

Green's relations and stability for subsemigroups

James East* and Peter M. Higgins†

Abstract

We prove new results on inheritance of Green's relations by subsemigroups in the presence of stability of elements. We provide counterexamples in other cases to show in particular that not all right-stable semigroups are embeddable in left-stable semigroups. This is carried out in the context of a survey of the various closely related notions of stability and minimality of Green's classes that have appeared in the literature over the last sixty years, and which have sometimes been presented in different forms.

We take as our starting point two well-known theorems from classical semigroup theory. Here is the first, which was proved by Green in [8, Theorem 3]:

Theorem 1. *In any finite semigroup, we have $\mathcal{D} = \mathcal{J}$.*

The conclusion of Theorem 1 is valid if “finite” is replaced by “periodic” or “group-bound” or “stable”. For the last, see [14, Theorem 1] or [20, Corollary A.2.5], but note that the definition of stability in [14] and other older papers such as [1] is slightly different from the modern definition (we will discuss this further at the end of the paper). We recall the meanings of these terms below, and discuss more general statements still.

Our second motivating theorem concerns the inheritance of Green's relations by subsemigroups. In all that follows, U will always denote a subsemigroup of a semigroup S . Letting \mathcal{G} stand for any of the five Green's relations, we shall denote \mathcal{G} on the semigroup U by \mathcal{G}' . We use a similar convention for the pre-orders $\leq_{\mathcal{G}'}$ (for $\mathcal{G} \neq \mathcal{D}$). Certainly for any \mathcal{G} we have

$$\mathcal{G}' \subseteq \mathcal{G}|_U,$$

where $\mathcal{G}|_U = \mathcal{G} \cap (U \times U)$ denotes the restriction of \mathcal{G} to U . While $\mathcal{G}' = \mathcal{G}|_U$ need not hold in general, the next result concerns a special case in which it does for the three smaller relations:

Theorem 2. *If U is a regular subsemigroup of S , and if \mathcal{G} is any of \mathcal{R} , \mathcal{L} or \mathcal{H} , then $\mathcal{G}' = \mathcal{G}|_U$.*

Theorem 2 is generally attributed to Hall, cf. [9, Result 9(ii)], and also [12, Lemma 1.2.13]. The first proof in the literature however is due to Anderson et. al. [1, Proposition 2] and that is recorded in the book [15, Proposition 2.4.3]. The following is essentially the argument in [7, Lemma 2.8]. We write $\text{Reg}(T)$ for the set of all regular elements of any semigroup T .

Lemma 3. *Let $x, y \in U$, a subsemigroup of S , with $y \in \text{Reg}(U)$. Then $x \leq_{\mathcal{R}} y \Rightarrow x \leq_{\mathcal{R}'} y$.*

Proof. We have $x = ya$ for some $a \in S^1$. Take $z \in U$ with $y = zzy$. Then $x = ya = yzya = yzx$. Since $zx \in U$, this shows that $x \leq_{\mathcal{R}'} y$. \square

*Centre for Research in Mathematics, School of Computing, Engineering and Mathematics, Western Sydney University, Locked Bag 1797, Penrith NSW 2751, Australia. Email: j.east@westernsydney.edu.au

†Department of Mathematical Sciences, University of Essex, Colchester CO4 3SQ, UK. Email: peteh@essex.ac.uk

From Lemma 3 (and its dual), it follows that if $x, y \in \text{Reg}(U)$, and if \mathcal{G} is any of \mathcal{R} , \mathcal{L} or \mathcal{H} , then $x \mathcal{G} y \Rightarrow x \mathcal{G}' y$. Theorem 2 then follows immediately.

As is well known, the conclusion of Theorem 2 does not extend to either the \mathcal{D} - or the \mathcal{J} -relation, even for a finite regular semigroup S as shown in the first of the following example set.

