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First- and Second-Order Necessary Conditions with Respect to Components for Discrete Optimal Control Problems

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Abstract

This paper is devoted to the study of discrete optimal control problems. We aim to obtain more constructive optimality conditions under weakened convexity assumptions. Based on a new approach introduced in this work, an optimality condition with respect to every component is obtained in the form of a global maximum principle. In addition, an optimality condition with respect to one of the components of a control in the form of the global maximum principle and with respect to another component of a control in the form of the linearized maximum principle are obtained. Furthermore, various second-order optimality conditions in terms of singular and quasi-singular controls with respect to the components are obtained on the fly.

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1. Introduction

The search for necessary optimality conditions for discrete optimal control problems (DOCPs) is one of the most attractive topics in control optimization theory. It was historically preceded by the discovery of the Pontryagin maximum principle [1] for continuous optimal control problems. The first discrete analogue of the maximum principle was obtained for linear DOCPs by Rozonoer [2]. In the same paper, Rozonoer argued that it may not be possible to extend the maximum principle to nonlinear DOCPs, and this argument was confirmed by subsequent studies [3, 4]. Soon after this work, extensive studies in this area were devoted to obtaining a number of first- and second-order optimality conditions in various forms. For example, the works [5–12] obtain optimality conditions in the form of a global maximum principle, while [3, 4, 13–18] obtain similar conditions in the form of a local maximum principle, the linearized maximum principle or the Euler equation. Moreover, second-order optimality conditions in terms of singular (in the sense of the discrete maximum principle) as well as quasi-singular controls are obtained in [14–16, 19–22].

At the same time, several results were also obtained in the discrete-time and infinite-horizon setting. Michel [23] was one of the first researchers to study the concave discrete-time infinite-horizon optimal control problem and obtained the necessary and sufficient conditions for optimality. Blot and Chebbi [24] extended the results of [2] to the infinite-horizon framework without concavity. A rigorous analysis of the infinite- horizon discrete-time optimal control theory based on several Pontryagin principles is provided in the book by Blot and Hayek [25]. In a recent study, Aseev, Krastanov, and Veliov [26] obtain the linearized discrete maximum principle by using a tangent cone to a set at a point for the discrete-time optimal control on the infinite horizon without requiring convexity. The problem of weakening of the latter in optimal control problems is examined thoroughly in the book by Zaslavski [27].

Generally, obtaining first- and second-order necessary optimality conditions for DOCPs is studied under various types of weakened convexity assumptions, such as a starlike set with respect to a point [28], starlike neighborhoods of a point [13], an e-convex set and convexity with respect to direction [6], a locally convex set in the neighborhood of a point [18, 20] and a tangent cone to a set at a point [29]. However, it can be argued that the use of these types of convexity assumptions does not enable one to obtain the necessary first-order optimality conditions that are satisfied for all elements of the set of control values. This implies that the results obtained under such convexity assumptions are less constructive than the discrete analogues of the Pontryagin maximum principle or its corollaries.

It can be claimed that it is essential to apply a more subtle approach that takes into account the specificity of the considered problem in the study of DOCPs. This is due to the fact that DOCPs have certain specific features: for example, the discrete analogue of the Pontryagin maximum principle is not always satisfied under the traditional assumptions for nonlinear DOCPs; the linearized discrete maximum principle and the discrete analogue of the Euler equation are not consequences of the discrete maximum principle, unlike for the continuous case; and the majority of the methods used to study continuous cases cannot be directly used for investigating DOCPs.

In light of all of the above points, the aim of this paper is to study DOCPs in the finite horizon setting to obtain more constructive optimality conditions under weakened convexity assumptions. To do this, we introduce a new approach that weakens the convexity assumptions. Using our approach that studies DOCPs with respect to the components of vector control, we obtain an optimality condition with respect to every component in the form of a global maximum principle (see Theorem 3.1). We also obtain an optimality condition with respect to one of the components of a control in the form of the global maximum principle and with respect to another component of a control in the form of the linearized maximum principle (see Theorem 3.2). Furthermore, we obtain various second-order optimality conditions in terms of singular and quasi-singular controls with respect to components (see Theorems 3.3 - 3.5). Consequently, this paper is the first that studies DOCPs with respect to the components, enabling us to obtain more constructive optimality conditions under a new type of weakened convexity assumptions, in contrast to the existing results, e.g., [13, 15, 18, 20, 22]. Our results have practical implications as they can be used in solving various problems. These include modelling economic, biomedical and chemical problems and optimizing complex technological systems in different issues of organization of production.

The paper is structured as follows. In Section 2, we introduce the optimization problem and assumptions. Section 3 shows the main results of the present paper for the explicit first- and second-order optimality conditions for DOCPs with respect to the components of vector control. In Section 4, we obtain various increment formulas of the objective functional with respect to the components by using various assumptions to prove the necessary optimality conditions in the next section. Section 5 shows the proofs of the theorems. Section 6 discusses perspectives for future research and some open problems. We give concluding remarks in the final section.

2. Problem Statement and Main Assumptions

Consider the following discrete optimization problem:

$$S(u(\cdot)) = \Phi\big(x(t_1)\big) \to \min, \tag{1}$$

$$x(t+1) = f\big(x(t), u(t), t\big),\ t \in I := \{t_0, t_0+1, ..., t_1-1\},\ x(t_0) = x^*, \tag{2}$$

$$u(t) \in U(t) \subseteq \mathbb{R}^r,\ t \in I. \tag{3}$$

Here, $\mathbb{R}^r$ is an r-dimensional Euclidean space, $x = (x_1, ..., x_n)^T \in \mathbb{R}^n$ is a state vector, $u = (u_1, ..., u_r)^T \in \mathbb{R}^r$ is a control vector, t is time (discrete), $x^* \in \mathbb{R}^n$ is a given vector, $\Phi(x) : \mathbb{R}^n \to \mathbb{R} :=]-\infty, +\infty[$ and $f(x, u, t) : \mathbb{R}^n \times \mathbb{R}^r \times I \to \mathbb{R}^n$ are given functions, $U(t_1 - 1)$ is an arbitrary given set, and $U(t) = V(t) \times W(t)$, $t \in I_{-1} := I \setminus \{t_1 - 1\}$, are given sets satisfying certain conditions, where $V(t) \subseteq \mathbb{R}^{r_0}, t \in I_{-1}$, and $W(t) \subseteq \mathbb{R}^{r_1}, t \in I_{-1}$, with $r_0 + r_1 = r$.

A control $u(\cdot)$ satisfying the condition (3) is said to be *admissible*. The pair $\big(u(\cdot), x(\cdot)\big)$ is said to be an *admissible process* if $u(t), t \in I$, is an admissible control and $x(t), t \in I \cup \{t_1\}$, is the corresponding trajectory of the system (2). We will find the minimum of the problem (1)-(3) from the set of admissible processes $\big(u(\cdot), x(\cdot)\big)$.

An admissible process $\big(\overline{u}(\cdot), \overline{x}(\cdot)\big)$ is said to be an *optimal process* if it is a solution to the problem expressed by (1)-(3). The components $\overline{u}(\cdot)$ and $\overline{x}(\cdot)$ of an optimal process $\big(\overline{u}(\cdot), \overline{x}(\cdot)\big)$ are said to be an *optimal control* and an *optimal trajectory*, respectively.

Existing studies (e.g., [4, 5, 8, 13]) that address the nonlinear problem expressed by (1)-(3) have shown that the validity of some necessary optimality conditions depends strongly on the structures of the sets $U(t)$, $t \in I$, and $f(x, U(t), t) := \{z : z = f(x, u, t), u \in U(t)\}$, $t \in I$, $x \in \mathbb{R}^n$. For instance, following [3, 4, 13], it is known that if along the optimal process $\big(\overline{u}(\cdot), \overline{x}(\cdot)\big)$, the set $f(\overline{x}(\theta), U(\theta), \theta)$ is not convex, then the discrete analogue of Pontryagin's maximum principle can be invalid at the point $\theta \in I$.

Remark 2.1 It should be emphasized that along an admissible process $(u^0(\cdot), x^0(\cdot))$, the convexity of the sets $f(x^0(\theta), V(\theta), w^0(\theta), \theta)$ and $f(x^0(\theta), v^0(\theta), W(\theta), \theta)$ does not always lead to the convexity of the set $f(x^0(\theta), V(\theta), W(\theta), \theta)$, where $u^0(\theta) = \big(v^0(\theta), w^0(\theta)\big)^T$, $v^0(\theta) \in V(\theta)$, $w^0(\theta) \in W(\theta)$ and $\theta \in I_{-1}$ (see Example 5.1).

Regarding Remark 2.1, it can be argued that the investigation of the problem expressed by (1)-(3) by components will be effective. Thus, the main aim of this paper is to study DOCPs with respect to the components of vector control.

Let us also recall some concepts that could be useful in studying the problem of (1)-(3).

Definition 2.1. [11] We call a set $Z \in \mathbb{R}^m$ $\gamma-$ *convex with respect to the point* $z_0 \in Z$ if for every $z \in Z$, there exists $\gamma = \gamma(z) \in]0,1]$ such that for all $\varepsilon \in [0,\gamma]$, the inclusion $z_0 + \varepsilon(z - z_0) \in Z$ is valid. If Z is γ-convex with respect to each of its points, then we call it γ-*convex.*

Definition 2.2. [28] We call a set $Z \in \mathbb{R}^m$ *starlike with respect to the point* $z_0 \in Z$ if for any point $z \in Z$, the segment connecting it to z_0 lies in Z.

It is important to remark here that every convex set as well as every open set is a γ-convex set, but the reverse is not always true. Indeed, for example, the set $Z = [-1,0[\cup]1,2]$ is γ-convex, but it is neither a convex nor an open set and is not even starlike with respect to any of its points.

To investigate the optimality of an admissible process $\left(u^0(\cdot), x^0(\cdot)\right)$, where $u^0(t) = \left(v^0(t), w^0(t)\right)^T$, $t \in I$, the following assumptions are used in the paper.

(A1) The functional $\varphi(\cdot)$ is continuously differentiable on $\mathbb{R}^n$;

(A2) The functional $\Phi(\cdot)$ is twice continuously differentiable on $\mathbb{R}^n$;

(B1) For every $t \in I$, the function $f(\cdot,t)$ and its partial derivative $f_x(\cdot,t)$ are continuous with respect to (x,u) on $\mathbb{R}^n \times \mathbb{R}^r$;

(B2) For every $t \in I_{-1}$, the partial derivative $f_w(\cdot,t)$ is continuous with respect to (x,u) on $\mathbb{R}^n \times \mathbb{R}^r$;

(B3) For every $t \in I$, the function $f(\cdot,t)$ and its partial derivatives $f_x(\cdot,t)$ and $f_{xx}(\cdot,t)$ are continuous with respect to (x,u) on $\mathbb{R}^n \times \mathbb{R}^r$;

(B4) For every $t \in I_{-1}$, the function $f(\cdot,t)$ and its partial derivatives $f_w(\cdot,t)$, $f_{ww}(\cdot,t)$, $f_{wx}(\cdot,t)$ and $f_{xw}(\cdot,t)$ are continuous with respect to (x,u) on $\mathbb{R}^n \times \mathbb{R}^r$;

(C1) For every $t \in I_{-1}$, the set $f(x^0(t), V(t), w^0(t), t)$ is γ-convex with respect to the point $x^0(t+1)$;

(C2) For every $t \in I_{-1}$, the set $f(x^0(t), v^0(t), W(t), t)$ is γ-convex with respect to the point $x^0(t+1)$;

(C3) For $\theta_1 = \theta + 1$, there exists $\delta > 0$ such that for all $x \in B_\delta(x^0(\theta_1))$, the set $f(x, v^0(\theta_1), W(\theta_1), \theta_1)$ is starlike with respect to the point $f(x, v^0(\theta_1), w^0(\theta_1), \theta_1)$, where $\theta \in \{t_0, t_0+1, ..., t_1 - 3\}$ and $B_\delta(x^0(\theta_1))$ is an open ball with radius $\delta > 0$ and center $x^0(\theta_1)$;

(C4) For $\theta \in I_{-1}$, there exists $\delta > 0$ such that for all $w \in B_\delta(w^0(\theta)) \cap W(\theta)$, the set $f(x^0(\theta), V(\theta), w, \theta)$ is starlike with respect to the point $f(x^0(\theta), v^0(\theta), w, \theta)$;

(C5) For every $t \in I_{-1}$, the set $W(t)$ is γ−convex with respect to the point $w^0(t)$.

