σ-FINITENESS OF ELLIPTIC MEASURES FOR QUASILINEAR ELLIPTIC PDE IN SPACE

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Abstract. In this paper we study the Hausdorff dimension of a elliptic measure μ_f in space associated to a positive weak solution to a certain quasilinear elliptic PDE in an open subset and vanishing on a portion of the boundary of that open set. We show that this measure is concentrated on a set of $\sigma-$finite $n-1$ dimensional Hausdorff measure for $p > n$ and the same result holds for $p = n$ with an assumption on the boundary.

We also construct an example of a domain in space for which the corresponding measure has Hausdorff dimension $\leq n-1-\delta$ for $p \geq n$ for some δ which depends on various constants including p.

The first result generalizes the authors previous work in [3] when the PDE is the $p-$Laplacian and the second result generalizes the well known theorem of Wolff in [24] when $p = 2$ and $n = 2$.

Contents

1. Introduction

In this paper we continue our study of the Hausdorff dimension of a measure associated with a certain positive weak solution, $u \geq 0$, to a PDE of p Laplace type. To introduce the PDE and the measure, we fix p, $1 < p < \infty$, and let $f : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$ be a real

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valued function with the following properties,

$$
\begin{aligned}
&a)\ f \text{ is homogeneous of degree } p \text{ on } \mathbb{R}^n \setminus \{0\}. \\
&\qquad \text{That is, } f(\eta) = |\eta|^p f\left(\tfrac{\eta}{|\eta|}\right) > 0 \text{ when } \eta \in \mathbb{R}^n \setminus \{0\}. \\
&b)\ \mathcal{D}f = (f_{\eta_1}, \dots, f_{\eta_n}) \text{ has continuous partial derivatives when } \eta \neq 0. \\
&c)\ f \text{ is uniformly convex on } B(0,1) \setminus \bar{B}(0,1/2). \\
&\qquad \text{That is, there exists } c_* \geq 1 \text{ such that for } \eta \in \mathbb{R}^n, 1/2 < |\eta| < 1, \\
&\qquad \text{and all } \xi \in \mathbb{R}^n \text{ we have } c_*^{-1}|\xi|^2 \leq \sum_{j,k=1}^{n} \tfrac{\partial^2 f}{\partial \eta_j \eta_k}(\eta)\xi_j\xi_k \leq c_*|\xi|^2.
\end{aligned} \tag{1.1}
$$

Put $f(0) = 0$. We next give examples of such f.

Example 1.2. From $a)$ in (1.1) it follows that $f(\eta) = \kappa(\eta)|\eta|^p$ when $\eta \in \mathbb{R}^n \setminus \{0\}$, where κ is homogeneous of degree 0. Using this fact one can show that if ϵ is sufficiently small, then $f(\eta) = |\eta|^p(1 + \epsilon\eta_1/|\eta|)$ satisfies (1.1). Such an f is not invariant under rotations.

From homogeneity of f and Euler's formula we have for a.e $\eta \in \mathbb{R}^n$ that

$$
\langle \mathcal{D}f(\eta), \eta\rangle = pf(\eta) \text{ and } \eta\,(\mathcal{D}^2 f(\eta)) = (p-1)\mathcal{D}f(\eta) \tag{1.3}
$$

where $\mathcal{D}^2 f(\eta) = (f_{\eta_j\eta_k})$ is an n by n matrix and η, $\mathcal{D}f(\eta)$ are regarded as $1 \times n$ row matrices.

Let O be an open set in $\mathbb{R}^n$ and $\hat{z} \in \partial O$. Let u be a positive weak solution in $O \cap B(\hat{z}, \rho)$ to the Euler-Lagrange equation

$$
\Delta_f u := \nabla \cdot \mathcal{D}f(\nabla u) = \sum_{j,k=1}^{n} f_{\eta_j\eta_k}(\nabla u) u_{x_k x_j} = 0 \tag{1.4}
$$

in $O \cap B(\hat{z}, \rho)$. That is, $u \in W^{1,p}(O \cap B(\hat{z}, \rho))$ and

$$
\int \langle \mathcal{D}f(\nabla u), \nabla\theta\rangle \mathrm{d}x = 0 \;\text{ whenever }\; \theta \in W_0^{1,p}(O \cap B(\hat{z}, \rho))
$$

where $\nabla\theta(x) = (\frac{\partial\theta}{\partial x_1}, \dots, \frac{\partial\theta}{\partial x_n})(x)$ whenever these partials exist in the distributional sense. We assume also that u has continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$. We continuously extend u (denoted with u also) to all $B(\hat{z}, \rho)$ by setting $u \equiv 0$ in $B(\hat{z}, \rho) \setminus O$. It is well known from [11, Theorem 21.2] that there exists a positive locally finite Borel measure μ_f on $\mathbb{R}^n$ associated with u. We call this measure as *elliptic measure* associated with a positive weak solution of (1.4). This measure has support contained in $\partial O \cap B(\hat{z}, \rho)$ with the property that

$$
\int \langle \mathcal{D}f(\nabla u), \nabla\phi\rangle \mathrm{d}x = -\int \phi \, \mathrm{d}\mu_f \text{ whenever } \phi \in C_0^\infty(B(\hat{z}, \rho)). \tag{1.5}
$$

Existence of μ_f follows from the maximum principle, basic Caccioppoli inequalities for u and the Riesz representation theorem for positive linear functional. Note that if ∂O and f are smooth enough then from an integration by parts in (1.5) and homogeneity in (1.3) we deduce that

$$
\mathrm{d}\mu_f = p\frac{f(\nabla u)}{|\nabla u|}\mathrm{d}\mathcal{H}^{n-1}|_{\partial O \cap B(\hat{z}, \rho)}.
$$

We next introduce the notion of the *Hausdorff dimension of a measure*. To this end, let λ be a real valued, positive, and increasing function on $(0, \infty)$ with $\lim_{r\to 0} \lambda(r) = 0$. For fixed

$0 < \delta$ and $E \subset \mathbb{R}^n$, we define $(\delta, \lambda)-$*Hausdorff content* of E in the usual way;

$$\mathcal{H}^{\lambda}_{\delta}(E) := \inf \left\{ \sum_i \lambda(r_i) \text{ where } E \subset \bigcup B(z_i, r_i),\ 0 < r_i < \delta,\ x_i \in \mathbb{R}^n \right\}. \tag{1.6}$$

Then the *Hausdorff measure* of E is defined by

$$\mathcal{H}^{\lambda}(E) := \lim_{\delta \to 0} \mathcal{H}^{\lambda}_{\delta}(E).$$

In case $\lambda(r) = r^{\alpha}$ we write $\mathcal{H}^{\alpha}$ for $\mathcal{H}^{\lambda}$. The Hausdorff dimension of μ_f, denoted by $\mathcal{H} - \dim \mu_f$, is defined by

$$\mathcal{H} - \dim \mu_f := \inf \left\{ \alpha : \ \exists \text{Borel set } E \subset \partial O \text{ with } \mathcal{H}^{\alpha}(E) = 0 \text{ and } \mu_f(\mathbb{R}^n \setminus E) = 0 \right\}.$$

Recall that μ is said to be *absolutely continuous* with respect to ν (if μ, ν, are positive Borel measures) provided that $\mu(E) = 0$ whenever E is a Borel set with $\nu(E) = 0$. Following standard notation, we write $\mu \ll \nu$. A set E is said to have $\sigma-$*finite* ν measure if

$$E = \bigcup_{i=1}^{\infty} E_i \text{ with } \nu(E_i) < \infty \text{ for } i = 1, \dots, \infty.$$

We note that if $f(\eta) = |\eta|^2$, then the Euler-Lagrange equation in (1.4) is the usual *Laplace equation*. In this case, if u is the *Green's function* for Laplace's equation with pole at some $z_0 \in \Omega$, then the measure corresponding to this harmonic function u as in (1.5) is *harmonic measure* relative to z_0 and will be denoted by ω.

The Hausdorff dimension of ω has been extensively studied in the last thirty five years in planar domains. In particular, in [8], Carleson proved that $\mathcal{H} - \dim \omega = 1$ when $\partial\Omega$ is a snowflake and $\mathcal{H} - \dim \omega \leq 1$ for any self similar Cantor set. In [20], Makarov proved that

Theorem A (Makarov)**.** *Let Ω be a simply connected domain in the plane and let $\lambda(r) := r \exp\{A\sqrt{\log \frac{1}{r} \log\log\log \frac{1}{r}}\}$. Then*

a) *ω is concentrated on a set of $\sigma-$finite $\mathcal{H}^1$ measure,*
b) *$\omega \ll \mathcal{H}^{\lambda}$ provided that A is large enough.*

We note that Theorem A implies $\mathcal{H} - \dim \omega = 1$ when Ω is a simply connected domain. For arbitrary domains in the plane, in [12], Jones and Wolff proved that $\mathcal{H} - \dim \omega \leq 1$ whenever $\Omega \subset \mathbb{R}^2$ and ω exists. In [23], Wolff improved this result by showing that ω is concentrated on a set of $\sigma-$finite $\mathcal{H}^1$ measure(see also [5, 13, 22]).

The Hausdorff dimension of harmonic measure in higher dimensions is considerably less understood than in the plane. When $n \geq 3$, in [7], Bourgain proved that $\mathcal{H} - \dim \omega \leq n - \tau$, where $\tau > 0$ depends only on the dimension n and the exact value of τ remains unknown. On the other hand, in [24], Wolff constructed examples in $\mathbb{R}^3$, we call *Wolff snowflakes*, for which the corresponding harmonic measures could have Hausdorff dimension either greater than 2 or less than 2. In [18], the second author, Verchota, and the third author proved a conjecture of Wolff in the affirmative: it was shown that both sides of a Wolff snowflake in $\mathbb{R}^n$ could have harmonic measures, say ω_1, ω_2, with either $\min(\mathcal{H} - \dim \omega_1, \mathcal{H} - \dim \omega_2) > n - 1$ or $\max(\mathcal{H} - \dim \omega_1, \mathcal{H} - \dim \omega_2) < n - 1$.

If $f(\eta) = |\eta|^p$ in (1.4), then the resulting PDE is called the $p-$*Laplace equation*:

$$\nabla \cdot \left[|\nabla u|^{p-2} \nabla u \right] = 0. \tag{1.7}$$

In this case, a solution u to (1.7) is called a $p-$*harmonic function* and the corresponding measure in (1.5) associated with u is called a $p-$*harmonic measure* and will be denoted by μ_p.

The nonlinearity and degeneracy of the $p-$Laplace equation makes it difficult to study the Hausdorff dimension of $p-$harmonic measure. The first result was obtained in [6], when Bennewitz and the second author studied the Hausdorff dimension of a $p-$harmonic measure, associated with a positive $p-$harmonic function u in $N \cap \Omega \subset \mathbb{R}^2$ with continuous boundary value 0 on $\partial\Omega$. In that result $\partial\Omega$ is a quasicircle and N is an open neighborhood of $\partial\Omega$. It was shown that all such measures, μ_p, corresponding to u, Ω, p as above, have the same Hausdorff dimension. Moreover,

$$\mathcal{H}-\dim\mu_p \geq 1 \text{ when } 1<p<2 \;\text{ while } \mathcal{H}-\dim\mu_p \leq 1 \text{ when } p>2.$$

After earlier studies in [6, 14, 16], the second author proved the following analogue of Theorem A in the $p-$harmonic setting (see [15]);

Theorem B (Lewis)**.** *Assume that $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain and N is a neighborhood of $\partial\Omega$. Let u be a positive p-harmonic in $\Omega \cap N$ with zero continuous boundary values on $\partial\Omega$. Let μ_p be the $p-$harmonic measure associated with u as described above. Let $\lambda(r)$ be as in Theorem A. Then*

a) *If $1<p<2$, there exists $A=A(p)\geq 1$, such that $\mu_p \ll \mathcal{H}^\lambda$.*
b) *If $2<p<\infty$, then μ_p is concentrated on a set of $\sigma-$finite $\mathcal{H}^1$ measure.*

A key fact used in [6, 14, 15, 16] is that if $\zeta = u$ or $\zeta = u_{x_i}$, $i=1,2$, then ζ is a weak solution to

$$L\zeta = \sum_{j,k=1}^{2} \frac{\partial}{\partial x_k}\left(b_{jk}\frac{\partial \zeta}{\partial x_j}\right) = 0 \tag{1.8}$$

where

$$b_{jk} = |\nabla u|^{p-4}\left[(p-2)u_{x_j}u_{x_k} + \delta_{jk}|\nabla u|^2\right].$$

Furthermore, if $v = \log|\nabla u|$ then $Lv \leq 0 (Lv \geq 0)$ when $1<p\leq 2 (2\leq p<\infty)$. Moreover, arguments in these papers also make heavy use of the fundamental inequality;

$$\frac{\hat{u}(z)}{d(z,\partial\Omega)} \approx |\nabla \hat{u}(z)| \text{ whenever } z \in \Omega\setminus \bar{B}(z_0,r_0). \tag{1.9}$$

where $\hat{u}$ is a certain "$p-$capacitary function" in $\Omega\setminus\bar{B}(z_0,r_0)$ for some fixed $z_0\in\Omega$ and $r_0 = d(z_0.\partial\Omega)/2$. The proof of (1.9) is highly nontrivial in a simply connected domain when $1<p\neq 2<\infty$, and in fact is the main result proved by the second author, Nyström, and Poggi-Corradini in [16, Theorem 1.5]. However if $p=2$, (1.9) is an easy consequence of the Koebe distortion estimates for a univalent function (use $\hat{u}$ = a Green's function for Ω). We also note that (1.9) can easily fail in arbitrary domains of $\mathbb{R}^n$ for $n\geq 2$.

Tools developed for p-harmonic functions in a series of papers by the second author and Nyström were used in [17] to obtain that μ_p is concentrated on a set of $\sigma-$finite $\mathcal{H}^{n-1}$ measure when $\partial\Omega\subset\mathbb{R}^n$ is sufficiently flat in the sense of Reifenberg, $u>0$ is p harmonic near $\partial\Omega$ and $p\geq n$. It was also shown in the same paper that if $p\geq n$ then all examples produced by Wolff's method had $\mathcal{H}-\dim\mu_p < n-1$, while if $p>2$, was near enough 2, then there existed a Wolff snowflake for which $\mathcal{H}-\dim\mu_p > n-1$. These examples provided the current authors with the necessary intuition to state and prove the following theorem in [3].

Theorem C (Akman, Lewis, Vogel)**.** *Let $O\subset\mathbb{R}^n$ be an open set and $\hat{z}\in\partial O$, $\rho>0$. Let $u>0$ be p-harmonic in $O\cap B(\hat{z},\rho)$ with continuous zero boundary values on $\partial O\cap B(\hat{z},\rho)$, and let μ_p be the p-harmonic measure associated with u. If $p>n$ then μ_p is concentrated* *on a set of $\sigma-$finite $\mathcal{H}^{n-1}$ measure. If $p = n$ the same conclusion is valid provided $\partial O \cap B(\hat{z}, \rho)$ is locally uniformly fat in the sense of $n-$capacity.*

The definition of a *locally uniformly fat set* will be given in section 2. We remark that Theorem C and the definition of $\mathcal{H} - \dim \mu_p$ imply that $\mathcal{H} - \dim \mu_p \leq n-1$ for $p \geq n$. A key lemma proved in this paper states that if $v = \log|\nabla u|$, then $Lv \geq 0$, weakly on $\{x : \nabla u(x) \neq 0\}$, when $p \geq n$. Here L is defined as in (1.8) with 2 replaced by n in the summation. Using this fact, some basic estimates for p harmonic functions, and a stopping stopping time argument as in [12, 23], we eventually arrived at Theorem C.

In [2], the authors studied the PDE (1.4), $\Delta_f u = 0$, and showed in $\mathbb{R}^2$ that if u, f are sufficiently smooth and $\nabla u(x) \neq 0$, then both u, u_{x_i}, $i = 1, 2$, satisfy

$$\tilde{L}\zeta := \sum_{k,j=1}^{2} \frac{\partial}{\partial x_k}\left(f_{\eta_j \eta_k}(\nabla u)\frac{\partial \zeta}{\partial x_j}\right) = 0. \tag{1.10}$$

in an open neighborhood of x. Furthermore, if $\tilde{v} = \log f(\nabla u)$ then pointwise in this neighborhood $\tilde{L}\tilde{v} \leq 0 (\tilde{L}\tilde{v} \geq 0)$ when $1 < p \leq 2 (2 \leq p < \infty)$. In [1] it was shown by the first author for general f as in 1.1 that $\tilde{L}\tilde{v} \leq 0 (\tilde{L}\tilde{v} \geq 0)$ weakly when $1 < p \leq 2 (2 \leq p < \infty)$. Using this fact and following the game plan of [6, 16], the first author proved in the same paper that

Theorem D (Akman). *Let $\Omega \subset \mathbb{R}^2$ be any bounded simply connected domain and let N be a neighborhood of $\partial\Omega$. Let u be a positive weak solution to (1.4) in $\Omega \cap N$ with zero continuous boundary values on $\partial\Omega$. Let μ_f be the measure associated with u as described above. Let* $\tilde{\lambda}(r) := r \exp\{A\sqrt{\log\frac{1}{r}\log\log\frac{1}{r}}\}$ for $0 < r < 10^{-6}$. *Then*

a) *If $1 < p \leq 2$, there exists $A = A(p, f) \geq 1$, such that $\mu_f \ll \mathcal{H}^{\tilde{\lambda}}$.*
b) *If $2 \leq p < \infty$, there exists $A = A(p, f) \leq -1$ such that μ_f is concentrated on a set of $\sigma-$finite $\mathcal{H}^{\tilde{\lambda}}$ measure.*

Note that Theorem D implies

$$\mathcal{H} - \dim \mu_f \begin{cases} \geq 1 & \text{when } 1 < p < 2, \\ = 1 & \text{when } p = 2, \\ \leq 1 & \text{when } 2 < p < \infty. \end{cases}$$

We also note that Theorem D is slightly weaker than Theorem A when $f(\eta) = |\eta|^2$, $\mu_f = \omega$, and Theorem B when $f(\eta) = |\eta|^p$, $1 < p \neq 2 < \infty$, $\mu_f = \mu_p$.

In this paper, we focus on the Hausdorff dimension of μ_f, in the same setting as in Theorem C. More specifically we prove

Theorem 1.11. *Let $O \subset \mathbb{R}^n$ be an open set and $\hat{z} \in \partial O$, $\rho > 0$. Let f be as in (1.1). Let $u > 0$ be a weak solution to $\Delta_f u = 0$ (see 1.4) in $O \cap B(\hat{z}, \rho)$ with continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$, and let μ_f be the measure associated with u as in (1.5). If $p > n$ then μ_f is concentrated on a set of $\sigma-$finite $\mathcal{H}^{n-1}$ measure. The same result holds when $p = n$ provided that $\partial O \cap B(\hat{z}, \rho)$ is locally uniformly fat in the sense of $n-$capacity.*

Remark 1.12. Theorem 1.11 and the definition of the Hausdorff dimension of a measure imply once again that $\mathcal{H} - \dim \mu_f \leq n - 1$ when $p \geq n$.