Examples 4. Remembering that $\mathcal{D} = \mathcal{J}$ for finite semigroups (cf. Theorem 1), let $S = \mathcal{M}^0[1; 2, 2]$ be the 2×2 combinatorial (meaning \mathcal{H} -trivial) Brandt semigroup, a representation of which is the set of 2×2 binary matrices with at most one non-zero entry, under the operation of multiplication:

$$S = \{0\} \cup \{a_{ij} : 1 \leq i, j \leq 2\},$$

where a_{ij} has 1 as its entry at position (i, j) . Observe that S is a five-element inverse semigroup consisting of $a = a_{12}$ and $b = a_{21}$, which are nilpotents, together with three idempotents 0 , $e = a_{11}$ and $f = a_{22}$. The multiplication follows the rule $a_{ij}a_{jk} = a_{ik}$ with all other products being zero. The semigroup S provides the following pair of counterexamples.

- (i) The subsemigroup $U = \{0, e, f\}$, being a semilattice, is \mathcal{J} -trivial and in particular $(e, f) \notin \mathcal{D}'$. However, $(e, f) \in \mathcal{D}|_U$ as $S \setminus \{0\}$ forms a \mathcal{D} -class of S , with a and b being mutual inverses and $e \mathcal{R} a \mathcal{L} f$ by virtue of the products $ea = a$, $ab = e$, $af = a$ and $ba = f$. Hence the conclusion of Theorem 2 is not valid if $\mathcal{G} \in \{\mathcal{D}, \mathcal{J}\}$.
- (ii) With the same containing semigroup S , take $U = \{0, e, a\}$, which is a subsemigroup of S in which the only non-zero products are $e^2 = e$ and $ea = a$. Note that $e \in \text{Reg}(U)$ and $e \mathcal{R} a$ in S , so certainly $e \leq_{\mathcal{R}} a$. But the equation $au = e$ has no solution for $u \in U^1$, so that $e \not\leq_{\mathcal{R}'} a$ (in fact, $a <_{\mathcal{R}'} e$). So Lemma 3 does not hold in general if y (taken here to be a) is not regular in U , despite x (taken here to be e) being regular in U .

The previous example shows that it is possible to have $x \leq_{\mathcal{R}} y$ with $x \in \text{Reg}(U)$ but $x >_{\mathcal{R}'} y$. Note that it is never possible to have $x <_{\mathcal{R}} y$ but $x \geq_{\mathcal{R}'} y$, as the latter implies $x \geq_{\mathcal{R}} y$. However, it is possible to have $x <_{\mathcal{R}} y$ but $x \not\leq_{\mathcal{R}'} y$ with $x \in \text{Reg}(U)$ as shown in the next example.

- (iii) Our semigroup S is I_3 , the symmetric inverse semigroup on the base set $\{1, 2, 3\}$. Let f be the (partial) mapping defined by $1f = 2$ and $2f = 3$, and let g be the transposition (12) . Here $3f$ and $3g$ are both undefined. Now $fg = e$ is the idempotent with domain $\{1\}$. Hence in I_3 we have $e <_{\mathcal{R}} f$. The inequality is certainly strict since $e <_{\mathcal{J}} f$, as $\text{rank}(e) = 1 < 2 = \text{rank}(f)$. Now let $U = \langle f, e \rangle$. Since the elements of U are non-decreasing and $1f > 1$ it follows that there is no solution $u \in U^1$ to the equation $fu = e$. Hence we have that $e \in \text{Reg}(U)$ and $e <_{\mathcal{R}} f$ but $e \not\leq_{\mathcal{R}'} f$.

Despite the first of the above examples, we do have the following, which concerns the case in which the regular elements of S form a subsemigroup. For a proof, see [7, Lemma 3.8], the proof of which uses Theorem 2.

Lemma 5. *If $U = \text{Reg}(S)$ is a subsemigroup of S , then $\mathcal{D}' = \mathcal{D}|_U$.*

From this we may quickly deduce the following:

Corollary 6. *If $U = \text{Reg}(S)$ is a subsemigroup of S , and if $\mathcal{D} = \mathcal{J}$, then $\mathcal{D}' = \mathcal{J}' = \mathcal{J}|_U$.*

Proof. We have $\mathcal{D}' \subseteq \mathcal{J}' \subseteq \mathcal{J}|_U = \mathcal{D}|_U = \mathcal{D}'$. \square

Now that we have returned somewhat to the theme of Theorem 1, let us work towards a result that extends it, and also gives some kind of analogue of Theorem 2 with respect to \mathcal{D} - and \mathcal{J} -classes.

