NOTE ON AN EIGENVALUE PROBLEM WITH APPLICATIONS TO A MINKOWSKI TYPE REGULARITY PROBLEM IN $\mathbb{R}^n$

MURAT AKMAN, JOHN LEWIS, AND ANDREW VOGEL

ABSTRACT. We consider existence and uniqueness of homogeneous solutions $u > 0$ to certain PDE of p-Laplace type, p fixed, $n-1 < p < \infty, n \geq 2$, when u is a solution in $K(\alpha) \subset \mathbb{R}^n$ where

$$K(\alpha) := \{x = (x_1, \dots, x_n) : x_1 > \cos\alpha\, |x|\} \quad \text{for fixed } \alpha \in (0, \pi],$$

with continuous boundary value zero on $\partial K(\alpha) \setminus \{0\}$. In our main result we show that if u has continuous boundary value 0 on $\partial K(\pi)$ then u is homogeneous of degree $1 - (n-1)/p$ when $p > n-1$. Applications of this result are given to a Minkowski type regularity problem in $\mathbb{R}^n$ when $n = 2, 3$.

CONTENTS

1. INTRODUCTION

Let $u > 0$ be a homogeneous p-harmonic function in the cone $K(\alpha) \subset \mathbb{R}^n$, $n \geq 2$, with continuous boundary value 0 on $\partial K(\alpha) \setminus \{0\}$ where

$$K(\alpha) := \{x = (x_1, \dots, x_n) : x_1 > \cos\alpha\, |x|\} \quad \text{for } \alpha \in (0, \pi].$$

2010 *Mathematics Subject Classification.* 35J60,31B15,39B62,52A40,35J20,52A20,35J92.

Key words and phrases. Eigenvalue problem, homogeneous solutions to $\mathcal{A}$-harmonic PDEs, Potentials, capacities, $\mathcal{A}$-harmonic Green's function, Minkowski problem, regularity in Monge-Ampère equation.

More specifically, for fixed $p, 1 < p < \infty$, u is a weak solution to $\nabla \cdot \left(|\nabla u|^{(p-2)} \, \nabla u \right) = 0$ in $K(\alpha)$ and

$$u(tx) = t^{\lambda} u(x) \text{ for some real } \lambda \text{ whenever } t > 0 \text{ and } x \in K(\alpha). \tag{1.1}$$

Given $x \in \mathbb{R}^n \setminus \{0\}$, introduce spherical coordinates $r = |x|$ and $x_1 = r \cos \theta$ for $0 \leq \theta \leq \pi$. If u as in (1.1) is p-harmonic in $K(\alpha)$ and $u(1, 0, \ldots, 0) = 1$ then using rotational invariance of the p-Laplace equation, it turns out that u has additionally the following form

$$u(x) = u(r, \theta) = r^{\lambda} \, \phi(\theta) \quad \text{for } 0 \leq \theta < \alpha \text{ and } r > 0 \tag{1.2}$$

with $\phi(0) = 1$ and $\phi(\alpha) = 0$ for some $\lambda(\alpha) = \lambda \in (-\infty, \infty)$ and $\phi \in C^{\infty}([0, \alpha])$.

It was first shown by Krol' and Maz'ya in [KM72] that if $1 < p \leq n - 1$ and $\alpha \in (0, \pi)$, α is near enough π, then there exists a unique solution to (1.1) in $K(\alpha)$ of the special form (1.2) with $\lambda(\alpha) > 0$. Tolksdorf in [Tol83] showed that given $\alpha \in (0, \pi)$, for $i = 1, 2$, there exist unique λ_i with $\lambda_2 < 0 < \lambda_1$ and ϕ_i where ϕ_i is infinitely differentiable on $[0, \alpha]$ satisfying $\phi_i(\alpha) = 0$ and $\phi_i(0) = 1$ and $u_i(r, \theta) = r^{\lambda_i} \phi_i(\theta)$ are solutions to the p- Laplace equation in $K(\alpha)$. Also Porretta and Véron gave another proof of Tolksdorf's result in [PV09]. A similar study was made in more general Lipschitz cones by Gkikas and Véron in [GV18].

Next we discuss what is known about "eigenvalues" λ in (1.2) for various α and n. Krol' in [Kro73] (see also [Aro86]) used (1.2) and separation of variables to show for u as in (1.2) that

$$\begin{aligned} 0 = & \frac{d}{d\theta} \left\{ [\lambda^2 \phi^2(\theta) + (\phi')^2(\theta)]^{(p-2)/2} \, \phi'(\theta) \, (\sin \theta)^{n-2} \right\} \\ & + \lambda [\lambda(p-1) + (n-p)] [\lambda^2 \phi^2(\theta) + (\phi')^2(\theta)]^{(p-2)/2} \phi(\theta) (\sin \theta)^{n-2}. \end{aligned}$$

Letting $\psi = \phi'/\phi$ in the above equation he obtained, the first order DE

$$\begin{aligned} 0 = & ((p-1)\psi^2 + \lambda^2) \, \psi' \\ & + (\lambda^2 + \psi^2) [(p-1) \, \psi^2 + (n-2) \cot \theta \, \psi + \lambda^2 (p-1) + \lambda(n-p)]. \end{aligned} \tag{1.3}$$

If $n = 2$ the cotangent term in the above DE goes out and variables can be separated in (1.3) to get

$$\frac{\lambda d\psi}{\lambda^2 + \psi^2} - \frac{(\lambda - 1) \, d\psi}{\lambda^2 + \psi^2 + \lambda(2-p)/(p-1)} + d\theta = 0.$$

The boundary conditions imply that ϕ is decreasing on $(0, \alpha)$ so $\psi(\alpha) = -\infty$ and $\psi(0) = 0$. Using this fact and integrating it follows that

$$\pm 1 - \frac{\lambda - 1}{\sqrt{\lambda^2 + \lambda(2-p)/(p-1)}} = \frac{2\alpha}{\pi} \tag{1.4}$$

where $+1$ is taken if $\lambda > 0$ and -1 if $\lambda < 0$. For later discussion we note that if $\alpha = \pi/2$, i.e., $K(\pi/2)$ is a half-space, then (1.4) gives

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = \frac{p - 3 - 2\sqrt{p^2 - 3p + 3}}{3(p-1)}.$$

We remark that $\lambda_1 = \lambda_1(\pi/2) = 1$ for $n \geq 2$ since $x_1 = r\cos\theta$ is p-harmonic for $1 < p < \infty$. Also if $\alpha = \pi$ and $n = 2$, i.e., $K(\pi) = \mathbb{R}^2 \setminus (-\infty, 0]$, then (1.4) yields

$$\lambda_1 = 1 - 1/p \quad \text{and} \quad \lambda_2 = (1/16)\left(7p - 16 - \sqrt{81p^2 - 288p + 288}\right)/(p-1).$$

For other values of $\lambda_2 = \lambda_2(\alpha)$ when $n = 2$, see [LV13]. For $n \geq 3$, $\alpha = \pi/2$, and $p = 2$, one can use the Kelvin transformation to get $\lambda_2(\pi/2) = 1 - n$ while if $p = n$, it follows from conformal invariance of the n-Laplacian that $\lambda_2(\pi/2) = -1$. Also if $p = (4n-2)/3$ then

$$-2\,\lambda_2(\pi) = \frac{p+1-n}{p-1} = \beta = \frac{n+1}{4n-5}$$

since $u(r,\theta) = r^{-\beta/2}(\cos(\theta/2))^\beta$ in (1.2) for $\alpha = \pi$. DeBlassie and Smits in [DS16] obtained estimates on $-\lambda_2(\pi/2)$, $1 < p \neq 2 < \infty$, by leaving out the cotangent term in (1.3). In fact their solution to the DE in (1.3) with the cotangent term omitted leads to a supersolution of the form (1.2) for the p-Laplace equation, so leads to a lower estimate for $-\lambda_2(\pi/2)$ in (1.3). Upper and lower estimates for $\lambda_2(\alpha)$ for $\alpha \in (0, \pi/2]$ were also obtained by these authors in [DS18], by finding p-harmonic subsolution and supersolution of the form $r^k\tilde{\phi}(\theta)$ where $k < 0$ and $\tilde{\phi}$ is the solution to (1.2) when $p = 2$ in $K(\alpha)$. Sub and super p-harmonic solutions of the form $r^k\cos\theta$ were also found in $K(\pi/2)$ by Llorente, Manfredi, Troy, and Wu in [LMTW19]. These estimates were then used to find upper and lower bounds for $\lambda_2(\pi/2)$ in $K(\pi/2)$. In [LMTW19], the authors also use shooting methods to give a strictly ODE proof for existence of a solution to (1.3) on $[0, \pi/2]$ satisfying $\psi(0) = 0$ and $\lim_{\theta\to\pi/2}\psi(\theta) = -\infty$ when p and n are fixed with $1 < p < \infty$ and $n \geq 2$.

In this paper we consider problems similar to the above for certain PDEs of p-Laplace type. Our results, when specialized to the p-Laplace equation for fixed $p > n-1$, give a unique solution u to (1.2) in $K(\pi)$ with continuous boundary value 0 on $\partial K(\pi)$ and $\lambda = \lambda_1(\pi) = 1 - (n-1)/p$ when $n \geq 3$ (compare with Krol's $n = 2$ and $\alpha = \pi$ result). To be more specific we need some notation. Put

$$B(z,r) = \{y \in \mathbb{R}^n : |z - y| < r\} \quad \text{whenever } z \in \mathbb{R}^n \text{ and } r > 0.$$

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^n$ and let $|y| = \langle y, y\rangle^{1/2}$ be the Euclidean norm of y. Let dy denote n-dimensional Lebesgue measure on $\mathbb{R}^n$ and let $\mathcal{H}^\gamma$, $0 < \gamma \leq n$, denote γ-dimensional *Hausdorff measure* on $\mathbb{R}^n$ defined by

$$\mathcal{H}^\gamma(E) = \lim_{\delta\to 0}\inf\left\{\sum_j r_j^\gamma;\ E \subset \bigcup_j B(x_j, r_j),\ r_j \leq \delta\right\}$$

where the infimum is taken over all possible δ-covering $\{B(x_j, r_j)\}$ of E. If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions h with distributional gradient $\nabla h = (h_{y_1}, \ldots, h_{y_n})$, both of which are q-th power integrable on O. Let

$$\|h\|_{1,q} = \|h\|_q + \|\,|\nabla h|\,\|_q$$

be the norm in $W^{1,q}(O)$ where $\|\cdot\|_q$ is the usual Lebesgue q norm of functions in the Lebesgue space $L^q(O)$. Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. Given $p, 1 < p < \infty$, suppose $f : \mathbb{R}^n \to [0, \infty)$ satisfies:

(1.5)

(a) $f(t\eta) = t^p f(\eta)$ when $t > 0$ and $\eta \in \mathbb{R}^n$.

(b) There exists $\tilde{a}_1 \geq 1$ such that if $\eta, \xi \in \mathbb{R}^n \setminus \{0\}$, then

$$\tilde{a}_1^{-1} |\xi|^2 |\eta|^{p-2} \leq \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial \eta_i \partial \eta_j}(\eta)\, \xi_i\, \xi_j \leq \tilde{a}_1 |\xi|^2 |\eta|^{p-2}.$$

(c) There exists $\tilde{a}_2 \geq 1$ such that for $\mathcal{H}^n$-almost every $\eta \in B(0,2) \setminus B(0,1/2)$,

$$\sum_{i,j,k=1}^{n} \left| \frac{\partial^3 f}{\partial \eta_i \partial \eta_j \partial \eta_k}(\eta) \right| \leq \tilde{a}_2.$$

To conform with the notation in [AGH+17] and [ALSV18] we put $\mathcal{A} = \nabla f$ for fixed $p > 1$ and given an open set O we say that v is $\mathcal{A}$-harmonic in O provided $v \in W^{1,p}(G)$ for each open G with $\bar{G} \subset O$ and

$$\int \langle \mathcal{A}(\nabla v(y)), \nabla \theta(y) \rangle \, dy = 0 \quad \text{whenever } \theta \in W_0^{1,p}(G). \tag{1.6}$$

As a short notation for (1.6) we write $\nabla \cdot \mathcal{A}(\nabla v) = 0$ in O. Note that if $f(\eta) = p^{-1}|\eta|^p$ then v as in (1.6) is p-harmonic in O. In this paper, we first prove

Theorem A. *Fix f as in (1.5), $n \geq 2, \alpha \in (0, \pi]$, and suppose $1 < p < \infty$ when $\alpha \in (0, \pi)$, while $p > n-1$ when $\alpha = \pi$. For $i = 1, 2$, there exists a unique $\mathcal{A}$-harmonic function $u_i > 0$ in $K(\alpha)$ with $u_i(1, 0, \dots, 0) = 1$ satisfying*

(+) *u_1 has continuous boundary value 0 on $\partial K(\alpha)$.*

(++) $\lim_{|x| \to \infty} u_2(x) = 0$ *and u_2 has continuous boundary value 0 on $\partial K(\alpha) \setminus \{0\}$.*

Moreover, (1.1) holds with $\lambda = \lambda_i(\alpha)$, for $i = 1, 2$, where $\lambda_2(\alpha) < 0 < \lambda_1(\alpha)$. Finally, $\lambda_1(\pi) = 1 - (n-1)/p$ for $p > n-1$ and

$$\lambda_1(\alpha) - 1 + \frac{n-1}{p} \approx (\pi - \alpha)^{\frac{p+1-n}{p-1}} \quad \text{as } \alpha \to \pi. \tag{1.7}$$

Remark 1.1. *We remark that if $1 < p \leq n-1$ then a slit has p-capacity zero in $\mathbb{R}^n$ for $n \geq 3$ and so one can show (see* [HKM06, chapter 2]*) that there are no solutions to (1.3). In fact, Krol' and Maz'ya in the paper mentioned earlier obtained that*

$$\lambda_1(\alpha) \approx \begin{cases} (\pi - \alpha)^{\frac{n-1-p}{p-1}} & \text{for } 1 < p < n-1 \\ -\frac{1}{\log(\pi - \alpha)} & \text{for } p = n-1 \end{cases} \qquad \text{as } \alpha \to \pi.$$

Here and in (1.7), $\approx$ means the ratio of the two functions is bounded above and below by positive constants depending only on p, n, and possibly $\tilde{a}_1$, $\tilde{a}_2$ in (1.5). We regard (1.7) as our main contribution in Theorem 1.2. For an outline of our efforts in trying *to prove this equality we refer the reader to* [ALV19]*. As mentioned above, our proof of existence and uniqueness in Theorem* A *for p-harmonic functions when* $0 < \alpha < \pi$ *is considerably less general than the proof in* [PV09] *given for "Lipschitz cones". Our proof, however, differs somewhat from the proof of these authors (even for p-harmonic functions). We include a proof in our setting mainly to facilitate the proof of* (1.7) *but also for completeness.*