We also construct for a given f some domains in $\mathbb{R}^n$ for which $\mathcal{H} - \dim \mu_f < n-1$ when $p \geq n$. To give the construction, let $0 < \alpha < \beta < 1/2$ be fixed numbers and let S be the cube in $\mathbb{R}^n$ with side length 1 and centered at 0. Let S' be the cube with side length $a_0 = 1/2$ and centered at 0 and set $\mathcal{C}_0 = S'$. Let $\tilde{Q}_{1,1}, \ldots, \tilde{Q}_{1,2^n}$ be the closed corner cubes of $\mathcal{C}_0$ of side length a_0a_1, $\alpha \le a_1 \le \beta$. Let $\mathcal{C}_1 = \bigcup_{i=1}^{2^n} \tilde{Q}_{1,i}$. Let $\{\tilde{Q}_{2,j}\}$, $j = 1, \dots, 2^{2n}$, be the closed corner cubes of each $\tilde{Q}_{1,i}$, $i = 1, \dots, 2^n$ of side length $a_0a_1a_2$, $\alpha \le a_2 \le \beta$. Let $\mathcal{C}_2 = \bigcup_{j=1}^{2^{2n}} \tilde{Q}_{2,j}$ (see figure 1.1).

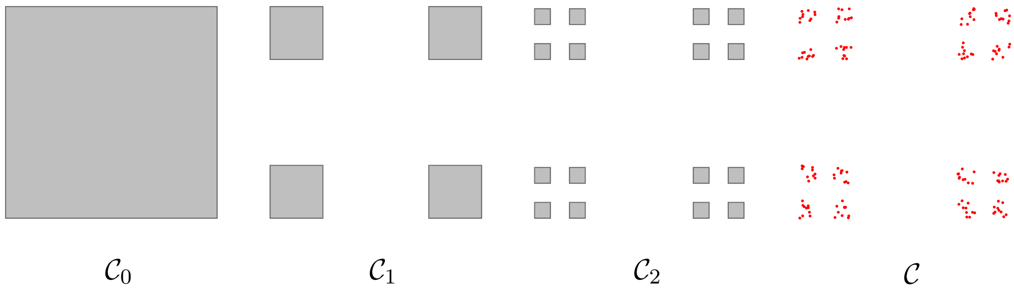


FIGURE 1.1. The sets $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}$ when $n = 2$.

Continuing recursively, at the m th step we get 2^{nm} closed cubes, $\tilde{Q}_{m,j}$, $j = 1, \dots, 2^{nm}$, of side length $a_0a_1a_2 \dots, a_m$, $\alpha \le a_m \le \beta$. Let $\mathcal{C}_m = \bigcup_{j=1}^{2^{nm}} \tilde{Q}_{m,j}$. Then $\mathcal{C}$ is obtained as the limit in the Hausdorff metric of $\mathcal{C}_m$ as $m \to \infty$.

Following an unpublished result of Jones and Wolff (see [10, Chapter IX, Theorem 2.1]), we prove

Theorem 1.13. *Let S be the unit cube and $\mathcal{C}$ be the set constructed above. Let u^∞ be a positive weak solution to (1.4) for fixed $p \ge n$ in $S \setminus \mathcal{C}$ with boundary values 1 on ∂S and 0 on $\mathcal{C}$. Let μ_f^∞ be the measure associated with u^∞ as in (1.5).*
Then $\mathcal{H} - \dim \mu_f^\infty \le n - 1 - \delta$ for some $\delta = \delta(p, n, c_*, \alpha, \beta, f) > 0$.

Moreover, $\delta \ge c^{-1}(p - n)$ where $c \ge 1$ can be chosen to depend only on n, α, β, and c_* in (1.1) when $p \in [n, n + 1]$.

If $f = g^p$ where g is homogeneous of degree 1, uniformly convex, and has continuous second partials, then δ can be chosen independent of $p \in [n, n + 1]$, so depends only on n, α, β, g.

In what follows, we state some regularity results for u in section 2. In section 3, we show that $\log f(\nabla u)$ is a weak sub solution to $\tilde{L}$ when $p \ge n$ where $\tilde{L}$ is as in (1.10) with 2 replaced by n in the summation. In section 4 we prove more advanced regularity results and essentially begin the proof of Theorem 1.11. In section 5, we prove a proposition and finish the proof of Theorem 1.11. In section 6, we prove Theorem 1.13.

In general to prove Theorem 1.11 we follow the proof of Theorem C which in turn made effective use of the proof scheme in [12, 23]. However the proof that $\log f(\nabla u)$ is a weak sub solution to $\tilde{L}$ is more involved, and in fact somewhat surprising to us, than the corresponding proof for $f(\nabla u) = |\nabla u|^p$, since in this case we could use rotational invariance of the p Laplace equation to considerably simplify the calculations. Also regularity results for $u, \nabla u, \log f(\nabla u)$, require more care than in [3] due to the nearly endpoint structural assumptions on f in (1.1).

Likewise to prove Theorem 1.13, we use the proof scheme in [10, chapter IX] only now we have little control over the zeros of ∇u. This lack of control forces us into an alternative finess type argument which produces the 'hodge podge' of results on δ in Theorem 1.13, rather than what we hoped to prove, namely $\delta > a > 0$ on $[n, n+1]$ (provided c_* in (1.1) is constant for $p \in [n, n+1]$).

2. Notation and Preparatory Lemmas

Let $x = (x_1, \ldots, x_n)$ denote points in $\mathbb{R}^n$ and let $\overline{E}$, ∂E, be the closure and boundary of the set $E \subset \mathbb{R}^n$. Let $\langle \cdot, \cdot \rangle$ be the usual inner product in $\mathbb{R}^n$ and $|x|^2 = \langle x, x \rangle$. Let $d(E, F)$ denote the distance between the sets E and F. Let $B(x, r)$ be the open ball centered at x with radius $r > 0$ in $\mathbb{R}^n$ and let dx denote Lebesque $n-$measure in $\mathbb{R}^n$. Given O' an open set $\subset \mathbb{R}^n$ and $q, 1 \leq q \leq \infty$, let $W^{1,q}(O')$ denote equivalence classes of functions $h : \mathbb{R}^n \to \mathbb{R}$ with distributional gradient $\nabla h = \langle h_{x_1}, \ldots, h_{x_n} \rangle$, both of which are q th power integrable on O' with Sobolev norm

$$\|h\|^q_{W^{1,q}(O')} = \int_{O'} (|h|^q + |\nabla h|^q) \mathrm{d}x.$$

Let $C_0^\infty(O')$ be the set of infinitely differentiable functions with compact support in O' and let $W_0^{1,q}(O')$ be the closure of $C_0^\infty(O')$ in the norm of $W^{1,q}(O')$.

Let $K \subset \overline{B}(x, r)$ be a compact set and let $\mathfrak{A} := \{\phi \in W_0^{1,n}(B(x, 2r)) : \ \phi \equiv 1 \text{ on } K\}$. We let

$$\text{Cap}(K, B(x, 2r)) := \inf_{\phi \in \mathfrak{A}} \int_{\mathbb{R}^n} |\nabla \phi|^n \mathrm{d}x. \tag{2.1}$$

We say that a compact set $K \subset \mathbb{R}^n$ is *locally* (n, r_0) *uniformly fat* or *locally uniformly* (n, r_0) *thick* provided there exists r_0 and c such that whenever $x \in K$ and $0 \leq r \leq r_0$,

$$\text{Cap}(K \cap \overline{B}(x, r), B(x, 2r)) \geq c > 0.$$

In the sequel, c will denote a positive constant ≥ 1 (not necessarily the same at each occurrence), which may depend only on p, n, c_* unless otherwise stated. In general, $c(a_1, \ldots, a_n)$ denotes a positive constant ≥ 1 which may depend only on $p, n, c_*, a_1, \ldots, a_n$ not necessarily the same at each occurrence. $A \approx B$ means that A/B is bounded above and below by positive constants depending only on p, n, c_*.

In this section, we will always assume that $2 \leq n \leq p < \infty$, and $r > 0$. We also assume that $\tilde{O}$ is an open set in $\mathbb{R}^n$ and $w \in \partial \tilde{O}$.

We begin by stating some interior and boundary estimates for a positive weak solution $\tilde{u}$ to (1.4) in $\tilde{O} \cap B(w, 4r)$. If $p = n$, we assume $\partial \tilde{O} \cap \overline{B}(w, 4r)$ is (n, r_0) uniformly fat as defined above using the capacity in (2.1). We assume that $\tilde{u}$ has zero boundary value on $\partial \tilde{O} \cap B(w, 4r)$ in the Sobolev sense and we extend $\tilde{u}$ as above by putting $\tilde{u} \equiv 0$ on $B(w, 4r) \setminus \tilde{O}$. Then as in (1.5) let $\tilde{\mu}_f$ be the positive Borel measure corresponding to $\tilde{u}$.

References for the proofs of Lemmas 2.2-2.6 can be found in [3] where these lemmas are stated for $f(\eta) = |\eta|^p$, however they also hold for f as in (1.1). Let c_* be as in (1.1).

Lemma 2.2. *Let $\tilde{O}, w, r, \tilde{u}, f, \tilde{\mu}_f$ be as above. Then there exists constant $c = c(p, n, c_*)$ such that*

$$\frac{1}{c} r^{p-n} \int_{B(w, \frac{r}{2})} f(\nabla \tilde{u}) \mathrm{d}x \leq \operatorname*{ess\,sup}_{B(w,r)} \tilde{u}^p \leq c \frac{1}{r^n} \int_{B(w, 2r)} \tilde{u}^p \mathrm{d}x.$$

If $B(z,2r') \subset \tilde{O} \cap B(w,4r)$ for some $r'>0$ then there is a constant $c = c(p,n,c_*)$ such that

$$\operatorname*{ess\,sup}_{B(z,r')} \tilde{u} \leq c \operatorname*{ess\,inf}_{B(z,r')} \tilde{u}.$$

Lemma 2.3. *Let $\tilde{O}, w, r, \tilde{u}, f$ be as in Lemma 2.2. Then there is $\alpha' = \alpha'(p,n,c_*) \in (0,1)$ and $c = c(p,n,c_*)$, such that $\tilde{u}$ has a Hölder continuous representative in $B(w,4r)$ (also denoted $\tilde{u}$). If $\tilde{w}, \hat{w} \in B(w,r)$ then*

$$|\tilde{u}(\tilde{w}) - \tilde{u}(\hat{w})| \leq c \left(\frac{|\tilde{w}-\hat{w}|}{r} \right)^{\alpha'} \operatorname*{ess\,sup}_{B(w,2r)} \tilde{u}\,.$$

Lemma 2.4. *Let $\tilde{O}, w, r, \tilde{u}, f, \tilde{\mu}_f$ be as in Lemma 2.2. Then there exists $c = c(p,n,c_*) \geq 1$ such that*

$$\tfrac{1}{c}\, r^{p-n} \tilde{\mu}_f(B(w, \tfrac{r}{2})) \leq (\operatorname*{ess\,sup}_{B(w,r)} \tilde{u})^{p-1} \leq c\, r^{p-n} \tilde{\mu}_f(B(w,2r)).$$

Remark 2.5. The left-hand side of the inequality in Lemma 2.4 is true for any open $\tilde{O}$ and $p \geq n$. However, the right-hand side of this inequality requires uniform fatness when $p=n$ and that is the main reason why the uniform fatness assumption appears in Theorem 1.11.

Lemma 2.6. *Let $\tilde{O}, w, r, \tilde{u}, f$ be as in Lemma 2.2. Then $\tilde{u}$ has a representative in $W^{1,p}(B(w,4r))$ with Hölder continuous derivatives in $\tilde{O} \cap B(w,4r)$. In particular, there exists α'', $0 < \alpha'' < 1$, and $c \geq 1$, depending only on p, n, c_*, with*

$$|\nabla \tilde{u}(x) - \nabla \tilde{u}(y)| \leq c \left(\frac{|x-y|}{\hat{r}} \right)^{\alpha''} \operatorname*{ess\,sup}_{B(\tilde{w},\hat{r})} |\nabla \tilde{u}| \leq \frac{c}{\hat{r}} \left(\frac{|x-y|}{\hat{r}} \right)^{\alpha''} \operatorname*{ess\,sup}_{B(\tilde{w},\hat{r})} \tilde{u}.$$

whenever $x, y \in B(\tilde{w}, \hat{r}/2)$, and $B(\tilde{w}, 4\hat{r}) \subset \tilde{O} \cap B(\tilde{w}, 4r)$.

Moreover,

$$\int\limits_{B(\tilde{w},\hat{r})} |\nabla \tilde{u}|^{p-2} \sum_{k,j=1}^{n} (\tilde{u}_{x_k x_j})^2 \mathrm{d}x \leq \frac{c}{\hat{r}^2} \int\limits_{B(\tilde{w},2\hat{r})} |\nabla \tilde{u}|^p \mathrm{d}x.$$

Lemma 2.7. *Let $\tilde{O}, w, r, \tilde{u}$ be as in Lemma 2.4. Suppose for some $z \in \mathbb{R}^n, t \geq 100r$, that $w \in \partial B(z,t)$ and*

$$B(w,4r) \setminus \bar{B}(z,t) = B(w,4r) \cap \tilde{O}.$$

Then there exists $\alpha''' = \alpha'''(p,n,f) \in (0,1)$ for which $\tilde{u}|_{\tilde{O} \cap B(w,3r)}$ has a $C^{1,\alpha'''} \cap W^{1,p}$ extension to the closure of $B(w,3r) \setminus \bar{B}(z,t)$ (denoted $\bar{u}$). Moreover,

$$\int\limits_{\tilde{O} \cap B(w,r/2) \cap \{|\nabla \bar{u}|>0\}} |\nabla \bar{u}|^{p-2} \sum_{j,k=1}^{n} \bar{u}^2_{x_j x_k} dx \leq \frac{c}{r^2} \int\limits_{\tilde{O} \cap B(w,2r)} |\nabla \bar{u}|^p \, dx$$

and if $y, \tilde{y} \in \tilde{O} \cap B(w,r/2)$, then

$$\tfrac{1}{c} |\nabla \bar{u}(y) - \nabla \bar{u}(\tilde{y})| \leq \left(\tfrac{|y-\tilde{y}|}{r} \right)^{\alpha'''} \max_{\tilde{O} \cap \bar{B}(w,r)} |\nabla \bar{u}| \leq \tfrac{c}{r} \left(\tfrac{|y-\tilde{y}|}{r} \right)^{\alpha'''} \max_{\tilde{O} \cap B(w,2r)} \bar{u}.$$

Proof. Lieberman in [19] essentially proves the above lemma. A careful reading of his paper gives the second estimate in this lemma as well as the fact that $|\nabla \bar{u}| \geq c^{-1}$ in $B(\zeta, r/c)$ whenever $\zeta \in \partial B(z,t) \cap B(w, 7r/2)$ where $c \geq 1$ depends only on p, n, and the structure constants for f. The first estimate then follows from Hölder continuity of derivatives, the fact that derivatives of $\bar{u}$ satisfy a uniformly elliptic PDE in divergence form near $\partial B(z,t) \cap B(w,3r)$ (see (3.2)), and a Caccioppoli inequality. $\square$

3. Sub solution estimate

Let $\tilde L$ be defined as in (1.10) with 2 replaced by n in the summation. That is,

$$\tilde L\zeta = \sum_{k,j=1}^{n} \frac{\partial}{\partial x_k}\left(f_{\eta_j\eta_k}(\nabla\tilde u)\frac{\partial\zeta}{\partial x_j}\right). \tag{3.1}$$

Let $\tilde v(x) = \log f(\nabla\tilde u(x))$ for $x \in \tilde O \cap B(w,4r)$. In this section we first show that $\tilde L\tilde v \geq 0$ weakly in a domain $\Omega \subset \tilde O \cap B(w,4r)$ when $p \geq n$ and $\nabla\tilde u \neq 0$ in Ω. To do so we note that Lemma 2.6 implies $\tilde u$ is locally in $W^{2,2}(\Omega)$ so (1.4) holds almost everywhere in Ω. It follows that for $l = 1, 2, \dots, n$,

$$\begin{aligned} 0 &= \int_\Omega \langle \mathcal{D}f(\nabla\tilde u), \nabla\phi_{x_l}\rangle \mathrm{d}x = -\int_\Omega \sum_{k=1}^{n} \frac{\partial(f_{\eta_k}(\nabla\tilde u))}{\partial x_l}\,\phi_{x_k}\mathrm{d}x \\ &= -\int_\Omega \sum_{k,j=1}^{n} f_{\eta_k\eta_j}(\nabla\tilde u)(\tilde u_{x_l})_{x_j}\phi_{x_k}\mathrm{d}x. \end{aligned} \tag{3.2}$$

whenever $\phi \in C_0^\infty(\Omega)$ and non-negative. Therefore, $\zeta = \tilde u_{x_l}$, $l = 1, \dots, n$, is a weak solution to (3.1). From (1.3) we also have

$$\int_\Omega \sum_{k,j=1}^{n} f_{\eta_j\eta_k}(\nabla\tilde u)\tilde u_{x_j}\phi_{x_k}\,\mathrm{d}x = (p-1)\int_\Omega \sum_{k=1}^{n} f_{\eta_k}(\nabla\tilde u)\phi_{x_k}\mathrm{d}x = 0. \tag{3.3}$$

From (3.3) we deduce that $\zeta = \tilde u$ is also a weak solution to (3.1). Let $\mathfrak{b}_{kj} = f_{\eta_k\eta_j}(\nabla\tilde u)$ and observe that for almost every $x \in \Omega$, where $\nabla\tilde u(x) \neq 0$,

$$\mathfrak{b}_{kj}\tilde v_{x_j} = \frac{\mathfrak{b}_{kj}}{f(\nabla\tilde u)}\sum_{m=1}^{n} f_{\eta_m}(\nabla\tilde u)\tilde u_{x_m x_j}. \tag{3.4}$$

Using (3.4) we find that

$$\begin{aligned} \int_\Omega \sum_{k,j=1}^{n} \mathfrak{b}_{kj}\tilde v_{x_j}\phi_{x_k}\,\mathrm{d}x &= \int_\Omega \sum_{k,j=1}^{n} \frac{\mathfrak{b}_{kj}}{f(\nabla\tilde u)}\sum_{m=1}^{n} f_{\eta_m}(\nabla\tilde u)\tilde u_{x_m x_j}\phi_{x_k}\,\mathrm{d}x \\ &= -\int_\Omega \sum_{m,k,j=1}^{n} \frac{\partial}{\partial x_k}\left(\frac{f_{\eta_m}(\nabla\tilde u)}{f(\nabla\tilde u)}\right)\mathfrak{b}_{kj}\tilde u_{x_m x_j}\,\phi\,\mathrm{d}x \end{aligned} \tag{3.5}$$

where to get the last line in (3.5) we have used

$$0 = \int_\Omega \sum_{m,k,j=1}^{n} \mathfrak{b}_{kj}\tilde u_{x_m x_j}\frac{\partial}{\partial x_k}\left(\frac{f_{\eta_m}(\nabla\tilde u)}{f(\nabla\tilde u)}\phi\right)\mathrm{d}x. \tag{3.6}$$

(3.6) is a consequence of (3.2) with $m = l$ and ϕ replaced by $\frac{f_{\eta_m}(\nabla\tilde u)}{f(\nabla\tilde u)}\phi$ as well as the fact that

$$\frac{f_{\eta_m}(\nabla\tilde u)}{f(\nabla\tilde u)} \in W^{1,2}_{\mathrm{loc}}(\Omega).$$

