

# EPIMORPHISMS, DOMINIONS AND $\mathcal{H}$ -COMMUTATIVE SEMIGROUPS

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**Abstract** In the present paper, a series of results and examples that explore the structural features of  $\mathcal{H}$ -commutative semigroups are provided. We also generalize a result of Isbell from commutative semigroups to  $\mathcal{H}$ -commutative semigroups by showing that the dominion of an  $\mathcal{H}$ -commutative semigroup is  $\mathcal{H}$ -commutative. We then use this to generalize Howie and Isbell's result that any  $\mathcal{H}$ -commutative semigroup satisfying the minimum condition on principal ideals is saturated.

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## 1. Introduction and Preliminaries

In this article, we are concerned with a series of results and examples that explore the class of semigroups  $S$  for which Green's relation  $\mathcal{H}$  is commutative:  $ab\mathcal{H}ba$  for all  $a, b$  in  $S$ . This definition of  $\mathcal{H}$ -commutativity was introduced by Tully in [13]. In ([11], Theorem 5.1), Nagy proposed a second definition of  $\mathcal{H}$ -commutativity: [Def. 1.1]. He then proved that the two characterizations coincide.

**Definition 1.1** ([11, Chapter V]) A semigroup  $S$  is called  $\mathcal{H}$ -commutative if for all  $a, b \in S$ , there exists  $x \in S^1$  such that  $ab = bxa$ .

Since such equations are always solvable in any group, we see at once that the collection  $HC$  of all  $\mathcal{H}$ -commutative semigroups represents an umbrella class for the classes of Groups and Commutative semigroups.

**Result 1.2** ([11, Theorem 5.2, Chapter V]) A semigroup  $S$  is  $\mathcal{H}$ -commutative if and only if Green's equivalence  $\mathcal{H}$  on  $S$  is a commutative congruence on  $S$ .

**Result 1.3** ([11, Theorem 5.3, Chapter V]) Every  $\mathcal{H}$ -commutative semigroup is decomposable into a semilattice of archimedean semigroups.

We now introduce dominions of semigroups. Dominions of permutative semigroups were studied in [8] by Khan and Shah. We restate here the presentation given in [8], adapted to the present context. Let  $S$  be any semigroup with a subsemigroup  $U$ . An element  $d \in S$  is said to be *dominated* by  $U$  if for every semigroup  $T$  and for all homomorphisms  $\alpha, \beta : S \rightarrow T$ ,  $u\alpha = u\beta$  for all  $u \in U$  implies that  $d\beta = d\alpha$ . The set of all elements of  $S$  dominated by  $U$  is called the *dominion* of  $U$  in  $S$  and will be denoted by  $\text{Dom}(U, S)$ . It may be easily checked that  $\text{Dom}(U, S)$  is a subsemigroup of  $S$  containing  $U$ . Any subsemigroup  $U$  of a semigroup  $S$  is said to be *closed* in  $S$  if  $\text{Dom}(U, S) = U$  and *absolutely closed* if it is closed in every containing semigroup  $S$ . Further a semigroup  $U$  is said to be *saturated* if  $\text{Dom}(U, S) \neq S$  for every properly containing semigroup  $S$  and *epimorphically embedded or dense* in  $S$  if  $\text{Dom}(U, S) = S$ .

A (semigroup) morphism  $\alpha : S \rightarrow T$  is said to be an *epimorphism* (*epi* for short) if for all morphisms  $\beta, \gamma$  with domain  $T$ ,  $\alpha\beta = \alpha\gamma$  implies  $\beta = \gamma$  (where the composition of morphisms is written from left to right). One may easily check that a morphism  $\alpha : S \rightarrow T$  is epi if and only if  $i : S\alpha \rightarrow T$  is epi and the inclusion map  $i : U \rightarrow S$  is epi if and only if  $\text{Dom}(U, S) = S$ . Every onto morphism is easily seen to be an epimorphism, but the converse is not true in general.

Semigroup dominions have been characterized by Isbell's zigzag theorem, which is as follows.

**Result 1.4** ([6, Theorem 2.3] or [4, Theorem VIII. 8.3.5]) Let  $U$  be a subsemigroup of a semigroup  $S$ . Then  $d \in \text{Dom}(U, S)$  if and only if  $d \in U$  or there exists a series of

factorizations of  $d$  as follows:

$d = a_0 t_1 = y_1 a_1 t_1 = y_1 a_2 t_2 = y_2 a_3 t_2 = \cdots = y_m a_{2m-1} t_m = y_m a_{2m}$ , where  $m \geq 1$ ,  $a_i \in U$ ,  $y_i, t_i \in S \setminus U$  and

$$\begin{aligned} a_0 &= y_1 a_1, & a_{2m-1} t_m &= a_{2m}; \\ a_{2i-1} t_i &= a_{2i} t_{i+1}, & y_i a_{2i} &= y_{i+1} a_{2i+1} \quad (1 \leq i \leq m-1). \end{aligned} \quad (1)$$

Such a series of factorizations is called a *zigzag* in  $S$  over  $U$  with value  $d$ , length  $m$  and spine  $u_0, u_1, \dots, u_{2m}$ .

A semigroup  $S$  is said to be *permutative* if it satisfies a permutation identity

$$x_1 x_2 \cdots x_n = x_{i_1} x_{i_2} \cdots x_{i_n} \quad (n \geq 2)$$

for some non-trivial permutation  $i$  of the set  $\{1, 2, \dots, n\}$ . Further  $S$  is called *semicommutative* if  $i_1 \neq 1$  and  $i_n \neq n$  and *left (resp. right) semicommutative* if  $i_1 \neq 1$  (resp.  $i_n \neq n$ ). Permutative semigroups are not saturated in general because commutative semigroups are not saturated. The infinite monogenic semigroup  $\langle a \rangle$  generated by  $a$  is not saturated since it is epimorphically embedded in the infinite cyclic group generated by  $a$  [4, Chapter VII Exercise 2(i)]. In [5], Howie and Isbell showed that commutative semigroups satisfying the minimum condition on principal ideals were saturated. In [7] Khan extended this result to semicommutative semigroups and in [8], Khan and Shah extended the theorem to right semicommutative semigroups (see [8, Theorem 2.1]).

The class of  $\mathcal{H}$ -commutative semigroups had been studied by several authors in one way or the other (see [9], [10], [11], [12] and [13] for example). The class of  $\mathcal{H}$ -commutative semigroups was first investigated by Tully [13]. He studied  $\mathcal{H}$ -commutative semigroups with the additional property that each congruence is uniquely determined by its kernel relative to a given element  $g$  and claimed that  $\mathcal{H}$ -commutative semigroups were precisely those semigroups with Green's relation  $\mathcal{H}$  a commutative congruence. Nagy [11] described archimedean properties of  $\mathcal{H}$ -commutative semigroups and showed that every  $\mathcal{H}$ -commutative semigroup is a semilattice of archimedean semigroups. Strecker [12], then, studied  $\mathcal{H}$ -commutative semigroups whose lattice of congruences was a chain. He also proved that a semigroup was an  $\mathcal{H}$ -commutative archimedean semigroup with an idempotent if and only if it was the ideal extension of a group by a commutative nilsemigroup. In [9], Mary studied semigroups whose set of completely regular elements was an  $\mathcal{H}$ -commutative set. Mary also gave equivalent characterizations of the condition that  $ab\mathcal{H}ba$  element-wise (for given  $a, b \in S$ ) without

assuming the whole semigroup  $S$  to be  $\mathcal{H}$ -commutative [10, Theorem 2.4] and of  $\mathcal{H}$ -commutative regular semigroups [10, Theorem 2.7]. For most of the principal results on  $\mathcal{H}$ -commutative semigroups, readers are referred to Nagy's book [11].