In order to give an application of Theorem A we need some background material. Let $E \subset \mathbb{R}^n$ be a convex set with nonempty interior. Then for $\mathcal{H}^{n-1}$ almost every $x \in \partial E$, there is a well defined outer unit normal, $\mathbf{g}(x,E)$ to ∂E. The function $\mathbf{g}(\cdot,E) : \partial E \mapsto \mathbb{S}^{n-1}$ (whenever defined) is called the Gauss map for ∂E. Let μ be a finite positive Borel measure on $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ satisfying

$$
\begin{aligned}
&(i) \int_{\mathbb{S}^{n-1}} |\langle \theta, \zeta\rangle| \, d\mu(\zeta) > 0 \quad \text{for all } \theta \in \mathbb{S}^{n-1}, \\
&(ii) \int_{\mathbb{S}^{n-1}} \zeta \, d\mu(\zeta) = 0.
\end{aligned}
\tag{1.8}
$$

Then in [ALSV18], it was shown that

Theorem 1.2. *Let* μ *be as in* (1.8)*,* f *as in* (1.5)*, and* p *fixed,* $n \le p < \infty$*. Then there exists a compact convex set* E *with nonempty interior and an* $\mathcal{A}$*-harmonic Green's function* U *for* $\mathbb{R}^n \setminus E$ *with pole at infinity satisfying*

(a) $\lim_{y \to x} \nabla U(y) = \nabla U(x)$ *exists for* $\mathcal{H}^{n-1}$*-almost every* $x \in \partial E$ *as* $y \in \mathbb{R}^n \setminus E$ *approaches* x *non-tangentially.*

(b) $\displaystyle\int_{\partial E} f(\nabla U(x)) \, d\mathcal{H}^{n-1} < \infty.$

(c) $\displaystyle\int_{\mathbf{g}^{-1}(K,E)} f(\nabla U(x)) \, d\mathcal{H}^{n-1} = \mu(K)$ *whenever* $K \subset \mathbb{S}^{n-1}$ *is a Borel set.*

(d) E *is the unique set up to translation for which* (c) *holds.*

Also in [AGH+17] the authors proved

Theorem 1.3. *Let* μ *be as in* (1.8) *and* f *be as in* (1.5)*. Then for fixed* p *with* $1 < p \neq n-1 < n$*, there exists a compact convex set* E *with non-empty interior and an* $\mathcal{A}$*-harmonic capacitary function,* $\tilde U$ *for* E*, satisfying* $(a)-(d)$ *of Theorem* 1.2 *with* $U = \tilde U$*. If* $p = n-1$*, then there exists a compact convex set* E *with nonempty interior having* $\mathcal{A}$*-capacity* 1*, and a corresponding* $\mathcal{A}$*-capacitary function* $\tilde U$ *for* E *satisfying* *(a) and (b) of Theorem 1.2 with $U = \tilde U$, as well as,*

(c') *There exists $\tilde b, 0 < \tilde b < \infty$, with*
$$\tilde b \int_{\mathbf{g}^{-1}(K,E)} f(\nabla \tilde U)\, d\mathcal{H}^{n-1} = \mu(K) \text{ whenever } K \subset \mathbb{S}^{n-1} \text{ is a Borel set.}$$

(d') *E is the unique set up to translation satisfying (c') with $\mathcal{A}$-capacity 1.*

The definition of $\mathcal{A}$-capacity, a $\mathcal{A}$-capacitary function, and of the $\mathcal{A}$-harmonic Green's function with pole at ∞ are given in section 2. As an application of Theorem A when $n = 2, 3$, we prove the regularity of the Minkowski problem.

Theorem B. *Let μ be as in (1.8) and f as in (1.5). Suppose also that $\hat\alpha \in (0,1)$, k is a non-negative integer, and $d\mu = \Theta\, d\mathcal{H}^{n-1}$ on $\mathbb{S}^{n-1}$ for some $0 < \Theta \in C^{k,\hat\alpha}(\mathbb{S}^{n-1})$. If $k \geq 1$, assume $f \in C^{k+2,\hat\alpha}(\mathbb{R}^n \setminus \{0\})$. Let E be the compact convex set with non-empty interior in Theorem 1.2 or Theorem 1.3 corresponding to μ. If either $n = 2, 3$, and $1 < p < \infty$, or $n \geq 4$ and $1 < p \leq 2$, then ∂E is locally the graph of a $C^{k+2,\hat\alpha}(\mathbb{R}^{n-1})$ function.*

Remark 1.4. *Theorems 1.3 and B were first proved by Jerison in [Jer96] for Laplace's equation (i.e., when $f(\eta) = |\eta|^2/2$) and after that generalized to p-harmonic functions when $1 < p < 2$ in [CNS+15] for $n > 2$. It will turn out that it suffices to assume that Θ is bounded above and below on $\mathbb{S}^{n-1}$ in order to conclude ∂E is strictly convex and locally the graph of a $C^{1,\epsilon}$ function where $\epsilon > 0$ depends on $\tilde a_1, \tilde a_2, p, n$, the eccentricity of E, and the bounds for Θ.*

1.1. **Outline of the proof of Theorems A and B.** Existence in Theorem A for $\alpha \in (0, \pi)$ follows easily from interior regularity results and Wiener type estimates for $\mathcal{A}$-harmonic functions listed in section 2. Uniqueness in Theorem A for $\alpha \in (0, \pi)$ follows from boundary Harnack inequalities, originally proved for positive p-harmonic functions vanishing on a portion of a Lipschitz domain in [LN07, LN10]. These inequalities were updated to $\mathcal{A}$-harmonic functions for fixed p with $1 < p < n$ in [AGH+17] and for $p \geq n$ in [ALSV18]. Uniqueness in the case $\alpha = \pi$ is somewhat more involved (since $K(\pi) \cap B(0, \rho)$ is not a Lipschitz domain), using not only the above boundary Harnack inequalities but also arguments from [LLN08] and [LN18]. To outline the proof of (1.7) we now write $u(\cdot, \alpha)$ and $\lambda(\alpha)$ for u_1 and λ_1 in Theorem A relative to $K(\alpha)$. First it follows easily from our existence and uniqueness results that $\lim_{\alpha \to \pi} \lambda(\alpha) = \lambda(\pi)$. From boundary Harnack inequalities for $\mathcal{A}$-harmonic functions, as well as an integral identity proved in [AGH+17] for $n - 1 < p < n$ and in [ALSV18] for $p \geq n$, we eventually obtain

$$\bar c(\delta)^{-1} \leq \int_{\partial K(\alpha) \cap \{x : x_1 \geq -1 + 4\delta\}} \sin(\pi - \alpha)\, f(\nabla u(y)) d\mathcal{H}^{n-1} \leq \bar c(\delta). \tag{1.9}$$

in (4.10) where
$$0 < \pi - \alpha << \delta << 1 \text{ and } \delta \text{ is fixed.}$$

Also $c(\delta) \geq 1$ is a positive constant depending only on p, n, and $\tilde{a}_1, \tilde{a}_2$ in (1.5). To estimate the integral in (1.9) we use a boundary Harnack inequality for $\mathcal{A}$-harmonic functions on lower dimensional sets from [LN18] to essentially obtain

$$|\nabla u(\cdot, \alpha)| \leq c' \, (\pi - \alpha)^{\frac{2-n}{p-1}} \quad \text{on } \partial K(\alpha) \cap [B(0,2) \setminus B(0,1/2)] \tag{1.10}$$

where c' depends on p, n, and $\tilde{a}_1, \tilde{a}_2$ in (1.5). From (1.9), (1.10), and homogeneity of $u(\cdot, \alpha)$ we finally get

$$\begin{aligned} c(\delta)^{-1} &\leq \left(\int_0^1 r^{(\lambda(\alpha)-1)p+n-2} dr \right) (\pi - \alpha)^{\frac{p-n+1}{p-1}} \\ &\leq \frac{c(\delta)}{(\lambda(\alpha)-1)p+n-1} (\pi - \alpha)^{\frac{p-n+1}{p-1}} \end{aligned} \tag{1.11}$$

where $c(\delta)$ has the same dependence as $\bar{c}(\delta)$ above and we have also used the fact that an element of surface area on $\partial K(\alpha)$ is of the form $[\sin(\pi - \alpha)]^{n-2} r^{n-2} dr$. From (1.11) and some arithmetic we conclude

$$\lambda(\alpha) \leq 1 - \frac{n-1}{p} + c^* \, (\pi - \alpha)^{\frac{p-n+1}{p-1}} \quad \text{as } \alpha \to \pi \tag{1.12}$$

for some $c^* = c^*(p, n, \tilde{a}_1, \tilde{a}_2)$ and so get the desired upper estimate for $\lambda_1(\alpha)$ in Theorem A. The lower estimate is similar. We note that a slightly different proof of Theorem A for p-harmonic functions when $n - 1 < p < n$ (with more details) is outlined in [ALV19].

As for the proof of Theorem B, armed with Theorems A, 1.2, and 1.3., we can follow closely the proof in [CNS+15], who in turn followed closely the proof in [Jer96]. Indeed, Jerison in [Jer96], first converts Theorem B into a regularity statement for the solution, say $\hat{u}$ to a Monge Ampère equation whose right-hand side corresponds to a measure $\hat{\mu}$ on $\mathbb{S}^{n-1}$. To show regularity of $\hat{u}$, he first generalized the Alexandrov-Bakelman inequality (see [Jer96, Lemma 7.3]) and then used this generalization to prove a certain integral inequality for $\hat{\mu}$ in Theorem 6.5 of [Jer96]. This inequality was then used to show that arguments in [Caf89, Caf90a, Caf91, Caf90b] could be used to eventually obtain Theorem B (see also [GH00]). Theorem A is used in Theorem B to prove the analogue of Theorem 6.5 in [Jer96] when $n = 2, 3$ and $p > 2$.

As for the plan of this paper, in section 2, we state some basic properties of $\mathcal{A}$-harmonic functions, give the definitions mentioned after Theorem 1.3, and prove existence in Theorem A. In section 3, we state several boundary Harnack inequalities and then apply these inequalities to prove uniqueness in Theorem A. In section 4 we state integral identities from [AGH+17, ALSV18] and then use these identities to prove Theorem A. Theorem B is proved in section 5.

2. Basic estimates and definitions for $\mathcal{A}$-harmonic functions

In this section we first introduce some notation and then state some fundamental estimates for $\tilde{\mathcal{A}} = \nabla \tilde{f}$-harmonic functions when p is fixed, $1 < p < \infty$, and $\tilde{f}$ satisfies (1.5) with $f = \tilde{f}$. Second, we define the $\mathcal{A}$-capacitary function when $1 < p < n$ and $\mathcal{A}$-harmonic Green's function with pole at ∞ when $p \geq n$ of a compact convex set E. Third, we show existence of u_i for $i = 1, 2$, in Theorem A relative to $K(\alpha)$ when $\alpha \in (0, \pi)$. Concerning constants, unless otherwise stated, in this section, and throughout the paper, c will denote a positive constant ≥ 1, not necessarily the same at each occurrence, depending at most on $p, n, \tilde{a}_1, \tilde{a}_2$, which sometimes we refer to as depending on the data. In general, $c(t_1, \dots, t_m)$ denotes a positive constant ≥ 1, which may depend at most on $p, n, \tilde{a}_1, \tilde{a}_2$ and $t_1, \dots, t_m$, not necessarily the same at each occurrence. Also, as in the introduction, if $B \approx C$ then B/C is bounded from above and below by constants which, unless otherwise stated, depend at most on the data. Let e_k be the n tuple with one in the kth position and zeros elsewhere. Let $d(E_1, E_2)$ denote the distance between the sets E_1 and E_2. For short we write $d(x, E_2)$ for $d(\{x\}, E_2)$. Also put $E_1 + E_2 = \{x + y : x \in E_1, y \in E_2\}$ and $\lambda E = \{\lambda x : x \in E\}$ for $\lambda > 0$. Let $\operatorname{diam}(E)$, $\bar{E}$, and ∂E denote the diameter, closure, and boundary of E respectively. We write $\max\limits_E \tilde{u}$, $\min\limits_E \tilde{u}$ to denote the essential supremum and infimum of $\tilde{u}$ on E whenever $E \subset \mathbb{R}^n$ and $\tilde{u}$ is defined on E.

Lemma 2.1. *Given $p, 1 < p < \infty, n \geq 2$, and $\tilde{f}$ as in (1.5), let $\tilde{u}$ be a positive $\tilde{\mathcal{A}} = \nabla \tilde{f}$-harmonic function in $B(w, 4r)$ for $r > 0$. Then*

$$
\begin{aligned}
&(i)\ r^{p-n} \int_{B(w,r/2)} |\nabla \tilde{u}|^p \, dy \,\leq\, c\,(\max_{B(w,r)} \tilde{u})^p, \\
&(ii)\ \max_{B(w,r)} \tilde{u} \leq c \min_{B(w,r)} \tilde{u}.
\end{aligned} \tag{2.1}
$$

Furthermore, there exists $\tilde{\sigma} = \tilde{\sigma}(p, n, \tilde{a}_1, \tilde{a}_2) \in (0, 1)$ such that if $x, y \in B(w, r)$, then

$$
(iii)\quad |\tilde{u}(x) - \tilde{u}(y)| \leq c \left(\frac{|x - y|}{r} \right)^{\tilde{\sigma}} \max_{B(w,2r)} \tilde{u}.
$$

Proof. A proof of this lemma can be found in [Ser64]. □

Lemma 2.2. *Let $p, n, \tilde{f}, \tilde{\mathcal{A}}, \tilde{u}, w, r$, be as in Lemma 2.1. Then $\tilde{u}$ has a representative locally in $W^{1,p}(B(w, 4r))$, with Hölder continuous partial derivatives in $B(w, 4r)$ (also denoted $\tilde{u}$), and there exist $\tilde{\beta} \in (0, 1]$ and $c \geq 1$, depending only on $p, n, \tilde{a}_1, \tilde{a}_2$, such that if $x, y \in B(w, r)$, then*

(2.2)

$$
(\hat{a})\quad c^{-1}\, |\nabla \tilde{u}(x) - \nabla \tilde{u}(y)| \,\leq\, (|x - y|/r)^{\tilde{\beta}} \max_{B(w,r)} |\nabla \tilde{u}| \leq\, c\, r^{-1}\, (|x - y|/r)^{\tilde{\beta}}\, \tilde{u}(w).
$$

$$
(\hat{b})\quad \int_{B(w,r)} \sum_{i,j=1}^{n} |\nabla \tilde{u}|^{p-2}\, |\tilde{u}_{x_i x_j}|^2 dy \leq c r^{(n-p-2)} \tilde{u}(w).
$$

Proof. A proof of Lemma 2.2 can be found in [Tol84]. □

Definition 2.3. *Fix $p, 1 < p < \infty$ and let $\tilde{f}$ be as in (1.5) with $f = \tilde{f}$. If $\tilde{K}$ is a compact subset of the connected open set D, define the $\tilde{\mathcal{A}} = \nabla \tilde{f}$-capacity of $\tilde{K}$ relative to D by*

$$
Cap_{\tilde{\mathcal{A}}}(\tilde{K}, D) = \inf \left\{ \int_D f(\nabla w(x)) dx : \ w \in C_0^\infty(D) \text{ and } w(x) \geq 1 \text{ for } x \in \tilde{K} \right\}.
$$

In case $\tilde f(\eta) = p^{-1}|\eta|^p$ for $\eta \in \mathbb{R}^n$, we write $\mathrm{Cap}_p(\tilde K, D)$ instead of $\mathrm{Cap}_{\tilde{\mathcal{A}}}(\tilde K, D)$. If $D = \mathbb{R}^n$ we also write $\mathrm{Cap}_{\tilde{\mathcal{A}}}(\tilde K)$ and $\mathrm{Cap}_p(\tilde K)$ for short. We note from (1.5) that

$$\mathrm{Cap}_p(\tilde K, D) \approx \mathrm{Cap}_{\tilde{\mathcal{A}}}(\tilde K, D) \quad \text{and} \quad \mathrm{Cap}_{\tilde{\mathcal{A}}}(\tau \tilde K + \{x_0\}) = \tau^{n-p}\mathrm{Cap}_{\tilde{\mathcal{A}}}(\tilde K) \tag{2.3}$$

for $\tau > 0$ and $x_0 \in \mathbb{R}^n$. Ratio constants depend only on the data. If $n \le p < \infty$ then $\mathrm{Cap}_{\tilde{\mathcal{A}}}(\tilde K) \equiv 0$.

