathematics, and Education (7OSME)	5th - 7th September 2018	Not Published at time of thesis submission	University of Oxford, Oxford, UK

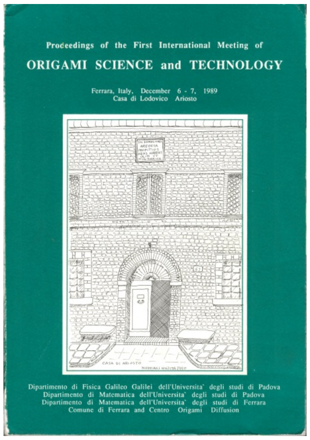
Table 2.1: A list of all OSME conferences including conference date, location and its proceedings' publication details

The OSME series is now a periodic academic conference which produces a series of books of peer-reviewed papers shown in Figure 2.1. Although there are many books and articles which are published outside of these conferences, this literature review focuses specifically on the OSME proceedings because, in the author's opinion, they represent the most complete and accurate collection of knowledge of Origami, science and mathematics.

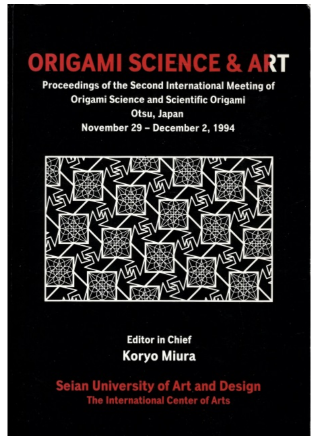
Although many of the papers in the OSME series fit into multiple categories, in this survey we will divide the papers into work in the following sub-fields. We will focus on those which have the strongest mathematical ties.

- The mathematics of Origami
- Computational Origami deconstruction, design and diagramming
- Exploration, design and colouring of Origami polyhedra
- Origami applications in science and technology
- Origami applications in art and design
- Origami in education and/or therapy
- History, Language and Psychology of Origami

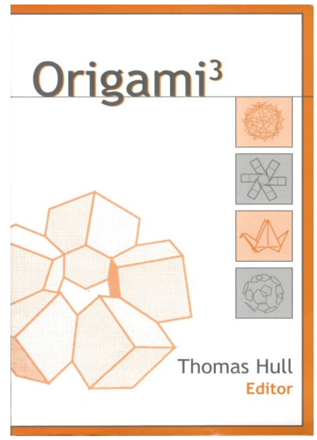
Because several different sets of categories have been used throughout the OSME series we put each of the papers into one of these categories based not on where



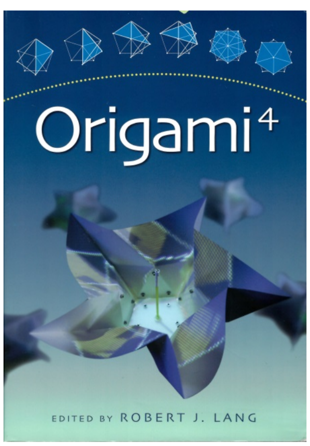
(a) Proceedings of the first international meeting of Origami Science and Technology



(b) Origami Science & Art, proceedings of the second international meeting of Origami Science and Scientific Origami

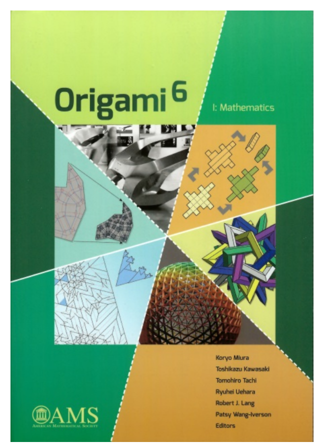


(c) Origami^3 proceedings of the third international meeting of Origami Science, Mathematics, and Education



(d) Origami^4 proceedings of the Fourth international conference on Origami Science, Mathematics, and Education

(e) Origami^5 proceedings of the fifth international conference on Origami Science, Mathematics, and Education



(f) Origami^6 proceedings of the sixth international conference on Origami Science, Mathematics, and Education

Figure 2.1: List of OSME proceedings with cover images

they were placed in the proceedings, but on what seems most appropriate after reviewing the contents. Where possible this survey briefly summarises the work done in each of the papers. However, the first two proceedings contained papers in several languages including French and Japanese. Thus a small number of papers were not summarised. These are listed at the end.

In the first few conferences the terminology used was fluid and many of the now well-known theorems were re-discovered by multiple folders working independently. Mainly through the existence of these conferences, this has now greatly reduced. However, many of the theorems have no one folder who can claim them as their own and are named after multiple individual contributors.

2.1.1 The mathematics of Origami

Although the fields of mathematical sciences and Origami are well established it is only in the last few decades that they have been linked. In the 1930's Beloch showed that paper folding is more powerful than straight edge and compass [25], this is also looked at in a paper by Hull in the OSME series [26]. However, many questions remain about how much Origami can actually do.

As the applications of Origami grew, the need for a definitive and complete model for it has become more apparent. Many have tried to produce a model for Origami and those which have succeeded consider only small areas or specific cases. The most well-known approach is taken through the one fold axioms which have been discovered by many independently; the first reference to these is by Justin. Lang later proved the completeness of this list of axioms. Alperin and Lang also take this further by looking at multi-fold axioms.

Maekawa and Kawasaki provide us with flat folding conditions which are named after them and others have looked at solving mathematical curiosities such as Alhazen's problem which was explored by Alperin, [27]. Table 2.2 summarises OSME papers presented on the mathematics of Origami.

Conference	Papers	Summary
1st meeting	"A possible example of system expansion in Origami geometry" H. Huzita [28]	Comparing Origami and Euclidean geometry. Unlike the latter, Origami is an open system and therefore can be expanded to solve higher order equations; it shows how Origami can theoretically be used to solve a range of problems. However, the author concludes that computer aided calculation is required to fully model these problems.
1st meeting	"A problem in the Kawasaki theorem" H. Huzita [29]	Looking at the Kawasaki problem, a condition which holds for any flat-foldable vertex in a crease pattern, and addressing the converse problem, that a vertex will always be flat-foldable when the Kawasaki condition is true. It also tries to enumerate the number of solutions.
1st meeting	"Arithmetic and analytic properties of paper folding sequences" M. Mendes france & A. Van Der Poorten [30]	Looks at how folds produced in paper-folding can produce infinite sequences. They show that the Fourier series found are almost periodic and thus are deterministic.
1st meeting	"Arithmetic properties of the solutions of a class of functional equations" J. Loxton & A. Van Der Poorten [31]	The authors look at the possible randomness of digits in a series of functions. This paper shows proofs for several theorems however it is not obviously linked to Origami.
1st meeting	"Aspects mathematiques du pliage de papier" J. Justin [32]	The title translates to 'Mathematical Aspects of Paper Folding' and covers an introduction to many areas. It covers the mathematics behind flexagons, mountain/valley assignments in folding strips of paper, geometric constructions, symmetries and more.
1st meeting	"Axiomatic development of Origami geometry" E. Frigerio & H. Huzita [33]	Presented is a proposal of 6 Origami axioms. Even without the 7th, found later by Lang, this paper is still able to show through comparison with Euclidean construction techniques that Origami is more powerful and simpler.

Table 2.2 – continued on next page

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Conference	**Papers**	**Summary**
1st meeting	“Complementary unit Origami - - maximum volume with minimal material - - More is less” Y. Kajikawa [34]	Explores the problems of partitioning the surface of a sphere and constructing spheres using intersecting circles. The author also looks at symmetries and dissymmetries of these partitions.
1st meeting	“Crystallographic flat Origami” T. Kawasaki & M. Yoshida [35]	An early attempt to mathematically model Origami tessellations. A group of Origami tessellations named crystallographic flat Origami is discussed.
1st meeting	“Draw of a regular heptagon by folding” B. Scimemi [36]	A collection of notes by a secretary of the conference as the original paper was not available. They were made from Scimemi’s presentation comparing Euclidean construction with folding, for construction of a regular pentagon, a shape which is not constructible in Euclidean geometry.
1st meeting	“Folding paper and thermodynamics” M. Mendes France [37]	An unusual approach to Origami and mathematics by using mountain/valley assignments formed when folding paper in half repeatedly as an analogue for the Bakers transformation. When these 180 degree folds are opened and folded at 90 degrees the curves produced are self-avoiding and when made infinitely long they make fractal patterns which it is shown can be modelled using equations from thermodynamics.
1st meeting	“Folds!” M. Dekking & M. Mendes France [38]	Dragon curves, the shapes made when folded strips of paper are unfolded such that all folds are at right angles, are explored including the sequences which can and cannot be dragon curves. Several well known sequences are found and also the concept of a dual to a dragon curve is explored.
1st meeting	“New relations in Origami geometry” E. Frigerio [39]	Notes from a conversation between the author and H. Huzita regarding some properties of triangles found by J. Justin. It looks at two folding properties of triangles.

Table 2.2 – continued on next page

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Conference	Papers	Summary
1st meeting	“On high dimensional flat Origamis” T. Kawasaki [40]	A notation based geometric model of Origami in three or more dimensions. For three dimensional Origami, creases are represented with faces intersecting at vertices or edges.
1st meeting	“On relation between mountain-crease and valley-crease of flat Origami” T. Kawasaki [41]	Notation for Origami and attempts to globalise foldability constraints for flat-foldable Origami.
1st meeting	“Origami and insanity” T. Yenn [42]	The author discusses several Origami models they have folded and includes some mathematical background about the models. The paper also looks at the definitions of Origami and insanity and makes comparisons between the two.
1st meeting	“Origami as an art of constraints” J. Smith [43]	This paper seeks to describe Origami with a set of constraints. Many types of Origami are considered. The properties of a bird base is included and models which include cutting, modular and polyhedral flattening are discussed.
1st meeting	“Origami geometry: Old and new” E. Frigerio [44]	Similar to Frigerio’s other paper, ‘New relations in Origami geometry’ in the same proceedings, Frigerio expands on that paper here to look at rules for mountain valley assignments in crease patterns.
1st meeting	“Recherches sur les moyens de reconnaitre si un probleme de geometrie peut se résoudre avec la reègle et le compas” M. Wansel [45]	This paper’s title translates as ‘Research on how to recognize if a geometry problem can be solved with a ruler and compass’ it looks at solving second degree equations and trigonometric functions.

Table 2.2 – continued on next page

Continued from previous page

Conference	Papers	Summary
1st meeting	"Resolution par le pliage de l'equation du troisieme degree' et applications geometriques" J. Justin [46]	This paper's title translates as 'Solution by folding to third degree equations and geometric applications'. It looks at the problems of trisecting angles, solving cubic equations and constructing polygons. This is the first paper referencing all 7 of the axioms.
1st meeting	"The algebra of paper-folding" H. Huzita & B. Scimemi [47]	An introduction to why Origami constructions are studied and how they are of use. It shows how, with no tools other than a piece of paper, we can create mathematical Origami constructions from trisecting angles and solving polynomial equations to constructing polygons.
1st meeting	"The elusive pentagon" R. Morassi [48]	A comparison of several pentagon construction methods looking at their precision, complexity and the size of the final pentagon. This work leads to the construction of an exact maximal pentagon.
1st meeting	"The Rudin-Shapiro sequence, Ising Chan, and paper-folding" M. Mendes France [49]	Looking at the dragon curves and sequences created through folding this paper shows how the Rudin-Shapiro sequence can be formed.
1st meeting	"The trisection of a given angle solved by the geometry of Origami" H. Huzita [50]	Looking at trisection of angles using H. Abe's method. Expanding to solving polynomial equations of degree three and providing a suggestion that fourth degree equations may also be solvable.
2nd meeting	"Artistic tiling problem by Origami" P. Forcher [51]	Presents Origami tiling models including a fish which can be an element of a mathematical tiling.
2nd meeting	"Fold - its physical and mathematical principles" K. Miura [52]	A study into the geometry of a fold. It asks why folded structures are characterised by rugged surfaces with convex and concave surfaces divided by sharp folds.
2nd meeting	"Four-dimensional Origami" K. Miyazaki [53]	Theoretical 4-dimensional folding done either by folding a 2-dimensional sheet or by folding a 3-dimensional shape through a 4-dimensional space is discussed using a selection of theoretical examples.

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Conference	Papers	Summary
2nd meeting	"Fujimoto successive method to obtain odd-number section of a segment or angle by folding operations" H. Huzita & S. Fujimoto [54]	Two iterative methods by Fugimoto to divide an edge or an angle into an odd number of equal segments are presented. The authors evaluate the speed at which these methods approach a solution which is found to be very rapid.
2nd meeting	"R(y)-1" T. Kawasaki [55]	Kawasaki concludes his study into the mathematics of Origami with the symbolic representation R(y)-1 which is referred to as the essence of Origami. A range of properties are shown to hold for several examples. However there are still certain things which are possible in the abstract model which cannot be realised in paper.
2nd meeting	"Right angle billiard games and their solutions by paper-folding" H. Huzita [56]	Using a selection of imaginary billiard games, such as a game of billiards where every collision with a cushion bounces back in a direction 90 degrees from its incoming one. This paper uses Origami to find solutions to these problems. Finally, it applies the Origami methods shown to draw a regular heptagon and also to trisect an angle.
2nd meeting	"Similarity in Origami" J. Maekawa [57]	Intended as a preface of the authors full study, the paper looks at similarity in the geometric sense beginning with the geometry of paper sizes and symmetries in traditional crease patterns and leading on to research into fractal/infinite folds.
2nd meeting	"Towards a mathematical theory of Origami" J. Justin [58]	This paper is a developed version of one of the earliest documents to discuss a mathematical theory of Origami, a handwritten paper titled "First ideas for a mathematical theory of paper folding". Initially splitting Origami into 2D, 3D and Curved Folds, this paper focuses on the first of these cases. It discusses the necessary and sufficient conditions without being overly rigorous and looks at several conditions including the non-crossing and no-twist conditions.

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Conference	Papers	Summary
2nd meeting	"Tridimensional transformation of paper by cutting and folding" R. Razani [59]	Suggested is a way in which cutting and folding to produce pop up structures could be modelled.
3OSME	"A mathematical model for non-flat Origami" S. Belcastro & T. Hull [60]	Based on the assumption that a fold is not necessarily flat the authors look at the necessary and sufficient conditions for a crease pattern to be foldable using a mapping from the two dimensional real plane to three dimensions. Necessary conditions are found, sufficiency conditions are shown to be more elusive.
3OSME	"Fold paper and enjoy math: Origamics" K. Haga [61]	An exploration into mathematical properties of folded paper, focusing initially on using a single fold on a square. Haga presents several special cases known as Haga's theorems and also looks at similar properties found with rectangular paper.
3OSME	"Just like young Gauss playing with a square: Folding the regular 17-gon" R. Geretschlager [62]	As a comparison between Euclidean and Origami constructions the author asks if it is possible to construct a regular 17 sided shape. A 17-gon is constructible using straight edge and compass as it only requires solving a quadratic equation; this paper shows it is also possible with Origami and a mathematically accurate method for this is presented.
3OSME	"Mathematical Origami: Another view of Alhazen's optical problem" R. Alperin [27]	This paper looks at the power of Origami constructions. A method for folding a heptagon and solving quartics is shown as well as an exploration of Alhazen's problem. The problem is to construct two lines, drawn from two points in a circle to one point on the circumference, such that they make equal angles with the normal vector at that circumference.
3OSME	"Origami with trignoometric functions" M. Kawamura [63]	Kawamura presents methods to accurately produce angles using Origami as a trigonometric function calculator. It also shows how the complexity of this can be simplified where approximate angle production will suffice.

Table 2.2 – continued on next page

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Conference	Papers	Summary
3OSME	"Paper-folding constructions in Euclidean geometry: An exercise in thrift" B. Scimemi [64]	Exploring several classic geometric problems in Euclidean geometry. As many have been previously shown Scimemi seeks solutions using Origami which have a minimal number of steps in order to minimise error build-up through folding and also to make the solutions more elegant.
3OSME	"Square cycles: An introduction to the analysis of flexagons" E. Berkove & J. Dumont [65]	This paper analyses the structure of cycles in flexagons based on several different methods for their creation. It poses open questions such as whether there is a link between the folding pattern and the structure diagrams and the task of enumeration of possible flexagons from a given net.
3OSME	"The combinatorics of flat folds: a survey" T. Hull [66]	Several classic results in the combinatorics of flat foldable Origami crease patterns are presented and attempts are made at generalising the Kawasaki and Maekawa theorems as well as counting valid valley-mountain assignments.
3OSME	"The definition of iso-area folding" J. Maekawa [67]	A definition for iso-area folding is given, stating if a fold has n-bar symmetry and the rotation axis passes perpendicularly through the plane of the unfolded paper, then the crease pattern has rotational symmetries and should be seen as iso-area; these symmetries might inverse crease direction. Maekawa's paper then explores some properties of some known and unpublished iso-area models.
4OSME	"An Excel-based solution to the one-cut folding problem" A. Huang [68]	Huang presents a variation on the straight skeleton algorithm, based on angle bisectors of adjacent sides and non-adjacent sides. This method is said to be easy to implement.
4OSME	"Configuration spaces for flat vertex folds" T. Hull [69]	This paper focuses on combinatorial issues with flat vertex folds. These are also shown to hold for folded cones.

Table 2.2 – continued on next page

Continued from previous page

Conference	Papers	Summary
4OSME	"Concepts and modelling of a tessellated molecule surface" E. Halloran [70]	Focusing on the surface of You's expandable Origami stent, this paper explores the crease pattern and its folded polyhedral surface without looking at its cylindrical structure.
4OSME	"Facet ordering and crease assignment in uniaxial bases" R. Lang & E. Demaine [71]	This work provides the first simple algorithm for crease alignment in a uniaxial base. A uniaxial base is a folded structure containing distinct flaps protruding from the base perpendicular to a line referred to as the axis of the base.
4OSME	"Folding curves" R. Geretschlager [72]	This paper looks at some basic results of curved crease folding. The algebraic expressions involved are shown to be quite complex and require a large amount of computing power for even the simplest of curves.
4OSME	"Fujimoto, number theory, and a new folding technique" T. Veenstra [73]	In search of a complete Fujimoto division, dividing the paper into nths providing divisions at all n, this paper looks at the numbers for which this is possible and provides several different viewpoints. The mathematical algorithm also provides an interesting application for expressing numbers in different bases.
4OSME	"Integer programming models for flat Origami" G. Konjevod [74]	The work tries to find bounds on the size of a $k * k$ chessboard which could be constructed using a unit square. Using integer programming the authors were able to define many constraints leading to a conjecture that a reduction factor of 2k would be required for this, but more research is required to prove it.
4OSME	"On the fish base crease pattern and its flat foldable property" H. Azuma [75]	This introductory approach provides ways to apply conic sections to the composition of flat foldable crease patterns. Several fish base patterns are shown to exist and be foldable.

Table 2.2 – continued on next page

Conference	Papers	Summary
4OSME	"One-, two-, and multi-fold Origami axioms" R. Alperin & R. Lang [76]	The authors extend the axioms to allow folding of two creases at once. Using the same approach as Lang's proof for the completeness of the seven 1-fold axioms, this paper finds 489 axioms and shows some mathematics which can be performed with them including solving a quartic. The paper finally considers higher level axioms and uses three fold axioms to solve a quintic.
4OSME	"Origami, isometries, and multilayer tangram" E. Frigerio [44]	This paper provides several mathematical designs which can be used to teach a variety of mathematics.
4OSME	"Surface transitions in curved Origami" J. Mosely [77]	With a mathematical focus based upon a sheet folding which must have zero Gaussian curvature at all points Mosely shows how it is possible to design models using curved creases where the folded shape can be predicted instead of by the usual experimentation method.
4OSME	"The method for judging rigid foldability" N. Watanabe & K. Kawaguchi [78]	Two methods are shown for judging rigid foldability, a diagram based method and a numerical method. Necessary conditions are shown and sufficient conditions are mentioned for future research.
4OSME	"The power of multifolds: folding the algebraic closure of the rational" T. Chow & C. Fan [79]	This paper looks at what numbers are constructible with Origami. Using multi fold axioms the paper shows that the algebraic closure of the rational numbers can be folded. It also shows irrational numbers cannot be constructed using axiomatic construction.
5OSME	"A combinatorial definition of 1D flat-folding" H. Kawasaki [80]	Using a combinatorial definition of one-dimensional Origami to provide an improved, more rigorous, proof of a theorem that states that any one-dimensional flat-foldable Origami can be folded using the local operations of crimping and end fold.

Table 2.2 – continued on next page

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Conference	Papers	Summary
5OSME	"A general method of drawing biplanar crease patterns" H. Cheng [81]	This paper provides a description of a prismatoid gadget algorithm to fold polyhedra with vertices in two planes: biplanars.
5OSME	"A note on operations of spherical Origami construction" T. Kawasaki [82]	This paper attempts to use set theory to describe Origami operations. 8 operations are found when Kawasaki creates operations to be used in spherical Origami construction. This spherical form of Origami is defined as folding a sphere in half through a great circle which divides two hemispheres.
5OSME	"Degenerative coordinates in 22.5 degree grid system" T. Tachi & E. Demaine [83]	Tachi and Demaine characterize the degeneracy of points which are constructible in a proposed 22.5 degree grid system. They show that using two types of operation any desired point in the grid system can be constructed.
5OSME	"Every spider web has a simple flat twist tessellation" R. Lang & A. Bateman [84]	A review of the mathematics governing Origami tessellations and the history of their development. It shows that every spider web, a graph with an orthogonal interior dual with no edge crossings, can be turned into a flat foldable shrink rotated tessellation using the vertices of the orthogonal interior dual for centres of rotations.
5OSME	"Flat vertex fold sequences" T. Hull & E. Chang [85]	A flat vertex fold is referred to as a single vertex from a crease pattern, as the mathematics of a flat vertex fold is almost fully understood the authors ask if it is completely determined by the sequence of angles.
5OSME	"Flat-unfoldability and woven Origami tessellations" R. Lang [86]	An algorithm used to create the effect of a woven structure is presented and implemented in Mathematica. Several examples are also shown.

Table 2.2 – continued on next page

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Conference	Papers	Summary
5OSME	"Introduction to the study of tape knots" J. Maekawa [87]	This paper describes a preliminary study of tape knots, shapes produced when a strip of paper is tied in a knot. The paper shows the range of shapes which can be created and enumerates the possible versions in terms of layering.
5OSME	"Origami alignments and constructions in the hyperbolic plane" R. Alperin [88]	Alperin defines axioms for Origami in the hyperbolic plane, these axioms are used to solve several mathematical problems such as solving quartics or quintics.
5OSME	"Precise division of rectangular paper into an odd number of equal parts without tools: an Origamics exercise" K. Haga [89]	Haga presents an alternative, independently generated, version of Lang's crossing diagonals technique. It enables a folder to find equal divisions of a sheet. This method works for any rectangle and presents some notable symmetries.
5OSME	"Stamp foldings with a given mountain-valley assignment" R. Uehara [90]	Work is done on enumerating folded states, and the number of ways of folding a crease pattern, is shown to be very large. Uehara proposes that finding the optimal solution could be NP-hard.
5OSME	"The speed of Origami constructions versus other construction tools" E. Tramuns [91]	Geometrography is the measure of the speed of Euclidean constructions. This paper defines a method for comparison with Origami construction which is done by creating an Origami tool.
5OSME	"Two folding constructions" R. Orndorff [92]	The author describes a folding method using Descartes' construction of a segment of length square root n. In addition a method is described for folding segments of length equal to square roots of the reciprocal of integers.
5OSME	"Universal hinge patterns for folding orthogonal shapes" N. Benbernou, E. Demaine, M. Demaine & A. Ovadya [93]	This paper presents an algorithmic proof that an $n * n$ square tilling of a simple hinge pattern can construct all face to face gluings of O(N) unit cubes.

Table 2.2 – continued on next page

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Conference	Papers	Summary
5OSME	"Variations on a theorem of Haga" E. Frigerio [94]	Frigerio presents work on Haga's theorem, including use of multiple creases or use of rectangles instead of squares.
6OSME	"A survey and recent results about common developments of two or more boxes" R. Uehara [95]	This paper summarises the work done on producing nets which can fold in different ways to produce different boxes. All possible variations of nets folding into 2 shapes with 30 squares are shown and the smallest net for 3 different boxes found contains over 500 unit squares.
6OSME	"A new scheme to describe twist-fold tessellations" T. Crain [96]	A language to describe simple twist fold arrangements is shown; this method can be used to provide new designs. It is also shown that there exists an infinite number of basket weave tessellations.
6OSME	"Abelian and non-abelian numbers via 3D Origami" J. Prieto & E. Tramuns [97]	This work aims to introduce folding axioms for 3D construction to increase the range of constructions possible with the one fold axioms. Some axioms are listed but the list is not comprehensive. These are shown to construct Abelian numbers and those numbers whose Galois group is not solvable.
6OSME	"Characterization of curved creases and rulings: Design and analysis of lens tessellations" E. Demaine, M. Demaine, D. Huffman, D. Koschitz & T. Tachi [98]	Some mathematical modelling of curved crease Origami is proposed. A general theory of curved Origami is stated. However, although still a long way off this work should enable mathematical design of curved crease works.
6OSME	"Colour symmetry approach to the construction of crystallographic flat Origami" L. De las Penas, E. Taganap & T. Rapanut [99]	In this paper the authors consider the unassigned crease pattern that is the orbit of its generating unit under a plane crystallographic group. On a flat foldable crease pattern a folding assignment is created using a colouring of the cease pattern.

Table 2.2 – continued on next page

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Conference	Papers	Summary
6OSME	"Colouring connections with counting mountain-valley assignments" T. Hull [100]	A survey of recent approaches for enumerating valid mountain-valley assignments which are linked to colouring problems. This is done for two colourable flat Origamis.
6OSME	"Equal division on any polygon side by folding" S. Chen [101]	'The median boundary method' is presented as an extension of the crossing-diagonals method. Although there is no advantage when used on a square, it can be expanded to any polygon.
6OSME	"Geometric and arithmetic relations concerning Origami" J. Guardia & E. Tramuns [102]	Presented is a general purpose formal language for the approximation of geometrical instruments. Introducing the concept of a tool formalises an instrument as a set of axioms, the geometric and virtual equivalence of tools is considered.
6OSME	"Graph paper for polygon-packed Origami design" R. Lang & R. Alperin [103]	Formal conditions are proposed for doubly periodic graph paper which is defined as 'periodic graph paper on which any uniaxial base can be designed with guaranteed non-dense axial conditions'. A new Origami grid, the Sterling grid, is presented.
6OSME	"Interactive construction and automated proof in Eos system with application to knot fold of regular polygons" F. Ghourabi, T. Ida & K. Takahashi [104]	Presented are constructions of knot folds for the regular pentagon and the regular heptagon which have been created by logically stating the geometric properties of the knots.
6OSME	"Locked rigid Origami with multiple degrees of freedom" Z. Abel, T. Hull & T. Tachi [105]	The authors look at the hypothesis that any triangular mesh origami can be continuously folded from an unfolded state, as this is true for many examples. They show that for one specific rigid origami model it is not foldable and thus show that triangulating a non-rigid fold pattern is not always enough to attain a rigid folding.

Table 2.2 – continued on next page

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Conference	**Papers**	**Summary**
6OSME	"On pleat rearrangements in Pureland tessellations" G. Konjevod [106]	A class of Origami constructions is described which creates woven effects in the paper from Pureland construction; a proposed folding sequence optimisation method is shown to be NP-hard.
6OSME	"Pentasia: An aperiodic Origami surface" R. Lang & B. Hayes [107]	A construction method for an aperiodic surface is presented. A section of this surface, Pentasia, is folded to demonstrate this method.
6OSME	"Rigid folding of periodic Origami tessellations" T. Tachi [108]	This paper shows that periodic triangulated crease patterns can be rigid foldable to a cylindrical form. They are shown to have two degrees of freedom, unlike the Miura-Ori fold which has one degree of freedom.
6OSME	"Rigidly foldable Origami twists" T. Evans, R. Lang, S. Magleby & L. Howell [109]	An evaluation method for rigid foldability is given. It is shown that a triangle twist cannot be rigidly foldable but a square twist can be. Also, a method is given to calculate twist angles for rigidly foldable regular polygonal twists.
6OSME	"Scaling any surface down to any fraction" E. Demaine, M. Demaine & K. Qaiser [110]	It investigates if any polyhedral surface can be scaled down by a specific fraction such that the shape is not deformed. Individual faces on the polyhedral structure are treated as connected thus pleats in the paper must align. This paper provides a method for scaling any quadrilateral but it is not known if this face can merge with other faces to solve the whole problem.
6OSME	"Spiderwebs, tilings, and flagstone tessellations" R. Lang [111]	Lang proposes an algorithm for the design of flagstone tessellations; this algorithm has been implemented in Mathematica and produces crease patterns and folded form rendering from a plane graph.

Table 2.2 – continued on next page

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Conference	Papers	Summary
6OSME	"Thick rigidly foldable structures realized by an offset panel technique" B. Edmondson, R. Lang, M. Morgan, S. Magleby & L. Howell [112]	The authors suggest an alternative solution to the tapered panels proposed by Tachi. They suggest using offset panels which hinge such that when they fold they have the same kinematics as the zero thickness models and the full range of motion. Several examples with advantages and limitations are shown.
6OSME	"Weaving a uniformly thick sheet from rectangles" E. Davis, E. Demaine, M. Demaine & J. Ramseyer [113]	The problem of weaving an infinite sixed sheet from finite length strips such that the resulting sheet has uniform thickness and the individual pieces are locked together is considered. Some examples are shown.

Table 2.2: Summaries of OSME papers presented in the area of: The Mathematics of Origami

2.1.2 Computational Origami deconstruction, design and diagramming

Origami initially consisted only of a small range of different foldable models. During the expansion of the field and due to the increased interest, designing Origami has become more accessible. To assist, many have looked to computing to enable more complex designs as well as to optimise designs for scientific applications.

Many traditional models are made from bases. These simple base shapes provide a number of manipulatable flaps which can be folded into a wide variety of models. The development of new bases was the key to unlocking the design potentials of Origami. This is done using a range of techniques such as circle packing. This area has been worked on by many Origamists most notably Lang with his software TreeMaker [114].

An entirely new approach was found based on tucking molecules and is presented by Tachi [115]. This provides a method which is hoped to be shown to be able to produce any 3D shape with the program Origamizer [115].

More bespoke programmes have been developed for specific design applications such as Tess, a tessellation design tool presented in 2001 at 3OSME by Bateman [116]. Others have looked at diagramming; Ida et al. presented Eos [117] at 4OSME in 2006 as a tool for diagram creation, at the same time as Fastag presented eGami [118], a similar program. More recently Akitaya et al. [119] show how they can generate diagrams direct from the crease patterns.

This area has received a particularly large amount of attention, the 'Computational Origami Movement' can be justified as starting with Bern and Hayes paper 'The complexity of flat origami' in the proceedings of the 7th Annual ACM-SIAM Symposium on Discrete Algorithms [120]. This grew into a quest to answer questions revolving around 'can a computer efficiently solve problems that arise from folding processes?'. More recently in 2007, Demaine published a book 'Geometric Folding Algorithms', this book is some of the only graduate level literature looking at mathematics and Origami which focuses on Computational Origami [16].

Conference	**Papers**	**Summary**
2nd meeting	"The technique to fold free flaps of formative art 'Origami'" F. Kawahata [121]	A method based on circle packing and similar to the tree method. The circles are placed around the edges of the square which makes crease patterns easier to work out. This method is similar to the tree method in that it also allows points to emanate from locations on the tree other than a central point.
2nd meeting	"The tree method of Origami design" R. Lang [114]	This paper presents the tree method, an extension of circle packing to enable crease pattern design such that not all flaps are emanating from a single location. The authors computer program is discussed which can find a local optimum for a crease pattern using this method. This must be one of the most important advances in computational Origami design.
3OSME	"A disk-packing algorithm for an Origami magic trick" M. Bern, E. Demaine, D. Eppstein & B. Hayes [122]	An algorithm using disk packing to find the crease patterns of solutions to the one-straight-cut theorem. The solutions found are foldable but the algorithm does not provide a folding method, the difficulty of finding this depends on the complexity of the shape to be cut. The authors also suggest simplifications to the method and its generalisation to cutting out an entire planar graph as well as a possible application to the problem of inside-outside non-obtuse triangulation and conforming Delaunay triangulation problems.
3OSME	"Computer tools and algorithms for tessellation design" A. Bateman [116]	Using a pre-existing algorithm to construct representations of folded states the author presents their software, 'Tess', an Origami tessellation generation and design program and they discuss how it uses this to generate the light pattern and crease lines of the folded state.

Table 2.3 – continued on next page

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Conference	**Papers**	**Summary**
3OSME	"Recent results in computational Origami" E. Demaine & M. Demaine [123]	A comprehensive introduction and explanation of many computational Origami theorems including a discussion of Robert Lang's TreeMaker software, which uses the tree method of Origami design, and several of the theorems worked on extensively by the authors such as the one-straight-cut theorem dating from 1721 and the silhouette folding problem first formally stated by Bern and Hayes which was proved to always hold by the authors in a previous paper. This paper also covers flattening polyhedra, foldability and map folding.
3OSME	"The foldinator modeller and document generator" J. Szinger [124]	The author presents their software 'Foldinator' which can be used to create folding instructions for Origami models. They include a discussion of the development and use of the program which models 2D sheet materials folding through a 3D space.
4OSME	"3D Origami design based on tucking molecules" T. Tachi [115]	Here Tachi provides a method which is hoped to be shown to be able to produce any 3D shape using pleats intersecting at tucking molecules.
4OSME	"Computational complexity of a pop-up book" R. Uchara & S. Teramoto [125]	The authors present a model for the construction of pop-up books and use this to show that the problems of opening and closing pop-up books are NP-hard.
4OSME	"Computational Origami system EOS" T. Ida, H. Takahashi, M. Marin, A. Kasem & F. Ghourabi [117]	This paper describes Eos, an E-Origami software environment for computational Origami. The software provides a simulation of the folds, shows some geometric properties and is also available in a partial form through a web browser.

Table 2.3 – continued on next page

Continued from previous page

Conference	Papers	Summary
4OSME	"Computer Origami simulation and the production of Origami instructions" T. Lam [126]	Based on a program for Origami simulation by S. Miyazaki this paper presents an extension which enables automatic diagram generation. The paper points out that the program cannot work on layers which are covered thus it is difficult to use this for complex models.
4OSME	"Construction of 3D virtual Origami models from sketches" H. Shimanuki, J. Kato & T. Watanabe [127]	Provides a method for three dimensional Origami design based on a two dimensional drawing. It generates a tree from the diagram and uses pre-existing algorithms to generate crease patterns.
4OSME	"eGami: Virtual paper folding and diagramming software" J. Fastag [118]	The authors present their software eGami; this enables a user to simulate the sequential folding of flat Origami models in real time. The programme simulates a real sheet of paper and comes with common inbuilt folds such as sink or reverse.
4OSME	"Graphics transformation of Origami models" L. Zamiatina [128]	The paper presents a method by which it is possible to use an Origami model as a seed to a computer generated artistic image. This is done by transforming the polygons in the image of the folded model.
4OSME	"Recognition, modelling, and rendering method for Origami using 2D bar codes" J. Mitani [129]	This paper proposes two methods for Origami modelling. The first constructs an Origami model and the second renders this to a screen by slightly perturbing the geometry to make it easier to see the configuration of the parts in the model.
4OSME	"Simulation of rigid Origami" T. Tachi [130]	It discusses the mathematics behind a proposed system for simulating folding motion using projection on to the constrained space based on a rigid Origami model using trajectories.

Table 2.3 – continued on next page

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Conference	**Papers**	**Summary**
5OSME	"A CAD system for diagramming Origami with prediction of folding processes" N. Tsuruta, J. Mitani, Y. Kanamori & Y. Fukui [131]	Proposed is a Computer Aided Design (CAD) system for flat foldable crease patterns. This programme also predicts possible future folds which could be made at each step. The system is shown to be difficult with complex Origami. However it predicts steps well with simpler models.
5OSME	"A simulator for Origami-inspired self-reconfigurable robots" S. Gray, N. Zeichner, M. Yim & V. Kumar [132]	The authors present a computer programme which can model Origami for use in programmable matter structures. It allows crease patterns to be viewed folding in real time, enabling identification of problems.
5OSME	"Circle packing for Origami design is hard" E. Demaine, S. Fekete & R. Lang [133]	Presented is a proof stating, within the general tree method for Origami design, the circle packing optimisation problem is NP-hard. This is shown based on a reduction of the 3-partition. This paper also shows that it is possible to guarantee the existence of a feasible solution if given a larger piece of paper.
5OSME	"Development of an intuitive algorithm for diagramming and 3D animated tutorial for folding crease patterns" H. Akitaya, M. Ribeiro, C. Koike & J. Ralha [134]	The authors describe their algorithm which folds a crease pattern generated using circle/river packing. It generates a vector image of the folded state of the model which can be edited for diagramming purposes.
6OSME	"Filling a hole in a crease pattern: Isometric mapping from prescribed boundary folding" E. Demaine & J. Ku [135]	The authors show that the "hole" problem will always have a solution for polygonal input boundaries folded at finitely many points if the input folding is non-expansive. Proposed is a polynomial time algorithm for finding isometric mappings consistent with prescribed boundary mappings.

Table 2.3 – continued on next page

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Conference	Papers	Summary
6OSME	“Simple flat Origami exploration system with random folds” N. Tsuruta, J. Mitani, Y. Kanamori & Y. Fukui [136]	In search of a design method for simple Origami models with only a few folds the authors present a random fold generator tool which creates a series of random Origami models from which specific designs can be chosen.
6OSME	“Unfolding simple folds from crease patterns” H. Akitaya, J. Mitani, Y. Kanamori & Y. Fukui [119]	This is a concerned with generating folding sequences from crease patterns. The authors identified methods for identifying possible instructions from a crease pattern including many simple folds such as outside and inside reverse folds.

Table 2.3: Summaries of OSME papers presented in the area of: Computational Origami deconstruction, design and diagramming

2.1.3 Exploration, design and colouring of Origami polyhedra

Polyhedra have been a mathematical curiosity for thousands of years with Pythagoras loosely credited for discovering the dodecahedron [137]. They require a high level of spatial skill to visualise. Even the simplest are difficult to construct. However, the creation of polyhedra using Origami can be unexpectedly simple. This enables them to be studied and explored with ease. Gurkewitz [138] presents ideas linking to a mathematics curriculum and shows how, with an Origami polyhedron model, it is possible to explore its properties to understand the geometric concepts behind it. Ishihara [139] and Cuccia [140] also provide examples of using polyhedra to better understand crystal and chemical structure respectively.