From (3.5) we have

$$\begin{aligned} \int_\Omega \sum_{k,j=1}^{n} \mathfrak{b}_{kj}\tilde v_{x_j}\phi_{x_k}\,\mathrm{d}x &= -\int_\Omega \sum_{m,k,j=1}^{n} \frac{\partial}{\partial x_k}\left(\frac{f_{\eta_m}(\nabla\tilde u)}{f(\nabla\tilde u)}\right)\mathfrak{b}_{kj}\tilde u_{x_m x_j}\,\phi\,\mathrm{d}x \\ &= -\int_\Omega (I' + I'')\phi\mathrm{d}x \end{aligned} \tag{3.7}$$

where (after taking the x_k derivative of the term)

$$
\begin{aligned}
I' &= \sum_{m,j,k,l=1}^{n} \tfrac{1}{f(\nabla\tilde{u})}\, \mathfrak{b}_{ml}\mathfrak{b}_{kj}\tilde{u}_{x_l x_k}\tilde{u}_{x_m x_j}, \\
I'' &= -\tfrac{1}{f^2(\nabla\tilde{u})} \sum_{m,j,k,l=1}^{n} \mathfrak{b}_{kj} f_{\eta_m}(\nabla\tilde{u}) f_{\eta_l}(\nabla\tilde{u})\tilde{u}_{x_l x_k}\tilde{u}_{x_m x_j}.
\end{aligned} \tag{3.8}
$$

To simplify computation in (3.8) we use matrix notation. If $f = f(\nabla\tilde{u})$, $f_{\eta_k}(\nabla\tilde{u}) = \mathfrak{b}_k$, $1 \leq k \leq n$, then we first observe by reordering the terms in (3.8) that

$$
(I' + I'')f = \sum_{m,j,k,l=1}^{n} [\mathfrak{b}_{nl}\tilde{u}_{x_l x_k}\mathfrak{b}_{kj}\tilde{u}_{x_j x_m} - \frac{1}{f}\,\mathfrak{b}_l\tilde{u}_{x_l x_k}\mathfrak{b}_{kj}\tilde{u}_{x_j x_m}\mathfrak{b}_m].
$$

Let $A = (\tilde{u}_{x_i x_j})$ and $B = (\mathfrak{b}_{ij})$, then for almost every $x \in \Omega$,

$$
(I' + I'')f = \text{tr } (BA)^2 - \frac{1}{f}\frac{1}{(p-1)^2}\nabla\tilde{u}BABAB(\nabla\tilde{u})^t \tag{3.9}
$$

where we have used (1.3) to replace $\mathfrak{b}_l$. We look at

$$
\frac{\zeta\, BABAB\,\zeta^t}{\zeta\, B\,\zeta^t} = \frac{\text{tr}(\zeta\, BABAB\,\zeta^t)}{\text{tr}(\zeta\, B\,\zeta^t)}.
$$

Observe from (1.1) that B is positive definite symmetric, A is symmetric, and from (1.4) that $\text{tr}(AB) = \text{tr}(BA) = 0$. Using these facts we see there exists $\mathcal{S}$ an orthogonal matrix so that $\mathcal{S}^t B\mathcal{S} = B_d$ is diagonal. Let $B'_d = B_d^{1/2}$ be the obvious square root of each component of B_d so that $B'_d\, B'_d = B_d$. With $A_1 = \mathcal{S}^t A\mathcal{S}$, it follows that

$$
\begin{aligned}
\frac{\zeta\, BABAB\,\zeta^t}{\zeta\, B\,\zeta^t} &= \frac{\zeta\,\mathcal{S}\mathcal{S}^t B\mathcal{S}\mathcal{S}^t A\mathcal{S}\mathcal{S}^t B\mathcal{S}\mathcal{S}^t A\mathcal{S}\mathcal{S}^t B\mathcal{S}\mathcal{S}^t\,\zeta^t}{\zeta\,\mathcal{S}\mathcal{S}^t B\,\mathcal{S}\mathcal{S}^t\zeta^t} \\
&= \frac{\zeta\mathcal{S}\, B_d A_1 B_d A_1 B_d\,\mathcal{S}^t\zeta^t}{\zeta\mathcal{S}\, B_d\,\mathcal{S}^t\zeta^t}.
\end{aligned}
$$

If $\xi = \zeta\mathcal{S} \neq 0$, then

$$
\frac{\zeta\, BABAB\,\zeta^t}{\zeta\, B\,\zeta^t} = \frac{\xi\, B_d A_1 B_d A_1 B_d\,\xi^t}{\xi\, B_d\,\xi^t} = \frac{\xi B'_d\, B'_d A_1 B'_d B'_d A_1 B'_d\, B'_d\xi^t}{\xi B'_d\, B'_d\xi^t}.
$$

Set $y = \xi B'_d \neq 0$, $E = B'_d A_1 B'_d$, and note that E is symmetric as B'_d, $A_1 = \mathcal{S}^t A\mathcal{S}$, and A are symmetric;

$$
\frac{\zeta\, BABAB\,\zeta^t}{\zeta\, B\,\zeta^t} = \frac{y\, B'_d A_1 B'_d B'_d A_1 B'_d\, y^t}{y\, y^t} = \frac{y\, EE\, y^t}{y\, y^t}.
$$

Now one can easily prove the following properties of trace;

$$
\begin{aligned}
&(i)\ \text{tr}(FGH) = F_{ij}G_{jk}H_{ki} = H_{ki}F_{ij}G_{jk} = \text{tr}(HFG), \\
&(ii)\ \text{tr}(P^{-1}GP) = \text{tr}(G)
\end{aligned}
$$

whenever F, G, H are matrices. Here (ii) follows easily from property (i) whenever P is an orthogonal matrix. From these properties (i)-(ii) we have

$$
\text{tr}(E) = \text{tr}(B'_d A_1 B'_d) = \text{tr}(B'_d B'_d A_1) = \text{tr}(B_d A_1) = \text{tr}(\mathcal{S}^t B\mathcal{S}\mathcal{S}^t A\mathcal{S}) = \text{tr}(BA).
$$

Therefore, we have $\text{tr}(E) = \text{tr}(BA) = \text{tr}(AB) = 0$. Similarly,

$$
\text{tr}(E^2) = \text{tr}((AB)^2)
$$

Now diagonalize E using another orthogonal matrix $\mathcal{S}_1$, so that $\mathcal{S}_1^t E \mathcal{S}_1 = E_d$ with the ijth entries given by $(E_d)_{ij} = e_i \delta_{ij}$. Then

$$\begin{aligned} \operatorname{tr}(E) &= \operatorname{tr}(E_d) = \sum_{i=1}^n e_i = 0, \\ \operatorname{tr}(E^2) &= \operatorname{tr}(\mathcal{S}_1^t E \mathcal{S}_1 \mathcal{S}_1^t E \mathcal{S}_1) = \operatorname{tr}(E_d^2) = \sum_{i=1}^n e_i^2. \end{aligned} \tag{3.10}$$

Moreover,

$$\frac{\zeta\, BABAB\, \zeta^t}{\zeta\, B\, \zeta^t} = \frac{y\mathcal{S}_1\mathcal{S}_1^t E \mathcal{S}_1 \mathcal{S}_1^t E \mathcal{S}_1 \mathcal{S}_1^t y^t}{y \mathcal{S}_1 \mathcal{S}_1^t y^t}$$

so that with $z = y\mathcal{S}_1 \neq 0$ we also have

$$\frac{\zeta\, BABAB\, \zeta^t}{\zeta\, B\, \zeta^t} = \frac{z E_d E_d z^t}{z z^t} = \frac{\sum_{i=1}^n e_i^2 z_i^2}{\sum_{i=1}^n z_i^2}. \tag{3.11}$$

Let $\kappa = z/|z|$ so that κ is a unit vector, then (3.11) implies

$$0 \leq \frac{\zeta\, BABAB\, \zeta^t}{\zeta\, B\, \zeta^t} = \sum_{i=1}^n e_i^2 \kappa_i^2. \tag{3.12}$$

Without loss of generality assume that e_1^2 is the largest of the e_k^2 then considering all possible unit vectors κ in (3.12) we see that

$$\frac{\zeta\, BABAB\, \zeta^t}{\zeta\, B\, \zeta^t} \leq \sup_{|\kappa|=1} \sum_{i=1}^n e_i^2 \kappa_i^2 = e_1^2. \tag{3.13}$$

Combining (3.9), (3.10), (3.12), and (3.13) we have

$$(I' + I'')f = \operatorname{tr}(BA)^2 - \frac{p}{p-1} \frac{\zeta\, BABAB\, \zeta^t}{\zeta\, B\, \zeta^t} \geq \sum_{i=1}^n e_i^2 - \frac{p}{p-1} e_1^2. \tag{3.14}$$

Now we can use (3.10) to get

$$e_1 = -(\sum_{i=2}^n e_i) \ \text{ and } \ e_1^2 = (\sum_{i=2}^n e_i)^2 \leq (n-1) \sum_{i=2}^n e_i^2. \tag{3.15}$$

Using (3.15) in (3.14) we have

$$(I' + I'')f \geq e_1^2 + \frac{1}{n-1} e_1^2 - \frac{p}{p-1} e_1^2 = e_1^2 \left(\frac{n}{n-1} - \frac{p}{p-1} \right) \tag{3.16}$$

Finally $\frac{t}{t-1}$ is decreasing on $t > 1$ so that for $p \geq n$ we see that $(I' + I'')f \geq 0$. Combining (3.7) and (3.16) we deduce that

$$\begin{aligned} \int_\Omega \sum_{k,j=1}^n \mathfrak{b}_{kj} \tilde{v}_{x_j} \phi_{x_k} \, \mathrm{d}x &= -\int_\Omega (I' + I'') \phi \mathrm{d}x \\ &\leq -\left(\frac{n}{n-1} - \frac{p}{p-1} \right) \int_\Omega \frac{e_1^2(x)}{f(\nabla \tilde{u}(x))} \phi \mathrm{d}x \\ &\leq 0. \end{aligned} \tag{3.17}$$

whenever $\phi \in C_0^\infty(\Omega)$ and non-negative. It follows from (3.17) that $\tilde{L}\tilde{v} \geq 0$ weakly in Ω when $p \geq n$.

Let δ_{jk} denote the Kronecker delta in the following lemma.

Lemma 3.18. *Let $\tilde{O}, w, r, \tilde{u}, f$ be as in Lemma 2.2. Let $-\infty < \theta \leq -1$. Let $\tilde{L}$ be defined as in (3.1) and $\tilde{v} = \log f(\nabla \tilde{u})$ when $x \in \tilde{O} \cap B(w, 4r)$ and $\nabla \tilde{u}(x) \neq 0$. Let $f_{\eta_j \eta_k} = \delta_{jk}$ when $\nabla \tilde{u}(x) = 0$ for $1 \leq j, k \leq n$. If $v' = \max\{\log f(\nabla \tilde{u}), \theta\}$ then $\zeta = v'$ is locally a weak sub solution to $\tilde{L}\zeta = 0$ in $\tilde{O} \cap B(w, 4r)$.*

Proof. From Lemma 2.6 we see that v' is locally in $W^{1,2}(\tilde{O} \cap B(w, 4r))$. Given $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, small, define

$$g(x) := (\max\{v'(x) - \theta - \varepsilon_1, 0\} + \varepsilon_2)^{\varepsilon_3} - \varepsilon_2^{\varepsilon_3}, \ x \in \tilde{O} \cap B(w, 4r).$$

It follows from (1.1) and $\tilde{L}v' \geq 0$ weakly at $x \in \tilde{O} \cap B(w, 4r)$ when $v'(x) \neq \theta$ (almost everywhere), that

$$\begin{aligned} 0 &\leq -\sum_{j,k=1}^{n} \int\limits_{\tilde{O} \cap B(w,4r)} f_{\eta_j \eta_k}(\nabla \tilde{u})(\phi g)_{x_j} v'_{x_k} \mathrm{d}x \\ &\leq -\sum_{j,k=1}^{n} \int\limits_{\tilde{O} \cap B(w,4r)} g f_{\eta_j \eta_k}(\nabla \tilde{u}) \phi_{x_j} v'_{x_k} \mathrm{d}x. \end{aligned} \tag{3.19}$$

whenever $\phi \in C_0^\infty(\tilde{O} \cap B(w, 4r))$ and non-negative. Using (3.19), the bounded convergence theorem, and letting first $\varepsilon_1 \to 0$, then $\varepsilon_2 \to 0$, and finally $\varepsilon_3 \to 0$, we get Lemma 3.18 as desired. □

4. Advanced Regularity Results

In this section we begin the proof of Theorem 1.11 by proving three lemmas. To this end, let $O, f, u, \hat{z}, \rho, \mu_f, p, n$ be as in Theorem 1.11.

Lemma 4.1. *There exists a constant $c = c(p, n, c_*)$ and a set $Q \subset \partial O \cap B(\hat{z}, \rho)$ such that*

$$\mu_f((\partial O \cap B(\hat{z}, \rho)) \setminus Q) = 0.$$

Moreover, for every $w \in Q$ there exists arbitrarily small $r = r(w)$, $0 < r \leq 10^{-10}$, such that

$$\overline{B}(w, 100r) \subset B(\hat{z}, \rho) \text{ and } \mu_f(B(w, 100r)) \leq c\, \mu_f(B(w, r)).$$

Proof. It follows from Lemma 2.4 that $\mu_f(B(x, t)) > 0$ whenever $x \in \partial O$ and $\partial O \cap B(x, t) \subset \partial O \cap B(\hat{z}, \rho)$. We show for $c > 0$ large enough that $\mu_f(\Theta) = 0$ where

$$\Theta := \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \liminf_{t \to 0} \frac{\mu_f(B(x, 100t))}{\mu_f(B(x, t))} \geq c \right\}.$$

Then the desired set Q in Lemma 4.1 will be the complement of Θ, i.e, $Q = (\partial O \cap B(\hat{z}, \rho)) \setminus \Theta$. To show that $\mu_f(\Theta) = 0$, we first see from the definition of Θ that for every $x \in \Theta$ there exists $t_0 = t_0(x)$ with

$$(c/2)\mu_f(B(x, t)) \leq \mu_f(B(x, 100t)) \text{ for every } t \in (0, t_0). \tag{4.2}$$

Then iterating (4.2) we obtain

$$\lim_{t \to 0} \frac{\mu_f(B(x, t))}{t^{n+1}} = 0 \text{ whenever } x \in \Theta$$

provided c in (4.2) is large enough. It follows that $\mu_f|_\Theta$ is absolutely continuous with respect to $\mathcal{H}^{n+1}$ measure. Since $\mathcal{H}^{n+1}(\mathbb{R}^n) = 0$ we conclude from our earlier remark that Lemma 4.1 is true. □

Next using translation and dilation invariance of (1.4), we work in a different domain. To this end, let

$$w \in Q \subset \partial O \cap B(\hat{z}, \rho)$$

be fixed and let $r = r(w)$ be a corresponding radius as in Lemma 4.1. We first set

$$u'(x) := \frac{u(w + rx)}{\operatorname*{ess\,sup}_{B(w,10r)} u} \text{ when } w + rx \in B(\hat{z}, \rho)$$

and define

$$\Omega' := \{x : \; w + rx \in O \cap B(\hat{z}, \rho)\}.$$

We observe that u' is a weak solution to (1.4) in Ω' as (1.4) is invariant under translation and dilation. Moreover, $u' > 0$ is continuous in $B(\zeta, \rho/r)$ with $u' \equiv 0$ on $B(\zeta, \rho/r) \setminus \Omega'$ provided that $\zeta = (\hat{z} - w)/r$. As in (1.5), there exists a finite Borel measure μ'_f on $\mathbb{R}^n$ with support in $\partial\Omega' \cap \overline{B}(\zeta, \rho/r)$ associated with u'.

We also note that

$$\mu'_f(E) = \frac{r^{p-n}}{\left(\operatorname*{ess\,sup}_{B(w,10r)} u\right)^{p-1}} \mu_f(\Xi(E))$$

whenever E is a Borel set and $\Xi(E) := \{w + rx : \; x \in E\}$.

As (1.4) is invariant under translation and dilation without loss of generality we can assume that $w = 0$, $r = 1$ with $B(0, 100) \subset B(\hat{z}, \rho)$. From Lemmas 2.4 and 4.1, we obtain for some $c = c(p, n, c_*) \geq 1$ and $2 \leq t \leq 50$ that

$$c^{-1} \leq \mu'_f(B(0,1)) \leq \operatorname*{ess\,sup}_{B(0,2)} u' \leq \operatorname*{ess\,sup}_{B(0,t)} u' \leq c\, \mu'_f(B(0,100)) \leq c^2. \tag{4.3}$$

By definition of u' and Hölder continuity of u near ∂O, it is easily seen that there exists some $\tilde{z} \in \partial B(0, 10)$ with $u'(\tilde{z}) = 1$, and

$$c_-^{-1} \leq d(\tilde{z}, \partial\Omega') \text{ for some } c_- = c_-(p, n, c_*) \geq 1. \tag{4.4}$$

Let M be a large number where we allow M to vary but shall fix it to satisfy several conditions after (5.8). After that we choose $s = s(M) > 0$ sufficiently small with $0 < s << e^{-M}$. Let δ, δ' be given such that $0 < \delta' < \min(\delta, 10^{-5})$ and choose $M > 0$ so large that

$$\text{if } \mu'_f(B(z,t)) = M t^{n-1} \text{ for some } t = t(z) \leq 1 \text{ then } t \leq \delta' \tag{4.5}$$

where $z \in \partial\Omega' \cap \overline{B}(0, 15)$. Existence of such $M = M(\delta') \geq 1$ follows from (4.3). Following [23], we observe from (4.5) for each $z \in \partial\Omega' \cap \overline{B}(0, 15)$ that there exists a largest t with $s \leq t \leq 1$ such that either

$$\begin{aligned} &(a)\ \mu'_f(B(z,t)) = M t^{n-1}\ , t > s \\ &\text{or} \\ &(b)\ t = s. \end{aligned} \tag{4.6}$$

Using the Besicovitch covering theorem (see [21]) we now obtain a covering $\{B(z_k, t_k)\}_{k=1}^N$ of $\partial\Omega' \cap \overline{B}(0, 15)$, where t_k satisfies either (a) or (b) in (4.6). Then each point of $\bigcup_{k=1}^N B(z_k, t_k)$ lies in at most $c = c(n)$ of $\{B(z_k, t_k)\}_{k=1}^N$. Let $\mathcal{G} = \mathcal{G}_M$ and $\mathcal{B} = \mathcal{B}_M$ be the set of all balls in this covering for which (a) and (b) in (4.6) hold respectively.