Furthermore, for the sake of convenience, we use the following notations. $f(t) := f(x^0(t), v^0(t), w^0(t), t)$, $f_x(t) := f_x(x^0(t), v^0(t), w^0(t), t)$, $H(t, \hat{u}) := \mathring{\psi}^T(t; \hat{u}) f(t)$, $H_x(t, \hat{u}) := \mathring{\psi}^T(t; \hat{u}) f_x(t)$, and $H_{xx}(t; \hat{u}) := \mathring{\psi}^T(t; \hat{u}) f_{xx}(t)$ (similarly, $f_w(t)$, $f_{xx}(t)$, $H_w(t; \hat{u})$, $H_{ww}(t; \hat{u})$, and $H_{xw}(t; \hat{u})$ are defined), where $\mathring{\psi}(\cdot; \hat{u})$ is the solution of (26) and $H(\psi, x, v, w, t) = \psi^T f(x, v, w, t)$ - is the Hamilton-Pontryagin function.

3. Statement of the Main Results

The main results of the present paper concern explicit first- and second-order optimality conditions for DOCPs with respect to the components of vector control. In this section, we present our main results and provide some illustrative examples to demonstrate the effectiveness of the obtained conditions.

3.1. First-Order Necessary Optimality Conditions by Components

The first-order optimality conditions with respect to the components can be summarized in the following theorems.

Theorem 3.1. *Let assumptions (A1), (B1), (C1) and (C2) hold along an admissible process* $\big(u^0(\cdot), x^0(\cdot)\big)$. *Then, for the admissible control*

$u^0(\cdot) = (v^0(\cdot), w^0(\cdot))^T$ to be optimal, it is necessary that the inequalities

$$\Delta_{\hat{u}} \Phi(f(t_1 - 1)) \geq 0, \ \forall \hat{u} \in U(t_1 - 1), \tag{4}$$

$$\Delta_{\tilde{v}} H(\theta; \hat{u}) \leq 0, \ \forall \hat{u} \in U_0(t_1 - 1), \ \forall (\theta, \tilde{v}) \in I_{-1} \times V(\theta), \tag{5}$$

$$\Delta_{\tilde{w}} H(\theta; \hat{u}) \leq 0, \ \forall \hat{u} \in U_0(t_1 - 1), \ \forall (\theta, \tilde{w}) \in I_{-1} \times W(\theta) \tag{6}$$

hold, where $\Delta_{\hat{u}} \Phi(f(t_1 - 1))$, $\Delta_{\tilde{v}} H(\theta; \hat{u})$, $\Delta_{\tilde{w}} H(\theta; \hat{u})$ and $U_0(t_1 - 1)$ are defined by (25)*,* (31)*,* (34) *and* (63)*, respectively.*

The proof of Theorem 3.1 is given in Section 5.

In fact, Theorem 3.1 gives an optimality condition with respect to every component in the form of a global maximum principle. This form of the maximum principle can be applied for a wider class of DOCPs than the discrete maximum principle obtained in [5, 7]. More specifically, it is obvious that for these DOCPs that if the latter is valid, the maximum principle by the components is also valid. However, the converse may not always be true. We illustrate this with the following example.

Example 3.1 Consider the following problem:

$$x_1(t+1) = v(t) \sin(\tfrac{\pi}{2} w(t)), \quad x_2(t+1) = v^2(t) \cos^2(\tfrac{\pi}{2} w(t)),$$

$$x_3(t+1) = x_1(t) + x_2(t) + x_3(t) + w^2(t) - v^2(t), \quad t \in I = \{0, 1\},$$

$$x_1(0) = x_2(0) = x_3(0) = 0, \ t_1 = 2, \ v \in V(t), \ w \in W(t), \ t \in \{0, 1\},$$

$$S(u(\cdot)) = \Phi(x(t_1)) = x_3(2) \to \min,$$

where $u(t) = (v(t), w(t))^T$, $t \in \{0, 1\}$, $V(0) = [-3, -2] \cup [-1, 0]$, $W(0) = [0, 1] \cup [2, 3]$, and $V(1) = W(1) = \{\pm\frac{1}{2}, 0, -1\}$.

One can calculate directly $S(u(\cdot)) = w^2(0) + w^2(1) - v^2(1)$. It is obvious that if $v^0(0) = w^0(0) = w^0(1) = 0$, $v^0(1) = -1$, $x^0(0) = x^0(1) = (0, 0, 0)^T$, and $x^0(2) = (0, 1, -1)^T$, then $(u^0(\cdot), x^0(\cdot))$ is an optimal process, where $u^0(\cdot) = (v^0(\cdot), w^0(\cdot))^T$, $x^0(\cdot) = (x_1^0(\cdot), x_2^0(\cdot), x_3^0(\cdot))^T$.

Furthermore, according to Definition 2.1, the sets

$$f(x^0(0), v^0(0), W(0), 0) = \{(0, 0, w^2(0))^T : w(0) \in [0,1] \cup [2,3]\} \text{ and}$$

$$f(x^0(0), V(0), w^0(0), 0) = \{(0, v^2(0), -v^2(0))^T : v(0) \in [-3,-2] \cup [-1,0]\}$$

are γ-convex with respect to the point $x^0(1)$. However, the set

$$f(x^0(0), V(0), W(0), 0) = \Big\{(v(0)\sin(\frac{\pi}{2}w(0)), v^2(0)\cos^2(\frac{\pi}{2}w(0)), \\ w^2(0) - v^2(0)) : v(0) \in [-3,-2] \cup [-1,0], w(0) \in [0,1] \cup [2,3]\Big\}$$

is not convex and is even not γ-convex with respect to the point $x^0(1)$.

Next, along an optimal process $(u^0(\cdot), x^0(\cdot))$, considering (25), (26), (31), (34) and (63), we have

$$\Delta_{\hat{u}}\Phi(f(1)) = \hat{w}^2 - \hat{v}^2 + 1, \hat{u} = (\hat{v}, \hat{w}) \in V(1) \times W(1);$$

$$U_0(1) = \{\hat{u} = (\hat{v}, \hat{w})^T : \hat{v} \in V(1), \hat{w} \in W(1), \hat{w}^2 - \hat{v}^2 + 1 = 0\} = \{(-1,0)^T\},$$

$$\text{i.e. } \hat{u} = u^0(1) = (-1,0)^T, \mathring{\psi}(1;\hat{u}) = (0,0,-1)^T, \ \mathring{\psi}(0;\hat{u}) = (0,-1,-1)^T;$$

$$\Delta_{\tilde{v}}H(0;\hat{u}) = 0, \ \tilde{v} \in V(0), \ \ \Delta_{\tilde{w}}H(0;\hat{u}) = -\tilde{w}^2, \ \tilde{w} \in W(0);$$

$$\Delta_{\tilde{u}}H(0;\hat{u}) = \tilde{v}^2\sin^2(\frac{\pi}{2}\tilde{w}) - \tilde{w}^2, \ \tilde{u} = (\tilde{v}, \tilde{w}) \in V(0) \times W(0).$$

Therefore, for $u^0(\cdot) = (v^0(\cdot), w^0(\cdot))^T$, all three statements of Theorem 3.1 are satisfied, namely, $\hat{w}^2 - \hat{v}^2 + 1 \geq 0, \ \forall(\hat{v}, \hat{w}) \in V(1) \times W(1)$; $0 \leq 0, \ \forall \hat{u} \in U_0(1), \forall \tilde{v} \in V(0)$; and $-\tilde{w}^2 \leq 0, \ \forall \hat{u} \in U_0(1), \ \forall \hat{w} \in W(0)$. However, along an optimal control $u^0(\cdot)$, the discrete maximum principle is not valid, such as for $\tilde{v} = -3 \in V(0)$ and $\tilde{w} = 1 \in W(0)$: $(\tilde{v}^2\sin^2(\frac{\pi}{2}\tilde{w}) - \tilde{w}^2)\mid_{(-3,1)} = 8 \leq 0$. Furthermore, the known local maximum principles are not effective (or effective but not applicable [15]) for investigating the optimal problem in Example 3.1 due to the fact that at the point $t = t_1 - 1$, they are valid only for those sets that consist of one element.

Consequently, Example 3.1 allows us to state that maximum principle with respect to the components is valid for a wider class of DOCPs compared to the discrete maximum principle. This implies that the method introduced in our paper which is based on studying DOCPs with respect to components have wider application areas.

Theorem 3.2. *Let assumptions (A1), (B1), (B2), (C1) and (C5) hold along an admissible process* $\big(u^0(\cdot), x^0(\cdot)\big)$. *Then, in order for the admissible control* $u^0(\cdot) = (v^0(\cdot), w^0(\cdot))^T$ *to be optimal, it is necessary that the inequalities* (4), (5) *and*

$$H_w^T(\theta;\hat{u})(\tilde{w} - w^0(\theta)) \leq 0, \;\; \forall \hat{u} \in U_0(t_1 - 1), \;\; \forall (\theta, \tilde{w}) \in I_{-1} \times W(\theta) \tag{7}$$

hold, where $U_0(t_1 - 1)$ *is defined by* (63).

The proof of this theorem is presented in Section 5.

In Theorem 3.2, we obtain an optimality condition with respect to one of the components of a control in the form of a global maximum principle and with respect to another component of a control in the form of the linearized maximum principle. Note that this theorem has its own application areas compared to Theorem 3.1, and the relevant results for the necessary optimality conditions are studied in [4, 13, 15].

Finally, we emphasize that the fulfillment of the first-order necessary optimality conditions (4)-(6) and (7) does not even guarantee the local minimum of the functional (1) in the presence of singularities (see [19]). The next section addresses such cases.

3.2. Second-Order Necessary Optimality Conditions by Components

In this section, we introduce the concepts of singular as well as quasi-singular controls with respect to the components, and for the optimality of such controls, various second-order necessary conditions are obtained.

Definition 3.1. An admissible control $u^0(\cdot) = (v^0(\cdot), w^0(\cdot))^T$ satisfying the conditions (4)-(6) is called *singular with respect to the vector component* $v\,\big(w\big)$ at the point $t = \theta \in I_{-1}$ with the parameter $(\hat{u}, V_0(\theta)) \subseteq U_0(t_1 - 1) \times V(\theta)$ $\big((\hat{u}, W_0(\theta)) \subseteq U_0(t_1 - 1) \times W(\theta)\big)$ if for all $\tilde{v} \in V_0(\theta)$ $\big(\tilde{w} \in W_0(\theta)\big)$, the following equality holds:

$$\Delta_{\tilde v} H(\theta;\hat u) = 0 \; \big(\Delta_{\tilde w} H(\theta;\hat u) = 0\big), \tag{8}$$

where $V_0(\theta) \setminus \{v^0(\theta)\} \neq \varnothing \; \big(W_0(\theta) \setminus \{w^0(\theta)\} \neq \varnothing\big)$.

Definition 3.2. An admissible control $u^0(\cdot) = (v^0(\cdot), w^0(\cdot))^T$ satisfying conditions (4), (5) and (7) is called *quasi-singular with respect to the vector component* w at the point $t = \theta \in I_{-1}$ with the parameter $(\hat u, W_0(\theta)) \subseteq U_0(t_1 - 1) \times W(\theta)$ if for all $\tilde w \in W_0(\theta)$, the following equality holds:

$$H_w^T(\theta;\hat u)\big(\tilde w - w^0(\theta)\big) = 0, \tag{9}$$

where $W_0(\theta) \setminus \{w^0(\theta)\} \neq \varnothing$, and $U_0(t_1 - 1)$ is defined by (63).

Now, we are in the position to present our main results for the second-order optimality conditions with respect to the components.

Theorem 3.3. *Let assumptions (A2), (B3), (C1) and (C3) hold along an admissible process* $\big(u^0(\cdot), x^0(\cdot)\big)$. *Moreover, let* $u^0(\cdot) = (v^0(\cdot), w^0(\cdot))^T$ *be singular with respect to the vector component* v *at the point* $t = \theta \in I_{-1} \setminus \{t_1 - 2\}$ *with the parameter* $(\hat u, V_0(\theta))$ *and be singular with respect to the vector component* w *at the point* $t = \theta_1$ *with the parameter* $(\hat u, W_0(\theta_1))$. *Then, for the admissible control* $u^0(\cdot)$ *to be optimal, it is necessary that for all* $\tilde v \in V_0(\theta)$, $\tilde w \in W_0(\theta_1)$ *and* $\alpha \in \mathbb{R}_+$, *the inequality*

$$\alpha^2 \mathring{M}\big((\theta,\tilde v);\hat u\big) + 2\alpha \mathring{N}\big((\theta_1,\tilde w);\hat u\big)\Delta_{\tilde v} f(\theta) + \mathring{M}\big((\theta_1,\tilde w);\hat u\big) \le 0 \tag{10}$$

holds where $\Delta_{\tilde v} f(\theta)$, $\mathring{M}(\cdot)$ *and* $\mathring{N}(\cdot)$ *are defined by (16), (48) and (49), respectively.*

Theorem 3.4. *Let assumptions (A2), (B3), (B4), (C4) and (C5) hold along an admissible process* $\big(u^0(\cdot), x^0(\cdot)\big)$. *Moreover, let* $u^0(\cdot) = \big(v^0(\cdot), w^0(\cdot)\big)^T$ *be singular with respect to the vector component* v *at the point* $t = \theta \in I_{-1}$ *with the parameter* $\big(\hat u, V_0(\theta)\big)$ *and be quasi-singular with respect to the vector component* w *at the point* $t = \theta$ *with the parameter* $\big(\hat u, W_0(\theta)\big)$. *Then, for the admissible* *control $u^0(\cdot)$ to be optimal, it is necessary that for all $\tilde{v} \in V_0(\theta)$, $\tilde{w} \in W_0(\theta)$ and $\alpha \in \mathbb{R}_+$, the inequality*

$$\begin{aligned} &\mathring{M}((\theta,\tilde{v});\hat{u}) + 2\alpha\mathring{\Omega}((\theta,\tilde{v});\hat{u})(\tilde{w} - w^0(\theta)) \\ &+\alpha^2(\tilde{w} - w^0(\theta))^T\mathring{G}(\theta;\hat{u}))(\tilde{w} - w^0(\theta)) \le 0 \end{aligned} \tag{11}$$

holds, where $\mathring{M}(\cdot)$, $\mathring{\Omega}(\cdot)$ and $\mathring{G}(\cdot)$ are defined by (48), (56) and (57), respectively.