Many of the OSME papers present an analysis of a particular module to explore which polyhedra can and cannot be made such as Strobl's paper [141] on knotted tape. This is often extended to explore if new and unknown polyhedra can be created such as Horii's work [142] on finding convex deltahedra. In particular, Lang's exploration [143] has led to a better understanding of polypolyhedra and most recently work on using curved creases to create polyhedral structures for architectural use.

Polyhedron flattening is also studied; in many real life situations polyhedra are flattened, from folding away a paper bag to folding an airbag in a car. This is not always possible using rigid Origami. However, Abel et al. [144] provide a possible solution using cuts and, taking a more applied approach, Balkcom et al. [145] analyse the folding of shopping bags.

The exploration of choosing aesthetically pleasing colourings for these polyhedra leads Morrow [146] to graph theory applications, and work by Hull [147] even suggests the possibility that Origami polyhedra may help in the search for a shorter and more elegant proof of the four colour theorem [148].

Due to the physical properties of paper, it is also possible to create inaccurate polyhedra in which the angles used are precise enough to construct the polyhedra but in fact are not accurate; Mosely [149] explored some examples of this.

Conference	Papers	Summary
2nd meeting	“Finding convex deltahedron through Origami” Y. Horii [142]	Considering deltahedron, convex polyhedral shapes with only equilateral triangles for faces, the author presents an analysis of their shape and how to enumerate each of them using Origami.
2nd meeting	“Modular Origami polyhedra” R. Gurkewitz [138]	Looking at Origami made from multiple units and how it is possible to adapt these units in a systematic way to make different polyhedra; this paper also discusses how this can be incorporated into a curriculum.
2nd meeting	“Molecular modelling of fullerenes with modular Origami” L. Cuccia, R. Lennox & F. Ow [140]	Some models of fullerenes made with modular Origami are considered. They are used as a way to explain the complex arrangements of these fullerene molecules.
2nd meeting	“Origami model of crystal structure, II. Spinel and corundum structures” S. Ishihara [139]	Models of crystal structures are presented. They can be made simply and demonstrate the complex nature of these structures in an easily understandable, visual way.
2nd meeting	“Planar graphs and modular Origami” T. Hull [147]	An introduction to basic graph theory concepts, in particular on colourability of polyhedral graphs. It uses these to show how knowledge of planar graph theory can lead to a deeper understanding of polyhedral structures and modular Origami.
2nd meeting	“Polyhedron Origami: a possible formulation by ‘simple units’” M. Kawamura [150]	Looks at units for modular Origami as belonging to several classes; one of these is a ’simple unit’. It defines different simple units using two angles and uses this to see the wide variety of polyhedra they can construct.
2nd meeting	“Symmetry in two and three-dimensional Origami with knotted tape” H. Strobl [141]	Looks at folding a long strip of paper to create polyhedral structures, which appear woven in nature, based on creating a knot in the paper.
2nd meeting	“The cube story told in modular-Origami language” T. Fuse [151]	This is a survey of the different ways to fold an Origami cube. It looks at ways using two to six sheets and then also looks at more complex models which make connected cubes or variations on cubes.

Table 2.4 – continued on next page

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Conference	Papers	Summary
2nd meeting	"The platonic solids and its interrelated solids" S. Fujimoto [152]	Provides photos of a collection of models of Origami structures of the platonic solids and molecular structures. The models are made from similar units which make up edges in the polyhedra and bonds in the molecular structures.
3OSME	"Circular Origami: A survey of recent results" E. Knoll [153]	This is a deceptively short survey. It addresses the use of folding circular sheets for both mathematics and education. Of note is the experimentation on an Origami 'endo-pentakis-icosidodecahedron', referred to as Geraldine, upon which the research was initially based.
3OSME	"Exploring the possibilities of a module" K. Burczyk & W. Burczyk [154]	This paper looks at polyhedral models made using R. Gurkewitz's edge module. Looking at variations of this module it asks which polyhedra can and cannot be constructed using this and similar modules.
3OSME	"Polypolyhedra in Origami" R. Lang [143]	This is an exploration of Origami models of multiple intersecting polyhedra with no intersecting edges called polypolyhedra. Through rigorous mathematical modelling 54 polypolyhedra are found and some of their properties are explored. The author also adapts a design for the five intersection tetrahedra to produce Origami models of the 4 homo-orbital polypolyhedra.
3OSME	"The validity of the Orb, an Origami model" J. Mosely [149]	Analyses the mathematical accuracy of Origami models of mathematical structures such as polyhedral models, looking specifically at a model of intersecting cubes and an orb.
3OSME	"Using graphs to colour Origami polyhedra" C. Morrow [146]	A method by which you can use Hamiltonian cycles in polyhedral graphs to help choose colours for Origami models that will be aesthetically pleasing.

Table 2.4 – continued on next page

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Conference	Papers	Summary
4OSME	"Folding paper shopping bags" D. Balkcom, E. Demaine, M. Demaine, J. Ochsendorf & Z. You [145]	The authors provide a mathematical analysis of several folding methods to flatten a paper shopping bag focusing on the durability of the bag given a specific folding sequence.
4OSME	"How many ways can you edge-colour a cube?" C. Morrow [155]	This paper looks at the problem of choosing colours for an Origami cube made with edge units. The author systematically looks as possibilities and categorises them based on their properties.
4OSME	"One-dimensional Origami: polyhedral skeletons in dance" K. Schaffer [156]	Written due to the increasing interest in DNA-Origami, but based on exploration of creation of polyhedra through dance routines, this paper explores how one dimensional Origami can construct a range of polyhedra.
4OSME	"The Celes family of modular Origami" M. Kawamura [157]	A discussion of Origami models of polyhedra made from strips of paper. Initially joined at the ends and expanding to longer strips with multiple connections, the paper presents a method for generating variations of these models.
6OSME	"Curve-folding polyhedra skeletons through smoothing" S. Chandra, S. Bhooshan & M. El-Sayed [158]	The authors present their work on folding the skeletons (edges) of polyhedra using curved creases in metals. For these polyhedra they show how several variables can change the output and describe the method by which they are generated.
6OSME	"Rigid flattening of polyhedra with slits" Z. Abel, R. Connelly, E. Demaine, M. Demaine, T. Hull, A. Lubiw & T. Tachi [144]	It is not always possible to flatten polyhedra in a rigid way; the authors provide a solution for a tetrahedron using a small slit in the shape which enables polyhedra flattening with one degree of freedom to be possible.

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Conference	**Papers**	**Summary**
6OSME	"Symmetric colourings of polypolyhedra" S. Belcastro & T. Hull [159]	Through exploration and analysis of the polypolyhedra found by Lang, the authors provide an analysis of some of the ways in which they can be symmetrically coloured.
6OSME	"Two calculations for geodesic modular works" M. Kawamura [160]	This paper looks at several colour combinations with implementation methods for creating modular geodesic spheres; it also looks at enumerating the number of edges of these polyhedra.

Table 2.4: Summaries of OSME papers presented in the area of: Exploration, Design and Colouring of Origami Polyhedra

2.1.4 Origami applications in science and technology

The applications of Origami are vast in number and range from space science to medicine. These applications use Origami techniques, to gain benefits in various ways, including; increasing strength in 3D structures, enabling microscopic constructions, providing economic and environmental benefits, increasing production speed and generating new shapes which provide better sound insulation.

Packaging is one of the most obvious and intuitive examples of an Origami application. Several authors have looked at how the techniques used in Origami as an art form can solve problems, such as the Eco-Origami pot design by Fuse [161], which folds and locks into place without the use of glue, and curved designs for packaging by Mitani [162], which can add both artistic design and strong structure to packaging.

Origami techniques have been explored to produce many structural benefits ranging from Filipov's [163] mathematical analysis of stiffness and flexibility in Origami structures to Klett [164] and Drechsler's [165] analysis of how these structures could be developed. Of special note is the Miura-Map fold which is an alternative method for folding a map which enables it to be opened and closed by holding only two opposite corners [166]. This turns out to have many applications as a structural crease pattern including in space science and, in a simpler form, it can be used to model biological folding of plants. This fold is also the catalyst for the study of rigid Origami which has been worked on by many including Tachi [167] who proposed the concept of tapered panels enabling rigidity to be maintained in thick Origami.

There has been much research into self-assembling autonomous folds for small scale folding including Mehner's work [168] on tissue engineering and the work of Ghosh [169] on folding thin metal sheets.

In this area there have been many publications outside of the OSME conferences, including in journals from engineering conferences and mathematical publications. There have also been several other conferences similar to OSME but focused on applications, one example being the workshop on Origami engineering held over three days in 2014 at the University of Illinois [170] although this is still a much smaller event than the OSME series.

Conference	Papers	Summary
1st meeting	"A note on intrinsic geometry of Origami" K. Miura [166]	Using mathematical modelling to explain the characteristics of the Miura map fold. To do this the author looks at a spherical representation of a vertex in a crease pattern and at convexity/concavity properties.
1st meeting	"Map fold a La Miura style, its physical character and application to the space science" K. Miura [171]	Analysis of the foldability and flexibility of the Miura map Origami fold and discussion of its application in solar sails and membrane design.
1st meeting	"Towards the realisation of the Miura fold by machine" W. Oosterbosch [172]	Comments about a Miura map folding machine created by the author including comments on the speed of folding and possible future developments.
1st meeting	"Wire bending" M. Mendes France [173]	Work on three dimensional curves formed by iteratively bending a wire. The paper proves that many of these curves are bounded and all are aperiodic.
2nd meeting	"Folded and unfolded Nature" B. Kresling [174]	Analysis of the geometry of both biological folding (leaves and wings) and folded structures used in space (sails) as well as several other examples in an attempt to better understand morphological principles and to define precise optimisation criteria in biology.
2nd meeting	"Folding of uniform plane Tessellations" T.Tarnai [175]	Presents results from introductory research into the possible folded solutions of the Origami $(3\hat{2}.4.3.4)$ tessellation as an analogue for the buckling patterns of axially compressed box columns.
2nd meeting	"How the Origami model explains the theory of Kikujutsu" T. Iwasaki [176]	Kikujutsu is introduced as a traditional Japanese carpenter's technique. This paper shows an Origami model which can be used to explain the theory behind this method.
2nd meeting	"Origamic architecture" M. Chatani [177]	This Japanese paper is presented with an English introduction describing pop-up postcards; the paper then proceeds to analyse the mathematics of pop-up structures.

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Conference	**Papers**	**Summary**
3OSME	"Folded tubes as compared to Kikko ("tortoise shell") bamboo" B. Kresling [178]	Research into the nature-inspired design of folded cylindrical hollow structures for engineering purposes. By looking at bamboo which is abnormally grown and recreating these shapes in paper, it presents nature-inspired rules for Origami design optimisation.
3OSME	"Origami pots" T. Fuse, A. Nagashima, Y. Ohara & H. Okumura [161]	A design for a glue-free, folded paper pot, which has been created in order to reduce waste and to replace the environmentally harmful polystyrene pots used in the fast food industry.
3OSME	"Self-assembling global shape using concepts from Origami" R. Nagpal [179]	The author suggests an Origami shape language for instructing a sheet of identically programmed flexible autonomous cells which fold into three dimensional structures. Some examples are also shown.
3OSME	"The application of Origami science to map and atlas design" K. Miura [180]	A summary of two separate solutions to problems with map design. This paper looks at the advantages of the Miura map fold for easy folding and unfolding as well as an alternative solution which overcomes the North-South navigation problem for multi-page maps.
3OSME	"To fold or to crumple?" B. DiDonna [181]	Through the observation that a crumpled sheet of paper follows the same rules of methodically folded Origami, the paper explores the physics of crumpling, using the geometry of a sheet and Hooke's law of elastic deformation along with the principles of conservation of energy. The paper attempts to show why crumpling and methodically folding produce similar results.
4OSME	"A brief history of Oribotics" M. Gardiner [182]	Combining Origami and Robotics this paper explores the work that has been done to create self-folding structures.
4OSME	"Airbag folding based on Origami mathematics" C. Cromvik & K. Eriksson [183]	The development on an algorithm which models the accurate geometry of a folded airbag is presented. Numerical examples and the mathematics behind the model are discussed.

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Conference	Papers	Summary
4OSME	“Expandable tubes with negative Poisson ratio and their application in medicine” Z. You & K. Kuribayashi [184]	This paper presents work on an Origami stent design which exhibits transverse expansion when stretched. Initially analysing a pre-existing method this paper also looks at helical arrangements which improve deployability.
4OSME	“Origamic architecture in the cartesian coordinate system” C. Cheong, H. Zainodin & H. Suzuki [185]	For the application of making paper models of buildings this paper develops formulas for the Cartesian coordinates of points modelling the creation of 3D Origami pop up constructions.
4OSME	“Origami-Inspired self-assembly” G. Pickett [186]	Looking at MMES, Micro-Electrical-Mechanical-Systems, this paper develops self-folding Origami tessellation structures.
4OSME	“The science of Miura-on: A review” K. Miura [187]	A review of the mathematics and physics research that has been done on the Miura-Origami map fold and other variants. It also looks at successful applications including maps, space technology and biology.
5OSME	“A design method for axisymmetric curved Origami with triangular prism protrusions” J. Mitani [162]	A method for wrapping an object using curved crease Origami which can be designed and visualised on a computer. This method can have many applications from packaging to lamp shade design.
5OSME	“Designing technical tessellations” Y. Klett & K. Drechsler [165]	The authors present work on the development of tessellations for use in structural applications from packaging to architecture, using both top down and bottom up strategies. Several examples are shown and analysed.

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Conference	Papers	Summary
5OSME	"Foldable parylene Origami sheets covered with cells: Toward applications in bio-implantable devices" K. Kuribayashi-Shigetomi & S. Takeuchi [188]	A description of a method to produce foldable micro-sized Origami structures by using poly p-xylyene which is biodegradable and flexible. These are to be used in highly bio-compatible implantable medical devices to minimise invasion in surgical procedures.
5OSME	"Folding a patterned cylinder by rigid Origami" K. Wang & Y. Chen [189]	This paper looks at several possible solutions for a one degree of freedom rigid foldable closed patterned cylinder. By mathematically modelling the crease patterns vertices as a spherical linkage the rigid transformations of the sheet are modelled. The authors suggest the use of this in applications in engineering such as energy absorbing devices.
5OSME	"Hands free microscale Origami" N. Bassik, G. Stern, A. Brafman, N. Atuobi & D. Gracias [190]	This paper provides techniques for producing millimetre or microscale Origami patterns which fold up autonomously and are made from thin metal sheets. They show how using thousands of these units a global shape can be achieved with greater complexity; however, production is currently challenging.
5OSME	"Origami folding: A structural engineering approach" M. Schenk & S. Guest [191]	Introduces folded textured sheets as a novel engineering application of Origami. This paper aims to extend the range of structural engineering applications of Origami. Some mechanics are explored for a pin jointed bar framework model.
5OSME	"Rigid-foldable thick Origami" T. Tachi [108]	Presented is a method which enables kinetic behaviour of an ideal Origami surface to be maintained in a rigid foldable Origami structure with thick panels. This is done using tapered panels with predefined maximum and minimum folding angles.

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Conference	Papers	Summary
5OSME	"The Origami crash box" J. Ma & Z. You [192]	Proposed is an Origami folding pattern for use as a high performance tubular energy absorber. It is shown to have increased mean crushing force and reduced peak crushing force when compared with conventional square tubes.
6OSME	"A study on crash energy absorption ability of lightweight structures with truss core panel" Y. Yang, X. Zhao, S. Tokura & I. Hagiwara [193]	This study looks at lightweight truss core panels and proposes the use of an additional reinforced part which would be used to maximise the crash energy absorption under a weight control condition. It is shown that with this addition the structure is able to absorb more energy compared with a structure of the same mass without the addition.
6OSME	"Comparison of compressive properties of periodic non-flat tessellations" Y. Klett, M. Grzeschik & P. Middendorf	Looking at the development of new core structures for application in sandwich constructions the authors show the results from a series of tests on the strength of these structures.
6OSME	"Configuration transformation and manipulation of Origami cartons" J. Dai [194]	Presented is a matrix operation model relating distinct topological figuration states during manipulation and folding of Origami cartons. The work aims to improve automation in packing applications.
6OSME	"Cosmological Origami: Properties of cosmic-web components when a non-stretchy dark-matter sheet folds" M. Neyrinck [195]	Exploring dark matter using Origami, a polyhedral collapse model is suggested in which nodes are created at the locations of galaxies and twist folds cause overlapping regions.
6OSME	"Demands on an adapted design process for foldable structures" S. Hoffmann, M. Barej, B. Gunther, M. Trautz, B. Corves & J. Feldhusen [196]	Analysing student projects, in this paper, a summary of several challenges to the construction of deployable folds are presented. Although deployable grid structures have been used for architecture, they are not commonplace due to the difficulties of implementing them.

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Conference	**Papers**	**Summary**
6OSME	"Deployable linear folded stripe structures" R. Maleczek [197]	Adapting the hexagonal network of folded strips, which is already in use, this paper aims to create 3D structures. The paper finds that these structures are rigid foldable only when the structures share one centre or a pair of centres and parallel stripes with a single extrusion direction.
6OSME	"Folding augmented: A design method to integrate structural folding in architecture" P. D'Acunto & J. Gonzalez [198]	Formalising the operations around which the proposed design method is built. The authors are able to minimise the complexity of the design process and include the ability to explore the spatial possibilities while controlling structural potentials.
6OSME	"Gravity and friction-driven self-organised folding" G. Filz, G. Grasser, J. Ladinig & R. Maleczek [199]	Ongoing research into self-organised folding of fibre cement and textile concrete sheet elements using gravity and friction as guiding concepts. This is shown to be similar to Origami.
6OSME	"Magnetic self-assembly of three-dimensional microstructures" E. Iwase & I. Shimoyama [200]	Beginning with a look at the theory of how torque acts on magnetic microplates the authors present a method for the construction of microscale 3D structures including actuators and sensors.
6OSME	"Numerical analysis of Origami structures through modified frame elements" K. Fuchi, P. Buskohl, J. Joo, G. Reich & R. Vaia [201]	A study looking at mechanical properties such as compression ratio and mechanical energy of adaptive Origami structures.
6OSME	"On the aesthetics of folding and technology: Scale, dimensionality, and materiality" M. Gardiner [202]	Exploring several examples from folding in DNA to folds in space, this paper is an exploration into how the scale, dimensionality and materiality of Origami change the aesthetics of the folded structure.

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Conference	**Papers**	**Summary**
6OSME	"ORICREATE: Modelling framework for design and manufacturing of folded plate structures" R. Chudoba, J. Van Der Woerd & J. Hegger [203]	By creating a generalised optimisation framework for the design of Oricreate folded plate structures, the author's show how it can be used to support a wide range of designs.
6OSME	"Planning motions for shape-memory alloy sheet" M. Ghosh, D. Tomkins, J. Denny, S. Rodriguez, M. Morales & N. Amato [169]	By adapting an existing motion planning algorithm the authors show it can be used for modelling collision free, gravitationally stable motions of shape metal alloys from a flat state to a three dimensional shape.
6OSME	"Screw algebra based kinematic and static modelling of Origami-inspired mechanisms" K. Zhang, C. Qiu & J. Dai [204]	Presented is an exploration of the kinematics and statics of Origami bases using the characteristics of a single crease and modelling the motions of panels and creases when folding using screw algebra.
6OSME	"Sound-insulating performance of Origami-based sandwich trusscore panels" S. Ishida, H. Morimura & I. Hagiwara [205]	By introducing a theory of sound insulation this paper compares traditional insulating panels with Origami inspired ones. They are both shown to have strengths in different areas.
6OSME	"Structural engineering applications of morphing sandwich structures" J. Gattas & Z. You [206]	The authors present an overview of a set of prototypes which show the capabilities of morphing sandwich structures. Covering the geometric design, fabrication and deployment of these structures, focusing on structural engineering applications.

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Conference	**Papers**	**Summary**
6OSME	"Thin-walled deployable grid structures" J. Ho & Z. You [207]	Presented is a new concept for deployable grid structures which uses the Tachi-Miura polyhedron. Experimentation with this pattern is shown to lead to several variations with a range of applications.
6OSME	"Toward engineering biological tissues by directed assembly and Origami folding" P. Mehner, T. Liu, M. Karimi, A. Brodeur, J. Paniagua, S. Giles, P. Richard, A.Nemtserova, S Liu, R. Alperin, S. Bhatia, M. Culpepper, R. Lang & C. Livermore [168]	Research is presented on the use of Origami to create 3D scaffolds with high controllability for biological tissue engineering. Two new fold patterns are shown which might provide structures suitable for representing tissue in the liver.
6OSME	"Toward optimization of stiffness and flexibility of rigid, flat-foldable Origami structures" E. Filipov, T. Tachi & G. Paulino [163]	The authors present ideas for optimising stiffness of rigid foldable structures based on an expansion of an established model by Schenk and Guest. This is used to study the rigidity of the Miura-Ori and is intended to be a tool allowing researchers to optimise stiffness in Origami systems.

Table 2.5: Summaries of OSME papers presented in the area of: Origami applications in science and technology

2.1.5 Origami applications in art and design

To most folders Origami is an art. Thousands of people enjoy paper-folding simply for its aesthetic qualities. Many papers from the OSME series are based on studies which have primarily artistic goals. Often these are based on a mathematical exploration of a model and its variations such as Demaine's systematic method for generating Origami mazes [208] or Mosely's work on connected cubes [209].

One area which has received great attention is the exploration of variations on the paper crane which have been mathematically modelled in several papers by Kawasaki [210–213]. Unlike in the mathematical Origami explorations where the aim is to push the boundaries of what can be created, many have explored how, by imposing additional rules on folds, an artist can challenge creativity. Others, however, have drawn from this mathematical understanding to design mathematically complex artworks such as Barreto's work on 'MARS' [214] and Sternberg's curved crease work [215]. On a slightly different note Smith suggested a database of all Origami designs [216] of which there are now several databases consisting of tens of thousands of designs [217].

Conference	Papers	Summary
2nd meeting	"Art, Origami and education" J. Smith [216]	Looking at Origami as an art of constraints, the author considers how models could be categorised into a database of all Origami designs. It also discusses the use of Origami in education.
2nd meeting	"Block Origami system" Y. Sato [218]	Written in Japanese, this paper presents the idea that folding one sheet of paper is similar to producing a single Chinese character; which is analogous to creating a word. If, however, we use multiple sheets the possibilities will be more varied, such as putting letters in combinations to make words. Using this, the paper aims to encourage the study and use of multi-sheet folding.
2nd meeting	"Breaking symmetry: Origami, architecture, and the forms of Nature" P. Engle [219]	Using asymmetric shapes found in Nature the author presents their approach to asymmetric Origami design. Four examples are given and discussed for different ways in which the symmetrical design can be broken.
2nd meeting	"Creative Origami 'snow crystals': Some new approaches to geometric Origami" K. Suzuki [220]	Studying several models of snowflakes folded from regular hexagons this paper attempts to classify and describe them. It shows how a seemingly infinite number of different snowflakes can be folded from a similar form. Of note is a figure of 128 possible snowflakes including a notation system to categorise them.
2nd meeting	"Discrete symmetric Origami structures: Dyssigami" M. Litvnov [221]	Suggested is a new type of Origami-like structure in which the paper is cut and then folded or transformed to make the final required shape. The paper is followed by an unindexed note on terminology in Origami.
2nd meeting	"Extruding and tessellating polygons from a plane" C. Palmer [222]	The author uses examples to show the similarities and differences between four different types of Origami tessellations. These are twist, collapse, iso-area and infinite progression.

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Conference	**Papers**	**Summary**
2nd meeting	"Form of Origami" Y. Momotani [223]	Origami models are folded to represent the basic shape of an object. This suggests that a greater understanding of what shapes represent an object might lead to better Origami design. Is it also suggested that this 'primitive and instinctive' ability is required to appreciate the model.
2nd meeting	"Hakari-ori reflective folding" J. Sadoka [224]	Defining methods for producing Hikari-Ori, the folding of crease patterns into low relief shapes which reflect light and cast shadows differently depending on location and viewpoint. Notably, they also state a folding rule they have discovered; this is a special case of Maekawa's theorem.
2nd meeting	"Image game" K. Kasahara [225]	A method for folding Origami models which are made by taking a crease pattern, folding it completely and then unfolding part way to open out the paper into a three dimensional model.
2nd meeting	"Lines meeting on a surface the "MARS" paper-folding" P. Barreto [214]	Presents a method for producing flat foldable tessellation-like designs based on grids which are not standard.
2nd meeting	"Origami fractal" K. Takei [226]	Presents the fundamentals of Origami and image engineering for generating fractals.
3OSME	"A study of twist boxes" N. Nagata [227]	Reviews several similar Origami box designs which are passed around a twist design. By analysing the angle of the twist in the design the author is able to categorise several families of boxes from their crease patterns.
3OSME	"The geometry of Orizuru" T. Kawasaki [210]	In-depth exploration of the mathematics behind the crease pattern of a paper crane, more specifically a bird base, and how it can be altered and still remain foldable.
4OSME	"A crystal map of the Orizuru world" T. Kawasaki [211]	The Orizuru world is defined as the set of all quadrilaterals with an inscribed circle, thus they can be folded into cranes. By creating a visual representation of the map of this set a crystal map is formed.

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Conference	**Papers**	**Summary**
4OSME	"A geometrical tree of fortune cookies" J. Maekawa [228]	Looks at the mathematics behind the shape of fortune cookies and properties such as being homeomorphic with a sphere and being a developable surface; meaning, like Origami, the surface does not stretch or shrink in the folding process.
4OSME	"Constructing regular n-gonal twist boxes" S. Belcastro & T. Veenstra [229]	An exploration of boxes which are constructed with a twist. It looks at the mathematical considerations needed to generalise these to any number of sides and gives an example of a seventeen sided box.
4OSME	"Curves and flats" S. Sternberg [215]	A discussion of using curved creases. Unlike folds made using straight creases, when you make curved creases you cannot flatten the model without unfolding. The author suggests a method for gaining a relative flatness using curved crease tessellations which notably can produce a Gausian curvature. The paper then looks at the restrictions on further folding when curved crease tessellations are used.
4OSME	"Fractal crease patterns" U. Ikegami [230]	A look at infinite folding models where the number of iterations of folding can, in theory, be increased infinitely. Starting with the Maekawa pyramid this paper looks at variations which increase the complexity further.
4OSME	"Orizuru deformation theory for unbounded quadrilaterals" T. Kawasaki & H. Kawasaki [212]	Looking at several different deformations of the base of a paper crane this paper mathematically analyses which variations are possible and what their restrictions are. It shows that this is only possible if the quadrilateral used has an inscribed circle.
4OSME	"Paper nautili: A model for three-dimensional planispiral growth" A. Lommel [231]	A new design method for smoothly curved 3D models of logarithmic spirals is presented. This paper also looks at some of the mathematics behind creating variations of this model with different growth rates.

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Conference	**Papers**	**Summary**
5OSME	"A systematic approach to twirl design" K. Burczyk & W. Burczyk [232]	Exploring variations on a twirl design by Goubergen in which paper tension is used to hold the structure of the shape. Many variations are explored by changing a range of 11 parameters.
5OSME	"Compression and rotational limitations of curved corrugations" C. Edison [233]	Through exploration of the properties of curved crease corrugations the author looks at how changing these curves can affect the compression and rotation ability of the folded structure.
5OSME	"Folding any orthogonal maze" E. Demaine, M. Demaine & J. Ku [208]	The authors present an algorithm to design a crease pattern for any specific maze based on a square grid design.
5OSME	"New collaboration on modular Origami and LED" M. Kawamura & H. Moriwaki [234]	The authors present a method to incorporate LEDs into Origami pieces which illuminate them from the inside creating x-ray like effects.
5OSME	"Oribotics: The future unfolds" M. Gardiner [235]	Using technology to enhance art, this paper describes work on an interactive lighting display and its development.
5OSME	"Polygon symmetry systems" A. Hudson [236]	Proposing alternative grid structures for use in Origami crease patterns, using a system for polygonal grids which are not the standard square, triangle or hexagon it shows some examples of folds for hendecagonal and decagonal crease patterns as well as a method for generalisation.
5OSME	"Reconstructing David Huffman's legacy in curved-crease folding" E. Demaine, M. Demaine & D. Koschitz [237]	A practical exploration of both published and unpublished Origami artworks, by the late David Huffman, which were made using curved creases. Working towards the development of a theory for curved crease Origami design.
5OSME	"Simulation of nonzero Gaussian curvature in Origami by curved-crease couplets" C. Leong [238]	The authors provide a method to produce curved creases as pleats which deform the shape of paper in such a way that it can give the impression of nonzero Gaussian curvature.

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Conference	**Papers**	**Summary**
5OSME	"Using the Snapology technique to teach convex polyhedra" F. Goldman [239]	Focusing on its simplest forms the authors look at polyhedra created from ribbons using the snapology technique. The paper also suggests that they could be used to teach the geometry of polyhedra.
6OSME	"A method to fold generalized bird bases from a given quadrilateral containing an inscribed circle" T. Kawasaki [213]	Here Kawasaki continues his work on paper cranes, this time looking at methods for folding cranes from a given quadrilateral containing an inscribed circle. Folding sequences can be found but the centres found are not always perfect centres for the shape given.
6OSME	"Base design of a snowflake curve model and its difficulties" U. Ikegami [240]	This paper looks at a generalised crease pattern for constructing a fractal snowflake model which is examined up to the 5th iteration. The number of creases required, however, is shown to grow extremely rapidly.
6OSME	"Crowdsourcing Origami sculptures" J. Mosely [209]	A series of artworks produced with connected simple cube models are presented. Models include a level three Menga-sponge and snowflake sponge and were folded collaboratively.
6OSME	"Design methods of Origami tessellations for triangular spiral multiple tilings" T. Sushida, A. Hizume & Y. Yamagishi [241]	The authors review three design methods of triangular spiral multiple tilings and also present another design method which produces these with one degree of freedom. A significant amount of mathematical effort is used to create these structures.
6OSME	"Extruding towers by serially grafting prismoids" H. Cheng [242]	This paper shows a design method by which a structure can be grafted onto another using pleats such as those commonly used in Origami tessellations. An example shown is of a house grafted onto the back of a turtle.

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Conference	**Papers**	**Summary**
6OSME	"Folding perspectives: Joys and uses of 3D anamorphic Origami" Y. Klett [243]	Unlike traditional Origami where the folded shape creates a visual representation as the primary impression of an Origami artwork, Klett proposes and provides examples of the introduction of printing on surfaces which can be viewed at multiple angles in a precisely engineered Origami piece to provide multiple visual impressions.
6OSME	"Master peace: An evolution of monumental Origami" K. Box & R. Lang [244]	Presenting a collection of examples this paper explores collaborative Origami models which have evolved from paper into monumental works enlarged, cast and fabricated in Origami.
6OSME	"Modelling vaults in Origami: A bridge between mathematics and architecture" C. Cumino, E. Frigerio, S. Gallina, M. Spreafico & U. Zich [245]	Working from a mathematical description of vaults, the authors successfully provide crease patterns for vaults which can be used to gain greater understanding of these architectural shapes.
6OSME	"Rotational Erection System (res): Origami extended with cut" Y. Miyamoto [246]	Exploring the geometry behind the creation of 3D structures which are created simply by cutting onto a sheet and folding. The paper presents several examples and design methods which can have a large range of applications including jewellery and furniture.
6OSME	"Wearable metal Origami" T. De Ruysser [247]	The author presents a textile-Metal laminate, developed over 16 years, to create tessellation Origami structures to be used in clothing. This paper describes the design process and shows some examples.

Table 2.6: Summaries of OSME papers presented in the area of: Origami applications in art and design

2.1.6 Origami in education and/or therapy

The act of folding paper is a kinaesthetic tactile process in which the shape of an object is manipulated, whilst the folded model is a 3D object which can be explored and manipulated further. Origami can be both a medium for exploration of purely artistic design or tightly restricted to challenge creativity. The geometric shapes created when folding provide a useful tool for mathematical investigation. Furthermore, as already seen in this survey, there is a plethora of mathematical treasures which can be studied and used in education. It is no wonder that Origami has been used as a teaching tool.

In the OSME proceedings many authors including, Hall [248] and Frigerio [249] provide lesson plans or work schemes designed to teach mathematics to a range of different audiences while Carter [250] provides a wider viewpoint and surveys the frequency of paper folding exercises in American mathematics textbooks. Due to its versatility Origami has been used for teaching many subjects, from visualising chemical structures to languages; Yee Ho [251] and Jackson [252] provide examples of using Origami to teach English and design, respectively. Notably, Edison [253] shows how Origami has also been used as a tool for engaging with disadvantaged student groups and Paparo [254] provides a case study of how Origami has helped drug addicts. This area had its own conference series COET, Conference on Origami in Education and Therapy, organised by John Smith who used the first OSME conference as a springboard for it [255]. Held in both 1991 and 1995, the proceedings of these, COET91 [256] and COET95 [257], provide a larger range of materials in this specific area.

Conference	Papers	Summary
1st meeting	"Origami-therapy applied to a drug addict" M. Paparo [254]	This is a note instead of a paper, due to the ill health of the original author. It follows the story of a boy, initially a drug addict with a limited vocabulary, who through Origami was able to progress away from drugs and subsequently enrol in the military.
2nd meeting	"Hypergami: A computational system for creating decorated paper constructions" M. Eisenberg & A. Nishioka [258]	Presented is an educational computer program designed to allow students to design a simple model by adjusting elements and applying colours. The results can be printed to enable real world construction.
2nd meeting	"One crease Origami: Less is More" P. Jackson [252]	An exploration of how Origami models with only one crease can be used as a teaching tool for university students studying design. The one-crease folds, which often include curves, are shown to be very varied in aesthetics and physical properties of the final output.
2nd meeting	"Origami as an aid to understanding symmetry groups" J. Nitta [259]	Several examples as a structure for teaching symmetry groups to students using modular Origamis. These examples can be taught with almost no budget and to a large group of students.
2nd meeting	"Physically handicapped Origami" S. Kase [260]	Facing the problem of an ageing population and an increasing number of people with disabilities in Japan, this study looks at how Origami can be used to help people with a range of disabilities and it provides advice and guidance from the authors' experiences.
2nd meeting	"Teaching Origami to develop visual/spatial perception" J. Hall [248]	This paper looks at both why it is important to teach visual/spatial skills and how this can be done using Origami providing discussion of some examples.
3OSME	"Application of Origami to the teaching of sophisticated communication techniques" D. Foreman-Takano [261]	Several lesson ideas and teaching resources for using Origami to teach the English language to students in Japan are presented. These have been developed through the authors experiences teaching English to Japanese university students.

Table 2.7 – continued on next page

Continued from previous page

Conference	Papers	Summary
3OSME	"In praise of the paper cup: Mathematics of Origami at the university" E. Frigerio [249]	A review of a university course for future primary school teachers demonstrating and exploring the mathematics which can be taught from a paper cup model.
3OSME	"Instances of Origami within mathematical content texts for preservice elementary school teachers" J. Carter & B. Ferrucci [250]	A review of the frequency of paper folding activities, examples or exercises in a selection of American school mathematics textbooks.
3OSME	"Origami and the adult ESL learner" L. Yee Ho [251]	Several lesson ideas and teaching structures for using Origami to teach ESL (English as a Second Language) to learners in the US.
3OSME	"Origami as a model for development in organisms" N. Budnitz [262]	Presenting an Origami analogue to the real-world series of events in genetics, with a crease pattern as an expression of the genome, reference folds represent genes used in early life which later do not play a part, and step by step instructions represent the making of proteins. Of note is the observation that similar crease patterns may fold into different models.
3OSME	"Using triangular boxes from rectangular paper to enrich trigonometry and calculus" V'Ann Cornelius & A. Tubis [263]	How quantitative mathematical analysis can be taught using Origami by adjusting the design of a triangular box. This example is presented as both an elegant Origami construction and also contains a fair amount of high school mathematics in its creation.
4OSME	"Modular Origami in the secondary geometry classroom" M. Cagle [264]	Motivated by a lack of attention given to 3D geometry in secondary mathematics, Cagle uses polyhedral models to enable students to fully understand this area as described by the Van Hiele model.
4OSME	"On the effective use of Origami in the mathematics classroom" V'Ann Cornelius & A. Tubis [265]	The authors provide several examples of simple Origami models which can be used to teach simple mathematics principles from algebraic experimentation with models to an introduction to geometry.

Table 2.7 – continued on next page

Continued from previous page

Conference	**Papers**	**Summary**
4OSME	“Origametria: A program to teach geometry and to develop learning skills using the art of Origami” M. Golan & P. Jackson [266]	Looking at the Origametria program this paper explains how it can be incorporated into a classroom setting and provides some examples.
4OSME	“The impact of Origami mathematics lessons on achievement and spatial ability of middle-school students” N. Boakes [267]	This study shows that in terms of spatial ability Origami had a positive benefit to students. The study also looked at the effect of gender on this and the effect of an Origami course on mathematical skill.
4OSME	“Understanding the effect of Origami practice, cognition, and language on spatial reasoning” M. Wilson, R. Flanagan, R. Gurkewitz & L. Skrip [268]	This study seeks to use Origami as a solution to the lack of mathematical understanding in US schools. With a sample of 37 middle school students, the study looked at factors including the students’ enjoyments of Origami, their enjoyment of mathematics, their attitude towards mathematics and their attitude to Origami.
4OSME	“Using Origami to promote problem solving, creativity, and communication in mathematics education” S. Pope & T. Lam [269]	The authors provide a summary of their work teaching students mathematics through three areas; whole class teaching where the whole group folds a model, reverse engineering of a model in groups and design of an Origami shape based on some basic geometric requirements.
5OSME	“Close observation and reverse engineering of Origami models” J. Morrow & C. Morrow [270]	Ways in which Close Observation and Reverse Engineering, CORE, of Origami models can be used to improve learning and problem solving skills as well as to promote creativity.
5OSME	“Hands-on geometry with Origami” M. Winckler, K. Wolf & H. Bock [271]	A series of learning units for grade 8 students are discussed. The units include work such as Haga’s theorem, trisecting angles and the Origami axioms. The course was tested with mathematically gifted students who responded well to the course.