Let c_- and $\tilde{z}$, be as in (4.4) and set $r_1 = (8c_-)^{-1}$. Choosing δ' smaller (so M larger) if necessary we may assume, thanks to (4.5), that

$$\bigcup_{k=1}^{N} \overline{B}(z_k, 6t_k) \cap B(\tilde{z}, 6r_1) = \emptyset. \tag{4.7}$$

Also put

$$\Omega'' = \Omega' \cap B(0,15) \setminus \bigcup_{k=1}^{N} \overline{B}(z_k, t_k) \text{ and } D = \Omega'' \setminus \overline{B}(\tilde{z}, 2r_1).$$

Let u'' be a positive weak solution to (1.4) in D with continuous boundary values,

$$u''(x) \equiv \begin{cases} 0 & \text{when } \; x \in \partial\Omega'' \\ \operatorname*{ess\,inf}_{\overline{B}(\tilde{z},2r_1)} u' & \text{when } x \in \partial B(\tilde{z}, 2r_1). \end{cases}$$

We extend u'' continuously to $\overline{B}(0,15)$ (also denoted u'') by putting

$$u''(x) \equiv \begin{cases} 0 & \text{when } \; x \in \overline{B}(0,15) \setminus \Omega'' \\ \operatorname*{ess\,inf}_{\overline{B}(\tilde{z},2r_1)} u' & \text{when } x \in \overline{B}(\tilde{z}, 2r_1). \end{cases}$$

We note that $u'' \leq u'$ on ∂D so by the maximum principle for weak solutions to (1.4) we have $u'' \leq u'$ in D. Also, ∂D is locally (n, r_0') uniformly fat where r_0' depends only on n and r_0 in Theorem 1.11 when $p = n$. Next we prove

Lemma 4.8. *For all $x \in D$ we have $|\nabla u''| \leq c M^{\frac{1}{p-1}}$ where $c = c(p, n, c_*)$.*

Proof. Let $x \in D$, and choose $y \in \partial D$ such that $|x - y| = d(x, \partial D) = d$. We first prove Lemma 4.8 when $y \in \partial B(z_k, t_k)$ and $x \in B(z_k, 2t_k)$. The same reasoning can be applied when $y \in \partial B(0, 15)$ or $y \in \partial B(z, 2r_1)$. To this end, let $\epsilon > 0$ be given and set

$$f^{\epsilon}(\eta) := \int_{\mathbb{R}^n} f(x) \psi_{\epsilon}(\eta - x) \, \mathrm{d}x$$

where $\psi \in C_0^{\infty}(B(0,1))$ with

$$\int_{\mathbb{R}^n} \psi \mathrm{d}x = 1 \text{ and } \psi_{\epsilon}(x) = \frac{1}{\epsilon^n} \psi\left(\frac{x}{\epsilon}\right) \text{ whenever } x \in \mathbb{R}^n.$$

We note that f^{ϵ} is no longer homogeneous but f^{ϵ} is infinitely differentiable. Moreover, whenever $\eta, \xi \in \mathbb{R}^n$ we have

$$c^{-1}(\epsilon + |\eta|)^{p-2} \, |\xi|^2 \; \leq \; \sum_{j,k=1}^{n} \frac{\partial^2 f^{\epsilon}}{\partial \eta_j \eta_k}(\eta) \xi_j \xi_k \leq \; c(\epsilon + |\eta|)^{p-2} |\xi|^2 \tag{4.9}$$

where $c = c(p, n, c_*) \geq 1$. Let u''_{ϵ} be a weak solution to (1.4) in D with f replaced by f^{ϵ} and the same continuous boundary values as u''. Then (1.5) holds with f, u replaced by $f^{\epsilon}, u''_{\epsilon}$. Using (4.9), an analogue of Lemma 2.6, and Schauder type estimates we see that u''_{ϵ} is infinitely differentiable in Ω'' and that $\zeta = u''_{\epsilon}$ is a pointwise solution to $L^{\star}\zeta = 0$ where

$$L^{\star}\zeta := \frac{1}{(\epsilon + |\nabla u''_{\epsilon}|)^{p-2}} \sum_{j,k=1}^{n} f^{\epsilon}_{\eta_j \eta_k}(\nabla u''_{\epsilon}) \zeta_{x_j x_k}. \tag{4.10}$$

Moreover, if we let

$$\tilde{\phi}(w) = \frac{e^{-\mathcal{N}|w - z_k|^2} - e^{-4\mathcal{N}t_k^2}}{e^{-\mathcal{N}t_k^2} - e^{-4\mathcal{N}t_k^2}}.$$

Then $L^\star\tilde\phi \geq 0$ in $B(z_k,2t_k)\setminus B(z_k,t_k)$ if $\mathcal{N}=\mathcal{N}(p,n,c_*)$ is sufficiently large. Thus if

$$\Phi(w) = (\operatorname*{ess\,sup}_{B(z_k,2t_k)} u')(1-\tilde\phi(w))$$

then $L^\star\Phi \leq 0$ in $B(z_k,2t_k)\setminus B(z_k,t_k)$. Using this fact, the maximum principle for solutions to (4.10), $u''\leq u'$, and comparing boundary values, we conclude that $u''_\epsilon \leq \Phi$ in $B(z_k,2t_k)\setminus \bar B(z_k,t_k)$. Letting $\epsilon\to 0$, we deduce from the usual variational type arguments and an analogue of Lemma 2.6 for u''_ϵ that subsequences of $\{u''_\epsilon\}, \{\nabla u''_\epsilon\}$ converge pointwise to $u'', \nabla u''$ in D and uniformly on compact subsets of D. Hence

$$u''\leq \Phi \text{ in } B(z_k,2t_k)\setminus \bar B(z_k,t_k). \tag{4.11}$$

Using (4.11) and applying Lemma 2.6 to u'' we see that

$$|\nabla u''(x)| \leq \frac{c}{d}\,u''(x) \leq \frac{c}{d}\,\Phi(x) \leq \frac{c^2}{t_k}\operatorname*{ess\,sup}_{B(z_k,2t_k)} u'. \tag{4.12}$$

where $d=d(x,D)$. Lemma 2.4 and (4.5)-(4.7) imply

$$t_k^{1-p}\operatorname*{ess\,sup}_{B(z_k,2t_k)}(u')^{p-1} \leq c\,t_k^{1-n}\,\mu'_f(B(z_k,4t_k)) \leq c^2 M. \tag{4.13}$$

Combining (4.12) and (4.13) we see that Lemma 4.8 holds for u'' at points in D which are also in $\bigcup B(z_k,2t_k)\setminus \bar B(z_k,t_k)$. Similar arguments also give this inequality at points near $\partial B(0,15)$ and $\partial B(\tilde z,2r_1)$. Thus there exists an open set W with $\partial D\subset W$ and $|\nabla u''|\leq cM^{1/(p-1)}$ in $W\cap D$ where $c=c(p,n,c_*)$. Applying Lemma 3.18 to u'', then a maximum principle for weak subsolutions to $\tilde L$ defined as in (3.1), we see that Lemma 4.8 holds for every $x\in D$. □

The proof of the next lemma is essentially the same as in [3, Lemma 8]. For completeness we give the arguments here.

Lemma 4.14. *The functions* $|\nabla u''|^{p-2}\,|u''_{x_jx_k}|$ *for* $1\leq j,k\leq n$ *are all integrable in* D.

Proof. Let $\Lambda\subset\partial\Omega''$ be the set of points where $\partial\Omega''$ is not smooth. Clearly $\mathcal{H}^{n-1}(\Lambda)=0$. If $\hat x\in\partial D\setminus\Lambda$, then $\hat x$ lies in exactly one of the finite number of spheres which contain points of ∂D. Let $d'(\hat x)$ denote the distance from $\hat x$ to the union of spheres not containing $\hat x$ but containing points of ∂D. If $d'=d'(\hat x)<s/100$, then from Lemma 2.7 applied to u'' we see that each component of $\nabla u''$ has a Hölder continuous extension to $B(\hat x,3d'/4)$. Also from Hölder continuity, Lemmas 2.7 and 4.8 we see that

$$\begin{aligned}
&\frac{1}{c}\sum_{j,k=1}^{n}\int\limits_{D\cap B(\hat x,\frac{d'}{8})}|\nabla u''|^{p-2}\,|u''_{x_jx_k}|\,\mathrm{d}x\\
&\leq (d')^{\frac{n}{2}}M^{\frac{p-2}{2(p-1)}}\sum_{j,k=1}^{n}\left(\int\limits_{D\cap B(\hat x,\frac{d'}{8})}|\nabla u''|^{p-2}\,|u''_{x_jx_k}|^2\,\mathrm{d}x\right)^{\frac12}\\
&\leq c(d')^{\frac{(n-2)}{2}}\,M^{\frac{p-2}{2(p-1)}}\left(\int\limits_{D\cap B(\hat x,\frac{d'}{2})}|\nabla u''|^{p}\,\mathrm{d}x\right)^{\frac12}\\
&\leq c^2M\,(d')^{(n-1)}.
\end{aligned} \tag{4.15}$$

To prove Lemma 4.14 we assume as we may that $B(z_l,t_l)\not\subset B(z_\nu,t_\nu)$ when $\nu\neq l$, since otherwise we discard one of these balls. Also from a well known covering theorem we get a covering $\{B(y_i, \frac{1}{20}d'(y_i))\}$ of $\partial D \setminus \Lambda$ with the property that $\{B(y_i, \frac{1}{100}d'(y_i))\}$ are pairwise disjoint. From (4.15) we find that

$$\begin{aligned}\sum_{i,j,k} \int\limits_{D\cap B(y_i,\frac{1}{8}d'(y_i))} |\nabla u''|^{p-2}|u''_{x_jx_k}|\,\mathrm{d}x &\le c\,M \sum_i (d'(y_i))^{n-1} \\ &\le c^2 M\,\mathcal{H}^{n-1}(\partial D).\end{aligned} \tag{4.16}$$

Let $d(x)$ denote $d(x,\partial D)$. We choose a covering $\{B(x_m, \frac{1}{2}d(x_m)\}$ of D with $\{B(x_m, \frac{1}{20}d(x_m)\}$, pairwise disjoint. We note that if $x \in D$ and $y \in \partial D$ with $|y-x| = d(x)$, then $y \in \partial D \setminus \Lambda$. Indeed otherwise y would be on the boundary of at least two balls contained in the complement of D and so by the no containment assumption above, would have to intersect $B(x, d(x))$, which clearly is a contradiction. Also we assert that if $d(x) \le 1000s$, then $d(x) \le \kappa\, d'(y)$ where κ can depend on various quantities including the configuration of the balls, $\{B(z_k, t_k)\}$ but is independent of $x \in D$ with $d(x) \le 1000s$. Indeed from the no containment assumption one deduces that otherwise there exists sequences $(x_m), (y_m), (y'_m)$, with $x_m \in D, y_m \in C_1, y'_m \in C_2$, where C_1, C_2 are spheres in $\{\partial B(z_j, r_j)\}_1^N$ with $C_1 \neq C_2$ and

$$\begin{aligned}&|x_m - y_m| = d(x_m), |y_m - y'_m| = d'(y_m) \text{ and} \\ &\text{as } m\to\infty, d(x_m)/d'(y_m)\to\infty, \text{ with } x_m, y_m, y'_m\to w \in C_1 \cap C_2 \subset \Lambda.\end{aligned} \tag{4.17}$$

From basic geometry we see that either $(i)\, C_1 \cap C_2 = w.$ or $(ii)\, C_1 \cap C_2$ is an $n-2$ dimensional sphere. If (i) holds then C_1, C_2 are tangent, so clearly for large m, $d(x_m) \le cd'(y_m)$. If (ii) holds then considering the tangent planes to C_1, C_2 through w we see for large m that

$$d(x_m) \le cd(x_m, C_1 \cap C_2) \le c^2 d'(y_m)$$

where c is independent of m. In either case we have reached a contradiction to (4.17). Hence our assertion is true.

From this analysis and our choice of covering of D we see that for a given $B(x_m, \frac{1}{2}d(x_m))$ with $d(x_m) < 1000s$, there exists $j = j(m)$ with $B(x_m, \frac{1}{2}d(x_m)) \subset B(y_j, \kappa' d'(y_j))$ for some $0 < \kappa' < \infty$ independent of m.

Let $S_l, l = 1, 2, 3,$ be disjoint sets of integers defined as follows.

$$\begin{cases} m \in S_1 & \text{if } d(x_m) \ge 1000s, \\ m \in S_2 & \text{if } m \notin S_1 \text{ and } \nexists\, j \text{ with } B(x_m, \frac{1}{2}d(x_m)) \subset B(y_j, \frac{1}{8}d'(y_j)), \\ m \in S_3 & \text{if } m \text{ is not in either } S_1 \text{ or } S_2. \end{cases}$$

Let

$$K_l = \sum_{m\in S_l} \int\limits_{D\cap B(x_m,\frac{1}{2}d(x_m))} |\nabla u''|^{p-2}|u''_{x_jx_k}|\mathrm{d}x \text{ for } l = 1, 2, 3.$$

Then

$$\int\limits_D |\nabla u''|^{p-2}|u''_{x_jx_k}|\mathrm{d}x \le K_1 + K_2 + K_3. \tag{4.18}$$

From Lemma 2.6 and the same argument as in (4.15) we see that

$$K_1 \le c\,M \sum_{m\in S_1} d(x_m)^{n-1} \le c^2 M s^{-1} \tag{4.19}$$

where we have used disjointness of our covering $\{B(x_m, \frac{1}{20}d(x_m))\}$. Using disjointness of these balls and (4.16) we get

$$K_3 \le c\,M\mathcal{H}^{n-1}(\partial D).$$

Finally if $m \in S_2$ then as discussed earlier there exists $j = j(m)$ with $d(x_m) \approx d'(y_j)$, where proportionality constants are independent of m, so $B(x_m, \frac{1}{2}d(x_m)) \subset B(y_j, \kappa' d'(y_j))$. From disjointness of $\{B(x_m, \frac{1}{20}d(x_m))\}$ and a volume type argument we deduce that each j corresponds to at most κ'' integers $m \in S_3$ where κ'' is independent of j. Using this fact, an argument as in (4.15), as well as disjointness of $\{B(y_i, \frac{1}{100}d'(y_i))\}$, we conclude that there is a $\tilde\kappa$ with $0 < \tilde\kappa < \infty$, satisfying

$$K_2 \le \tilde\kappa M \sum_{m\in S_2} d(x_m)^{n-1} \le \tilde\kappa^2 M \sum_j d'(y_j)^{n-1} \le \tilde\kappa^3 M\, \mathcal{H}^{n-1}(\partial D). \tag{4.20}$$

Using (4.19)-(4.20) in (4.18) we find that Lemma 4.14 is valid. □

We next show that there exists $c = c(p, n, c_*) \ge 1$ such that

$$c^{-1} \le \mu''_f(\partial\Omega'' \cap B(0,10)) \le \mu''_f(\partial\Omega'') \le c. \tag{4.21}$$

To prove (4.21), it follows from Lemmas 2.2-2.4, (4.7), and the fact $u'(\tilde z) = 1$ that $u'' \ge 1/c$ on $\partial B(\tilde z, 4r_1)$ for some $c = c(p,n,c_*) \ge 1$. Let l denote the line from the origin through $\tilde z$ and let ζ_1 be the point on this line segment in $\partial B(\tilde z, 4r_1) \cap B(0,10)$. Let ζ_2 be the point on the line segment from ζ_1 to the origin with $d(\zeta_2, \partial\Omega'') = \frac{1}{20}r_1$ while $d(\zeta, \partial\Omega'') > \frac{1}{20}r_1$ at all other points on the line segment from ζ_1 to ζ_2. Then from (4.4), Lemma 2.2, and the above discussion we see that $u''(\zeta_2) \ge 1/c'$ for some $c' = c'(p,n,c_*) \ge 1$. Also, $B(\zeta_2, \frac{1}{2}r_1) \subset B(0,10)$. Let $\hat\zeta$ be the point in $\partial\Omega''$ with $|\hat\zeta - \zeta_2| = d(\zeta_2, \partial\Omega'')$. Applying Lemma 2.4 with $w = \hat\zeta, r = 2d(\zeta_2, \partial\Omega'')$, we deduce that the left hand inequality is valid. The right hand inequality in this claim follows once again from Lemma 2.4 and $u'' \le u'$.

Using Lemmas 4.8-4.14 and (4.21) we prove the following lemma.

Lemma 4.22. *There exists $c = c(p,n,c_*)$ such that*

$$\int_{\partial D} |\log f(\nabla u'')| \frac{f(\nabla u'')}{|\nabla u''|}\, d\mathcal{H}^{n-1} \le c \log M.$$

Proof. Let

$$\log^+ t := \max\{\log t, 0\} \text{ and } \log^- t := \log^+(1/t) \text{ for } t \in (0,\infty).$$

We first give a proof of Lemma 4.22 for $\log^+ f(\nabla u'')$. To this end, we observe from Lemma 2.7 that

$$d\mu''_f = p\frac{f(\nabla u'')}{|\nabla u''|} d\mathcal{H}^{n-1} > 0 \text{ on } \partial\Omega'' \setminus \Lambda. \tag{4.23}$$

It follows from Lemma 4.8, (4.21), (4.23), and $\mathcal{H}^{n-1}(\Lambda) = 0$ that

$$\int_{\partial\Omega''} \log^+(f(\nabla u''))\frac{f(\nabla u'')}{|\nabla u''|}\, d\mathcal{H}^{n-1} \le c \log M\, \mu''_f(\partial\Omega'') \le c^2 \log M. \tag{4.24}$$

To prove Lemma 4.22 for $\log^- f(\nabla u'')$, fix ξ, $-\infty < \xi < -1$, and set $v''(x) = \max(\log f(\nabla u''), \xi)$ when $x \in \overline{D} \setminus \Lambda$. Given small $\theta > 0$ we set

$$\Lambda(\theta) = \{x \in D : d(x,\Lambda) \le \theta\} \text{ and } D(\theta) = D \setminus \Lambda(\theta). \tag{4.25}$$

Observe from Lemmas 4.8-4.14 and (4.21) that

$$\tilde L u''(x) = \nabla \cdot \big(\mathcal{D} f(\nabla u''(x))\big) = 0$$

exists pointwise for almost every $x \in D(\theta)$ and is integrable on $D(\theta)$. Put

$$
\begin{aligned}
I(\theta) &= \int\limits_{D(\theta)} \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(f_{\eta_j\eta_k}(\nabla u'')u''_{x_k}\right) v''\, \mathrm{d}x + \int\limits_{D(\theta)} \sum_{j,k=1}^{n} f_{\eta_j\eta_k}(\nabla u'')u''_{x_k}\frac{\partial v''}{\partial x_j}\mathrm{d}x \\
&= I_1(\theta) + I_2(\theta)
\end{aligned}
\tag{4.26}
$$

From (4.25) and $p-1$ homogeneity of derivatives of f we see that $I_1(\theta) = 0$. To handle $I_2(\theta) = I(\theta)$, we first use a barrier argument as in Lemma 4.8, and then we use Lemma 2.7 to deduce that there exists some $c = c(p, n, c_*) \geq 1$, such that

$$
c^{-1} \leq |\nabla u''| \leq c \text{ on } \bar{B}(\tilde{z}, 2r_2) \setminus B(\tilde{z}, 2r_1) \text{ where } r_2 = (1 + c^{-1})r_1. \tag{4.27}
$$

Let ϕ, $0 \leq \phi \leq 1$, be an infinitely differentiable function in $\mathbb{R}^n$ with $\phi \equiv 1$ on $\mathbb{R}^n \setminus B(\tilde{z}, 2r_2)$, $|\nabla\phi| \leq c\, r_1^{-1}$, and $\phi \equiv 0$ on an open set containing $B(\tilde{z}, 2r_1)$. From (4.4) and the definition of r_1 we have $|\nabla\phi| \leq c^2$. Rearranging $I_2(\theta)$ and writing $f_{\eta_j\eta_k}$ for $f_{\eta_j\eta_k}(\nabla u'')$ we have

$$
\begin{aligned}
I_2(\theta) &= \int\limits_{D(\theta)} \sum_{j,k=1}^{n} f_{\eta_j\eta_k}(\phi u'')_{x_k} v''_{x_j}\mathrm{d}x + \int\limits_{D(\theta)} \sum_{i,k=1}^{n} f_{\eta_j\eta_k}((1-\phi)u'')_{x_k} v''_{x_j}\mathrm{d}x \\
&= I_{21}(\theta) + I_{22}(\theta).
\end{aligned}
$$

It follows from Lemmas 2.6-2.7, (4.27), and an argument similar to (4.15) that

$$
\begin{aligned}
|I_{22}(\theta)| \leq & \int\limits_{B(\tilde{z},2r_2)} \sum_{j,k=1}^{n} |f_{\eta_j\eta_k}|(1-\phi)|\nabla u''|\, |v''_{x_j}|\mathrm{d}x \\
& + \int\limits_{B(\tilde{z},2r_2)\setminus B(\tilde{z},2r_1)} \sum_{j,k=1}^{n} |f_{\eta_j\eta_k}||\nabla\phi|u''\, |v''_{x_j}|\mathrm{d}x \\
\leq & \; c,
\end{aligned}
\tag{4.28}
$$

where c is independent of θ. From (4.27) and Lemmas 2.6, 2.7, 4.8 and 4.14 we see that the integrand in the integral defining $I_{21}(\theta)$ is bounded by an integrable function independent of θ. Using this fact and the Lebesgue dominated convergence theorem we find that

$$
\lim_{\theta \to 0} I_{21}(\theta) = \int\limits_{D} \sum_{j,k=1}^{n} f_{\eta_j\eta_k}(\phi u'')_{x_k} v''_{x_j}\mathrm{d}x =: I'_{21}. \tag{4.29}
$$