Recall that a semigroup S is *periodic* if every element has finite order, while S is *group-bound* if for every element x of S , some power of x belongs to a subgroup of S . So S is periodic (or

group-bound) if for all $x \in S$, there exists $k \geq 1$ such that $x^k = x^{2k}$ (or $x^k \mathcal{H} x^{2k}$), respectively, cf. [13, Theorems 1.2.2 and 2.2.5].

An element x of a semigroup S is called *right-stable* (or *left-stable*) if for all $y \in S$, we have $x \mathcal{J} xy \Rightarrow x \mathcal{R} xy$ (or $x \mathcal{J} yx \Rightarrow x \mathcal{L} yx$), respectively. An element is *stable* if it is both left- and right-stable. A semigroup is *stable* if each of its elements is stable. Similarly, we may speak of left- or right-stable semigroups.

Consider now the following sequence of statements, concerning a semigroup S :

- (I) If S is finite, then $\mathcal{D} = \mathcal{J}$.
- (II) If S is periodic, then $\mathcal{D} = \mathcal{J}$.
- (III) If S is group-bound, then $\mathcal{D} = \mathcal{J}$.
- (IV) If S is stable, then $\mathcal{D} = \mathcal{J}$.
- (V) If $x, y \in S$, and if at least one of x, y is stable, then $x \mathcal{J} y \Rightarrow x \mathcal{D} y$.

Statement (I) is Theorem 1, and the others are also well known, cf. [13, Proposition 2.1.4], [10, Theorem 1.2(vi) and Remark 1.7], [20, Corollary A.2.5] and [15, Propositions 2.3.7 and 2.3.9], respectively. Moreover we have (V) \Rightarrow (IV) \Rightarrow (III) \Rightarrow (II) \Rightarrow (I), since

$$\text{finite} \Rightarrow \text{periodic} \Rightarrow \text{group-bound} \Rightarrow \text{stable}. \quad (1)$$

Indeed, only the last of these implications is not obvious, so we give a short proof for convenience (for a proof that group-bound semigroups satisfy an alternative formulation of stability, see [10, Theorem 1.2(vi)]):

Proposition 7. *Any group-bound semigroup is stable.*

Proof. Let S be a group-bound semigroup, and let $x, y \in S$. Clearly it suffices to show that $x \mathcal{J} xy \Rightarrow x \leq_{\mathcal{R}} xy$ and $x \mathcal{J} yx \Rightarrow x \leq_{\mathcal{L}} yx$. By duality, it suffices to prove only the first of these implications. So suppose $x = axyb$ for some $a, b \in S^1$. Some power of yb belongs to a subgroup of S , say $(yb)^k$. Let z be the inverse of $(yb)^k$ in this subgroup. Then $x = a^k x (yb)^k = a^k x (yb)^k (yb)^k z = x (yb)^k z \leq_{\mathcal{R}} xy$. \square

The converse of Proposition 7 does not hold. For example, the semigroup of natural numbers under addition is stable (as is any \mathcal{J} -trivial semigroup) but not group-bound. It is easy to find examples to show that the other implications in (1) are also non-reversible in general.

The next result is a generalisation of statement (V) above (take $U = S$ in the statement):

Theorem 8. *Let x and y be elements of a semigroup S with y stable, and suppose x and y belong to some subsemigroup U of S for which $\mathcal{L}' = \mathcal{L}|_U$ and $\mathcal{R}' = \mathcal{R}|_U$. If $x \mathcal{J} y$ and $x \leq_{\mathcal{J}'} y$, then $x \mathcal{D}' y$.*

Proof. Since $x \leq_{\mathcal{J}'} y$, we have $x = ayz$ for some $a, z \in U^1$. It then follows that

$$x = ayz \leq_{\mathcal{J}} ay \leq_{\mathcal{J}} y \mathcal{J} x,$$

so that all the above elements are \mathcal{J} -related. In particular, $y \mathcal{J} ay$, and so stability of y gives $y \mathcal{L} ay$, and hence $y \mathcal{L}' ay$ (as $y, ay \in U$ and $\mathcal{L}' = \mathcal{L}|_U$). A similar calculation gives $y \mathcal{R}' yb$. Since \mathcal{R}' is a left congruence, it follows that $ay \mathcal{R}' ayb = x$. Thus, $x \mathcal{R}' ay \mathcal{L}' y$, and so $x \mathcal{D}' y$. \square