Table 2.7 – continued on next page

Continued from previous page

Conference	Papers	Summary
5OSME	"My favourite Origamics lessons on the volume of solids" Shi-Pui Kwan [272]	A collection of examples, of how to find the volume of a solid, for use in teaching. Ranging from a cubic box to antiprisms.
5OSME	"Narratives of success: Teaching Origami in low income urban communities" C. Edison [253]	A series of accounts collected by the author where Origami has been used for disadvantaged students in inner city and or low income environments. This includes examples of how students who had not engaged with the education system, possibly having spent time in jail, can be taught to follow instructions, build self-confidence and engage with learning.
5OSME	"Origametria and the Van Hiele theory of teaching geometry" M. Golan [273]	Looking at Origametria, a programme of Origami in the curriculum in Israeli preschools, this paper looks at how it follows the van Hiele theory of geometric teaching.
5OSME	"Origami and learning mathematics" S. Pope & T. Lam [274]	Pope and Lam provide several examples of how mathematical principles such as proofs by mathematical reasoning can be taught using Origami.
5OSME	"Origami and spatial thinking of college-age students" N. Boakes [275]	Continuing the work by the author in 4OSME this paper presents results from another study this time focusing on a course of 75 College age students. The results show promising links between engaging in Origami related study and spatial skills.
5OSME	"Student teachers introduce Origami in kindergarten and primary schools: Froebel revisited" M. Fiol, N. Dasquens & M. Prat [276]	This paper presents work from 10 years of exploration with teaching teachers how to use Origami while teaching primary school mathematics. It also reviews other work in this area.
6OSME	"Area and optimization problems" E. Frigerio & M. Spreafico [277]	Based on work on an ongoing 5 year program of Origami classes for Italian elementary pupils, the authors provide lesson structures which have engaged students who would not normally engage with the syllabus. This is shown also to improve cooperation and concentration skills.

Table 2.7 – continued on next page

Continued from previous page

Conference	Papers	Summary
6OSME	"Mathematics and art through the cuboctahedron" Shi-Pui Kwan [278]	Looking specifically at several models of the cuboctahedron, Kwan looks at the integration of mathematics, art and Origami.
6OSME	"Origami-inspired deductive threads in pre-geometry" A. Tubis [279]	Tubis provides a rough preliminary outline of a programme incorporating Origami into pre-geometry mathematics education such as providing folding verifications for the standard formulae for areas of shapes.
6OSME	"The kindergarten Origametria programme" M. Golan & J. Oberman [280]	Working with the curriculum of the Israeli ministry of education, this paper discusses the content of a course for kindergarten teachers on how exploration of folding in Origami can be used as a learning tool.
6OSME	"Using Origami to enrich mathematical understanding of self-similarity and fractals" A. Bahmani, K. Sharif & A. Hudson [281]	Presented is a programme developed for schools in Iran introducing and covering concepts including infinity. These courses use Origami fractals such as Palmer's flower towers and Fujimoto's hydrangea. Experiences with this programme are also discussed.
6OSME	"Using paper folding to solve problems in school geometry" Y. Huang & P. Lee [282]	Using the Origami Axioms, Huang and Lee present several Origami based demonstrations of school geometry problems looking at parallel lines, triangles, quadrilaterals and circles.
6OSME	"Using the Fujimoto approximation technique to teach chaos theory to high school students" L. Poladian [283]	Poladian shows how this Origami technique can capture a student's interest when studying chaos theory covering a range of mathematical principles from manipulating simple fractions to some of the deepest results in number and chaos theory.

Table 2.7: Summaries of OSME papers presented in the area of: Origami in education and/or therapy

2.1.7 The study of the history, language and psychology of Origami

Although the study of the history of paper-folding is not one of the aims of the OSME conferences there have been several papers presented there exploring this area. Here we look at those papers which concern either the study of the history, language or psychology of Origami. Many of these are linked and look at questions such as ‘Why did people in the past choose to fold paper?’ or ‘What is the history of the word Origami?’

This area of research is made complex by the lack of a universally accepted and consistent definition of Origami and the questionable reliability of supporting evidence for some theories. Thus, there are many theories of the history of paper-folding and its origins. Paper is often said to originate in China in the second century BCE [11]. Although it is likely that the first to fold paper were those that invented it, this should not be considered as Origami. Lister [4] and Hatori [11] show us evidence to support the theory that it developed independently in both the east and the west leading Hatori to conclude that “Origami has never been a Japanese art” [11]. In contrast the word Origami is definitely Japanese, based on ‘Ori’ meaning to fold and ‘kami’ meaning paper. Okamura [284] provides a background on how and why the word was developed discussing the motivation behind this choice.

In addition, some researchers have looked at the more recent history of Origami such as in Tateishi's work, [285], analysing how cross-linguistic differences have affected Origami instructions, while Tubis and Mills, [286], looked at myths including Origami, such as the myth surrounding the origin of the five pointed stars on the American flag.

Some authors have looked to the future. Early in the OSME series, Van Goubergen suggested how a mix of computer aided design and traditional approaches would be needed in order to not alienate folders [287]. Brill suggests that folders will need to move away from traditions of the past such as using bases [288].

Of special note is a discussion of the psychology behind Origami design work by Yoshizawa who is considered to be the grandmaster of Origami, credited with creating over 50,000 models in his lifetime [289]. Yoshizawa's paper is one of the only mentions of the psychology of Origami and it may be that he is one of a very few to look at this field.

There are many sources of information in this field outside of the OSME series. Lister, a founding member of BOS, was perhaps the world's leading Origami historian and wrote many papers which have been made available through the BOS website [290]. There is also a book "Notes on the History of Origami" [291] by Smith which provides a more in-depth look at much of the history of Origami, However, it dates from 1972 which is before many of the papers here. In addition Lister has published a book "The History of Paper Folding in Britain" [292].

Conference	Papers	Summary
2nd meeting	"Another view of the word 'Origami'" M. Okamura [284]	Presented is the history of the word "Origami" based on evidence from several sources. Showing references to paper-folding under different names, the authors discuss how the word Origamizaiku was created as a name for paper-folding based on its femininity, lack of homonyms and that it conveyed the superiority of Japanese paper folding.
2nd meeting	"Movement of nature, folding line structures and others" A. Yoshizawa [293]	Yoshizawa, the founder of modern Origami, presents a discussion of the psychology behind Origami design. This paper describes how, for Origami design, you need not only folding knowledge and skills but also good observation skills as, when we make an Origami representation of a flower we are not actually folding a flower, just something which captures an abstracted geometry of a flower.
2nd meeting	"Origami and motivation" T. Yenn [42]	Beginning with a unique definition of some Origami terminology Yenn recounts personal experiences with Origami and uses these to understand why a person might choose to fold paper or study Origami.
2nd meeting	"Some observations on the history of paper-folding in Japan and the west - a development in parallel" D. Lister [4]	A summary of the history of paper-folding in Japan and in the west explaining the need for evidence based sources. The author dismisses some theories of paper-folding in Japan which are not based on evidence, until such evidence can be found.
2nd meeting	"The roots of Origami and its cultural background" K. Ohashi [294]	Provided is a somewhat different approach to the history of Origami in Japan which looks at the atributes of the Japanese people and their culture that may have made them decide to fold paper.

Table 2.8 – continued on next page

Continued from previous page

Conference	Papers	Summary
2nd meeting	“Thoughts of the future of Origami design: Something old, something new…” H. Van Goubergen [287]	A collection of possible future developments in Origami. Suggestions ranging from computer based mathematical design to design through creativity and exploration of shaping. The author warns that there is a risk of these schools diverging as far as to alienate the folder and asks if the future may hold more folds from the best of both schools or increased use of restricted folding techniques, such as in Pureland Origami.
2nd meeting	“Traditional and technical” D. Brill [288]	Considering the use of traditional bases and the single uncut square in the history of Origami design, this paper asks how these rules can be broken. It leaves a final thought that folders should leave traditional prejudices behind and in doing so greater possibilities of folds will emerge.
4OSME	“Redundancy of verbal instructions in Origami diagrams” K. Tateishi [285]	Through several examples of both diagrammatic and verbal instructions Tateishi shows how the diagram language used in Origami is sufficient to provide all necessary instruction and is also better suited than word instructions.
5OSME	“Betsy Ross revisited: general fold and one-cut regular and star polygons” A. Tubis & C. Mills [286]	A critical look at the Origami related origin story of the 5 point star on the (US) American flag and replication of the method to produce a one straight cut solution. This method is then expanded to methods to produce other stars.
5OSME	“Deictic properties of Origami technical terms and translatability: cross-linguistic differences between English and Japanese” K. Tateishi [295]	A review of how cultural attitudes to language have made Origami instructions difficult to translate due to differing concepts of an instruction in the English and Japanese languages.

Table 2.8 – continued on next page

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Conference	Papers	Summary
5OSME	"History of Origami in the east and the west before interfusion" K. Hatori [11]	An evidence based review of different opinions on the beginnings and history of Origami presenting alternative viewpoints for both the argument that Origami originated in China and that Origami is a Japanese art.
6OSME	"Computational problems related to paper crane in the Edo period" J. Maekawa [296]	Maekawa provides evidence that the mathematical study of Origami is at least 200 years old by providing examples of Origami problems found in Japanese manuscripts and sangakus.
6OSME	"Mitate and Origami" K. Hatori [297]	Mitate, to liken something to something else, is stated as being essential for Origami, where the geometric shape of the model is likened to the geometric shape of the object it represents. The history of Mitate and how it is used in almost all Origami is explored with a few examples of early Origami which did not use Mirate.

Table 2.8: Summaries of OSME papers presented in the area of: The study of the history, language and psychology of Origami

2.2 Summary

In this chapter we have reviewed the literature on Origami and its different aspects and uses. We have concentrated on the papers presented in the OSME conference series proceedings, of which there are six so far. The next OSME conference is due this year taking place in September at Oxford University. This represents an almost complete overview of the work in this field.

The next chapter recalls the mathematical background of Origami including the One Fold Axioms. Basic but essential definitions are given before mathematical questions are addressed. The most advanced question to be considered is an attempt at devising an Origami-inspired alternative, and much more succinct proof to the 4-colour theorem [148]. This will be the subject of the chapter after next.

3

Mathematics of Origami

This chapter reviews the basic mathematical concepts which underpin Origami constructions and attempts to show how these concepts can be used to tackle serious mathematical questions. Background definitions are recalled and then used to state the fundamental axioms of single fold Origami constructions. These are then used to formulate solutions to some well known mathematical problems. The chapter ends by revisiting and recalling popular and serious results, such as Maekawa's theorem, Kawasaki's theorem and the One Straight Cut theorem due to Demaine [122].

Serious work on the Mathematics of Origami has been done since the 1930's, for example the work of Beloch [25], however, much of this work had been done in isolation, with many people reinventing work already done by others. It is only in the last three decades that this has been more coordinated and attention paid to the work that had been done previously leading many to state that the serious work in this field has been done in this time [16]. There is already a plethora of information sources on the subject. Research has been and is being carried out into many areas of Origami, to understand, for instance, the underlying rules and properties that restrict Origami when making a single fold, such as work by Alperin [76], and what is and what is not constructible using single fold axioms.

After presenting the fundamental Origami concepts, this chapter looks at the work that has been done to solve mathematical problems using Origami constructions. It then compares the scope of this construction approach with that of straight edge and compass construction. In particular, it looks at the three ancient problems of trisecting angles, doubling the cube, and squaring the circle.

Expanding on this, the chapter shows ways in which the restrictions of the single fold Origami constructions can be removed by introducing multi-fold axioms. This extension helps solve polynomial equations and uses curved creases to construct transcendental numbers, for instance. It then explores the theorems underpinning flat foldable Origami crease patterns including their colourability and expands the colourability theorems to cover 3d constructions.

3.1 Mathematical Origami construction

From now on it is assumed that the material properties of the piece of paper to be folded are ignored. This concern is entirely with its representation as a mathematical object. To this end, we begin by considering the one-fold construction approach and investigate its capabilities as a solution tool. First however, we present the necessary mathematical foundations for our model similar to that from Euclidean Geometry.

It is important to note that the expressions ‘to crease’ and ‘to fold’ are often used loosely in literature. This thesis will refer to a fold as the action of bending and pressing flat a piece of paper which produces a crease; thus a crease is the mark left by the action of folding.

Following are some definitions that lead to the complete set of 1-Fold Origami Axioms, also known as the Huzita-Justin Axioms (HJAs), [298].

Definition 3.1.1. Let ***a piece of paper*** represent a simply connected, bounded subset of the $\mathbb{R}^2$ plane without holes and homeomorphic to a disk.

The edges of the paper are the boundaries of this disc. Making a fold produces a crease in the paper equivalent to a line in the plane.

Definition 3.1.2. A ***straight line***, $L = (a, b, c)$, in the $\mathbb{R}^2$ plane is represented by coefficients a, b and c of equation $ax + by + c = 0$. Lines represent either a crease or the edge of the paper.

We note here that similar to straight edge and compass constructions it is possible to construct the reflected image of a line thus it is not necessary to define this separately, however, we show this later on in this chapter and thus will allow the existence of such reflections as lines or points to be assumed.

Intersections of lines produce points. Endpoints of lines need not be explicitly defined; edges of the paper and other folds will inevitably create them. Note that, in folding constructions, lines are primal. This means, lines, by intersecting, define points, unlike in standard Euclidean Geometry, [16].

Definition 3.1.3. A ***point*** $P = (X, Y)$ is represented by its coordinates X and Y in $\mathbb{R}^2$. Points are only created where lines intersect.

Having the concepts of point and line formally defined, another fundamental concept, that of alignment, is now provided.

Definition 3.1.4. An ***alignment*** is the process of bringing together, i.e. aligning or folding, a point to a point, a point to a line, or a line to a line, [19]. After the paper is pressed flat a crease is produced. In other words, alignments are means for deciding where a fold is going to be made.

When the paper is unfolded back to the original plane from which it was folded, the crease is represented by a new line through the plane. Marked on the paper is the intersection of this infinite line and the paper.

Definition 3.1.5. An ***Origami Construction*** is produced as the piece of paper has creases placed on it, it contains lines representing creases and the edges of the paper as well as vertices where these intersect; together these separate the paper into convex polygons forming faces such as on a map.

We now break down the alignments into different types. A line or a point can be folded onto themselves, creating a fold along a line or through a point. Folding along a line gives a fold line in the same place as the original line, thus not constructing any new lines. This is, therefore, removed from our set of alignments. This gives us four possible alignments [299].

A single alignment is not always enough to uniquely describe a crease as for example there are an infinite number of folds which can be placed through a single point. Thus in order to use these for one-fold constructions we must use them in combination. To make a single fold in the plane, we combine alignments in such a way that only a finite number of solutions exist, usually only one. These combinations are commonly known as the Huzita-Justin one-fold axioms or constructions, [19]. We note here that they are not axioms in the mathematical sense as they are not always possible.

Definition 3.1.6. The ***Huzita-Justin 1-fold constructions*** are all possible types of fold, generating a single crease that use a minimal combination of alignments. There are seven such types giving the following constructions, [16]:

Axiom 1: Folding through 2 points: Given 2 points A and B, fold A onto A and B onto B, Figure 3.1(a).

Axiom 2: Folding 2 points together: Given 2 points A and B, fold A onto B, Figure 3.1(b).

Axiom 3: Folding a line onto another line: Given 2 lines a and b, fold a onto b, Figure 3.1(c).

Axiom 4: Folding a line onto itself so that the crease intersects a point: Given a point A, and a line b, fold A onto A and b onto b, Figure 3.1(d).

Axiom 5: Folding a point onto a line so that the resulting crease goes through another point: Given 2 points A and B, and one line c, if possible fold A onto c and B onto B, Figure 3.1(e).

Axiom 6: Folding 2 points onto 2 lines: Given 2 points A and B, and 2 lines c and d, if possible fold A onto c and B onto d, Figure 3.1(f).

Axiom 7: Folding a point to a line so that the resulting crease is perpendicular to another line: Given a point A, and 2 lines b and c, fold A onto b and c onto c, Figure 3.1(g).

We note that Axiom 5 and 6 are not always possible however the other axioms are always possible given a large enough sheet of paper [16].

3.2 1-fold axiomatic Origami constructions

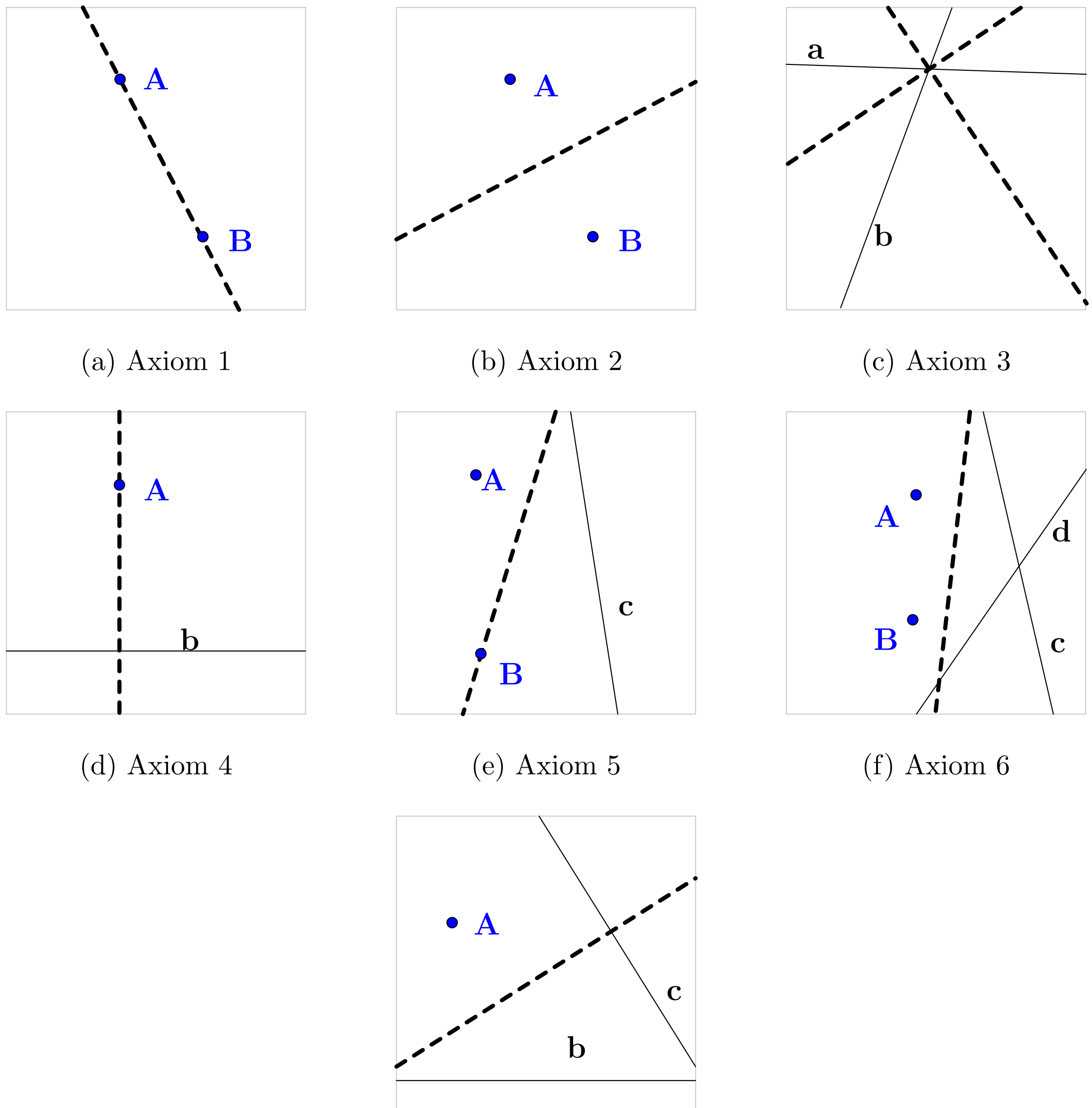
With the seven 1-fold axioms given above, we now have a basis for simple Origami constructions. We ask what can be constructed with these axioms? Can we construct/generate numbers, for instance? If so, which numbers? Integers? Irrationals? Complex? In the following, these questions and others will be considered.

3.2.1 Constructing reflections

It is not clear from the existing literature if construction of reflections can always be done using the axioms. However, this seems to be obvious in those sources although they do not provide a formal statement to the effect. Here, we provide such a statement in the form of a theorem followed by a proof.

Theorem 3.2.1. *The reflection of any Origami constructable object, point or line, with respect to another line is always constructable using the Origami constructions.*

Proof. First we note that to prove this theorem, it is sufficient to show that the reflection of any point is always constructable. A separate proof for a line is



(a) Axiom 1 (b) Axiom 2 (c) Axiom 3

(d) Axiom 4 (e) Axiom 5 (f) Axiom 6

(g) Axiom 7

Figure 3.1: Origami Axioms

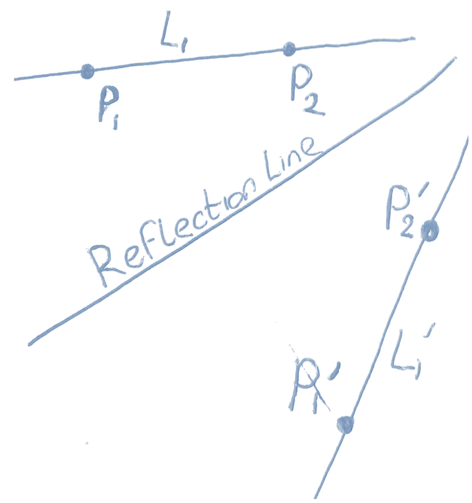


Figure 3.2: Reflection of a line using the reflections of two points

not required since, if we can construct the reflection of a point with respect to a line, lines will follow. Note, with Euclidean algebra any line can be defined by two points; with an Origami construction approach a line can be constructed which connects two points. Thus, given any line, two arbitrary points can always be chosen on that line whereby the reflections of those points can be used to construct the reflection of the original line. This is shown in figure 3.2 Where given a line L_1 and a reflection line, two arbitrary points, P_1 and P_2, on L_1 are chosen. The reflections of these points P_1' and P_2' respectively can be used to produce the reflection of L_1 which is shown as L_1'.

We now proceed to construct the reflection of a point.

We note here that as Axioms 5 and 6 do not always have a solution, we must make the construction of a reflection of a point using the other five axioms. In fact we will only need axioms 3 and 4. The following steps are shown in figure 3.3.

Step 1: Axiom 4 gives us that we can always fold through a point such that the crease is perpendicular to a given line, thus we can construct such a line through A and perpendicular to b, this is shown in subfigure 3.3a.

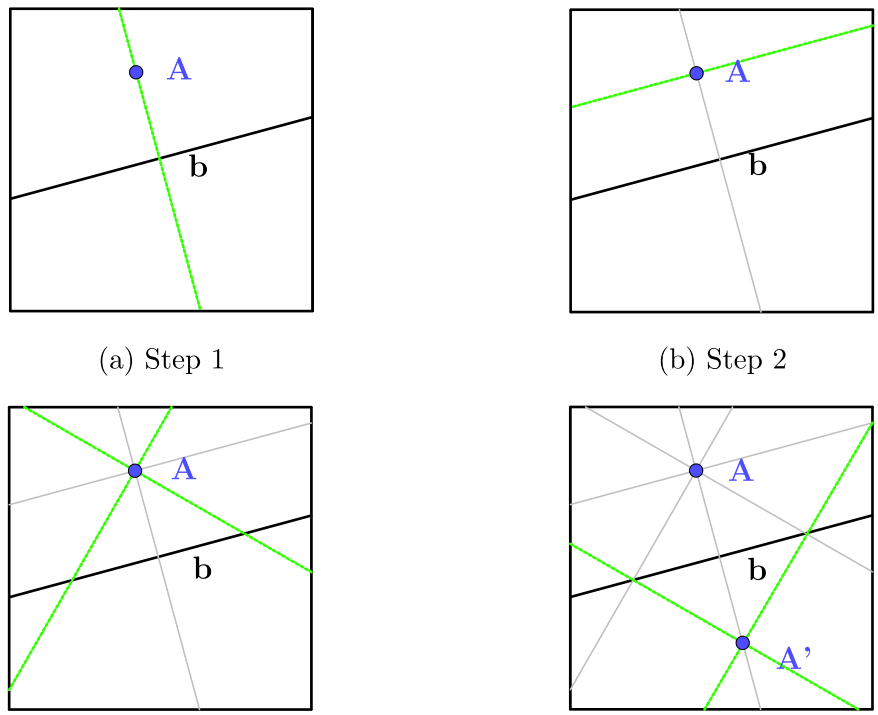
Step 2: Again using axiom 4, we can now construct a perpendicular line to the line just constructed again through point A. This will be parallel to line b and is shown in subfigure 3.3b.

Step 3, Using axiom 3, to fold a line onto a line, the two lines we have constructed in steps 1 and 2 can be aligned with each other creating two angle bisectors. These will intersect the line b at $45°$ as shown in subfigure 3.3c.

Step 4, We can again use axiom 4 to fold perpendicular lines to those just constructed passing through the points where those lines intersect b, this is shown in subfigure 3.3d.

These final two lines will also be at $45°$ to the fold line, thus their intersection will be at the location of the the reflection of point A, we shall call this A'.

This shows a method that is always possible for construction of the reflection of as point. As this is possible it follows that the construction of the reflection of any



(a) Step 1 (b) Step 2

(c) Step 3 (d) Step 4

Figure 3.3: Origami Construction of reflections

constructable object in the plane is constructable using the one fold axioms. $\square$

We note here that only one of the angle bisectors is needed as they will intersect the first crease we made perpendicular to our reflection line b.

From here on we shall assume reflections of points or lines are constructable and not show this in step by step instructions for a construction.

3.2.2 Manipulating lengths

Trigonometry allows one to consider angles as equivalent to lengths, thus we can manipulate numbers in Origami either as distances between two points or angles between two lines. The initial dimensions of the paper are irrelevant. A unit square for instance, will suffice. The interest is in dividing or multiplying the angles and distances within it. This removes the need for measuring tools other than the paper itself.

The HJAs allow us to create reference points in order to measure accurately. In the 1970s, a group of mathematicians, John Montroll amongst them, [300], strived to make Origami more precise. For instance, where many people would make approximate folds or folds "to taste", he tried to find precise folding sequences that would give the desired result. However, where a crease needs to be made that is not easily aligned with pre-existing landmarks, the folder has to create those landmarks, using a sequence of folds which generate reference points that can be used in future creases. An important landmark used is the mark corresponding to a certain quotient along an edge or crease. This could be $\frac{1}{3}$ or $\frac{1}{7}$ of the way along an edge. It is obvious that one can keep halving the edges to get $\frac{1}{2^n}$. But it is not immediately apparent if it is possible to get other fractions such as $\frac{1}{3}$, or $\frac{1}{7}$, Haga's theorem is one such method producing a range of fractions precisely, several methods to create landmarks can be found in [301]. Some of these can become particularly complex and often simpler imprecise methods can be found which approximate solutions very closely, the Fugimoto method is one of these iterative methods. The Fujimoto method and Haga's theorems, are recalled in the following sections [302].

3.2.2.1 Approximate methods using iterative approaches: the Fujimoto method

The Fujimoto method [54] is an iterative process. It is applied here to find thirds, as illustrated in Figure 3.4:

1. Given edge AB of a piece of paper, fold through an arbitrary point C perpendicular to the edge;

2. Fold point B to meet point C producing point D;

3. Fold point A to meet point D producing point E;

4. Fold point B to meet point E producing point F.

In general, continue to fold alternating edges to point N to produce point $N + 1$ until the required level of precision is achieved.

Assuming that one edge of the paper is of unit length, in effect, you have split the paper into X and $1 - X$. Folding the other end of the edge to meet the crease results in generating X and two lots of $\frac{1-X}{2}$. Folding the first edge to the last

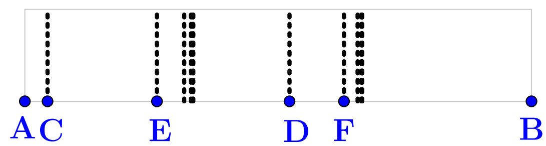


Figure 3.4: The Fujimoto method

crease gives one edge of length $\frac{1-X}{2}$ and two of length $\frac{X+1}{4}$.

Continuing this process leads to a sequence which will tend to $\frac{1}{3}$. If we calculate the errors at each step, given crease N is made with an initial error of $\pm\epsilon$, we construct crease $N+1$ with error $\frac{\epsilon}{2}$. Thus as we remake each crease the error is $\frac{1}{4}$ of the previous error. This method tends to generate accurate solutions quickly.

The approach can be extended and combined with some other iterative methods so that we can generate any odd fractional division [302], thus we can produce all prime divisions of an edge. The resulting edge from a prime division can be divided again to produce any rational division of an edge using combinations of these methods. However, it becomes impracticable to generate more complex divisions. A direct and, more importantly, accurate approach may be more practical.

3.2.2.2 Procedures based on Haga's theorems

Can the ratio 1/3 be directly and accurately generated, i.e. without resorting to an iterative process? Haga [301] first discovered that this can indeed be done simply by placing one of the bottom corners on the centre mark of the top edge.

Theorem 3.2.2 (Haga's first theorem [18])**.** *By simply placing the lower left vertex C of a square onto, D, the midpoint of the upper side, AB, and marking all the intersections, each edge of the square is divided in a fixed ratio, as follows: see Figure 3.5a.*

1. $\frac{3}{8}$ *and* $\frac{5}{8}$ *are produced thus the left edge is divided in the ratio* $3 : 5$
2. $\frac{4}{6}$ *and* $\frac{2}{6}$ *are produced thus the right edge is divided in the ratio* $2 : 1$
3. $\frac{21}{24}$ *and* $\frac{3}{24}$ *are produced thus the right edge is also divided in the ratio* $7 : 1$
4. $\frac{5}{6}$ *and* $\frac{1}{6}$ *are produced thus the lower edge is divided in the ratio* $1 : 5$

In addition, the length of the crease produced through the page is $\frac{\sqrt{5}}{2}$*.*

This fold results in three Pythagorean triangles with sides forming a Pythagorean triple, making it a useful construction tool. From this one fold you can construct sixths, eights and twentyfourths. Haga also proposed two other variations on this fold which produce useful results and allow the instant creation of more ratios.

Theorem 3.2.3 (Haga's second theorem [18])**.** *Mark the midpoint of the upper edge of a square piece of paper, and make a crease through its midpoint and the lower right vertex. A right triangular flap is formed. If the line of the shorter leg of the flap is extended to intersect the left edge of the square, the intersection* *point divides the left edge into two parts, the shorter part equals $\frac{1}{3}$ of the whole edge (see Figure 3.5b), .*

This second theorem produces a lot of useful divisions:

1. $\frac{1}{2}$ is produced by the intersection of the extension of BE and the left edge.

2. $\frac{1}{3}$ and $\frac{2}{3}$ are produced by the intersection of the extension of DE and the left edge.

3. $\frac{1}{4}$ and $\frac{3}{4}$ are produced by the intersection of the extension of CE and the left edge.

4. $\frac{1}{5}$ and $\frac{4}{5}$ are produced by the intersection of a vertical line through E and the top edge.

5. $\frac{2}{5}$ and $\frac{3}{5}$ are produced by the intersection of a horizontal line through E and the left edge.

Using this method one can produce in only a few folds any number of fifths; this is much simpler than using straight edge and compass construction, [61]. Haga's third theorem is a variant on the first theorem.

Theorem 3.2.4 (Haga's third theorem [18])**.** *Mark the midpoint of the upper edge of a square of paper. Fold the paper to place the lower right corner on the left edge, then shift upward or downward until the right edge of the paper passes* *through the marked midpoint. Make a crease. The crease formed divides the left edge into two parts, the shorter part being $\frac{1}{3}$ of the whole edge. See Figure 3.5c.*

In this variant of the first theorem, the fold brings a corner point to a given point $\frac{1}{3}$ of the way along an opposite edge. However, due to the 5^{th} axiom we can do this without first measuring to find $\frac{1}{3}$. This fold also produces three Pythagorean triangles. From this one fold you can construct sixths, eights and twentyfourths [18]. In addition, the length of the crease produced through the page is $\frac{\sqrt{10}}{3}$.

Haga's first theorem fold has been extended in several ways including looking at the same fold made on silver rectangles rather than squares [61] and explored other variations, [94]. They have also been generalised by choosing an arbitrary point of intersection with the top edge rather than the midpoint. See Figure 3.5d.

1. Take a square and label the corners $ABCD$ clockwise;

2. Fold D up to an arbitrary point, E on line AB;

3. Mark the intersection of the folded position of DC with BC as F;

4. In the resulting diagram, if the length AE is $\frac{1}{n}$ then BF is $\frac{1}{n+1}$.

This method allows any rational division of an edge.

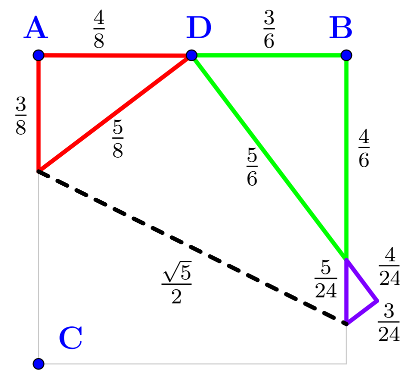
There are programs designed to give the simplest possible folding sequence to find reference points. One such program is Lang's "Reference Finder". Although unlike the above method, not all of its solutions are exact. The program gives an output sequence for folding to find the required reference point and provides the necessary accuracy. This leads to the question of whether it is possible to construct any irrational division of an edge.

3.2.2.3 Silver, bronze and gold ratios

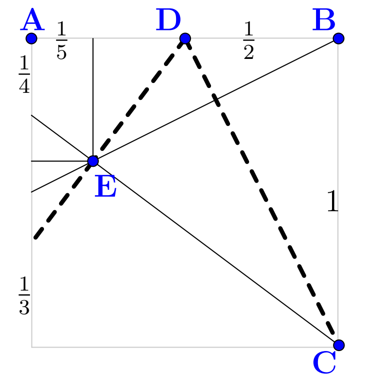
It is perhaps not surprising that one can produce rational numbers with Origami. What about irrational numbers? We have already seen that Haga's first theorem allowed to fold $\frac{\sqrt{5}}{2}$. Further examples of constructions of irrational numbers are provided below.

The bronze ratio is $1 : \sqrt{3}$. It refers to the fact that when a bronze rectangle is cut into thirds parallel to the short side, then the pieces are also bronze rectangles. A procedure to construct a bronze rectangle [303] is as follows.

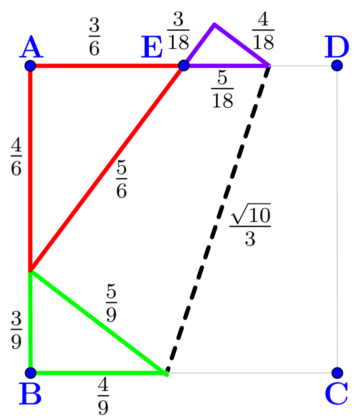
1. Fold a square in half with a horizontal crease creating AB;
2. Fold the top right corner to meet this fold, at C, such that the crease passes through the lower right corner, mark the intersection of this crease with the top edge as D;
3. Fold a crease through D perpendicular to the top edge;



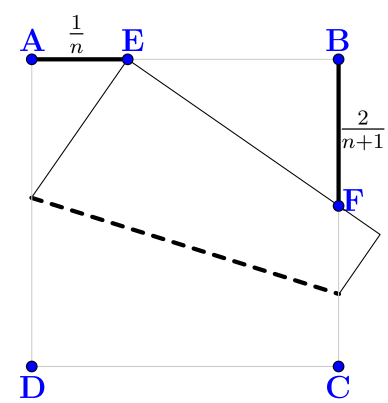
(a) First theorem



(b) Second theorem



(c) Third theorem



(d) Generalised first theorem

Figure 3.5: Haga's theorems

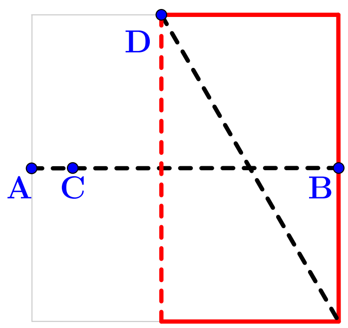
4. This fold defines the edge of a bronze rectangle, as shown in Figure 3.6a.

The silver ratio is $1 : \sqrt{2}$. It refers to the fact that when a rectangle of silver ratio is cut in half parallel to the shorter edge it will leave 2 rectangles that are also of silver ratio. Paper in this ratio is referred to as the A-Paper; specific sizes are referred to as $A0, A1, ..., An$, where An is $\frac{1}{2}$ the area of $A(n-1)$. [304]. A procedure to fold silver rectangles is given below [303]:

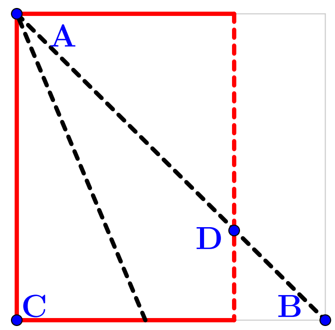
1. Fold the diagonal between 2 opposite corners of a square, A and B;
2. Fold a third corner, C, to meet this fold and mark the intersection with D;
3. Fold a crease through D perpendicular to the edge CB;
4. This fold defines the edge of a silver rectangle, as shown in Figure 3.6b.

The golden ratio is $1 : \frac{1+\sqrt{5}}{2}$, which is possibly the most elegant of the three named ratios shown here as when a rectangle of golden ratio has a square removed, the remaining strip is also of golden ratio. A method to construct a rectangle of golden ratio is given below [303]:

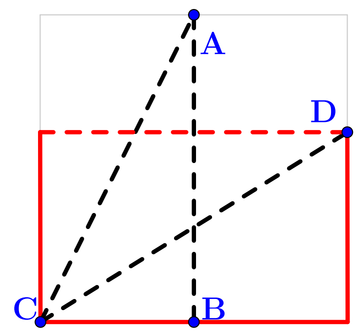
1. Fold a square in half with a vertical crease creating AB;
2. Fold a crease through the bottom left corner, we shall label this C, and A;
3. Fold the lower edge to meet this crease bisecting the angle at C;



(a) Bronze rectangle



(b) Silver rectangle



(c) Golden rectangle

Figure 3.6: Construction of rectangles

4. Where this third crease intersects the edge of the paper at D make a horizontal crease through D;

5. This fold defines the edge of a golden rectangle, as shown in Figure 3.6c.

We have shown 3 irrational ratios which can be constructed using the axioms. It is easy to generalise this to produce other ratios. For example, given that we can construct any rational number $\frac{m}{n}$, we can therefore construct any rectangle with rational edge lengths, $\frac{m}{n}$ and $\frac{o}{p}$, thus the diagonal across this rectangle can be any number of the format, $\sqrt{\frac{mp^2+on^2}{np^2}}$ where m, n, o and p are integers. This can be simplified to the square root of the sum of two quotients of two squares $\sqrt{\frac{m^2}{n^2}+\frac{o^2}{p^2}}$. This alone is a large group of irrational numbers which can be produced. We note here that we cannot construct every irrational number however we can construct solutions to any quartic equation as shown in Hull [305]. A notable exception which we cannot construct is Pi, this is not constructable using the axioms [299].