Definition 2.4. *Let $p, \tilde f, \tilde A$, be as in Definition 2.3. A compact set $\tilde K \subset \mathbb{R}^n$ is called uniformly (r_0, p)-fat if there exists $\hat c \ge 1$ such that*

$$\frac{Cap_p(\tilde K \cap \bar B(w, r), B(w, 2r))}{Cap_p(\bar B(w, r), B(w, 2r))} \ge \hat c^{-1}$$

for all $0 < r \le r_0$ and $w \in \tilde K$.

Lemma 2.5. *Let $p, \tilde f, \tilde{\mathcal{A}}$, be as in Definition 2.4 and suppose that $\tilde K$ is a uniformly (r_0, p)-fat compact set wirh $\tilde K \cap B(z, 3\rho) \neq \emptyset$, where $r_0 = diam(\tilde K)$. Let $\zeta \in C_0^\infty(B(z, 4\rho))$ with $\zeta \equiv 1$ on $B(z, 3\rho)$. If $0 \le \tilde u$ is $\tilde{\mathcal{A}}$-harmonic in $B(z, 4\rho) \setminus \tilde K$, and $\tilde u\zeta \in W_0^{1,p}(B(z, 4\rho) \setminus \tilde K)$, then $\tilde u$ has a continuous extension to $B(z, 3\rho)$ obtained by putting $\tilde u \equiv 0$ on $\tilde K \cap B(z, 3\rho)$. Moreover, if $0 < r < \min\{r_0, \rho\}$ and $w \in \tilde K \cap B(z, 2\rho)$, then*

$$(i) \quad r^{p-n} \int\limits_{B(w,r/2)} |\nabla \tilde u|^p \, dy \, \le \, c_1 \left(\max_{B(w,r)} \tilde u \right)^p . \tag{2.4}$$

where c_1 depends only on $p, n, \tilde a_1, \tilde a_2$, and the uniform (r_0, p)-fatness constant for $\tilde K$. Furthermore, there exist $\hat\sigma \in (0, 1)$ and $c_2 \ge 1$, having the same dependence as c_1, such that

$$(ii) \quad |\tilde u(x) - \tilde u(y)| \le c_2 \left(\frac{|x - y|}{r} \right)^{\hat\sigma} \max_{B(w,r)} \tilde u$$

whenever $x, y \in B(w, r/2)$ and $0 < r < \min\{r_0, \rho\}$.

Proof. Here (i) in (2.4) is a standard Caccioppoli inequality and (ii) for $y \in \tilde K$ follows from uniform (r_0, p)-fatness of $\tilde K$ and essentially Theorem 6.18 in [HKM06]. Combining this fact with (2.1) (iii) we obtain (ii). □

Lemma 2.6. *Let $\tilde{\mathcal{A}}, p, \tilde f, \tilde K, r_0, z, \rho, \tilde u$ be as in Lemma 2.5. Then there exists a unique finite positive Borel measure $\tilde\nu$ with support contained in $\tilde K \cap B(z, 3\rho)$ such that*

$$\int \langle \tilde{\mathcal{A}}(\nabla \tilde u(y)), \nabla \phi(y) \rangle \, dy \; = \; - \int \phi \, d\tilde\nu \quad \textit{whenever } \phi \in C_0^\infty(B(z, 2\rho)). \tag{2.5}$$

Moreover, there exists $\bar c \ge 1$, with the same dependence as c_1 in Lemma 2.5, for which

$$\bar c^{-1} \, r^{p-n} \tilde\nu(B(w, r/2)) \le \max_{B(w,r)} \tilde u^{p-1} \le \bar c r^{p-n} \tilde\nu(B(w, 2r)) \tag{2.6}$$

whenever $0 < r < \min\{r_0, \rho\}$ and $w \in \tilde K \cap B(z,\rho)$. Furthermore, suppose for some constant $\Lambda \geq 1$ that if $w \in \tilde K \cap B(z,\rho)$, and $0 < s < r$, there exists $a_s(w) \in B(w,r)\setminus \tilde K$ with

$$\Lambda\, d(a_s(w), \partial[B(z,2\rho)\setminus \tilde K]) \geq s.$$

Suppose also that whenever $w_1, w_2 \in B(z,2r)\setminus \tilde K$ and $0 < r \leq \rho/\Lambda$, there exists a rectifiable curve $\tau : [0,1] \to B(z,2\rho)\setminus \tilde K$ with $\tau(0) = w_1$ and $\tau(1) = w_2$, and such that

$$\begin{aligned} &(2.7)\\ &\quad (a)\quad \mathcal{H}^1(\tau) \,\leq\, \Lambda\,|w_1 - w_2|,\\ &\quad (b)\quad \min\{\mathcal{H}^1(\tau([0,t])),\, \mathcal{H}^1(\tau([t,1]))\,\} \,\leq\, \Lambda\, d(\tau(t), \partial[B(z,2\rho)\setminus\tilde K]),\ t\in(0,1). \end{aligned}$$

If $w \in B(z, r/2)\cap \tilde K$ then

$$[r^{p-n}\tilde\nu(B(w,2r))]^{1/(p-1)} \approx \tilde u(a_r(w)) \approx \max_{B(w,r)} \tilde u \approx [r^{p-n}\tilde\nu(B(w,r/2))]^{1/(p-1)}. \tag{2.8}$$

Ratio constants depend only on the data, the uniform fatness constant for $\tilde K$, and Λ.

Proof. For the proof of (2.5), see [HKM06, Theorem 21.2] The left-hand inequality in (2.6) follows from (2.5), (1.5), and Hölder's inequality, using a test function, ϕ, with $\phi \equiv 1$ on $\bar B(w,r/2)$. The proof of the right-hand inequality in (2.6) follows from [KZ03] (see also [EL91]). Here (2.7) is equivalent to a Harnack chain condition used in the definition of an non-tangentially accessible domain (see [JK82]). The proof of the middle inequality in (2.8) follows from an argument often attributed to Carleson (see [AS05]) and just uses (2.4) (ii), (2.1) (ii), and (2.6). The first and last inequalities in (2.8) give the "doubling property" of ν measure. □

Remark 2.7. *Uniform (r_0,p)-fatness of $\mathbb{R}^n\setminus D$ for some $r_0 > 0$ is a sufficient condition for solvability of the Dirichlet problem for $\tilde{\mathcal{A}}$-harmonic PDEs in a bounded domain D in the sense that if ϕ is a continuous function on ∂D, then there exists an $\tilde{\mathcal{A}}$-harmonic function Φ in D with continuous boundary values equal to ϕ on ∂D. In fact, if every point in $\mathbb{R}^n\setminus D$ is uniformly (r_0,p)-fat, then*

$$\int_0^{r_0}\left[\frac{Cap_p((\mathbb{R}^n\setminus D)\cap \bar B(w,r), B(w,2r))}{Cap_p(\bar B(w,r), B(w,2r))}\right]^{\frac{1}{(p-1)}}\frac{dr}{r} = \infty.$$

That is, uniform (r_0,p)-fatness implies Wiener regularity (see [HKM06, Theorem 6.33]). We also remark that if $E\subset B(0,\rho)$ is a convex set with $diam(E) = 1$ and $\mathcal{H}^k(E) > 0$ for some positive integer $k > n-p$ then E is $(1,p)$-uniformly fat and $Cap_{\tilde{\mathcal{A}}}(E, B(0,2\rho)) \approx 1$ with ratio constants depending only on the data when $1 < p < n$ while for $p \geq n$ these constants depend on the data and also ρ. On the other hand, if $\mathcal{H}^k(E) < \infty$ for some positive integer $k \leq n-p$ then $Cap_{\tilde{\mathcal{A}}}(E) = 0$ (see [HKM06, chapter 2]).

2.1. **Definition of $\mathcal{A}$-capacitary and $\mathcal{A}$-harmonic Green's functions.**

Definition 2.8. *Let $1 < p < n$ and f be as in (1.5) and let E be a compact convex set with $Cap_{\mathcal{A}}(E) > 0$. Then the $\mathcal{A}$-capacitary function of E, say $\tilde{U}$, is the unique continuous function $\tilde{U} \not\equiv 1, 0 < \tilde{U} \leq 1$, on $\mathbb{R}^n$ satisfying*

$$
\begin{aligned}
&(a) \quad \tilde{U} \text{ is } \mathcal{A}\text{-harmonic in } \mathbb{R}^n \setminus E. \\
&(b) \quad \tilde{U} \equiv 1 \text{ on } E \text{ and } \tilde{U}(x) \to 0 \text{ uniformly as } |x| \to \infty. \\
&(c) \quad |\nabla \tilde{U}| \in L^p(\mathbb{R}^n) \text{ and } \tilde{U} \in L^{p^*}(\mathbb{R}^n) \quad \text{for } p^* = \frac{np}{n-p}. \\
&(d) \quad Cap_{\mathcal{A}}(E) = \int_{\mathbb{R}^n} \langle \mathcal{A}(\nabla \tilde{U}), \nabla \tilde{U} \rangle \, dy.
\end{aligned}
\tag{2.9}
$$

For existence and uniqueness of $\tilde{U}$ see Lemma 4.1 in [AGH+17]. We note that if $\tilde{\nu}$ denotes the measure associated with $\tilde{U}$ as in Lemma 2.6 then $\tilde{\nu}(E) = \mathrm{Cap}_{\mathcal{A}}(E)$ (see [AGH+17, Lemma 4.2]). Therefore, if $E \subset B(0,1)$ with $\mathrm{diam}(E) \geq 1/2$ and $n - 1 < p < n$ then from (2.8) and Remark 2.7 we have

$$
c^{-1} \leq \mathrm{Cap}_{\mathcal{A}}(E) \leq c \max_{B(0,2)} (1 - \tilde{U}) \tag{2.10}
$$

where c depends only on the data.

In order to define an $\mathcal{A}$-harmonic Green's function with pole at ∞ when $p \geq n$, we first have to define a fundamental solution, say F, with pole at 0 in $\mathbb{R}^n$ when $p \geq n$. Definitions for $p = n$ and $n < p < \infty$ are different and we start with $p = n$.

Definition 2.9. *If $p = n$ we say that F is a fundamental solution to $\nabla \cdot \mathcal{A}(\nabla F) = 0$ in $\mathbb{R}^n$ with pole at 0 if*

$$
\begin{aligned}
&(i) \quad F \text{ is } \mathcal{A}\text{-harmonic in } \mathbb{R}^n \setminus \{0\}, \\
&(ii)\; F \in W^{1,l}_{loc}(\mathbb{R}^n) \text{ for } 1 < l < n, \; F(e_1) = 1, \text{ and} \\
&\qquad |F(x)| = O(\log |x|) \text{ in a neighborhood of } \infty, \\
&(iii) \quad \int \langle \mathcal{A}(\nabla F(z)), \nabla \theta(z) \rangle \, dz = -\theta(0) \quad \text{whenever } \theta \in C_0^\infty(\mathbb{R}^n).
\end{aligned}
\tag{2.11}
$$

If $p > n$ we say that F is a fundamental solution to $\nabla \cdot \mathcal{A}(\nabla F) = 0$ in $\mathbb{R}^n$ with pole at 0 if

$$
\begin{aligned}
&(i) \quad F \text{ is } \mathcal{A}\text{-harmonic in } \mathbb{R}^n \setminus \{0\}, \\
&(ii) \; F \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n), \; F \text{ is continuous in } \mathbb{R}^n, \; F(0) = 0, \; F > 0 \text{ in } \mathbb{R}^n \setminus \{0\}, \\
&(iii) \int \langle \mathcal{A}(\nabla F(z)), \nabla \theta(z) \rangle \, dz = -\theta(0) \quad \text{whenever } \theta \in C_0^\infty(\mathbb{R}^n).
\end{aligned}
\tag{2.12}
$$

Existence and uniqueness of F in (2.11) and (2.12) are proved in Lemma 4.4 and Lemma 4.6 of [ALSV18], respectively.

Definition 2.10. *Given a compact, convex set $E \subset \mathbb{R}^n$ we say that U is the $\mathcal{A}$-harmonic Green's function for $\mathbb{R}^n \setminus E$ with pole at ∞, if $U : \mathbb{R}^n \setminus E \to (0, \infty)$*

has continuous boundary value 0 *on* ∂E*,* U *is* $\mathcal{A}$*-harmonic in* $\mathbb{R}^n \setminus E$*, and* $U(x) = F(x) + k(x)$ *where* $k(x)$ *is a bounded function in a neighborhood of* ∞*.*

Remark 2.11. *In* [ALSV18] *the authors show that* U *exists and is unique if and only if the convex compact set* E *is either (a) nonempty when* $p > n$ *or (b) contains at least two points when* $p = n$*. If* U *exists then it was also shown that* $k \leq 0$ *in* $\mathbb{R}^n \setminus E$ *and* k *is Hölder continuous in a neighbourhood of* ∞ *with* $\lim\limits_{x\to\infty} k(x) = k(\infty)$*. They then define*

$$\mathcal{C}_{\mathcal{A}}(E) := \begin{cases} e^{-k(\infty)/\gamma} & \text{when } p = n, \\ (-k(\infty))^{p-1} & \text{when } p > n. \end{cases}$$

If E *is a single point and* $p = n$ *(so* U *does not exist), set* $\mathcal{C}_{\mathcal{A}}(E) := 0$*. Here* γ *is a constant depending only on the data which occurs in the asymptotic expansion of* $F(x)$ *as* $x \to \infty$*. From the definition of* $\mathcal{C}_{\mathcal{A}}(E)$ *and translation, dilation invariance of* $\mathcal{A}$*-harmonic functions it follows as in* (2.3) *that if* $x_0 \in \mathbb{R}^n$*,* $r > 0$*, and* E *is a convex compact set then*

$$\mathcal{C}_{\mathcal{A}}(rE + \{x_0\}) = \begin{cases} r\,\mathcal{C}_{\mathcal{A}}(E) & \text{when } p = n, \\ r^{p-n}\mathcal{C}_{\mathcal{A}}(E) & \text{when } p > n. \end{cases} \tag{2.13}$$

Also if ν *is the measure associated with* U *as in Lemma 2.6 then (see Lemmas 5.2, 5.3 in* [ALSV18]*),* $\nu(E) = 1$*. Hence if* $E \subset B(0,1)$ *with* $1/2 \leq diam(E)$ *it follows from* (2.8) *that*

$$\max_{B(0,2)} U \approx 1 \text{ where the proportinal constants depend only on the data.} \tag{2.14}$$

Finally, if $E_1 \subset E_2$ *are compact convex sets and* U_1 *and* U_2 *the corresponding* $\mathcal{A}$*-harmonic Green's functions with pole at* ∞ *then*

$$U_1 \geq U_2 \text{ in } \mathbb{R}^n \quad \text{so} \quad \mathcal{C}_{\mathcal{A}}(E_1) \leq \mathcal{C}_{\mathcal{A}}(E_2). \tag{2.15}$$

2.2. **Existence in Theorem A.** To show existence and uniqueness for u_1 and u_2 in Theorem A we shall also need the following lemma.