Our results naturally fall into three parts, which are presented as Sections 2, 3 and 4 respectively. In Section 2, we show that the structural features of commutative semigroups are mirrored in the class  $HC$  in that, for any  $S \in HC$ , all five Green's relations are equal and correspond to the mutual divisibility of elements. Moreover  $S$  is a semilattice of archimedean components. As a consequence of this, it may be easily deduced that the regular members of  $HC$  comprise the class of semilattices of groups. We also show that the  $HC$  condition on  $S$  is equivalent to the requirement that  $\mathcal{H}$  is a congruence on  $S$  and  $S/\mathcal{H}$  is commutative. Section 3 is devoted to the class  $HM$  of  $\mathcal{H}$ -commutative monoids and provides some examples and remarks showing the distinction between the classes  $HC$  and  $HM$ .

In the last section, we prove that the dominion of a  $\mathcal{H}$ -commutative semigroup is  $\mathcal{H}$ -commutative; this generalizes Isbell's result, from commutative semigroups to  $\mathcal{H}$ -commutative semigroups (see [6, Corollary 2.5]). Finally we show that any  $\mathcal{H}$ -commutative semigroup satisfying the minimum condition on principal right ideals is saturated which extends Howie and Isbell's result (see [5, Theorem 3.1]) from commutative semigroups to  $\mathcal{H}$ -commutative semigroups.

Throughout the remainder of the paper  $S$  will denote an  $\mathcal{H}$ -commutative semigroup unless otherwise indicated. Background material and facts on semigroups that are assumed in what follows can be found in Clifford and Preston [2], Higgins [3] and Howie [4] and will be used throughout without explicit mention. For a comprehensive survey on the topic of semilattice decompositions of semigroups, there is the text [1].

## 2. Structure of $\mathcal{H}$ -commutative semigroups

First we examine the general character of  $\mathcal{H}$ -commutative semigroups.

**Proposition 2.1.** The idempotents of any  $\mathcal{H}$ -commutative semigroup  $S$  are central.

**Proof.** Let  $a \in S$  and  $e \in E(S)$ . Then, for some  $x \in S^1$ , we have  $ea = axe$ , whence  $ea = axe = axe = ea$ . Similarly, for some  $y \in S^1$ , we have  $ae = eya$ , whence  $ae = eya = e^2ya = eya = ae$ . Therefore  $ea = eae = ae$ . Hence each idempotent commutes with every member of  $S$ . ■

**Proposition 2.2.** Let  $S$  be an  $\mathcal{H}$ -commutative semigroup. Then  $aS = Sa$  ( $\forall a \in S$ ).

**Proof.** Take any  $a \in S$ . Then  $ab \in aS$  ( $b \in S$ ). As  $S$  is  $\mathcal{H}$ -commutative, for some  $x \in S^1$ , we have  $ab = bxa \in Sa$ . Thus  $aS \subseteq Sa$ . By symmetry we also have the reverse inclusion  $Sa \subseteq aS$  whence it follows that  $aS = Sa$  for all  $a \in S$ , as required. ■

**Theorem 2.3.** All five Green's relations coincide on an  $\mathcal{H}$ -commutative semigroup  $S$ .

**Proof.** For any  $a \in S$ , by Proposition 2.2, we have  $aS = Sa$  whence it follows that  $aS^1 = S^1a$  for all  $a \in S$ . Hence, for any  $a, b \in S$ ,

$$a\mathcal{L}b \Leftrightarrow (aS^1 = bS^1) \Leftrightarrow (S^1a = S^1b) \Leftrightarrow a\mathcal{R}b.$$

So we infer that  $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D}$  in  $S$ . What is more we have  $S^1aS^1 = (S^1)^2a = S^1a$ , whence it follows that  $\mathcal{L} = \mathcal{J}$  also and, therefore, all five of Green's relations coincide on  $S$ . ■

**Remark 2.4.** In writing  $\mathcal{H}$ , therefore, we have a symbol that may denote any one of the five Green's relations on  $S$ , noting that  $a\mathcal{H}b$  if and only if each of  $a$  and  $b$  are mutually *divisible*, meaning that each is a factor of the other. In this context there is no need to distinguish between left factors, right factors, or interior factors.

Since  $\mathcal{L}$  is a right congruence and  $\mathcal{R}$  is a left congruence in any semigroup  $S$ , it follows that in a  $\mathcal{H}$ -commutative semigroup,  $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D}$  is a congruence.

It is the case that if we take any surjective homomorphism  $\phi : S \rightarrow T$  from an  $\mathcal{H}$ -commutative semigroup  $S$ , then  $T$  is also  $\mathcal{H}$ -commutative since for any  $a\phi, b\phi \in T$

$(a, b \in S)$  either  $ab = ba$ , in which case  $a\phi b\phi = (ab)\phi = (ba)\phi = b\phi a\phi$ , or there exists an  $x \in S$  such that  $ab = bxa$ , in which case  $a\phi b\phi = (ab)\phi = (bxa)\phi = b\phi x\phi a\phi$ .

We generalize this result in our last section to epimorphisms of  $\mathcal{H}$ -commutative semigroups where we show that dominion of any  $\mathcal{H}$ -commutative semigroup is  $\mathcal{H}$ -commutative i.e. If  $U$  is any  $\mathcal{H}$ -commutative subsemigroup of a semigroup  $S$ , then  $\text{Dom}(U, S)$  is also  $\mathcal{H}$ -commutative.

**Theorem 2.5.**

- (a) ([11, Theorem 5.2, Chapter V]) A semigroup  $S$  is  $\mathcal{H}$ -commutative if and only if  $\mathcal{H}$  is a congruence and  $S/\mathcal{H}$  is commutative.
- (b) If  $S$  is  $\mathcal{H}$ -commutative, then  $S/\mathcal{H}$  is the greatest *combinatorial* (meaning  $\mathcal{H}$ -trivial) image of  $S$ .

**Proof** (a) By Theorem 2.3 and Remark 2.4, given that  $S$  is  $\mathcal{H}$ -commutative, then  $\mathcal{H}$  is a congruence on  $S$ . Moreover, for any  $a, b \in S$ , there exist  $x, y \in S^1$  such that  $ab = bxa = xyba$ , thus showing that  $ba|ab$ ; by symmetry we have  $ab|ba$  likewise so that  $ab\mathcal{H}ba$ , whence in  $S/\mathcal{H}$ , we have  $H_aH_b = H_{ab} = H_{ba} = H_bH_a$ , so that  $S/\mathcal{H}$  is commutative.