Theorem 3.5. *Let assumptions (A2), (B3), (B4), (C1) and (C5) hold along an admissible process $\left(u^0(\cdot), x^0(\cdot)\right)$. Moreover, let $u^0(\cdot) = \left(v^0(\cdot), w^0(\cdot)\right)^T$ be singular with respect to the vector component v at the point $t = \theta \in I_{-1} \setminus \{t_1 - 2\}$ with the parameter $(\hat{u}, V_0(\theta))$ and be quasi-singular with respect to the vector component w at the point $t = \theta_1$ with the parameter $(\hat{u}, W_0(\theta_1))$. Then, for the admissible control $u^0(\cdot)$ to be optimal, it is necessary that for all $\tilde{v} \in V_0(\theta)$, $\tilde{w} \in W_0(\theta_1)$ and $\alpha \in \mathbb{R}_+$, the inequality*

$$\begin{aligned} &\mathring{M}((\theta,\tilde{v});\hat{u}) + 2\alpha\Delta_{\tilde{v}}f^T(\theta)\mathring{P}(\theta_1;\hat{u})(\tilde{w} - w^0(\theta_1)) \\ &+\alpha^2(\tilde{w} - w^0(\theta_1))^T\mathring{G}(\theta_1;\hat{u}))(\tilde{w} - w^0(\theta_1)) \le 0 \end{aligned} \tag{12}$$

holds, where $\Delta_{\tilde{v}}f(\theta)$, $\mathring{M}(\cdot)$, $\mathring{G}(\cdot)$ and $\mathring{P}(\cdot)$ are defined by (16), (48), (57) and (62), respectively.

The proofs of these theorems are presented in Section 5.

Consequently, we obtain second-order necessary optimality conditions by components in the forms of (10), (11), and (12). Although these conditions have various areas of application, the applications of (10) and (11) are less constructive relative to those of (12). This is because assumptions (C3) and (C4) are required for the validity of optimality conditions (10) and (11), and it is generally difficult to determine whether these assumptions are true. Hence, there may be a question, for instance, of whether it is possible to weaken (C4). In other words, it is unclear whether Theorem 3.4 is valid if (C4) holds only at one point. The following example provides the answer to this question.

Example 3.2 Consider the following optimization problem:

$$x_1(t+1) = x_1(t) + \sqrt{2}v(t)w(t),\ \ x_2(t+1) = -x_1^2(t) + x_2(t) + v^2(t)(w(t) - 1),$$
$$x_1(0) = x_2(0) = 0,\ \ I = \{0,1\},\ \ t_1 = 2,\ \ u(t) = (v(t), w(t)) \in V(t) \times W(t),$$
$$t \in \{0,1\},\ \ V(0) = [-1,0],\ W(0) = [-2,2],\ \ V(1) = W(1) = \{0, \pm 1\},$$

$$S(u(\cdot)) = \Phi(x(2)) = -x_2(2) \to \min.$$

Let us calculate $S(u(\cdot))$:

$S(u(\cdot)) = v^2(0)[2w^2(0) - w(0) + 1] + v^2(1)[1 - w(1)] \to \min,$

where $(v(0), w(0)) \in [-1,0] \times [-2,2]$ and $(v(1), w(1)) \in \{0, \pm 1\} \times \{0, \pm 1\}$.

Clearly, $u^0(t) = (v^0(t), w^0(t))^T = (0,1)^T$, $t \in \{0,1\}$, is an optimal control, and $x^0(t) = (0,0)^T$, $t \in \{0,1,2\}$, is an optimal trajectory. Moreover, assumptions (A2), (B3), (B4) and (C5) hold for this example, but (C4) is satisfied only at the point $w^0 = w^0(0) = 1$, i.e., the set $f(x^0(0), V(0), w^0, 0)\ |_{w^0=1} = \{(\sqrt{2}v(0), 0)^T : v(0) \in [-1,0]\}$ is convex. Now, let us check the condition (11) along an optimal process $(u^0(\cdot), x^0(\cdot))$ at the point $t = 0$.

By (25), (26), (31), (51) and (63), one can write the following calculations:

$$\Delta_{\hat{u}}\Phi(f(1)) = \hat{v}^2(1 - \hat{w}),\ \ \hat{u} = (\hat{v}, \hat{w}) \in \{0, \pm 1\} \times \{0, \pm 1\}, \Delta_{\tilde{v}} f(0) = (\sqrt{2}\tilde{v}, 0)^T,$$

$$f_w(0) = (0,0)^T,\ \ U_0(1) = \{(0, \hat{w})^T : \hat{w} \in \{0, \pm 1\}\} \cup \{(\hat{v}, 1)^T : \hat{v} \in \{0, \pm 1\}\},$$

$$\mathring{\psi}(1; \hat{u}) = \mathring{\psi}(0; \hat{u}) = (0,1)^T,\ \ \Delta_{\tilde{v}} H(0; \hat{u}) = 0, \forall \hat{u} \in U(1),$$

$$H_w(0; \hat{u}) = 0,\ \ H_{ww}(0; \hat{u}) = 0,\ \ \Delta_{\tilde{v}} H_w(0; \hat{u}) = \tilde{v}^2,$$

$$H_{xx}(\psi^0(1; \hat{u}), x^0(1), \hat{u}, 1) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix},\ \mathring{\Psi}(1; \hat{u}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ \mathring{\Psi}(0; \hat{u}) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Next, by (49), (56) and (57), in a similar fashion, one can obtain

$$\mathring{M}((0; \tilde{v}); \hat{u}) = -4\tilde{v}^2,\ \mathring{\Omega}((0, \tilde{v}); \hat{u}) = \tilde{v}^2,\ \mathring{G}(0; \hat{u}) = 0. \tag{13}$$

Taking into account the above expressions for $\Delta_{\hat{u}}\Phi(f(1))$, $\Delta_{\tilde{v}} H(0; \hat{u})$ and $H_w(0; \hat{u})$, we obtain that the optimal control $u^0(t) = (0,1)^T$, $t \in \{0,1\}$, is singular with respect to the vector component v at the point $t = 0$ with parameter $(u, V(0))$, where $\hat{u} \in U_0(1)$, and is quasi-singular with respect to the vector component w at the point $t = 0$ with parameter $(\hat{u}, W(0))$, where $\hat{u} \in U_0(1)$. Thus, taking into account (13), the condition (11) takes the form

$$-4\tilde{v}^2 + 2\alpha\tilde{v}^2(\tilde{w} - 1) \le 0, \;\; \forall(\alpha, \tilde{v}, \tilde{w}) \in R_+ \times [-1, 0] \times [-2, 2].$$

This inequality for $\alpha = 3$, $\tilde{v} = -1$, $\tilde{w} = 2$ is not satisfied: $2 < 0$.

Thus, Example 3.2 enables us to state that assumption (C4) is essential for the validity of Theorem 3.4 and generally cannot be weakened.

4. Various Increment Formulas of the Objective Functional by Components

In this section, considering separate cases, first- and second-order increment formulas of the objective functional (1) with respect to the components are obtained along an admissible process $(u^0(\cdot), x^0(\cdot))$, where $u^0(t) = (v^0(t), w^0(t))^T$, $v^0(t) \in V(t)$, $t \in I$, and $w^0(t) \in W(t)$, $t \in I$. The results of this section are auxiliary and play an important role in the proof of the theorems in the following section.

4.1. First-Order Increment Formulas

To obtain first order increment formulas, we consider the following various cases.

Case 1.1 Assumptions (A1), (B1) and (C1) hold true.

Let $(\theta, \tilde{v}, \hat{u}) \in I_{-1} \times V(\theta) \times U(t_1 - 1)$ be any fixed point. Consider the special variation of the admissible control $u^0(\cdot) = (v^0(\cdot), w^0(\cdot))^T$ in the form

$$u(t; p_1, \varepsilon) = \begin{cases} u^0(t), & t \in I \setminus \{\theta, t_1 - 1\}, \\ (v(\varepsilon), w^0(\theta))^T, & t = \theta, \\ \hat{u}, & t = t_1 - 1. \end{cases} \tag{14}$$

Here, $p_1 := (\theta, \tilde{v}, \hat{u})$, and the vector function $v(\varepsilon) :]0, \tilde{\gamma}_1] \to V(\theta)$ is the solution of the following equation:

$$f(x^0(\theta), v(\varepsilon), w^0(\theta), \theta) - f(\theta) = \varepsilon \Delta_{\tilde{v}} f(\theta), \varepsilon \in \,]0, \tilde{\gamma}_1], \tag{15}$$

where $\tilde{\gamma}_1 := \gamma(\tilde{u}) \in]0,1]$ exists by Definition 2.1, and

$$\Delta_{\tilde{v}} f(t) := f(x^0(t), \tilde{v}, w^0(t), t) - f(x^0(t), v^0(t), w^0(t), t). \tag{16}$$

Note that the existence of $v(\varepsilon) :]0, \tilde{\gamma}_1] \to V(\theta)$ follows from (C1) and Definition 2.1, and it is clear that for every $\varepsilon \in]0, \tilde{\gamma}_1]$, the function $u(\cdot; p_1, \varepsilon)$ is an admissible control.

Consider an admissible process $(u(\cdot; p_1, \varepsilon), x(\cdot; p_1, \varepsilon))$. It is obvious that the increment $x(t; p_1, \varepsilon) - x^0(t) =: \Delta x(t; p_1, \varepsilon)$, $t \in I \cup \{t_1\}$, $\varepsilon \in]0, \tilde{\gamma}_1]$, is a solution to the system

$$\begin{cases} \Delta x(t+1; p_1, \varepsilon) = f(x^0(t) + \Delta x(t; p_1, \varepsilon), u(t; p_1, \varepsilon), t) - f(t), \\ \Delta x(t; p_1, \varepsilon) = 0, \quad t \in \{t_0, t_0 + 1, \ldots, \theta\}. \end{cases} \tag{17}$$

Considering (14) and (15), the system (17) can be written in a clearer manner:

$$\Delta x(t+1; p_1, \varepsilon) = \begin{cases} 0, & t_0 - 1 \le t < \theta, \\ \varepsilon \Delta_{\tilde{v}} f(\theta), & t = \theta, \\ f(x(t; p_1, \varepsilon), u^0(t), t) - f(t), & \theta < t < t_1 - 1, \\ \Delta_{\hat{u}} f(t_1 - 1) & \\ \quad + \Delta_{x(t_1 - 1; p_1, \varepsilon)} f(x^0(t_1 - 1), \hat{u}, t_1 - 1), & t = t_1 - 1, \end{cases} \tag{18}$$

where $\varepsilon \in]0, \tilde{\gamma}_1]$, and

$$\begin{aligned} &\Delta_{\hat{u}} f(t_1 - 1) := f(x^0(t_1 - 1), \hat{u}, t_1 - 1) - f(t_1 - 1), \\ &\Delta_{x(t_1 - 1; p_1, \varepsilon)} f(x^0(t_1 - 1), \hat{u}, t_1 - 1) \\ &\quad := f(x(t_1 - 1; p_1, \varepsilon), \hat{u}, t_1 - 1) - f(x^0(t_1 - 1), \hat{u}, t_1 - 1). \end{aligned} \tag{19}$$

Let us apply the steps method. Then, using Taylor's formula considering (B1), we obtain from (18)

$$\|\Delta x(t; p_1, \varepsilon)\| \le \tilde{K}(p_1)\varepsilon, \ t \in I, \ \varepsilon \in]0, \tilde{\gamma}_1], \ \tilde{K}(p_1) > 0, \tag{20}$$

where $\|\cdot\|$ is the Euclidean norm and $\tilde{K}(p_1)$ is some number.