We will however look at Pi again in section 4

3.2.3 Generating angles

In a similar way to generating lengths, angles can be produced. As with an arbitrary length, we can perform some similar operations on an angle such as to find $\frac{1}{3}$ of it. We note this is not exactly the same as with lengths as we cannot for example find $\frac{1}{5}$ of an angle as we can with lengths. It is however more likely that we would need to construct a specific angle. We can do this using operations on 360° which we have at any point where creases meet. As an example, one can create a 90° angle by folding a crease perpendicular to any other. Since, as already seen, one can create lengths through creating rectangles with these edge lengths, it is thus possible to create any angle using the expression $\tan^{-1}(\frac{K}{K'})$ where K and K' are constructible lengths. In fact, many of the methods which are used to construct lengths can be used to construct angles including both exact methods and approximate iterative ones. For example, an equivalent of the iterative Fugimoto method can be used to produce divisions of angles [299].

3.2.4 Solving historical geometric problems

It has been known for a long time that angle trisection, doubling the cube and squaring the circle cannot be solved with only a straight edge and compass, [306].

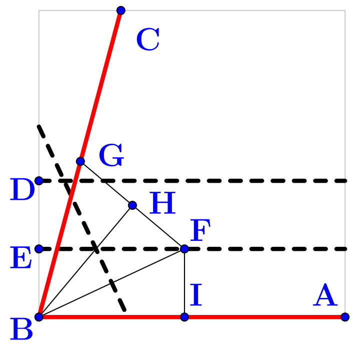


Figure 3.7: Trisecting an angle

Let us see if Origami 1-fold constructions can fare any better.

3.2.4.1 Trisecting angles

Trisecting an angle is possible with Origami 1-fold constructions, [299], using the following procedure illustrated in Figure 3.7. Here we look at the case of an acute angle. This, however, can be generalised.

1. Start with a rectangle and mark the bottom corners as A and B. With an arbitrary point, C, on the top edge of the paper, we provide the angle ABC to be trisected;

2. First create a crease perpendicular to the vertical edge through another arbitrary point, D,on the left edge, this uses axiom 4;

3. Then fold point D to B. Mark the intersections of this crease and the left

edge with E, this uses axiom 2;

4. Using axiom 6, align D on line BC and simultaneously align B on the fold line from E. The position of B when this fold is made is marked F, the position of D becomes G, and the position of E becomes H when folded;

5. Finally point I is marked as the intersection of a vertical crease through F and the bottom edge.

We now have three similar triangles, BGH, BHF and BFI, thus the three angles at A must be similar and the angle is trisected [19, p. 33].

3.2.4.2 Doubling the cube

The problem can be described as follows. Given a cube of side length 1, construct a length such that a cube with this side length would have double the volume of the given cube. The Origami approach to solving this problem starts by splitting the paper into thirds, then placing the bottom left corner on the top edge so that the bottom of the left hand crease is laid on the right hand crease, (see Figure 3.8). This results in BF being the length the cube of which is twice the cube of AE. This was originally shown by Messer, [307] and is also presented in several other sources [16].

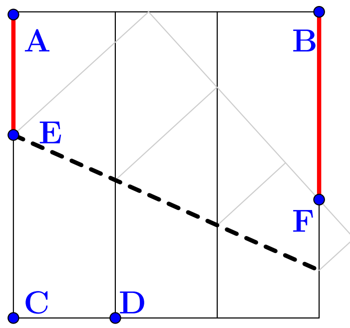


Figure 3.8: Doubling the cube

3.2.4.3 Pythagoras theorem

This is probably the most well known result in Mathematics. It has a number of proofs (hundreds) including some using a straight edge and compass only, [308], and indeed some based on Origami, [309]. Let us recall the proof due to Hisashi Abe, [309], (see Figure 3.9).

1. Begin with a square and mark an arbitrary reference point, called E, on the right hand edge;

2. Fold through the bottom left hand corner, B, and the arbitrary point E;

3. Fold through the top left corner, A, perpendicular to the line just made, creating points D and F;

4. Fold through the bottom right corner C perpendicular to the first crease;

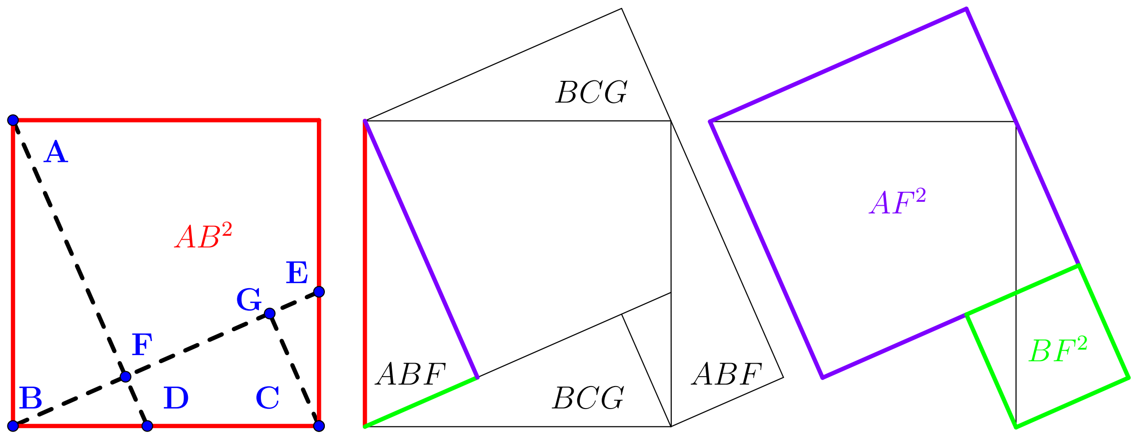


Figure 3.9: Pythagorean theorem

only part of this crease is needed. Mark the intersection with G.

By moving the triangles BCG and ABF we get a visual proof of the theorem. [309]

3.2.5 Trigonometry and algebra with Origami

Origami can be used to solve trigonometric equations. Kawamura's methods, for instance, uses the sine, cosine and tangent functions to compute specific angles [63]. Equally, Origami can be used to solve algebraic equations. In [76] and in [305] it has been shown that 1-fold constructions can solve quadratic and cubic equations respectively. Note that it has been shown that it is not possible to extend this to handle quintic equations [76].

3.2.6 2-, multi-, and curved fold constructions

So far we have been limited in our construction methods to only using the well defined 1-fold constructions. These construction are, however, far from a complete representation of the scope of all Origami. Unlike Euclidean geometry, Origami is an open system. In reality, multiple creases can be folded at any one time; one can fold creases and leave them folded, fold curved creases and cut or tear the paper. Consequently, 1-fold construction axioms have been extended to 2-fold constructions. This allows two folds to be made in one move and any combination of alignments between these two folds and the unfolded part of the paper. This is more powerful as it allows for angle trisection with only one move. However, although it is well defined, 489 axioms are needed to define all the possible combinations of folds. This shows how quickly the complexity of Origami increases when we allow more complex operations. But, it can be used to solve algebraic equations up to septics [310]. Theoretically this can be expanded to n^{th} degree equations if n-fold axioms are used.

Note that n-fold axioms cannot be used to construct transcendental numbers such as π. Therefore, the problem of squaring the circle can only be approximated. However, allowing curved creases, this is possible as was shown by Hull, [26]. It is important to note that although 1- and 2-fold constructions are well defined, Origami in general is not well defined for even 3d folding with straight creases let alone curved creases [26].

3.3 Flat-foldable Origami construction

With one-fold constructions we make each fold and unfold, the paper is our tool acting as a calculator. In most real world Origami the model is produced without unfolding after every step. Many folds are therefore combined to produce a finished model. This creates more challenges than 1-fold construction, some of these are explored here. Note that when a fold is made in axiomatic construction the interest is only in its location. Now, however, the concern is with the angle between the faces bordering a crease and the direction of the crease. We now give some definitions.

We note here that not all creases in these models go through the whole paper and only a part of the line representing the crease is needed.

Definition 3.3.1. A ***segment*** of a straight line consists of all points along the line between two given endpoints.

Definition 3.3.2. A ***Crease Pattern, CP,*** is a collection of creases, represented by lines and segments, on the plane / piece of paper. A crease pattern contains all the folds required to produce a model but does not contain detail about the order of folding or final layering orders.

In Origami when we make a fold we can fold the paper in one of 2 directions, thus the lines on the CP represent either a fold towards the observer or away from them.

Definition 3.3.3. A ***mountain*** fold is a fold in the plane which has the crease pointing towards the observer and the folded sections going away.

Definition 3.3.4. A ***valley*** fold is the opposite of a mountain fold with creases pointing away from the observer.

From behind, mountain and valley folds are valley and mountain folds, respectively [120].

Definition 3.3.5. A CP is considered to be an ***assigned CP*** when the creases are expressly assigned either as mountains or valleys.

Definition 3.3.6. A CP is considered to be an ***unassigned CP*** when creases can be folded in either direction.

Definition 3.3.7. The ***fold angle*** is the angle change about the crease, where there is no fold there is no change thus the fold angle is $0°$, where the paper is folded onto itself, one side of the paper is rotated $180°$ about the fold line, thus a fold angle $\alpha \in \mathbb{R}$, such that $\alpha \leq 180°$ and $\geq -180°$. An unfolded fold has an angle of $0°$.

Definition 3.3.8. Given a CP, its ***foldability*** is a binary property which states if the creases defined in the CP are realisable and can be folded. Foldability can be looked at globally across a whole CP or locally at a vertex.

Definition 3.3.9. A ***flat foldable Origami*** is one where the CP has a flat folded state. More specifically, in a flat foldable Origami, all folds can be made where the creases are folded with a fold angle of $\pm 180°$ thus all the sections of folded paper lie either face up or face down and the folded model lies in a plane.

Rigid foldability looks at the process of folding and if the creases can fold as if they were hinges connecting rigid faces.

We now focus specifically on flat foldable Origami. Before stating the main results of this type of folding, some restrictive assumptions are presented.

Assumption 3.3.1: The paper is continuous and cannot intersect itself, thus it cannot pass through itself or create knots.

Assumption 3.3.2: The paper does not stretch, thus the distance between two points along the paper is invariable.

These assumptions mean that, for instance, saddle points cannot be created through bending or curving the paper as this would stretch it. Let us now state the main theorems of flat foldable Origami.

Theorem 3.3.1 (Maekawa 1985 [311])**.** *In a flat foldable Origami construction, at any vertex made by intersecting creases on the CP, the number of mountains and valleys differs by 2, and the total number of creases is at least 4.*

A proof of this can be found in [16].

Theorem 3.3.2 (Kawasaki [311])**.** *A vertex in a CP is flat foldable if and only if the sum of alternating angles at that vertex is* $180°$*.*

A proof of this can be found in [16].

Theorem 3.3.3 (The one straight cut, OSC, theorem, [16])**.** *Any series of polygons, any map with straight edges, can be folded so that all edges, and only the edges, lie on a single line in the folded state of a CP. It is then possible to cut them all with one straight cut.*

The proof is a bit involved. It can be found in [16]. To illustrate the theorem, it is for example, possible to cut out a star or a swan with one cut. We can see clearly where the name comes from.

Flat foldable Origami has not been used for mathematical exploration as has been done with 1-fold construction. We have seen many examples of how Origami can be used to solve mathematical problems which are not all simple. They are often, however, curiosities. Trisecting angles and doubling cubes are not top of the preoccupations of today's mathematicians. The question we want to raise here is whether Origami can be applied to tackle modern mathematical issues. Can it be used to solve open mathematical problems, for instance?

3.4 Summary

We have recalled the Mathematical notions necessary to formalise Origami. Important results and some complex problems have been visited. In the following we revisit the famous 4-colour theorem and build an Origami-inspired framework for a new proof to be attempted.

4

Colourability of graphs and maps: the 4-colour theorem revisited

This chapter revisits the famous 4-colour theorem and constructs a framework inspired by Origami in which a new proof is attempted. We begin with several definitions which we will use throughout the chapter.

Definition 4.0.1. A ***planar graph*** representing a map is a graph $G = (V, E)$ consisting of a set of vertices V and a set of edges $E = (v_1, v_2)$ connecting these vertices where $v_1, v_2 \in V$.

Definition 4.0.2. The ***faces*** of a planar graph are the areas which are surrounded by edges.

We have seen in chapter 3 that a CP is a collection of straight lines constructed on the plane. Now we have our above definition, a CP can be represented as map or a straight-line planar graph embedded in the paper.

Definition 4.0.3. The ***degree*** of a vertex is the number of edges for which that vertex is an endpoint. Thus odd and even degree vertices connect to an odd or even number of edges.

Definition 4.0.4. Vertices connected by an edge are referred to as ***adjacent vertices***.

Definition 4.0.5. Faces which share an edge are referred to as ***adjacent faces***.

Definition 4.0.6. A ***path*** is a sequence of vertices with the property that each vertex in the sequence is adjacent to the vertex next to it.

Definition 4.0.7. A ***circuit*** is a path which starts and ends at the same vertex.

Definition 4.0.8. A ***connected graph*** is a graph where there exists a path between any pair of vertices.

Definition 4.0.9. A ***bridge*** is an edge, the removal of which would separate the graph and stop it from being connected.

Definition 4.0.10. The ***dual*** graph of our planar graph or map is a graph constructed with one vertex for each face in the map and an edge between adjacent faces. This is shown in figure 4.8 with the green graph being the dual.

Definition 4.0.11. A ***face colouring*** on a map is one where we colour the faces

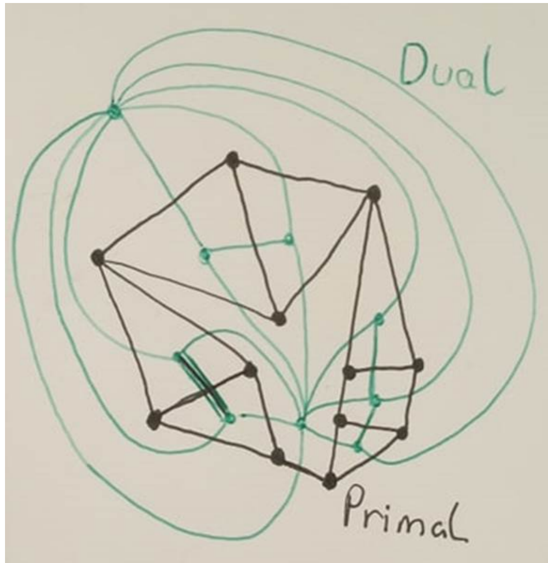


Figure 4.1: Example of a dual graph

of the graph such that no two adjacent faces are assigned the same colour. Where n colours are required to colour the graph it is referred to as an $n-$colouring.

Definition 4.0.12. A ***vertex colouring*** on the dual is one where we colour the vertices of the graph such that no two adjacent vertices are assigned the same colour.Where n colours are required to colour the graph it is referred to as an $n-$colouring.

When we look at crease patterns, we will be discussing face colourings. Therefore, a crease pattern is $n-$colourable if at most n colours are required to colour it such that no adjacent regions/faces have the same colour.

There is a well known result for any flat foldable CP. This result is

Theorem 4.0.1 (2−colour theorem, [311])**.** *Maps of flat foldable crease patterns are* 2−*colourable.*

Proof. Take a flat foldable Origami object in its folded state and consider the faces on the CP. Each of these faces is created in one of two orientations, face up or face down. Moving over any crease will take you to a neighbouring polygon, as you have moved over a crease this polygon will be the opposite way up. Therefore, you can colour all polygons based on which way up they are in the folded state. If you label all of these and open it up you will have a two colour pattern for the model. □

The 4-Colour Theorem, is a well-known and popular result in modern mathematics and goes back a long way. The conjecture was first raised by Francis Guthrie, when he was a student at UCL in 1852, studying under De Morgan [312]. It can be stated as follows.

Theorem 4.0.2. *Given any separation of a plane into regions, no more than four colours are required to colour the regions so that no adjacent regions have the same colour.*

A first proof of the 4-Colour Theorem was due to Alfred Bray Kempe who announced it in the journal Nature in 1879. It was, however, later shown to be flawed by a certain Percy John Heawood. This meant that after just 11 years, the 4-Colour Theorem became the 4-Colour Conjecture again. It resisted the assaults of many great mathematicians of the late 19^{th} and 20^{th} centuries including De la Vallee Poussin, Heawood, Veblen and Birkhoff, on both sides of the Atlantic. It was not until 1976 that a proof has at last been put together. That was by Appel and Haken, [313]. Their proof consisted of an exhaustive analysis of discrete cases which was completed by a computer meaning that the proof is criticised by some [148]. Although shorter proofs have been found since, all require a computer to check through hundreds if not thousands of cases thus, there still is the possibility that a more elegant, short proof exists.

Here we discuss the connections between the n-colour problems and Origami. Before attempting to prove such theorems with Origami let us start by looking at the links between Origami and colourability. Recall from early in this thesis that a CP can be considered as a graph where creases define edges and vertices their intersections. It is also a map where regions/faces are bounded by the creases or edges. We propose a way to link the two and sketch an alternative proof.

4.1 An Origami based exploration of the colourability of maps

In the following we relate a map to a Connected Planar Graph, CPG, drawn on the plane of a piece of paper. Using the OSC theorem, we know it must be possible to flat fold our paper in such a way as to align all the edges of the CPG onto a line. We can also fold along this line and produce a flat foldable crease pattern, CP, which is also a map but now has added-edges. From the 2-colour theorem for flat foldable Origami we know that the map of the CP is $2-$colourable. We then show ways in which we can remove added-edges from this map and retain the $2-$colourability property. This is shown to be possible until a point at which we prove no more of our added-edges can be removed without requiring an additional colour. Finally, we look at the properties of the remaining added-edges and present some ideas about how they could be removed in such a way that we will only ever require four colours.

4.1.1 Observations of colourability of graphs

We note that one of the novel approaches we have explored looks at addition and removal of vertices and edges; the aim is to produce a new graph for which we know the colouring and then subsequently reverse the changes while maintaining knowledge of the graphs colourability. This process is unusual in graph theory as often one would look for properties of the graph they were presented with; here we look for properties of another graph and how those change when we alter the graph.

Here we make some simple observations which show that it is sometimes difficult to see how changing a graph would affect its face colourability. This is followed by some examples to demonstrate that each of these cases are possible,

1. There exist n-colourable connected planar graphs which require more colours when one or more edges are removed.

2. There exist n-colourable connected planar graphs which require fewer colours when one or more edges are removed.

3. There exist n-colourable connected planar graphs in which the removal of some edges has no effect on n.

Examples:

1. Example 1, figure 4.2, requires three colours, the diagram demonstrates how removing edge A has no effect on the colourability, removing edge C reduces the requirement of number of colours to two, and removing edge B increases the requirement to four.

2. Example 2, figure 4.3, requires 2 colours; we remove edge A and now require

three colours.

3. Example 3, figure 4.4, requires 4 colours, removal of edge A reduces the requirement to three.

From the above we show that edges can be of three types:

1. Those that when removed leave the $n-$colouring intact.

2. Those that affect it upwards, i.e. more colours are needed.

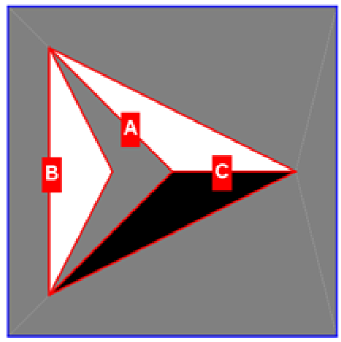
3. Those that affect it downwards i.e. fewer colours are needed.

Note that these observations would hold for adding edges as it is simply the process in reverse. Note also that the effect of adding or removing edges is, at least visually, difficult to predict.

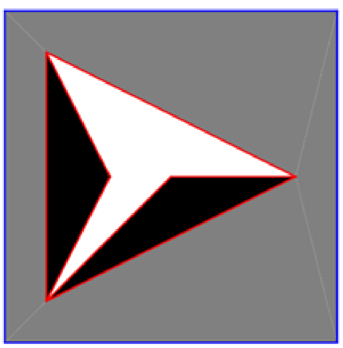
4.2 An Origami approach

Given a map to colour, this map is equivalent to a CPG, $G = (V, E)$ consisting of a set of vertices $V = (1, \ldots, v)$ and a set of edges $E = (1, \ldots, e)$. Edges create a set of faces for which we aim to provide a colouring. We also colour the outside of the map. At this stage we know the following:

1. The map is connected; if it were not then we can colour it as if it were 2

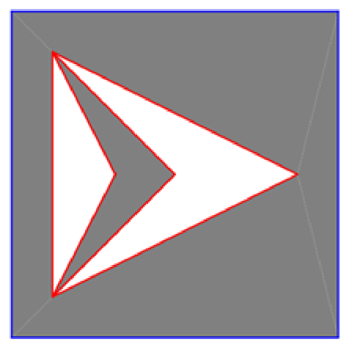


(a) Example 1



(b) Example 1 with A removed

(c) Example 1 with B removed



(d) Example 1 with C removed

Figure 4.2: Examples of removing edges from graphs

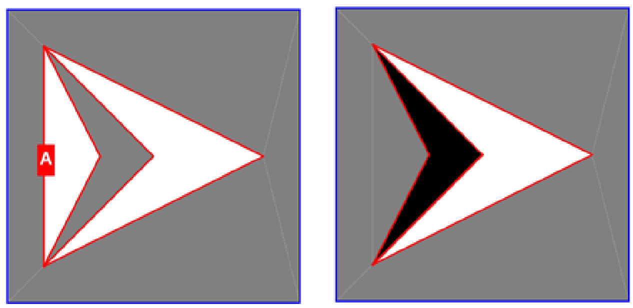


Figure 4.3: Example 2

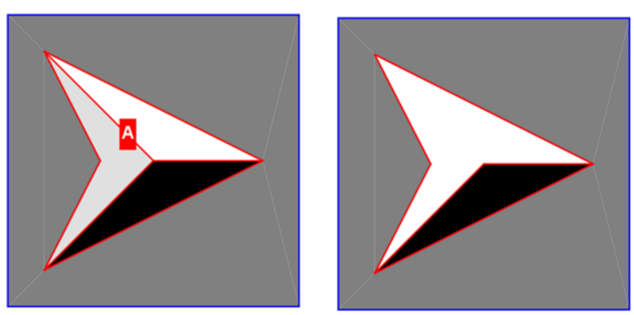


Figure 4.4: Example 3

independent maps.

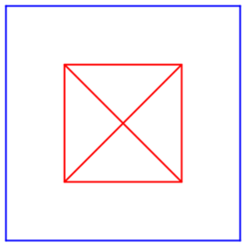
2. It has no vertices which are only connected to a single edge, as this does not separate regions on the map.

We assume that it is possible to construct an isomorphic graph to this map such that it has straight edges. We will refer to these edges, vertices and faces as the initial edges, vertices and faces, of the graph/map. We draw our map on the plane of a piece of paper and apply the OSC theorem, which stipulates that there is always an Origami method which will fold the edges of a graph onto a single line. We note here that the outside of the map is included thus for any colouring problems we look at the outside as a region to be coloured.

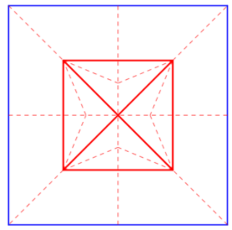
It is hard to tell which side of this line each face of the CPG will be, however we can fold along it placing all the faces on one side of it and maintaining a flat folding.

Unfolding results in a CP which consists of the initial CGP plus a number of new edges and vertices which we refer to as added edges and added vertices. Since this CP is flat foldable it is $2-$colourable by the Origami $2-$colour theorem. Figure 4.5 is an example of this.

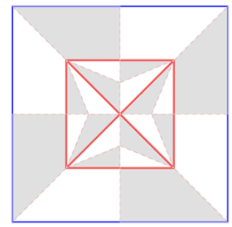
To recap, we begin with a CPG which is equivalent to a map, which leads to a CP also equivalent to a map. We have created more regions but the map is now $2-$colourable. The application of the OSC theorem results in an explosion



(a) Initial CGP



(b) CP with added edges



(c) A 2-colouring of the CP

Figure 4.5: Example of the Edge adding process

of regions as their number has vastly increased.

4.2.1 Removing edges and maintaining 2-colourability

The aim is to strip away added edges of this extended map; we show that one can remove specific edges without requiring any additional colours. Some added edges of the extended map, when they form paths across the paper or cycles, can be removed without removing 2−colourability, as the following lemma states.

Lemma 4.2.1. *We can remove added edges from this map where they make a path from one edge of the paper to another or where there is a cycle.*

Proof. We first look at the case of cycles. Where we have a cycle, there exists both a sub-map of the map inside the cycle and another sub-map outside it. No faces can exist both inside and outside this sub-map without being separated by an edge. If we invert the colouring of the internal sub-map, A becomes B and B becomes A, then the colouring inside will still be valid, as we remove all edges at the boundary of the sub-maps the bordering faces join thus maintaining the $2-$colourability property. We can also do this for paths between edges of the paper as they also separate the map into two sub-maps, one side of this can have colouring flipped and the bordering faces join. □

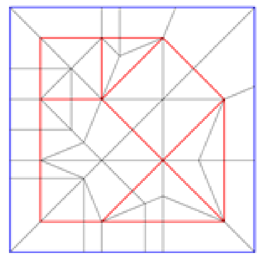
We note here that we may not keep the property of foldability when we remove these edges. Figure 4.6 gives examples of how edges can be removed differently. Edge removal has been grouped into steps so that it is clear which edges are being removed and whether they are cycles or not.

In Trial 1 we are given a 3-colourable CPG, in Trial 2 we are given a CPG which requires four colours.

As we keep stripping edges there comes a point where we can no longer remove edges without requiring additional colours.

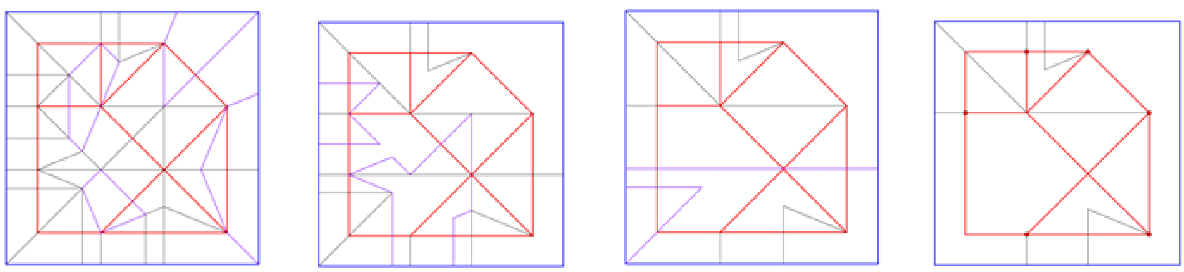
Lemma 4.2.2. *The resulting map from removing all possible added edges while maintaining $2-$colourability cannot be reduced further without requiring additional colours.*

Proof. By OSC theorem our CP is flat foldable, thus Maekawa's theorem must hold at all vertices not at the edge of the paper. A consequence of Maekawa's theorem is that the number of folds at each vertex must be an even number.



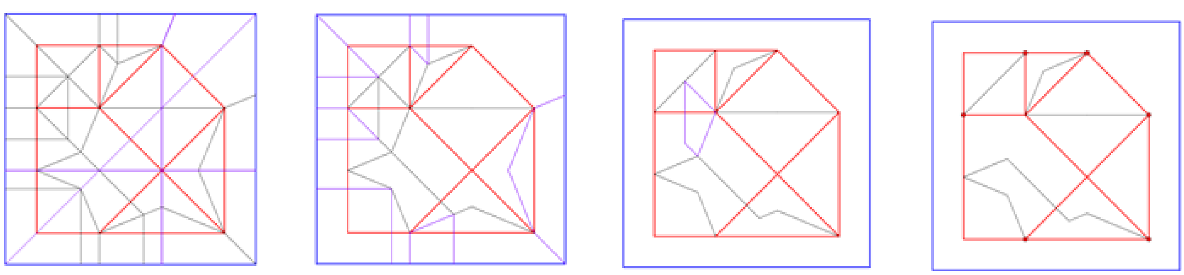
(a) Initial CP

Trial 1



(b) Step 1 (c) Step 2 (d) Step 3 (e) Final Solution

Trial 2



(f) Step 1 (g) Step 2 (h) Step 3 (i) Final Solution

Figure 4.6: Examples of Crease Removal

When we remove paths or cycles they will always remove two edges from a vertex unless it is at the edge of the paper, thus all vertices in the remaining map will still have an even number of initial and added edges. If cycles of added edges exist, we remove them. If paths of added edges exist from one edge to another edge, we remove them. Thus once we have removed all of these, the added edges we have remaining will make a forest. We can now show why we cannot remove any of these edges. If we attempt to remove an edge in the middle of a tree where at each endpoint there are other edges, we must remove an edge connected to it. If we did not then we would leave this vertex with an odd number of edges, thus also an odd number of regions meeting at a point in our map. This would require at least three colours. Thus we must remove an edge connected to it. We have to do this for both ends of the edge unless we hit the edge of the paper and we know we have no paths between two edges of the paper as well as no cycles. Thus we will always have at least one endpoint of our path where the edge cannot be removed without adding an additional colour. $\square$

Assuming the initial CGP to be $2-$colourable, we have the following lemma:

Lemma 4.2.3. *If the original CPG was $2-$colourable we must be able to remove all added edges from our CP, then all added edges can be removed without altering its 2-colourability.*

Proof. If we are given a $2-$colourable map it can be shown that it must have an even number of edges meeting at any vertex. By Maekawa's theorem our CP also has this property apart from at the edges of the paper. When we remove edges we remove paths or cycles, thus apart from at the edge of the paper we maintain the even number of edges at any vertex. Thus there cannot exist any vertex which has an irremovable endpoint as they are only irremovable if the path terminates at a vertex with an odd number of added edges. □

There are many possible ways we can choose loops or edge to edge paths to remove. Thus there are many possible sets of remaining edges which cannot be removed. We now look at these remaining added edges and try to characterise their properties and quantity.

We have already shown that as all cycles are removed we are left with a forest. Although we do not know the position of the edges we find we do know where the endpoints are.

Lemma 4.2.4. *In our reduced CP, after all removable edges have been removed maintaining* $2-$*colourability, the endpoints of the trees made by the added edges are located at each point where an odd number of edges in the initial CPG met.*

Proof. Given that our CP is flat foldable, before any creases are removed, by Maekawa's theorem, we know there must be an even number of creases at each vertex which is not at the edge of the paper. Thus there must be an odd number of edges added at any point where an odd number of edges meet in the CPG. □

4.2.2 Removing the remaining edges

We now look to see if we can remove any other edges in this CPG. If we can find a method to remove the remaining added edges, and can do this using only two colours we would have the basis for a proof of the four colour theorem.

Lemma 4.2.5. *If a tree exists such that it only passes through a single face on the initial map, and all the adjacent faces on the initial map are free from added edges then it can be removed at the cost of requiring a third colour.*

Proof. This is obvious as this face is only surrounded by faces of the initial two colours, thus it is possible to use the third colour without affecting the rest of the map. □

This can be extended as follows.

Lemma 4.2.6. *If a tree exists such that it passes through a group of faces on the initial map which can be $n-$coloured, and the adjacent faces on the initial map do not contain any added edges, then the tree can be removed and n colours can be added as a $2-$colouring of this section.*

Proof. This is obvious in the same way as for the previous lemma since this face is only surrounded by faces of the initial two colours, it is possible to use the third and fourth colours without affecting the rest of the map. □

This is equivalent to saying if we can show that only two colours are required to colour the faces with remaining added edges then we can construct a proof of the $4-$colour theorem.

Lemma 4.2.7. *Once we remove an added edge which forces us to require a third colour we cannot remove any other combination of the remaining added edges to reduce the colourability back to two.*

Proof. We know we will now have a vertex with an odd number of edges, all of which will be initial edges. Thus we cannot remove any further edges to make even, and thus 2-colourtability cannot be regained. □

It might be possible to optimise the removal process at the $2-$colourable stage such that the remaining edges are easier to remove. Perhaps this could consist of minimising crossings, maximising the space between the edges, attempting to empty specific faces of the CGP from added edges.

Another attempt is made to find a proof for the 4-colour theorem in Appendix A, some these look promising, however the work on these is not complete.

This includes, attempting to get remaining edges to fall into a 2-colourable region using moated faces. Exploring other ways of producing the added edges to make the CPG 2-colourable and exploring circuits which separate the region into two, 2-colourable regions.

4.3 A statement equivalent to the 4-colour theorem

During our Origami inspired exploration of the 4-colour theorem, we searched for an equivalent statement to the famous problem. Many have searched for statements that are equivalent to the 4-colour theorem as these may shed extra light on the problem and can be used to help inspire new proofs. One example of this is

Is it possible to colour the edges of a cubic planar graph with only three colours.

This is stated as equivalent to the 4-colour problem in an OSME paper by Hull [147] and although it is not one of the main focuses of Hull's work, the authors comment, "Thus the possibility exists that by three colouring cubic modular works, further light might be thrown on the search for a short proof of the 4-colour theorem", provided motivation for some of our exploration.

Following is a statement which we will later prove, providing an equivalence to the 4-colour theorem.

Theorem 4.3.1. *Given any connected planar graph corresponding to a map, the faces of the map are 4 colourable if and only if a circuit or multiple circuits can be found covering all odd vertices in the map, such that each circuit contains an even number of odd vertices and no edge is contained in more than one circuit.*

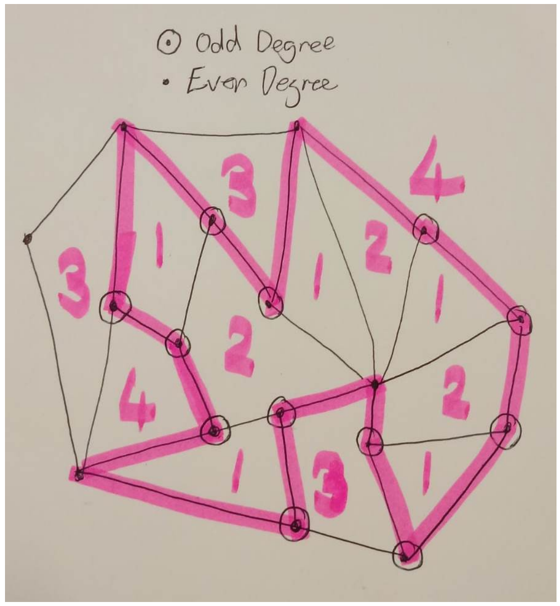


Figure 4.7: Example of a circuit covering all odd vertices

An example of this is given in figure 4.7 where the circuit is represented by the pink line.

Our literature search did not find this equivalence statement in the pre-existing literature and we believe this is new. It is, however, somewhat similar to the Tait colouring which was used in an unsuccessful attempt to prove the 4-colour theorem. We show how this is different to this colouring later in this section.

We note here that we are interested mainly in face colourings but we will also be looking at the vertex colouring of the dual graph to the map, this is a view-point which is potentially less explored than that of vertex colouring alone and is certainly not the norm for exploration of statements equivalent to the 4-colour theorem.

We shall begin with some definitions:

Definition 4.3.1. A ***planar graph*** representing a map is a graph $G = (V, E)$ consisting of a set of vertices V and a set of edges $E = (v_1, v_2)$ connecting these vertices where $v_1, v_2 \in V$.

Definition 4.3.2. The ***degree*** of a vertex is the number of edges for which that vertex is an endpoint. Thus odd and even degree vertices connect to an odd or even number of edges.

Definition 4.3.3. Vertices connected by an edge are referred to as ***adjacent vertices***.

Definition 4.3.4. Faces which share an edge are referred to as ***adjacent faces***.

Definition 4.3.5. A ***path*** is a sequence of vertices with the property that each vertex in the sequence is adjacent to the vertex next to it.

Definition 4.3.6. A ***circuit*** is a path which starts and ends at the same vertex.

Definition 4.3.7. A ***connected graph*** is a graph where there exists a path between any pair of vertices.

Definition 4.3.8. A ***bridge*** is an edge, the removal of which would separate the graph and stop it from being connected.

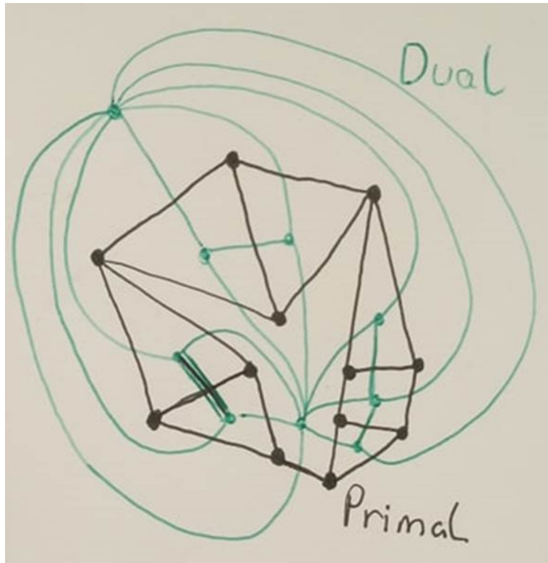


Figure 4.8: Example of a dual graph

Definition 4.3.9. The ***dual*** graph of our planar graph or map is a graph constructed with one vertex for each face in the map and an edge between adjacent faces. This is shown in figure 4.8 with the green graph being the dual.

Definition 4.3.10. A ***face colouring*** on a map is one where we colour the faces of the graph such that no two adjacent faces are assigned the same colour.

Definition 4.3.11. A ***vertex colouring*** on the dual is one where we colour the vertices of the graph such that no two adjacent vertices are assigned the same colour.

For our proof of theorem 4.3.1, the map is the primal graph and we look to colour its faces, where we refer to the dual we will be looking to colour the vertices.

In the definition of a dual we note that the property which is maintained is the adjacency of faces and vertices. This means that a colouring problem will be the same in the dual or the primal.

