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FURTHER RIGID TRIPLES OF CLASSES IN G_2

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Abstract. We establish the existence of two rigid triples of conjugacy classes in the algebraic group G_2 in characteristic 5, complementing results of the second author with Liebeck and Marion. As a corollary, the finite groups $G_2(5^n)$ are not $(2,4,5)$-generated, confirming a conjecture of Marion in this case.

1. Introduction

Let G be a connected simple algebraic group over an algebraically closed field K, and let C_1, ...,C_s be conjugacy classes of G. Following [16], we say the s-tuple $\mathbf{C} = (C_1, \dots, C_s)$ is *rigid* in G if the set

$$\mathbf{C}_0 \stackrel{\text{def}}{=} \{(x_1, \dots, x_s) \in C_1 \times \cdots \times C_s \, : \, x_1 x_2 \cdots x_s = 1\}$$

is non-empty and forms a single orbit under the action of G by simultaneous conjugation.

Some well-known examples of rigid tuples of classes in simple algebraic groups are the Belyi triples and Thompson tuples, defined in [18]. Other rigid triples are known, see for instance [2, 3, 5, 9, 17]. Rigid tuples of classes are interesting in the context of the inverse Galois problem [12], and also arise naturally in the theory of ordinary differential equations [7].

Recall that a group is (a,b,c)-generated if it is generated by elements x, y and z, of respective orders a, b and c, such that $xyz = 1$. The group is then called an (a,b,c)-group, and the triple (x,y,z)

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is called an (a,b,c)-triple of the group. The theory of (a,b,c)-generation of finite groups has close connections to rigidity, for instance it is a basic observation that given a rigid tuple $\mathbf{C}$ of classes of G, all subgroups $\langle x_1,\ldots,x_s\rangle$ for $(x_1,\ldots,x_s)\in\mathbf{C}_0$ are conjugate in G, so that there is at most one $r>0$ such that the finite subgroup $G(p^r)$ is generated by elements in such an s-tuple.

Let $K=\bar{\mathbb{F}}_5$ be the algebraic closure of the field of five elements. In [9] it is shown that the simple algebraic group $G=G_2(K)$ has a rigid triple of conjugacy classes of elements of orders 2, 5 and 5, and any triple of elements (x_1,x_2,x_3) in the corresponding set $\mathbf{C}_0$ generates a copy of Alt_5. This is then used to show that none of the groups $G_2(5^n)$, $SL_3(5^n)$ or $SU_3(5^n)$ is a $(2,5,5)$-group.

Here we produce two further rigid triples of classes in $G=G_2(K)$, closely related to the triple above. Recall from [1] that G has a unique class of involutions, with representative t, say, and $C_G(t)=A_1\tilde{A}_1$ is a central product of two subgroups $SL_2(K)$, where A_1 (resp. $\tilde{A}_1$) is generated by a long (resp. short) root subgroup of G. There also exist two classes of elements of order 4, with representatives s_1 and s_2, such that $C_G(s_1)=A_1T'$ and $C_G(s_2)=\tilde{A}_1T''$, where T' and T'' are 1-dimensional tori. Finally, recall from [8] that G has three classes of unipotent elements of order 5: the long and short root elements, and the class labelled $G_2(a_1)$, with representative $u=x_\beta(1)x_{3\alpha+\beta}(1)$, where α (resp. β) is the short (resp. long) simple root of G. From [11, Table 22.1.5], the centraliser $C_G(u)=U_4.\mathrm{Sym}_3$, where U_4 is a 4-dimensional connected unipotent group.

Theorem 1.1.

(i) *The triples of classes* $\mathbf{C}=(t^G,s_1^G,u^G)$ *and* $\mathbf{D}=(t^G,s_2^G,u^G)$ *are rigid in* $G=G_2(K)$.

(ii) *Every triple of elements in* $\mathbf{C}_0$ *or* $\mathbf{D}_0$ *generates a subgroup isomorphic to the symmetric group* Sym_5.

(iii) *None of the groups* $G_2(5^n)$ *are a* $(2,4,5)$*-group for any* n. *Neither are the groups* $SL_3(5^n)$ *or* $SU_3(5^n)$.

Remark 1.2.

(1) *Each subgroup* Sym_5 *in part (ii) here contains a subgroup* Alt_5 *arising from* [9, Theorem 1(ii)].