Furthermore, taking into account (19),(20) and (B1), for $\Delta_{x(t_1-1;p_1,\varepsilon)} f(x^0(t_1-1),\hat{u},t_1-1)$, we easily obtain the followings:

$$\begin{aligned}\Delta_{x(t_1-1;p_1,\varepsilon)} f(x^0(t_1-1),\hat{u},t_1-1) &= \\ &f_x(x^0(t_1-1),\hat{u},t_1-1)\Delta x(t_1-1;p_1,\varepsilon)+o(\varepsilon),\end{aligned} \tag{21}$$

$$\|\Delta_{x(t_1-1;p_1,\varepsilon)} f(x^0(t_1-1),\hat{u},t_1-1)\| \le K(p_1)\varepsilon,\; K(p_1)>0. \tag{22}$$

Here and throughout the paper, we will use $\varepsilon^{-m}o(\varepsilon^m)\to 0$ as $\varepsilon\to 0$, with $m>0$.

Let us now calculate the increment $S(u(\cdot,p_1,\varepsilon)) - S(u^0(\cdot)) =: \Delta S(u^0(\cdot);p_1,\varepsilon)$, where $u(t;p_1,\varepsilon)$, $t\in I$, is defined by (14). Since $x^0(t_1)=f(t_1-1)$, by (18) and (19), the following equality holds:

$$\begin{aligned}&\Delta S(u^0(\cdot);p_1,\varepsilon) = \Phi(x^0(t_1)+\Delta x(t_1;p_1,\varepsilon)) - \Phi(x^0(t_1)) \\ &= \Phi(f(x^0(t_1-1),\hat{u},t_1-1) + \Delta_{x(t_1-1;p_1,\varepsilon)} f(x^0(t_1-1),\hat{u},t_1-1)) \\ &- \Phi(f(t_1-1)),\;\varepsilon\in\,]0,\tilde{\gamma}_1].\end{aligned} \tag{23}$$

From (23), considering (22) and (A1) and using the Taylor expansion at the point $f(x^0(t_1-1),\hat{u},t_1-1)$, we obtain

$$\begin{aligned}&\Delta S(u^0(\cdot);p_1,\varepsilon) = \Delta_{\hat{u}}\Phi(f(t_1-1)) + \\ &\Phi_x^T(f(x^0(t_1-1),\hat{u},t_1-1))\Delta_{x(t_1-1;p_1,\varepsilon)} f(x^0(t_1-1),\hat{u},t_1-1)) + o_1(\varepsilon),\end{aligned} \tag{24}$$

where

$$\Delta_{\hat{u}}\Phi(f(t_1-1)) := \Phi(f(x^0(t_1-1),\hat{u},t_1-1)) - \Phi(f(t_1-1)). \tag{25}$$

Following [15], we introduce the vector function $\mathring{\psi}(t;\hat{u})$, $t\in I$, as the solution of the linear discrete system

$$\begin{cases}\mathring{\psi}(t-1;\hat{u}) = f_x^T(t)\mathring{\psi}(t;\hat{u}), \quad t\in\{t_0+1,...,t_1-2\},\\ \mathring{\psi}(t_1-2;\hat{u}) = f_x^T\big(x^0(t_1-1),\hat{u},t_1-1\big)\mathring{\psi}(t_1-1;\hat{u}),\\ \mathring{\psi}(t_1-1;\hat{u}) = -\Phi_x\big(f\big(x^0(t_1-1),\hat{u},t_1-1\big)\big).\end{cases} \tag{26}$$

Let us continue the calculation of $\Delta S(\cdot)$ by considering (21) in (24). Then, by (26), the expansion (24) takes the form

$$
\begin{aligned}
\Delta S(u^0(\cdot);p_1,\varepsilon) &= \Delta_{\hat{u}}\Phi(f(t_1-1)) \\
&- \mathring{\psi}^T(t_1-2;\hat{u})\Delta x(t_1-1;p_1,\varepsilon) + o_{\Sigma}(\varepsilon), \varepsilon \in]0,\tilde{\gamma}_1].
\end{aligned}
\tag{27}
$$

Here and throughout the paper, we denote $o_{\Sigma}(\varepsilon)$ as a total remainder term.

Let us now calculate the second term in (27). Let $t \in \{\theta_1,...,t_1-2\}$. Then, from (18), taking into account (20) and applying Taylor's formula, we obtain

$$
\Delta x(t+1;p_1,\varepsilon) = f_x(t)\Delta x(t;p_1,\varepsilon) + o_2(\varepsilon;t), \quad t \in \{\theta_1,...,t_1-2\}. \tag{28}
$$

Consider (28) in the following identity:

$$
\begin{aligned}
&\mathring{\psi}^T(t_1-2;\hat{u})\Delta x(t_1-1;p_1,\varepsilon) = \mathring{\psi}^T(\theta;\hat{u})\Delta x(\theta_1;p_1,\varepsilon) \\
&+ \sum_{t=\theta_1}^{t_1-2}\left[\mathring{\psi}^T(t;\hat{u})\Delta x(t+1;p_1,\varepsilon) - \mathring{\psi}^T(t-1;\hat{u})\Delta x(t;p_1,\varepsilon)\right].
\end{aligned}
$$

Then, by (26), for $\mathring{\psi}^T(t_1-2;\hat{u})\Delta x(t_1-1;p_1,\varepsilon)$, we easily obtain the following representation:

$$
\mathring{\psi}^T(t_1-2;\hat{u})\Delta x(t_1-1;p_1,\varepsilon) = \mathring{\psi}^T(\theta;\hat{u})\Delta x(\theta_1;p_1,\varepsilon) + o_{\Sigma}(\varepsilon). \tag{29}
$$

Therefore, taking into account (29) in (27) and considering the equality $\Delta x(\theta_1;p_1,\varepsilon) = \varepsilon\Delta_{\tilde{v}}f(\theta)$ (see (19)) and the definition of the function $H(\cdot)$, for $\Delta S(u^0(\cdot);p_1,\varepsilon)$, we obtain the first-order increment formula of the form

$$
\Delta S(u^0(\cdot);p_1,\varepsilon) = \Delta_{\hat{u}}\Phi(f(t_1-1)) - \varepsilon\Delta_{\tilde{v}}H(\theta;\hat{u}) + o_{\Sigma}(\varepsilon), \ \varepsilon \in]0,\tilde{\gamma}_1]. \tag{30}
$$

where

$$
\Delta_{\tilde{v}}H(t;\hat{u}) = \mathring{\psi}^T(t;\hat{u})\Delta_{\tilde{v}}f(t). \tag{31}
$$

Case 1.2. Assumptions (A1), (B1) and (C2) hold true.

Let $p_2 = (\theta,\tilde{w},\hat{u}) \in I_{-1} \times W(\theta) \times U(t_1-1)$ be an arbitrary fixed vector parameter. Similar to (14), let us define a variation (with respect to the component w) of the admissible control $u^0(\cdot) = (v^0(\cdot),w^0(\cdot))^T$ as follows:

$$
u(t;p_2,\varepsilon) = \begin{cases} u^0(t), & t \in I \setminus \{\theta,t_1-1\}, \\ (v^0(\theta),w(\varepsilon))^T, & t = \theta, \\ \hat{u}, & t = t_1-1. \end{cases} \tag{32}
$$

Here, the vector function $w(\varepsilon) :]0, \tilde{\gamma}_2] \to W(\theta)$ is the solution of the following equation:

$$f(x^0(\theta), v^0(\theta), w(\varepsilon), \theta) - f(\theta) = \varepsilon \Delta_{\tilde{w}} f(\theta),\ \varepsilon \in]0, \tilde{\gamma}_2] \subset]0, 1],$$

where $\Delta_{\tilde{w}} f(\theta)$ is defined similarly to (16). Note that the existence of $w(\varepsilon) :]0, \tilde{\gamma}_2] \to W(\theta)$ follows from (C2) and Definition 2.1.

In this case, using (32) and applying step by step the scheme used to obtain the formula (30), for increment $S(u(\cdot; p_2, \varepsilon)) - S(u^0(\cdot)) =: \Delta S(u^0(\cdot); p_2, \varepsilon)$, we easily obtain

$$\Delta S(u^0(\cdot); p_2, \varepsilon) = \Delta_{\hat{u}} \Phi(f(t_1 - 1)) - \varepsilon \Delta_{\tilde{w}} H(\theta; \hat{u}) + o_{\Sigma}(\varepsilon),\ \varepsilon \in]0, \tilde{\gamma}_2], \tag{33}$$

where

$$\Delta_{\tilde{w}} H(t; \hat{u}) = \mathring{\psi}^T(t; \hat{u}) \Delta_{\tilde{w}} f(t). \tag{34}$$

Case 1.3 Assumptions (A1), (B1), (B2) and (C5) hold true.

Consider the special variation of the admissible control $u^0(\cdot) = (v^0(\cdot), w^0(\cdot))^T$ in the form

$$u(t; p_3, \varepsilon) = \begin{cases} u^0(t), & t \in I \setminus \{\theta, t_1 - 1\}, \\ (v^0(\theta), w(\varepsilon))^T, & t = \theta, \\ \hat{u}, & t = t_1 - 1. \end{cases} \tag{35}$$

Here, $p_3 := (\theta, \tilde{w}, \hat{u})$, where $\theta \in I_{-1}$, $\tilde{w} \in W(\theta)$ and $\hat{u} \in U(t_1 - 1)$ are arbitrary fixed points, $w(\varepsilon) = w^0(\theta) + \varepsilon(\tilde{w} - w^0(\theta)) \in W(\theta)$, $\varepsilon \in]0, \tilde{\gamma}_3] \subset]0, 1]$, where the existence of $\tilde{\gamma}_3 = \gamma(\tilde{w})$ follows from (C5) by considering Definition 2.1.

Consider an admissible process $(u(\cdot; p_3, \varepsilon), x(\cdot; p_3, \varepsilon))$. Similar to (18), considering (B2), for the increment $x(\cdot; p_3, \varepsilon) - x^0(\cdot) =: \Delta x(\cdot; p_3, \varepsilon)$, we can write

$$\Delta x(t+1; p_3, \varepsilon) = \begin{cases} 0, & t_0 - 1 \le t < \theta, \\ \varepsilon f_w(\theta)(\tilde{w} - w^0(\theta)) + o(\varepsilon), & t = \theta, \\ f(x(t; p_3, \varepsilon), u^0(t), t) - f(t), & \theta < t < t_1 - 1, \\ \Delta_{\hat{u}} f(t_1 - 1) & \\ \quad + \Delta_{x(t_1 - 1; p_3, \varepsilon)} f(x^0(t_1 - 1), \hat{u}, t_1 - 1), & t = t_1 - 1, \end{cases} \tag{36}$$

where $\varepsilon \in \left]0, \tilde{\gamma}_3\right]$; $\Delta_{\hat{u}} f(\cdot)$ and $\Delta_{x(t_1-1;p_3,\varepsilon)} f(\cdot)$ are defined similarly to (19). From (36), similar to (20) - (22), we obtain

$$\begin{aligned} &\|\Delta x(t;p_3,\varepsilon)\| \leq \tilde{K}(p_3)\varepsilon, t \in I, \varepsilon \in \left]0, \tilde{\gamma}_3\right], \tilde{K}(p_3) > 0, \\ &\Delta_{x(t_1-1;p_3,\varepsilon)} f(x^0(t_1-1), \hat{u}, t_1-1) = \\ &\qquad\qquad f_x(x^0(t_1-1), \hat{u}, t_1-1)\Delta x(t_1-1;p_3,\varepsilon) + o(\varepsilon), \\ &\|\Delta_{x(t_1-1;p_3,\varepsilon)} f(x^0(t_1-1), \hat{u}, t_1-1)\| \leq \hat{K}(p_3)\varepsilon, \hat{K}(p_3) > 0. \end{aligned} \tag{37}$$

Following the scheme used to obtain formula (30) and taking into account (25), (26), (35)-(37), (A1) and (B1), for increment $S(u(\cdot;p_3,\varepsilon)) - S(u^0(\cdot)) =: \Delta S(u^0(\cdot);p_3,\varepsilon)$, we have

$$\Delta S(u^0(\cdot);p_3,\varepsilon) = \Delta_{\hat{u}}\Phi(f(t_1-1)) - \mathring{\psi}^T(\theta;\hat{u})\Delta x(\theta+1;p_3,\varepsilon) + o_{\Sigma}(\varepsilon).$$

Therefore, considering $\Delta x(\theta+1;p_3,\varepsilon) = \varepsilon f_w(\theta)(\tilde{w} - w^0(\theta)) + o(\varepsilon)$ (see (36)) in the last equality, we obtain the following first-order increment formula:

$$\Delta S(u^0(\cdot);p_3,\varepsilon) = \Delta_{\hat{u}}\Phi(f(t_1-1)) - \varepsilon H_w^T(\theta;\hat{u})(\tilde{w} - w^0(\theta)) + o_{\Sigma}(\varepsilon),\ \varepsilon \in \left]0, \tilde{\gamma}_3\right]. \tag{38}$$

4.2. Second-Order Increment Formulas

We next consider the following various cases for obtaining the second-order increment formulas.