Conversely suppose that  $\mathcal{H}$  is a congruence on  $S$  and that  $S/\mathcal{H}$  is commutative. Then, for any  $a, b \in S$ , we have in  $S/\mathcal{H}$  that  $H_{ab} = H_aH_b = H_bH_a = H_{ba}$ , and  $ab\mathcal{H}ba$ .

(b) Suppose that  $H_a\mathcal{H}H_b$  in the quotient semigroup  $T = S/\mathcal{H}$ . Then, since  $T$  is also  $\mathcal{H}$ -commutative, we have  $H_a|H_b$  in  $T$  so that, for some  $c \in S$ , we have  $H_b = H_aH_c = H_{ac}$ . Hence  $a|ac|b$  so that  $a|b$  in  $S$  and, by symmetry,  $b|a$  also so that  $a\mathcal{H}b$  in  $S$ , which is to say that  $H_a = H_b$ . Therefore  $\mathcal{H}$  is trivial in  $T$ . Hence  $T = S/\mathcal{H}$  is combinatorial, as required.

Conversely let  $\mu$  be any congruence on  $S$  such that  $S/\mu$  is combinatorial. Take any  $a, b \in S$  such that  $a\mathcal{H}b$ . Then, since homomorphisms preserve Green's relations, we have  $a\mu\mathcal{H}b\mu$  in  $S/\mu$ . Since  $S/\mu$  is combinatorial, it then follows that  $a\mu = b\mu$ , whence we infer that  $\mathcal{H} \subseteq \mu$ . Therefore  $\mathcal{H}$  is the least combinatorial congruence on  $S$  (which is equivalent to saying that  $S/\mathcal{H}$  is the maximum combinatorial image of  $S$ ), as required. ■

**Theorem 2.6.**

(a) For a semigroup  $S$ , the following are equivalent:

- (i)  $S$  satisfies the equations  $\forall a, b \exists x, y \in S : (a = axa) \wedge (ab = bya)$ .
- (ii)  $S$  is  $\mathcal{H}$ -commutative and regular.
- (iii)  $S$  is  $\mathcal{H}$ -commutative and  $S/\mathcal{H}$  is regular.
- (iv)  $S$  is a semilattice of groups.
- (v)  $\mathcal{H} = \eta$ , where  $\eta$  is the least semilattice congruence on  $S$ .

(b) If  $S$  is an  $\mathcal{H}$ -commutative semigroup, then,  $\text{Reg}(S)$ , the set of all regular elements of  $S$ , if non-empty, is an  $\mathcal{H}$ -commutative subsemigroup of  $S$  which is itself a semilattice of groups.

**Proof** (a) (i)  $\Rightarrow$  (ii). The first equation ensures that  $S$  is regular, for then  $axa \in V(a)$ , while the second ensures that  $S$  is  $\mathcal{H}$ -commutative.

(ii)  $\Rightarrow$  (iii). By Theorem 2.5,  $\mathcal{H}$  is a congruence and, so,  $S/\mathcal{H}$  is also regular.

(iii)  $\Rightarrow$  (iv). By Theorem 2.5,  $S/\mathcal{H}$  is commutative and combinatorial and, since  $S/\mathcal{H}$  is also regular,  $S/\mathcal{H}$  consists entirely of idempotents and so  $S/\mathcal{H}$  is a semilattice. Again, for each  $a \in S$ , we have  $a^2 \in H_a$ . Thus  $H_a$  is a group and  $S$  is, therefore, a semilattice of groups.

(iv)  $\Rightarrow$  (i). Since  $S$  is regular, any  $x \in V(a)$  is a solution to the first equation. Take any  $a, b \in S$ , whence we may write  $H = H_{ab} = H_{ba} = H_e$ , where  $e$  is the identity element of the group  $\mathcal{H}$ . Hence  $be, ea \in H$ . Put  $y = (be)^{-1}ab(ea)^{-1}$ , where inversion is in the group  $H$ , noting also that  $y \in H$ . Then

$$bya = b(eye)a = (be)y(ea) = (be)(be)^{-1}ab(ea)^{-1}ea = e(ab)e = ab;$$

thereby proving that  $S$  is an  $\mathcal{H}$ -commutative semigroup.

(iv)  $\Rightarrow$  (v). In any semigroup, we have  $\mathcal{J} \subseteq \eta$ , so that  $\mathcal{H} \subseteq \eta$  is always true. (Indeed, since  $S$  is regular, we have  $\eta = \mathcal{J}^*$ , the least congruence containing  $\mathcal{J}$ ). Conversely, since each  $\eta$ -class  $a\eta$  is a group, it follows that  $a\eta \subseteq H_a$ , a group  $\mathcal{H}$ -class, so that  $\eta \subseteq \mathcal{H}$  and we conclude that  $\mathcal{H} = \eta$ .

(v)  $\Rightarrow$ (iv). Since  $\mathcal{H} = \eta$ , it follows that, for any  $\mathcal{H}$ -class  $H$  of  $S$  and  $a \in H$ , we have  $a^2 \in H$ , whence  $H$  is a group. Since  $\mathcal{H} = \eta$ , we have that each  $\eta$ -class is a group, and so  $S$  is a semilattice of groups.

(b) By Proposition 2.1, idempotents commute with each other whence it follows that  $\text{Reg}(S)$  is a subsemigroup as for any  $a' \in V(a)$ ,  $b' \in V(b)$  we have  $b'a' \in V(ab)$ . Again by Proposition 2.1 it now follows that  $\text{Reg}(S)$  is a semilattice of groups, whence from (a) it follows that  $\text{Reg}(S)$  is an  $\mathcal{H}$ -commutative subsemigroup of  $S$ .  $\blacksquare$

**Definition 2.7.** A semigroup  $S$  is called *archimedean* if for each  $a, b \in S$ , there exists  $n \geq 1$  such that  $a|b^n$ , both as a left divisor and a right divisor.

**Remark 2.8.** There is no loss of generality in taking the same value of  $n$  for the left and right divisors, for suppose that  $b^n = ax$  and  $b^m = ya$  ( $x, y \in S^1$ ). Then  $b^{mn} = ax(ax)^{m-1} = (ya)^{n-1}ya$ , so that  $a$  is both a left and right divisor of a common power of  $b$ .

**Lemma 2.9.** Let  $S$  be an  $\mathcal{H}$ -commutative semigroup with  $a, b, a_1, a_2, b_1, b_2 \in S$  and  $n \in \mathbb{Z}^+$ . Then

- (a) the relation of divisibility is compatible with multiplication, meaning that if  $a_1|b_1$  and  $a_2|b_2$ , then  $a_1a_2|b_1b_2$ .
- (b) if  $a|b$ , then  $a^n|b^n$ ;
- (c) if  $a\mathcal{H}b$ , then  $a^n\mathcal{H}b^n$ ;
- (d)  $(bc)^n\mathcal{H}b^nc^n$ ;
- (e)  $a|b^nc^n$  if and only if  $a|(bc)^n$ .