We have already noted that Theorem 1 follows from Theorem 8, as do each of statements (I)–(V) above. We also have the following:

Corollary 9. Let $x, y \in U$, a regular subsemigroup of a stable semigroup S , with $x \mathcal{J} y$ and $x \leq_{\mathcal{J}'} y$. Then $x \mathcal{D}' y$.

Proof. By Theorem 2, we have $\mathcal{L}' = \mathcal{L}|_U$ and $\mathcal{R}' = \mathcal{R}|_U$. Since S is stable, Theorem 8 applies. \square

We may also infer (by taking $U = S$ in Theorem 8) the following fact, proved in [15, Proposition 2.3.9], which also of course implies that $\mathcal{D} = \mathcal{J}$ in any stable semigroup.

Corollary 10. In any semigroup, a \mathcal{J} -class containing a stable element is a \mathcal{D} -class.

It was also proved in [15, Proposition 2.3.7] that every element of such a \mathcal{J} -class is stable.

Example 11. The stability assumption on y cannot be dropped in Theorem 8, due to the fact that any semigroup U may be embedded in a (regular) bisimple (i.e., \mathcal{D} -universal) monoid S : see [19, Theorem 2] and also [11, Corollary 1], cf. [12, Corollary 1.2.15]. If U here is regular, then certainly $\mathcal{L}' = \mathcal{L}|_U$ and $\mathcal{R}' = \mathcal{R}|_U$, cf. Theorem 2. It follows that the two-element semilattice $U = \{0, e\}$ (which is certainly regular) may be embedded in a bisimple semigroup S and then $0 \mathcal{J} e$ and $0 \leq_{\mathcal{J}'} e$, yet $(0, e) \notin \mathcal{J}' = \mathcal{D}'$, contrary to the conclusion of the theorem. Incidentally, this shows that any bisimple semigroup embedding U is not stable.

As a concrete example that illustrates the previous remark we may take S to be the bicyclic monoid, which has presentation $S = \langle a, b : ab = 1 \rangle$. Here S is a bisimple inverse semigroup, which contains an infinite descending chain of idempotents: $1 > ba > b^2a^2 > \dots$ (here $>$ is the natural partial order defined by $e \leq f \Leftrightarrow e = fef$), and S possesses infinitely many copies of the two-element semilattice. As noted at the end of the previous paragraph, it follows that the bicyclic monoid S is not stable. It is also easy to see this directly, as for example $(1, 1 \cdot b) \in \mathcal{J} \setminus \mathcal{R}$ and $(1, a \cdot 1) \in \mathcal{J} \setminus \mathcal{L}$.

Although every semigroup is embeddable in a bisimple semigroup, not every semigroup is embeddable in a stable semigroup, or even in a left- or right-stable semigroup. The next result (cf. [7, Lemma 2.9]) will help clarify this.

Proposition 12. If U is a regular subsemigroup of a left- or right-stable semigroup S , then U is left- or right-stable, respectively.

Proof. If S is right-stable (the left-stable case is similar), then for any $x, y \in U$,

$$x \mathcal{J}' xy \Rightarrow x \mathcal{J} xy \Rightarrow x \mathcal{R} xy \Rightarrow x \mathcal{R}' xy,$$

using the stability of S for the second implication, and Theorem 2 for the third. \square

Examples 13. (i) The regularity assumption on U cannot be removed in Proposition 12. Indeed, we can see this by means of a construction from [13, page 60, Exercise 1]. Let S be the monoid of 2×2 real matrices under ordinary matrix multiplication. It is well known that S is stable, cf. [6, Lemma 3.1]. Let U be the subsemigroup of S consisting of all matrices of the form $\begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$ with $a, b > 0$. Then U is simple (\mathcal{J} -universal) but \mathcal{D} -trivial (and hence \mathcal{R} -, \mathcal{L} - and \mathcal{H} -trivial), and has no idempotents. It follows that for any $x \in U$, $x^2 \mathcal{J} x$ holds, yet neither $x^2 \mathcal{R} x$ nor $x^2 \mathcal{L} x$ holds. Thus, U is neither left- nor right-stable.