We claim that $I'_{21} \leq 0$. To verify this claim let $u^* = u^*(\delta) = \max(u'' - \delta, 0)$. Convoluting ϕu^* with an approximate identity and taking limits we see from Lemma 3.18 that

$$
\int\limits_{D} \sum_{j,k=1}^{n} f_{\eta_j\eta_k}(\phi u^*)_{x_k} v''_{x_j} dx \leq 0.
$$

Moreover, once again from Lemmas 4.8 and 4.14, we observe that the above integrand is dominated by an integrable function independent of δ. From this fact, the above inequality, and the Lebesgue dominated convergence theorem we get assertion $I'_{21} \leq 0$. Using (4.28), (4.29), and above claim we conclude that

$$
\lim_{\theta \to 0} I(\theta) \leq c. \tag{4.30}
$$

On the other hand from [9, chapter 5] and (4.26) we see that integration by parts can be used to get

$$
I(\theta) = \int\limits_{\partial D(\theta)} v'' \sum_{j,k=1}^{n} f_{\eta_j\eta_k} u''_{x_k}\mathfrak{n}^j \mathrm{d}\mathcal{H}^{n-1} = (p-1)\int\limits_{\partial D(\theta)} v'' \sum_{j=1}^{n} f_{\eta_j}\mathfrak{n}^j \mathrm{d}\mathcal{H}^{n-1}
$$

where $\mathfrak{n}$ is the outer unit normal to $\partial D(\theta)$. From Lemma 4.8, the dominated convergence theorem, and the definition of $D(\theta)$, we have

$$\int_{\partial D(\theta)\setminus\partial B(\tilde{z},2r_1)} v'' \sum_{j=1}^{n} f_{\eta_j}\,\mathfrak{n}^j \mathrm{d}\mathcal{H}^{n-1} \to \int_{\partial\Omega''\setminus\Lambda} v'' \sum_{j}^{n} f_{\eta_j}\,\mathfrak{n}^j \mathrm{d}\mathcal{H}^{n-1} \text{ as } \theta\to 0. \tag{4.31}$$

From (1.3), (4.30), and (4.31) we deduce

$$\begin{aligned}\int_{\partial\Omega''\setminus\Lambda} v'' \sum_{j=1}^{n} f_{\eta_j}\mathfrak{n}^j \mathrm{d}\mathcal{H}^{n-1} &= -p \int_{\partial\Omega''\setminus\Lambda} v'' \frac{f(\nabla u'')}{|\nabla u''|}\,\mathrm{d}\mathcal{H}^{n-1}\\ &\leq p(p-1)^{-1}\lim_{\theta\to 0} I(\theta) + c \leq 2c.\end{aligned} \tag{4.32}$$

where we have also used the fact that $\mathfrak{n} = -\frac{\nabla u''}{|\nabla u''|}$ and

$$\left| \int_{\partial B(\tilde{z},2r_1)} v'' \sum_{j=1}^{n} f_{\eta_j}\mathfrak{n}^j \mathrm{d}\mathcal{H}^{n-1} \right| \leq c = c(p,n,c_*).$$

Letting $\xi\to-\infty$ in (4.32) and using the monotone convergence theorem we see that (4.32) holds with v'' replaced by $\log f(\nabla u'')$. Finally from (4.32) for $\log f(\nabla u'')$ and (4.24) we conclude the validity of Lemma 4.22. □

5. Proof of Theorem 1.11

In this section we first give a proposition which will be a consequence of lemmas we obtained in section 4 and then we prove Theorem 1.11. To this end let $O, f, u, \hat{z}, \rho, \mu_f$ be as in Theorem 1.11. Let w, r, Q be as in Lemma 4.1 and let $\hat{\lambda}$ be a positive non-decreasing function on $(0,1]$ with $\lim\limits_{t\to 0}\frac{\hat{\lambda}(t)}{t^{n-1}} = 0$.

Proposition 5.1. *There is a compact set $F = F(w,r) \subset \partial O\cap B(w,20r)$ such that*

$$\mathcal{H}^{\hat{\lambda}}(F) = 0 \text{ and } \mu_f(B(w,100r)) \leq c\mu_f(F).$$

Proof. We first note from Lemma 2.4 and the fact $u''\leq u'$ that for given j, $1\leq j\leq N$

$$t_j^{1-n}\,\mu''_f(\overline{B}(z_j,t_j)) \leq c\,t_j^{1-p} \operatorname*{ess\,sup}_{B(z_j,2t_j)} (u')^{p-1} \leq c^2\, t_j^{1-n}\,\mu'_f(B(z_j,4t_j)). \tag{5.2}$$

where N is the constant defined after (4.6). For given $A >> 1$, we see from (4.6) that $\{1,2,\dots,N\}$ can be divided into disjoint subsets: the good set $\mathfrak{G}$, the bad set $\mathfrak{B}$, and the ugly set $\mathfrak{U}$ as follows,

$$\begin{cases} \mathfrak{G} := \{j : t_j > s\},\\ \mathfrak{B} := \{j : t_j = s \text{ and } \frac{f(\nabla u'')(x)}{|\nabla u''|(x)} \geq M^{-A} \text{ for some } x\in\partial\Omega''\cap\partial B(z_j,s)\setminus\Lambda\},\\ \mathfrak{U} := \{j : t_j = s \text{ and } j\notin\mathfrak{B}\}.\end{cases}$$

Let $t'_j = t_j$ when $j\in\mathfrak{G}$ and $t'_j = 4s$ when $j\in\mathfrak{B}$. We define

$$E := \partial\Omega'\cap \bigcup_{j\in\mathfrak{G}\cup\mathfrak{B}} B(z_j,t'_j). \tag{5.3}$$

We first show for some $c = c(p,n,c_*)\geq 1$ and given $\epsilon > 0$ that

$$\mathcal{H}^{\hat{\lambda}}_{\delta'}(E) \leq \epsilon \text{ and } c^{-1}\leq \mu'_f(E) \tag{5.4}$$

where δ' is as in (4.5) and $\mathcal{H}^{\hat\lambda}_{\delta'}(E)$ is the Hausdorff content of E defined in (1.6). Proposition 5.1 will essentially follow from (5.4). To show (5.4), observe that if

$$x \in \bigcup_{j\in\mathfrak{G}\cup\mathfrak{B}} B(z_j,t'_j) \text{ then } x \text{ lies in at most } c=c(n) \text{ of } \{B(z_j,t'_j)\}. \tag{5.5}$$

This observation can be proved using $t_j \geq s, 1\leq j\leq N$, a volume type argument, and the fact that $\{B(z_j,t_j)\}_1^N$ is a Besicovitch covering of $\partial\Omega'\cap\bar B(0,15)$.

We first consider $j\in\mathfrak{B}$. Using (4.12), (4.13), the definition of $\mathfrak{B}$, and (5.5) we find for some $c=c(p,n,c_*)\geq 1$ that

$$M^{-A} \leq \frac{f(\nabla u'')(x)}{|\nabla u''(x)|} \leq c(t'_j)^{1-n}\mu'_f(B(z_j,t'_j)) \text{ whenever } j\in\mathfrak{B}. \tag{5.6}$$

Rearranging this inequality, summing over $j\in\mathfrak{B}$, and using (5.5), we see that

$$\sum_{j\in\mathfrak{B}}(t'_j)^{n-1} \leq \tilde c\, M^A\mu'_f\left(\bigcup_{j\in\mathfrak{B}}B(z_j,t'_j)\right) \leq (\tilde c)^2\, M^A$$

provided $\tilde c=\tilde c(p,n,c_*)$ is large enough. Now since $t'_j=4s$ for all $j\in\mathfrak{B}$ we may for given A,M,ϵ choose $s>0$ so small that

$$\frac{\hat\lambda(4s)}{(4s)^{n-1}} \leq \frac{\epsilon}{2(\tilde c)^2M^A}$$

where we have used the definition of $\hat\lambda$. Using this choice of s in (5.6) we get

$$\sum_{j\in\mathfrak{B}}\hat\lambda(t'_j) \leq \frac{\epsilon}{2}. \tag{5.7}$$

On the other hand, we may suppose δ' in (4.5) is so small that $\hat\lambda(t'_j)\leq(t'_j)^{n-1}$ for $1\leq j\leq N$. Then from (4.3), (4.6), and (5.5), we see that

$$\sum_{j\in\mathfrak{G}}\hat\lambda(t'_j) \leq \sum_{j\in\mathfrak{G}}(t'_j)^{n-1} \leq \frac{1}{M}\sum_{j\in\mathfrak{G}}\mu'_f(B(z_j,t_j)) \leq \frac{\epsilon}{2} \tag{5.8}$$

provided $M=M(\epsilon)$ is chosen large enough. Fix M satisfying all of the above requirements. In view of (5.7), (5.8), and the definition of Hausdorff content we have $\mathcal{H}^{\hat\lambda}_{\delta'}(E)\leq\epsilon$ for E as in (5.3). This finishes the proof of the left hand inequality in (5.4). To prove the right hand inequality in (5.4), we use (1.1), Lemma 4.22, and the definition of $\mathfrak{U}$ to obtain

$$\begin{aligned}
\mu''_f\left(\partial\Omega'\cap\bigcup_{j\in\mathfrak{U}}\bar B(z_j,t_j)\right) &\leq \mu''_f\left(\left\{x\in\partial\Omega'' : \frac{f(\nabla u'')(x)}{|\nabla u''(x)|}\leq M^{-A}\right\}\right)\\
&\leq \frac{c}{(p-1)A\log M}\int_{\partial\Omega''}|\log f(\nabla u'')|\,\frac{f(\nabla u'')}{|\nabla u''|}\mathrm{d}\mathcal{H}^{n-1}\\
&\leq \frac{c'}{A}.
\end{aligned} \tag{5.9}$$

Choosing $A=A(p,n,c_*)$ large enough we have from Lemma 4.22 and (5.9),

$$\mu''_f\left(\bigcup_{j\in\mathfrak{G}\cup\mathfrak{B}}B(0,10)\cap\overline{B}(z_j,t_j)\right) \geq \mu''(B(0,10)) - \mu''\left(\bigcup_{j\in\mathfrak{U}}\bar B(z_j,t_j)\right) \geq \frac{1}{c} \tag{5.10}$$

for some $c(p,n,c_*) \geq 1$. Finally from (5.2)-(5.5) and (5.10),we get for some $c = c(p,n,c_*) \geq 1$ that

$$\frac{1}{c^3} \leq \frac{1}{c^2} \sum_{j\in\mathfrak{G}\cup\mathfrak{B}} \mu''_f(\bar{B}(z_j,t_j)) \leq \frac{1}{c} \sum_{j\in\mathfrak{G}\cup\mathfrak{B}} \mu'_f(\bar{B}(z_j,t'_j)) \leq \mu'_f(E).$$

For $j \in \mathfrak{G}$ we have used the definition of t_j so that

$$\mu'_f(B(z_j,4t_j)) < M4^{n-1}t_j^{n-1} = 4^{n-1}\mu'_f(B(z_j,t_j)) = 4^{n-1}\mu'_f(B(z_j,t'_j))$$

Thus (5.4) is valid. To finish the proof of Proposition 5.1, we note that we can choose E_m relative to $\delta' = \epsilon = 2^{-m}$ for $m = m_0, m_0+1, \dots$ with

$$\mathcal{H}^{\hat{\lambda}}_{\delta'}(E_m) \leq 2^{-m} \text{ and } c^{-1} \leq \mu'_f(E_m). \tag{5.11}$$

From (5.11) and measure theoretic arguments we see that if we set

$$E' = \bigcap_{k=m_0} \left(\bigcup_{m=k} E_m \right)$$

then it follows from regularity of μ'_f that there exists a compact set $F \subset E'$ satisfying $\mathcal{H}^{\hat{\lambda}}(F) = 0$ and $c^{-1} \leq \mu'_f(F)$ where c is as in (5.11). In view of these two estimates we conclude that the proof of Proposition 5.1 is now complete □

We next give an easy consequence of Lemma 4.1 and Propositions 5.1. Let Q be as in Lemma 4.1 and let $\hat{\lambda}$ be as in Proposition 5.1. We first prove that there exists a Borel set $Q_1 \subset Q$ with

$$\mu_f(\partial O \cap B(\hat{z},\rho) \setminus Q_1) = 0 \ \text{ and } \ \mathcal{H}^{\hat{\lambda}}(Q_1) = 0. \tag{5.12}$$

To prove (5.12) we first observe that if $\mu_f(\partial O \cap B(\hat{z},\rho)) < \infty$ then it follows from Lemma 4.1, Proposition 5.1, a Vitali type covering argument, and induction that there exists compact sets $\{F_l\}$ such that $F_l \subset Q$, $F_k \cap F_l = \emptyset$ for $k \neq l$ and $\mu_f(F_1) > 0$ with

$$\mu_f(Q \setminus \bigcup_{l=1}^{m} F_l) \leq c'\mu_f(F_{m+1}), \ m = 1,2,\dots$$

for some $c' = c'(p,n,c_*) \geq 1$. Moreover $\mathcal{H}^{\hat{\lambda}}(F_l) = 0$ for all l. Then it follows from measure theoretic arguments that $Q_1 = \bigcup_{l=1}^{\infty} F_l$ has the desired properties in (5.12). In case $\mu_f(\partial O \cap B(\hat{z},\rho)) = \infty$, we can write $\partial O \cap B(\hat{z},\rho)$ as a union of countable Borel sets with finite μ_f measure and apply the same argument in each set. Therefore we conclude that there exists a Borel set Q_1 in Q satisfying (5.12).

We now prove Theorem 1.11. To this end, we let

$$P := \left\{ x \in \partial O \cap B(\hat{z},\rho) : \ \limsup_{t\to 0} \frac{\mu_f(B(x,t))}{t^{n-1}} > 0 \right\}.$$

We first show that

$$\mu_f(\partial O \cap B(\hat{z},\rho) \setminus P) = 0. \tag{5.13}$$

From Lemma 4.1 we have $\mu_f(\partial O \cap B(\hat{z},\rho) \setminus Q) = 0$. Therefore, it suffices to prove (5.13) with Q replacing $\partial O \cap B(\hat{z},\rho)$. To do this we argue by contradiction and thus assume $\mu_f(Q \setminus P) > 0$. Then, by Egoroff's theorem there exists a compact set $K \subset Q \setminus P$ with

$$\mu_f(K) > 0 \text{ and } \lim_{t\to 0} \frac{\mu_f(B(x,t))}{t^{n-1}} = 0 \text{ uniformly for } x \in K. \tag{5.14}$$

Set $\alpha_0 = 1$ and choose $\alpha_k \in (0,1), k = 1, 2, \dots,$ such that

$$\alpha_{k+1} < \frac{\alpha_k}{2} \text{ and } \sup_{0<t\leq\alpha_k} \frac{\mu_f(B(x,t))}{t^{n-1}} \leq 2^{-2k} \text{ for all } x \in K.$$

Define $\hat{\lambda}_0(t)$ on $(0,1]$ in the following way: put $\hat{\lambda}_0(0) = 0$,

$$\hat{\lambda}_0(\alpha_k) = 2^{-k}(\alpha_k)^{n-1} \text{ for } k = 0, 1, \dots,$$

and define $\hat{\lambda}_0(t)$ when $t \in [\alpha_{k+1}, \alpha_k]$ in such a way that

$$\frac{\hat{\lambda}_0(t)}{t^{n-1}} \text{ is linear for } t \in [\alpha_{k+1}, \alpha_k] \text{ whenever } k = 0, 1, \dots.$$

Clearly,

$$\frac{\hat{\lambda}_0(t)}{t^{n-1}} \to 0 \text{ as } t \to 0.$$

Moreover, we observe that

$$\begin{aligned} &\hat{\lambda}_0(2t) \leq 2^{n+1}\hat{\lambda}_0(t) \text{ for } 0 < t < 1/2, \\ &\frac{\mu_f(B(x,t))}{\hat{\lambda}_0(t)} \leq 2^{1-k} \text{ whenever } \alpha_{k+1} \leq t \leq \alpha_k \text{ and } x \in K. \end{aligned} \tag{5.15}$$

Let Q_1 be as in (5.12) relative to $\hat{\lambda}_0$. Then for a given positive integer m it follows from (5.12) that there is a covering $\{B(x_j, r_j)\}$ of $K \cap Q_1$ with

$$r_j \leq \frac{\alpha_m}{2} \text{ for all } j \text{ and } \sum_j \hat{\lambda}_0(r_j) \leq 1.$$

We may assume that there is an $x_j' \in K \cap B(x_j, r_j)$ for each j since otherwise we discard $B(x_j, r_j)$. Then from (5.15) we find that

$$\mu_f(K \cap Q_1) \leq \sum_j \mu_f(B(x_j', 2r_j)) \leq 2^{1-m} \sum_j \hat{\lambda}_0(2r_j) \leq 2^{n+2-m}.$$

Since m is arbitrary, we have $\mu_f(K \cap Q_1) = 0$. Using this equality and (5.12) we find that $\mu_f(K) = \mu_f(Q \setminus Q_1) + \mu_f(K \cap Q_1) = 0$ and so we have reached a contradiction in (5.14). Hence, $\mu_f(Q \setminus P) = 0$ and (5.13) holds.

We next show that the set P has $\sigma-$ finite $\mathcal{H}^{n-1}$ measure. To this end, once again we may assume $\mu_f(\partial O \cap B(\hat{z}, \rho)) < \infty$. Let m be an arbitrarily fixed positive integer and define

$$P_m := \left\{ x \in P : \limsup_{t\to 0} \frac{\mu_f(B(x,t))}{t^{n-1}} > \frac{1}{m} \right\}.$$

Given $\hat{\delta} > 0$ we choose a Besicovitch covering $\{B(y_i, r_i)\}$ of P_m with

$$y_i \in P_m,\ r_i \leq \hat{\delta},\ B(y_i, r_i) \subset B(\hat{z}, \rho), \text{ and } r_i^{n-1} < m\, \mu_f(B(y_i, r_i)).$$

It follows that

$$\sum_i r_i^{n-1} < m \sum_i \mu_f(B(x_i, r_i)) \leq c m \mu_f(\partial O \cap B(\hat{z}, \rho)) < \infty. \tag{5.16}$$

Letting $\hat{\delta} \to 0$ and using the definition of $\mathcal{H}^{n-1}$ measure we conclude from (5.16) that $\mathcal{H}^{n-1}(P_m) < \infty$. As m is arbitrary we conclude that P has $\sigma-$finite $\mathcal{H}^{n-1}$ measure. In view of this observation, (5.13) and Lemma 4.1, the proof of Theorem 1.11 is now complete. $\square$

6. Proof of Theorem 1.13

This section is dedicated to the proof of Theorem 1.13. Before giving a proof we recall our setting from section 1; let

$$\tilde{\Gamma} = \left\{ \tilde{Q}_{k,j};\ k = 1, \dots, \text{ and } j = 1, \dots, 2^{kn} \right\}$$

denote the set of cubes defined in section 1 and let $\mathcal{C}$ be the corresponding Cantor set. Also as in section 1 let $\mathcal{S}$ be the cube in $\mathbb{R}^n$ with side length 1 centered at the origin. and let u^∞ be the positive weak solution to

$$\Delta_f u^\infty = \nabla \cdot \mathcal{D}f(\nabla u^\infty) = 0 \text{ in } \mathcal{S} \setminus \mathcal{C}$$

with continuous boundary values 1 on $\partial\mathcal{S}$ and 0 on $\mathcal{C}$. Let μ_f^∞ be the measure associated with u^∞ as in (1.5). For ease of notation, we write μ, u for μ_f^∞, u^∞. Next let α, β be the constants as in section 1 and $s(\tilde{Q}_{k,j}) = a_0 a_1 a_2 \dots a_k < 2^{-(k+1)}$ denote the side length of $\tilde{Q}_{k,j}$ where $\alpha \leq a_i \leq \beta < 1/2$ for every $i = 1, 2, \dots$. Let c_* be the constant as in (1.1).