**Proof.** (a) Since  $a_i|b_i$  ( $i = 1, 2$ ), we may write for some  $c_i \in S^1$  that  $b_i = c_ia_i$ . Then we have, for some  $x \in S^1$ , that

$$b_1b_2 = c_1a_1c_2a_2 = (c_1c_2x)(a_1a_2)$$

so that  $a_1a_2|b_1b_2$ , as required.

- (b) Follows by induction on  $n$  upon taking  $a_1 = a_2 = a$  and  $b_1 = b_2 = b$ .
- (c) Follows as, by Theorem 2.5,  $\mathcal{H}$  is a congruence on  $S$ .



(d) Follows as, by Theorem 2.5,  $S/\mathcal{H}$  is a commutative semigroup.

(e) This follows from (d) and the transitivity of the relation of divisibility. ■

Consider the *archimedean division relation*  $\lambda$  on  $S$  whereby  $a\lambda b$  if  $a|b^n$  for some  $n \geq 1$ . Clearly  $\lambda$  is reflexive. To see that  $\lambda$  is transitive, suppose further that  $b|c^m$  for some  $m \geq 1$ . Then, by Lemma 2.9(b), we have  $a|b^n|c^{mn}$ , so that  $a\lambda c$ , showing that  $\lambda$  is transitive. Hence  $\lambda$  is a quasi-order on  $S$ , which then induces an equivalence relation  $\rho$  on  $S$  defined by  $a\rho b$  if and only if  $a\lambda b$  and  $b\lambda a$ . Indeed  $\rho$  is a congruence on  $S$ ; for suppose that  $a\rho b$  so that  $b^n = ay$  say, and take any  $c \in S^1$ . Then, for some  $x, y \in S^1$ , we have  $b^nc^n = aycc^n = aycc^{n-1} = acxyc^{n-1}$  so that  $ac|b^nc^n$  whence, by Lemma 2.9(e), we infer that  $ac|(bc)^n$ . Exchanging the roles of  $a$  and  $b$  in the argument gives that  $bc$  divides some power of  $ac$  and so  $ac\rho bc$ . Hence  $\rho$  is a right congruence and by the left-right symmetry of the relation of division, it follows that  $\rho$  is also a left congruence and, therefore,  $\rho$  is a congruence on  $S$ .

Recall that for any relation  $R \subseteq S \times S$ ,  $R^*$  denotes the least congruence on  $S$  that contains the relation  $R$ . A particular case of this is that  $\eta = \eta_0^*$ , where  $\eta_0 = \{(a, a^2), (ab, ba) : a, b \in S\}$ , is the least semilattice congruence on any semigroup  $S$ . Our discussion has led to the following result, which directly generalizes the well-known theorem for commutative semigroups [2, p136].

**Theorem 2.10.** Let  $S$  be an  $\mathcal{H}$ -commutative semigroup. Then

- (a) the relation  $\rho$  on  $S$  defined by  $a\rho b$  if and only if each of  $a$  and  $b$  divides a power of the other is the least semilattice congruence  $\eta$  on  $S$ ;
- (b) each subsemigroup  $a\eta$  ( $a \in S$ ) of  $S$  is archimedean;
- (c)  $a\eta$  is a union of  $\mathcal{H}$ -classes of  $S$  and  $a\eta$  contains at most one idempotent.

**Proof** (a) From the fact that  $a|a^2$  and  $a^2|a^2$ , we may conclude that  $a\rho a^2$  for all  $a \in S$ . Next  $ab = bxa = xyba$  for some  $x, y \in S^1$  so that  $ab|ba$ . By symmetry  $ba|ab$  and, so,  $ab\rho ba$  for all  $a, b \in S$ . Since  $\eta$  is the least congruence containing all pairs of the form  $(a, a^2)$  and  $(ab, ba)$ , it follows that  $\eta \subseteq \rho$  and  $\rho$  is itself a semilattice congruence on  $S$ .

Conversely, suppose that  $a\rho b$  so that  $a|b^k$  and  $b|a^n$  say. Then, for some  $x, y \in S^1$ ,

we have  $a^n = bx$  and  $b^k = ya$ , which yields:

$$a \eta a^n = bx \eta b^2 x = ba^n \eta ba \eta b^k a = ya^2 \eta ya = b^k \eta b$$

whence it follows that  $\rho \subseteq \eta$ . Therefore  $\rho = \eta$  as claimed.

(b) Since  $S/\eta$  is a band (indeed a semilattice), each class  $a\eta$  is a subsemigroup of  $S$ . Take any  $a, b \in S$  such that  $a\eta b$  and  $x \in S^1$  such that  $ab = bxa$ .

We have for some  $t \in S^1$  and  $n \geq 1$  that  $b^n = at$ . Then  $b^{2n} = a(tat) \in a\eta$ , whence  $x = tat \in a\eta$  is such that  $ax = b^{2n}$ . By symmetry we conclude that each of  $a$  and  $b$  divides a power of the other, on the left and on the right, in the semigroup  $a\eta$ . Therefore each  $\eta$ -class  $a\eta$  is an archimedean semigroup.

(c) For any  $a \in S$ , we have  $a\mathcal{H}b$  if and only if  $a|b$  and  $b|a$ , whence  $a\eta b$  so that  $H_a \subseteq a\eta$ . Therefore each  $\eta$ -class is a union of Green's classes of  $S$ . Finally, if for two idempotents  $e, f \in E(S)$ , we have  $e \eta f$ , then it follows by idempotency that  $e\mathcal{H}f$ , which implies that  $e = f$  (and that  $H_e$  is the unique maximal subgroup of  $S$  contained in  $e\eta$ ). ■

### 3. More Results and Examples

The fact that the  $\mathcal{H}$ -commutative condition on a semigroup  $S$  is defined by the first order sentence  $(\forall a, b \in S, \exists x \in S), ab = bxa$  or  $ab = ba$  allows us to produce further examples. For the moment, we first confine ourselves to the class of Monoids.

**Theorem 3.1.** The class  $HM$  of  $\mathcal{H}$ -commutative monoids is closed under the taking of homomorphic images, direct products, and regular submonoids.

**Proof.** That  $HM$  is closed under the taking of homomorphic images and direct products follows from  $HM$  being defined by the first order sentence  $(\forall a, b \in M, \exists x \in M), ab = bxa$ .

Next let  $N$  be a regular submonoid of  $M$ . Then, by Theorem 2.6(b),  $\text{Reg}(M)$  is itself a semilattice of groups, whence the same is true of  $N \subseteq \text{Reg}(M)$ , and, so, by Theorem 2.6(b),  $N$  is an  $\mathcal{H}$ -commutative submonoid of  $M$ . ■

The distinction between monoids and semigroups is important. Moreover  $HM$  does not constitute a variety of monoids. Both these conclusions follow from the following examples. First another closure lemma is proved.