Lemma 2.12. *Fix* p *with* $1 < p < \infty$ *and* $\alpha \in (0, \pi)$ *and suppose* $0 < r \leq R/10$*. Let* v *be the* $\mathcal{A}$*-harmonic function in* $D = [K(\alpha) \setminus \bar{B}(re_1, \frac{r\alpha}{100})] \cap B(0,R)$ *with continuous boundary values* $v \equiv 1$ *on* $\partial B(re_1, \frac{r\alpha}{100})$ *and* $v \equiv 0$ *on* $[\partial B(0,R) \cap K(\alpha)] \cup [\partial K(\alpha) \cap B(0,R)]$*. Then there exists* $c \geq 1$ *such that*

$$-c\,\langle \nabla v(x), \tfrac{x - re_1}{|x - re_1|} \rangle \geq v(x) \quad \text{whenever } x \in D. \tag{2.16}$$

Here c *depends on the data and* α *if* $1 < p \leq n-1$*, while* c *depends only on the data if* $p > n-1$*.*

Proof. Let $\hat{D} = \{y : y + re_1 \in D\}$ and define $\hat{v}$ on $\hat{D}$ by $\hat{v}(y) = v(y + re_1)$ for $y \in \hat{D}$. Given λ with $1 < \lambda < 1001/1000$, set $\hat{D}(\lambda) = \{y \in \hat{D} : \lambda y \in \hat{D}\}$. From the definition of $D, \hat{D}, v, \hat{v}$, and translation and dilation invariance of $\mathcal{A}$-harmonic functions we see that $y \mapsto \hat{v}(y)$ and $y \mapsto \hat{v}(\lambda y)$ are both $\mathcal{A}$-harmonic in $\hat{D}(\lambda)$. If

$$h(y) := \frac{\hat{v}(y) - \hat{v}(\lambda y)}{\lambda - 1} \quad \text{for } y \in \hat{D}(\lambda)$$

we claim that

$$\breve{c}\, h(y) \geq \hat{v}(y) \quad \text{for } y \in \hat{D}(\lambda) \tag{2.17}$$

where $\breve{c} \geq 1$ has the same dependence as c in Lemma 2.12. Using the boundary maximum principle for $\mathcal{A}$-harmonic functions and continuity of h and $\hat{v}$ we see that it suffices to prove (2.17) when $y \in \partial\hat{D}(\lambda)$. To do this we note from the definition of $\hat{D}(\lambda)$ that if $y \in \partial\hat{D}(\lambda)$, then either $y = z/\lambda$ for some $z \in \partial\hat{D}$ with $\hat{v}(z) = 0$ or $y \in \partial\hat{D}$ and $\hat{v}(y) = 1$. In the first case we see that $\hat{v}(\lambda y) = 0$ so (2.17) is trivially true. In the second case let $\tilde{f}(\eta) = f(-\eta)$, and note that $1 - \hat{v}$ is $\tilde{\mathcal{A}}$-harmonic in $\hat{D}$. Using this note, uniform fatness of $K(\alpha) \cap B(0, R)$, the definition of $\hat{D}$, (2.4) (ii) for $\hat{v}$, and Harnack's inequality we deduce that if $r' = \frac{2\alpha r}{100}$, then $1 - \hat{v} \geq c_*^{-1}$ on $\partial B(0, r')$ for some $c_* \geq 1$ with the same dependence as c in the statement of Lemma 2.12. Thus $c_*(1 - \hat{v}) \geq 1$ on $\partial B(0, r')$ and $(1 - \hat{v}) \equiv 0$ on $\partial B(0, r'/2)$. Also this function is $\tilde{\mathcal{A}}$-harmonic in $T = B(0, r') \setminus \bar{B}(0, r'/2)$.

Using these facts and a barrier type argument as in [AGH$^+$17, section 7] or [ALSV18, (4.6)-(4.9)], it follows (since $|y| = r'/2$) that

$$\hat{v}(y) - \hat{v}(\lambda y) = 1 - \hat{v}(\lambda y) \geq (\lambda - 1)/(\bar{c}\, c_*) \tag{2.18}$$

where $\bar{c} \geq 1$ depends only on the data. From (2.18) we conclude that (2.17) also holds in the second case when $y \in \partial\hat{D}(\lambda)$. Thus (2.17) holds on $\partial\hat{D}(\lambda)$ so by the above maximum principle is valid in $\hat{D}(\lambda)$. Letting $\lambda \to 1$ in (2.17) and using (2.2) $(\hat{a})$, as well as the chain rule, we get

$$-c\langle \nabla\hat{v}(y), y/|y|\rangle \geq \hat{v}(y)$$

for $y \in \hat{D}$. Clearly this inequality implies (2.16). □

To begin the proof of existence in Theorem A for u_1, let v and D be as in Lemma 2.12 and put $R = l, r = l/10$, for $l = 2, 3, \ldots$. Set $v_l = M_l\, v$ where $M_l > 0$ is chosen so that $v_l(e_1) = 1$. Extend v_l to a continuous function in $\bar{B}(0, l)$ by defining $v_l \equiv 0$ on $[\bar{B}(0, l) \setminus K(\alpha)] \cup \partial B(0, l)$ while $v_l \equiv M_l$ on $\bar{B}(\frac{le_1}{10}, \frac{l\alpha}{1000})$. Using Lemmas 2.1, 2.2, 2.5 and letting $l \to \infty$ it follows from Ascoli's theorem that a subsequence of (v_l), also denoted (v_l), converges uniformly to u_1, an $\mathcal{A}$-harmonic function in $K(\alpha)$ that is also Hölder continuous in $\mathbb{R}^n$ with $u_1 \equiv 0$ on $\mathbb{R}^n \setminus K(\alpha)$.

To construct u_2, we let $r = 1/l$, $R = l$, and let $v_l = \hat{M}_l\, v$ for $l = 2, 3, \ldots$, where $\hat{M}_l$ is chosen so that $v_l(e_1) = 1$. Extend v_l to a continuous function on $\bar{B}(0, l)$ by putting $v_l \equiv 0$ on $[B(0, l) \setminus K(\alpha)] \cup \partial B(0, l)$ and $v_l \equiv \hat{M}_l$ on $\bar{B}(e_1/l, \frac{\alpha}{100l})$. Also from Lemmas 2.1, 2.2, 2.5 and (2.8) we deduce for $l > \rho > 2/l$, that there exists $c \geq 1$ and $\breve{\beta} \in (0, 1)$

such that

$$\max_{B(0,l)\setminus B(0,\rho)} v_l \le c v_l(\rho e_1) \le c^2 \rho^{-\breve{\beta}}. \tag{2.19}$$

Here c and $\breve{\beta}$ depend on the data and α if $1 \le p \le n-1$ while these constants depend only on the data if $p > n-1$. Letting $l \to \infty$, it follows from the above lemmas, and Ascoli's theorem that a subsequence of (v_l), also denoted (v_l), converges uniformly to u_2, an $\mathcal{A}$-harmonic function in $K(\alpha)$ that is locally Hölder continuous in $\mathbb{R}^n \setminus \{0\}$ with $u_2 \equiv 0$ on $\mathbb{R}^n \setminus (K(\alpha) \cup \{0\})$. Moreover, (2.19) holds with v_l replaced by u_2 and from (2.16) we have

$$-c\,\langle \nabla u_2(x), x/|x| \rangle \ge u_2(x) \quad \text{whenever } x \in K(\alpha). \tag{2.20}$$

3. Boundary Harnack inequalities and uniqueness in Theorem A

To prove that u_1 and u_2 are unique and satisfy (1.1) in Theorem A we use a variety of boundary Harnack inequalities, mostly in Lipschitz domains. To set the stage for these inequalities, let $K \subset \mathbb{R}^{n-1}, n \ge 2$, be a nonempty compact set and recall that $\phi : K \to \mathbb{R}$ is said to be Lipschitz on K provided there exists $\hat{b}, 0 < \hat{b} < \infty$, such that

$$|\phi(z') - \phi(w')| \le \hat{b}\,|z' - w'| \quad \text{whenever } z', w' \in K. \tag{3.1}$$

The infimum of all $\hat{b}$ such that (3.1) holds is called the Lipschitz norm of ϕ on K, denoted by $\|\phi\hat{\|}_K$. It is well-known that if $K \subset \mathbb{R}^{n-1}$ is compact, then ϕ has an extension to $\mathbb{R}^{n-1}$ (also denoted by ϕ) which is differentiable almost everywhere in $\mathbb{R}^{n-1}$ and

$$\|\phi\hat{\|}_{\mathbb{R}^{n-1}} = \|\, |\nabla \phi|\, \|_\infty \le c \|\phi\hat{\|}_K.$$

Now suppose that D is an open set, $w \in \partial D, \hat{r} > 0$, and

$$\begin{aligned} \partial D \cap B(w, 4\hat{r}) &= \{ y = (y', y_n) \in \mathbb{R}^n : y_n = \phi(y') \} \cap B(w, 4\hat{r}), \\ D \cap B(w, 4\hat{r}) &= \{ y = (y', y_n) \in \mathbb{R}^n : y_n > \phi(y') \} \cap B(w, 4\hat{r}) \end{aligned} \tag{3.2}$$

in an appropriate coordinate system for some Lipschitz function ϕ on $\mathbb{R}^{n-1}$ with $\phi(w') = w_n$. Note from elementary geometry that if $\zeta \in \partial D \cap B(w, 2\hat{r})$ and $0 < s < \hat{r}$, we can find points

$$a_s(\zeta) \in D \cap B(\zeta, s) \quad \text{with} \quad d(a_s(\zeta), \partial D) \ge c^{-1} s$$

for a constant c depending on $\|\phi\hat{\|}$. In the following, we let $a_s(\zeta)$ denote one such point. Also let $\Delta(w, r) = \partial D \cap B(w, r), r > 0$, and if $\zeta \in \Delta(w, 2\hat{r})$ and $t > 1$ let

$$\Gamma(\zeta) = \Gamma(\zeta, t) = \{ y \in D \cap B(w, 4\hat{r}) : |y - \zeta| < t\, d(y, \partial D) \}.$$

Unless otherwise stated we always assume that t is fixed and so large that $\Gamma(\zeta)$ contains the inside of a truncated cone with vertex at ζ, height $\hat{r}$, axis along the positive e_n axis, and of angle opening $\theta = \theta(t) > 0$. We note for D and $\hat{r}$ as above that $\mathbb{R}^n \setminus (D \cap B(w, \hat{r}))$ is uniformly $(\hat{r}, p)$-fat for $1 < p < \infty$. Thus, if v satisfies the same hypotheses as $\tilde{u}$ in Lemmas 2.5 and 2.6, then these Lemmas are valid with $\tilde{u}$ replaced by v in the above D. It follows that (see [ALSV18, Section 8] and [AGH+17, section 10] there exists $\bar{c} \geq 1$, depending only on the data and $\|\hat{\phi}\|$, such that if $0 < r \leq \hat{r}$ and $\bar{r} = r/\bar{c}$, then

$$\bar{r}^{p-n} \int\limits_{B(w,\bar{r})} |\nabla v|^p dx \leq \bar{c}(v(a_{\bar{r}}(w)))^p. \tag{3.3}$$

Moreover, there exists $\hat{\sigma} \in (0,1)$, depending only on the data and $\|\hat{\phi}\|$, such that

$$|v(x) - v(y)| \leq \bar{c} \left(\frac{|x-y|}{\bar{r}} \right)^{\hat{\sigma}} v(a_{\bar{r}}(w)) \quad \text{whenever } x, y \in B(w, \bar{r}). \tag{3.4}$$

Finally, there exists a unique finite positive Borel measure ν on $\mathbb{R}^n$, with support contained in $\bar{\Delta}(w,r)$, such that

$$\begin{aligned} &(a) \quad \int \langle \nabla f(\nabla v), \nabla \psi \rangle dx = -\int \psi \, d\nu \quad \text{whenever } \psi \in C_0^\infty(B(w,r)), \\ &(b) \quad \bar{c}^{-1}\, \bar{r}^{p-n} \nu(\Delta(w,\bar{r})) \leq (v(a_{\bar{r}}(w)))^{p-1} \leq \bar{c}\, \bar{r}^{p-n} \nu(\Delta(w,\bar{r})). \end{aligned} \tag{3.5}$$

Also in [AGH+17, section 10] for $1 < p < n$ and in [ALSV18, section 8] for $p \geq n$ we updated to $\mathcal{A}$-harmonic functions the following Lemmas proved in [LN07], [LN10], for p-harmonic functions when $1 < p < \infty$.

Lemma 3.1. *Let $D, \hat{r}, w, \phi$ be as in (3.2), p fixed, $1 < p < \infty$, and $0 < r \leq \hat{r}$. Also let v be $\mathcal{A}$-harmonic in $D \cap B(w,r)$ and continuous in $B(w,r)$ with $v \equiv 0$ on $B(w,r) \setminus D$. There exists $c_\star \geq 1$, depending only on the data and $\|\hat{\phi}\|$, such that if $4\tilde{r} = r/c_\star$ and $x \in B(w,\tilde{r}) \cap D$, then*

$$\begin{aligned} &(a) \quad c_\star^{-1} \frac{v(x)}{d(x,\partial D)} \leq \langle \nabla v(x), e_n \rangle \leq |\nabla v(x)| \leq c_\star \frac{v(x)}{d(x,\partial D)}\,, \\ &(b) \quad \lim_{\substack{x \to y \\ x \in \Gamma(y) \cap B(w,2r)}} \nabla v(x) \overset{def}{=} \nabla v(y) \ \textit{exists} \quad \textit{for } \mathcal{H}^{n-1}\textit{-almost every } y \in \Delta(w,\tilde{r}). \end{aligned} \tag{3.6}$$

Moreover, $\Delta(w,\tilde{r})$ has a tangent plane for $\mathcal{H}^{n-1}$-almost every $y \in \Delta(w,\tilde{r})$. If $\mathbf{n}(y)$ denotes the unit normal to this tangent plane pointing into $D \cap B(w,2\tilde{r})$, then

$$\nabla v(y) = |\nabla v(y)|\, \mathbf{n}(y) \quad \textit{for } \mathcal{H}^{n-1}\textit{-almost every } y \in \Delta(w,2\tilde{r}) \tag{3.7}$$

and

$$\frac{d\nu}{d\mathcal{H}^{n-1}}(y) = p \frac{f(\nabla v(y))}{|\nabla v(y)|} \quad \textit{for } \mathcal{H}^{n-1}\textit{-almost every } y \in \Delta(w,2\tilde{r}). \tag{3.8}$$

Finally, there exists $q > p/(p-1)$ and $c_{\star\star}$ with the same dependence as $c_\star$ such that

$$\int_{\Delta(w,\tilde{r})} \left(\frac{f(\nabla v)}{|\nabla v|} \right)^q d\mathcal{H}^{n-1} \leq c_{\star\star}\, r^{(n-1)(1-q)} \left(\int_{\Delta(w,\tilde{r})} \frac{f(\nabla v)}{|\nabla v|}\, d\mathcal{H}^{n-1} \right)^q. \tag{3.9}$$

To prove uniqueness for u_1 in Theorem A we need the following boundary Harnack inequality.