Lemma 4.3.2. *Given our definition of the dual a face colouring on the primal graph is equivalent to the vertex colouring on the dual.*

Proof. This is given by our definition which defined each vertex in the dual as representing a face in the primal graph, and vertices in the dual are adjacent, if and only if, a face is adjacent in the primal graph. □

Where our primal graph has several dis-connected components and we colour the faces, the outside face is shared but, as this is the only shared face, it does not affect how many colours are required for the overall graph. In the case of vertex colouring a dual graph with disconnected components, there will also be no effect on colouring as no paths exist between the disconnected regions of the graph, so each can be coloured as a separate entity. Because of this, we only need to concern ourselves with connected graphs.

We ask now if our definition of the dual can produce all connected planar graphs.

Lemma 4.3.3. *Given any connected planar graph we can produce a dual of this graph which is also a connected planar graph. Also, this, the dual of the dual, is equivalent to the original graph.*

Proof. It is obvious that we can always place a vertex in every face on a connected planar graph.

Given that the graph is planar, when constructing a dual, we can always construct an edge from one of the vertices placed in a face to any location along an edge bordering that face in the primal graph where that edge would continue to the point placed in the neighbouring face. Therefore we can always construct the edges for our dual.

Vertices in the dual are created in each face of the primal graph, thus the number of vertices in the dual is equal to the number of faces in the primal graph. For every edge in the primal graph an edge is constructed in the dual thus the number of edges is unchanged, thus by Euler's Formula "$V - E + F = 2$" the number of faces in the dual is equal to the number of vertices in the dual.

Given any graph, to find a vertex colouring we can find the dual of this graph as a map for which we need to provide a face colouring. □

As the 4-colour problem's traditional statement refers to a vertex colouring on any planar graph, we note that when vertex colouring it is not required to look at graphs with edges which connect a vertex to itself. This would mean that a vertex would need to not have the same colour as itself which is obviously not possible. The dual situation to this is a tree or edge in the primal graph where both sides are in the same face. Therefore all vertices of degree $leq1$ can be ignored along with any edges which connect a vertex to itself.

4.3.1 Proof that our circuits imply 4-colourability

In order to prove theorem 4.3.1 we will first show that, if circuits can be found covering all odd vertices in the map, such that each circuit contains an even number of odd vertices and no edge is contained in more than one circuit, then the map is 4 colourable. We begin with a standard graph theory result.

Lemma 4.3.4. *The Handshake lemma: Every graph has an even number of vertices of odd degree*

The proof for this is straight forward and this is a standard graph theory result. Each edge connects exactly 2 vertices, thus sum of the degrees of all vertices must be even. This implies that the number of vertices of odd degree must also be even.

We now look at a single circuit, or non intersecting circuits, on our connected planar graph. We note that a circuit which does not repeat vertices clearly separates the plane upon which the graph is drawn into two regions; we shall refer to these as the inside and the outside although they are interchangeable. An example of this is shown in figure 4.9.

Where a circuit passes through a vertex more than once there are several different ways in which the circuit can be traversed. Note, every circuit contains a Euler

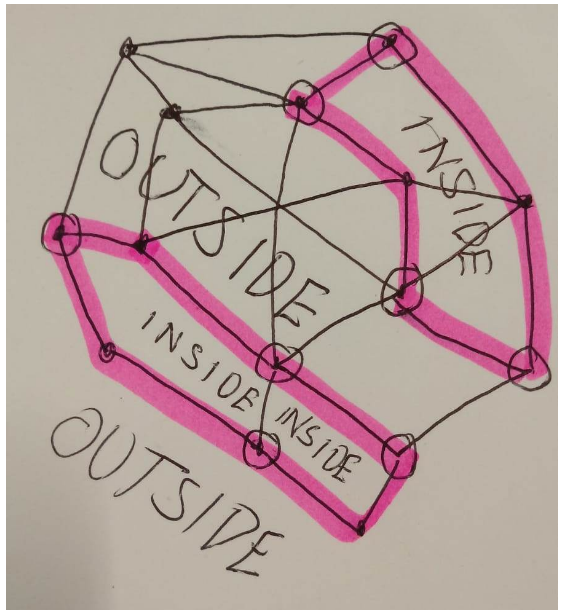


Figure 4.9: Example of two circuits separating the plane into two regions

cycle.

Lemma 4.3.5. *Given a single circuit, it always separates the faces of the graph into two regions; an inside and an outside.*

We note here that this has similarities, to work in knot theory on the colouring of link diagrams but we cannot find a graph theory statement/proof for this and have provided a proof of this below; an example of this is shown in figure 4.10.

Proof. If the circuit does not have any repeated vertices then a group of one or more faces will clearly be bounded by this circuit. By following the circuit with the edges of the circuit on our left we can define everything to our left as inside

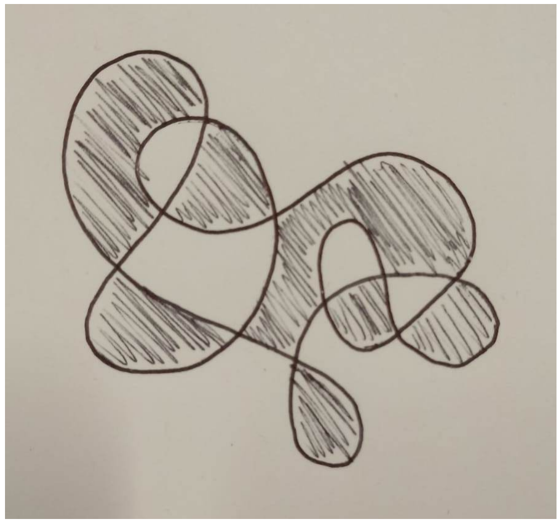


Figure 4.10: Similarities between a knot and a circuit

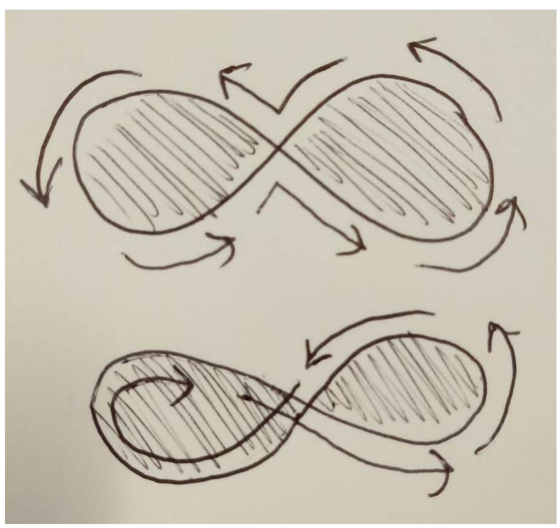


Figure 4.11: Two ways to move round a circuit

the circuit and everything to our right as outside.

Where we have intersections we need to be more careful as there are multiple ways we can travel around the circuit as shown in figure 4.11.

One way to do this is traversing the circuit in the same way with the edges of the circuit to the left and defining everything to our left as inside and everything to our right as outside; however, whenever we pass one of the edges of the circuit which we do not follow, we change our definition so that the left is now the outside and the right is the inside. We do this each time we pass an edge in the circuit for both the left and the right.

If we look only at the edges and vertices contained in our circuit, the degree of every vertex must be even by definition; therefore the difference between the number of edges not followed on the left and the right when passing a single vertex will always be ± 2. Thus there will never be a case where the left and right sides of the circuit will either be both inside or both outside. $\square$

We also now provide another, slightly simpler proof for this.

Proof. Irrespective of how many intersections our circuit has, we know that when looking only at the edges and vertices in the circuit the degree of every vertex must be even. Also there must be a Euler cycle covering it. Therefore the face colouring of the graph of this circuit is bipartite and is two colourable. $\square$

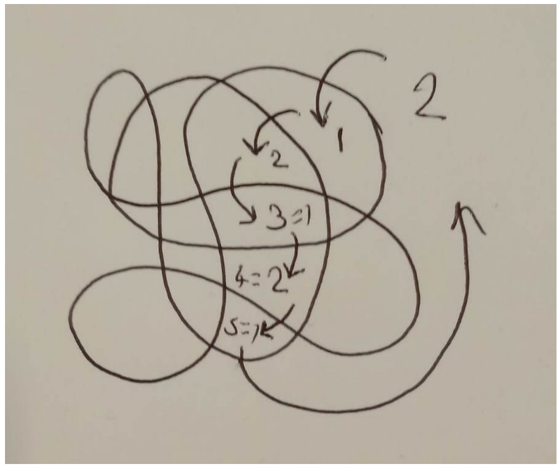


Figure 4.12: Method to produce a 2-colouring

We also note that given all vertices have even degree it is possible to colour the faces by counting how many edges are crossed to get to the outer face, this will be either even or odd and thus you can colour the faces with two colours. figure 4.12 shows this.

Above we noted that every circuit contains a Euler cycle, we now also note that this also applies to any interconnected circuits which intersect at only vertices and do not share edges. These will also contain a Euler cycle as at any vertex where they intersect you can follow one cycle then the other to complete a Euler cycle on both circuits.

Lemma 4.3.6. *Given a collection of circuits, that do not share edges, they will always separate the faces of the graph into two regions; an inside and an outside.*

Proof. Given that the edges of any individual circuit will separate the faces on

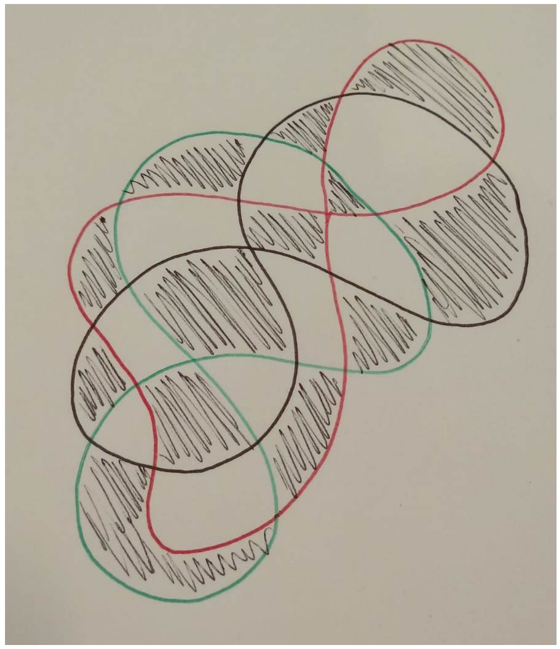


Figure 4.13: Faces between multiple circuits still can be 2-colourable

the plane into two regions, when adding a second circuit which does not share edges we only need to invert the colouring on the inside of the new circuit. This is always possible unless an edge is shared when it would cause two adjcent faces to have the same colour. It follows that this process can be repeated and any number of circuits will split the faces in the plane the same way. □

So far we have shown that the circuits as defined in theorem 4.3.1 will always separate the faces into two regions. We refer to these as inside and outside although they are interchangeable.

Looking at one of these regions, the "inside", we aim to colour the faces inside this region with only 2 colours.

Definition 4.3.12. ***Internal vertices*** shall be those verticies which are not part of the outer face.

When we look at the graph representing the region bounded by a circuit, or part of a circuit, all internal vertices will have even degree. We note that this is because any vertices of odd degree will be on a circuit.

We now look to colour the internal faces of this graph. Here we only look at those edges on the border or the inside of the region, there may be edges extending to other parts outside of our region but we ignore these.

Lemma 4.3.7. *A face colouring of a connected planar graph, such that all internal vertices have even degree, requires only 2 colours if the outer face is ignored and the number of edges which extend from the vertices of the outer face is even.*

We note here that the above theorem is so natural should be known, this may already exist in some other form however it was not obvious from a literature search if it had been looked at before and thus it may be new. An proof has been given bellow.

Proof. The edges of the outer face have an even number of endpoints, this is obvious as each edge has two endpoints. All the remaining edges extend from the outer face into the region. These edges separate all the outermost faces of the graph.

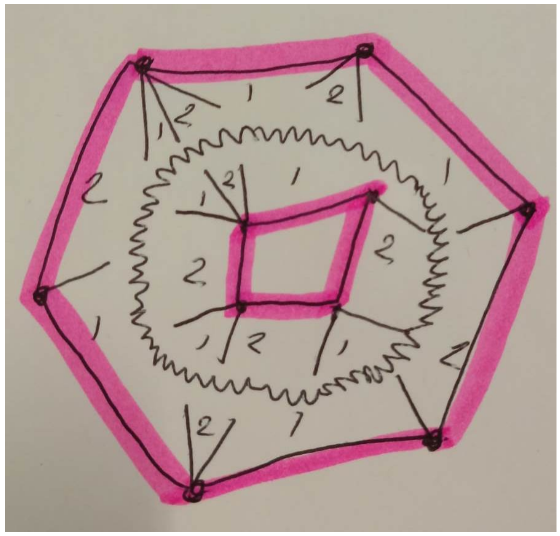


Figure 4.14: An example of circuits within circuits

We now look at vertex colouring of the dual graph. As all internal vertices on the primal graph are even degree, a cycle with an odd length can only exist if it passes through the vertex representing the outer face. If this vertex is removed all cycles have even length therefore the graph is bipartite and vertices are 2-colourable.

This implies that the internal faces on the planar graph are also 2-colourable. □

There may exist circuits within circuits as in 4.14, this is not an issue as long as both the boundaries have an even number of edges extending from them into the region. As this is equivalent to the region having multiple outside faces.

It just remains to show that there will be an even number of edges extending into any region of faces from the circuits which form a boundary to that region.

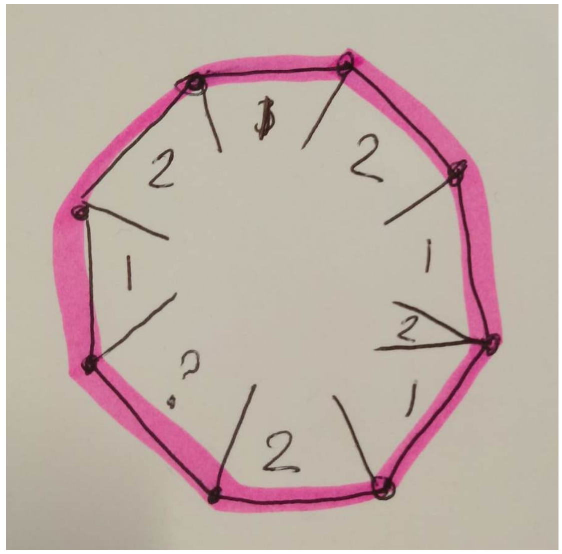


Figure 4.15: counting an odd vertex

Every region in every circuit must have an even number of edges extending into it from the boundary of the region; this can be seen in figure 4.15 and figure 4.16. To ensure this property is true it is sufficient to require that each circuit has an even number of vertices of odd degree.

We note here a counting problem. Where an odd degree vertex is passed more than once, such as A_1 in figure 4.17, does it count for one or more than one circuit and which circuit does it count for? We solve this by insisting that either it counts only for 1 circuit but moreover where circuits intersect they can be referred to as one circuit. Thus we do not count the number of times an odd vertex is passed by a circuit; instead we count how many odd degree vertices are in a given circuit or connected group of circuits.

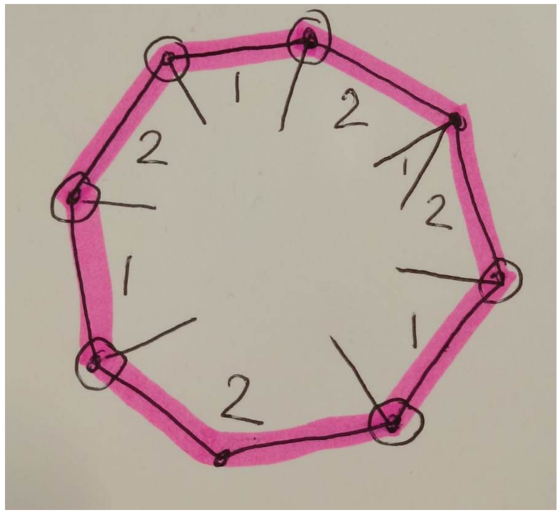


Figure 4.16: counting an odd vertex

Here we note that if the number of edges extending into the region from the boundary is odd, it would not be possible to find a 2-colouring and there would also be an odd number of edges extending out from this region, as there is an even number of edges in total.

If we assume the existence of an internal region which has an odd number of edges extending into it, then the outside would also be odd, this would leave an uneven number of odd points outside this and so on. At the outermost points and the innermost parts of the graph this would not be rectifiable as odd degree vertices would be uncovered.

Thus to ensure no odd degree vertices or odd number of edges inside a region we ensure all circuits have an even number of odd degree vertices. Therefore as

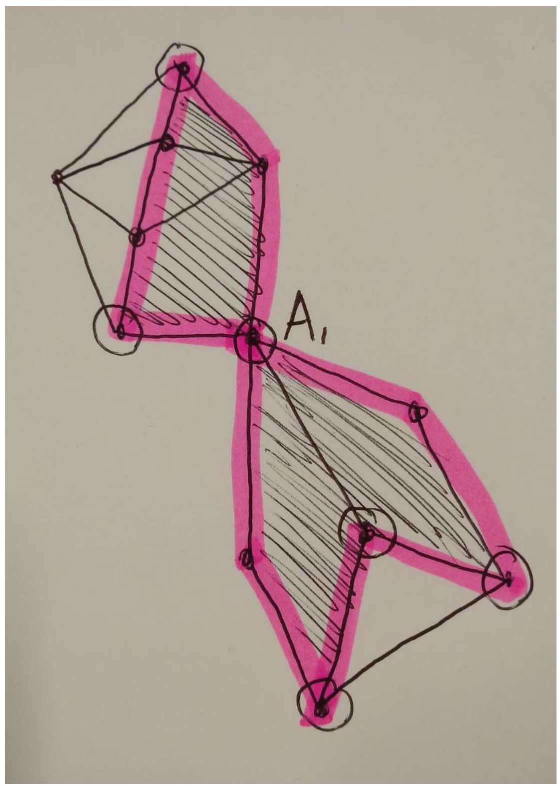


Figure 4.17: counting an odd vertex

every region has an even number of edges extending from it, this must hold for the whole graph.

4.3.2 Proof that 4-colourable maps contain our circuits

We aim to show that if given a planar graph with a face colouring then circuits exist.

If the map is 4 (face) colourable (a,b,c,d), then it is always possible to choose 2 pairs of 2 colours and separate the map into 2 distinct regions. (a,b – b,c or a,c-b,d or . . .)

These regions must be bounded by edges which construct a circuit. This is obvious as there is always an edge separating the regions and they are bounded by this; the outside of any region can always be followed by a circuit. We shall call these the "border edges" as they form the border between the two regions

Looking at one of these 2 regions, (refer to this as the inside), it can not contain any odd degree vertices which are not on the "border edges" of the region.

There will be an even number of edges extending into any region from these borders as otherwise it would need more than 2 colours internally.

Finally, as the inside and the outside of each circuit will have an even number of edges extending form it, there must be an even number of odd edges in that circuit.

Above, we have shown the construction of a proof for theorem 4.3.1.

To conclude this section, we note that these circuits are similar to patterns found in Tait's colouring [314] which was used for a false proof of the 4-colour theorem. The conjecture stated:

Every 3-connected planar cubic graph has a Hamiltonian cycle through all its vertices

The conjecture was disproved using the "Tutte graph" to provide a counter example.

In figure 4.18 we show the Tutte graph and our circuits showing that this is not a counter example for our theorem.

4.4 Colourability of non flat-foldable crease patterns

Recall that, in section 4.2, as we removed edges we maintained the $n-$colourability of the CP, however, we did not conserve foldability. We know that for flat Origami

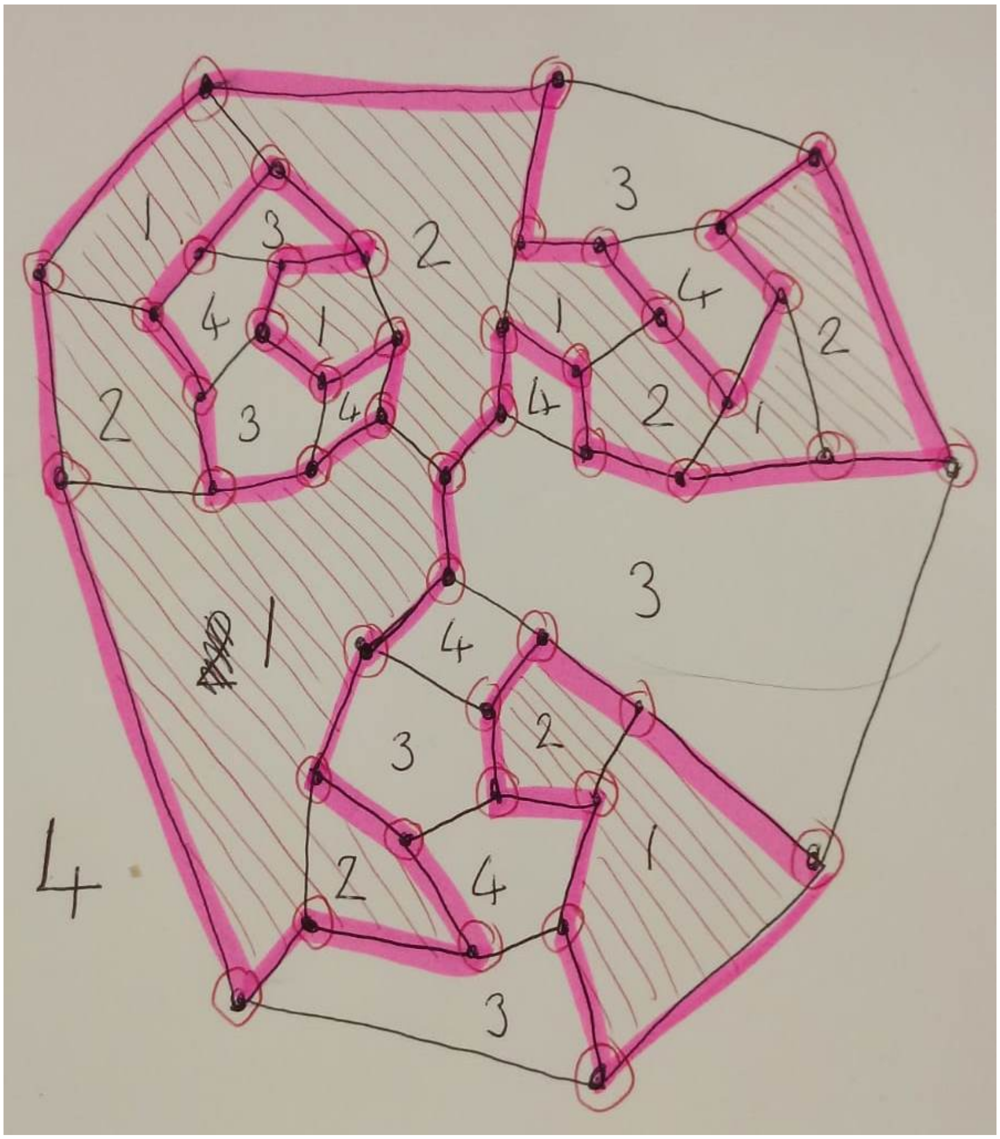


Figure 4.18: The Tutte graph

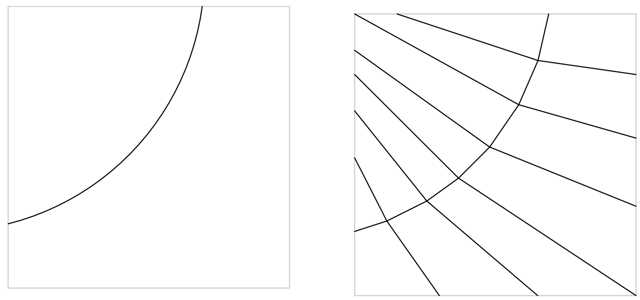


Figure 4.19: Straight line approximation to a curved fold

the CP is 2−colourable. However, does 2−colourability of a CP imply flat foldability and what is the colourability of non-flat Origami?

Now we conjecture that the maps of any foldable crease patterns are 3−colourable. To prove this, we need some definitions. Note that the statement covers all crease patterns including non-flat foldable constructions and those with curved creases. Recall that it was assumed that the paper cannot stretch and so one can only fold curved creases in such a way that there exists a cross section with a straight line. To deal with CPs with curved creases, consider that a crease is placed along all cross sections of a curved crease that results in a straight line, thus a curve in the paper can be approximated by a series of very short straight creases, (see Figure 4.19). At each of these creases and their intersections the paper behaves in the same way as with any other Origami using straight creases. Thus, the proof of the conjecture is only required for Origami with straight creases and rigid faces in the CP.

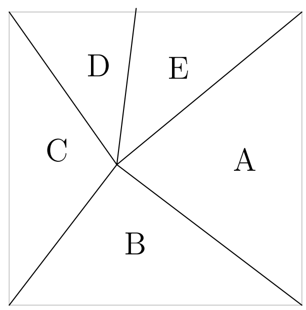


Figure 4.20: A not flat-foldable, not 2−colourable crease pattern

Lemma 4.4.1. *There are CPs whose maps require more than two colours.*

Proof. Figure 4.20 provides such a CP. Note that it is foldable but not flat foldable. We now attempt to colour this CP with 2 colours. It can be seen that colouring region A with colour 1 means that regions B and E must have colour 2, which implies that region C and region D must both be coloured with colour 1. As they are adjacent a third colour is needed. □

Let us now show that to colour the maps of foldable CPs, flat-foldable or otherwise, at most 3 colours are needed.

Definition 4.4.1. The ***Dual Graph (DG)*** of a CP map is a graph whose vertices correspond to the regions/faces of the map and whose edges link adjacent regions. (See Figure 4.21.)

We will apply our colouring to the DG and look at the properties of this graph to prove it only requires 3 colours.

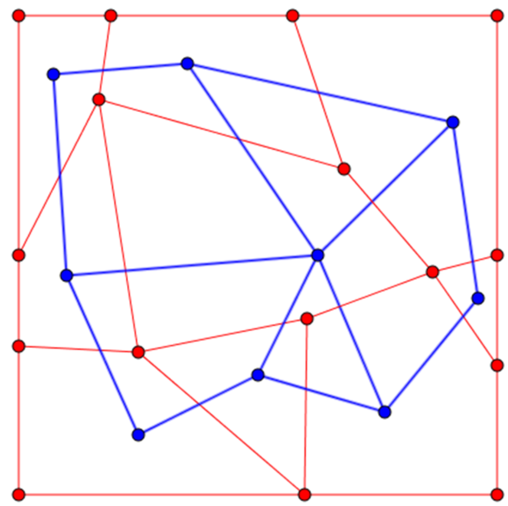


Figure 4.21: CP to DG example; CP in red, DG in blue

Definition 4.4.2. A ***Vertex colouring*** on a DG of a crease pattern is made when we colour each of the vertices in the DG so no two adjacent vertices have the same colour.

Vertices in the DG are adjacent if they are joined by an edge, faces in the CP are adjacent if they are separated by an edge, as these are equivalent, the definitions for colouring are also equivalent. Thus, a DG vertex colouring is similar to a CP face colouring. Let us now show that all DGs are planar.

Definition 4.4.3. A ***planar graph*** is a graph that can be drawn on the plane in such a way that its edges intersect only at their endpoints.

Lemma 4.4.2. *DGs are always planar.*

Proof. It is obvious that all CPs are planar, due to our definitions of points and lines that make up the CP. In a CP produced on a plane, wherever creases are made to intersect, vertices are always formed. Therefore, a CP is always planar. When constructing a DG we place a vertex in each face of the CP. In order for edges to cross on the produced DG there must be two pairs of faces connected by edges in the same location, which is not possible as the CP is planar. □

We now introduce a definition for a triangle free graph and aim to show that DGs are also triangle free.

Definition 4.4.4. A ***triangle free graph*** is one in which no three vertices form a triangle of edges. Equivalently, it is a graph with no 3-cliques.

Note: A triangle free graph is equivalently defined as a graph with clique number less than or equal to two, a graph with girth less than or equal to four, a graph with no induced 3-cycle or a locally independent graph. This is illustrated in Figure 4.22.

Lemma 4.4.3. *The DG must be triangle free.*

In order to prove that the DG is triangle free we attempt to create a triangle in a DG and show that this is not possible. In other words we take the proof by contradiction approach.

Proof. A triangle in the DG represents three faces in the CP which are connected

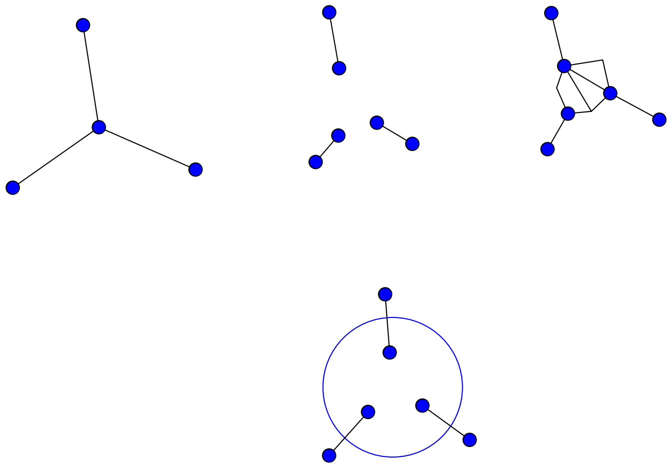


Figure 4.22: Possible ways to create a triangle in the dual graph

in pairs by edges which meet at edges in the CP. These edges will either meet at a vertex in the crease pattern, will continue to a hole in the folding material or will join with a more complex crease pattern in the centre. In each case a band is produced in which there are no creases other than the three radiating from the centre, this is seen in Figure 4.22. If the first of the three creases is created as a valley of fold angle θ such that $180^\circ \geq \theta > 0^\circ$ then both sections either side of this fold will be oriented towards the observer. The two other creases are connected by one section of paper which is not folded. However, this piece will no longer fit without bending as both creases have been brought closer together. The same applies for a a mountain fold, thus it is not possible to orient these creases such that they are folded unless two of the edges are in the same location or one of the sections is curved. $\square$

We introduce one final theorem which we will use in our proof.

Theorem 4.4.4. *Grötzsch [315]) Every triangle-free planar graph is 3-colourable,*

.

Using the above we are now able to prove our generalised Origami colourability theorem.

Conjecture 4.4.5. The CP of any rigid foldable Origami structure is 3-colourable.

Proof. By our example, Figure 4.20, we know some CPs may require three colours, thus it only remains to show that we never need more than three colours. We know all DGs are planar and are also triangle free thus by Grötzsch's theorem all DGs are at most 3-colourable. By virtue of the equivalence of the DG and the CP for colourability we have that CPs are at most 3-colourable. □

This is equivalent to saying that a 4-colourable crease pattern cannot be folded.

We note here that it would be nice to have a proof of this which did not refer to the Grötzsch's theorem as this would potentially lead to an origami based proof of the Grötzsch's theorem. We believe this may be possible but will leave it for future work.

4.5 Summary

In this chapter we have explored aspects of the relationship between Origami and graph theory. In particular we have looked at the equivalence between maps and CPs and that between CPs and planar graphs. We have then given some definitions and drawn results on 2-colourability and 3-colourability. We have also shown how, by removing edges introduced in a CP after the application of the OSC theorem, we may devise a new and succinct proof of the 4-colour theorem. The next chapter is on the concept of error modelling and estimation in Origami. When using Origami as a mathematical tool, minimising errors is important. We will show how errors crop up in all forms of Origami; a model is presented for these errors. This model can be used to choose better folding sequences.

5 Error modelling in Origami

As discussed in chapter 3, the Fujimoto method uses an iterative approach to reduce error for the specific application of approximating lengths, this is not however, a generic model for errors in folding processes. It is possible that errors in specific applications of Origami have been explored before, however, any formal mathematical error modelling does not appear to have been published. With applications of Origami becoming increasingly more commonplace and ever more complex, it would be helpful to have a mathematical basis for error modelling and measurement.

Although most Origami artists will provide advice on how to make creases more accurately, many of these techniques have not been analysed formally. A model that captures the accuracy of a crease and the process that produced it is therefore desirable. This model may also help choose between crease making processes in terms of accuracy and potentially efficiency. Here we look at Origami constructions from the ground up to analyse the basic concept of an Origami instruction. By introducing an arbitrary margin of error to an alignment, it is possible to model each of the 1-fold Huzita-Justin Axioms (HJAs) [298], creating a model for any Origami construction which shows the factors that most affect the accuracy of a crease. Although this Origami construction approach only includes the creation of a single crease per iteration, it is possible to use the same method to look at more complex moves such as squashes and sinks.

Building upon the 1-fold Origami constructions, the previous results are used to study how errors made in one crease affect the formation of subsequent creases in any Origami structure. This shows there exist creases in which errors have a greater impact on the final result than others and how, by changing the order of instructions, one can create the same end product with a greater accuracy. This chapter compares the results with those of some pre-existing examples of tips to improve accuracy.

Two methods to apply these results have been provided. Firstly, a method which minimises the dependency of creases on each other and thus the opportunity for build up of error in a given crease pattern. Secondly, a more in-depth calculation of which alignments should be made where there is a choice from many.

The chapter concludes with some interesting results where iterations of creases act to increase the accuracy and reduce the error in a future crease. Noting that these have similar properties to the Fujimoto Approximation method [54], it may be possible to define mathematically the properties of a crease which will have the effect of reducing the error level in future steps.

5.1 The importance of accuracy

Many Origami textbooks provide tips and guidance on how to fold neatly. An extensive literature exists providing such information [316–323]. The instructions are often varied and it is the formal analysis which is lacking.

Although machines work within tolerances often too small to be noticed without measurement, people are not as accurate. An interesting example is that of 1000 paper cranes made by a selection of members of the public at an event organised by the University of Essex Origami Society. Even though all folders were making almost the exact same model, ignoring the final judgement fold, some of the finished cranes looked completely different to others. This may be due to the skill of the specific person folding each model, of course, but it could also be due to slight differences in the instructions or sequences of folds carried out by each person.

Accuracy in Origami is very important, not just when an artist is making a model for an exhibition or display, but also when a machine is producing a folded product.

In general, for a specific model there will be bespoke requirements for it to count as successful. For example, if a machine is producing an Origami box then its stacking ability or “stackability” will be affected by inaccuracies in its creases. Also, if creating a box which holds itself together with a lock, how level the top of the box is may not be as important as the amount of paper available to produce the lock. However, with a model of an animal the importance may be on the legs coming to crisp points rather than angles elsewhere. This implies that accuracy in some aspects of a model may be more important than in others. Systematic ways for approaching accuracy through error estimation is needed.

The Fujimoto approximation method, as shown in chapter 3, is one of the only mathematical Origami techniques where accuracy plays a fundamental role. It is an iterative way to make errors in folding decrease exponentially. This approach however, only looks at the construction of accurate creases and the only error incurred is in choosing an incorrect starting point; there are, of course, additional errors added at every fold which are not accounted for in the Fujimoto method. This chapter looks at modelling the random errors introduced in any fold.

5.2 Error modelling in Origami constructions

Paper folding, like all real applications, is hardly error free. In Origami, folds are performed based on aligning creases or predefined points with others. With a complex folding sequence it is likely that the instructions assume there are no errors, and thus are generic. For example, it may ask the folder to fold such that a point is reflected to align with an intersection of several creases. Due to errors in previous folds, these creases may not be perfectly aligned. This raises the question, which intersection should be chosen?

In Origami, the most basic form of instruction is given by simply providing the folder with a diagram of a crease pattern, the aim is to construct all the creases; the collapsing process of folding is then left to the folder. Although the folder is only given a diagram of the crease pattern and on their paper there are no starting creases, the process is referred to as folding from a crease pattern. With a crease pattern, unlike step by step instructions, the folder is not given a specific order to construct the creases. With this approach, once the creases are put into the paper, much of the accuracy of the end result has already been defined; creases are only moved if they were not accurate enough when first made. Constructing the crease pattern before folding is referred to as pre-creasing, this is also often said to be the most accurate way to produce creases.

For the purposes of our model, all fold lines are straight; curved folding is not considered. Thickness, strength and other material properties of the paper are not considered. There are some complex moves which, when performed in the folding process, create a series of interconnected creases. This chapter will not examine moves such as squashes, sinks and rabbit ears but it is possible they could be included in the model at a later date simply by modelling them as rigidly connected structure on an inextensible surface, where the inaccuracies in one fold define the locations of several creases.

Judgement folds and measured folds [317, 318, 320, 321] will not be examined. Judgement folds have no accurately defined position and thus have no accurate measurement for error. Measured folds are the opposite; creases are marked not by a folding process, but by measuring and scoring or by folding along a mark already drawn on the paper. It may be the case that for some machines, creases are made using measurements from an edge or from a corner, but this would not be the same for different machines. It is again possible that our model could be extended to look at measured creases. However, because the accuracy is due to the measurement process and not to the folding one, this is not currently included.

Note, Appendix B looks at the ideas presented in this chapter and applies them to measurement which may relate to producing measured folds using other Origami related tools.

In reality, there are many different ways to make a crease. We give some examples for making a fold that puts a crease through two points:

- With the paper on the table, lift one side of the paper over the other creating a curved section between the upper and lower flap, attempt to align the curved part of the paper through the two points and increase pressure until it becomes a crease. This makes a straight but probably inaccurate crease.

- Make a short crease through each point in the rough direction of the other. Extend them from each end and hope they go through the other, adjusting if needed. Thus can often produce a non-straight crease or one which has greater accuracy at one end.

- make pinch marks as in the second method, prior to doing the first method. Feel for when the paper folds more easily and this should be when the paper is folding through the pre-existing creases.

Including an individual's skill and personal technique there is no mathematical way to capture all the possible methods to construct a crease. To simplify this we will create a model where the error in an alignment will be capped. After construction, the errors in these alignments could be measured and if a crease is made with an error larger than this the crease can be remade. For this approach we can produce mathematical bounds.

Considering this we look to define a simple mathematical way to look at errors uniformly. To begin with, the chapter creates a model from the 1-fold constructions of the HJAs [298, 299].

A set of uniform maximum errors for each of the primary alignments is introduced in order to analyse how these affect the location of creases for each one-fold construction. This standardised error allows accurate definition of a region in which any fold can be made such that it satisfies the one-fold construction to within the given tolerance. With these in mind, given a selection of one-fold constructions to produce a crease, a choice is made based on accuracy.

To produce the model we consider a square region in the plane to represent the square piece of paper we are folding which has no thickness, does not stretch and is finite. For each of the possible alignments we define a region in which a crease can be constructed such that the maximum possible error is ϵ. Combining these allows the creation of an error model for the HJAs.