Let $\tilde{Q} \in \tilde{\Gamma}$ for some k with $k \geq 10^5$ and $j = 1, \dots, 2^{kn}$. We first show that

$$\mu(100\tilde{Q}) \leq c \max_{\partial 2\tilde{Q}} u^{p-1} \leq c^2 \mu(\tilde{Q}) \text{ for some } c = c(p, n, c_*, \alpha, \beta) \tag{6.1}$$

where once again c_* is as in (1.1). To prove (6.1) note from the geometry of $\mathcal{C}$ that there exists a smallest $\tilde{Q}' \in \tilde{\Gamma}$ with

$$100\tilde{Q} \subset (1+\theta)\tilde{Q}'$$

where $\theta = \frac{1}{100n} \min(\alpha, 1/2 - \beta)$. Covering $\tilde{Q}' \cap \mathcal{C}$ by balls of radius $\approx s(\tilde{Q})$ and applying Lemma 2.4 in each ball we deduce that

$$\mu(100\tilde{Q}) \leq \mu(\tilde{Q}') \leq c \max_{\partial(1+\theta)\tilde{Q}'} u^{p-1} \tag{6.2}$$

where $c = c(p, n, c_*, \alpha, \beta)$. Using Harnack's inequality, basic geometry and once again Lemma 2.4 we also see that

$$(s(\tilde{Q}))^{n-p} \max_{\partial(1+\theta)\tilde{Q}'} u^{p-1} \leq \tilde{c}(s(\tilde{Q}))^{n-p} \max_{\partial(1+\theta)\tilde{Q}} u^{p-1} \leq \tilde{c}^2 \mu(\tilde{Q}) \tag{6.3}$$

where $\tilde{c}$ has the same dependence as c. Combining (6.2) and (6.3) we obtain (6.1). From Hölder continuity of $1 - u$ near $\partial\mathcal{S}$, Harnack's inequality, and Lemma 2.4 we also find that

$$\mu(\mathcal{C}) \approx 1 \tag{6.4}$$

where proportionality constants depend only on p, n, c_*, α, β. Analogous to Proposition 5.1 we prove

Proposition 6.5. *Let $\tilde{Q} \in \tilde{\Gamma}$ be a given cube. Then there exists $\delta' > 0$ with the same dependence as δ in Theorem 1.13, $c = c(p, n, c_*, \alpha, \beta) \geq 1$, and a compact set $F \subset \mathcal{C} \cap \tilde{Q}$ with*

$$\mathcal{H}^{n-1-\delta'}(F) = 0 \textit{ and } \mu(\tilde{Q}) \leq c\mu(F).$$

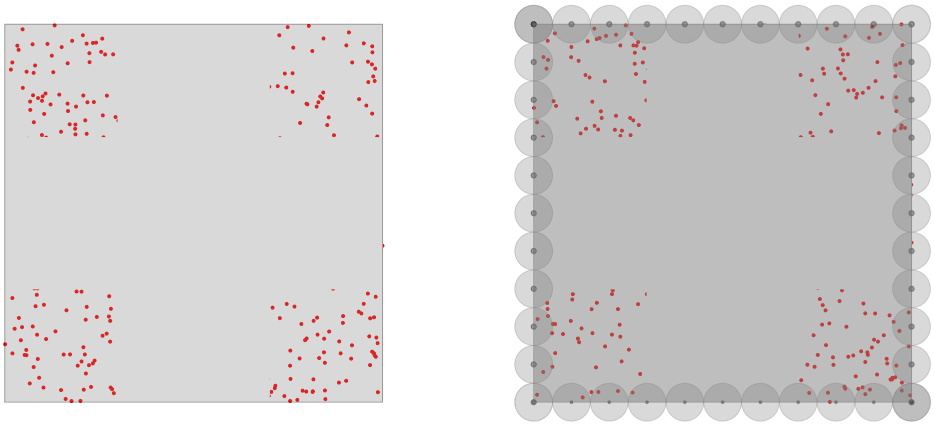
Proof. We shall only show that the conclusion of Proposition 6.5 is valid with $\tilde{Q}$ replaced by $\tilde{Q}_0$ = the closed cube with side length 1/2 and center at 0 (denoted $\mathcal{C}_0$ in section 1). The general version of Proposition 6.5 is proved in a similar way, as one sees from using (6.1) and arguing as in the construction of u' in (4.3), With this understanding, we simplify the proof of Proposition 6.5 further by noting that if $\lambda(r) = r^{n-1-\delta'}$, $0 < r \leq 1$, then from measure theoretic type arguments it suffices to show for given $\epsilon, \tau > 0$, that there exists δ', c as above and a compact set $F \subset \mathcal{C}$ with

$$\mathcal{H}^\lambda_\tau(F) \leq \epsilon \text{ and } \mu(F) \geq 1/c. \tag{6.6}$$

To prove (6.6) and in view of the proof of Theorem 1.11 we shall need some more notation: Let $\{B(x_l, \frac{\theta}{10})\}_1^{N_1}, x_l \in \partial\tilde{Q}_0$, be a Besicovitch covering of $\partial\tilde{Q}_0$ and set

$$Q_0 := \tilde{Q}_0 \cup \left(\bigcup_{l=1}^{N_1} B(x_l, \frac{\theta}{10}) \right)$$

where $\theta = \frac{1}{100n}\min(\alpha, 1/2 - \beta)$ as earlier (see figure 6.1).



$\tilde{Q}_0$ Q_0

FIGURE 6.1. The cubes $\tilde{Q}_0$ and Q_0.

If $\tilde{Q}$ is a cube with center z let $\gamma Q = \{x = z + 2\gamma s(\tilde{Q})y : y \in Q_0\}$. We write Q for $1Q$. From our constructions we have for $k = 1, 2, \dots,$

$$\begin{aligned} &\tilde{Q}_{k,j} \subset Q_{k,j} \subset (1+\theta/2)\tilde{Q}_{k,j} \\ &(1+\theta)\tilde{Q}_{k,j} \cap (1+\theta)\tilde{Q}_{k,j'} = \emptyset \text{ for } j \neq j' \text{ and} \\ &\text{either } \tilde{Q}_{k,j} \subset \tilde{Q}_{k',j'} \text{ or } \tilde{Q}_{k,j} \cap \tilde{Q}_{k',j'} = \emptyset, \text{ when } \ k > k'. \end{aligned} \tag{6.7}$$

Let $\tilde{\Lambda}$ be a finite disjoint covering of $\mathcal{C}$ by cubes in $\tilde{\Gamma}$ and let Λ be the collection of all $Q_{k,j}$ with $\tilde{Q}_{k,j} \in \tilde{\Lambda}$.

Remark 6.8. Note that cubes $\tilde{Q}_{k,j} \in \tilde{\Lambda}$ are closed cubes whereas the cubes $Q_{k,j} \in \Lambda$ are open. Moreover, figure 6.2 tells us (as an example) that $\tilde{Q}_{k-1,j} \notin \tilde{\Lambda}$ and $\tilde{Q}_{k,j_2} \notin \tilde{\Lambda}$. On the other hand, $\tilde{Q}_{k,j_1} \in \tilde{\Lambda}$ and $\tilde{Q}_{k+1,j_1} \in \tilde{\Lambda}$ and therefore by definition of Λ, $Q_{k,j_1} \in \Lambda$ and $Q_{k+1,j_1} \in \Lambda$.

Let $\bar{u}$ be the positive weak solution to

$$\Delta_f \bar{u} = \nabla \cdot \mathcal{D}f(\nabla \bar{u}) = 0 \text{ in } \Omega = B(0,n) \setminus \bigcup_{Q \in \Lambda} \bar{Q}$$

with boundary values 1 on $\partial B(0,n)$ and 0 on ∂Q for every $Q \in \Lambda$. Extend $\bar{u}$ to $B(0,n)$ by putting $\bar{u} = 0$ on every $Q \in \Lambda$. Let $\bar{\mu}$ be the measure associated with $\bar{u}$ as in (1.5). Let $\bar{v} = \log f(\nabla \bar{u})$ and define $\tilde{L}$ as in Lemma 3.18 relative to $\bar{u}$. Recall from this lemma that $\max(\bar{v}, \eta)$ is a weak sub solution to $\tilde{L}$ whenever $\eta \in (-\infty, \infty)$. Then $\tilde{L}\bar{v} = \nu$ weakly, where ν is a locally finite positive Borel measure on $\Omega \cap \{x : \ |\nabla \bar{u}| > 0\}$. In case $p = 2, n = 2$, we shall see that ν is a locally finite atomic measure on Ω.

Next we state a key lemma.

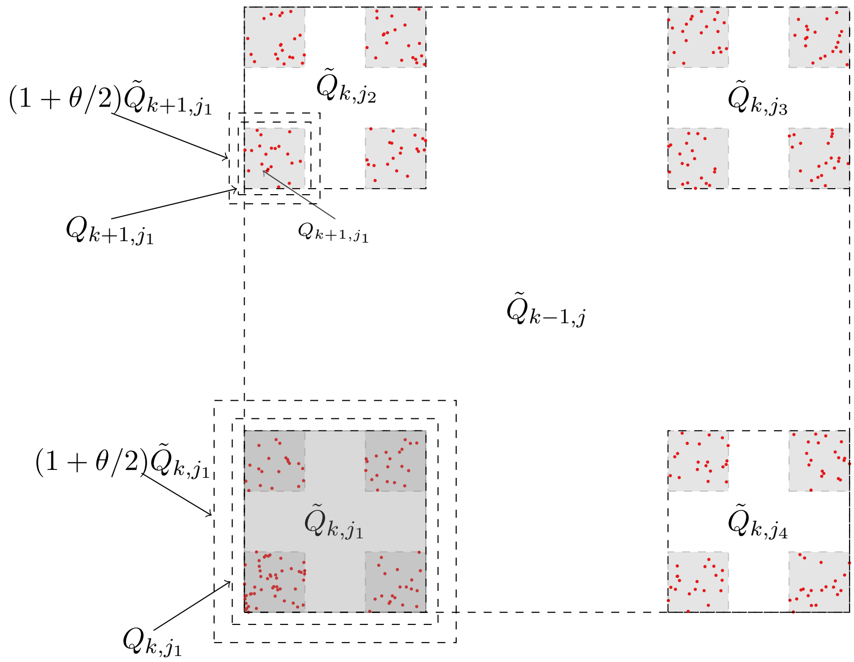


FIGURE 6.2. The cubes $\tilde{Q}_{k-1,j_1}$, $\tilde{Q}_{k,j_1}$, $\tilde{Q}_{k+1,j_1}$, Q_{k,j_1}, and Q_{k+1,j_1}.

Lemma 6.9. *Let $\Omega, \bar{u}, \bar{\mu}, \bar{v}, \nu$, be as above and suppose $\tilde{Q} \in \tilde{\Gamma} \setminus \tilde{\Lambda}$. There exists $c_2, c_3, c_4 \geq 10^5$, such that if $\tilde{Q}' \subset \tilde{Q}$, $\tilde{Q}' \in \tilde{\Lambda}$, and $c_2 s(\tilde{Q}') \leq s(\tilde{Q})$, then*

$$\int_O \bar{u} \mathrm{d}\nu \geq c_3^{-1} \bar{\mu}(\bar{Q}) \quad \textit{where} \quad O = \left\{ x \in (1+\theta)\tilde{Q} : \ d(x, \partial\Omega) \geq \frac{s(\tilde{Q})}{c_4} \right\}$$

and $Q \in \Lambda$ is the cube associated with $\tilde{Q} \in \tilde{\Lambda}$. Here c_2, c_4 depend only on p, n, c_*, α, β, and can be chosen independent of $p \in [n, n+1]$ provided c_* in (1.1) is constant on this interval. Also $c_3^{-1} \geq (p-n)c_5^{-1}$, where c_5 has the same dependence as c_2.

Moreover, if $f = g^p$, where g is as in Theorem 1.13, then c_3^{-1} can be chosen to depend only on n, g, c_*, α, β, when $p \in [n, n+1]$.

Proof. Let ξ be the minimum of $\bar{u}$ on $\partial(1+\theta)\tilde{Q}$ and let

$$G = \{x : \ \bar{u}(x) < \xi/2\} \cap (1+\theta)\tilde{Q}.$$

We note that

$$d(\partial G, \partial\Omega) \geq c^{-1} s(\tilde{Q}) \text{ and } \xi \geq c^{-1} \max_{(1+\theta)\tilde{Q}} \bar{u}$$

thanks to Harnack's inequality and Hölder continuity of $\bar{u}$ near $\partial\Omega$ (Lemmas 2.2 and 2.3). From this note and our hyphothesis, we deduce that if c_2 is large enough, then a component of G, say G', contains two disjoint cubes, $(1+\theta)\tilde{Q}_1, (1+\theta)\tilde{Q}_2$, with $\tilde{Q}_1, \tilde{Q}_2 \in \tilde{\Gamma}$ and

$$(c')^{-1} s(\tilde{Q}_i) \leq s(\tilde{Q}) \leq c' s(\tilde{Q}_i) \text{ for } i = 1, 2,$$

where c' has the same dependence as c_2 in Lemma 6.9. Let ξ_1 be the minimum of $\bar{u}$ on $\partial(1+\theta)\tilde{Q}_1 \cup \partial(1+\theta)\tilde{Q}_2$. Then from our construction, the maximum principle for solutions to (1.4), and once again Harnack's inequality - Hölder continuity of $\bar{u}$ near $\partial\Omega$, we see that G' contains at least two components of $G_1 = \{x : \ \bar{u}(x) < \xi_1/2\}$. Moreover,

$$d(\partial G_1, \partial\Omega) \geq c^{-1} s(\tilde{Q}) \text{ and } \xi_1 \geq c^{-1} \max_{(1+\theta)\tilde{Q}} \bar{u}. \tag{6.10}$$

Let t_0, $\xi_1/2 \leq t_0 < \xi/2$, be the largest t for which there are at least two components of $\{x: \ \bar u(x) < t\}$ contained in G'. Then there exists $\hat x \in G' \cap \{x: \ \bar u(x) = t_0\}$ such that $\hat x$ lies on the boundary of two different components of $\{x: \ \bar u(x) < t_0\}$ in G'. Also,

$$\bar u(\hat x) = t_0, \ \nabla \bar u(\hat x) = 0, \ \ d(\hat x, \partial\Omega) \geq c_6^{-1} s(\tilde Q), \text{ and } t_0 \geq c_6^{-1} \max_{(1+\theta)\tilde Q} \bar u, \tag{6.11}$$

where $c_6 \geq 1$ has the same dependence as c_4 in Lemma 6.9. Indeed observe from Lemma 2.6 that $\bar u$ has Hölder continuous derivatives in an open neighborhood of $\hat x$. So if $\nabla \bar u(\hat x) \neq 0$ we easily obtain a contradiction to the definition of t_0, using the implicit function theorem and the definition of a component. From this contradiction we conclude that $\nabla \bar u(\hat x) = 0$. Existence of c_6 depending on p, n, c_*, α, β follows from (6.10) which in turn was proved using Lemmas 2.3 and 2.2. Also it is easily checked from references providing proofs of these lemmas (see section 2) that constants may be chosen to depend only on n, α, β when $p \in [n, n+1]$ provided c_* in (1.1) is chosen independent of p in this interval.

For $\hat x, t_0$ as in (6.11) we now choose

$$z \in \partial\Omega \setminus \partial B(0,n) \text{ with } d(\hat x, \partial\Omega \setminus \partial B(0,n)) = |z - \hat x|.$$

Let z_1 be the first point on this line segment starting from $\hat x$ with $\bar u(z_1) = (1/2)t_0$. Let $[\hat x, z_1]$ denote the line segment from $\hat x$ to z_1. Then

$$(1/2)t_0 \leq \int_{[\hat x, z_1]} |\nabla \bar u| \mathrm{d}\mathcal{H}^1$$

so there exists z_2 on $[\hat x, z_1]$ and $c_7 = c_7(p, n, c_*, \alpha, \beta)$ with

$$(1/2)t_0 \leq |\nabla \bar u(z_2)||\hat x - z_1| \text{ while } d(z_2, \partial\Omega) \geq c_7^{-1} s(\tilde Q) \tag{6.12}$$

where the last inequality follows from our choice of z_1, basic geometry, and Lemma 2.3. From (6.11), (6.12), we find ρ such that

$$\rho = \rho(p, n, c_*, \alpha, \beta) \geq c^{-1} s(\tilde Q) \text{ with } B(\hat x, \rho), B(z_2, \rho) \subset \Omega.$$

Let Ω' denote the convex hull of $B(\hat x, \rho/2)$ and $B(z_2, \rho/2)$. Then from Harnack's inequality, Lemma 2.6, a Poincare type inequality, and (6.11), (6.12), we have

$$\begin{aligned} c^{-1} s(\tilde Q)^{n-p} (\max_{(1+\theta)\tilde Q} \bar u)^{p-1} &\leq \ s(\tilde Q)^{n-p} \bar u(z_2)^{p-1} \\ &\leq c \int_{\Omega' \cap \{|\nabla \bar u| > 0\}} \bar u |\nabla \bar u|^{p-2} \, |\nabla \bar v|^2 \, \mathrm{d}x \\ &\leq c^2 \int_{\Omega' \cap \{|\nabla \bar u| > 0\}} \bar u |\nabla \bar u|^{p-4} \sum_{i,j=1}^n \bar u_{x_i x_j}^2 \, \mathrm{d}x. \end{aligned} \tag{6.13}$$

Using Lemma 2.4 and (6.13), it follows that

$$\bar\mu(\bar Q) \leq c \int_{\Omega' \cap \{|\nabla \bar u| > 0\}} \bar u |\nabla \bar u|^{p-4} \sum_{i,j=1}^n \bar u_{x_i x_j}^2 \, \mathrm{d}x. \tag{6.14}$$

Next we revisit the proof of Lemma 3.18 in order to estimate ν.