Case 2.1 Assumptions (A2), (B3), (C1) and (C3) hold true.

Define the vector parameter in the form $c_1 = (\alpha, \theta, \theta_1, \tilde{v}, \tilde{w}, \hat{u})$, where $\alpha \in \mathbb{R}_+ := \left]0, +\infty\right[$, $\theta \in \{t_0, t_0+1, ..., t_1-3\}$, $\theta_1 = \theta + 1$, $\tilde{v} \in V(\theta)$, $\tilde{w} \in W(\theta_1)$ and $\hat{u} \in U(t_1-1)$ are any fixed points. Consider an admissible process $(u(\cdot;c_1,\varepsilon), x(\cdot;c_1,\varepsilon))$ and the increment $x(\cdot;c_1,\varepsilon) - x^0(\cdot) =: \Delta x(\cdot;c_1,\varepsilon)$, where $u(\cdot;c_1,\varepsilon)$ and $\Delta x(\cdot;c_1,\varepsilon)$ are defined as follows:

$$u(t;c_1,\varepsilon) = \begin{cases} u^0(t), & t \in I \setminus \{\theta, \theta_1, t_1-1\}, \\ (v(\varepsilon), w^0(\theta))^T, & t = \theta, \\ (v^0(\theta_1), w(\varepsilon))^T, & t = \theta_1, \\ \hat{u}, & t = t_1 - 1, \end{cases} \tag{39}$$

$$\begin{cases} \Delta x(t+1;c_1,\varepsilon) = f(x^0(t)+\Delta x(t;c_1,\varepsilon), u(t;c_1,\varepsilon), t) - f(t), \\ \Delta x(t;c_1,\varepsilon) = 0, \quad t \in \{t_0, t_0+1, ..., \theta\}. \end{cases} \tag{40}$$

Here, the vector functions $v(\varepsilon) :]0,\gamma_1] \to V(\theta)$ and $w(\varepsilon) :]0,\gamma_1^*] \to W(\theta_1)$ are defined implicitly as follows:

(a) $v(\varepsilon) :]0,\gamma_1] \to V(\theta)$ is a solution of the following equation:

$$f(x^0(\theta), v(\varepsilon), w^0(\theta), \theta) - f(\theta) = \varepsilon\alpha\Delta_{\tilde{v}} f(\theta), \varepsilon \in]0,\gamma_1], \tag{41}$$

where $\Delta_{\tilde{v}} f(\theta)$ is defined by (16), and $\gamma_1 = (1+\alpha)^{-1}\gamma, \quad \gamma = \gamma(\tilde{v}) \in]0,1]$.

First, by assumption (C1) and Definition 2.1, the solution of equation (41) as a vector function $v(\cdot)$ exists; second, by considering (41) and equality $u(\theta;c_1,\varepsilon) = (v(\varepsilon), w^0(\theta))^T$ from (40), we have

$$\Delta x(\theta_1;c_1,\varepsilon) = \varepsilon\alpha\Delta_{\tilde{v}} f(\theta), \ \varepsilon \in]0,\gamma_1], \tag{42}$$

$$\|\Delta x(\theta_1;c_1,\varepsilon)\| \le K\varepsilon, \ \varepsilon \in]0,\gamma_1], \ K = K(c_1) > 0, \tag{43}$$

$$x^0(\theta_1) + \Delta x(\theta_1;c_1,\varepsilon) \in B_\delta(x^0(\theta_1)), \varepsilon \in]0,\gamma_1^*], \ \gamma_1^* = \min\{\gamma_1, \frac{\delta}{K+1}\}. \tag{44}$$

(b) $w(\varepsilon) :]0,\gamma_1^*] \to W(\theta_1)$ is a solution of the following equation:

$$\begin{aligned} & f\big(x(\theta_1;c_1,\varepsilon), v^0(\theta_1), w(\varepsilon), \theta_1\big) - f\big(x(\theta_1;c_1,\varepsilon), v^0(\theta_1), w^0(\theta_1), \theta_1\big) \\ & \qquad = \varepsilon\Delta_{\tilde{w}} f\big(x(\theta_1;c_1,\varepsilon), v^0(\theta_1), w^0(\theta_1), \theta_1\big), \ \varepsilon \in]0,\gamma_1^*], \end{aligned} \tag{45}$$

where $x(\theta_1;c_1,\varepsilon) = x^0(\theta_1) + \Delta x(\theta_1;c_1,\varepsilon)$ and

$$\begin{aligned} & \Delta_{\tilde{w}} f\big(x(\theta_1;c_1,\varepsilon), v^0(\theta_1), w^0(\theta_1), \theta_1\big) \\ & := f\big(x(\theta_1;c_1,\varepsilon), v^0(\theta_1), \tilde{w}, \theta_1\big) - f\big(x(\theta_1;c_1,\varepsilon), v^0(\theta_1), w^0(\theta_1), \theta_1\big). \end{aligned} \tag{46}$$

By (C3), (44), (46) and Definition 2.2, the solution of equation (45) as a vector function $w(\cdot)$ exists.

In this case, for $S(u(\cdot;c_1,\varepsilon)) - S(u^0(\cdot)) =: \Delta S(u^0(\cdot);c_1,\varepsilon)$, the following second-order increment formula holds:

$$\begin{aligned}
&\Delta S(u^0(\cdot);c_1,\varepsilon) = \Delta_{\hat{u}}\Phi(f(t_1-1)) - \varepsilon[\alpha\Delta_{\tilde{v}}H(\theta;\hat{u}) + \Delta_{\tilde{w}}H(\theta_1;\hat{u})] \\
&- \frac{\varepsilon^2}{2}[\alpha^2\mathring{M}((\theta,\tilde{v});\hat{u}) + 2\alpha\mathring{N}((\theta_1,\tilde{w});\hat{u})\Delta_{\tilde{v}}f(\theta) + \mathring{M}((\theta_1,\tilde{w});\hat{u})] \\
&+ o_{\Sigma}(\varepsilon^2), \varepsilon \in \left]0,\gamma_1^*\right].
\end{aligned} \tag{47}$$

Here, $\Delta_{\hat{u}}\Phi(f(t_1-1))$, $\Delta_{\tilde{v}}H(\theta;\hat{u})$ and $\Delta_{\tilde{w}}H(\theta_1;\hat{u})$ are defined by (25), (31) and (34), respectively, and

$$\mathring{M}((\tau,p);\hat{u}) := \Delta_p f^T(\tau)\mathring{\Psi}(\tau;\hat{u})\Delta_p f(\tau), \ (\tau,p) \in \{(\theta,\tilde{v}),(\theta_1,\tilde{w})\}, \tag{48}$$

$$\mathring{N}((\theta_1,\tilde{w});\hat{u}) := \Delta_{\tilde{w}}H_x^T(\theta_1;\hat{u}) + \Delta_{\tilde{w}}f^T(\theta_1)\mathring{\Psi}(\theta_1;\hat{u})f_x(\theta_1), \tag{49}$$

$$\Delta_{\tilde{w}}H_x(\theta_1;\hat{u}) := \mathring{\psi}^T(\theta_1;\hat{u})[f_x(x^0(\theta_1),v^0(\theta_1),\tilde{w},\theta_1) - f_x(\theta_1)], \tag{50}$$

where the matrix function $\mathring{\Psi}(t;\hat{u}),\ t \in I$, is defined as the solution of the linear discrete system [15]

$$\begin{cases}
\mathring{\Psi}(t-1;\hat{u}) = f_x^T(t)\mathring{\Psi}(t;\hat{u})f_x(t) + H_{xx}(t;\hat{u}), \quad t \in \{t_0+1,...,t_1-2\}, \\
\mathring{\Psi}(t_1-2;\hat{u}) = f_x^T\big(x^0(t_1-1),\hat{u},t_1-1\big)\mathring{\Psi}(t_1-1;\hat{u}) \\
\times f_x\big(x^0(t_1-1),\hat{u},t_1-1\big) + H_{xx}(\mathring{\psi}(t_1-1;\hat{u}),x^0(t_1-1),\hat{u},t_1-1), \\
\mathring{\Psi}(t_1-1;\hat{u}) = -\Phi_{xx}\big(f(x^0(t_1-1),\hat{u},t_1-1)\big).
\end{cases} \tag{51}$$

It should be noted that $\mathring{\psi}(t;\hat{u}),\ t \in I$, defined by (26) and $\mathring{\Psi}(t;\hat{u}), t \in I$, correspond to the admissible control $\hat{u}(t), t \in I$, where $\hat{u}(t_1-1) = \hat{u} \in U(t_1-1)$, and $\hat{u}(t) = u^0(t) = (v^0(t),w^0(t))^T, t \in I_{-1}$.

The proof of (47) is presented in Appendix A.

Case 2.2 Assumptions (A2), (B3), (B4), (C4) and (C5) hold true.

Again, we start with a vector parameter $c_2 = (\alpha,\theta,\tilde{v},\tilde{w},\hat{u})$, where $\alpha \in \mathbb{R}_+$, $\theta \in I_{-1}$, $\tilde{v} \in V(\theta)$, $\tilde{w} \in W(\theta)$ and $\hat{u} \in U(t_1-1)$ are arbitrary fixed points. Consider also the variations of the admissible control $u^0(\cdot) = (v^0(\cdot), w^0(\cdot))^T$ of the form

$$u(t;c_2,\varepsilon) = \begin{cases} u^0(t), & t \in I \setminus \{\theta, t_1 - 1\}, \\ (v(\varepsilon), w(\varepsilon))^T, & t = \theta, \\ \hat{u}, & t = t_1 - 1. \end{cases} \tag{52}$$

Here,

$$w(\varepsilon) = w^0(\theta) + \varepsilon\alpha(\tilde{w} - w^0(\theta)) \in B_\delta(w^0(\theta)) \cap W(\theta), \ \varepsilon \in]0, \gamma_2], \tag{53}$$

where $\gamma_2 := \min\{(1+\alpha)^{-1}\gamma(\tilde{w}), (1+\alpha)^{-1}(1+\|\tilde{w} - w^0(\theta)\|)^{-1}\delta\}$ (the scalar $\gamma(\tilde{w}) \in]0,1]$ exists by (C5) and Definition 2.1) and it is clear that $\gamma_2 \in]0,1]$; the vector function $v(\varepsilon) :]0,\gamma_2] \to V(\theta)$ is a solution of the following equation:

$$\begin{aligned} &f(x^0(\theta), v(\varepsilon), w(\varepsilon), \theta) - f(x^0(\theta), v^0(\theta), w(\varepsilon), \theta) = \\ &= \varepsilon[f(x^0(\theta), \tilde{v}, w(\varepsilon), \theta) - f(x^0(\theta), v^0(\theta), w(\varepsilon), \theta)], \varepsilon \in]0, \gamma_2]. \end{aligned} \tag{54}$$

Note that the existence of $w(\varepsilon) :]0, \gamma_2] \to W(\theta)$ follows from (C5) and Definition 2.1, and the existence of $v(\varepsilon) :]0, \gamma_2] \to V(\theta)$ follows from (C4) by considering (53) and Definition 2.2. Obviously, for every $\varepsilon \in]0, \gamma_2]$, the function $u(\cdot; c_2, \varepsilon)$ is an admissible control.

In this case, for $S(u(\cdot;c_2,\varepsilon)) - S(u^0(\cdot)) =: \Delta S(u^0(\cdot);c_2,\varepsilon)$, the following second-order increment formula holds:

$$\begin{aligned} \Delta S(u^0(\cdot);c_2,\varepsilon) = \\ &\Delta_{\hat{u}}\Phi(x^0(t_1 - 1)) - \varepsilon[\Delta_{\tilde{v}}H(\theta;\hat{u}) + \alpha H_w^T(\theta;\hat{u})(\tilde{w} - w^0(\theta))] \\ &- \frac{\varepsilon^2}{2}[\mathring{M}((\theta,\tilde{v});\hat{u}) + 2\alpha\mathring{\Omega}((\theta,\tilde{v});\hat{u})(\tilde{w} - w^0(\theta)) \\ &+ \alpha^2(\tilde{w} - w^0(\theta))^T\mathring{G}(\theta;\hat{u})(\tilde{w} - w^0(\theta))] + o_\Sigma(\varepsilon^2), \ \varepsilon \in]0, \gamma_2]. \end{aligned} \tag{55}$$

Here, $\mathring{M}((\theta,\tilde{v});\hat{u})$ is defined by (48), and

$$\mathring{\Omega}((\theta,\tilde{v});\hat{u}) := \Delta_{\tilde{v}}H_w^T(\theta;\hat{u}) + \Delta_{\tilde{v}}f^T(\theta)\mathring{\Psi}(\theta;\hat{u})f_w(\theta), \tag{56}$$

$$\mathring{G}(\theta;\hat{u}) := f_w^T(\theta)\mathring{\Psi}(\theta;\hat{u})f_w(\theta) + H_{ww}(\theta;\hat{u}), \tag{57}$$

where $\Delta_{\tilde{v}} H_w^T(\theta;\hat{u})$ is analogously defined by (50).