5.3 Alignments

As with Chapter 3, let a piece of paper represent a simply connected, bounded subset of the $\mathbb{R}^2$ plane without holes and homeomorphic to a disk with the edges of the paper, marked as segments, creating the region boundaries. A crease in the paper is also represented as a segment. The size of any paper used in the real world is finite thus the concept of a line of infinite length is not strictly required. Recall that, in folding constructions, lines are primal. This means that lines define points, unlike in standard Euclidean geometry where points define lines [16] and all lines are in fact segments. For simplicity in this chapter we shall refer to lines and segments as lines.

Folding to extend a crease is not usually included in the HJA's, as mathematically in a perfect model it does not create anything new, however, it is included here as there is a possibility that one may wish to extend a crease and this may be done with some error. This gives the possible alignments [298], which are needed for the constructions, and an additional alignment for extending creases.

When making a fold, the folder takes everything to one side of the required crease and reflects it about the crease. An alignment is made where an object on one side of the crease is reflected to the same location as an object on the other side.

We will use the term 'folding' to mean 'making a fold which aligns', thus folding point P_1 to point P_2 implies that we make a fold which reflects point P_1 such that it is aligned with point P_2. We now define the following mathematical terms.

Definition 5.3.1. Every ***fold*** made shall be represented by the fold function $f : \mathbb{R}^2 \to \mathbb{R}^2$, mapping the plane to itself.

Thus the folded position of a point P_1 is $f(P_1)$, a fold which aligns P_1 onto P_2 perfectly will be represented by saying $f(P_1) = P_2$.

We can align points and lines with each other giving three types of alignment:

Definition 5.3.2. A ***point to point alignment***, of points P_1 and P_2, is any fold made such that $f(P_1) = P_2$

Definition 5.3.3. A ***line to line alignment***, of lines l_1 and l_2, is any fold made such that $\exists \subset of points P_n \in l_1$ such that $f(P_n) \in P_m$ where $P_m \subset l_2$.

Definition 5.3.4. A ***point to line alignment***, of point P_1 and line l_1, is any fold made such that $\exists P_n \in l_1$ such that $f(P_1) = P_n$.

We can also fold a line or point onto itself, meaning we crease through a line or a point. There are two possibilities for folding a line onto itself:

Definition 5.3.5. A ***line to itself alignment, creating a perpendicular crease*** through line l_1, is any fold made such that $\forall$ points $P_n \in l_1$ that can be aligned with l_1, $\exists P'_n \in l_1$ such that $f(P_n) = P'_n$ and $P_n \neq P'_n$ except where $P_n \in$ the fold line.

Definition 5.3.6. A ***line to itself alignment, creating an extension of the crease line*** l_1, is any fold made such that $\forall$ points $P_n \in l_1$, $f(P_n) = P_n$.

Note the subtle difference between the two definitions above, in folding through a crease the points are aligned with themselves, with a perpendicular crease they are folded to other points on the line, apart from where the creases intersect.

We note in the above definitions we are not too concerned with the parts of the lines which do not overlap, it is obvious that at some point a longer crease will need to be aligned to a shorter crease. This will be explored more later in the chapter.

Consider a fold making machine. To achieve a crease, the machine is assumed to make alignments with a tolerance say ϵ. Each possible alignment is considered individually in this analysis and remains defined by the tolerated error and where the target crease is specified. It is assumed that creases made in reality would be distributed with many having smaller errors however the maximum error provides the bounds for this.

The error is defined as the distance between the points and lines which are aligned when the crease is folded, thus for imperfect alignments we define a distance function d.

Definition 5.3.7. $d(A, B)$ is the ***Euclidean distance*** between the two points A and B.

This allows us to say that where there is a specific error in an alignment we can measure this as $d(P_1, P_2) = k$ where k is the measure of the size of the error.

For a fold to be made aligning two points with a maximum error of ϵ we would have a constraint that $d(f(p_1), p_2) \leq \epsilon$

For alignments with lines this is a bit more complex; we have P_a as a point on the line a, and P_a' as the point on the line b which P_a is aligned with, thus for a fold to be made aligning with a maximum error of ϵ we would have a constraint that $\forall P_a \in a \exists P_a' \in b$ such that $d(f(P_a), P_a') \leq \epsilon$.

We note here that not all points on an object are required to be aligned as aligning part of one segment onto part of another is acceptable. Careful attention must be paid to which parts of a line actually can be aligned.

Further to the above definitions we can define C as what we shall refer to as the 'circle' around an object.

Definition 5.3.8. $C(A, x)$ is the region, which we shall call a ***'circle'***, around the object A in the $\mathbb{R}^2$ plane such that the maximum distance a point within that region can be from any point in A is x. For a point this gives a circle centred at A of radius x.

With this we can state more generally that for an alignment to be made folding P_1 onto P_2 with maximum error ϵ, $f(P_1) \in C(P_2, \epsilon)$.

Unlike before, for alignments with lines or regions this is not as complex; we have P_a as an aligned point on the line a, and the line b, for a fold to align P_a onto b with a maximum error of ϵ we have the similar constraint that $f(P_1) \in C(b, \epsilon)$.

Our 'circle' notation is notably simpler when aligning objects which are not points.

To simplify this further we shall make one more definition:

Definition 5.3.9. $d_A(A, B)$ shall be a measure of the maximum error when aligning A onto B, thus the maximum euclidean distance between two aligned points on the objects such that no point on either object is closer to the other than the aligned points. Whenever an accurate alignment is made $d_A(A, B) = 0$

We note that, when aligning an object with itself, it may not be possible to measure the distance an object has been moved. It may be just as appropriate for a model to measure the distance between the object and the resulting crease. For our model we have chosen to be consistent with ϵ as the error in alignments.

We now locate the regions in which any possible crease resides, that will satisfy these conditions. The boundaries of these regions are found; they will be referred to as the Crease Boundaries (CB) and a crease made without error will be referred to as a Target Crease (TC). Note that in the diagrams the inaccuracies are overly exaggerated to help display the results.

Note that in the following sections we have produced several conic sections using creases. Showing that conic sections can be produced was first done by Lokta in 1907 [324]. The aim of the work by Lokta was to produce such conic sections, however, the motivation behind the following sections was not in fact to produce these, but was to produce the boundaries of the regions of creases when an error is made. The work of Lokta was not studied while undergoing research in this section of the thesis; it was only after working out which conic sections had been produced that the author was made aware of this work. the work of Lokta [324] is cited throughout where we have concluded the same results.

5.3.1 Folding to align a point onto another point

Consider first the crease resulting from pressing the paper after aligning a point to another. Because there are an infinite number of ways to bring the two points to within ϵ from each other there is also an infinite number of resulting creases. To capture them all the region that contains them all must be described. To that end the folds which take the first point to somewhere on the circle that is centred about the target point with radius ϵ is calculated as this will produce the creases which are furthest from he target crease, TC. The region bounding these will be the crease boundary, CB, of all possible folds.

We begin by folding such that we align a point to another point thus we want to find the region of all the possible folds that will take that point to within ϵ another one. Figure 5.1 shows several creases resulting from folding P_1 onto the points on the circle of radius ϵ centred about P_2. All the possible creases will be within the hyperbola.

Thus the CB's of the region are determined by a hyperbola [324]. In Figure 5.2 the blue dashed line is the TC which satisfies the alignment of P_1 onto P_2. The hyperbolic curves, CB_1 and CB_2, mark CB's which satisfy the alignment with maximum error ϵ. Therefore, any crease which lies entirely in this region will produce an error at the alignment of ϵ or less. EC is an example crease which touches CB_1 as the error is equal to ϵ. When folded, EC aligns $P1$ onto the circle

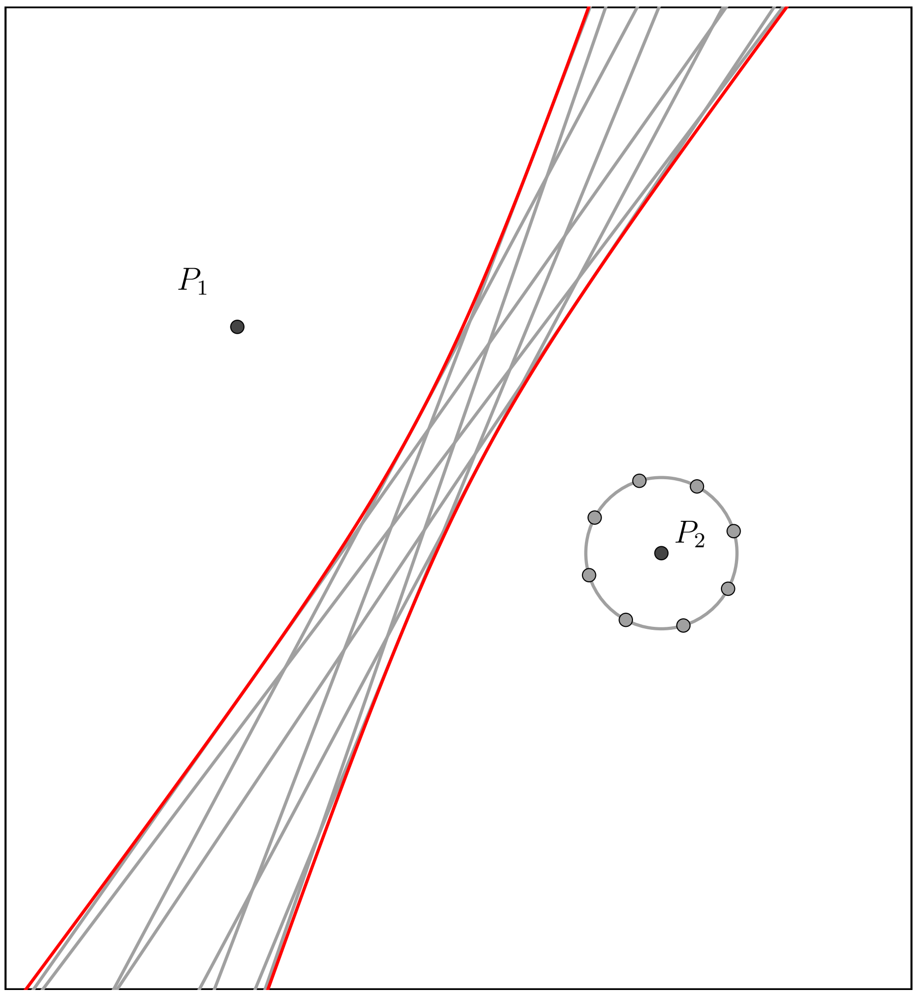


Figure 5.1: Folding to align a point onto points on a circle, radius ϵ, produces creases tangent to a hyperbolic curve

about $P2$. Figure 5.2 is the complete error model for this alignment.

Lemma 5.3.1. *For a crease to be constructed such that it folds a point P_1 to within the circle $C(P_1, \epsilon)$ the following must hold. That crease must either not intersect the hyperbola, constructed from tangents to all the creases made when folding the point P_1 to the boundary of the circle $C(P_1, \epsilon)$, or be tangent to it.*

Proof. We know from Lokta [324] that the creases produced when folding a point P_1 onto a circle are tangent to a hyperbola. Aligning to any point inside this circle will produce a crease which is tangent to another hyperbola which shares the same foci as the boundary parabola but is entirely inside it. As any crease made aligning the point P_1 to a point inside, and not on the boundary, of the circle about P_2, this crease is tangent to a hyperbola which does not cross through the outer hyperbola it is not possible for any crease made in this way to intersect the outer parabola. This is shown in figure 5.3. $\square$

5.3.2 Folding to align a point onto a line

Folding to align a single point to all locations along a line produces a series of creases which are all tangents to a parabola [325] whose focus is P_1 and directrix is L_1. This is shown by Figure 5.4 where resulting creases are shown for folds taking P_1 onto a series of points along L_1. These creases are all tangent to the blue parabola [325].

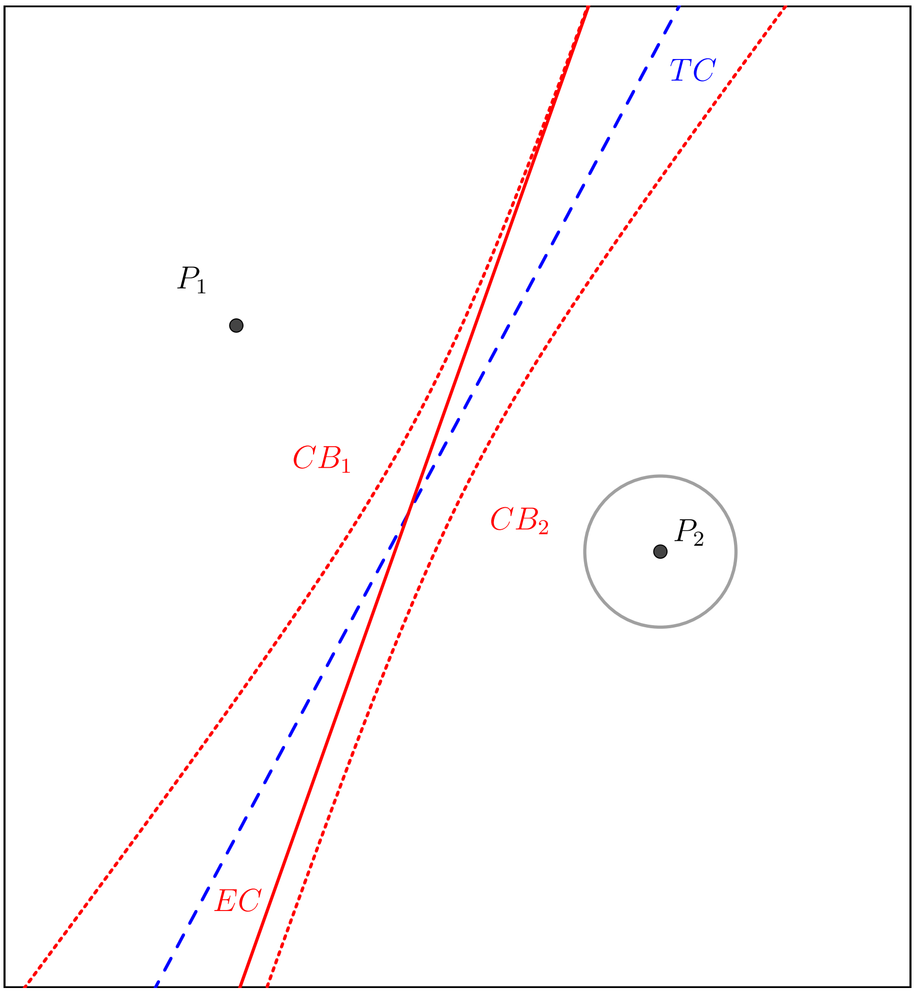


Figure 5.2: A model for folding a point to a point including the Target Crease, TC, and an Example Crease, EC, with error of ϵ

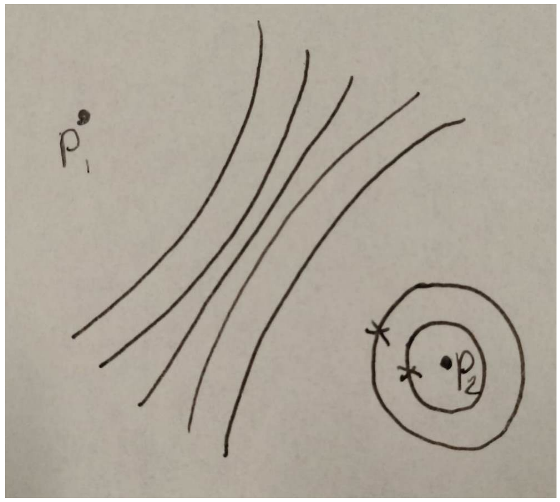


Figure 5.3: Concentric Hyperbolas

When we fold a point, P_1, to align that point to within ϵ of a given line line L_1 we must fold to align with a point inside the circle, $C(L_1, \epsilon)$, about that line. For an infinite line this would be bounded by two straight lines parallel to the target line. Any alignment onto this boundary produces a crease which is tangent to one of two parabolas BP_1 and BP_2.

Lemma 5.3.2. *For a fold to be made which satisfies the alignment, the crease must either intersect or be a tangent to the outer parabola BP_2 and must either not intersect the inner parabola BP_1 or it must be a tangent to it. This can be simplified by saying that any line which is a tangent to a parabola between the two bounding parabolas will align the point to within ϵ of the line.*

Proof. Between the two parallel straight lines which are located either side of our target line L_1, there exists an infinite number of other straight lines.

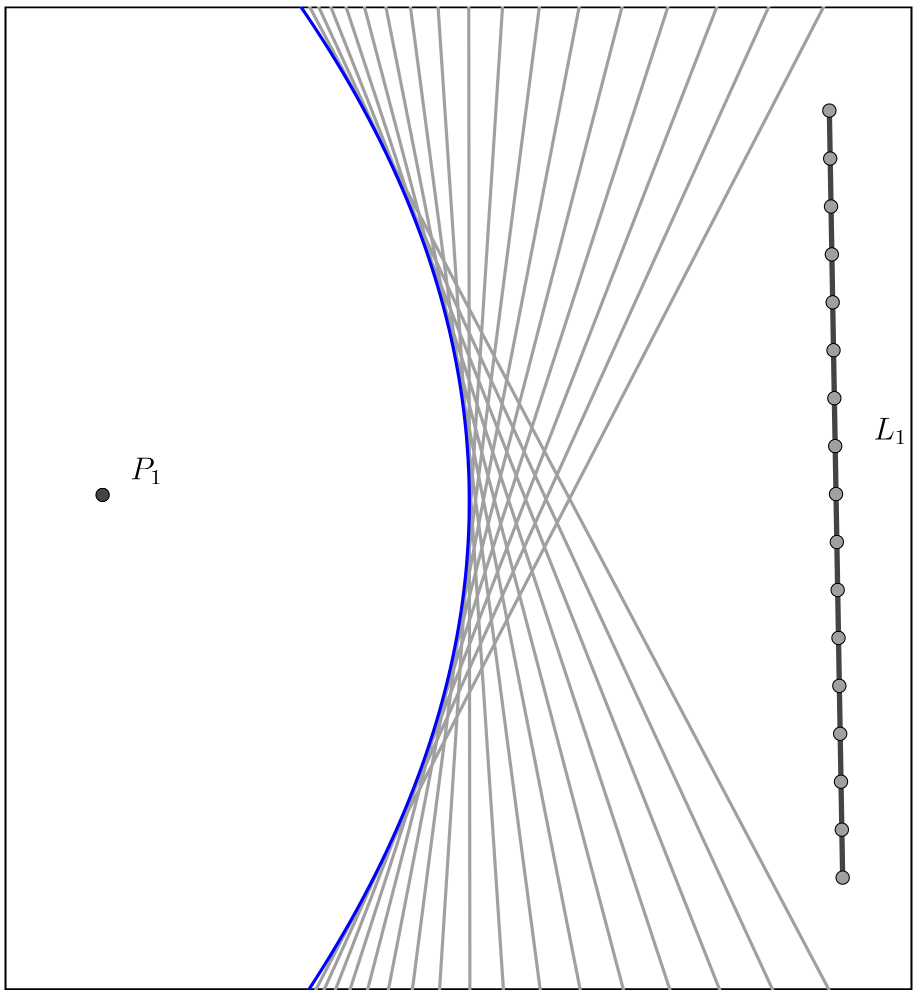


Figure 5.4: Folding to align a point onto points along a line produces creases tangent to a parabola

For each of these lines we imagine a corresponding parabola formed such that it is tangent to each of the creases representing a fold of the point, P_1, onto a point on that line.

Therefore, we have an infinite number of parabolas which could be produced between the inner parabola BP_1 and the outer parabola BP_2. None of these parabolas will intersect as they each fold to a separate one of the parallel lines.

As the parabolas do not cross they are all also bounded by those parabolas BP_1 and BP_2.

Any potential crease line will be tangent to exactly one parabola as it folds the point P_1 onto exactly one of these parallel lines; thus it must pass through a point between or on the two boundary parabolas BP_1 and BP_2.

Therefore, if a crease aligns with the furthermost line from P_1 then it will be tangent to BP_2, any other crease will be tangent to a parabola which at all points is nearer to P_1 than BP_2 and thus will cross BP_2 in two places.

Finally, if a fold aligns with the near-most line from P_1 then it will be tangent to BP_1, any other fold will be tangent to a parabola which at all points is further away from P_1 than BP_1 and thus any crease tangent to it will never cross BP_1. $\square$

The outer boundary parabolas are shown in figure 5.5 where an accurate fold taking P_1 onto L_1 would produce a crease tangent to TP. The two bounding parabolas are shown as BP_1 and BP_2.

Note that it is not possible to align a point to a line beyond the endpoint of the line; this can be taken into account by considering the end of a line as a point, thus as the region for this has already been defined for point-to-point alignments it is possible to combine these. Figure 5.7 shows the boundary parabolas for folding P_1 onto L_1 in green, these are the same as those shown in Figure 5.5. Figure 5.7 also shows the hyperbolic bounds of possible creases taking P_1 onto the endpoints of L_1, P_2 or P_3. These are shown in Blue and red respectively.

We now note an interesting fact about the positions of the intersections of these bounds.

Theorem 5.3.3. *The intersections of the hyperbolic and parabolic lines $I_1, I_2, ..., I_6$, as in figure 5.8 can be found on two lines which are perpendicular to the target line L_1 and pass through its endpoints.*

For the proof of this theorem we shall require a definition for our parabola.

Definition 5.3.10. A ***parabola*** is the set of all points whose distance from a certain point, the focus, is equal to the distance from a certain line, the directrix.

Proof. First we note that the intersection of the accurate creases, representing the folds aligning P_1 onto P_2 and P_3 respectively will of course be tangent to the accurate parabola. We now look to see precisely where these creases will be tangent to this parabola.

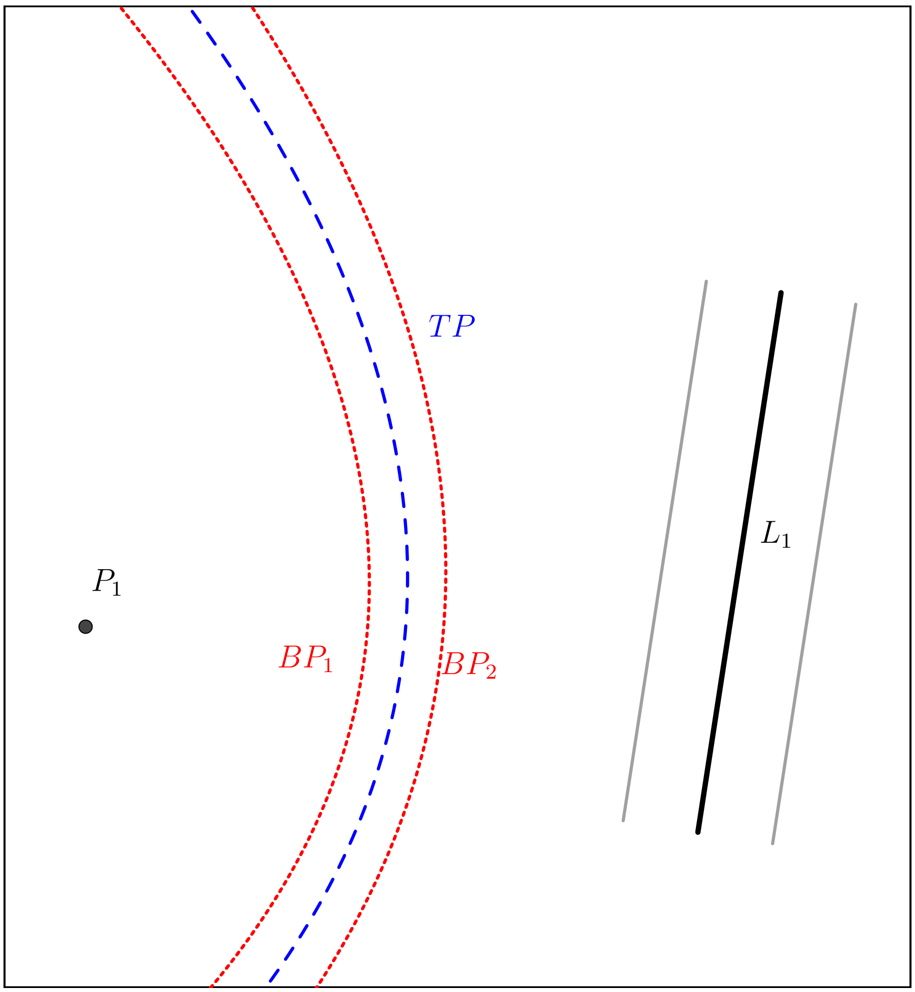


Figure 5.5: Boundary parabolas produced when aligning a point with lines ϵ either side of a line

We note that for any crease which represents a fold aligning a point P_1 onto another point P_2, that crease also represents a fold which constructs the angle bisector of the lines from any point along that crease to P_1 and P_2 This is shown in figure 5.6.

We note from our definition of a parabola, that the point on the parabola, where the crease will be tangent to it, is an equal distance from the point P_1, the focus, and the line L_1, the directrix.

We note that, the distance between a point and a line is the same as the distance between that point and the point on the line which is closest to it. A line through these points will always be perpendicular to the original line.

Therefore the point on the parabola where the crease is tangent to it, and thus is the same distance from the focus P_1 to the specific point on the directrix L_1, is also on a line perpendicular to the directrix.

This implies that the intersection of the accurate creases, representing the folds aligning P_1 onto P_2 and P_3 respectively with the accurate parabola will be at points which intersect the perpendicular lines passing through the endpoints of L_1.

As a single crease will only align a point to a single location, where the boundary parabolas intersect the boundary hyperbola there will be exactly one crease which will align P_1 to exactly one point. This will be the intersection of the circles about

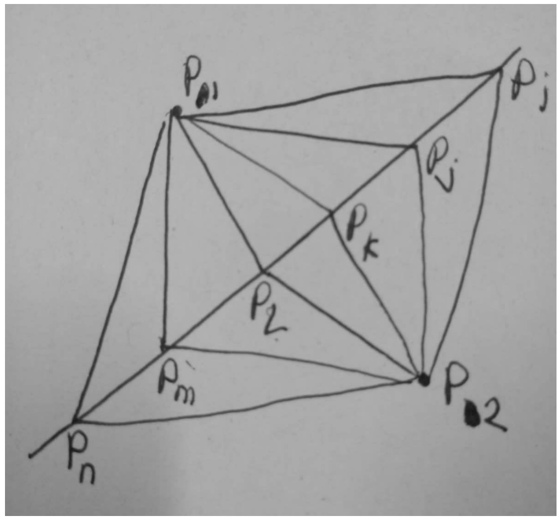


Figure 5.6: Folding a point to another point is similar to folding many angle bisectors

P_2 and P_3 with the lines ϵ either side of L_1.

This intersection also takes place on the perpendicular line to L_1 which passes through its endpoints. Therefore, the points where the creases which fold to these points will be tangent to a boundary parabola will also be on this perpendicular line.

The same follows for the hyperbola, however, it is obvious that at the intersections of the hyperbola and parabolas they will be parallel as they can only produce one crease at that point.

Therefore $I_1, I_2, ..., I_6$ can be found on two lines which are perpendicular to the target line L_1 and pass through its endpoints. $\square$

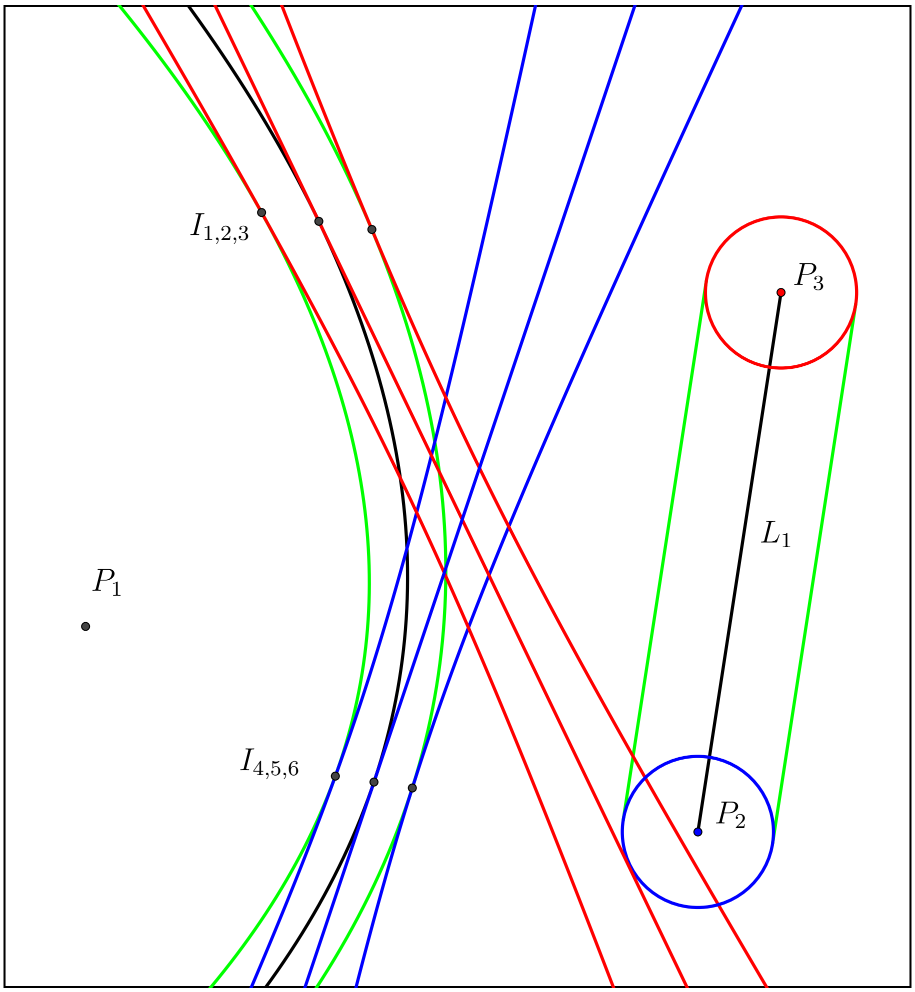


Figure 5.7: The intersection of parabola formed when folding to allign a point to a line and hyperbolas formed when folding the same point to the endpoints on the line

It is possible to define our model using a combination of the parabolic approach shown in Figure 5.5 and the hyperbolic approach. This is shown by Figure 5.8. In this diagram CB_1 and CB_5 are hyperbolic lines formed folding P_1 onto P_3, CB_3 and CB_4 are the same for P_2. CB_2 completes the boundaries and is the smaller of the two bounding parabolas. We note here that not all lines within this region satisfy the alignment within an error of ϵ thus the region is not 'full'.

As almost all of the possible creases will use the parabolas, for simplicity Figure 5.9, our final model, only includes the parabolas; the lines cropping the parabolas mark where this approach should change to the hyperbolic lines from before.

In Figure 5.9 P_1 is folded onto L_1; this produces the target parabola, TP. The two lines either side of L_1 are ϵ either side of it. TC is an example of a Target Crease which is tangent to TP; EC is an example crease which is tangent to the inner parabola.

Folding to align a point to a line could be considered as folding a point to another point with an additional degree of freedom.

5.3.3 Folding to align a line onto another line

Folding to align two lines can be considered as aligning a series of points on one line with a series of points on another. It could be assumed that it is only necessary

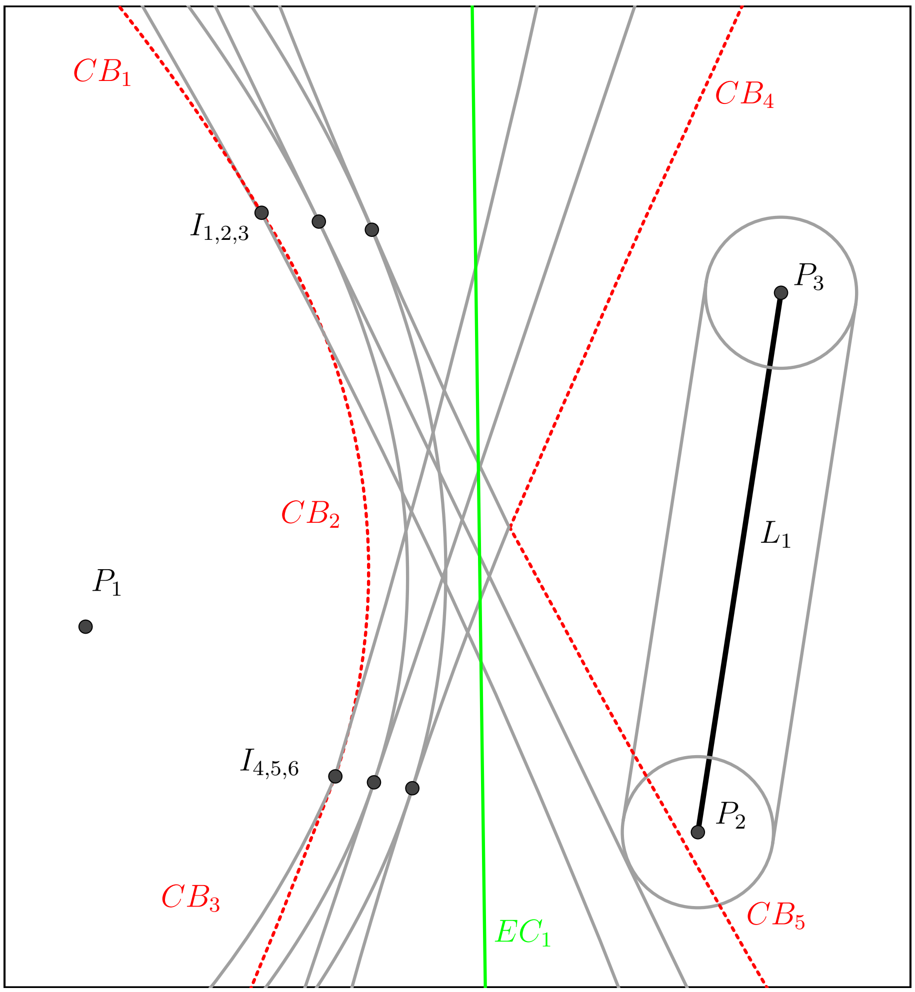


Figure 5.8: Full crease boundaries for folding to align a point to a line with unsatisfactory Example crease, EC

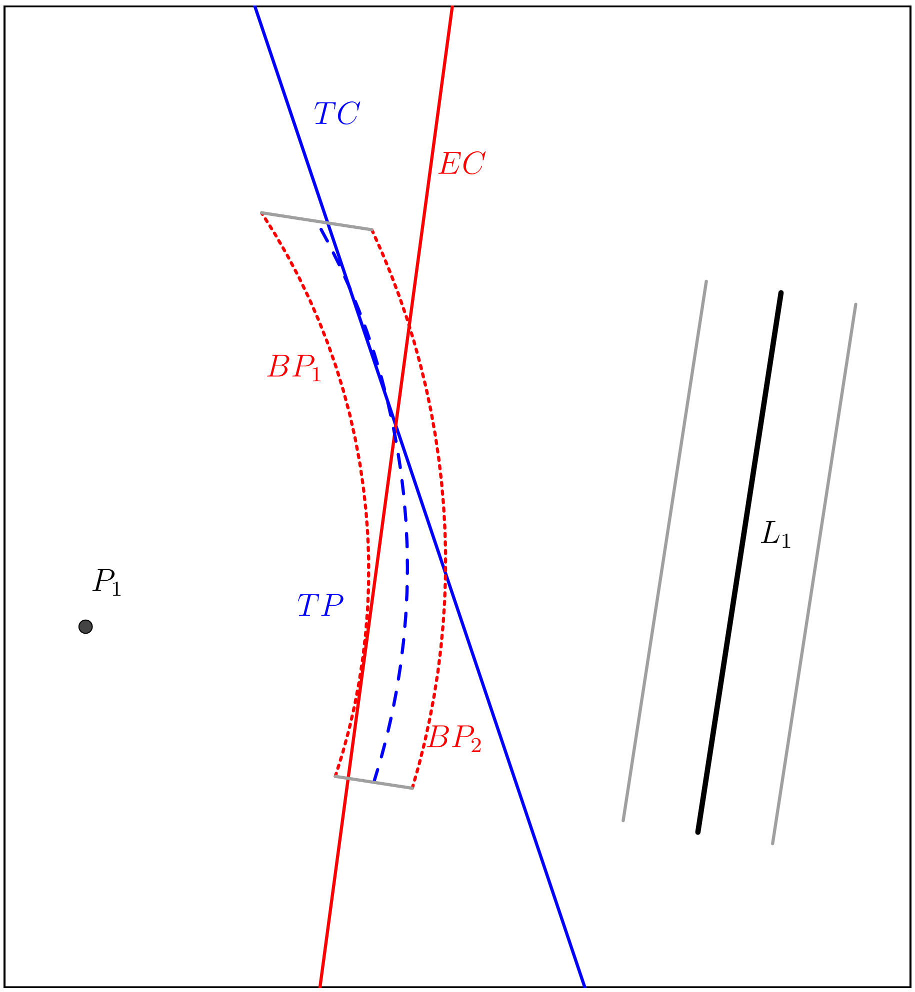


Figure 5.9: A simplified model for folding to align a point with a line including a sample Target Crease, TC, and an Example Crease, EC, with error of ϵ

to look at the parts of the lines which overlap. However, the folder is not told which parts of the line will overlap as marked points showing which part of each do not exist thus there is no specific part of a line which must align with another.

As we have mentioned before there may be cases where both lines are not the same length, it is only possible to measure the error where both lines exist, alignment beyond the endpoints is not possible. Because of this we can only insist that those points which have been aligned are done so with maximum error epsilon.

We define which points have been aligned as the maximum distance between any two points on the two lines, such that they are both no closer to any other point on their opposite line, is ϵ when folded as there is nothing to align on the overhanging parts. Thus, when folded, the two shorter endpoints are no more than ϵ away from the other line.

Figure 5.10 shows six CB's. CB_3 and CB_4 are created by taking a reflection of L_2 onto the grey line ϵ past L_1 and moving one end of the line until it touches the nearer grey line. These lines represent the maximum clockwise rotation and are the same if calculated by starting on the nearer line and rotating until the line intersects the furthest grey line. CB_1 and CB_6 are made the same way, however anticlockwise rotation is applied. CB_2 and CB_5 show the creases when L_2 is folded to align onto each of the grey lines beside L_1. Any crease which satisfies this alignment with error of ϵ or less will not cross any of the CB's.

Note although P_1, P_2 and P_3 are on the lines either side of L_1, P_3 is beyond the endpoint of L_1 thus is not required to align.