6.1. **The case $p \geq n > 2$.** In this case from (3.8)-(3.9) we see for $n > 2$ and $p \geq n$ that if $\nabla \bar u(x) \neq 0$, then

$$\tilde L \bar v = h \text{ weakly} \tag{6.15}$$

where

$$h = f^{-1}(I' + I'') = f^{-1} \left[\mathrm{tr}(BA)^2 - \frac{1}{f} \frac{1}{(p-1)^2} \nabla \bar u BABAB (\nabla \bar u)^t \right]$$

and $A = (\bar u_{x_i x_j}), B = (f_{\eta_i\eta_j})$ are $n \times n$ matrices. If $p > n$ we see from (3.10) and (3.16) that

$$\operatorname{tr}(BA)^2 - \frac{1}{f}\frac{1}{(p-1)^2}\nabla\bar u BABAB(\nabla\tilde u)^t \geq \frac{p-n}{n(n-1)(p-1)}\operatorname{tr}(AB)^2. \tag{6.16}$$

Moreover,

$$\hat c\ \operatorname{tr}(AB)^2 \geq |\nabla\bar u|^{2p-4}\sum_{i,j=1}^n \bar u^2_{x_i x_j} \tag{6.17}$$

for some $\hat c \geq 1$ depending only on p, n, α, β, and c_* in (1.1), as follows from positive definiteness and $p-2$ homogeneity of B as well as symmetry of A. Combining (6.15)-(6.17) we conclude for almost every x with $\nabla\bar u(x) \neq 0$ that

$$h \geq c^{-1}(p-n)|\nabla\bar u|^{p-4}\sum_{i,j=1}^n \bar u^2_{x_i, x_j} \tag{6.18}$$

where $c \geq 1$ depends only on p, n, α, β, and c_* in (1.1). Combining (6.18), (6.15), and (6.14) we get

$$(p-n)\bar\mu(\bar Q) \leq \tilde c\int_{\Omega'\cap\{|\nabla\bar u|>0\}}\bar u\, \mathrm{d}\nu \leq \tilde c\int_O \bar u \mathrm{d}\nu \tag{6.19}$$

where $\tilde c = \tilde c(p, n, c_*, \alpha, \beta)$ and this constant can be chosen independent of p on $[n, n+1]$. From the definition of Ω' and (6.19) we see that the first part of Lemma 6.9 is true when $p > n$.

To handle values of p near n, $n \geq 3$, we need to examine the case when $h = 0$ (so $p = n$) in (6.15). Indeed, from (3.10) - (3.16) we see for $p = n$ that

$$fh = \left[\ \operatorname{tr}\ (E^2) - \frac{n}{n-1}\frac{yE^2y^t}{yy^t}\right] = g(y) \geq 0,$$

where $E = B_d' A_1 B_d'$, $y = \nabla\bar u\mathcal{S}B_d'$, $A_1 = \mathcal{S}^tA\mathcal{S}$, $B_d = \mathcal{S}^tB\mathcal{S}$, and $B_d = B_d'B_d'$. Also $\mathcal{S}$ is an orthogonal matrix and B_d a diagonal matrix as in section 3. If $g(y) = 0, y \neq 0$, and $E \neq 0$, then since E is symmetric, it follows from basic matrix theory that y is an eigenvector of E, so $yE = \mathcal{V}y$ for some $\mathcal{V} \neq 0$. Thus,

$$(\nabla\bar u\mathcal{S}B_d')B_d'A_1B_d' = \mathcal{V}(\nabla\bar u\mathcal{S}B_d')$$

so since $\mathcal{S}, B_d'$ are invertible it follows that at x we have $\nabla\bar uBA = \mathcal{V}\nabla\bar u$. If we rewrite this in terms of f and $\bar u$ we get

$$(n-1)\nabla f(\nabla\bar u(x)) = \mathcal{V}\nabla\bar u(x) \tag{6.20}$$

where we have used the $n-1$ homogeneity of $\mathcal{D}f$. On the other hand at almost every x where $E = 0$ we have

$$A = (\bar u_{x_i x_j}) = 0 \text{ since } B_d' \text{ and } \mathcal{S} \text{ are invertible.} \tag{6.21}$$

Assume that either (6.20) or (6.21) hold almost everywhere in $B(w, r) \subset \Omega$ for some $w \in \Omega$ and $r > 0$. If $B(w, r) \cap \{x : \bar u(x) = t\} \neq \emptyset$, we assert that

$$f(\nabla\bar u) \text{ is constant on each component of } B(w, r) \cap \{x : \bar u(x) = t\}\,. \tag{6.22}$$

To prove this assertion let $x' \in B(w, r)$ and suppose that $\nabla\bar u(x') \neq 0$. Then

$$\pm\bar u_{x_i}(x') \geq n^{-1}|\nabla\bar u(x')| \text{ for some } 1 \leq i \leq n.$$

Assume for example that $i = n$ so that $\bar u_{x_n}(x') \geq |\nabla\bar u|(x')/n$. Consider the mapping, $\Psi(x_1, \dots, x_n) = (x_1, \dots, \bar u(x_1, ..., x_n))$. From the inverse function theorem and Lemma

2.6 we see that in a neighborhood of $\Psi(x'), \Psi$ has a $C^{1,\alpha''}$ inverse Φ and $f(\nabla \bar u(x))$ is in $W^{1,2}(B(x',\rho))$ for some small $\rho>0$. We claim that

$$H = f(\nabla \bar u)\circ \Phi \in W^{1,2}(B(\Psi(x'),\rho')) \text{ for small } \rho'>0. \tag{6.23}$$

Here H is considered as a function of $x_1,\dots,x_{n-1},\bar u$. One can prove (6.23) for example by, (a) approximating $f(\nabla\bar u)$ in the $W^{1,2}(B(x',\rho))$ norm by a sequence, (q_j) of $C^\infty(\mathbb{R}^n)$ functions, (b) using the chain rule and change of variables theorem to show that $H_j = q_j\circ\Phi \in W^{1,2}(B(\Psi(x'),\rho'))$ with norms bounded by a constant independent of j, (c) showing that $H_j \to H$ in the norm of $W^{1,2}(B(\Psi(x'),\rho'))$.

From (6.23) and well known properties of Sobolev functions it follows that H is "absolutely continuous on most lines". Therefore, in our situation, if $\hat z = (\Psi_1(x'),\dots,\Psi_{n-1}(x'))$, then for almost every t with $|t-\Psi_n(x')|<\rho'/2$ it is true that in a neighborhood of $\hat z$, we have $H(\cdot,t)\in W^{1,2}$ as a function of $x_1,\dots,x_{n-1}$. Let

$$\hat e_i = (0,\dots,0,1,0,\dots,0,-\bar u_{x_i}/\bar u_{x_n}(x_1,\dots,x_{n-1},t))$$

denote the vector with 1 in the i th position and $-\bar u_{x_i}/\bar u_{x_n}(x_1,\dots,x_{n-1},t)$, in the n th position, for $1\le i\le n-1$,. Then from either (6.20) or (6.21) we have for $\mathcal{H}^{n-1}$ almost every $(x_1,\dots,x_{n-1})$ in a neighborhood of $\hat z$ that

$$\frac{\partial H}{\partial x_i}(\cdot,t) = \nabla f(\nabla \bar u)\cdot \hat e_i = 0. \tag{6.24}$$

Transferring this information to $f(\nabla\bar u)$ we see first for almost every t that $f(\nabla \bar u)$ is constant on $\{x:\bar u(x)=t\}\cap B(x',\rho/2)$. Second from continuity of $f(\nabla\bar u)$ and $\bar u$, we then conclude this statement for every t. Finally, the definition of a component and continuity of $f(\nabla\bar u)$, $\bar u$ imply assertion (6.22).

Armed with (6.22) we can show for G', t_0, as in (6.11) and ξ_1 as in (6.10) that if O' is an open set in Ω containing $\hat K = \{x\in G' : \xi_1/2 \le \bar u(x)\le t_0\}$ then

$$\nu(O')>0. \tag{6.25}$$

Indeed otherwise, by our construction, n homogeneity of f, (6.11), and (6.22) we have $\nabla\bar u = 0$ on $\partial G'\cap\{x: \bar u(x)=t_0\}$ which easily leads to a contradiction by a barrier argument. In fact, if

$$B(y,\hat r)\subset \hat K\cap\{u<t_0\} \text{ with } \hat y\in\partial B(y,\hat r)\cap\{x:\ \bar u(x)=t_0\}$$

then from the Hopf maximum principle $|\nabla\bar u(\hat y)|>0$. From this contradiction we conclude that (6.25) is valid when $p=n\ge 3$.

6.2. **The case $n=p=2$.** In this case we note from (1.1) and the computation in Lemma 3.18 that $\tilde L\bar v=0$ weakly on $\{x:\nabla\bar u(x)\neq 0\}$ and $\tilde L$ is uniformly elliptic where

$$\tilde L\bar v = \sum_{k,j=1}^{2}\frac{\partial}{\partial x_k}\Big(f_{\eta_k\eta_j}\frac{\partial \bar v}{\partial x_j}\Big)$$

as in section 1. To analyze this case let $\hat x\in\Omega$ be any point with $\nabla\bar u(\hat x)=0$. We temporarily use complex notation and write $\bar u_z = (1/2)(\bar u_{x_1} - i\bar u_{x_2})$ where $i=\sqrt{-1}$. We note that $\bar u_z$ is a $k-$quasiregular mapping of Ω, where $k=k(p,n,c_*)$ (see [4, 16.4.3] for this fact and more on quasiregular mappings in the plane). From properties of quasiregular mappings we see that the zeros of u_z in Ω are isolated. Next we note from the factorization theorem for quasiregular mappings (see [4, Corollary 5.5.4]) that $\bar u_z = \mathfrak{t}\circ\mathfrak{s}$ where $\mathfrak{t}$ is analytic in $\mathfrak{s}(\Omega)$, $\mathfrak{s}$ is a quasiconformal mapping of $\mathbb{R}^2$, and $\mathfrak{s}(\hat x)=0$. From local properties of analytic functions, and $\mathbb{R}^2$ quasiconformal mappings, as well as (1.1), it follows that there exists $\tilde r>0$ such that $B(\hat x, 8\tilde r)\subset\Omega$ and if $0<\rho\le 2\tilde r$, then

$$0<c_-^{-1} f(\nabla\bar u(x))\le f(\nabla \bar u(y))\le c_- f(\nabla\bar u(x)) \tag{6.26}$$

whenever $x, y \in B(\hat{x}, 2\rho) \setminus \bar{B}(\hat{x}, \rho/4)$. Here $c_- \geq 1$ may depend on various quantities but is independent of ρ . Using (6.26), standard Caccioppoli type estimates for linear divergence form PDE, and Hölder's inequality we find that

$$
\begin{aligned}
c^{-1}\left(\int_{B(\hat{x},\rho)\setminus\bar{B}(\hat{x},\rho/2)} |\nabla\bar{v}|\mathrm{d}x\right)^2 &\leq \rho^2 \int_{B(\hat{x},\rho)\setminus\bar{B}(\hat{x},\rho/2)} |\nabla\bar{v}|^2\mathrm{d}x \\
&\leq c\int_{B(\hat{x},2\rho)\setminus\bar{B}(\hat{x},\rho/4)} |\bar{v}-\bar{v}(\hat{x}+(\rho,0))|^2\mathrm{d}x \\
&\leq c^2\rho^2
\end{aligned}\tag{6.27}
$$

where again $c \geq 1$ is a positive constant independent of ρ. Putting $\rho = 2^{-l}\tilde{r}$ in (6.27) and summing over $l = -1, 0, \dots,$ we find that

$$
\int_{B(\hat{x},2\tilde{r})} |\nabla\bar{v}|\mathrm{d}x < c^*\tilde{r} < \infty. \tag{6.28}
$$

In view of (6.26)-(6.28) we can now use a more or less standard argument to show that if $0 \leq \chi \in C_0^\infty(B(\hat{x}, 2\tilde{r}))$ then

$$
\int_{B(\hat{x},2\tilde{r})} \sum_{k,j=1}^{2} f_{\eta_k\eta_j}(\nabla\bar{u})\,\frac{\partial\bar{v}}{\partial x_j}\,\frac{\partial\chi}{\partial x_k}\mathrm{d}x = -\hat{a}\,\chi(\hat{x}) \tag{6.29}
$$

for some $\hat{a} > 0$. For completeness we give the proof of (6.29) here. To do this let $\sigma \in C_0^\infty(B(\hat{x}, 2\tilde{r}))$ with $\sigma = 1$ on $\bar{B}(\hat{x}, \tilde{r})$. If $\phi \in C^\infty(B(\hat{x}, 2\tilde{r}) \setminus \bar{B}(\hat{x}, \rho/2))$ we first show that for $\mathcal{H}^1$ almost every ρ with $0 < \rho < \tilde{r}$,

$$
\int_{B(\hat{x},2\tilde{r})\setminus\bar{B}(\hat{x},\rho)} \sum_{k,j=1}^{2} f_{\eta_k\eta_j}(\nabla\bar{u})\frac{\partial\bar{v}}{\partial x_j}\frac{\partial(\phi\sigma)}{\partial x_k}\mathrm{d}x = \int_{\partial B(\hat{x},\rho)} \sum_{k,j=1}^{2} f_{\eta_k\eta_j}(\nabla\bar{u})\frac{\partial\bar{v}}{\partial x_j}\xi_k\,\phi\mathrm{d}\mathcal{H}^1 \tag{6.30}
$$

where $\xi = (\xi_1, \xi_2)$ denotes the inward unit normal to $\partial B(\hat{x}, \rho)$. To verify (6.30) for small $\hat{\delta} > 0$, let $\psi \in C_0^\infty([\rho - \hat{\delta}, \infty))$ with $\psi \equiv 1$ on $[\rho, \infty)$. Put $\hat{\psi}(x) = \psi(|x - \hat{x}|), x \in \mathbb{R}^2$, and replace $\phi\sigma$ by $\phi\hat{\psi}\sigma$ on the left hand side of (6.30). Then the resulting integral is now zero since $\tilde{L}\bar{v} = 0$ weakly in $B(\hat{x}, 2\tilde{r}) \setminus \{\hat{x}\}$. Using this fact, the Lebesgue differentation theorem, letting $\hat{\delta}\to 0$, and doing some arithmetic, we eventually obtain (6.30). Next from (6.27) and a weak type estimate we see there exists ρ' with $\rho/2 \leq \rho' \leq \rho$ such that

$$
\int_{\partial B(\hat{x},\rho')} |\nabla\bar{v}|\,\mathrm{d}\mathcal{H}^1 \leq c' \tag{6.31}
$$

where c' is independent of ρ. Using (6.28), (6.30), and (6.31), we find for a sequence (ρ_l) with $\lim_{l\to\infty}\rho_l = 0$ and $0 \leq \chi \in C_0^\infty(B(\hat{x}, 2\tilde{r}))$ that

(6.32)

$$
\begin{aligned}
\int_{B(\hat{x},2\tilde{r})} \sum_{k,j=1}^{2} f_{\eta_k\eta_j}(\nabla\bar{u})\frac{\partial\bar{v}}{\partial x_j}\frac{\partial(\chi\sigma)}{\partial x_k}\mathrm{d}x &= \lim_{l\to\infty} \int_{B(\hat{x},2\tilde{r})\setminus\bar{B}(\hat{x},\rho_l)} \sum_{k,j=1}^{2} f_{\eta_k\eta_j}(\nabla\bar{u})\frac{\partial\bar{v}}{\partial x_j}\frac{\partial(\chi\sigma)}{\partial x_k}\mathrm{d}x \\
&= \lim_{l\to\infty} \int_{\partial B(\hat{x},\rho_l)} \sum_{k,j=1}^{2} f_{\eta_k\eta_j}(\nabla\bar{u})\frac{\partial\bar{v}}{\partial x_j}\xi_k\,\chi\mathrm{d}\mathcal{H}^1 \\
&= \chi(\hat{x})\lim_{l\to\infty}\int_{\partial B(\hat{x},\rho_l)} \sum_{k,j=1}^{2} f_{\eta_k\eta_j}(\nabla\bar{u})\frac{\partial\bar{v}}{\partial x_j}\xi_k\,\mathrm{d}\mathcal{H}^1 \\
&= -\chi(\hat{x})\hat{a}
\end{aligned}
$$

for some real $\hat{a}$. Now (6.29) follows from (6.32) and the observation that $\chi\sigma$ can be replaced in (6.32) by χ since $\chi(1-\sigma)$ has compact support in $B(\hat{x},2\tilde{r})\setminus\{\hat{x}\}$ and $\tilde{L}\bar{v}=0$ weakly in $B(\hat{x},2\tilde{r})\setminus\{\hat{x}\}$. Finally to show $\hat{a}>0$ we note that (6.30) remains true if ϕ is replaced by $\bar{v}$, as follows from approximating $\bar{v}$ in the $W^{1,2}(B(\hat{x},2\tilde{r}))\setminus\bar{B}(\hat{x},\rho/2))$ norm by smooth functions and taking limits using Lemma 2.6. Doing this we deduce from the left hand integral in (6.30) that

$$\int\limits_{B(\hat{x},2\tilde{r})\setminus\bar{B}(\hat{x},\rho_l)}\sum_{k,j=1}^{2} f_{\eta_k\eta_j}(\nabla\bar{u})\frac{\partial\bar{v}}{\partial x_j}\frac{\partial(\bar{v}\sigma)}{\partial x_k}\mathrm{d}x \geq c^{-1}\int\limits_{B(\hat{x},\tilde{r})\setminus\bar{B}(\hat{x},\rho_l)}|\nabla\bar{v}|^2\mathrm{d}x - c\int\limits_{B(\hat{x},2\tilde{r})\setminus B(\hat{x},\tilde{r})}|\nabla\bar{v}||\bar{v}||\nabla\sigma|\mathrm{d}x$$

where c depends only on p,n,c_*,α,β. Moreover from the right hand integral in this inequality and (6.26) we find that

$$\int\limits_{\partial B(\hat{x},\rho_l)}\sum_{k,j=1}^{2} f_{\eta_k\eta_j}(\nabla\bar{u})\frac{\partial\bar{v}}{\partial x_j}\xi_k\,\bar{v}\mathrm{d}\mathcal{H}^1 = \bar{v}(\hat{x}+(\rho_l,0))\int\limits_{\partial B(\hat{x},\rho_l)}\sum_{k,j=1}^{2} f_{\eta_k\eta_j}(\nabla\bar{u})\frac{\partial\bar{v}}{\partial x_j}\xi_k\,\mathrm{d}\mathcal{H}^1 + T_l$$

where $|T_l|\leq\bar{c}$ and $\bar{c}$ is independent of l. If $\hat{a}=0$ in (6.32), then from the above estimates it follows easily that $\bar{v}\in W^{1,2}(B(\hat{x},\tilde{r}))$. However then linear elliptic PDE theory yields that $\bar{v}$ is bounded in $B(\hat{x},\tilde{r}/2)$, which is a contradiction. Thus $\hat{a}\neq 0$. Using this fact and comparing the above inequalities we see that

$$+\infty = \lim_{l\to\infty}\bar{v}(\hat{x}+(\rho_l,0))\int\limits_{\partial B(\hat{x},\rho_l)}\sum_{k,j=1}^{2} f_{\eta_k\eta_j}(\nabla\bar{u})\frac{\partial\bar{v}}{\partial x_j}\xi_k\,\mathrm{d}\mathcal{H}^1.$$

Since

$$\bar{v}(\hat{x}+(\rho_l,0))\to-\infty \text{ as } l\to\infty$$

it follows that necessarily $\hat{a}>0$. From (6.29) we see that $\tilde{L}\bar{v}$ may be regarded weakly as an atomic measure on Ω when $p=2, n=2$ and hence (6.25) is also valid when $n=2, p=2$.

We now are in a position to finish the proof of Lemma 6.9 when $p=n$ and for a general f, as well as when $f=g^p, p\in[n,n+1]$, and g is as in Theorem 1.13. We consider first the case when $f=g^p$, as the compactness argument in either case is essentially the same.