Incidentally, this example shows that the classes of left-stable semigroups and right-stable semigroups are not closed under taking subsemigroups. On the other hand, examining the properties listed in (1), it is clear that the classes of finite semigroups and periodic semigroups are closed under taking subsemigroups. The class of group-bound semigroups is not, however. Indeed, the monoid of natural numbers is a subsemigroup of the group of integers, and the former is not group-bound.

(ii) In Example 11 we observed that the bicyclic semigroup $S = \langle a, b : ab = 1 \rangle$ is neither left- nor right-stable. Since S is regular, it follows from Proposition 12 that S does not embed in any left- or right-stable semigroup.

Corollary 2.2 of [1] says that the bicyclic semigroup cannot be embedded in a stable semigroup, but again we note that the definition of stability used in [1] is different from (and stronger than, cf. Example 18) the modern definition we have been using.

(iii) It is even possible for a right-stable semigroup not to be embeddable in a left-stable semigroup (and a dual statement also holds). For example, consider the Baer-Levi semigroup U , consisting of all injective maps $f : \mathbb{Z} \rightarrow \mathbb{Z}$ with $\mathbb{Z} \setminus \text{im}(f)$ infinite [2]. By [4, Theorem 8.2], U is right-simple (i.e., \mathcal{R} -universal), and hence right-stable. The same result in [4] also shows that U is right-cancellative and without idempotents. By contrast, U is not left-cancellative. Indeed, any left-cancellative and right-simple semigroup is a right group (i.e., the direct product of a group with a right-zero semigroup, cf. [3, page 39]), but U is without idempotents. The impossibility of embedding U in a left-stable semigroup then follows from the next general fact:

Proposition 14. *If U is a right-simple semigroup, then the following are equivalent:*

- (i) U embeds in a left-stable semigroup,
- (ii) U is left-stable,
- (iii) U is left-cancellative,
- (iv) U is a right group.

Proof. (i) \Rightarrow (iii). Suppose U embeds in a left-stable semigroup S . As usual, we write Green's relations on S and U by \mathcal{G} and \mathcal{G}' , respectively. Let $a \in U$. Then

$$a \mathcal{R}' a^2 \Rightarrow a \mathcal{R} a^2 \Rightarrow a \mathcal{J} a^2 \Rightarrow a \mathcal{L} a^2,$$

using left-stability of S in the last step. Thus, $a \mathcal{H} a^2$. It follows that H_a is a group \mathcal{H} -class of S for all $a \in U$, cf. [12, Corollary 1.2.6].

Now suppose $a, b, c \in U$ are such that $ab = ac$. Let a^{-1} be the inverse of a in the group H_a , and let e be the identity of this group. Since $b \mathcal{R}' a$, we have $b \mathcal{R} a \mathcal{R} e$, so it follows that $b = eb$. Similarly $c = ec$. But then $b = eb = a^{-1}ab = a^{-1}ac = ec = c$.

(iii) \Rightarrow (iv). We have already noted that this follows from [3, page 39].

(iv) \Rightarrow (ii). Suppose $U = G \times R$, where G is a group and R a right-zero semigroup. Then $xy \mathcal{L} y$ for all $x, y \in U$. Indeed, writing $x = (g, s)$ and $y = (h, t)$, we have $(g^{-1}, s) \cdot xy = y$.

(ii) \Rightarrow (i). This is clear. □

We next classify the stable (bi)simple semigroups. Recall that a semigroup is *completely regular* if it is a union of groups: i.e., each \mathcal{H} -class is a group. A semigroup is *completely simple* if it is simple and completely regular. (This is not the standard definition of completely simple, but see for example [13, Theorem 3.3.2] for a proof of equivalence with the standard definition, which we will not reproduce here.) Any completely simple semigroup is isomorphic to a Rees matrix semigroup (without zero) over a group. See [13, Theorem 3.3.1] for a proof of this fact, and also the definition of Rees matrix semigroups. Since any Rees matrix semigroup over a group is bisimple (i.e., \mathcal{D} -universal), so too is any completely simple semigroup.