Lemma 3.2. *Let $D, \hat r, w, \phi, p$, be as in Lemma 3.1 and $0 < r \leq \hat r$. Also let v_i, for $i = 1, 2$ be $\mathcal{A}$-harmonic in $D \cap B(w, r)$ and continuous in $B(w, r)$ with $v_1 \equiv v_2 \equiv 0$ on $B(w, r) \setminus D$. Then there exist $\beta_+ \in (0, 1)$ and $c_+ \geq 1$, depending only on the data and $\|\hat{\phi}\|$, such that if $r^+ = r/c^+$ then*

$$\left| \frac{v_1(x)}{v_2(x)} - \frac{v_1(y)}{v_2(y)} \right| \leq c_+ \left(\frac{|x-y|}{r^+} \right)^{\beta_+} \frac{v_1(x)}{v_2(x)} \tag{3.10}$$

whenever $x, y \in D \cap B(w, r^+)$.

3.1. **Uniqueness in Theorem A for $0 < \alpha < \pi$.** To prove uniqueness for u_1 when $p, 1 < p < \infty$, and $\alpha \in (0, \pi)$ are fixed, suppose $\hat u > 0$ in $K(\alpha)$ and is also $\mathcal{A}$-harmonic as well as continuous in $\mathbb{R}^n$ with $\hat u \equiv 0$ on $\mathbb{R}^n \setminus K(\alpha)$ and $\hat u(e_1) = 1$. Using Lemma 3.2 with $D = B(0, R) \cap K(\alpha), v_1 = u_1, v_2 = \hat u$, and $w = 0, \hat r = R/2$, we find that

$$\left| \frac{u_1(x)}{\hat u(x)} - \frac{u_1(y)}{\hat u(y)} \right| \leq c_+ \left(\frac{|x-y|}{R} \right)^{\beta_+} \frac{u_1(x)}{\hat u(x)} \tag{3.11}$$

in $B(0, \frac{R}{2c_+})$ for some c_+ and β_+ depending only on the data and Lipschitzness of $\partial K(\alpha)$. Fixing x, y, and letting $R \to \infty$ it follows that $u_1 = \hat u$. To show that u_1 has the form (1.2) observe that for fixed $t > 0$, the function $x \mapsto u_1(tx)$ for $x \in K(\alpha)$ is positive, $\mathcal{A}$-harmonic, and has boundary value 0 on $\partial K(\alpha)$, so by uniqueness of u_1, we have

$$u_1(tx) = u_1(te_1) u_1(x) \quad \text{whenever } x \in K(\alpha). \tag{3.12}$$

Differentiating (3.12) with respect to t (permissible by Lemma 2.2) and evaluating at $t = 1$ we see that

$$\langle x, \nabla u_1(x) \rangle = \langle e_1, \nabla u_1(e_1) \rangle u_1(x) \quad \text{whenever} \quad x \in K(\alpha).$$

If we put $\rho = |x|, x/|x| = \omega \in \mathbb{S}^{n-1}$, in this identity we obtain that

$$\rho \, (u_1)_\rho(\rho\omega) = \langle e_1, \nabla u_1(e_1) \rangle u_1(\rho\omega).$$

Dividing this equality by $\rho u_1(\rho\omega)$, integrating with respect to ρ, and exponentiating, we find that $u_1(r\omega) = r^{\lambda_1} u_1(\omega)$ whenever $\omega \in \mathbb{S}^{n-1}$ where $\lambda_1 = \langle e_1, \nabla u_1(e_1) \rangle$.

To prove uniqueness for u_2 in $K(\alpha)$ with p and α fixed with $0 < \alpha < \pi, 1 < p < \infty$, we let $0 < \hat u$ be $\mathcal{A}$-harmonic in $K(\alpha)$ with continuous boundary value 0 on $\partial K(\alpha) \setminus \{0\}, \hat u(e_1) = 1$, and

$$\lim_{|x| \to \infty} \hat u(x) = 0. \tag{3.13}$$

From Lemma 3.2 we see that if $w \in \partial K(\alpha) \setminus \{0\}$, $r = |w|/4$, $v_1 \neq v_2 \in \{\hat u, u_2\}$, then (3.10) in Lemma 3.2 is valid. Now (3.10) for $\hat u, u_2$, (3.13), (2.19) for u_2, Harnack's inequality and the maximum principle for $\mathcal{A}$-harmonic functions yield that

$$c^{-1} \leq \frac{u_2(x)}{\hat u(x)} \leq c \quad \text{for } x \in K(\alpha) \tag{3.14}$$

where $c \geq 1$ depends only on the data. Indeed, if for example

$$\liminf_{\substack{x\to 0\\ x\in K(\alpha)}} \frac{u_2(x)}{\hat{u}(x)} = 0$$

then the above program first gives $u_2(x)/\hat{u}(x) \to 0$ as $x \to 0$ in $K(\alpha)$ and second that $u_2 \equiv 0$, clearly a contradiction.

Now (3.14), (3.6) (a) for $\hat{u}$ and u_2 when $w \in \partial K(\alpha) \setminus \{0\}$ and $r = |w|/4$, and (2.20) imply that there exist $c_* \geq 1$ and $\hat{\beta} \in (0,1)$, depending only on the data and α, such that

$$\left| \frac{u'(x)}{u''(x)} - \frac{u'(y)}{u''(y)} \right| \leq c_* \frac{u'(x)}{u''(x)} \left(\frac{\rho}{\min\{|x|,|y|\}} \right)^{\hat{\beta}} \quad \text{for } x, y \in \mathbb{R}^n \setminus B(0, c_* \rho) \tag{3.15}$$

whenever $0 < \rho < 1/c_*$ and $u' \neq u'' \in \{\hat{u}, u_2\}$. Fixing x, y, and letting $\rho \to 0$ we conclude that $\hat{u} = u_2$. The proof of (3.15) is quite similar to the proof of (3.10) (given the above assumptions) only arguments are made in $\mathbb{R}^n \setminus B(0,\rho)$ rather than $B(0,r)$. For the proof of a somewhat stronger inequality than (3.15) when $\hat{u}$ and u_2 are p-harmonic functions, see the proof of Theorem 3 and Corollary 5.25 in [LN10]. The proof of (3.15) when $\hat{u}$ and u_2 are $\mathcal{A}$-harmonic is essentially unchanged, so we omit the details. Homogeneity of u_2, i.e., (1.1), assuming uniqueness, is proved in the same way as for u_1 when $\alpha \in (0,\pi)$.

3.2. **Existence and uniqueness in Theorem A for $\alpha = \pi$.** It remains to show existence and uniqueness in Theorem A when $\alpha = \pi$ and $p > n-1$. To do this, for $i = 1, 2$, we temporarily write

$$u_i(tx, \alpha) = t^{\lambda_i(\alpha)} u_i(x, \alpha) \quad \text{for } x \in K(\alpha) \text{ and } \alpha \in (0,\pi)$$

for the functions in Theorem A corresponding to $K(\alpha)$. From the maximum principle for $\mathcal{A}$-harmonic functions it follows that if $0 < \alpha_1 < \alpha_2 < \pi$, then $u_1(\cdot,\alpha_1) \leq \bar{c}\, u_1(\cdot,\alpha_2)$ in $K(\alpha_1) \cap B(0,1)$ so necessarily

$$0 < \lambda_1(\alpha_2) \leq \lambda_1(\alpha_1).$$

Also strict inequality must hold since otherwise from (1.1) it would follow that $u_1(\cdot,\alpha_1)/u_1(\cdot,\alpha_2)$ has an absolute maximum in $K(\alpha_1)$ which again leads to a contradiction by way of the maximum principle for $\mathcal{A}$-harmonic functions. Similarly , if $0 < \alpha_1 < \alpha_2 < \pi$, then $u_2(\cdot,\alpha_1) \leq \bar{c}\, u_2(\cdot,\alpha_2)$ in $K(\alpha_1) \setminus B(0,1)$ and $\lambda_2(\alpha) < 0$ for $\alpha \in (0,\pi)$, thanks to (2.19) for $u_2(\cdot,\alpha)$. Thus

$$0 < -\lambda_2(\alpha_2) \leq -\lambda_2(\alpha_1).$$

Moreover, strict inequality holds in this equation since otherwise we could get a contradiction by the same argument as above. We conclude from our considerations for $i = 1, 2$, that

$$|\lambda_i(\alpha)| \text{ is decreasing on } (0,\pi). \tag{3.16}$$

For $i = 1, 2$, let

$$\lambda_i(\pi) = \lim_{\alpha \to \pi} \lambda_i(\alpha).$$

We note that if $\alpha \in (0,\pi]$ and $n-1 < p < \infty$ then Lemmas 2.1, 2.2, 2.5, and (2.8) are valid for u_1 in $K(\alpha) \cap B(0,\rho)$ with constants depending only on the data as follows from uniform (ρ,p)-fatness of $(\mathbb{R}^n \setminus K(\alpha)) \cap B(0,\rho)$ when $n-1 < p < \infty$. Using these facts and Ascoli's theorem we find that as $m \to \infty$, a subsequence of $\{u_1(\cdot, \pi - 1/m)\}$, converges uniformly on compact subsets of $\mathbb{R}^n$ to $u_1(\cdot,\pi)$, a Hölder continuous function on $\mathbb{R}^n$ which is $\mathcal{A}$-harmonic in $K(\pi)$ with $u_1 \equiv 0$ on $\partial K(\pi)$. Similarly, Lemmas 2.1, 2.2, 2.5, (2.8), (2.20), and (2.19) (with v_l replaced by u_2) are valid for $u_2(\cdot,\alpha)$ in $K(\alpha) \cap B(w,\rho)$ whenever $w \in \partial K(\alpha) \setminus \{0\}$ and $\rho < |w|/4$. All constants depend only on the data for $n-1 < p < \infty$. Using these facts as above, we obtain $u_2(\cdot,\pi)$, a uniform limit on compact subsets of $\mathbb{R}^n \setminus \{0\}$, of a subsequence of $(u_2(\cdot,\pi - 1/m))$ as $m \to \infty$. Also $u_2(\cdot,\pi)$ is $\mathcal{A}$-harmonic in $K(\alpha) \setminus \{0\}$ and locally Hölder continuous on $\mathbb{R}^n \setminus \{0\}$. Moreover, (2.19), (2.20) hold with v_l, u_2, replaced by $u_2(\cdot,\pi)$. From (1.1) for $\alpha \in (0,\pi)$ and (3.16) we deduce for $i = 1,2$, that

$$u_i(tx,\pi) = t^{\lambda_i(\pi)} u_i(x,\pi) \quad \text{whenever } x \in \mathbb{R}^n \setminus \{0\}. \tag{3.17}$$

To prove uniqueness of $u_i(\cdot,\pi)$ for $i = 1,2$, we need several Lemmas analogous to Lemmas 3.1 and 3.2 for Lipschitz domains.

Lemma 3.3. *Fix p with $n-1 < p < \infty$, $n > 2$, $t > 0$, and let $\breve{I}$ be the line segment with endpoints $-3te_1/2$ and $-te_1/2$. Let $0 < v$ be $\mathcal{A} = \nabla f$-harmonic in $B(-te_1, t/2) \setminus \breve{I}$ with continuous boundary value 0 on $\breve{I}$. Then there exists $c \geq 4$, depending only on the data, such that*

$$c^{-1} \frac{v(x)}{d(x,\breve{I})} \leq |\nabla v(x)| \leq c \frac{v(x)}{d(x,\breve{I})} \tag{3.18}$$

for $x \in B(-te_1, t/c) \setminus \breve{I}$.

Proof. See Lemma 7.1 in [LN18]. □

Lemma 3.4. *Let $p, n, f, t, \breve{I}$, be as in Lemma 3.3. For fixed $\rho, 0 < \rho < t/2$, let $0 < v_i$, $i = 1,2$, be $\mathcal{A}$-harmonic in $B(-te_1,\rho) \setminus \breve{I}$. There exist $c_* \geq 1$ and $\beta_* \in (0,1)$, depending only on the data, such that*

$$\left| \frac{v_1(x)}{v_2(x)} - \frac{v_1(y)}{v_2(y)} \right| \leq c_* \frac{v_1(x)}{v_2(x)} \left(\frac{|x-y|}{\rho} \right)^{\beta_*} \tag{3.19}$$

whenever $x, y \in B(-te_1, \rho/c_*) \setminus \breve{I}$.

Proof. See Lemma 6.2 in [LN18]. □

We now prove uniqueness of $u_1(\cdot,\pi)$ when $p > n-1$ and $n \geq 2$. Suppose $0 \leq \hat{u}$ is also $\mathcal{A}$-harmonic in $K(\pi)$ with continuous boundary value 0 on $\partial K(\pi)$ and $\hat{u}(e_1) = 1$. Then from (3.19), Harnack's inequality, and the maximum principle for $\mathcal{A}$-harmonic functions we deduce for $n \geq 3$ as in (3.14) that

$$\tilde{c}^{-1} \leq \frac{u_1(x,\pi)}{\hat{u}(x)} \leq \tilde{c} \quad \text{whenever } x \in \mathbb{R}^n \setminus \partial K(\pi) \tag{3.20}$$

where $\tilde{c}$ depends only on the data. To prove (3.20) for $n = 2$ we note that both components of $B(-te_1, \rho) \setminus \breve{I}$ are Lipschitz domains so we can use the boundary Harnack inequality for Lipschitz domains (Lemma 3.2) to estimate the ratio of $u_1(\cdot, \pi)/\hat{u}$ in $B(-te_1, t/c) \setminus \breve{I}$. Doing this and using Harnack's inequality, the maximum principle for $\mathcal{A}$-harmonic functions, once again, it follows that Lemma 3.4 and (3.20) are also valid when $n = 2$. Next observe from homogeneity of $u_1, u_1 \geq 0$, and Lemmas 2.1, 2.2, that given $0 < \delta < \pi$ there exists $c(\delta) \geq 1$, depending only on the data and δ, such that

$$c(\delta)^{-1} \frac{u_1(x, \pi)}{d(x, \partial K(\pi))} \leq |\nabla u_1(x, \pi)| \leq c(\delta) \frac{u_1(x, \pi)}{d(x, \partial K(\pi))} \tag{3.21}$$

for $x \in K(\pi - \delta)$. Using Lemma 3.3 for $u_1(\cdot, \pi)$ when $n > 2$ and (3.1) (a) on both sides of $\partial K(\pi)$ when $n = 2$, we deduce for fixed $\delta = \delta_0$ near enough π that (3.21) is valid when $x \in K(\pi)$ for some $c(\delta_0)$, depending only on the data. Finally (3.20), (3.21), and Lemmas 3.3, 3.4, can be used for $n \geq 3$ as in [LN18, subsection 4.2, Assumption 1] and for $n = 2$ as in [LLN08] to show first that (3.21) with u_1 replaced by $\hat{u}$ holds when $x \in K(\pi)$ for some $0 < \delta = \delta_1 < \delta_0$. Second that (3.19) of Lemma 3.4 is valid for x, y in $K(\pi) \cap B(0, \rho)$ whenever $v_1 = u_1, v_2 = \hat{u}$. Letting $\rho \to \infty$ in this inequality it follows that $\hat{u} = u_1(\cdot, \pi)$ so u_1 is unique.