We shall need some more notation. For fixed α,β, let

$$\tilde{\Gamma}_m=\{\tilde{Q}^{(m)}\},\ m=1,2,\dots,$$

be collections of cubes with side lengths defined as in section 1 with $a_1,a_2,...,$ replaced by $a_1^{(m)},a_2^{(m)},\dots,$ where $0<\alpha\leq a_k^{(m)}<\beta$, for $k,m=1,\dots$. Let $\mathcal{C}_m$ denote the corresponding Cantor set and suppose $\tilde{\Lambda}_m$ is a finite covering of $\mathcal{C}_m$ by disjoint cubes in $\tilde{\Gamma}_m$. Define $Q^{(m)}$ relative to $\tilde{Q}^{(m)}$ in the same way that Q_0 below (6.6) is defined relative to $\tilde{Q}_0$ and set

$$\Lambda_m := \{Q^{(m)}:\ \tilde{Q}^{(m)}\in\tilde{\Lambda}_m\} \text{ and } \Omega_m := B(0,n)\setminus\bigcup_{Q^m\in\Lambda_m}\bar{Q}^{(m)}.$$

Suppose (p_m) is a sequence of points in $[n,n+1]$ with $\lim_{m\to\infty}p_m=\hat{p}$. Let $f_m=g^{p_m}$ and let $\bar{u}_m$ be the weak solution to (1.4) relative to f_m on Ω_m with continuous boundary values, 1 on $\partial B(0,n)$ and 0 on $\partial Q^{(m)}$ for every $Q^{(m)}\in\Lambda_m$. Extend $\bar{u}_m$ to $B(0,n)$ by putting $\bar{u}_m=0$ on $Q^{(m)}$ for every $Q^{(m)}\in\Lambda_m$. Let $\bar{\mu}_m$ be the measure associated with $\bar{u}_m$ as in (1.5) and let $\bar{v}_m=\log f_m(\nabla\bar{u}_m)$. Finally define $\tilde{L}_m\bar{v}_m=\nu_m$, weakly as in Lemma 3.18 relative to $\bar{u}_m, f_m$, on $\{x:\nabla u_m(x)\neq 0\}$ when $n\geq 3$ and on Ω_m when $n=p=2$ (see (6.29)).

From (6.25) and (6.19) we see that if Lemma 6.9 is false for c_4 sufficiently large and $n \geq 3$, then there exists $\tilde{Q}^{(m)} \in \tilde{\Gamma}_m$ with

$$0 < \int_{O_m \cap \{|\nabla \bar{u}_m| > 0\}} \bar{u}_m \mathrm{d}\nu_m = b_m \mu_m(\bar{Q}^{(m)}) \tag{6.33}$$

where

$$O_m = \left\{ x \in (1+\theta)\tilde{Q}^{(m)} : \, d(x, \partial\Omega_m) \geq \frac{s(\tilde{Q}^{(m)})}{c_4} \right\} \text{ and } 0 < b_m \to 0 \text{ as } m \to \infty.$$

Let z_m denote the center of $\tilde{Q}^{(m)}$ and let $\hat{\Omega}_m = \{y : z_m + s(\tilde{Q}^m)y \in \Omega_m\}$. Put

$$\hat{u}_m(y) = \frac{\bar{u}_m(z_m + s(\tilde{Q}^m)y)}{\max\limits_{2\tilde{Q}^{(m)}} \bar{u}_m} \text{ whenever } y \in \hat{\Omega}_m.$$

Using translation and dilation invariance of (1.1) we see that $\hat{u}_m$ is a weak solution to (1.4) in $\hat{\Omega}_m$. Let $\hat{\mu}_m$ denote the measure corresponding to $\hat{u}_m$ with f and u replaced by f_m and $\hat{u}_m$. Then from Lemma 2.4 and Harnack's inequality we find from estimates similar to those in (6.1) that

$$c^{-1} \leq \hat{\mu}_m(\mathcal{S}) \leq \max_{1000\mathcal{S}} \hat{u}_m \leq c\hat{\mu}_m(2000\mathcal{S}) \leq c^2 \tag{6.34}$$

where $c \geq 1$ is independent of $f_m, p_m \in [n, n+1]$ for fixed c_* in (1.1). Once again we emphasize that this independence follows from the fact that the constants in Lemmas 2.2-2.6 can be chosen independent of these quantities. Let $\hat{v}_m = \log f_m(\nabla \hat{u}_m)$. Then $\hat{v}_m$ is a weak sub solution to $\hat{L}_m$ in the interior of $1000\mathcal{S} \cap \{x : \nabla \hat{u}_m \neq 0\}$ where $\hat{L}_m$ is defined as in Lemma 3.18 relative to $\hat{u}_m, f_m$. Let $\hat{\nu}_m$ be the corresponding measure. From (6.33)-(6.34) we deduce that if $\hat{O}_m = \{y : z_m + s(\tilde{Q}_m)y \in O_m\}$, then

$$\int_{\hat{O}_m} \hat{u}_m \mathrm{d}\hat{\nu}_m \to 0 \text{ as } m \to \infty. \tag{6.35}$$

Using (6.34), Lemmas 2.2-2.6, the fact that $d(\cdot, \partial\hat{\Omega}'_m)$ is Lipschitz, and Ascoli's theorem we see there exists sub sequences, $(\hat{\Omega}'_m)$ of $(\hat{\Omega}_m)$ such that $\hat{\Omega}'_m \cap B(0,R)$ converges to $\hat{\Omega} \cap B(0,R)$ for each $R > 0$ in the Hausdorff distance metric and $(\hat{u}'_m)$ of $(\hat{u}_m)$ with $(\hat{u}'_m)$ converging uniformly to $\hat{u}$ in the interior of $1000\mathcal{S}$. We also can choose the sub sequence so that $\nabla \hat{u}'_m$ converges uniformly to $\nabla \hat{u}$ on compact subsets of $\hat{\Omega}$. Using these facts it is easily seen that $\hat{u}$ is a weak solution to (1.4) with $\hat{f} = g^{\hat{p}}$ in the interior of $1000\mathcal{S} \cap \hat{\Omega}$ and $\hat{u}$ is continuous in the interior of $1000\mathcal{S}$ with $\hat{u} = 0$ on $\partial\hat{\Omega} \cap 1000\mathcal{S}$. Let $\hat{\mu}$ be the measure corresponding to $\hat{u}$ and let $\hat{\nu}$ be the measure corresponding to $\hat{v}$. Then for $n \geq 3$ we may also assume that $\hat{L}'_m \hat{v}'_m = \hat{\nu}'_m$ converges weakly to $\hat{L}\hat{v} = \nu$ as measures on compact subsets in the interior of $\mathcal{S} \cap \{x : \nabla \hat{u}(x) \neq 0\}$. Indeed from the definition of f_m and uniform convergence of $(\nabla \hat{u}'_m)$ we see that $(f_m)_{\eta_k \eta_j}(\nabla \hat{u}'_m), 1 \leq k, j \leq n$, converges uniformly on compact subsets in the interior of $\mathcal{S} \cap \{x : \nabla \hat{u}(x) \neq 0\}$. Also from Lemma 2.6 we deduce that for large m, $\hat{v}'_m$ is uniformly bounded in $W^{1,2}$ on an open set with compact closure in $\mathcal{S} \cap \{x : \nabla \hat{u}(x) \neq 0\}$. Using these facts and well known theorems on weak convergence in $W^{1,2}$ we see that if $n \geq 3$, then a sub sequence of $(\hat{v}'_m)$ (also denoted $(\hat{v}'_m)$) yields,

$$
\begin{aligned}
-\lim_{m\to\infty}\int \phi \mathrm{d}\hat{\nu}'_m &= \lim_{m\to\infty}\int \sum_{j,k=1}^{n} (f_m)_{\eta_k\eta_j}(\nabla \hat{u}'_m)(\hat{v}'_m)_{x_j}\phi_{x_k}\mathrm{d}x \\
&= \int \sum_{j,k=1}^{n} \hat{f}_{\eta_k\eta_j}(\nabla \hat{u})\hat{v}_{x_j}\phi_{x_k}\mathrm{d}x \qquad (6.36)\\
&= -\int \phi d\hat{\nu}
\end{aligned}
$$

whenever ϕ is infinitely differentiable with compact support in $1000\mathcal{S}\cap\hat{\Omega}\cap\{x : \nabla\hat{u}(x) \neq 0\}$. If $n = 2$ we claim that $\hat{\nu}'_m$ converges weakly to $\hat{\nu}$ on compact subsets in the interior of $1000\mathcal{S} \cap \hat{\Omega}$. To see this we note from the discussion preceding (6.26) that there exists t_m analytic in $s_m(\hat{\Omega}'_m)$ and s_m quasiconformal in $\mathbb{R}^2$ with $(u'_m)_z = t_m \circ s_m$ in $\hat{\Omega}'_m$. From normal family type arguments for $\mathbb{R}^2$ quasiconformal mappings and analytic functions we see that there exist subsequences of $(t_m), (s_m)$ (also denoted $(t_m), (s_m)$) with (s_m) converging to s a quasiconformal mapping of $\mathbb{R}^2$, uniformly on compact subsets of $\mathbb{R}^2$, and t_m converging uniformly to t analytic, uniformly on compact subsets in the interior of $s(1000\mathcal{S} \cap \hat{\Omega})$. Using these facts and the argument principle for analytic functions we conclude that the constants in (6.26) - (6.28) can be chosen independent of m. From this conclusion, uniform convergence of $(\nabla \hat{u}'_m)$ and simple estimates in (6.29) we obtain (6.36) for ϕ infinitely differentiable with compact support in $1000\mathcal{S} \cap \hat{\Omega}$. Let

$$
\hat{O} = \left\{x \in (1+\theta)\mathcal{S} : \ d(x, \partial\hat{\Omega}) > c_4^{-1}\right\}.
$$

Then from (6.36) and (6.35) we have

$$
\hat{\nu}(\hat{O}) = 0. \tag{6.37}
$$

On the other hand, we can essentially repeat the argument from (6.11)-(6.32) since the same constants in Lemmas 2.2-2.6 as earlier can also be used for $\hat{u}$. Moreover, since $1000\mathcal{S} \cap \hat{\Omega}_m$ converges in the Hausdorff distance sense to $1000\mathcal{S} \cap \hat{\Omega}$ the Harnack chains used to obtain the analogue of (6.25) can all be chosen in $\hat{\Omega}_m$ for m large enough. A more cut to the chase type argument is to observe that if $\hat{x}_m, t_0^{(m)}, G'_m$ denote the sets in (6.11), and ξ_1^m is as in (6.10) relative to $\hat{u}'_m$ in $(1+\theta)\mathcal{S} \cap \hat{\Omega}'_m$, then these sequences converge pointwise and in the Hausdorff distance sense to $\hat{x}', \hat{t}_0, \hat{\xi}_1, \hat{G}' \subset (1+\theta)\mathcal{S}$. Moreover (6.10),(6.11) are now valid for $\hat{u}$ in this symbology. Repeating the argument leading to (6.19) we see that in order to avoid a contradiction to (6.37) we must have $\hat{p} = n$. Now repeating the argument from (6.18) to (6.32) we also rule out the case $\hat{p} = n$ and so for c_2, c_4 large enough, obtain $\hat{\nu}(\hat{O}) > 0$, a contradiction to (6.37). The proof of Lemma 6.9 is now complete when $f = g^p$. For a general f it follows from (6.19) that we need only consider the case $p = n$. If $p = n$, we again argue by contradiction and use a compactness argument similar to the above to get a contradiction. We omit the details. □

Following [10, Chapter IX, Theorem 2.1], we continue the proof of Proposition 6.5 by repeating the stopping time argument in Theorem 1.11 only with cubes in $\tilde{\Gamma}$ rather than balls. First let $M >> 1$ be so large that if $\tilde{Q} \in \tilde{\Gamma}$ and $\mu(\tilde{Q}) \geq Ms(\tilde{Q})^{n-1}$, then $s(\tilde{Q}) \leq \min(\tau, 10^{-5})$. This choice is possible as we see from (6.4). Let $s << \tau$ and choose a covering $\tilde{\Lambda}_M = \mathcal{B}_M \cup \mathcal{G}_M$ of $\mathcal{C}$ by cubes in $\tilde{\Gamma}$, according to the following recipe. Either $x \in \mathcal{C}$ lies in a cube in

$$
\mathcal{G}_M := \left\{\tilde{Q} \in \tilde{\Gamma} : \ s(\tilde{Q}) > s, \ \mu(\tilde{Q}) \geq Ms(\tilde{Q})^{n-1}, \text{ and } \tilde{Q} \text{ is maximal}\right\}
$$

or no such cube exists and x lies in a cube in

$$\mathcal{B}_M := \{\tilde{Q} \in \tilde{\Gamma} : \ s(\tilde{Q}) \le s \text{ and } \tilde{Q} \text{ is maximal}\}.$$

Note that $\mathcal{B}_M \cup \mathcal{G}_M$ is a disjoint covering of $\mathcal{C}$. As earlier let $\Lambda_M = \{Q : \tilde{Q} \in \tilde{\Lambda}_M\}$ and define $\bar{u}$ as below (6.7) relative to Λ_M. Then $\bar{u}$ is a solution to (1.4) in $\Omega = B(0,n) \setminus \cup_{Q\in\Lambda_M}\bar{Q}$ and continuous in $B(0,n)$ with $\bar{u} = 0$ on $\cup_{Q\in\Lambda_M}\bar{Q}$ while $\bar{u} = 1$ on $\partial B(0,n)$. Let $\bar{\mu}$ be the measure associated with $\bar{u}$ as in (1.5). From the maximum principle for solutions to (1.4) we see that $\bar{u} \le u$ in $\mathcal{S}$ and as in (6.1) and (6.4) that for $\tilde{Q} \in \tilde{\Lambda}_M$,

$$\begin{aligned} &s(\tilde{Q})^{1-n}\bar{\mu}(\bar{Q}) \le c\, s(\tilde{Q})^{1-p} \max_{(1+\theta)\tilde{Q}} u^{p-1} \le c^2\, s(\tilde{Q})^{1-n}\, \mu(2\tilde{Q}) \\ &\bar{\mu}(B(0,n)) \approx 1. \end{aligned} \tag{6.38}$$

where $Q \in \Lambda_M$ corresponds to $\tilde{Q} \in \tilde{\Lambda}_M$. Let

$$\mathcal{E} := \left\{\tilde{Q} \in \tilde{\Gamma} \setminus \tilde{\Lambda}_M \text{ for which there exists } \tilde{Q}' \in \tilde{\Lambda}_M \text{ with } \tilde{Q}' \subset \tilde{Q} \text{ and } c_2\, s(\tilde{Q}') \le s(\tilde{Q})\right\}.$$

For c_2, c_4 as in Lemma 6.9 and $\tilde{Q} \in \mathcal{E}$ we also define

$$O := O(\tilde{Q}) = \left\{x \in (1+\theta)\tilde{Q} \text{ with } d(x,\partial\Omega) \ge \frac{s(\tilde{Q})}{c_4}\right\}.$$

We note that each point in

$$\bigcup_{\tilde{Q}\in\mathcal{E}} O(\tilde{Q}) \text{ lies in at most } \hat{N} \text{ of the } \tilde{Q} \in \mathcal{E}$$

where $\hat{N}$ has the same dependence as c_4. Using this observation and Lemma 6.4 it follows for $n \ge 3$ that

$$\sum_{\tilde{Q}\in\mathcal{E}} \bar{\mu}(\bar{Q}) \le \breve{c} \int\limits_{\Omega\cap\{x:\nabla\bar{u}\neq 0\}} \bar{u}\mathrm{d}\nu \le 2\breve{c} \int\limits_{\Omega\cap\{x:|\nabla\bar{u}|>\delta''\}} \bar{u}\mathrm{d}\nu. \tag{6.39}$$

provided $\delta'' > 0$ is small enough.

If $p = 2 = n$ the integral on the right hand side of (6.39) is taken over Ω. In general $\breve{c}$ depends on p, n, α, β, c_* but in view of Lemma 6.4 we have $1 \le \breve{c} \le c(p-n)^{-1}$, where c can be chosen to depend only on n, α, β, c_* when $p \in [n, n+1]$ while if $f = g^p$ then $\breve{c}$ can be chosen to depend only on n, α, β, c_* when $p \in [n, n+1]$.

We now essentially repeat the argument leading to Lemma 4.22. Choose $\eta \in (-\infty, \infty)$ so small that if $|\xi| \le \delta''$ then $\log f(\xi) \le \eta$. Using (6.39) and arguing as (4.26)-(4.32) we obtain for $n \ge 3$ and $v' = \max(\bar{v}, \eta)$ that

$$\sum_{\tilde{Q}\in\mathcal{E}} \bar{\mu}(\bar{Q}) \le 2\breve{c} \int\limits_{\Omega\cap\{x:|\nabla\bar{u}|>\delta''\}} \bar{u}\mathrm{d}\bar{\nu} \le -\int_\Omega \sum_{k,j=1}^n f_{\eta_k\eta_j}(\nabla\bar{u}) v'_{x_j}\bar{u}_{x_k}\mathrm{d}x \le c\breve{c}\log M. \tag{6.40}$$

To estimate the left hand side of (6.40), given $\tilde{Q}' \in \tilde{\Lambda}_M$, we let $\sigma(\tilde{Q}')$ be the number of cubes $\tilde{Q} \in \mathcal{E}$ with $\tilde{Q}' \subset \tilde{Q}$ and $c_2 s(\tilde{Q}') \le s(\tilde{Q})$. From our construction we see for τ small enough that

$$\sigma(\tilde{Q}') \ge -c^{-1}\log(s(\tilde{Q}')) \tag{6.41}$$

From (6.40) and (6.41) we get

$$-\sum_{\tilde{Q}'\in\tilde{\Lambda}_M} \log(s(\tilde{Q}'))\bar{\mu}(\bar{Q}') \ \le\ c\sum_{\tilde{Q}\in\mathcal{E}} \bar{\mu}(\bar{Q}) \le c^2\breve{c}\log M \tag{6.42}$$

where $c \geq 1$ in (6.40), (6.41), and (6.42) has the same dependence as c_2 in Lemma 6.9. From (6.38) and (6.42) we see that if c is large enough with

$$\tilde{\Lambda}_1 := \{\tilde{Q} \in \tilde{\Lambda}_M : s(\tilde{Q}) \leq M^{-c^3\check{c}}\} \text{ and } \Lambda_1 := \{Q : \tilde{Q} \in \tilde{\Lambda}_1\}$$

then

$$\sum_{Q \in \Lambda_1} \bar{\mu}(\bar{Q}) \leq (1/2)\tilde{\mu}(B(0,n)). \tag{6.43}$$

Finally choosing $s << \min(M^{-c^3\check{c}}, \tau)$, we see that $\mathcal{B}_M \subset \tilde{\Lambda}_1$. Let

$$F := \mathcal{C} \cap \left(\bigcup_{\tilde{Q} \in \tilde{\Lambda}_M \setminus \tilde{\Lambda}_1} \tilde{Q} \right).$$

Then from (6.1), (6.38), and (6.43) we deduce for c having the same dependence as in (6.40)-(6.43) that

$$c^{-1} \leq \bar{\mu} \left(\bigcup_{Q \in \Lambda_M \setminus \Lambda_1} \bar{Q} \right) \leq c\mu(F). \tag{6.44}$$

Moreover, if $\delta' = \frac{1}{2\check{c}c^3}$, where c is as in the definition of $\tilde{\Lambda}_1$, then since $\tilde{\Lambda}_M \setminus \tilde{\Lambda}_1 \subset \mathcal{G}_M$, we have

$$\sum_{\tilde{Q} \in \tilde{\Lambda}_M \setminus \Lambda_1} s(\tilde{Q})^{n-1-\delta'} \leq cM^{-1/2} \sum_{\tilde{Q} \in \tilde{\Lambda}_M \setminus \Lambda_1} \mu(\tilde{Q}) \leq M^{-1/4} \leq \epsilon \tag{6.45}$$

provided $M \geq M_0$ is large enough. In view of our earlier calculations we conclude that δ' has the same dependence as in (6.6). Moreover if $f = g^p, g$ as in Theorem 1.13, then M_0 can be chosen independent of p in $[n, n+1]$. It follows from (6.44) and (6.45) that (6.6) is true. From our earlier remarks we conclude that Proposition 6.5 holds which finishes proof of Theorem 1.13. $\square$

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