Proposition 15. *If S is a simple (i.e., \mathcal{J} -universal) semigroup, then the following are equivalent:*

- (i) S is stable,
- (ii) S is completely regular,
- (iii) S is completely simple,
- (iv) $x \mathcal{R} xy \mathcal{L} y$ for all $x, y \in S$.

Proof. Since S is simple, $x \mathcal{J} xy \mathcal{J} y$ for all $x, y \in S$, so clearly (i) \Leftrightarrow (iv). We have also noted that the completely simple semigroups are precisely the completely regular simple semigroups, so (ii) \Leftrightarrow (iii).

(iii) \Rightarrow (iv). Suppose S is completely simple, and let $x, y \in S$ be arbitrary. Since S is bisimple, $L_x \cap R_y$ is nonempty, and hence an \mathcal{H} -class. It is therefore a group (by complete regularity), and hence contains an idempotent. It follows from [13, Proposition 2.3.7] that $xy \in R_x \cap L_y$.

(iv) \Rightarrow (ii). Take $x = y$ to deduce that $x \mathcal{H} x^2$ for all $x \in S$, so that H_x is a group for all x , cf. [13, Theorem 2.2.5]. \square

Remark 16. For a simple semigroup S , the condition “ S embeds in a stable semigroup” is not equivalent to conditions (i)–(iv) from Proposition 15. Indeed, the matrix semigroup denoted U in Example 13(i) is simple and non-stable, but embeds in a stable semigroup.

Above we have noted more than once that the older definition of stability [1, 14] is not exactly the same as the modern one we have used. We conclude with some comments regarding this. Following Koch and Wallace’s older definition [14], we will say a semigroup S is *KW-stable* if for all $x, y \in S$,

$$Sx \subseteq Sxy \Rightarrow Sx = Sxy \quad \text{and} \quad xS \subseteq yxs \Rightarrow xS = yxs. \quad (2)$$

We will also say that S is *KW¹-stable* if for all $x, y \in S$ (or equivalently, all $x, y \in S^1$),

$$S^1x \subseteq S^1xy \Rightarrow S^1x = S^1xy \quad \text{and} \quad xS^1 \subseteq yxs^1 \Rightarrow xS^1 = yxs^1. \quad (3)$$

Clearly (2) and (3) are equivalent if $S^1x = Sx$ and $xS^1 = xS$ for all $x \in S$ (which occurs for example if S is regular and/or a monoid). The latter pair of equalities is equivalent to having $x \in xS \cap Sx$ for all $x \in S$. Note also that (3) simply says that for all $x, y \in S$,

$$x \leq_{\mathcal{L}} xy \Rightarrow x \mathcal{L} xy \quad \text{and} \quad x \leq_{\mathcal{R}} yx \Rightarrow x \mathcal{R} yx. \quad (4)$$

Part (i) of the following was proved in [15, Proposition 2.3.10]. Part (ii) was stated in [14, footnote 2]. We provide a simple proof (of (ii)) for completeness.

Proposition 17. (i) *KW¹-stability is equivalent to stability.*

(ii) *KW-stability implies (KW¹-)stability.*

Proof. To prove (ii), suppose S is KW-stable. By duality, it suffices to demonstrate the first implication in (4), so suppose $x, y \in S$ are such that $x \leq_{\mathcal{L}} xy$. We must show that $x \geq_{\mathcal{L}} xy$. This is clear if $x = xy$, so suppose otherwise. Since $x \leq_{\mathcal{L}} xy$, we have $x \in Sxy$. Then $Sx \subseteq SSxy \subseteq Sxy$, so in fact $Sx = Sxy$ by KW-stability. Since $x \in Sxy = Sx$, it follows that $xy \in Sxy = Sx$, which gives $xy \leq_{\mathcal{L}} x$. \square

Although KW-stability implies (KW¹-)stability, the converse does not hold in general. Indeed, the following elegant counterexample was constructed by O’Carroll in [18, Section 3]:

Example 18. Writing $\mathbb{N} = \{0, 1, 2, \dots\}$, define $A = -\mathbb{N} \cup (2\mathbb{N} + 1)$ and $B = -\mathbb{N}$. Also define $f, g \in \mathcal{T}_{\mathbb{Z}}$ (the full transformation semigroup on \mathbb{Z}) by

$$f : \begin{cases} a \mapsto 0 & \text{for } a \in A \\ 2x \mapsto -x & \text{for } x \geq 1 \end{cases} \quad \text{and} \quad g : \begin{cases} b \mapsto b + 1 & \text{for } b \in B \\ 2x \mapsto 2x + 1 & \text{for } x \geq 1 \\ 2x - 1 \mapsto 2x + 1 & \text{for } x \geq 1. \end{cases}$$

Since g is injective on $A = \text{im}(g)$, it follows that g is a group element of $\mathcal{T}_{\mathbb{Z}}$. Let G be the infinite cyclic group generated by g . Also write K for the kernel of $\mathcal{T}_{\mathbb{Z}}$, which consists of all constant mappings $\mathbb{Z} \rightarrow \mathbb{Z}$. Then by [18, Theorem 3.5], $S = G \cup fG \cup K$ is a (KW¹-)stable semigroup that is not KW-stable.

Consider again the (modern) definition of stability, which says that for all $x, y \in S$,

$$x \mathcal{J} xy \Rightarrow x \mathcal{R} xy \quad \text{and} \quad x \mathcal{J} yx \Rightarrow x \mathcal{L} yx. \quad (5)$$

It follows quickly from the first of these implications that if x and z are two elements of some common \mathcal{J} -class of S , then $R_z \leq_{\mathcal{R}} R_x \Rightarrow R_z = R_x$ (if $R_z \leq_{\mathcal{R}} R_x$, then $z = xy$ for some $y \in S^1$, so $x \mathcal{J} xy$ and we apply (5)). That is, if S is right-stable, then for any \mathcal{J} -class J of S , all \mathcal{R} -classes contained in J are minimal in the $\leq_{\mathcal{R}}$ -ordering on \mathcal{R} -classes contained in J . This clearly implies the following condition:

M_R^* : For each \mathcal{J} -class J of S , the set of all \mathcal{R} -classes contained in J has a minimal element.

The reverse implication ($M_R^* \Rightarrow$ right-stability) was proved in [17, Lemma 2.2]. Of course left-stability is equivalent to condition M_L^* , defined dually in terms of \mathcal{L} -classes. See also [10], [4, Section 6.6] and [12, Section 1.2] for further discussions of these and other minimality conditions.

Condition M_R^* is weaker than the condition known as M_R . The semigroup S satisfies M_R if every nonempty set of \mathcal{R} -classes of S contains an element minimal in the $\leq_{\mathcal{R}}$ order, or equivalently that there are no infinite descending chains of \mathcal{R} -classes. (Condition M_L is defined dually with respect to \mathcal{L} -classes.) Green's main extension of Theorem 1 to classes containing infinite semigroups is [8, Theorem 8], which states that $\mathcal{D} = \mathcal{J}$ if S satisfies M_R and M_L (or, in Green's notation, that $\mathfrak{d} = \mathfrak{f}$ if S satisfies \mathcal{M}_r and \mathcal{M}_l). Since we have noted that the conjunction of M_R^* and M_L^* is equivalent to stability, we have the following well-known result (cf. [4, Theorem 6.45]):

Theorem 19. If S satisfies M_R^* and M_L^* , then $\mathcal{D} = \mathcal{J}$.

On the other hand, there are no one-sided versions of the results discussed in the previous paragraph. Indeed, if S is a so-called *Croisot-Teissier semigroup*, as defined in [4, Section 8.2], and first studied in [5, 21], then S satisfies M_R (and hence M_R^*), but not M_L^* (and hence not M_L), yet $\mathcal{D} \neq \mathcal{J}$ in general. This all follows from [4, Theorem 8.11] and [16, Theorems 6, 11 and 12], cf. [10, Section 1].

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