The proof of (55) is presented in Appendix B.

Case 2.3 Assumptions (A2), (B3), (B4), (C1) and (C5) hold true.

Consider the variations of the admissible control $u^0(\cdot) = (v^0(\cdot), w^0(\cdot))^T$ of the form

$$u(t;c_3,\varepsilon) = \begin{cases} u^0(t), & t \in I \setminus \{\theta, \theta_1, t_1-1\}, \\ (v(\varepsilon), w^0(\theta))^T, & t = \theta, \\ (v^0(\theta_1), w(\varepsilon))^T, & t = \theta_1, \\ \hat{u}, & t = t_1 - 1. \end{cases} \tag{58}$$

Here,

(a) $c_3 = (\alpha, \theta, \theta_1, \tilde{v}, \tilde{w}, \hat{u})$, where $\alpha \in \mathbb{R}_+$, $\theta \in I_{-1} \setminus \{t_1-2\}$, $\tilde{v} \in V(\theta)$, $\tilde{w} \in W(\theta_1)$ and $\hat{u} \in U(t_1-1)$ are arbitrary fixed points;

(b) the vector function $v(\varepsilon) :]0, \gamma(\tilde{v})] \to V(\theta)$ is a solution of the equation

$$f(x^0(\theta), v(\varepsilon), w^0(\theta), \theta) - f(\theta) = \varepsilon \Delta_{\tilde{v}} f(\theta),\ \varepsilon \in \left]0, \gamma(\tilde{v})\right], \tag{59}$$

where $\Delta_{\tilde{v}} f(\theta)$ is defined by (16), and the existence of $v(\cdot)$ follows from (C1) by considering Definition 2.1; and

(c) the vector function $w(\varepsilon) :]0, \overline{\gamma}_3] \to W(\theta_1)$ is defined as

$$w(\varepsilon) = w^0(\theta_1) + \alpha\varepsilon(\tilde{w} - w^0(\theta_1)), \varepsilon \in \left]0, \overline{\gamma}_3\right], \tag{60}$$

where $\overline{\gamma}_3 := (1+\alpha)^{-1}\gamma(\tilde{w})$, the existence of $w(\cdot)$ and $\gamma(\tilde{w}) \in]0,1]$ follows from (C5) and Definition 2.1.

For every vector parameter c_3 and for all $\varepsilon \in]0, \gamma_3]$, where $\gamma_3 = \min\{\gamma(\tilde{v}), \overline{\gamma}_3\}$, the function $u(t;c_3,\varepsilon)$, $t \in I$ is an admissible control.

In this case, for $S(u(\cdot;c_3,\varepsilon)) - S(u^0(\cdot)) =: \Delta S(u^0(\cdot);c_3,\varepsilon)$, the following second-order increment formula holds:

$$
\begin{aligned}
\Delta S(u^0(\cdot); c_3, \varepsilon) = \Delta_{\hat{u}} \Phi(f(t_1 - 1)) - \\
\varepsilon[\Delta_{\tilde{v}} H(\theta; \hat{u}) + \alpha H_w^T(\theta_1; \hat{u})(\tilde{w} - w^0(\theta_1))] \\
- \frac{\varepsilon^2}{2}[\mathring{M}((\theta, \tilde{v}); \hat{u}) + 2\alpha \Delta_{\tilde{v}} f^T(\theta) \mathring{P}(\theta_1; \hat{u})(\tilde{w} - w^0(\theta_1)) \\
+ \alpha^2 (\tilde{w} - w^0(\theta_1))^T \mathring{G}(\theta_1; \hat{u})(\tilde{w} - w^0(\theta_1))] + o_\Sigma(\varepsilon^2), \ \varepsilon \in\ ]0, \gamma_3],
\end{aligned}
\tag{61}
$$

where $\mathring{M}(\cdot)$ and $\mathring{G}(\cdot)$ are defined by (48) and (57), respectively, and

$$
\mathring{P}(\theta_1; \hat{u}) := H_{xw}(\theta_1; \hat{u}) + f_x^T(\theta_1) \mathring{\Phi}(\theta_1; \hat{u}) f_w(\theta_1). \tag{62}
$$

The proof of (61) is given in Appendix C.

5. Proofs of Theorems

Recall that

$$
U_0(t_1 - 1) := \{\hat{u} : \hat{u} \in U(t_1 - 1), \ \ \Delta_{\hat{u}} \Phi(f(t_1 - 1)) = 0\}, \tag{63}
$$

where $\Delta_{\hat{u}} \Phi(f(t_1 - 1))$ is defined by (25).

Here, we present proofs of Theorems 3.1 - 3.5.

Proof. of Theorem 3.1 By the conditions of the theorem, the increment formulas (30) and (33) are valid. Then, along an optimal process $(u^0(\cdot), x^0(\cdot))$, for every $p_1 = (\theta, \tilde{v}, \hat{u})$ and $p_2 = (\theta, \tilde{w}, \hat{u})$ and for all $\varepsilon \in\ ]0, \min\{\tilde{\gamma}_1, \tilde{\gamma}_2\}[$, the following inequalities hold

$$
\Delta S(u^0(\cdot); p_1, \varepsilon) = \Delta_{\hat{u}} \Phi(t_1 - 1) - \varepsilon[\Delta_{\tilde{v}} H(\theta; \hat{u}) + \varepsilon^{-1} o_\Sigma(\varepsilon)] \geq 0, \tag{64}
$$

$$
\Delta S(u^0(\cdot); p_2, \varepsilon) = \Delta_{\hat{u}} \Phi(t_1 - 1) - \varepsilon[\Delta_{\tilde{w}} H(\theta; \hat{u}) + \varepsilon^{-1} o_\Sigma(\varepsilon)] \geq 0, \tag{65}
$$

where $\theta \in I_{-1}$, $\tilde{v} \in V(\theta)$, $\tilde{w} \in W(\theta)$ and $\hat{u} \in U(t_1 - 1)$. The inequality (4) follows from (64) and the arbitrariness of $\varepsilon \in\ ]0, \min\{\tilde{\gamma}_1, \tilde{\gamma}_2\}[$. Furthermore, the inequality (5) follows from (64) considering (63), the arbitrariness of $\varepsilon \in\ ]0, \min\{\tilde{\gamma}_1, \tilde{\gamma}_2\}[$ and the definition of $o_\Sigma(\varepsilon)$. Similarly, we obtain the proof of the inequality (6) from (65). $\square$

Proof. of Theorem 3.2 Since (A1), (B1) and (C1) hold, formula (30) is valid. Then, along the optimal process $(u^0(\cdot), x^0(\cdot))$, inequality (64) holds. Thus, as in the proof of Theorem 3.1, we obtain the validity of inequalities (4) and (5).

Moreover, since (A1), (B1), (B2) and (C5) hold, formula (38) can be used. Then, for every $p_3 = (\theta, \tilde{w}, \hat{u}) \in I_{-1} \times W(\theta) \times U(t_1 - 1)$ and for all $\varepsilon \in]0, \tilde{\gamma}_3]$, the increment (38) is nonnegative along the optimal process $(u^0(\cdot), x^0(\cdot))$. Therefore, considering (63), the arbitrariness of $\varepsilon \in]0, \tilde{\gamma}_3]$ and the definition of $o_{\sum}(\varepsilon)$, we obtain the validity of inequality (7). □

Proof. of Theorem 3.3 By the conditions of the theorem, for every vector parameter $c_1 = (\alpha, \theta, \theta_1, \tilde{v}, \tilde{w}, \hat{u})$ and for all $\varepsilon \in]0, \gamma_1^*]$, formula (47) is valid. Then, considering (63) and Definition 3.1, for $\hat{u} \in U_0(t_1 - 1)$ and for all $\tilde{v} \in V_0(\theta)$, $\tilde{w} \in W_0(\theta_1)$, $\alpha \in \mathbb{R}_+$ and $\varepsilon \in]0, \gamma_1^*]$, formula (47) takes the form

$$\begin{aligned}\Delta S(u^0(\cdot); c_1, \varepsilon) = -\frac{\varepsilon^2}{2}\big[\alpha^2 \mathring{M}\big((\theta, \tilde{v}); \hat{u}\big)\\ + 2\alpha \mathring{N}\big((\theta_1, \tilde{w}); \hat{u}\big)\Delta_{\tilde{v}} f(\theta) + \mathring{M}\big((\theta_1, \tilde{w}); \hat{u}\big) + \varepsilon^{-2} o_{\sum}(\varepsilon^2)\big].\end{aligned} \tag{66}$$

Thus, since along the optimal control $u^0(\cdot)$, the increment $\Delta S(u^0(\cdot); c_1, \varepsilon)$ is nonnegative, taking into account the arbitrariness of $\varepsilon \in]0, \gamma_1^*]$ and the definition of $o_{\sum}(\varepsilon^2)$, we easily obtain the validity of (10) from (66). □

Proof. of Theorem 3.4 By the conditions of this theorem, for every $c_2 = (\alpha, \theta, \tilde{v}, \tilde{w}, \hat{u})$ and for all $\varepsilon \in]0, \gamma_2]$, formula (55) holds. Then, by Definition 3.1 and (9), taking into account (63), for $\hat{u} \in U_0(t_1 - 1)$ and for all $\tilde{v} \in V_0(\theta)$, $\tilde{w} \in W_0(\theta)$, $\alpha \in \mathbb{R}_+$ and $\varepsilon \in]0, \gamma_2]$, formula (55) takes the form

$$\begin{aligned}\Delta S(u^0(\cdot); c_2, \varepsilon) = -\frac{\varepsilon^2}{2}[\mathring{M}((\theta, \tilde{v}); \hat{u}) + 2\alpha \mathring{\Omega}((\theta, \tilde{v}); \hat{u})(\tilde{w} - w^0(\theta))\\ + \alpha^2 (\tilde{w} - w^0(\theta))^T \mathring{G}(\theta; \hat{u})(\tilde{w} - w^0(\theta)) + \varepsilon^{-2} o_{\sum}(\varepsilon^2)].\end{aligned} \tag{67}$$

Hence, since along the optimal control $u^0(\cdot)$, the increment $\Delta S(u^0(\cdot); c_2, \varepsilon)$ is nonnegative, considering the arbitrariness of $\varepsilon \in]0, \gamma_2]$ and the definition of $o_{\sum}(\varepsilon^2)$, we easily obtain the validity of (11) from (67). □

Proof. of Theorem 3.5 Since assumptions (A2), (B3), (B4), (C1) and (C5) are satisfied, for every $\theta \in I_{-1} \setminus \{t_1 - 2\}$, $\tilde{v} \in V(\theta)$, $\tilde{w} \in W(\theta_1)$ and $\hat{u} \in U(t_1 - 1)$ and for all $\varepsilon \in]0,\gamma_3]$, formula (61) holds. Then, by Definitions 3.1 and 3.2, taking into account (63), for $\hat{u} \in U_0(t_1 - 1)$ and for all $\tilde{v} \in V_0(\theta)$, $\tilde{w} \in V_0(\theta_1)$, $\alpha \in \mathbb{R}_+$ and $\varepsilon \in]0,\gamma_3]$, formula (61) takes the form

$$\begin{aligned}\Delta S(u^0(\cdot);c_3,\varepsilon) = -\frac{\varepsilon^2}{2}\big[&\mathring{M}((\theta,\tilde{v});\hat{u}) + 2\alpha\Delta_{\tilde{v}} f^T(\theta)\mathring{P}(\theta_1;\hat{u})(\tilde{w} - w^0(\theta_1)) \\ &+ \alpha^2(\tilde{w} - w^0(\theta_1))^T\mathring{G}(\theta_1;\hat{u})(\tilde{w} - w^0(\theta_1)) + \varepsilon^{-2}o_{\Sigma}(\varepsilon^2)\big].\end{aligned} \tag{68}$$

Thus, since along the optimal control $u^0(\cdot)$, the increment $\Delta S(u^0(\cdot);c_3,\varepsilon)$ is nonnegative, considering the arbitrariness of $\varepsilon \in]0,\gamma_3]$ and the definition of $o_{\Sigma}(\varepsilon^2)$, we easily obtain the validity of (12) from (68). □

6. Perspectives and Open Problems

In this section, we provide a short discussion regarding the prospects that are open to the researchers of optimal control problems when using the new approach to optimality conditions introduced in this paper.

First, to demonstrate the application of studying DOCPs with respect to the components of vector control, we have considered a simple discrete optimal problem. However, we believe that our approach may be applied to more complicated discrete optimal control problems, such as the problems with terminal equality and inequality constraints, problems with a delay, and infinite horizon discrete time optimal control problems. Future research may examine whether our approach can indeed be applied to such optimal control problems.

Second, in this study, we have obtained optimality conditions with respect to the components of vector control in the form of a global maximum principle by using assumptions (C1), (C2), and (C5). However, these assumptions may not hold for some DOCPs. In this case, first- and second-order necessary optimality conditions with respect to components can be obtained in the form of a local maximum principle.