**Lemma 3.2.** The 0-direct union  $V = S \dot{\cup} T \dot{\cup} \{0\}$  of two  $\mathcal{H}$ -commutative semigroups  $S$  and  $T$  is  $\mathcal{H}$ -commutative.

**Proof.** Let  $a, b \in V$ . If  $a, b \in S$  or if  $a, b \in T$ , then there exists  $x \in S^1$  or  $x \in T^1$  as the case may be such that  $ab = bxa$ . Otherwise  $ab = ba = 0$ . So it follows that  $V$  satisfies the condition to be  $\mathcal{H}$ -commutative. ■

**Example 3.3.** The class  $HC$  is not closed under the taking of direct products or even under the taking of direct powers of one of its members. What follows is an example of an  $\mathcal{H}$ -commutative semigroup  $S$  with ten elements such that  $S \times S$  is not  $\mathcal{H}$ -commutative. Let  $C_3 = \langle a : a^3 = a^4 \rangle = \{a, a^2, a^3 = z\}$ , where  $z$  denotes the zero element of this monogenic semigroup. Let  $S_3$  be the symmetric group on  $\{1, 2, 3\}$  and let  $S$  be the 0-direct union  $S = C_3 \cup S_3 \cup \{0\}$ . Then  $C_3$  and  $S_3$  are each  $\mathcal{H}$ -commutative (as  $C_3$  is commutative and  $S_3$  is a group) and, so, by Lemma 3.2, is their 0-direct union  $S$ . We note that  $|S| = 3 + 6 + 1 = 10$ . Take the transpositions  $t_1 = (2\ 3)$  and  $t_2 = (3\ 1)$  of  $S_3$ , noting that  $t_1 t_2 = (1\ 3\ 2) \neq (1\ 2\ 3) = t_2 t_1$ . Consider the product  $(a, t_1)(a, t_2)$  in  $S \times S$  and suppose that  $x \in (S \times S)^1$  were such that

$$(a, t_1)(a, t_2) = (a, t_2)x(a, t_1). \quad (2)$$

If we had  $x = (u, v) \in S \times S$  then, since  $aua = z$  for all  $u \in C_3$ , equation (2) takes the form:

$$\begin{aligned} (a, t_1)(a, t_2) &= (a, t_2)(u, v)(a, t_1) \\ \Rightarrow (a^2, t_1 t_2) &= (z, t_2 v t_1), \end{aligned}$$

which is a contradiction as  $a^2 \neq z$ . On the other hand if we put  $x = 1$ , then (2) becomes  $(a^2, t_1 t_2) = (a^2, t_2 t_1)$ , which is also false as  $t_1 t_2 \neq t_2 t_1$ . Therefore, although  $S$  is a finite  $\mathcal{H}$ -commutative semigroup,  $S \times S$  is not  $\mathcal{H}$ -commutative. In particular this shows that the  $\mathcal{H}$ -commutative property cannot be defined by the condition that  $S$  satisfies some set of equations with solutions in  $S$  (as opposed to solutions in  $S^1$ ).

**Example 3.4.** Although  $HC$  is closed under the taking of regular subsemigroups, this is not the case for arbitrary subsemigroups, even if the initial  $\mathcal{H}$ -commutative semigroup happens to be a monoid or indeed a group. To see this, we need look no further than the free group  $G$  on  $\{g, h\}$ , as  $G$  contains the free monoid  $M$  on the same pair of generators and  $M$  is clearly not  $\mathcal{H}$ -commutative.

**Example 3.5.** We produce an example of a finite  $\mathcal{H}$ -commutative semigroup  $T$  with a subsemigroup  $S$  that is not  $\mathcal{H}$ -commutative. Let  $U$  be the 0-disjoint union  $U = C_3^1 \cup S_3 \cup \{0\}$ , so that  $U = S \cup \{1\}$ , where  $S$  is as in Example 3.3. Then  $S \times S$  is not  $\mathcal{H}$ -commutative and is a subsemigroup of the finite semigroup  $T = U \times U$ . What is more,  $T$  is  $\mathcal{H}$ -commutative: for take any  $(p, q), (u, v) \in T$ . We have  $(p, q)(u, v) = (pu, qv)$ . Let  $x = (y, z)$  where, if  $p, u \in S_3$  we put  $y = u^{-1}pup^{-1}$  and otherwise put  $y = 1$ ; similarly put  $z = 1$  unless  $q, v \in S_3$  in which case we put  $z = v^{-1}qvq^{-1}$ . Then we obtain  $(p, q)(u, v) = (u, v)(y, z)(p, q)$  as  $pu = uyp$  and  $qv = vzbq$ , thereby verifying the  $\mathcal{H}$ -commutative property.

**Example 3.6.** The archimedean components of the maximum semilattice image  $T/\eta$  of an  $\mathcal{H}$ -commutative semigroup  $T$  are not necessarily themselves  $\mathcal{H}$ -commutative, as we may verify for the case of the  $\mathcal{H}$ -commutative semigroup  $T = U \times U$  of Example 3.5 as follows.

For  $i = 1, 2$ , take any  $a_i = (u_i, v_i) \in C_3 \times S_3$ . Then

$$a_1^3 = (z, v_1^3) = (u_2, v_2)(z, v_2^{-1}v_1^3) = a_2(z, v_2^{-1}v_1^3);$$

from which we conclude that every member  $a_2 \in C_3 \times S_3$  divides some power of every member  $a_1 \in C_3 \times S_3$  in the semigroup  $C_3 \times S_3$ .

Next consider any factorization of  $a_1^k$  over  $T = U \times U$  of the form  $a_1^k = (a^k, v_1^k) = (u_2, v_2)(u_3, v_3)$  ( $k \geq 1$ ). Then  $u_2, u_3 \in C_3^1$  and  $v_2, v_3 \in S_3$ . Hence if  $a_2 | a_1^k$  for some  $k \geq 1$ , then  $a_2 \in C_3^1 \times S_3$ . Suppose however that  $a_2 = (1, v_2)$ . Then for any  $k \geq 1$ , we have  $a_2^k = (1, v_2^k) \notin a_1T$  as  $a_1 \in C_3 \times S_3$ . Therefore we conclude that  $C_3 \times S_3$  is an  $\eta$ -class of  $T$ .

As in Example 3.3, we now take  $(a, t_1), (a, t_2) \in a_1\eta$ . However, since  $(a_1\eta)^1 = (C_3 \times S_3)^1 \subseteq (S \times S)^1$  and, as shown in Example 3.3, there is no  $x \in (S \times S)^1$  such that  $(a, t_1)(a, t_2) = (a, t_2)x(a, t_1)$ , it follows that  $a_1\eta$  is not itself an  $\mathcal{H}$ -commutative semigroup.

## 4. Epimorphisms and Dominions

We now generalize Isbell's result [6, Corollary 2.5] from commutative semigroups to  $\mathcal{H}$ -commutative semigroups.