To briefly outline the strategy in the proof of Lemma 3.4, assuming (3.21) for $\hat{u}, u_1$, in $\mathbb{R}^n \setminus \partial K(\pi)$, when $n \geq 3$, suppose $a, b \in (0, \infty)$. Then using Lemmas 2.1, 2.2, and (3.21), one can show that $\chi(x) = (a\,|\nabla \hat{u}(x)| + b\,|\nabla u_1(x)|)^{p-2}$ is an A_2 weight on $\mathbb{R}^n$ with A_2 constant $\leq c$ where c depends only on the data. That is,

$$\left(\int_{B(y,r)} \chi dx \right) \cdot \left(\int_{B(y,r)} \chi^{-1} dx \right) \leq c\, r^{2n} \text{ whenever } y \in \mathbb{R}^n \text{ and } r > 0.$$

Also $\zeta = a\, u_1 - b\, \hat{u}$ is a weak solution to the degenerate elliptic divergence form PDE,

$$L\zeta = \sum_{i,j=1}^{n} \frac{\partial (b_{ij}(x) \zeta_{x_j})}{\partial x_i} = 0 \tag{3.22}$$

where

$$b_{ij}(x) = \int_0^1 f_{\eta_i \eta_j}(ta\, u_1(x, \pi) + (1 - t) b\, \hat{u}(x))\, dt \tag{3.23}$$

whenever $x \in K(\pi)$. Moreover, for some $c \geq 1$ depending only on the data,

$$c^{-1} \chi(x) |\xi|^2 \leq \sum_{i,j=1}^{n} b_{ij}(x) \xi_i \xi_j \leq c |\xi|^2 \chi(x) \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\}. \tag{3.24}$$

Using (3.22)-(3.24) the authors then use boundary Harnack inequalities from divergence form linear degenerate elliptic PDE whose degeneracy is given in terms of an A_2 weight to get Lemma 3.4 (see section 4 in [LN18]) . On the other hand, (3.21) for $\hat{u}$ is proven by a perturbation type argument as in (4.42)-(4.45) of [LN18].

Uniqueness of $u_2(\cdot, \pi)$ is proved similarly. Indeed suppose $\hat{u}$ is also $\mathcal{A}$-harmonic in $K(\pi)$ with continuous boundary value 0 on $\partial K(\pi) \setminus \{0\}$, and $\lim_{|x| \to \infty} \hat{u}(x) = 0$.

Then (3.20) and (3.21) in $K(\pi)$ are valid with $u_1(\cdot,\pi)$ replaced by $u_2(\cdot,\pi)$ by the same argument as the one we gave for $u_1(\cdot,\pi)$. These inequalities can then be used as outlined above to show that for some $\bar{c}^* \geq 1$ and $\beta^* \in (0,1)$, depending only on the data, that

$$\left| \frac{u_2(x,\pi)}{\hat{u}(x)} - \frac{u_2(y,\pi)}{\hat{u}(y)} \right| \leq c^* \frac{u_2(x,\pi)}{\hat{u}(x)} \left(\frac{\rho}{\min\{|x|,|y|\}} \right)^{\beta^*} \tag{3.25}$$

whenever $|x|,|y| \geq 2\rho$. Letting $\rho \to 0$ we then get $u_2(\cdot,\pi) = \hat{u}$. This completes the proof of uniqueness for $u_1(\cdot,\pi)$ and $u_2(\cdot,\pi)$.

4. Proof of (1.7) in Theorem A

To show $\lambda(\pi) = 1 - (n-1)/p$ for fixed $p > n-1, n \geq 2$, and f as in (1.5), we let $0 < \delta < 10^{-100}$ be a small but fixed number. Also $\epsilon > 0$, $0 < \epsilon << \delta^{1000}$ is allowed to vary. Put

$$E = \{x : x_1 \geq -1\} \setminus K(\pi - \epsilon).$$

Given $\eta \in \mathbb{R}^n \setminus \{0\}$, let $\hat{f}(\eta) = f(-\eta)$ when $n-1 < p < n$. If $n-1 < p < n$ let $U = 1 - \hat{U}$ where $\hat{U}$ is the $\hat{\mathcal{A}} = \nabla \hat{f}$-capacitary function for E as in Theorem 1.3, so U is $\mathcal{A} = \nabla f$-harmonic. If $n \leq p < \infty$ let U be the $\mathcal{A} = \nabla f$-harmonic Green's function for E as in Theorem 1.2. We also write u and λ for $u_1(\cdot, \pi-\epsilon)$ and $\lambda_1(\pi - \epsilon)$ in Theorem A when there is no chance of confusion. We shall need the following lemma (see Definition 2.8, Remark 2.11 for notation).

Lemma 4.1. *We have*

$$p\int_{\partial E} \langle x + e_1, \mathbf{n}\rangle f(\nabla U) d\mathcal{H}^{n-1} = \begin{cases} \frac{p(n-p)}{p-1} Cap_{\hat{\mathcal{A}}}(E) & \text{when } n-1 < p < n \\ \gamma & \text{when } p = n \\ \frac{p-n}{p-1} \mathcal{C}_{\mathcal{A}}(E)^{1/(p-1)} & \text{when } p > n \end{cases} \approx c$$

where $\mathbf{n}(x)$ denotes the outer unit normal at $x \in \partial E$ and c depends only on the data.

Proof. Lemma 4.1 is proved in [ALV19] (see Remark 11.3) for $p \geq n$ and in [AGH⁺17] (see Remark 13.4) for $n-1 < p < n$, using the Hadamard variational formula. The integral in these remarks is defined in terms of a measure on $\mathbb{S}^{n-1}$ obtained by way of the Gauss map, so for example as in (c) of Theorem 1.2 for $p \geq n$, and the support function of a convex set relative to zero rather than $-e_1$. However, using (1.8) (ii) and the definition of a support function it is easily seen that both integrals are equal. □

To obtain estimates on U near ∂E we note that in [LN18, Lemma 5.3], it was shown that for fixed p with $n-1 < p < \infty$ and $n \geq 3$, a continuous function w on $\mathbb{R}^n$ exists with $w \equiv 0$ on T where

$$T := \{x : \ x_k = 0 \text{ for } 2 \leq k \leq n \text{ and } -\infty < x_1 < \infty\}.$$

Also w is $\mathcal{A}$-harmonic in $\mathbb{R}^n \setminus T$ and for $x \in \mathbb{R}^n$,

$$w(x) \approx |x - x_1 e_1|^\theta \quad \text{where } \theta = \frac{p+1-n}{p-1}. \tag{4.1}$$

Ratio constants depend only on the data. We use (4.1) to show that there exists $\tilde{c}_1 \geq 1$ depending only on the data with

$$\tilde{c}_1\, U(x) \geq w(x) \quad \text{when } x \in B(0,2) \cap \{y : |y - y_1 e_1| \geq \tilde{c}_1 \epsilon\}. \tag{4.2}$$

To prove (4.2) observe from Lemma 4.1, (2.10) with $\mathcal{A}$ replaced by $\hat{\mathcal{A}}$ for $n-1 < p < n$, and (2.14) when $p \geq n$ that $w \leq c' U$ on $\partial B(0,2)$. Using (4.1) and the boundary maximum principle for $\mathcal{A}$-harmonic functions it follows that for some $c'' \geq 1$,

$$w \leq c' U + c'' \epsilon^{\theta} \quad \text{in } K(\pi - \epsilon) \cap B(0,2) \tag{4.3}$$

where constants depend only on the data. Using (4.1) in (4.3) we see for $\epsilon > 0$, sufficiently small, that (4.2) is valid. Next we show for some $\tilde{c}_2 \geq 4$, depending only on the data that

$$U/w \leq \tilde{c}_2 \quad \text{in } B(-\tfrac{1}{2}e_1, \tfrac{1}{\tilde{c}_2}) \setminus T. \tag{4.4}$$

To prove (4.4) let v_1 be the $\mathcal{A}$-harmonic function in $B(-\frac{1}{2}e_1, \frac{1}{4}) \setminus T$ with continuous boundary values $v_1 = u$ on $\partial B(-\frac{1}{2}e_1, \frac{1}{4})$ and $v_1 \equiv 0$ on $T \cap B(-\frac{1}{2}e_1, \frac{1}{4})$. Comparing boundary values of u and v_1, we see from the maximum principle for $\mathcal{A}$-harmonic functions that $u \leq v_1$ in $B(-\frac{1}{2}e_1, \frac{1}{4}) \setminus T$. This inequality, (4.1), and Lemma 3.4 with $v_2 = w$, give (4.4) since $u(-\frac{1}{2}e_1 + \frac{1}{4}e_n) \approx w(-\frac{1}{2}e_1 + \frac{1}{4}e_n)$.

Let $S = E \cap \{y : y_1 \geq -1 + 4\delta\}$ and let V be the $\mathcal{A}$-harmonic Green's function for the complement of $S_1 = E \cap \{x : x_1 \leq -1 + 4\delta\}$ with a pole at infinity when $p \geq n$ while $V = 1 - \hat{V}$ where $\hat{V}$ is the $\hat{\mathcal{A}}$-harmonic capacitary function for S_1 if $n-1 < p < n$. We note from (2.15) that $V \geq U$ in $\mathbb{R}^n$ when $p \geq n$. Using this note, (3.6) (b), (3.7), and the Hopf boundary maximum principle we deduce for $n < p < \infty$ that

$$\begin{aligned}
&\int_{\partial S_1 \cap \partial E} \langle y + e_1, \nabla U \rangle f(\nabla U(y)) |\nabla U(y)|^{-1} d\mathcal{H}^{n-1} \\
&\qquad \leq \int_{\partial S_1} \langle y + e_1, \nabla V(y) \rangle f(\nabla V(y)) |\nabla V(y)|^{-1} d\mathcal{H}^{n-1} \\
&\qquad \leq c\delta^{\frac{|p-n|}{p-1}}
\end{aligned} \tag{4.5}$$

thanks to Lemma 4.1 with E replaced by S_1 and (2.13), where c depends only on the data. If $1 < p < n$ we see from (2.9) (b) that $U(x), V(x) \to 1$ as $|x| \to \infty$ so $U \leq V$ in $\mathbb{R}^n$, by the maximum principle for $\mathcal{A}$-harmonic functions. In view of this fact and (2.3) we conclude that (4.5) remains valid when $1 < p < n$ if $\delta^{\frac{|p-n|}{p-1}}$ is replaced by $\delta^{|p-n|}$. If $p = n$ it follows from (2.4) (ii), (2.8), for U with $w = -e_1$, and (2.14), (2.8), dilation invariance and Harnack's inequality for $\mathcal{A}$-harmonic functions, as applied to V, that for some $c \geq 1, \hat{\sigma} \in (0,1)$, depending only on the data,

$$\max_{\partial B(-e_1, 8\delta)} U \leq c\delta^{\hat{\sigma}} \leq c^2 \delta^{\hat{\sigma}} \min_{\partial B(-e_1, 8\delta)} V.$$

Then by the boundary maximum principle for $\mathcal{A}$-harmonic functions,

$$U \leq c^2 \delta^{\hat{\sigma}} V \quad \text{in } B(-e_1, 8\delta) \setminus S_1. \tag{4.6}$$

Using (4.6) and arguing as above it follows for some $c \geq 1$ that

$$\int_{\partial S_1 \cap \partial E} \langle y + e_1, \nabla U \rangle f(\nabla U(y)) |\nabla U(y)|^{-1} d\mathcal{H}^{n-1} \leq c\delta^{n\hat{\sigma}}. \tag{4.7}$$

From (4.5), (4.7), Lemma 4.1, we see for $\delta > 0$ sufficiently small that

$$\begin{aligned} &\int_{\partial E} \langle y + e_1, \nabla U \rangle f(\nabla U(y)) |\nabla U(y)|^{-1} d\mathcal{H}^{n-1} \\ &\qquad \approx \int_{\partial S \cap \partial E} \langle y + e_1, \nabla U \rangle f(\nabla U(y)) |\nabla U(y)|^{-1} d\mathcal{H}^{n-1} \end{aligned} \tag{4.8}$$

where constants depend only on the data. Finally, we claim for some $c(\delta) \geq 1$, depending only on the data and δ that

$$c(\delta)^{-1} \leq \frac{u}{U} \leq c(\delta) \quad \text{in } B(0, 1-2\delta) \cap K(\pi - \epsilon). \tag{4.9}$$

Once (4.9) is proved we get Theorem A as follows. Note that $S \subset B(0, 1-2\delta)$ and in Lemma 4.1, $\langle x + e_1, \mathbf{n}(x) \rangle = \sin \epsilon$ when $x \neq 0$ and $x \in \partial S \cap \partial E$. Using this note, Lemma 4.1, (4.8). (4.9), and the Hopf boundary maximum principle we find that for some $\bar{c}(\delta) \geq 1$, depending only on the data and δ,

$$\bar{c}(\delta)^{-1} \leq \int_{\partial S \cap \partial E} \sin(\epsilon) f(\nabla u(y)) d\mathcal{H}^{n-1} \leq \bar{c}(\delta). \tag{4.10}$$

We also note that $\partial E \cap B(-1/2, 1/4)$ is Lipschitz on a scale of $\epsilon/100$. That is, if $z \in \partial E \cap \bar{B}(-1/2, 1/4)$, there exists $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\|\phi\hat{\|} \leq 100$ such that after a possible rotation of coordinates,

$$\begin{aligned} E \cap B(z, \epsilon/100) &= \{x = (x', x_n) : x_n > \phi(x')\} \cap B(z, \epsilon/100), \\ \partial E \cap B(z, \epsilon/100) &= \{x = (x', x_n) : x_n = \phi(x')\} \cap B(z, \epsilon/100). \end{aligned} \tag{4.11}$$