Note, the bounded nature of the creases play a role in alignment; alignments of longer lines and lines further apart both yield more accurate folds.

5.3.4 Folding through a point

Folding to align a point with itself is the same as folding through a point. Given an error of ϵ one might assume we should fold through the circle that is centred about the target point with radius ϵ. However, this would allow one to align the point with a position 2ϵ away from the original location, thus we fold passing through the circle centred about the point with radius $\frac{\epsilon}{2}$. This means any line passing through the circle also passes within $\frac{\epsilon}{2}$ of the point.

Figure 5.11 shows that for examples of the point P_1 being folded onto points ϵ away from itself produce creases tangent to the circle radius $\frac{\epsilon}{2}$.

In Figure 5.12 the aim is to fold a crease through P_1. There are an infinite number of creases which can be folded through P_1, ETC is an example of this. EC_1 and EC_2 both have errors: EC_2 has the maximum permitted error as the crease is tangent to CB.

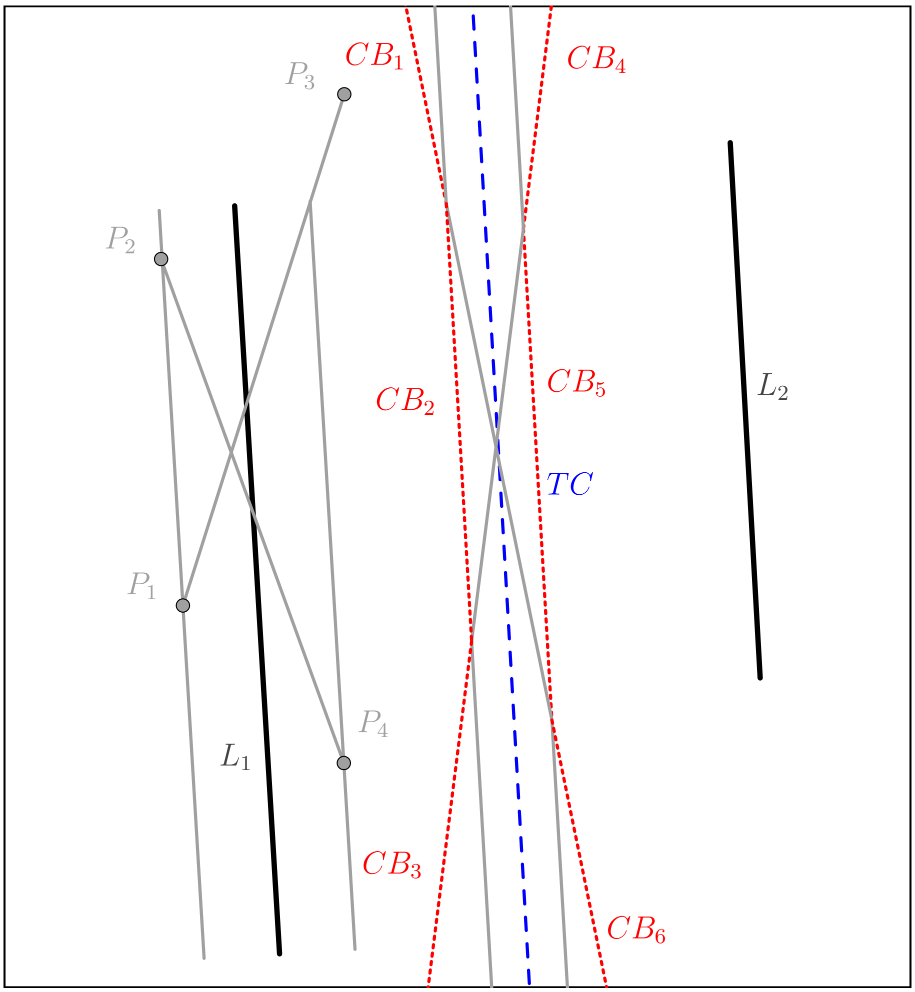


Figure 5.10: Crease boundaries produced when folding to align a line onto another line

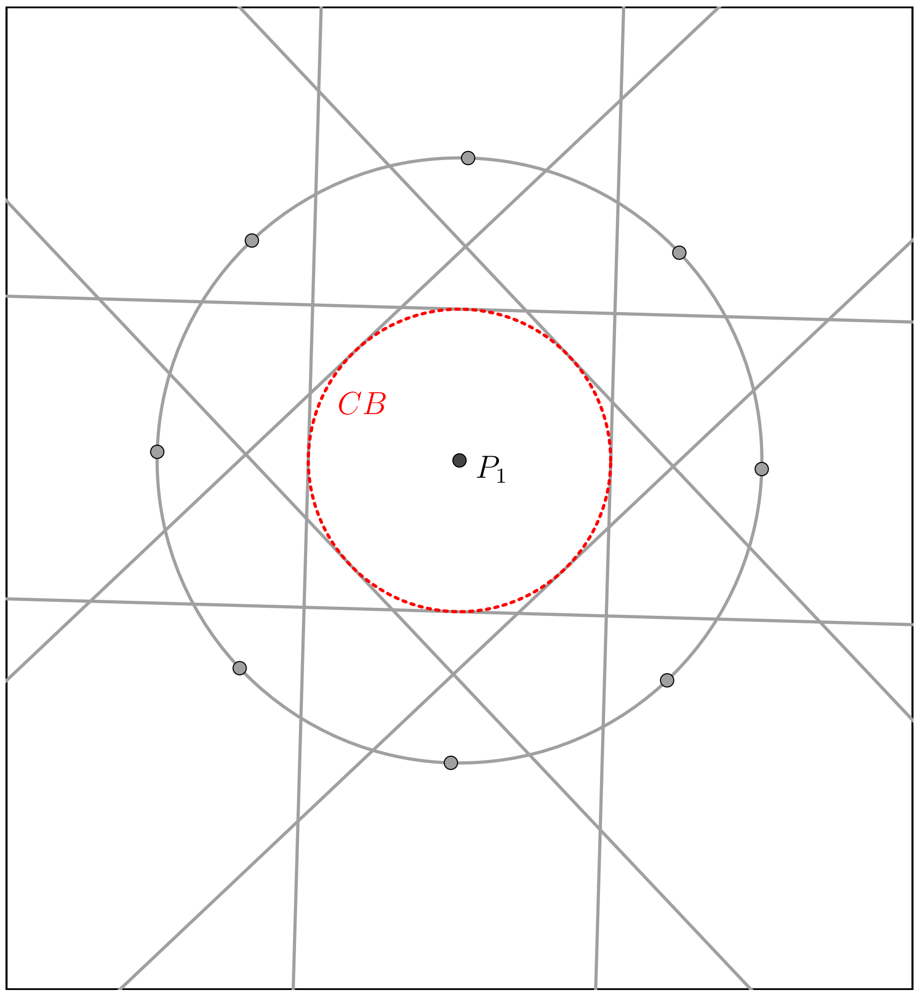


Figure 5.11: Folding a point onto points on the circle radius ϵ from it (Note this is shown 4 times larger than the other examples)

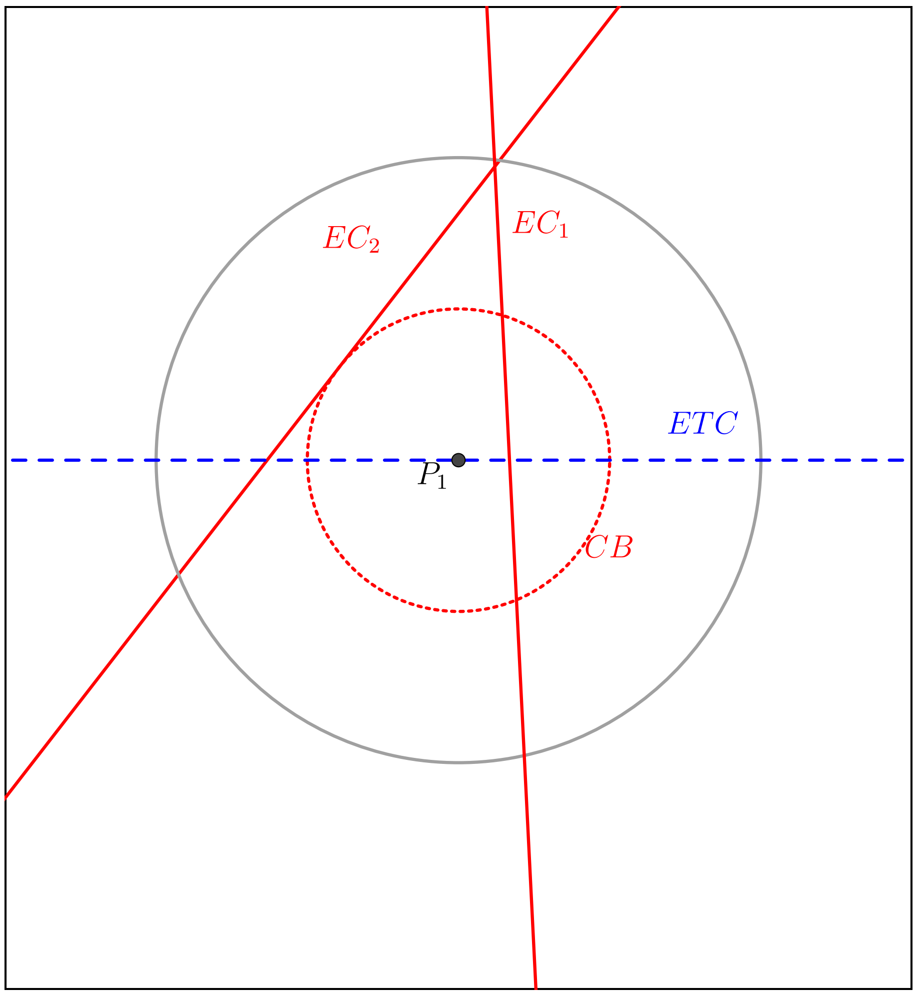


Figure 5.12: A model for folding to align a point with itself including an example the Target Crease, TC, and two Example Creases, EC (Note as with Figure 5.11 this is shown 4 times larger than the other examples)

5.3.5 Folding a perpendicular crease through a line

A crease is made perpendicular to a line, thus aligning the line with itself. The crease is constructed such that at the end of the line closest to the crease, the error is not more than ϵ. The shorter endpoint is the last point at which we have both parts to align. If the crease is made in the middle of the line then either endpoint can be used to measure the error. Figure 5.13 shows the possible CB's for creases through a given line. An arbitrary intersection point AP, is picked as this alignment does not define a specific crease. CB_1 and CB_2 are the CB's which satisfy the alignment. They are generated by reflecting line L_1 such that its endpoint is exactly ϵ from the line.

5.3.6 Folding through a line to extend a crease

If we make a crease which aligns a line with itself and is parallel to that line we have a fold so that any point on that line is not aligned more than ϵ away from its original position. As with folding through a point, here we need a crease which can be at most $\frac{\epsilon}{2}$ from the line. This is important and it should be noted that the circles are not radius ϵ as that would allow an error of twice ϵ.

In Figure 5.14 CB_4 and CB_3 represent the maximum clockwise error which can occur, CB_1 and CB_6 represent the anticlockwise maximum error, CB_2 and CB_5

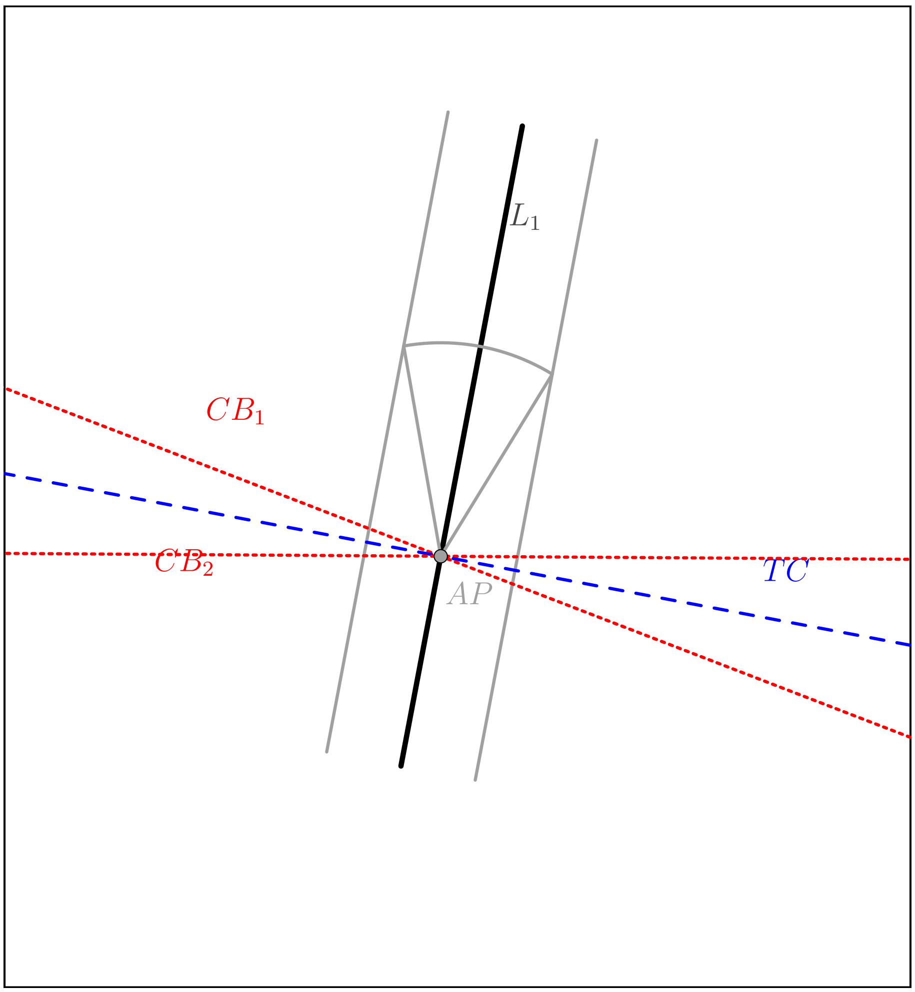


Figure 5.13: Crease boundaries produced when folding to produce a crease perpendicular to align a line

are the furthest a parallel crease could be from the line to be acceptable. Together, these define the region in which any crease can lie.

5.4 An error model for the Origami HJA's

We now go through all seven of the HJA's and produce models of the CB's of the regions where the error is less than or equal to ϵ. We also calculate the maximum translation and rotation of a crease inside this region.

5.4.1 Axiom 1: Fold a crease through two points

Given P_1 and P_2, fold so that P_1 is aligned onto P_1 and P_2 onto P_2. From the alignments for P_1 and P_2 we have that the crease must pass through both the circles centred at these points with radius ϵ. Thus we can represent the boundary of the region of possible creases.

In Figure 5.15 $CB2$ and $CB5$ represent creases made at the maximum distance from the TC while remaining parallel with the accurate crease. We get the other four CBs from the tangents to the circles.

It is worth noting that the points where the CBs cross are not on the circles as

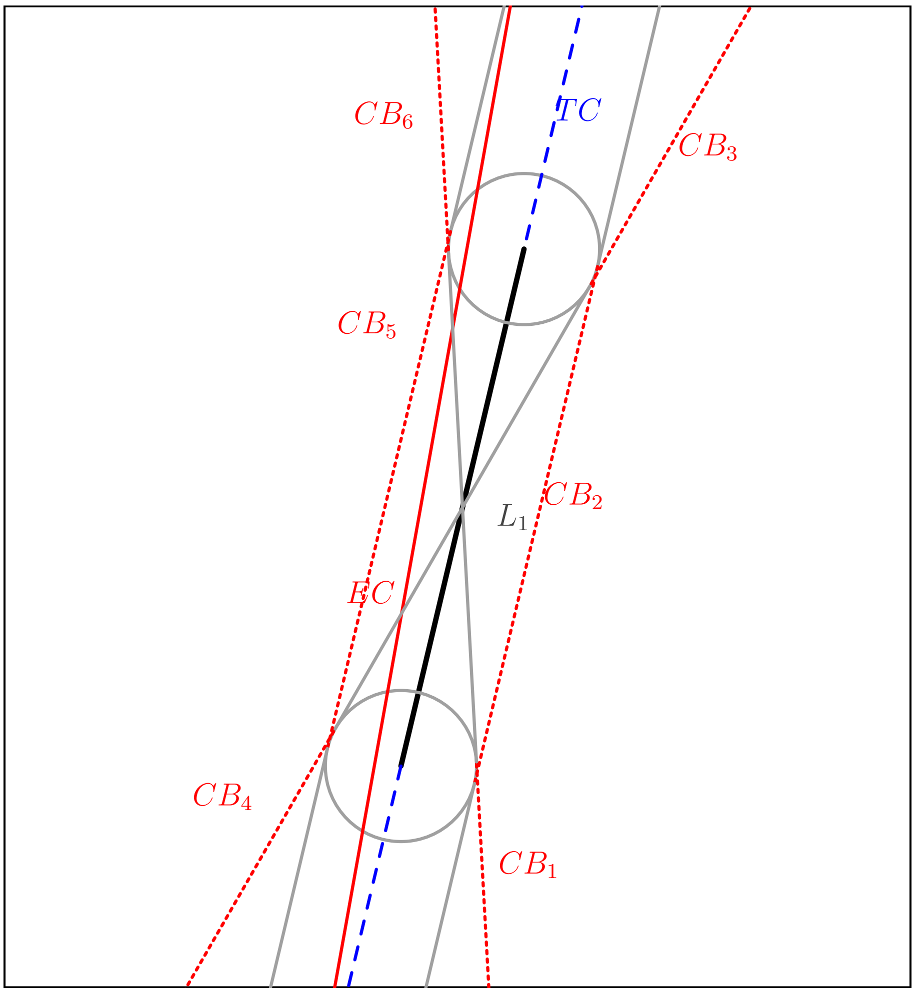


Figure 5.14: Folding through a line to extend a crease (Note this is shown 2 times larger than the other examples)

they are tangents to them. Given $P_1 = (X_1, Y_1)$ we have the circle

$$(X - X_1)^2 + (Y - Y_1)^2 = \epsilon^2. \tag{5.1}$$

Given $P_1 = (X_1, Y_1)$ and $P_2 = (X_2, Y_2)$, the maximum rotation of a crease is:

$$\theta = \pm \sin^{-1}\left(\frac{\frac{\epsilon}{2}}{\sqrt{(\frac{X_1 - X_2}{2})^2 + (\frac{Y_1 - Y_2}{2})^2}}\right) \tag{5.2}$$

The maximum translation is $\pm\frac{\epsilon}{2}$.

We note here that folding through two points is found to be the same as the alignment for folding to extend a line; thus no additional axiom is required.

5.4.2 Axiom 2: Fold a crease to align two points

Given two points, P_1 and P_2, fold so that P_1 is aligned with P_2. This has also been considered, see Figure 5.2. But, here we develop the model as follows.

Let the two points be (X_1, Y_1) and (X_2, Y_2) then the hyperbola has foci at each of these two points and passes through a point on the line between the two foci $\frac{\epsilon}{2}$

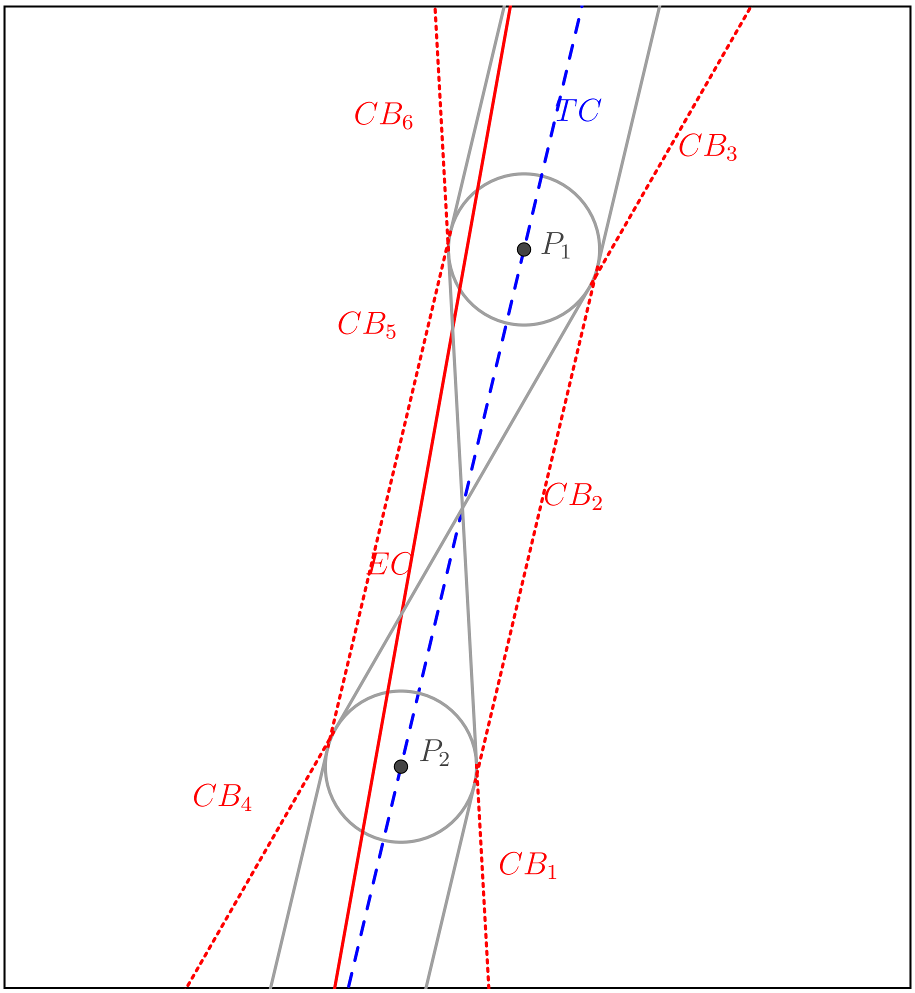


Figure 5.15: Axiom 1 - Folding through 2 points

from the midpoint. This point is given by:

$$(\frac{X_1+X_2}{2}+\frac{\epsilon}{2}\frac{X_2-X_1}{\sqrt{(X_2-X_1)^2+(Y_2-Y_1)^2}},\frac{Y_1+Y_2}{2}+\frac{\epsilon}{2}\frac{Y_2-Y_1}{\sqrt{(X_2-X_1)^2+(Y_2-Y_1)^2}}) \tag{5.3}$$

From this we should get a hyperbola of the form:

$$1=\frac{(x\cos(\theta)-y\sin(\theta)-h)^2}{a^2}-\frac{(x\sin(\theta)-y\cos(\theta)-k)^2}{b^2}. \tag{5.4}$$

The expansion of this equation with points (X_1, Y_1) and (X_2, Y_2) is extremely long and difficult to simplify; it is shown in Figure 5.16 after it has been through simplification in the computer program 'Maple'. Given these points the maximum rotation of a crease is:

$$\theta=\pm\sin^{-1}(\frac{\epsilon}{\sqrt{(\frac{X1-X2}{2})^2+(\frac{Y1-Y2}{2})^2}}) \tag{5.5}$$

and the maximum translation is $\pm\epsilon$.

5.4.3 Axiom 3: Fold a crease to align two lines

Given two lines, L_1 and L_2, fold so that L_1 is aligned onto L_2. This has already been considered, see Figure 5.10.

$$
\begin{aligned}
&-2\,e^4+4\,e^2\,x_1^2-8\,e^2\,x_1\,x+8\,e^2\,x^2-8\,e^2\,x\,x_2+4\,e^2\,x_2^2+4\,e^2\,y_1^2-8\,e^2\,y_1\,y+8\,e^2\,y^2-8\,e^2\,y\,y_2+4\,e^2\,y_2^2-2\,x_1^4+8\,x_1^3\,x-8\,x_1^2\,x^2-8\,x_1^2\,x\,x_2+4\,x_1^2\,x_2^2-4\,x_1^2\,y_1^2\\
&+8\,x_1^2\,y_1\,y+8\,x_1^2\,y^2-24\,x_1^2\,y\,y_2+12\,x_1^2\,y_2^2-4\,x_1^2\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|+\\
&16\,x_1\,x^2\,x_2-8\,x_1\,x\,x_2^2+8\,x_1\,x\,y_1^2-32\,x_1\,x\,y_1\,y+16\,x_1\,x\,y_1\,y_2+32\,x_1\,x\,y\,y_2-24\,x_1\,x\,y_2^2+8\,x_1\,x\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|\\
&\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|+16\,x_1\,x_2\,y_1\,y-16\,x_1\,x_2\,y_1\,y_2-16\,x_1\,x_2\,y^2+16\,x_1\,x_2\,y\,y_2-8\,x^2\,x_2^2+8\,x^2\,y_1^2-16\,x^2\,y_1\,y_2+8\,x^2\,y_2^2-\\
&8\,x^2\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|+8\,x\,x_2^3-24\,x\,x_2\,y_1^2+32\,x\,x_2\,y_1\,y+16\,x\,x_2\,y_1\,y_2\\
&-32\,x\,x_2\,y\,y_2+8\,x\,x_2\,y_2^2+8\,x\,x_2\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|-2\,x_2^4+12\,x_2^2\,y_1^2\\
&-24\,x_2^2\,y_1\,y+8\,x_2^2\,y^2+8\,x_2^2\,y\,y_2-4\,x_2^2\,y_2^2-4\,x_2^2\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|-2\,y_1^4+8\,y_1^3\\
&y-8\,y_1^2\,y^2-8\,y_1^2\,y\,y_2+4\,y_1^2\,y_2^2-4\,y_1^2\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|+16\,y_1\,y^2\,y_2-8\,y_1\,y\,y_2^2+\\
&8\,y_1\,y\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|-8\,y^2\,y_2^2-8\,y^2\\
&\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|+8\,y\,y_2^3+8\,y\,y_2\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|\\
&\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|-2\,y_2^4-4\,y_2^2\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|+\\
&\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|^3\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|+\left|e+\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|\\
&\left|e-\sqrt{x_1^2-2\,x_1\,x_2+x_2^2+y_1^2-2\,y_1\,y_2+y_2^2}\right|^3=0
\end{aligned}
$$

Figure 5.16: Generic equation for CBs when folding a crease to align two points

5.4.4 Axiom 4: Fold perpendicular to a line and crease through a point

Given L_1, and P_1, fold so that L_1 is aligned onto L_1 and P_1 onto P_1.

In Figure 5.17 each of the four CB's will fold the endpoint of L_1 to align with the grey lines ϵ either side of $L1$; they are also tangent to the circle of radius $\frac{\epsilon}{2}$ about P_1.

It is important to note here that as P_1 is being aligned with itself the circle is radius $\frac{\epsilon}{2}$ not ϵ.

5.4.5 Axiom 5: Fold a crease through a point and align a point onto a line

Given P_1, P_2 and L_1, crease so that P_1 is aligned onto L_1 and P_2 onto P_2.

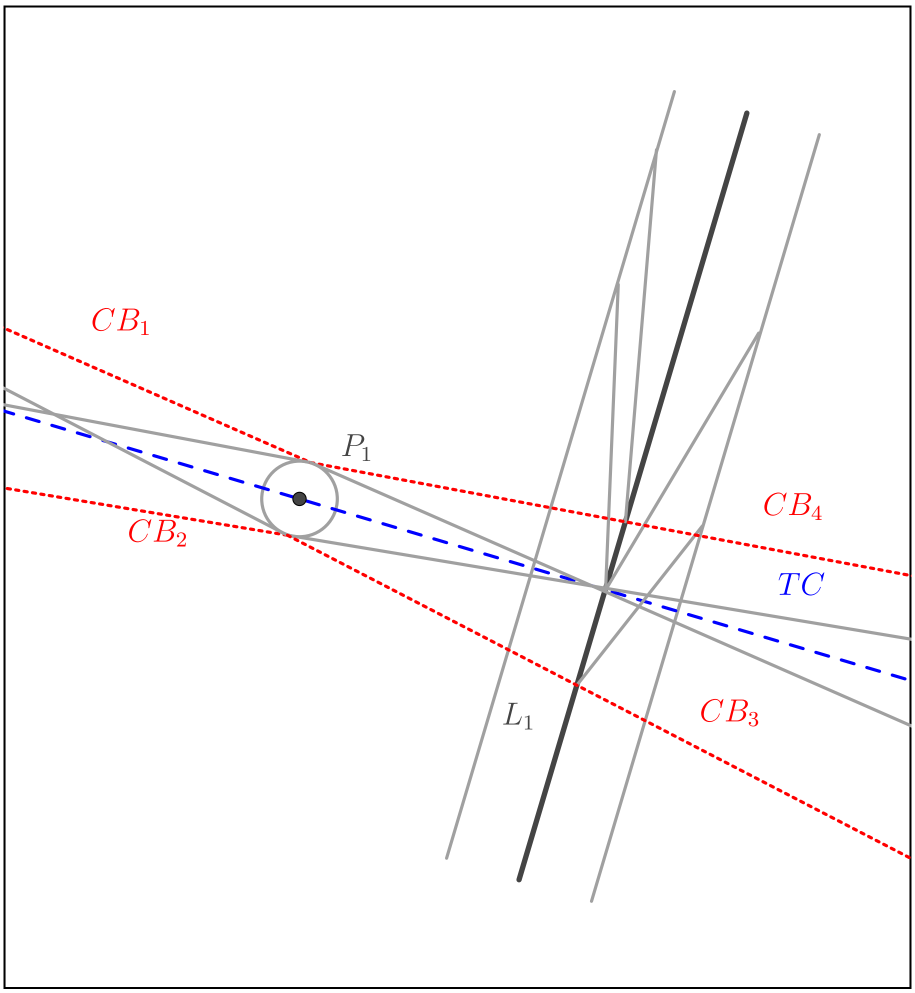


Figure 5.17: Axiom 4 - Folding a line to itself so the crease passes through a point

Here we fold a point to a line, thus we get two bounding parabolas and at the same time we have that the resulting creases must go through a circle of radius $\frac{\epsilon}{2}$ about the second point. We can take, the tangents to the circle and each of the parabolas create the region of creases.

In Figure 5.18, CB_3 and CB_4 represent the fold made with the maximum clockwise rotation. It is tangent to the inner parabola and the circle. CB_1 and CB_6 represent the crease made with the maximum anticlockwise rotation. They are tangent to the outer parabola and the circle. CB_2 and CB_5 represent the maximum movement a crease can make perpendicular to the TC without rotation.

5.4.6 Axiom 6: Fold a crease to align two points with two lines

Given P_1, P_2, L_1 and L_2, fold so that P_1 is aligned onto L_1 and P_2 onto L_2.

Here we fold a point to a line twice, thus we get two bounding parabolas twice. We can take the tangents to one parabola from each alignment to create the CB's.

In Figure 5.19, CB_1 and CB_4 represent the crease made with the maximum clockwise rotation, they are tangent to the inner parabola one alignment and the outer parabola for the other. CB_3 and CB_6 represent the crease made with the maximum anticlockwise rotation. CB_2 and CB_5 represent the maximum

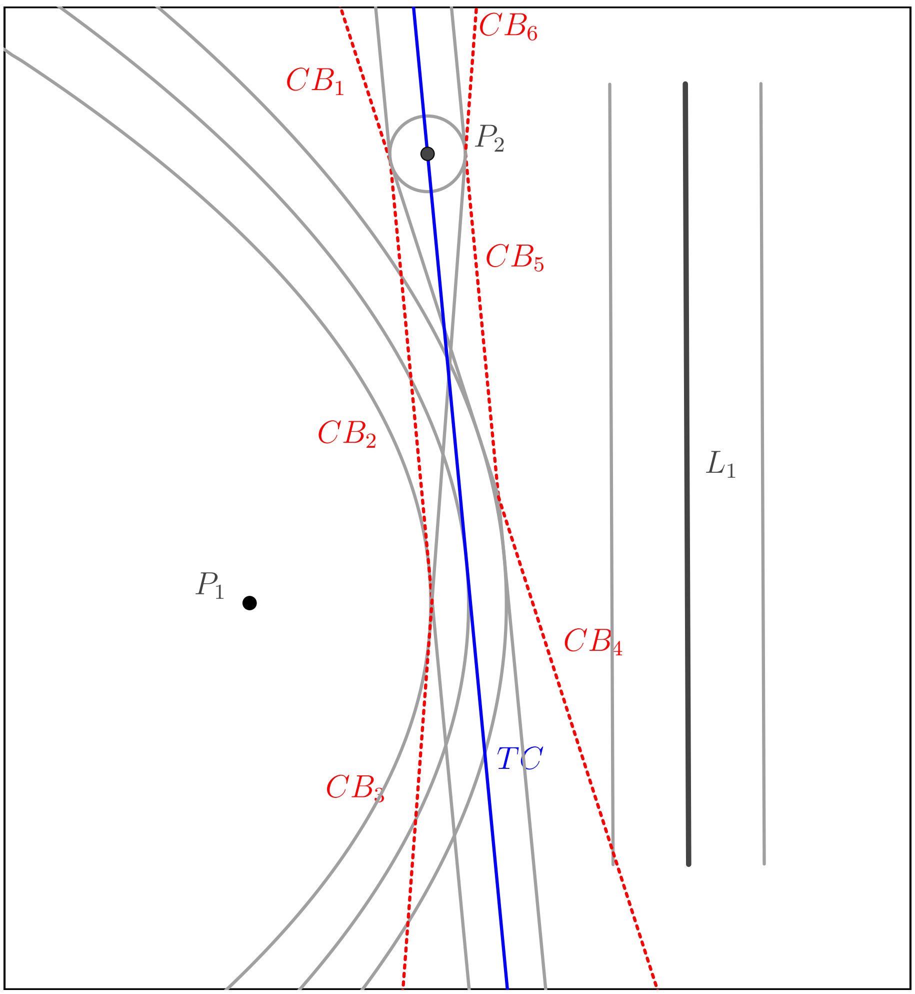


Figure 5.18: Axiom 5 - Folding a point to a line so that the fold passes through a point

movement a crease can make perpendicular to the TC without rotation; they are tangent to the equivalent parabolas for each alignment.

5.4.7 Axiom 7: Fold a point to a line making a crease perpendicular to a line

Given P_1, L_1 and L_2, crease so that P_1 is aligned onto L_1 and L_2 is aligned onto L_2. Thus we have the two bounding parabolas for the first alignment, which define the movement perpendicular to the TC, and two further parabolas from the second alignment. Together these provide four CB's.

In Figure 5.20 CB_1 and CB_3 represent creases made with the maximum clockwise rotation. CB_2 and CB_4 represent creases made with the maximum anticlockwise rotation.

5.5 Illustration: Folds on a square

To illustrate the modelling methodology we have just introduced, we apply it in simple structures to help choose the best HJA to use through error considerations in order to generate simple creases.

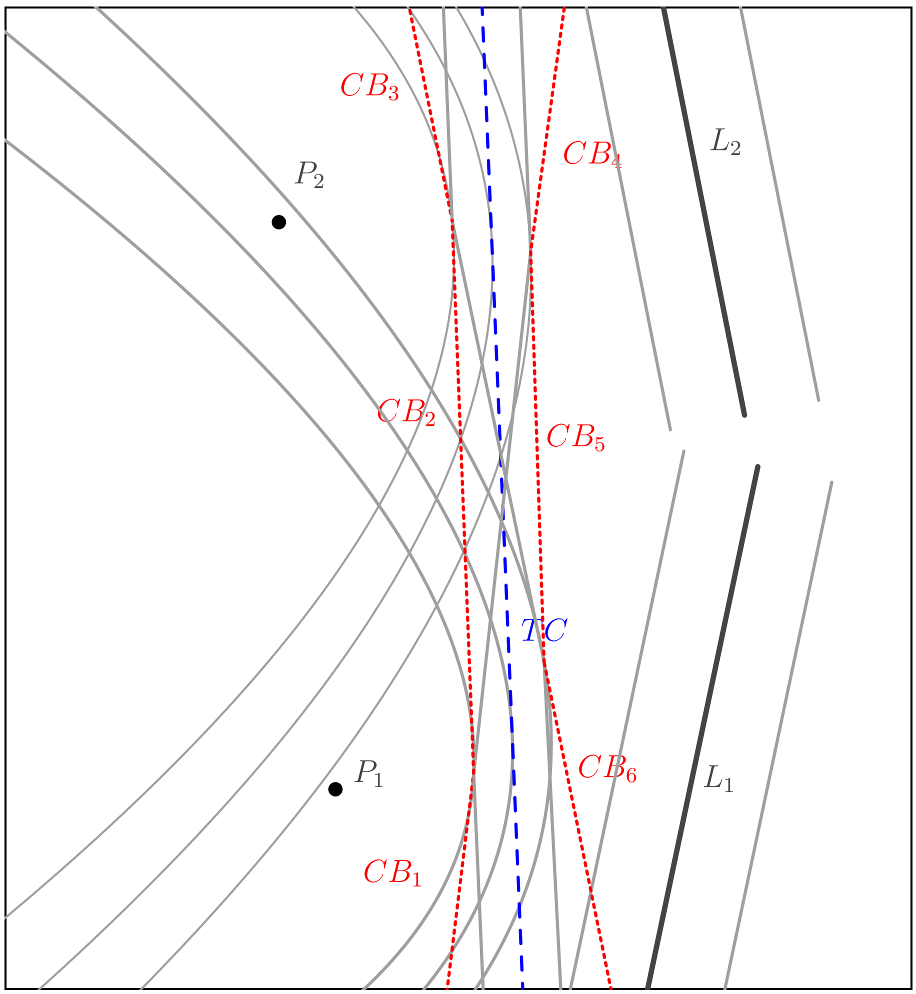


Figure 5.19: Axiom 6 - Folding two points to two lines

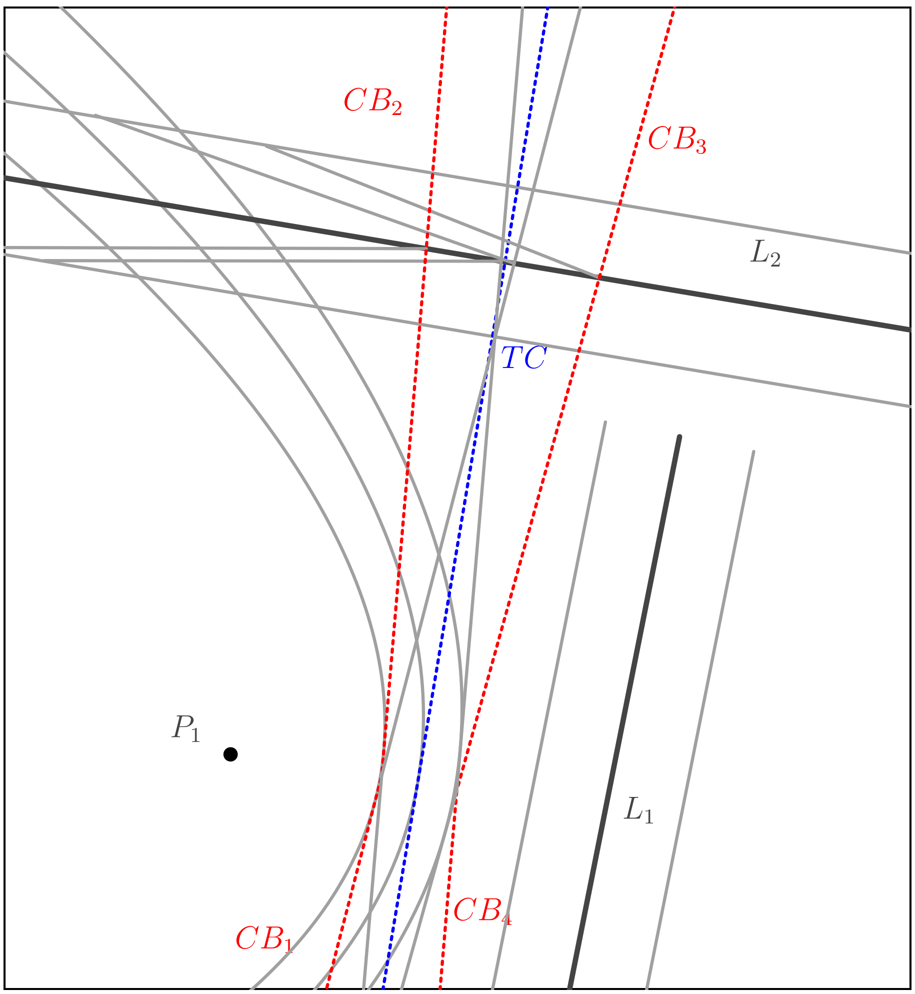


Figure 5.20: Axiom 7 - Folding a point to a line and a line to itself

5.5.1 Folding in half diagonally

When folding in half diagonally, several options arise. We compare folding the two opposite corners together as in Figure 5.21, folding two adjacent edges onto each other as in Figure 5.23 and folding through the two corner points as in Figure 5.22. Comparing these in Figure 5.24 we can see that the region of solutions for folding through two points is smaller than both the other regions. Thus we conclude that this is the better axiom choice as it minimises the sixe of the region within the bounds of the paper.