**Theorem 4.1.** Let  $U$  be an  $\mathcal{H}$ -commutative subsemigroup of a semigroup  $S$ . Then

$\text{Dom}(U, S)$  is  $\mathcal{H}$ -commutative.

**Proof.** Let  $U$  be any  $\mathcal{H}$ -commutative subsemigroup of a semigroup  $S$ . Then we have to show that  $\text{Dom}(U, S)$  is also  $\mathcal{H}$ -commutative; i.e., for all  $d, h \in \text{Dom}(U, S)$  there exists some  $w \in \text{Dom}(U, S)^1$  such that  $dh = hwd$ .

**Case (i):** If both  $d, h \in U$ , then, trivially  $dh = hwd$  for some  $w \in U^1$ .

**Case (ii):** Let  $d \in U$  and  $h \in \text{Dom}(U, S) \setminus U$ . Then, by the zigzag theorem, there exists a series of factorizations of  $h$  as follows:

$h = a_0y_1 = x_1a_1y_1 = x_1a_2y_2 = x_2a_3y_2 = \cdots = x_ma_{2m-1}y_m = x_ma_{2m}$ , where  $m \geq 1$ ,  $a_i \in U$ ,  $x_i, y_i \in S \setminus U$  and

$$\begin{aligned} a_0 &= x_1a_1, & a_{2m-1}y_m &= a_{2m}; \\ a_{2i-1}y_i &= a_{2i}y_{i+1}, & x_i a_{2i} &= x_{i+1}a_{2i+1} \quad (1 \leq i \leq m-1). \end{aligned} \quad (3)$$

Now

$$\begin{aligned} dh &= da_0y_1 && \text{(by zigzag equations (3))} \\ &= a_0w_1dy_1 && \text{(for some } w_1 \in U^1 \text{ as } U \text{ is } \mathcal{H}\text{-commutative)} \\ &= x_1a_1w_1dy_1 && \text{(by zigzag equations (3))} \\ &= x_1w_1dw_2a_1y_1 && \text{(for some } w_2 \in U^1 \text{ as } U \text{ is } \mathcal{H}\text{-commutative)} \\ &= x_1w_1dw_2a_2y_2 && \text{(by zigzag equations (3))} \\ &= x_1a_2w_3w_1dw_2y_2 && \text{(for some } w_3 \in U^1 \text{ as } U \text{ is } \mathcal{H}\text{-commutative)} \\ &= x_2a_3w_3w_1dw_2y_2 && \text{(by zigzag equations (3))} \\ &= x_2w_3w_1dw_2w_4a_3y_2 && \text{(for some } w_4 \in U^1 \text{ as } U \text{ is } \mathcal{H}\text{-commutative)} \\ &\vdots \\ &= x_mw_{2m-1}w_{2m-3} \cdots w_3w_1dw_2w_4 \cdots w_{2m-2}w_{2m}(a_{2m-1}y_m) \\ &= x_mw_{2m-1}w_{2m-3} \cdots w_3w_1dw_2w_4 \cdots w_{2m-2}w_{2m}a_{2m} && \text{(by zigzag equations (3))} \\ &= x_ma_{2m}w_{2m+1}w_{2m-1}w_{2m-3} \cdots w_3w_1dw_2w_4 \cdots w_{2m-2}w_{2m} \\ &\quad \text{(for some } w_{2m+1} \in U^1 \text{ as } U \text{ is } \mathcal{H}\text{-commutative)} \end{aligned}$$

$$\begin{aligned}
&= hw_{2m+1}w_{2m-1}w_{2m-3}\cdots w_3w_1w_2w_4\cdots w_{2m-2}w_{2m}w_{2m+2}d \\
&\quad (\text{for some } w_{2m+2} \in U^1 \text{ as } U \text{ is } \mathcal{H}\text{-commutative and by zigzag equations (3)}) \\
&= hwd \text{ (where } w = w_{2m+1}w_{2m-1}w_{2m-3}\cdots w_3w_1w_2w_4\cdots w_{2m-2}w_{2m}w_{2m+2} \in U^1),
\end{aligned}$$

as required.

**Case (iii):** Let  $d \in \text{Dom}(U, S) \setminus U$  and  $h \in U$ .

The proof in this case is the left-right dual to that of Case (ii).

**Case (iv):** Let  $d, h \in \text{Dom}(U, S) \setminus U$ .

Let (3) be a zigzag for  $h \in \text{Dom}(U, S)$  in  $S$  over  $U$ . Now

$$\begin{aligned}
dh &= da_0y_1 && \text{(by zigzag equations (3))} \\
&= a_0w_1dy_1 && \text{(by Case(ii) for some } w_1 \in U^1) \\
&= x_1a_1w_1dy_1 && \text{(by zigzag equations (3))} \\
&= x_1w_1dw_2a_1y_1 && \text{(by Case(ii) for some } w_2 \in U^1 \text{ as } w_1d \in \text{Dom}(U, S)) \\
&= x_1w_1dw_2a_2y_2 && \text{(by zigzag equations (3))} \\
&= x_1a_2w_3w_1dw_2y_2 && \text{(by Case(ii) for some } w_3 \in U^1 \text{ as } w_1dw_2 \in \text{Dom}(U, S)) \\
&= x_2a_3w_3w_1dw_2y_2 && \text{(by zigzag equations (3))} \\
&= x_2w_3w_1dw_2w_4a_3y_2 && \text{(by Case(ii) for some } w_4 \in U^1 \text{ as } w_3w_1dw_2 \in \text{Dom}(U, S)) \\
&\vdots \\
&= x_mw_{2m-1}w_{2m-3}\cdots w_3w_1dw_2w_4\cdots w_{2m-2}w_{2m}a_{2m-1}y_m \\
&= x_mw_{2m-1}w_{2m-3}\cdots w_3w_1dw_2w_4\cdots w_{2m-2}w_{2m}a_{2m} && \text{(by zigzag equations (3))} \\
&= x_ma_{2m}w_{2m+1}w' && \text{(by Case(ii) for some } w_{2m+1} \in U^1 \text{ as } w' \in \text{Dom}(U, S), \\
&\quad \text{where } w' = w_{2m-1}w_{2m-3}\cdots w_3w_1dw_2w_4\cdots w_{2m-2}w_{2m}) \\
&= hw_{2m+1}w_{2m-1}w_{2m-3}\cdots w_3w_1w_2w_4\cdots w_{2m-2}w_{2m}w_{2m+2}d \\
&\quad (\text{for some } w_{2m+2} \in U^1 \text{ as } U \text{ is } \mathcal{H}\text{-commutative,} \\
&\quad w' = w_{2m-1}w_{2m-3}\cdots w_3w_1dw_2w_4\cdots w_{2m-2}w_{2m} \text{ and by zigzag equations (3)}) \\
&= hwd \text{ (where } w = w_{2m+1}w_{2m-1}w_{2m-3}\cdots w_3w_1w_2w_4\cdots w_{2m-2}w_{2m}w_{2m+2} \in U^1),
\end{aligned}$$

as required. Thus  $\text{Dom}(U, S)$  is  $\mathcal{H}$ -commutative. ■

**Corollary 4.2.** Let  $\phi : S \rightarrow T$  be epi. If  $S$  is  $\mathcal{H}$ -commutative, then  $T$  is  $\mathcal{H}$ -commutative.