(2) *Keeping track of details in the proof in* [9] *shows that* $G_2(K)$ *has a unique class of subgroups* Alt_5. *These subgroups have centraliser* Sym_3, *and by Lang's theorem these split into three classes in* $G_2(5^r)$, *with centraliser orders* 6, 3 *and* 2. *Similarly, if* S *and* S' *are representatives of the two subgroup classes in part (ii) here, then* $C_G(S)\cong\mathrm{Sym}_3$, *while* $C_G(S')$ *is cyclic of order* 2. *It follows that the class of* S *(resp.* S'*) splits into* 3 *(resp.* 2*) classes of subgroups in* $G_2(5^r)$, *with centralisers of order* 6, 3, 2 *(resp.* 2 *and* 2*).*

(3) *A conjecture of Marion* [14] *states that, for a simple algebraic group* G *in characteristic* p, *if* δ_i *denotes the dimension of the variety of elements of* G *of order* i *and if* $\delta_a+\delta_b+\delta_c=2\dim(G)$, *then at most finitely many of the finite groups* $G(p^r)$ *are* (a,b,c)*-groups. For* G *of type* G_2, *this criterion holds precisely when* $(a,b,c)=(2,4,5)$ *or* $(2,5,5)$. *Hence part (iii), together with* [9, Theorem 1(iii)], *verifies the conjecture for* $G=G_2$ *in characteristic* 5. *A non-constructive proof*

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of this fact is given in [6, Proposition 3.7(i)], *where it is shown that every* $(2,4,5)$*-subgroup and* $(2,5,5)$*-subgroup of* $G_2(K)$ *is reducible on the natural* 7*-dimensional module, by considering the dimensions of* $SL_7(K)$*-conjugacy classes of elements in the relevant* (a,b,c)*-triples.*

2. Proof of the Theorem

We proceed in the manner of [9]. Let $G = G_2(K)$ and t, u, s_1, $s_2 \in G$ as above. If σ is a Frobenius morphism of G induced from the field map $x \mapsto x^5$ of K, then

$$G = \bigcup_{n=1}^{\infty} G_{\sigma^n} = \bigcup_{n=1}^{\infty} G_2(5^n).$$

The element $u = x_\beta(1)x_{3\alpha+\beta}(1)$ is a regular unipotent element in a subgroup $A_2 = SL_3(K)$ of G generated by long root groups, and therefore lies in a subgroup $\Omega_3(5) \cong \mathrm{Alt}_5$ of G, which we denote by A. Now, let $S = N_{A_2}(A) = SO_3(5) \cong \mathrm{Sym}_5$. Following the proof given in [9] we find that $N_G(A) = S \times C_G(A)$ and $C_G(A) = \langle z, \tau\rangle \cong \mathrm{Sym}_3$, where $\langle z\rangle$ is the centre of A_2 and τ is an outer involution in $N_G(A_2) = A_2.2$. Note that $C_{A_2}(\tau) = SO_3(K)$, so $\tau \in C_G(S)$.

Let v be an involution in $S \setminus A$, so that $S = \langle A, v\rangle$, and define $S' = \langle A, v\tau\rangle$, so that $S' \cong \mathrm{Sym}_5$ also. Then $C_G(S)$, $C_G(S') \leq C_G(A) = \langle z, \tau\rangle$ and therefore

$$C_G(S) = \langle z, \tau\rangle\,, \tag{2.1}$$

$$C_G(S') = \langle \tau\rangle\,. \tag{2.2}$$

In particular S and S' are not conjugate in G.

Next consider the set of $(2,4,5)$-triples of Sym_5. It is straightforward to show that there are exactly 120 such triples, and that Sym_5 acts transitively on these by simultaneous conjugation.

Now let $\mathbf{C} = (t^G, s_1^G, u^G)$ and $\mathbf{D} = (t^G, s_2^G, u^G)$, and for q a fixed power of 5 let $\mathbf{C}_0(q) = \mathbf{C}_0 \cap G_2(q)^3$ and $\mathbf{D}_0(q) = \mathbf{D}_0 \cap G_2(q)^3$. We now show that $|\mathbf{C}_0(q)| = |\mathbf{D}_0(q)| = |G_2(q)|$. For this we require the character table of $G_2(q)$, given in [1] and available in the CHEVIE [4] computational package. Since $C_G(u)/C_G(u)^\circ = S_3$, an application of Lang's theorem [13, Theorem 21.11] shows that $u^G \cap G_2(q)$ splits into three classes, with representatives denoted in [1] by u_3, u_4 and u_5, and respective centraliser orders $6q^4$, $3q^4$ and $2q^4$. For $x, y, z \in G_2(q)$ let a_{xyz} be the corresponding class algebra constant. Calculations with the character table show that