From (4.11), (3.8), (1.5) (a), (3.9) with q replaced by $p/(p-1)$ (permissible by Hölder's inequality), (3.5) (b), and Harnack's inequality all applied to u and U we see that

$$\int_{\partial E \cap B(z, \frac{\epsilon}{1000})} f(\nabla v) d\mathcal{H}^{n-1} \approx \epsilon^{n-p-1} v^p(z + 10\epsilon e_n) \quad \text{whenever } v = u \text{ or } v = U \tag{4.12}$$

where ratio constants depend only on the data. Using this inequality, (4.9), (4.2), (4.4), (4.1), and the Hopf boundary maximum principle once again, we obtain that

$$\int_{\partial E \cap B(z, \frac{\epsilon}{1000})} f(\nabla u) d\mathcal{H}^{n-1} \approx \epsilon^{n-p-1} w(z + 10\epsilon e_n)^p \approx \epsilon^{\frac{(p+1-n)}{p-1}} \tag{4.13}$$

where ratio constants depend on the data and δ. Integrating (4.13) over $z \in \partial E \cap B(-1/2, 1/4)$, and interchanging the order of integration or giving a covering argument, we conclude after some arithmetic that

$$\int_{\partial E \cap B(-1/2, 1/4)} f(\nabla u) d\mathcal{H}^{n-1} \approx \epsilon^{\frac{2-n}{p-1}}. \tag{4.14}$$

Using (4.14), $(\lambda(\pi-\epsilon)-1)p$-homogeneity of $f(\nabla u)$, and $0<\lambda(\pi-\epsilon)<1$ in (4.10) we arrive at

$$
\begin{aligned}
1 &\approx \int_{\partial S} \sin(\epsilon) f(\nabla u(y))\, d\mathcal{H}^{n-1} \\
&\approx \epsilon^{\frac{(p+1-n)}{p-1}} \int_0^1 t^{p(\lambda-1)+n-2} dt \\
&\approx \frac{1}{p(\lambda-1)+n-1}\epsilon^{\frac{(p+1-n)}{p-1}}
\end{aligned} \tag{4.15}
$$

where for brevity we have written λ for $\lambda(\pi-\epsilon)$. Also ratio constants depend only on δ and the data. Clearly (4.15) implies that

$$\lambda - 1 + (n-1)/p \approx \epsilon^{\frac{(p+1-n)}{p-1}}.$$

So if $\alpha=\pi-\epsilon$ and we use the notation in Theorem A it follows from this inequality that there exist δ_0 and ϵ_0 with $0<\epsilon_0<<\delta_0$, and a positive constant $k\geq 1$ depending on δ_0 and the data such that if $\pi-\epsilon_0<\alpha<\pi$, then

$$k^{-1}(\pi-\alpha)^{\frac{(p+1-n)}{p-1}} \leq \lambda_1(\alpha) - 1 + (n-1)/p \leq k\,(\pi-\alpha)^{\frac{(p+1-n)}{p-1}}. \tag{4.16}$$

Thus Theorem A is true once we prove claim (4.9).

Claim (4.9) is easily proved for $n=2$ using (3.10) on both sides of $\partial K(\pi-\epsilon)$ in each of the Lipschitz domains obtained from removing the positive x_1 axis from $B(0,1-2\delta)\cap\partial K(\pi-\epsilon)$, as well as Harnack' s inequality and $u(e_1)\approx U(e_1)\approx 1$. Thus we assume $n>2$. In this case we give an argument which was first used in [BL05, Lemma 2.16] and later in [LN18, section 6.1]. To begin note that (4.9) follows from

$$c^{-1}\frac{u((\delta-1)e_1+\delta e_n)}{U((\delta-1)e_1+\delta e_n)} \leq \max_{B((\delta-1)e_1,\delta/200)\setminus E} \frac{u}{U} \leq c\frac{u((\delta-1)e_1+\delta e_n)}{U((\delta-1)e_1+\delta e_n)} \tag{4.17}$$

for some $c\geq 1$, depending only on the data, as we see from $0<\epsilon<<\delta<<1$, $U(e_1)\approx u(e_1)\approx 1$, and Harnack's inequality for $\mathcal{A}$-harmonic functions. To prove the right-hand inequality in (4.17) let

$$C_t := \{x=(x_1,x')\in\mathbb{R}^n : (\delta-t-1)<x_1<(\delta+t-1),\ |x'|<\delta/1000\}$$

when $t\in[\delta/200,\delta/10]$ and suppose that $u/U>\zeta$ at some point in $\partial C_{\delta/200}\setminus E$. Given $t\in(\delta/200,\delta/100)$ observe from Harnack's inequality and the maximum principle for $\mathcal{A}$-harmonic functions that either we have $u(y)/U(y)>\zeta$ at some y in $\partial C_t\setminus E$ with $y_1=\delta\pm t-1$ or the right-hand inequality in (4.17) holds. If ζ is large enough this observation implies that there exists $I=[\delta/200,\delta/100]$ or $I=[-\delta/100,-\delta/200]$ such that for all $t\in I$ there is $y''=y''(t)$ with $0<|y''|<\delta/1000$ and

$$\frac{u(\delta+t-1,y'')}{U(\delta+t-1,y'')} > \zeta. \tag{4.18}$$

If for example there exists $t'\in[-\delta/100,-\delta/200]$ such that for all $y=(\delta+t'-1,y'')$ in $\partial C_{-t'}\setminus E$ we have $u(y)/U(y)\leq\zeta$, then we can apply the above analysis in

$$\{x=(x_1,x')\in\mathbb{R}^n : (\delta+t'-1)<x_1<(\delta+t-1), |x'|<\delta/1000\}$$

whenever $t \in [\delta/200, \delta/100]$ to conclude the existence of $I = [\delta/200, \delta/100]$.

Let ν and τ denote the measures associated with u and U restricted to $C_{\delta/10}$. We observe from (3.5) (b) that ν and τ are doubling measures in the sense that if $z \in C_{\delta/100} \cap \partial E$ and $0 < s < \delta/200$, then

$$\theta(B(z, 10s)) \leq c\, \theta(B(z, s)) \tag{4.19}$$

for $\theta \in \{\nu, \tau\}$ and some $c \geq 1$ depending only on the data.

Given $t \in I$, choose $y''(t)$ as in (4.18). If $|y''(t)| > 4\epsilon$ we put $\rho(t) = |y''(t)|$. Otherwise since as noted earlier $\partial E \cap B(-1, 1/4)$ is Lipschitz on a scale of $\epsilon/100$, we deduce from (3.10) of Lemma 3.2 that there exists $\hat y'' = \hat y''(t)$ with $|\hat y''| = 4\epsilon$ and

$$\zeta < \frac{u(\delta + t - 1, y'')}{U(\delta + t - 1, y'')} \leq c \frac{u(\delta + t - 1, \hat y'')}{U(\delta + t - 1, \hat y'')}.$$

In this case we put $\rho(t) = 4\epsilon$. Set $\tilde y''(t) = y''(t)$ when $|y''(t)| > 4\epsilon$ while $\tilde y''(t) = \hat y''(t)$, otherwise. Using (4.19) and (3.5) (b) once again it follows that

$$\zeta^{p-1} \leq c \left(\frac{u(\delta + t - 1, \tilde y''(t))}{U(\delta + t - 1, \tilde y''(t))} \right)^{p-1} \leq c^2 \, \frac{\nu(B((\delta + t - 1)e_1, \rho(t)))}{\tau(B((\delta + t - 1)e_1, \rho(t)))}. \tag{4.20}$$

Next using a standard covering lemma we see there exists $\{t_j\}, t_j \in I$, for which (4.20) holds with $t, \tilde y''(t), \rho(t)$, replaced by $t_j, \tilde y''(t_j), \rho(t_j)$. Also if $\kappa(t_j) = (\delta + t_j - 1)e_1$, then

$$\begin{aligned} L := \{y : y_1 = \delta + t - 1, t \in I\} \cap \partial E \subset \bigcup_j B(\kappa(t_j), \rho(t_j)), \\ B(\kappa(t_k), \rho(t_k)/5) \cap B(\kappa(t_l), \rho(t_l)/5) = \emptyset \quad \text{when } l \neq k. \end{aligned} \tag{4.21}$$

From (4.19), (4.20), (4.21), (3.5), and Harnack's inequality it follows, for some $c \geq 1$, depending only on the data, that

$$\begin{aligned} \zeta^{p-1} \tau(L) &\leq \zeta^{p-1} \tau \left(\bigcup_j B(\kappa(t_j), \rho(t_j)) \right) \\ &\leq \zeta^{p-1} \sum_j \tau(B(\kappa(t_j), \rho(t_j))) \\ &\leq c \sum_j \nu(B(\kappa(t_j), \rho(t_j)/5)) \leq c^2 \nu(L). \end{aligned} \tag{4.22}$$

Also from (2.8) and Harnack's inequality we see that

$$\delta^{p-n} \tau(L) \approx U((\delta - 1)e_1 + \delta e_n)^{p-1} \quad \text{and} \quad \delta^{p-n} \nu(L) \approx u((\delta - 1)e_1 + \delta e_n)^{p-1}$$

where ratio constants depend only on the data. Using these inequalities in (4.22) we find that

$$\zeta \leq c \frac{u((\delta - 1)e_1 + \delta e_n)}{\hat U((\delta - 1)e_1 + \delta e_n)}.$$

The righthand inequality in (4.17) follows from this display and Harnack's inequality for $\mathcal{A}$-harmonic functions with $2\zeta = \max(u/U)$ on $\partial C_{\delta/200} \setminus E$. Interchanging the roles of u and U in this argument we get the left-hand inequality in (4.17). This completes the proof of claim (4.9) and so also of Theorem A.

5. Proof of Theorem B

We begin this section with a discussion of some familiar concepts from convex geometry which were used in [Jer96, CNS+15] to prove analogues of Theorem B. Let $E \subset \mathbb{R}^n$ be a compact convex set with nonempty interior. Translating and dilating E if necessary we may assume that $\bar{B}(0,1) \subset E$ is a ball with largest radius contained in E while $\bar{B}(0,\tilde{R}_0)$ is the ball with smallest radius and center at the origin containing E for some $\tilde{R}_0 > 0$. Then $\tilde{e} = 1/\tilde{R}_0$ is called the eccentricity of E. From basic geometry one sees that if $w \in \partial E$ there exists $\hat{c} = \hat{c}(n) \geq 1$, depending only on n such that ∂E can be covered by at most $N = \hat{c}^2(\tilde{e})^{1-n}$ balls, $B(w,\tilde{r})$, with $w \in \partial E, \tilde{r} \geq 1/8$, and the property that after a possible change of coordinates there exists a real valued convex function ϕ on

$$\bar{B}'(w',\tilde{r}) := \{x' = (x_1,\dots,x_{n-1}) : |x'-w'| \leq \tilde{r}\}$$

which extends to a Lipschitz function on $\mathbb{R}^{n-1}$ with $\|\hat{\phi\|} \leq \hat{c}/\tilde{e}$. Moreover, if we let $w = (w', w_n)$ and $\phi(w') = w_n$, after a possible change of coordinates, we also have

$$\begin{aligned} &\{(x',x_n) : x_n = \phi(x') \text{ and } x' \in \bar{B}'(w',\tilde{r})\} \subset \partial E, \\ &\{x = (x',x_n) : x_n > \phi(x')\} \cap B(w,\tilde{r}) \subset E \setminus B(0,1/2). \end{aligned} \tag{5.1}$$

Definition 5.1. *Let ψ be a real valued convex function on a bounded convex open set $\Omega \subset \mathbb{R}^{n-1}$. If $x' \in \Omega$ we write $\theta = (\theta_1,\dots,\theta_{n-1}) \in \partial\psi(x')$ provided $\psi(y') \geq \psi(x') + \langle \theta, y'-x' \rangle$ whenever $y' \in \Omega$. If τ is a finite positive Borel measure on Ω then ψ is said to be a solution to the Monge-Ampère equation*

$$det(\nabla^2\psi) = \tau \quad on\ \Omega \tag{5.2}$$

in the sense of Alexandrov provided that

$$\mathcal{H}^{n-1}\left(\bigcup_{x'\in K} \partial\psi(x')\right) = \mathcal{H}^{n-1}\left(\partial\psi(K)\right) = \tau(K) \quad \text{for each Borel set } K \subset \Omega. \tag{5.3}$$

Let $\mathbf{g}$ denote the Gauss function for ∂E, suppose (5.1) is valid, and set $\Omega = B'(w',\tilde{r})$ and $\phi = \psi$. If $x' \in \Omega$ then one can define

$$\mathbf{g}(x',\phi(x')) = \left\{ \frac{(\theta,-1)}{(1+|\theta|^2)^{1/2}} : \ \theta \in \partial\phi(x') \right\}. \tag{5.4}$$

We note that the mapping $x' \mapsto \frac{\langle x',-1\rangle}{(1+|x'|^2)^{1/2}} = \xi$ is one to one from $\mathbb{R}^{n-1}$ onto $\mathbb{S}^{n-1} \cap \{\xi : \ \xi_n < 0\}$. Moreover, the inverse of this mapping has Jacobian $|\xi_n|^{-n}$ at ξ with $|\xi| = 1$ and $\xi_n < 0$. Using this fact, it follows from (5.2), (5.3), and (5.4) that if $K \subset B(w',\tilde{r})$ is a Borel set and $\tilde{K} := \{(x',\phi(x')) : x' \in K\}$ then

$$\mathcal{H}^{n-1}(\partial\phi(K)) = \tau(K) = \int_{\mathbf{g}(\tilde{K})} |\xi_n|^{-n} d\mathcal{H}^{n-1} \tag{5.5}$$

in the sense of Alexandrov. Next suppose for fixed p with $1 < p < \infty$ that U, E, and μ are as in Theorem 1.2 or $U = 1 - \tilde{U}$ where $\tilde{U}$ is as in Theorem 1.3. Then for $\mathcal{H}^{n-1}$-almost every $y \in \partial E$ we see from Theorems 1.2 and 1.3 that

$$\mathbf{g}(y) = \frac{\nabla U(y)}{|\nabla U(y)|}\,. \tag{5.6}$$

Thus $\mathbf{g}$ is well defined by (5.6) on a Borel set $E_1 \subset E$ with $\mathcal{H}^{n-1}(E \setminus E_1) = 0$. If also $d\mu = \Theta d\mathcal{H}^{n-1}|_{\mathbb{S}^{n-1}}$ and there exists $\tilde{a}_3 \geq 1$ such that

$$0 < \tilde{a}_3^{-1} \leq \Theta(\xi) \leq \tilde{a}_3 \quad \text{for } \mathcal{H}^{n-1}\text{-almost every } \xi \in \mathbb{S}^{n-1}, \tag{5.7}$$

then from $\||\nabla\phi|\|_\infty \leq c/\tilde{e} < \infty$, finiteness and positivity of μ, as well as the Radon-Nikodym theorem we conclude for τ and K as in (5.5) that

$$\begin{aligned} \tau(K) &= \int_{\mathbf{g}(\tilde{K})} |\xi_n|^{-n}\, d\mathcal{H}^{n-1}\xi \\ &= \int_K \frac{(1 + |\nabla\phi|^2(x'))^{(1+n)/2} f(\nabla U(x', \phi(x'))}{\Theta(\mathbf{g}(x', \phi(x'))} dx'. \end{aligned} \tag{5.8}$$

Thus to prove regularity of ∂E, we study the Monge-Ampère equation in domains of the form $\Omega = B'(w', \tilde{r})$ with measure as in (5.8). To outline some of the work of previous authors on the Monge-Ampère equation we need several definitions.