**Proof.** As  $\phi : S \rightarrow T$  be epi, the inclusion morphism  $i : S\phi \rightarrow T$  is epi. Thus  $\text{Dom}(S\phi, T) = T$ . As  $S$  is  $\mathcal{H}$ -commutative, by Remark 2.4,  $S\phi$  is  $\mathcal{H}$ -commutative. Therefore, by Theorem 4.1,  $T$  is  $\mathcal{H}$ -commutative, as required. ■

In Propositions 4.3 through 4.7, we assume that  $S$  is an  $\mathcal{H}$ -commutative semigroup and  $T$  is a semigroup containing  $S$  such that  $\text{Dom}(S, T) = T$ . We also assume that  $K$  is a right ideal of  $S$  satisfying the minimum condition on principal right ideals and such that  $\forall d \in T \setminus S, \forall u \in T$ , if  $s = du \in S$ , then  $s \in K$ .

**Proposition 4.3.** For any  $a \in K$ , there exists  $c \in K$  and a positive integer  $r$  such that  $a^r c$  is idempotent and  $a^r \mathcal{H} a^r c$ .

**Proof.** Consider the descending sequence  $aK^1 \supseteq a^2K^1 \supseteq \dots$  of principal right ideals  $aK^1, a^2K^1$  etc generated by  $a, a^2, \dots$ . By the hypothesis, the above descending sequence must stabilize. Therefore,

$$a^r K^1 = a^{2r} K^1 \quad \text{for some } r, \text{ and so } a^r = a^{2r} c, \quad (4)$$

for some  $c \in K$ . Then

$$a^r = a^{2r} c = a^r a^r c = a^r a^{2r} c c = a^{3r} c^2 = \dots = a^{(k+1)r} c^k. \quad (5)$$

Hence  $a^r c = a^{kr} c^k$ , for all  $k \geq 1$ .

Now put  $k = 2$ . We obtain

$$\begin{aligned} a^r c &= a^{2r} c^2 = a^r a^r c^2 = a^r c^2 x a^r \quad (\text{for some } x \in S) \\ &= a^r c^2 x a^{2r} c = a^r c^2 x a^r a^r c = a^r a^r c^2 a^r c \quad (\text{for some } x \in S) \\ &= a^{2r} c^2 a^r c = (a^r c)(a^r c) = (a^r c)^2 \end{aligned} \quad (6)$$

Therefore,  $a^r c$  is an idempotent. Now we show that  $a^r \mathcal{H} a^r c$ . As

$$a^r c = a^{2r} c^2 \quad (\text{by equation (6)})$$

$$\begin{aligned}
&= a^r a^r c^2 = a^r c^2 x a^r \quad (\text{for some } x \in S) \\
&= (a^r c^2 x) a^r
\end{aligned}$$

and  $a^r(a^r c) = a^{2r} c = a^r$ , we have  $a^r \mathcal{L} a^r c$ . Since, by Theorem 2.3, all Green's relations are equal on  $\mathcal{H}$ -commutative semigroups, we have  $a^r \mathcal{H} a^r c$ , as required.  $\blacksquare$

**Proposition 4.4.** For each  $b \in T \setminus S$ , there exists an idempotent  $f \in K$  such that  $b = bf (= fb)$ .

**Proof.** As  $\text{Dom}(S, T) = T$ , by the zigzag theorem,  $b = a_0 x_1 = y_1 a_1 x_1$ , for some  $a_0, a_1 \in S$ . As  $y_1 a_1 = a_0 \in S$ , by hypothesis,  $y_1 a_1 = a_0 \in K$ . Hence  $b$  has a left divisor  $a_0 \in K$ . Let  $B$  be the set of all left divisors of  $b$  in  $K$ . Then  $B \neq \emptyset$ . Let  $\mathcal{B}$  be the set of all principal right ideals of  $K$  generated by the elements of  $B$ . Let  $k \in B$  be such that the principal right ideal of  $K$  generated by  $k$  is minimal in  $\mathcal{B}$ . Then  $b = kz$  for some  $z \in T \setminus S$ . By the same argument used in the factorization of  $b$ , it follows that  $z = k'z'$  for some  $z' \in T \setminus S$  and  $k' \in K$ . As the principal right ideal of  $K$  generated by  $kk'$  is contained in the principal right ideal generated by  $k$ , we have  $k = kk'l = k(k'l)^2 = k(k'l)^q$  ( $l \in K$ ) for all  $q = 1, 2, 3, \dots$ . Now, consider the descending sequence  $(k'l)K^1 \supseteq (k'l)^2 K^1 \supseteq \dots$  of principal right ideals  $(k'l)K^1, (k'l)^2 K^1$  etc generated by  $k'l, (k'l)^2, \dots$ . As  $K$  satisfies the minimum condition on principal right ideals,  $(k'l)^r = (k'l)^{2r} k''$  for some  $k'' \in K$  and some positive integer  $r$ . Thus, as in the proof of Proposition 4.3,  $(k'l)^r$  is a multiple of an idempotent  $f = (k'l)^r k''$ .

Hence

$$k = k(k'l)^r = k(k'l)^{2r} k'' = k(k'l)^r (k'l)^r k'' = k(k'l)^r f.$$

Now

$$b = kz = k(k'l)^r f z = k(k'l)^r f z f \quad (\text{by Proposition 2.1})$$

$$= (kz)f = bf, \quad \text{as required.} \quad \blacksquare$$

For  $b \in T \setminus S$  take  $e = f \in K$  as in Proposition 4.4. Then  $b = eb \in eT$ . However  $eK \subseteq K \subseteq S$  and  $b \notin S$  so that  $eb \in eT \setminus eK$ . Hence  $eK$  is properly contained in  $eT$ .

**Proposition 4.5.** For any idempotent  $e \in K$ ,  $\text{Dom}(eK, eT) = eT$ .



**Proof.** Take any  $ed \in eT$  for any  $d \in T \setminus S$ . Since  $d \in \text{Dom}(S, T)$ , by the zigzag theorem,  $d$  has a zigzag in  $T$  over  $S$ . Hence we may write

$$\begin{aligned}
ed &= ea_0x_1 = (ea_0)(ex_1) && \text{(by Proposition 2.1)} \\
&= (ey_1)(ea_1)(ex_1) = (ey_1)(ea_2)(ex_2) && \text{(by zigzag equations and Proposition 2.1)} \\
&\vdots \\
&= (ey_m)(ea_{2m})
\end{aligned}$$

which is clearly a zigzag over  $eS = eK$  with value  $ed$ . Therefore,  $\text{Dom}(eK, eT) = eT$ . ■

Recall the natural partial order  $\leq$  of the idempotents of a semigroup whereby  $e \leq f \Leftrightarrow ef = fe = e$ . An idempotent  $e \in K$  is said to be a right (left) divisor of  $y \in T \setminus S$  if  $y = xe$  ( $y = ex$ ) for some  $x \in T$ . Then necessarily  $x \in T \setminus S$  as  $e \in S$ , and  $S$  is a subsemigroup of  $T$ , and since  $e$  is an idempotent, we can take  $x = y$ . By Proposition 2.1, if  $S$  is  $\mathcal{H}$ -commutative, then  $e \leq f \Leftrightarrow ef = e$  and  $e$  is a right divisor of  $y \Leftrightarrow e$  is a left divisor of  $y \Leftrightarrow y = ey = ye$ .