5.5.2 Folding in half horizontally

Using the 1-fold model we have two ways of folding in half with a horizontal crease; we can either look at folding two parallel edges of the paper together, or folding either of the other edges to meet themselves. The latter is equivalent to folding two adjacent corners of the paper to meet each other. In Figure 5.25 we show the outcome for the CB's for both methods. This shows how the side which is aligned is accurate however the further you get away from the alignment the greater the size of the maximum error. In contrast the alignment of the entire top edge is much more accurate. Aligning edge-to-edge is the obvious choice. It is expected to produce the best solution in this case.

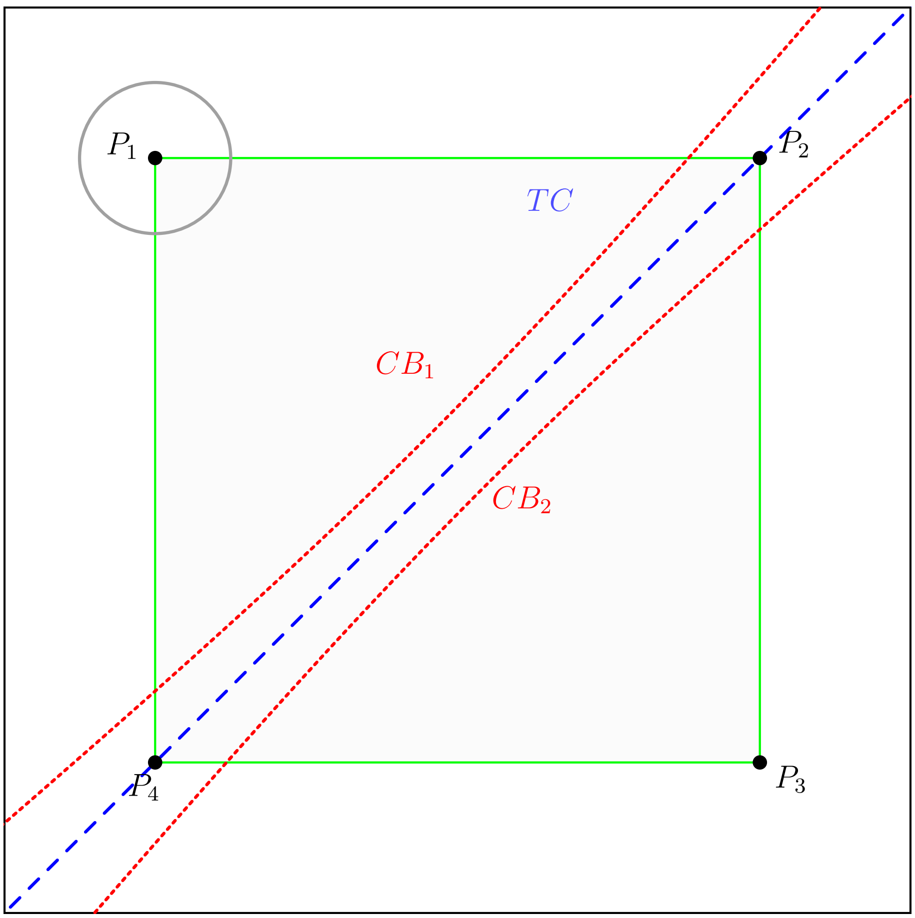


Figure 5.21: Folding a diagonal through a square by folding opposite corners together

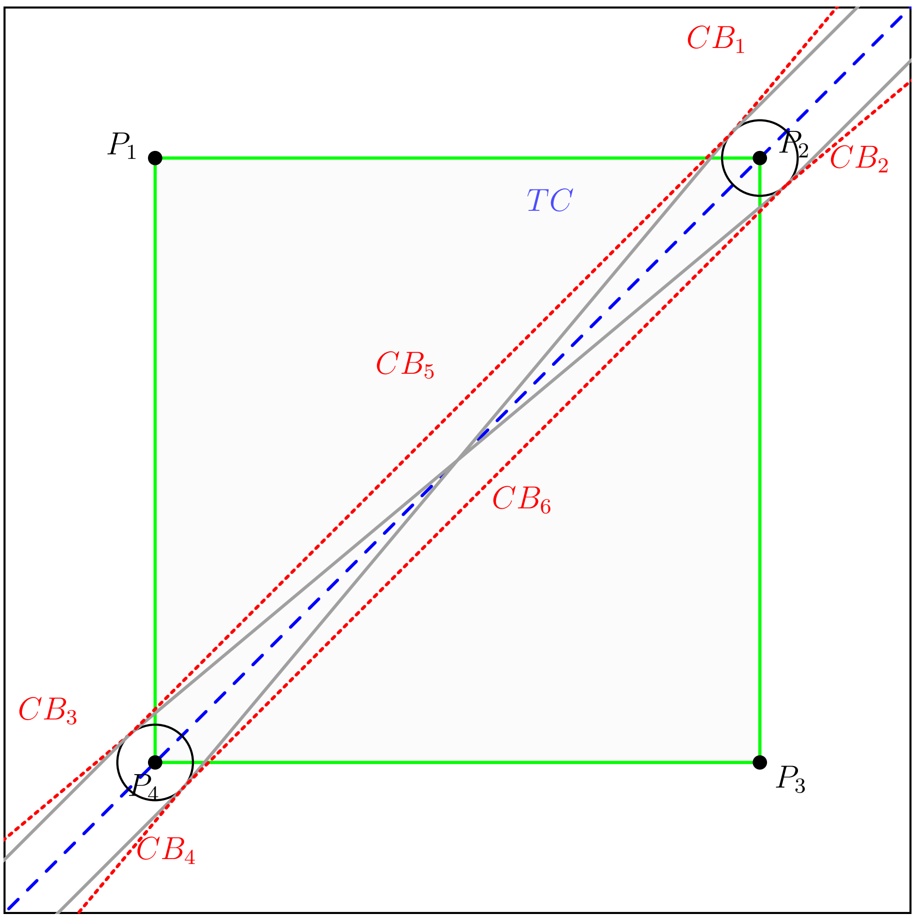


Figure 5.22: Folding a diagonal through a square by folding through opposite corners

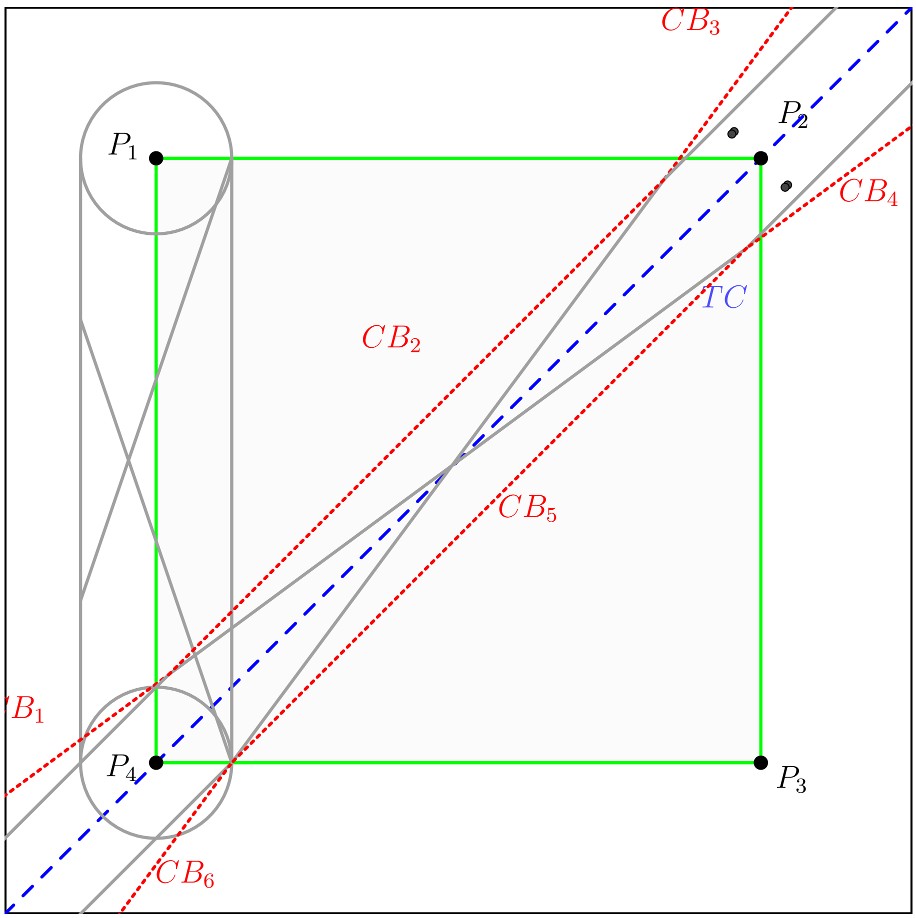


Figure 5.23: Folding a diagonal through a square by folding two edges together

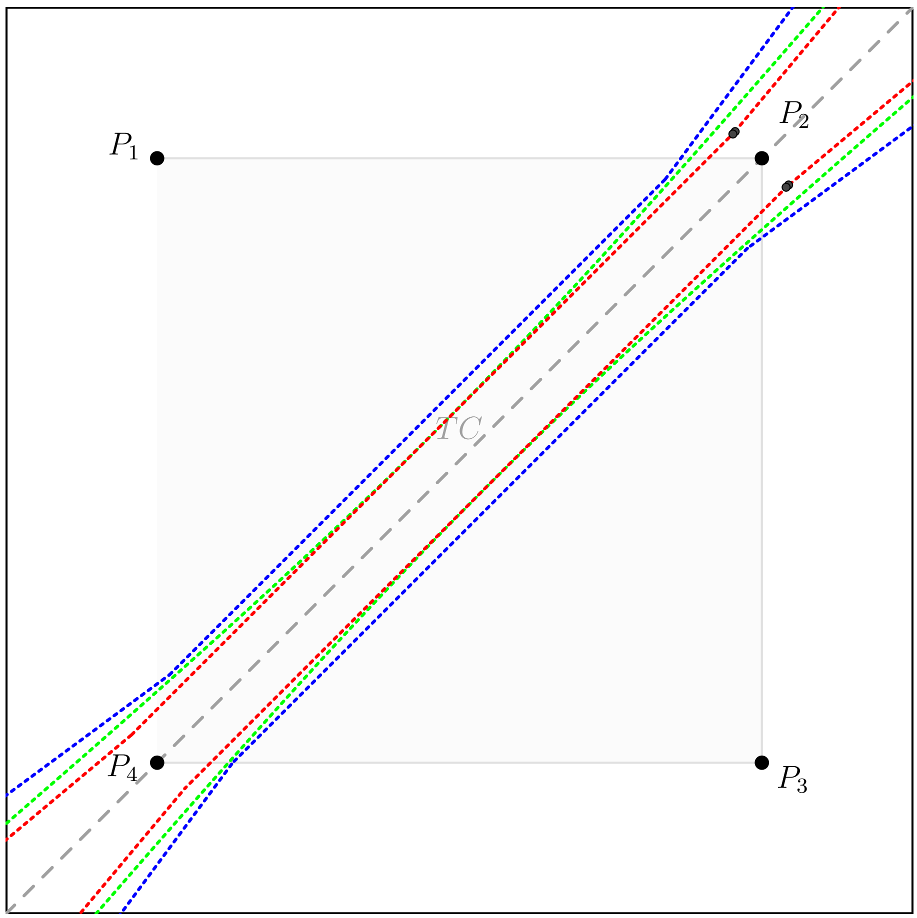


Figure 5.24: Comparison of all three methods for folding a diagonal through a square: folding through two points in red, folding two points together in green and folding edges together in blue

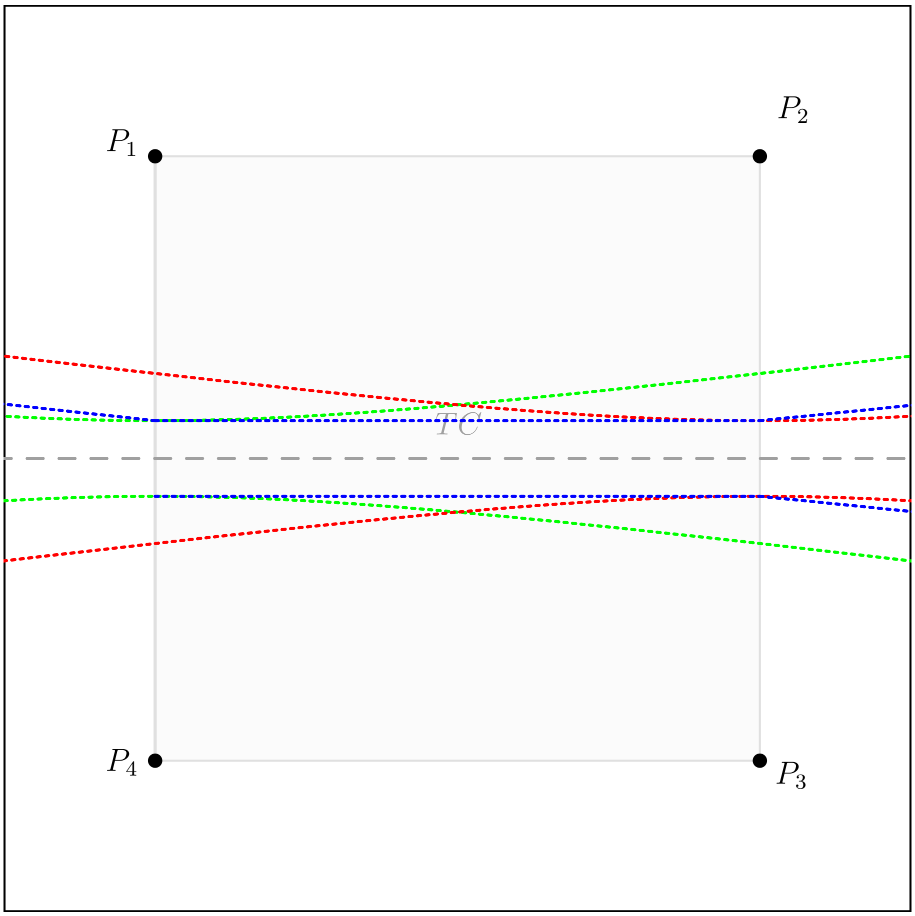


Figure 5.25: Comparison of all three methods for folding horizontally through a square: folding to align P_2 and P_3 in red, folding to align P_1 and P_4 in green, folding to align the top and bottom edge in blue

5.5.3 Folding to locate the centre

Which method is best to find the centre of a square? If we limit ourselves to just one pair of intersecting lines then the edge-to-edge horizontal and vertical folds generate errors of the same size as folding point-to-point for both diagonals. However, the region in which the centre point resides is approximately square of edge length ϵ in the former, while it is the same but rotated diagonally in the latter. It is conceivable that more lines could be constructed and the centres of all intersections taken as the middle. However this would be using unnecessary creases and multiple points would be found.

5.6 Compounding of errors through multiple creases

Even simple Origami objects involve multiple creases. Error compounding therefore is unavoidable. Here we consider modelling the effect of axiom choice on a crease pattern with multiple folds. We note that this would become increasingly difficult as instead of having a nicely defined line or point to create the crease we may be starting from a point or line for which we only know the region in which it lies.

For example if we had found the centre of a square we would have a region in which this point could lie. If we wanted to fold a corner to the centre and find the region in which that fold could lie we would need to fold that point to a point within ϵ of that region. With lines this is even harder as we would need to align with all possible lines in a region.

We note that using the "circle" and distance notation, as we continue to make steps constructions which compound errors, the shape of the circle can become arbitrarily complex however we can still define it and are still able to describe the requirement to align one complex region with another.

We look instead at ways of quantifying the possible error at each stage and giving it a value which can be used to help choose axioms to apply. Here we compare some possibilities for this quantification of the error.

1. One solution is to quantify the maximum angle of rotation a crease can have. We can build a general model of this for each axiom. This is a much simpler way to model the error as for each crease one only gets one error based on the points and lines which are to be aligned. However, this does not take into account the movement of a crease perpendicular to the TC. Thus we could have a crease which can be folded in the correct direction but which is folded off the edge of the paper.

2. If we aim to define the maximum shift perpendicular to the crease we get a measure of how spread out the creases can be.

3. Another solution is to define the maximum distance from the crease we make

to the ideal one. We can measure this by finding the distance between the creases at the boundary of the paper. This method takes into account the shape of the paper as well as the location of points and lines to be aligned. However, this does not help where we make creases which do not extend to the boundary of the paper.

4. If we look to define the area of the entire region we get a single value for the error. But this cannot be found without knowing the exact boundaries of the paper.

5. Rather than looking for a specific value we look at what affects the error for each axiom. For example with the Axiom from Section 5.4.2, the further the two points are apart the smaller the error, but the further the paper is from the line between those two points the larger the error at the boundaries. If we compare this to Axiom 1, the further apart the two points are the smaller the error but the further the paper extends past the two points in the direction of the line between those two points, the larger the error at the boundaries.

5.6.1 Error control: A simple approach

We have seen that when folding we are making alignments, thus, error is caused both by the limited accuracy with which we can align objects and the pre-existing

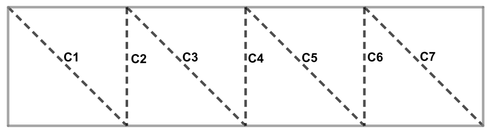


Figure 5.26: A simple crease pattern

errors in them. With this in mind we can assume that, where several creases rely on each other in order to be constructed, the error will be compounded. We now look to minimise the compounding of errors without looking at the specific errors involved in each step. Here we make one fold at a time and each time unfold; thus layers do not increase and all folds are made with the same random error. For each crease we will assign an error state based on the folding sequence. Let us first define what we mean by error state.

Definition 5.6.1. An ***error state*** is a measure of how many times the error has been compounded in order to construct a particular crease. For a crease x_i constructed by aligning objects $x_1, \ldots, x_n$, we assign an error state $\sigma \in \mathbb{Z}$ such that $\sigma(x_i) = 1 + \max_{1 \leq j \leq n}(\sigma(x_j))$. Also, $\sigma(x_i) = 0$ for all initial elements such as paper borders.

Figure 5.26 is a simple crease pattern which consists of a 1×4 rectangle divided into 8 similar triangles. Three possible folding sequences to construct this crease pattern are compared below.

Method 1: Folding each crease in sequence from $C1$ to $C7$. First, $C1$ is created by aligning the left edge with the top edge. Then $C2$ can be created by folding perpendicular to the bottom edge at the point where it intersects $C1$. Next $C3$ is created by aligning $C2$ with the top edge; the process repeats in this way for the remaining creases $C4$ to $C7$.

It is obvious that Method 1 is inaccurate, as $C7$ is affected by the errors in all the other creases which have been compounded at each stage.

Method 2: Begin with the vertical creases, then add the diagonals. First, $C4$ is created by folding in half. Then $C2$ and $C6$ are created by folding each of the edges to $C4$. Finally, the diagonals are added by folding a crease through the two points made by the intersections of the vertical and horizontal creases with the paper edge.

This is the most likely way one would naturally construct the crease pattern. It is easy to see why it is more accurate than the first sequence. However, it is not obvious to see if it can be improved upon.

Method 3: In each step, fold each crease that can be constructed immediately; make each crease independently of the others in this step. Then repeat this process until all creases are constructed. Thus, we can initially create $C1$, $C4$ and $C7$. Then we can create $C2$, $C3$, $C5$ and $C6$.

Method 3 minimises the compounding of errors in Figure 5.1. We can see that the maximum error state has been reduced from 7 to 2. A method is presented which insures that the error state of each crease is minimised. Call it the Least Error State or LES method.

Table 5.1: Error states of creases made using methods 1 to 3

Error states	$C1$	$C2$	$C3$	$C4$	$C5$	$C6$	$C7$
Method 1	1	2	3	4	5	6	7
Method 2	3	2	3	1	3	2	3
Method 3	1	2	2	1	2	2	1

Definition 5.6.2. The ***LES method*** constructs creases in a sequence of rounds. In round 1, a set of creases M_1 are made, where each crease in M_1 can be made using nothing but the boundary of the paper for alignments. Then recursively, in round k, a new set of creases M_k are made, where each crease in M_k can be made using only the creases in $\bigcup_{i=1}^{k-1} M_i$ and the boundary of the paper for alignments.

Lemma 5.6.1. *The LES method returns the lowest possible error state for any crease.*

Proof. Given a crease, see if it could be made with a lower error state. Given that a crease x_i is in the set M_k it has been constructed in round k and has error state $\sigma = k$. Thus, all creases used in alignments to construct x_i have error state $\sigma = k - 1$ or less. In each round $1, 2, 3, ..., k - 1$ of the method it must not have been possible to construct x_i, or any other crease with error state $\sigma = K$ or higher. Thus it is not possible to construct x_i until round k; its Least Error State is $\sigma = k$. □

It is likely that, as all creases have the lowest error state, and compounding of errors is minimised, the result is most accurate. However, there is a possibility that in each step we are limited to axiom choices which are not the most accurate. It is possible that by choosing other more accurate axioms, with a higher error state, we could get a better result. We conjecture that LES runs in polynomial time.

Theorem 5.6.2. *The LES method runs in polynomial time.*

Proof. Each Axiom constructs a new crease from a pre-defined number of other objects. We shall set the maximum number of objects needed to be a. For each round, we compute all k_i^a possible creases from the k_i objects with error state $\sigma \leq i$, then compare each against all creases which are still to be constructed. There are at most $O(n)$ error states, so LES should be computable in $O(n^{a+2})$ time. □

5.7 Summary

In this chapter we have developed a framework in which errors incurred when realising creases can be estimated. A model of the error in terms of the extent to which a crease can vary, assuming some distance ϵ between the points we want to align in order to realise the crease, is given for each of the 1-Fold axioms or HJA's. The model defines an area on a square piece of paper on which a crease is made. This area is defined by straight lines or curved ones as parabolas or hyperbolas. It is also shown how estimating this area for different ways of making a crease allows one to choose the fold that will result in a more accurate crease. Crease accuracy is important in origami; this has also been illustrated. Finally an algorithm to minimise the compounding of errors due to successive crease making has been given.

Note that the content of this chapter has now appeared as an OSME paper in the 7th OSME proceedings [326].

The next chapter is an overall conclusion of this thesis.

6 Conclusion

This thesis has looked at several interesting and related questions which arise from an in depth exploration of Origami science and its underpinning mathematics. Through a thorough literature survey we have presented a comprehensive review of work that has been carried out in the last few decades on Origami. Our emphasis has been on the mathematical aspects, however, work concerned with Psychology, Teaching and Art has also been covered. In fact our survey concerns to a large extent the papers that appeared at the OSME conferences series. The proceedings of all six conferences to date have been consulted and all papers have been briefly summarised. We believe that there are other important papers presented outside the OSME conferences; several of those we are aware of and we consider to be important have also been mentioned.

This survey led to many fascinating open or unexplored questions of which several are looked at in this thesis. Before exploring new mathematical aspects of Origami, a solid foundation on the already existing theorems and knowledge of the Mathematics of Origami was required.

Chapter three has shown how we can use Origami to solve problems such as trisecting angles and doubling the cube. It has also proved the new result that any Origami crease pattern is at most 3-colourable. We later built on this to show how the proof of the 4-colour theorem can be explored by converting the task of map colouring into an Origami problem.

It is clear that if it can be shown that it is always possible to 4-colour a map using an Origami technique then a proof of this would be equivalent to a proof of the 4-colour theorem. Many attempts were made to prove this theorem. Although the thesis does not provide a complete method, it demonstrates how Origami methods could become a tool for finding proofs of mathematical principles more complex than just trisecting angles and solving polynomials. It is yet to be seen if this might help to find solutions to these problems.

In Chapter five we have shown that, given a standardised error model for each of the one-fold constructions, differences between the sizes of the regions containing the resulting creases point to differences in accuracy. This allows axiom choices to be made which minimise the overall error. This approach has been illustrated on some simple examples and several ways to quantify the error regions have been suggested.

We have also introduced LES, an algorithm to reduce the compounding of errors. Applying it is by far the easiest way to improve overall accuracy in Origami construction.

In conclusion this chapter has shown that given a standardised error model, we can both choose the most accurate alignments to make; as well as dramatically reducing how the compounding of errors using a polynomial time algorithm.

6.1 Future Work

In future work we will consider both development of the model and further exploration for mathematical problems solvable with Origami techniques. Perhaps most desirable would be a proof of the 4 colour theorem, although this may be still just out of reach it may be achievable. More realistically we hope to explore if other problems may be approachable with Origami techniques.

We also hope to expand the error model to allow more complex error regions and to generalise the approach to other than just the 1-fold construction axioms. Although the error model is helpful for single crease models it is possibly over complex to expand into a multi-fold process, thus looking at the maximum translations, rotation or distance between the fold and target fold, give a value which can be used more effectively and built into an accuracy optimisation method.

Exploration of the more practical side of potential folding machines may also lead to further operations which are not Origami constructions but produce measured creases.

There are also other questions still remaining. can more advanced mathematics be done if we do not limit ourselves to a single fold or straight folds? This question is similar to that seen with constructing π using the somewhat undefined area of folding involving curved creases. We also do not know if there are any open mathematical questions that we can solve using Origami? It is possible that with further research into folding in general, we could answer these questions.

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Appendices

A

Notes on experimentation with the 4-colour problem

A considerable amount of time on my thesis was spent exploring possible solutions to finding a proof for the 4-colour Theorem, some more are briefly listed below.

A.1 Generating added edges without use of Origami

Having been inspired by the OSC theorem and having used Origami to add edges to a CPG it is obvious that looking at other ways of producing these edges, without Origami, might be of interest.

We look at what's special about the remaining edges when those that are easy to remove are removed. We note that the endpoints of the chains made my added edges are where there were odd degree vertices. More specifically the odd vertices on a graph are where they are odd verticies on the graph consisting only of the added edges (if the initial CPG is removed).

A.2 Expansion of exploration of moated faces

We have seen that if the remaining added edges, which cannot be removed with cycles or paper-edge to paper-edge paths, are contained in a single face, they can be dealt with if they only exist in a single face in the CPG. It is also possible to look at pairs of faces or even multiple disconnected pairs of edges. These regions of added edges need to be far enough away from each other as to be not share any edges.

We shall refer to regions of faces which do not share any edges as moated from each other. If a solution can be found where moated regions contain only individual or pairs of faces then a 4-colouring possible.

Unfortunately this approach was not successful and will still always will leave a problem as it is unlikely all examples will be able to do this.

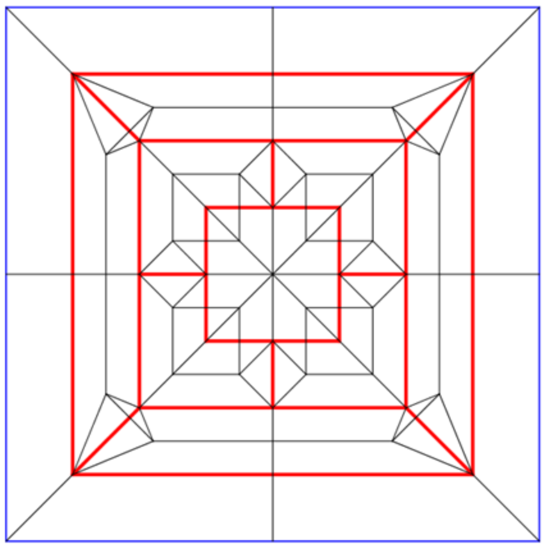
However, these moated regions have a boundary, if we can find this boundary we can guarantee to find a colouring

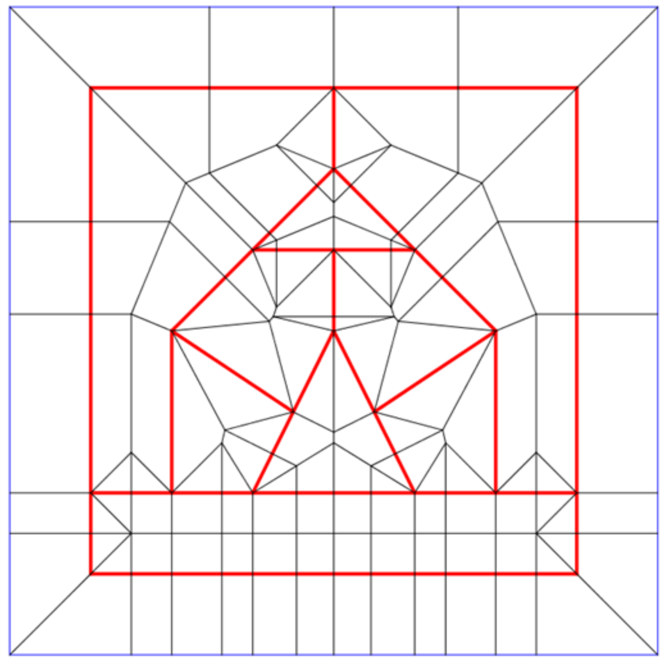
Looking at the odds degree vertices and where they are in the graph shows there are strong links between them. You can you get an isolated odd vertex as the number of odd degree vertices must be even. (This is a common result by the Handshake theorem)

We remember that the aim is to separate all the faces in the CPG into 2 groups each being 2 colourable. Thus perhaps we should look at the border between these two regions. Notably, it covers all the odd vertices.

through exploration of this we also found an equilivant statement to the 4-colour theorem.

Lemma A.2.1. *Given any planar graph corresponding to a map, the map is 4 colourable if and only if a circuit or multiple circuits can be found covering all odd vertices such that each circuit contains an even number of odd vertices.*





B

Notes on possible Origami approaches which could be used to optimise precision of Non Origami tools

In this note we look at the precision associated with measuring tools and methods by which we can use these tools such that their precision is increased, we then relate this to Origami and show how Origami tools can further increase this precision. The concepts looked at here are similar to the analysis of our error modelling in Chapter 5

Understanding Origami constructions is very useful for an Origami artist as they can use these techniques to efficiently construct Origami models. It is also useful to anyone using applications of Origami, for the same reasons. However, if you are not using Origami in your application one would rarely use folded paper as a tool.

Due to the speed of constructing Origami tools and their easily replaceable and disposable nature this note suggests a method by which Origami tools might become useful.

We look at the precision associated with measuring tools, using, among others, the example of finding the most accurate way of measuring the area of a square given a specific tool, then looking at how this might apply to other real world examples.

Finally, we take these techniques and attempt both to apply them to integrate Origami techniques to improve these methods.

B.1 Measuring tools and errors

Errors can affect both the accuracy (correctness of the output), and precision (size of the range of the output) of a result. Here we are only looking at the precision of a result and we are not taking into account factors such as operator error. Given a specific measuring tool, there is a fixed tolerance or precision associated with it. Using this tool we will only be able to take measurements accurately to within this pre-set tolerance. This affects how precisely you can take a measurement. However, here we ask, are there ways in which we can reduce the tolerances in our output based solely on how we use the tool. Here we give an example of how we can achieve this.

We now provide a sample object which is $60cm$ high and $60cm$ wide. The measurer is informed that the shape is perfectly square however does not know the dimensions and is asked to, as precisely as possible, calculate the area using only a meter stick.

The meter stick is marked with $1cm$ divisions thus we can measure from $1cm$ to $100cm$ with a precision of $\pm 0.5cm$. If we wish to use this tool to measure the area of a square we might initially consider measuring the height/width of this square and then calculate the area. We now attempt this initial method:

- Measuring the height and width gives us measurements of $60cm$ which with our precision have maximums of $60.5cm$ and minimums of $59.5cm$.
- Our maximum area, calculated by multiplying both diagonals is $3660.25cm^2$ and the minimum is $3540.25cm^2$.
- The difference between these two values is $120cm^2$.

However this might not be the most precise method, we now suggest a different method where we measure instead the diagonals, the actual lengths of the diagonals are $\sqrt{7200}cm$ which is $84.8528cm$ to six significant figures.

- Measuring the diagonals gives measurements of $85cm$ which with our precision have maximums of $85.5cm$ and minimums of $84.5cm$.

- Our maximum area, calculated by multiplying both diagonals and dividing by 2 is $3655.125cm^2$ and the minimum is $3570.125cm^2$.
- The difference between these two values is $85cm^2$.

The actual area of the square was $3660cm^2$ which is contained in the range of measurements from both methods, however, the latter method provides tighter bounds, thus a higher confidence, on the final measurement.

As we are told that our shape is square we can even take this one step further as square rooting our maximum and minimum areas give $60.46cm$ and $59.75cm$ respectively, to four significant figures, as bounds for our height and width which is more precise than measuring directly, albeit only slightly.

This shows that the precision of a measurement is not only affected by our tool but how we use them.

The reason for the increase in precision of the final result is due to the effect of measuring a longer part of the object, thus relative to this measurement the tolerance has a smaller effect. Perhaps if we had a measuring tape of the same precision as the meter stick we would produce an even more precise measurement using the perimeter.

B.2 Using Origami as a measuring tool to reduce errors

We now relate these to Origami techniques.

If we take our meter rule and measure a piece of paper which is $100cm$ long we will have a resulting piece of paper which is between $99.5cm$ and $100.5cm$.

There is of course a random error associated in folding. If however, we fold that paper into tenths, we will have a piece of paper that should be between $9.95cm$ and $10.05cm$. If we were to repeat this for pieces of paper that were measured by the meter stick to be from $100cm$ to $1cm$ in steps of $1cm$ we would theoretically be able to use these to mark up a $10cm$ ruler with $1mm$ precision.

In reality this method would be impractical and it is likely that random errors would outweigh the increase in precision, however if, rather than measuring an object, we wish to mark a measurement on an object, then we can use Origami techniques to produce a template device possibly with more precision and accuracy than a measuring device.

We now give another example of this. We begin by attempting to mark on an object a measurement of $25\sqrt{5}cm$ along an edge. Using our meter rule again we will first need to convert this to a decimal measurement which is $55.9017cm$ to six significant figures. Thus we can measure a length of $56cm$ and we will get a marked lenth of between $55.5cm$ and $56.5cm$.

Instead we measure and mark a square piece of paper, using diagonal measurements, to get a square of edge length $50cm$, (Here we skip over the logistics of doing this as it is the same as for the 60cm square) we would get a square with edge lengths between $50.56cm$ and $49.85cm$. By marking the midpoint of the top edge using a fold and then folding one of the bottom corners up to the top edge we will produce a fold of length between $56.53cm$ and $55.73cm$ which is a range of $0.8cm$ compared with $1cm$. This is again ignoring random errors however there are fewer places in which they can occur.

As a final example this time without specific measurements, if you wish to cut an object in half we compare measuring through traditional methods with Origami methods. We are given an object to cut in half, we simplify this problem to marking the middle of the object as both would use the same cutting tool.

Initially we measure the object to within a tolerance of $\pm E$. We then measure along the object half that of the measured length of the object and we make a mark in this location. This can now be up to $\pm 2E$ away from the actual midpoint.

We attempt this with Origami: We mark the length of the object on a piece of paper, this is marked with a random error of $\pm O$, , where O is our error associated with marking the object in the specific place, this time marking the paper. When we fold this in half we have a piece of paper which is accurate to within $\pm(\frac{O}{2}+F)$, where F is the fold error, thus when we mark it we end up with a measurement with error of $\pm(\frac{3O}{2} + F)$ from the centre of our object.

The question now remains is it possible to reduce $\pm(\frac{3O}{2} + F)$ below $\pm 2E$

B.3 Conclusion

In conclusion this paper has shown that there exist ways in which we can use tools to increase their precision and there are methods we can take to reduce the quantity of random errors which can be introduced.

It is perhaps surprising to see that those methods which are most obvious for performing simple tasks can be improved by performing them in an unusual way, such as measuring the diagonals of a square rather than the height and width

This would also scale to estimating the area of larger objects such as a building site or a building, by measuring the diagonals we get a more precise result.

We now plan to look at these methods and see how they relate to axiomatic Origami construction and provide steps for increasing the accuracy of Origami. one possibly route will be to see if it is possible to create a model for Origami construction methods including error modelling which might create an output for any method of the types and quantities of errors associated with such a method.