Finally, we use assumption (C3) to prove Theorem 3.3. As noted in section 3, it is not easy to determine whether (C3) holds in the application of Theorem 3.3. Therefore, it is interesting to investigate whether assumption (C3) is essential for the validity of Theorem 3.3.

7. Conclusions

In this paper, we have established more constructive first- and second-order necessary optimality conditions under lightened convexity assumptions. These results are obtained by introducing a new approach that weakens such assumptions. This approach studies optimal control problems with respect to the components of vector control, and it is more characteristic for discrete rather than the continuous optimal control problems.

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Appendices

AppendixA. Proof of Formula (47)

Consider an admissible process $(u(\cdot;c_1,\varepsilon),x(\cdot;c_1,\varepsilon))$, where an admissible control $u(\cdot;c_1,\varepsilon)$ is defined by (39), (41) and (45). Then, taking into account (39), (42), (45) and the inequality $\gamma_1^* \le \gamma_1$, we can write the system (40) as follows:

$$\Delta x(t+1;c_1,\varepsilon) = \begin{cases} 0, & t_0-1 \le t < \theta, \\ \varepsilon\alpha\Delta_{\tilde{v}} f(\theta), & t=\theta, \\ \varepsilon\Delta_{\tilde{w}} f\big(x(\theta_1;c_1,\varepsilon),v^0(\theta_1),w^0(\theta_1),\theta_1\big) & \\ \qquad +\Delta_{x(\theta_1;c_1,\varepsilon)} f(\theta_1), & t=\theta_1, \\ f(x(t;c_1,\varepsilon),u^0(t),t)-f(t) & \theta_1 < t < t_1-1, \\ \Delta_{\hat{u}} f(t_1-1)+\Delta_{x(t_1-1;c_1,\varepsilon)} f(x^0(t_1-1),\hat{u},t_1-1), & t=t_1-1. \end{cases} \tag{A.1}$$

Here, $\varepsilon \in]0, \gamma_1^*]$, $x(t; c_1, \varepsilon) =: x^0(t) + \Delta x(t; c_1, \varepsilon)$, $\Delta_{\tilde{v}} f(\theta)$ is defined by (16), $\Delta_{\hat{u}} f(\cdot)$ and $\Delta_{x(t_1-1;c_1,\varepsilon)} f(\cdot)$ are analogously defined by (19), and $\Delta_{\tilde{w}} f(\cdot)$ is defined by (46).

From (A.1), similar to (20)-(22), taking into account assumption (B3) and (43), we obtain

$$\|\Delta x(t; c_1, \varepsilon)\| \leq K^*(c_1)\varepsilon,\ t \in I,\ \varepsilon \in]0, \gamma_1^*],\ K^*(c_1) \geq K(c_1). \tag{A.2}$$

$$\begin{aligned}\Delta_{x(t_1-1;c_1,\varepsilon)} f(x^0(t_1-1), \hat{u}, t_1-1) = \\ f_x(x^0(t_1-1), \hat{u}, t_1-1)\Delta x(t_1-1; c_1, \varepsilon) + o(\varepsilon),\end{aligned} \tag{A.3}$$

$$\|\Delta_{x(t_1-1;c_1,\varepsilon)} f(x^0(t_1-1), \hat{u}, t_1-1)\| \leq \hat{K}(c_1)\varepsilon,\ \varepsilon \in]0, \gamma_1^*],\ \hat{K}(c_1) > 0. \tag{A.4}$$

Now, let us calculate the increment $\Delta S(u^0(\cdot); c_1, \varepsilon)$. Similar to (23), we can write

$$\begin{aligned}\Delta S(u^0(\cdot); c_1, \varepsilon) =& \Phi(f(x^0(t_1-1), \hat{u}, t_1-1) + \Delta_{x(t_1-1;c_1,\varepsilon)} f(x^0(t_1-1), \hat{u}, t_1-1)) \\ &- \Phi(f(t_1-1)).\end{aligned} \tag{A.5}$$

From (A.5), considering (A.4) and assumption (A2) and using the Taylor expansion at the point $f(x^0(t_1-1), \hat{u}, t_1-1)$, we obtain

$$\Delta S(u^0(\cdot); c_1, \varepsilon) = \Delta_{\hat{u}} \Phi(f(t_1-1)) + \Delta^{(1)} S(u^0(\cdot); c_1, \varepsilon) + \frac{1}{2}\Delta^{(2)} S(u^0(\cdot); c_1, \varepsilon) + o(\varepsilon^2), \tag{A.6}$$

where $\Delta_{\hat{u}} \Phi(f(t_1-1))$ is defined by (25), and

$$\Delta^{(1)} S(u^0(\cdot); c_1, \varepsilon) := \Phi_x^T\big(f(x^0(t_1-1), \hat{u}, t_1-1)\big)\Delta_{x(t_1-1;c_1,\varepsilon)} f(x^0(t_1-1), \hat{u}, t_1-1), \tag{A.7}$$

$$\begin{aligned}\Delta^{(2)} S(u^0(\cdot); c_1, \varepsilon) := \Delta_{x(t_1-1;c_1,\varepsilon)} f^T(x^0(t_1-1), \hat{u}, t_1-1)\times \\ \times \Phi_{xx}\big(f(x^0(t_1-1), \hat{u}, t_1-1)\big)\Delta_{x(t_1-1;c_1,\varepsilon)} f(x^0(t_1-1), \hat{u}, t_1-1).\end{aligned} \tag{A.8}$$

By (26) and the definition of the function $H(\cdot)$, the formula (A.7) takes the form

$$\Delta^{(1)}S(u^0(\cdot);c_1,\varepsilon) = -\Delta_{x(t_1-1;c_1,\varepsilon)}H(\mathring{\psi}(t_1-1;\hat{u}),x^0(t_1-1),\hat{u},t_1-1).$$

From the last equality, according to (A.2), assumption (B3), and Taylor's formula, we have the following representation for $\Delta^{(1)}S(u^0(\cdot);c_1,\varepsilon)$:

$$\begin{aligned}\Delta^{(1)}S(u^0(\cdot);c_1,\varepsilon) &= -H_x^T(\mathring{\psi}(t_1-1;\hat{u}),x^0(t_1-1),\hat{u},t_1-1)\Delta x(t_1-1;c_1,\varepsilon)\\ &-\frac{1}{2}\Delta x^T(t_1-1;c_1,\varepsilon)H_{xx}(\mathring{\psi}(t_1-1;\hat{u}),x^0(t_1-1),\hat{u},t_1-1)\Delta x(t_1-1;c_1,\varepsilon)+o_1(\varepsilon^2).\end{aligned} \tag{A.9}$$

Furthermore, substitute (A.3) into (A.8). Then, by (51), we have

$$\begin{aligned}\Delta^{(2)}S(u^0(\cdot);c_1,\varepsilon) &= -\Delta x^T(t_1-1;c_1,\varepsilon)f_x^T(x^0(t_1-1),\hat{u},t_1-1)\mathring{\Psi}(t_1-1;\hat{u})\times\\ &\times f_x(x^0(t_1-1),\hat{u},t_1-1)\Delta x(t_1-1;c_1,\varepsilon)+o_2(\varepsilon^2),\ \varepsilon\in\,]0,\gamma_1^*].\end{aligned} \tag{A.10}$$

Substituting (A.9) and (A.10) into (A.6) and taking into account (26) and (51), we obtain

$$\begin{aligned}\Delta S(u^0(\cdot);c_1,\varepsilon) &= \Delta_{\hat{u}}\Phi(f(t_1-1))-\mathring{\psi}^T(t_1-2;\hat{u})\Delta x(t_1-1;c_1,\varepsilon)\\ -\frac{1}{2}\Delta x^T(t_1-1;c_1,\varepsilon)&\mathring{\Psi}(t_1-2;\hat{u})\Delta x(t_1-1;c_1,\varepsilon)+o_\Sigma(\varepsilon^2),\varepsilon\in\,]0,\gamma_1^*].\end{aligned} \tag{A.11}$$

Let us now calculate the second term in (A.11). According to the definition of $H(\cdot)$ and (A.1), we have the followings:

$$\begin{aligned}&\mathring{\psi}^T(\theta_1;\hat{u})\Delta x(\theta_2;c_1,\varepsilon) = \varepsilon\Delta_{\tilde{w}}H(\mathring{\psi}(\theta_1;\hat{u}),x(\theta_1,c_1,\varepsilon),v^0(\theta_1),w^0(\theta_1),\theta_1)\\ &+\Delta_{x(\theta_1;c_1,\varepsilon)}H(\theta_1;\hat{u}),\quad \mathring{\psi}^T(\theta;\hat{u})\Delta x(\theta_1;c_1,\varepsilon)=\varepsilon\alpha\Delta_{\tilde{v}}H(\theta;\hat{u}),\\ &\mathring{\psi}^T(t;\hat{u})\Delta x(t+1;c_1,\varepsilon)=\Delta_{x(t;c_1,\varepsilon)}H(t;\hat{u}),\ t\in\{\theta_2,\dots t_1-2\}.\end{aligned} \tag{A.12}$$

We, first, consider (A.12) in the following identity

$$\begin{aligned}\mathring{\psi}^T(t_1-2;\hat{u})\Delta x(t_1-1;c_1,\varepsilon) &= \mathring{\psi}^T(\theta;\hat{u})\Delta x(\theta_1;c_1,\varepsilon)+\mathring{\psi}^T(\theta_1;\hat{u})\Delta x(\theta_2;c_1,\varepsilon)\\ &+\sum_{t=\theta_2}^{t_1-2}\mathring{\psi}^T(t;\hat{u})\Delta x(t+1;c_1,\varepsilon)-\sum_{t=\theta_1}^{t_1-2}\mathring{\psi}^T(t-1;\hat{u})\Delta x(t;c_1,\varepsilon).\end{aligned}$$

Then, taking into account (26), (50), assumption (B3) and the Taylor formula, we easily obtain

$$\mathring{\psi}^T(t_1-2;\hat u)\Delta x(t_1-1;c_1,\varepsilon)=\varepsilon[\alpha\Delta_{\tilde v}H(\theta;\hat u)+\Delta_{\tilde w}H(\theta_1;\hat u)]$$
$$+\varepsilon^2\alpha\Delta_{\tilde w}H_x^T(\theta_1;\hat u)\Delta_{\tilde v}f(\theta)+\frac{1}{2}\sum_{t=\theta_1}^{t_1-2}\Delta x^T(t;c_1,\varepsilon)H_{xx}(t;\hat u)\Delta x(t;c_1,\varepsilon)+o_{\Sigma}(\varepsilon^2). \tag{A.13}$$

Next, we calculate the third term in (A.11). From (A.1), taking into account (A.2) and Taylor formula, we have the following decompositions:

$$\Delta x(\theta_1;c_1,\varepsilon)=\varepsilon\alpha\Delta_{\tilde v}f(\theta),\quad \Delta x(\theta_2;c_1,\varepsilon)=\varepsilon\Delta_{\tilde w}f(\theta_1)+f_x(\theta_1)\Delta x(\theta_1;c_1,\varepsilon)+o_{\Sigma}(\varepsilon;\theta_1),$$
$$\Delta x(t+1;c_1,\varepsilon)=f_x(t)\Delta x(t;c_1,\varepsilon)+o(\varepsilon;t),\quad t\in\{\theta_2,...,t_1-2\}. \tag{A.14}$$

Let us consider (A.14) in the following identity:

$$\Delta x^T(t_1-1;c_1,\varepsilon)\mathring{\Psi}(t_1-2;\hat u)\Delta x(t_1-1;c_1,\varepsilon)=\Delta x^T(\theta_1;c_1,\varepsilon)\mathring{\Psi}(\theta;\hat u)\Delta x(\theta_1;c_1,\varepsilon)$$
$$+\Delta x^T(\theta_2;c_1,\varepsilon)\mathring{\Psi}(\theta_1;\hat u)\Delta x(\theta_2;c_1,\varepsilon)+\sum_{t=\theta_2}^{t_1-2}\Delta x^T(t+1;c_1,\varepsilon)\mathring{\Psi}(t;\hat u)\Delta x(t+1;c_1,\varepsilon)$$
$$-\sum_{t=\theta_1}^{t_1-2}\Delta x^T(t;c_1,\varepsilon)\mathring{\Psi}(t-1;\hat u)\Delta x(t;c_1,\varepsilon),\ \ \varepsilon\in\,]0,\gamma_1^*].$$

Then, by (51), we obtain

$$\Delta x^T(t_1-1;c_1,\varepsilon)\mathring{\Psi}(t_1-2;\hat u)\Delta x(t_1-1;c_1,\varepsilon)=\varepsilon^2[\alpha^2\Delta_{\tilde v}f^T(\theta)\mathring{\Psi}(\theta;\hat u)\Delta_{\tilde v}f(\theta)$$
$$+2\alpha\Delta_{\tilde w}f^T(\theta_1)\mathring{\Psi}(\theta_1;\hat u)f_x(\theta_1)\Delta_{\tilde v}f(\theta)+\Delta_{\tilde w}f^T(\theta_1)\mathring{\Psi}(\theta_1;\hat u)\Delta_{\tilde w}f(\theta_1)]$$
$$-\sum_{t=\theta_1}^{t_1-2}\Delta x^T(t;c_1,\varepsilon)H_{xx}(t;\hat u)\Delta x(t;c_1,\varepsilon)+o_{\Sigma}(\varepsilon^2). \tag{A.15}$$

As a result, let us consider (A.13) and (A.15) in (A.11). Then, taking into account (48) and (49), for $\Delta S(u^0(\cdot);c_1,\varepsilon)$, we obtain formula (47).