$$a_{ts_iu_j} = \begin{cases} q^4 & \text{if } i = 1,\ j \in \{3,4,5\} \text{ or } i = 2,\ j = 4, \\ 3q^4 & \text{if } i = 2,\ j = 3, \\ 0 & \text{if } i = 2, j = 5. \end{cases}$$

and it follows that

$$|\mathbf{C}_0(q)| = \sum_{j=3}^{5} |u_j^{G_2(q)}| a_{ts_1u_j} = |G_2(q)| \left(\frac{q^4}{6q^4} + \frac{q^4}{3q^4} + \frac{q^4}{2q^4} \right) = |G_2(q)|,$$

$$|\mathbf{D}_0(q)| = \sum_{j=3}^{5} |u_j^{G_2(q)}| a_{ts_2u_j} = |G_2(q)| \left(\frac{3q^4}{6q^4} + \frac{q^4}{2q^4} \right) = |G_2(q)|.$$

Now let $\mathbf{E}$ denote (resp. $\mathbf{E}'$) denote the set of triples $(x_1, x_2, x_3) \in \mathbf{C}_0 \cup \mathbf{D}_0$ which generate a conjugate of S (resp. a conjugate of S'). Then G is transitive on both $\mathbf{E}$ and $\mathbf{E}'$, since if $\langle x_1, x_2, x_3 \rangle = \langle y_1, y_2, y_3 \rangle^g$ are each isomorphic to Sym_5, then (x_1^g, x_2^g, x_3^g) and (y_1, y_2, y_3) are $(2,4,5)$ triples in a fixed copy of Sym_5, hence conjugate in Sym_5 by the observation above. Moreover both $\mathbf{E}$ and $\mathbf{E}'$ are non-empty, since S and S' each contain $(2,4,5)$-triples and a unique conjugacy class of unipotent elements, whose elements are conjugate to an element of A and therefore are conjugate to u. By (2.1) and (2.2) the stabiliser of a point in $\mathbf{E}$ is isomorphic to Sym_3, and the stabiliser of a point in $\mathbf{E}'$ is cyclic of order 2. Hence applying Lang's theorem shows that $\mathbf{E}(q) = \mathbf{E} \cap G_2(q)^3$ splits into three $G_2(q)$-orbits, of orders $|G_2(q)|/r$ for $r = 2, 3, 6$, and similarly $\mathbf{E}'(q) = \mathbf{E}' \cap G_2(q)^3$ splits into two orbits, each of order $|G_2(q)|/2$. Therefore,

$$|\mathbf{E}(q)| + |\mathbf{E}'(q)| = |G_2(q)| \left(\frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \right) = |\mathbf{C}_0(q)| + |\mathbf{D}_0(q)|$$

and it follows that $\mathbf{C}_0(q) \cup \mathbf{D}_0(q) = \mathbf{E}(q) \cup \mathbf{E}'(q)$ for each q. Therefore

$$\mathbf{C}_0 \cup \mathbf{D}_0 = \bigcup_{n=1}^{\infty} \mathbf{C}_0(5^n) \cup \mathbf{D}_0(5^n) = \bigcup_{n=1}^{\infty} \mathbf{E}(5^n) \cup \mathbf{E}'(5^n) = \mathbf{E} \cup \mathbf{E}'$$

Hence G has exactly two orbits on $\mathbf{C}_0 \cup \mathbf{D}_0$. A triple in $\mathbf{C}_0$ cannot lie in the same orbit as a triple in $\mathbf{D}_0$ since the corresponding elements of order 4 are not G-conjugate, and it follows that the two G-orbits are $\mathbf{C}_0$ and $\mathbf{D}_0$.

This proves parts (i) and (ii) of the Theorem. For part (iii), suppose that $G_2(5^n)$, $SL_3(5^n)$ or $SU_3(5^n)$ is a $(2,4,5)$-group, with corresponding set of generators x_1, x_2, x_3. Since $L(G_2) \downarrow A_2$ is a direct sum of $L(A_2)$ and two 3-dimensional irreducible A_2-modules (cf. [10, Table 8.5]), it follows that $C_{L(G_2)}(x_1, x_2, x_3) = 0$. An application of a result of Scott [15] to the module $L(G)$, as in the proof of [16, Corollary 3.2], then yields

$$\dim(x_1^G) + \dim(x_2^G) + \dim(x_3^G) \geq 2\dim(G) = 28,$$

implying $(x_1^G, x_2^G, x_3^G) = \mathbf{C}$ or $\mathbf{D}$, which contradicts part (ii) of the Theorem. □

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