**Proposition 4.6.** For each  $b \in T \setminus S$ , there exists a smallest idempotent  $e \in K$  such that  $b = be$  ( $= eb$ ).

**Proof.** By Proposition 4.4, there exist  $e_1 \in E(K)$  such that  $b = be_1$ . Suppose that  $e_2 \in E(K)$  is such that  $b = be_2$ . Then  $b = be_2 = be_1e_2$  and it follows that the set  $F$  of all idempotent divisors of  $b$  in  $K$  is a subsemilattice of the semilattice  $E(K)$  of all idempotents in  $K$ . As  $K$  satisfies the minimum condition on principal right ideals and idempotents are central,  $F$  cannot have an infinite descending chain and so there must be a least element  $e$  in  $F$ . ■

Now consider any principal right ideal  $B$  of  $eK$ , for any idempotent  $e$ , generated by any element  $b = ek \in eK$ . Then  $B = \{b\} \cup b(eK) = \{b\} \cup bK = bK^1$ . Therefore  $B$  is equal to the principal right ideal of  $K$  generated by  $b$  whence  $eK$ , as does  $K$ , satisfies the minimum condition on principal right ideals. This allows us to apply the argument of Proposition 4.4 to  $eK$  in the the following proposition.

Moreover  $\forall d \in eT \setminus eS, \forall u \in eT$  such that  $s = du \in eS$ , then we have  $d \in T \setminus S, u \in T$  and  $s = du \in S$ . Therefore  $s = du \in K$ . Now  $s = du = edu \in eK$ . Thus the conditions that we assumed throughout as regards to factorizations also apply to  $eK$  in  $eT$  as well.

**Proposition 4.7.** Let  $b \in T \setminus S$  and let  $e \in E(K)$  be a smallest idempotent such that  $b = be$ , as provided through Proposition 4.6. If  $S$  is properly contained in  $T$ , then there exist an element  $z \in eT \setminus eK$  whose only divisors in  $eK$  are the elements of  $H_e$ , the  $\mathcal{H}$ -class of the element  $e$ .

**Proof.** As  $b \in T \setminus S, b \in eT \setminus eK$ . Since  $\text{Dom}(eK, eT) = eT$ , by the zigzag theorem,  $b = a_0x_1 = y_1a_1x_1$ , for some  $a_0, a_1 \in eK$  and  $x_1, y_1 \in eT \setminus eK$ . As in Proposition 4.4, let  $B$  be the set of all divisors of  $b$  in  $eK$  and let  $\mathcal{B}$  be the set of all principal right ideals of  $eK$  generated by the elements of  $B$ . Let  $k \in B$  be such that the principal right ideal of  $eK$  generated by  $k$  is minimal in  $\mathcal{B}$ . Then  $b = kz$ , where  $z \in eT \setminus eK$ . Let  $k'$  be an arbitrary factor of  $z$  in  $eK$  so that  $z = k'z'$  for some  $z' \in eT \setminus eK$ . Now, as in the proof of Proposition 4.4, there exists  $k^* = l(k'l)^{n-1}k'' \in eK$  such that  $k'k^* = (k'l)^nk'' = f$ , which is an idempotent in  $eK$ . This  $f$  is an idempotent factor of  $z$ , and thus also of  $b$ . Thus  $e \leq f$ . Hence  $k'(k^*e) = fe = e$ . As  $k'(k^*e) = e$  and  $ek' = k'$ , we have  $k'\mathcal{R}e$ . Since, by Theorem 2.3, all Green's relations are equal on  $\mathcal{H}$ -commutative semigroups, we have  $k'\mathcal{H}e$ . Thus  $k' \in H_e$ . Hence  $z$  is the required element.  $\blacksquare$

**Theorem 4.8.** Let  $S$  be any  $\mathcal{H}$ -commutative semigroup and let  $T$  be any semigroup containing  $S$  such that  $\text{Dom}(S, T) = T$ . Let  $K$  be any right ideal of  $S$  satisfying the minimum condition on principal right ideals and such that  $\forall d \in T \setminus S, \forall u \in T, if  $s = du \in S$ , then  $s \in K$ . Then  $S$  is saturated.$

**Proof.** Suppose on the contrary that  $S$  is not saturated. Then there exists a semigroup  $T$  containing  $S$  properly and such that  $\text{Dom}(S, T) = T$ . Then, by Propositions 4.5,  $\text{Dom}(eK, eT) = eT$  for each idempotent  $e \in K$ . By Proposition 4.7, let  $z \in eT \setminus eK$  be such that the only divisors of  $z$  in  $eK$  are members of  $H_e$ . Since  $T$  (and not just  $S$ ) is  $\mathcal{H}$ -commutative, the spine factors of  $z \in eK$  (for any zigzag of  $z \in eT$  over  $eK$ ) are also left (and right) factors of  $z$ . Therefore the zigzag of  $z \in eT$  over  $eK$  is in fact a zigzag over  $H_e$ , a contradiction as  $H_e$ , being a group, is absolutely closed [5, Theorem 2.3].  $\blacksquare$

Now, if we take  $K = S$ , then the assumption “ $\forall d \in T \setminus S, \forall u \in T, \text{ if } s = du \in S, \text{ then } s \in K$ ” is trivially satisfied. Thus we get the following theorem as a corollary to Theorem 4.8.

**Theorem 4.9.** Any  $\mathcal{H}$ -commutative semigroup satisfying the minimum condition on principal right ideals is saturated.

We provide a class of examples of a saturated  $\mathcal{H}$ -commutative semigroup as an application of Theorem 4.9.

**Theorem 4.10.** Every  $\mathcal{H}$ -commutative archimedean semigroup containing an idempotent element is saturated.

**Proof.** Let  $S$  be any  $\mathcal{H}$ -commutative archimedean semigroup containing an idempotent element. By Theorem 4.9, it is sufficient to show that  $S$  satisfies the minimal condition on principal right ideals. By [12, Theorem 4],  $S$  is an ideal extension of a group  $G$  by a commutative nilsemigroup. Now, for any  $a \in S$ , consider a descending chain  $aS^1 \supseteq a^2S^1 \supseteq \dots$  of principal right ideals of  $S$ . Then for some  $n \geq 1$ , we have  $a^n \in G$  whence  $a^{n+k}S^1 = G$  for all  $k \geq 0$ . In particular the descending chain stabilizes and the result follows by Theorem 4.9. ■

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