□

AppendixB. Proof of Formula (55)

Consider an admissible process $(u(\cdot;c_2,\varepsilon),x(\cdot;c_2,\varepsilon))$, where $u(\cdot;c_2,\varepsilon)$ is defined by (52)-(54). Then, similar to (A.1), by (52)-(54) and considering (19), for the increment $x(\cdot;c_2,\varepsilon)-x^0(\cdot)=:\Delta x(\cdot;c_2,\varepsilon)$, $\varepsilon\in]0,\gamma_2]$, we can write

$$\Delta x(t+1;c_2,\varepsilon)=\begin{cases}0, & t_0-1\le t<\theta,\\ \varepsilon[f(x^0(\theta),\tilde{v},w(\varepsilon),\theta) & \\ -f(x^0(\theta),v^0(\theta),w(\varepsilon),\theta)]+\Delta_{w(\varepsilon)}f(\theta), & t=\theta,\\ f(x(t;c_2,\varepsilon),u^0(t),t)-f(t) & \theta<t<t_1-1,\\ \Delta_{\hat{u}}f(t_1-1)+\Delta_{x(t_1-1;c_2,\varepsilon)}f(x^0(t_1-1),\hat{u},t_1-1), & t=t_1-1.\end{cases} \tag{B.1}$$

From (B.1), similar to (20)-(22), taking into account assumptions (B3), (B4) and applying Taylor's formula, we obtain the followings:

$$\|\Delta x(t;c_2,\varepsilon)\|\sim O(\varepsilon),\quad (t,\varepsilon)\in I\times]0,\gamma_2],$$

$$\Delta_{x(t_1-1;c_2,\varepsilon)}f(x^0(t_1-1),\hat{u},t_1-1)=$$
$$f_x(x^0(t_1-1),\hat{u},t_1-1)\Delta x(t_1-1;c_2,\varepsilon)+o(\varepsilon),$$
$$\|\Delta_{x(t_1-1;c_2,\varepsilon)}f(x^0(t_1-1),\hat{u},t_1-1)\|\sim O(\varepsilon),\quad \varepsilon\in\,]0,\gamma_2]\,.$$

These will be used to obtain formulas below.

Applying an approach similar to the scheme used to obtain (A.11), by (25), (26), (51), (52), (B.1) and assumptions (A2) and (B3), for $\Delta S(u^0(\cdot);c_2,\varepsilon)$, we obtain a decomposition in the form

$$\begin{aligned}&\Delta S(u^0(\cdot);c_2,\varepsilon)=\Delta_{\hat{u}}\Phi(f(t_1-1))-\mathring{\psi}^T(t_1-2;\hat{u})\Delta x(t_1-1;c_2,\varepsilon)\\&-\frac{1}{2}\Delta x^T(t_1-1;c_2,\varepsilon)\mathring{\Psi}(t_1-2;\hat{u})\Delta x(t_1-1;c_2,\varepsilon)+o_{\Sigma}(\varepsilon^2),\varepsilon\in\,]0,\gamma_2]\,.\end{aligned} \tag{B.2}$$

Let us now, similar to (A.13), calculate the second term in (B.2). Using the definition of the function $H(\cdot)$ and the identity

$$\begin{aligned}\mathring{\psi}^T(t_1-2;\hat{u})\Delta x(t_1-1;c_2,\varepsilon)=&\mathring{\psi}^T(\theta;\hat{u})\Delta x(\theta_1;c_2,\varepsilon)+\sum_{t=\theta_1}^{t_1-2}\mathring{\psi}^T(t;\hat{u})\Delta x(t+1;c_2,\varepsilon)\\&-\sum_{t=\theta_1}^{t_1-2}\mathring{\psi}^T(t-1;\hat{u})\Delta x(t;c_2,\varepsilon),\end{aligned}$$

considering (26), (52), (53), (B.1) and assumptions (B3) and (B4) and applying Taylor's formula, we obtain

$$
\begin{aligned}
&\mathring{\psi}^T(t_1-2;\hat u)\Delta x(t_1-1;c_2,\varepsilon) = \varepsilon\Delta_{\tilde v}H(\theta;\hat u) + \varepsilon\alpha H_w^T(\theta;\hat u)(\tilde w - w^0(\theta)) + \\
&\varepsilon^2\alpha\Delta_{\tilde v}H_w^T(\theta;\hat u)(\tilde w - w^0(\theta)) + \frac{\varepsilon^2\alpha^2}{2}(\tilde w - w^0(\theta))^T H_{ww}(\theta;\hat u)(\tilde w - w^0(\theta)) \\
&+\frac{1}{2}\sum_{t=\theta_1}^{t_1-2}\Delta x^T(t;c_2,\varepsilon)H_{xx}(t;\hat u)\Delta x(t;c_2,\varepsilon) + o_{\sum}(\varepsilon^2).
\end{aligned}
\tag{B.3}
$$

We next, similar to (A.15), calculate the third term in (B.2) using the identity

$$
\begin{aligned}
&\Delta x^T(t_1-1;c_2,\varepsilon)\mathring{\Psi}(t_1-2;\hat u)\Delta x(t_1-1;c_2,\varepsilon) = \Delta x^T(\theta_1;c_2,\varepsilon)\mathring{\Psi}(\theta;\hat u)\Delta x(\theta_1;c_2,\varepsilon) \\
&+\sum_{t=\theta_1}^{t_1-2}\Delta x^T(t+1;c_2,\varepsilon)\mathring{\Psi}(t;\hat u)\Delta x(t+1;c_2,\varepsilon) - \sum_{t=\theta_1}^{t_1-2}\Delta x^T(t;c_2,\varepsilon)\mathring{\Psi}(t-1,\hat u)\Delta x(t;c_2,\hat u)
\end{aligned}
$$

and considering (51), (53), (B.1) and assumptions (B3) and (B4). As a result, we have the following decomposition:

$$
\begin{aligned}
&\Delta x^T(t_1-1;c_2,\varepsilon)\mathring{\Psi}(t_1-2;\hat u)\Delta x(t_1-1;c_2,\varepsilon) = \\
&\quad = \varepsilon^2[\Delta_{\tilde v}f^T(\theta)\mathring{\Psi}(\theta;\hat u)\Delta_{\tilde v}f(\theta) + 2\alpha\Delta_{\tilde v}f^T(\theta)\mathring{\Psi}(\theta;\hat u)f_w(\theta)(\tilde w - w^0(\theta)) \\
&\quad + \alpha^2(\tilde w - w^0(\theta))^T f_w^T(\theta)\mathring{\Psi}(\theta;\hat u)f_w(\theta)(\tilde w - w^0(\theta))] \\
&\quad - \sum_{t=\theta_1}^{t_1-2}\Delta x^T(t;c_2,\varepsilon)H_{xx}(t;\hat u)\Delta x(t;c_2,\varepsilon) + o_{\sum}(\varepsilon^2),\ \varepsilon\in\,]0,\gamma_2].
\end{aligned}
\tag{B.4}
$$

Then, substituting (B.3) and (B.4) into (B.2) and considering (48), (56) and (57), we obtain formula (55).

□

AppendixC. Proof of Formula (61)

Consider an admissible process $(u(\cdot;c_3,\varepsilon),x(\cdot;c_3,\varepsilon))$, where $u(\cdot;c_3,\varepsilon)$ is defined by (58)-(60). Then, similar to (A.1), by (58) and (59) and considering (16) and (19), for the increment $x(\cdot;c_3,\varepsilon)-x^0(\cdot) =: \Delta x(\cdot;c_3,\varepsilon)$, $\varepsilon\in]0,\gamma_3]$, the following equality is valid:

$$\Delta x(t+1; c_3, \varepsilon) = \begin{cases} 0, & t_0 - 1 \le t < \theta, \\ \varepsilon \Delta_{\tilde{v}} f(\theta), & t = \theta, \\ f(x^0(\theta_1) + \Delta x(\theta_1; c_3, \varepsilon), v^0(\theta_1), w(\varepsilon), \theta_1) - f(\theta_1), & t = \theta_1, \\ f(x(t; c_3, \varepsilon), u^0(t), t) - f(t), & \theta_1 < t < t_1 - 1, \\ \Delta_{\hat{u}} f(t_1 - 1) + \Delta_{x(t_1-1; c_3, \varepsilon)} f(x^0(t_1 - 1), \hat{u}, t_1 - 1), & t = t_1 - 1. \end{cases} \tag{C.1}$$

Using (C.1), let us step by step apply an approach similar to the scheme used to obtain formula (A.11). Then, taking into account (25), (26), (51) and assumptions (A2) and (B3), for $\Delta S(u^0(\cdot); c_3, \varepsilon)$, we obtain

$$\begin{gathered} \Delta S(u^0(\cdot); c_3, \varepsilon) = \Delta_{\hat{u}} \Phi(f(t_1 - 1)) - \mathring{\psi}^T(t_1 - 2; \hat{u}) \Delta x(t_1 - 1; c_3, \varepsilon) \\ -\frac{1}{2} \Delta x^T(t_1 - 1; c_3, \varepsilon) \mathring{\Psi}(t_1 - 2; \hat{u}) \Delta x(t_1 - 1; c_3, \varepsilon) + o_{\Sigma}(\varepsilon^2),\ \varepsilon \in\ ]0, \gamma_3]\,. \end{gathered} \tag{C.2}$$

Let us now calculate the second and third terms in (C.2). First, similar to (A.13), considering (16), (26), (3[illegible]), (C.1) and assumptions (B3) and (B4) and applying Taylor's formula, we easily obtain

$$\begin{aligned} &\mathring{\psi}^T(t_1 - 2; \hat{u}) \Delta x(t_1 - 1; c_3, \varepsilon) = \varepsilon [\Delta_{\tilde{v}} H(\theta; \hat{u}) + \alpha H_w^T(\theta_1; \hat{u})(\tilde{w} - w^0(\theta_1))] \\ &+ \frac{\varepsilon^2}{2} [\alpha^2 (\tilde{w} - w^0(\theta_1))^T H_{ww}(\theta_1; \hat{u}) + 2\alpha \Delta_{\tilde{v}} f^T(\theta) H_{xw}(\theta_1; \hat{u})](\tilde{w} - w^0(\theta_1)) \\ &\quad + \frac{1}{2} \sum_{t=\theta_1}^{t_1-2} \Delta x^T(t; c_3, \varepsilon) H_{xx}(t; \hat{u}) \Delta x(t; c_3, \varepsilon) + o_{\Sigma}(\varepsilon^2),\ \varepsilon \in\ ]0, \gamma_3]\,. \end{aligned} \tag{C.3}$$

Furthermore, similar to (A.15), considering (16), (51), (C.1) and assumptions (B3) and (B4), we obtain

$$\begin{aligned}\Delta x^T(t_1-1;c_3,\varepsilon)\mathring{\Psi}(t_1-2;\hat{u})\Delta x(t_1-1;c_3,\varepsilon) &= \varepsilon^2[\Delta_{\tilde{v}} f^T(\theta)\mathring{\Psi}(\theta;\hat{u})\Delta_{\tilde{v}} f(\theta)\\ &+ 2\alpha\Delta_{\tilde{v}} f^T(\theta) f_x^T(\theta_1)\mathring{\Psi}(\theta_1;\hat{u}) f_w(\theta_1)(\tilde{w}-w^0(\theta_1))\\ &+ \alpha^2(\tilde{w}-w^0(\theta_1))^T f_w^T(\theta_1)\mathring{\Psi}(\theta_1;\hat{u}) f_w(\theta_1)(\tilde{w}-w^0(\theta_1))]\\ &- \sum_{t=\theta_1}^{t_1-2}\Delta x^T(t;c_3,\varepsilon)H_{xx}(t;\hat{u})\Delta x(t;c_3,\varepsilon) + o_{\sum}(\varepsilon^2),\ \ \varepsilon\in\,]0,\gamma_3]\,.\end{aligned} \tag{C.4}$$

Thus, we substitute (C.3) and (C.4) into (C.2). Then, considering (48), (57) and (62), for $\Delta S(u^0(\cdot);c_3,\varepsilon)$, we obtain formula (61).

□

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