Definition 5.2. *Given ψ, Ω as in Definition 5.1 and $x' \in \Omega$, $t > 0$, $\theta \in \partial\psi(x')$, we put*

$$S(x', \theta, t) := \{y' \in \Omega : \ \psi(y') - \psi(x') - \langle \theta, y' - x' \rangle < t\} \tag{5.9}$$

and call $S = S(x', \theta, t)$ a cross section of ψ. Define the reduced distance $\delta(\cdot, S)$ on $S(x', \theta, t)$ by

$$\delta(z', S) = \min\left\{ \frac{|z' - \hat{x}|}{|z' - \hat{y}|} : \ \hat{x}, \hat{y} \in \partial S \text{ and } z' \text{ lies on the line segment from } \hat{x} \text{ to } \hat{y} \right\}.$$

Note from convexity of ψ that $S(x', \theta, t)$ is a convex set. Let $\bar{x}'$ denote the centroid of S and for $0 < \lambda < 1$, set

$$S(x', \theta, t, \lambda) := \{z' : z' = \lambda(y' - \bar{x}') + \bar{x}', \text{ such that } y' \in S(x', \theta, t)\}.$$

For ease of writing, for $y' \in \Omega$, we put

$$\begin{aligned} S_\lambda &:= S(x', \theta, t, \lambda), \\ l_{x',\theta}(y') &:= \psi(x') + \langle \theta, y' - x' \rangle, \\ \tilde{\psi}(y') &:= \psi(y') - l_{x',\theta}(y') - t \end{aligned} \tag{5.10}$$

when x', θ, t are understood. Then from a theorem of John (see [Fig17, A.3.2]) it follows that there exists a unique ellipsoid, $\mathcal{E}$ of maximum volume with $\mathcal{E} \subset S \subset (n-1)\mathcal{E}$. Using this fact and basic geometry we deduce the existence of a positive constant $\beta(n)$ and an affine mapping of the form $Tz' = A(z' - \bar{x}')$ for $z' \in \mathbb{R}^{n-1}$ where A is an $n-1 \times n-1$ nonsingular matrix with $T(\bar{x}') = 0$ and

$$B'(0, \beta(n)) \subset T(S) \subset B'(0, 1).$$

Here $T(S)$ is said to be a normalization of S. Note that $\delta(z', T(S)) = \delta(T^{-1}z', S)$ and if $\Psi(z') = \psi(T^{-1}z')$ for $z' \in T(S)$ then Ψ is convex and

$$\partial\psi(x') = A^t\,\partial\Psi(Tx')$$

where A^t is the transpose of A. Also Ψ is a solution to the Monge-Ampère equation in $T(S)$ with measure $\mathcal{T}$ where

$$\mathcal{T}(T(K)) = (\det\, A^{-1})\ \tau(K)$$

whenever $K \subset S$ is a Borel set. Finally, let $\tilde{\Psi}(Ty') = \tilde{\psi}(y')$ for $y' \in \Omega$.

Using the above normalizations it was shown in [Jer96, Lemma 7.3] that

Lemma 5.3. *Let $\Omega, \psi, \tau, t, x', \theta$, be as in Definitions 5.1, 5.2, and suppose $\bar{S}(x', \theta, t) \subset \Omega$. Then given $0 < \epsilon \leq 1$, there is a positive constant $C(\epsilon, n)$ such that*

$$|\tilde{\Psi}(z')|^{n-1} \leq C(\epsilon, n)\,\delta(z', T(S))^{\epsilon} \int_{T(S)} \delta(y', T(S))^{1-\epsilon} d\mathcal{T}(y') \tag{5.11}$$

whenever $z' \in T(S)$.

Proof. See Lemma 7.3 in [Jer96]. □

Our goal is to show for $\psi = \phi$, $\Omega = B'(w', \tilde{r})$, and $\Theta, \tilde{a}_3$ as in (5.7), (5.8), that there exists $\epsilon_0 \in (0, 1]$ for which

$$\int_S \delta(y', S)^{1-\epsilon_0} d\tau(y') \leq \hat{C}\tau(S_{\frac{1}{2}}) \tag{5.12}$$

whenever $\bar{S} = \bar{S}(x', \theta, t) \subset \Omega$ where ϵ_0 and $\hat{C}$ depend on the data, $\tilde{e}$, and $\tilde{a}_3$ in (5.7).

Before proving (5.12) we show as in [Jer96] and [GH00], how (5.12) can be used to prove Theorem B. Indeed, normalizing this problem we deduce first that if $\tilde{\Psi}(Tz') = \tilde{\psi}(z')$, then from (5.12) it follows as in Proposition 2.10 of [GH00] that

$$\int_{T(S)} \delta(z', T(S))^{1-\epsilon_0} d\mathcal{T}(z') \leq \hat{C}\mathcal{T}(S_{\frac{1}{2}}) \leq C'\,|\min_{T(S)} \tilde{\Psi}|^{n-1} = C't^{n-1}. \tag{5.13}$$

Using this inequality in (5.11) of Lemma 5.3 with $\epsilon = \epsilon_0$ and $z' = T(x')$, $\tilde{\Psi}(T(x')) = -t$, we deduce

$$1 \approx d(T(x'), \partial T(S)) \tag{5.14}$$

where ratio constants have the same dependence as $\hat{C}$ in (5.12). Using (5.12)-(5.14), it follows that

Lemma 5.4. *Let ψ be a real valued convex function on the convex open set Ω, and continuous on $\bar{\Omega}$. If $\psi \geq l$ on $\partial\Omega$ where l is an affine function and $\psi(z') = l(z')$ at some point $z' \in \Omega$, then either $\{y' : \psi(y') = l(y')\} = \{z'\}$ or this set has no extremal points in Ω.*

Proof. See Theorem 1 in [Caf91] or Theorem 4.1 in [GH00]. The proof in either paper is by contradiction and uses invariance of (5.11) and (5.12) under affine mappings as well as the following result. Suppose that $\hat{\psi}_j$ for $j = 1, 2, \dots,$ are convex functions and solutions to the Monge-Ampère equation with measures $\hat{\tau}_j$ in an open set $\hat{\Omega}$. If $(\hat{\psi}_j)$ converges uniformly on compact subsets of $\hat{\Omega}$ to $\hat{\psi}$, a solution to the Monge-Ampère equation in $\hat{\Omega}$ with measure $\hat{\tau}$, then $\hat{\tau}_j \rightharpoonup \hat{\tau}$ weakly in $\hat{\Omega}$ (see [Gut01, Lemma1.2.2]). $\square$

Applying Lemma 5.4 with $\psi = \phi$ and $\Omega = B'(w', \tilde{r})$ as in (5.1) we see that ∂E is strictly convex since otherwise it would follow from repeated application of Lemma 5.4 to balls (as in (5.1)), with nonempty intersection, that ∂E contains a line segment of infinite length. From this contradiction we conclude that ∂E is strictly convex. Now given $w = (w', \phi(w')) \in \partial E$, with $x' \in \bar{B}(w', \tilde{r}/4)$, where $\phi, \tilde{r}$ are as in (5.1), we choose t so that for $S(x', \theta, t)$ as in (5.10) we have

$$S(x', \theta, t) \subset B'(x', \tilde{r}/2) \quad \text{and} \quad \bar{S}(x', \theta, t) \cap \partial B'(x', \tilde{r}/2) \neq \emptyset.$$

Geometrically this means there is a point $z = (z', z_n) \in \partial E$ with $z' \in \bar{S}(x', \theta, t) \cap \partial B'(x', \tilde{r}/2)$ which lies at most t distance from the support plane $y_n = l_{x',\theta}(y')$ to ∂E at $(x', \phi(x'))$. We claim that $t \geq t_0 > 0$, where t_0 has the same dependence as $\hat{C}$ in (5.12). Indeed, otherwise using a compactness argument, the above convergence result, and Lemma 5.4 we could obtain a contradiction to the strict convexity of ∂E. Finally, we observe from Lipschitzness of ϕ as in (5.1) that there exists $r_1 \geq r_0 > 0$ where r_0 has the same dependence as t_0 with $B'(x', r_1) \subset S$. Next if $\tilde{\phi}$ is as in (5.10) with $\phi = \psi$, we claim there is a $\sigma > 2$, with the same dependence as t_0, r_0, satisfying

$$0 \leq t + \tilde{\phi}(y') \leq \sigma^{-l}\, t \quad \text{when } l = 1, 2, \dots, \text{ and } y' \in \hat{S}_{1/2^l}. \tag{5.15}$$

Here $\hat{S}_\lambda$ is defined in the same way as S_λ in (5.10) only with $\bar{x}'$ replaced by x'. Indeed, this inequality holds for $l = 1$ since otherwise we could use Lemma 5.4 and a compactness argument, as above, to contradict the strict convexity of ∂E. Iterating this inequality we obtain (5.15). From (5.15) and arbitrariness of $x' \in B'(w', \tilde{r}/4)$, we get first that

$$\begin{aligned} &\nabla\phi(x') \text{ exists for } x' \in B'(w', \tilde{r}/4), \\ &|\phi(y') - \phi(x') - \langle \nabla\phi(x'), y' - x' \rangle| \leq \hat{C}_1 |y' - x'|^{1+\alpha'} \end{aligned} \tag{5.16}$$

whenever $y' \in B'(w', \tilde{r}/4)$, where $\hat{C}_1 \geq 1, \alpha' \in (0,1)$, depend on the data, $\tilde{e}$, and $\tilde{a}_3$ in (5.7). Also from convexity and uniform Lipschitzness of ϕ we deduce the existence of $\hat{C}_2 \geq 1$, having the same dependence as $\hat{C}_1$ for which

$$\hat{C}_2\, |\langle \nabla\phi(y') - \nabla\phi(x'), y' - x' \rangle| \geq |\nabla\phi(y') - \nabla\phi(x')||y' - x'| \tag{5.17}$$

whenever $x', y' \in B'(w', \tilde{r}/4)$. Combining (5.16), (5.17), and using the triangle inequality, we find that

(5.18)

∂E is locally $C^{1,\alpha'}$ with norm constants depending only on the data, $\tilde{e}$, and $\tilde{a}_3$.

Using (5.18) and results from [Lie88] we see that ∇U when $1 < p \leq n$ or $\nabla \tilde{U}$ when $p > n$ has a $C^{1,\beta'}$ extension to ∂E for some $\beta' \in (0,1)$ having the same dependence as α'. Also from [Lie88] or (5.23) (to be proved) we have $\min\{|\nabla U|, |\nabla \tilde{U}|\} > 0$ on ∂E where constants depend only on the data and $\tilde{e}$. In view of this information and (5.2), (5.5), (5.8), we find that if $0 < \Theta \in C^{0,\hat{\alpha}}(\mathbb{S}^{n-1})$, then for some $0 < s_1, s_2, \alpha_*$, having the same dependence as α',

$$s_1 < \frac{d\tau}{d\mathcal{H}^{n-1}} < s_2 < \infty \quad \text{and} \quad \frac{d\tau}{d\mathcal{H}^{n-1}} \in C^{0,\alpha_*}(\Omega). \tag{5.19}$$

From the above remarks, (5.19), and [Caf89, Caf90a, Caf90b], we conclude that $\phi \in C^{2,\hat{\alpha}}(\Omega)$. Further applications of [Caf89, Caf90a, Caf90b] also give the higher order smoothness results in Theorem B.

It remains to prove (5.12) in order to complete the proof of Theorem B. Throughout the proof of this inequality we let $C \geq 1$ be a positive constant which may depend only on the data, $\tilde{e}$, and $\tilde{a}_3$, not necessarily the same at each occurrence. Also if $A \approx B$, proportionality constants may depend on the data, $\tilde{e}$, and $\tilde{a}_3$. Let $\hat{f}(\eta) = f(-\eta)$ when $1 < p < n$ and $\hat{f}(\eta) = f(\eta)$ when $p \geq n$. Also set $\hat{U} = 1 - \tilde{U}$ when $1 < p < n$ and $\hat{U} = U$ when $p \geq n$. Then $\hat{U}$ is $\hat{\mathcal{A}} = \nabla \hat{f}$-harmonic in $\mathbb{R}^n \setminus E$ with continuous boundary value 0 on ∂E. Observe from the discussion above (5.1), (5.7), and (5.8) that if $K \subset \Omega$ is a compact set then

$$\tau(K) \approx \int_K |\nabla \hat{U}(x', \phi(x'))|^p dx' = \chi(K). \tag{5.20}$$

Thus we only prove (5.12) for $\chi(\cdot)$. Recall that $\bar{S} = \bar{S}(x', \theta, t) \subset \Omega = B'(w', \tilde{r})$. We see that

$$F = \{(y', \phi(y')) : y' \in \bar{S}\}$$

is the part of ∂E that lies below or on the plane

$$\Sigma_1 = \{(y', y_n) : y_n - \phi(x') - \langle \theta, y' - x' \rangle = t\}$$

and above or on the support plane $\{(y', y_n) : y_n - \phi(x') - \langle \theta, y' - x' \rangle = 0\}$ to ∂E at $x = (x', \phi(x'))$. Then S can be viewed as the projection of F onto the plane $y_n = 0$ by lines parallel to e_n or the y_n axis. To simplify the geometry in what follows and for use in adapting the work in [Jer96] to our situation we also project F onto Σ_1 by lines parallel to e_n. More specifically, given $y = (y', \phi(y')) \in F$, let $\pi(y) \in \Sigma_1$ be that point with

$$\langle \pi(y), e_i \rangle = y_i' \quad \text{for } 1 \leq i \leq n-1.$$

Let $\tilde{S} = \pi(F)$, and note that $\tilde{S}$ is convex. Define the reduced distance $\delta(\cdot, \tilde{S})$ as in Definition 5.2 with S and $\mathbb{R}^{n-1}$, replaced by $\tilde{S}$ and Σ_1 respectively. From (5.1) and the discussion above this display we deduce that

$$\delta(x', S) \approx \delta(\pi(x), \tilde{S}) \quad \text{whenever } x \in F \tag{5.21}$$

where ratio constants depend only on $n, \tilde{e}$. Let $\delta(x,F) = \delta(\pi(x), \tilde{S})$ when $x \in F$. Then from (5.20) and (5.21) we conclude that to prove (5.12) it suffices to show,

$$\int_F \delta^{1-\epsilon_0}(x,F)\,|\nabla \hat{U}(x)|^p d\mathcal{H}^{n-1} \leq C \min\{|\nabla \hat{U}|^p : y \in F\}\ \mathcal{H}^{n-1}(F). \tag{5.22}$$

Remark 5.5. *We note that in* [Jer96, section 6], *the analogue of F is projected onto Σ_1 by rays through the origin. If $P(y)$ denotes this radial projection of $y \in F$ onto Σ_1, then in* [Jer96] *the reduced distance of $y \in F$ is defined to be equal to $\delta(P(y), \tilde{S})$. Using the definition of reduced distance and* (5.1) *it is easily verified as in* (5.21) *that $\delta(P(y), \tilde{S}) \approx \delta(\pi(y), \tilde{S})$ where proportionality constants depend only on n and $\tilde{e}$. Thus* (5.22) *implies the corresponding inequality in* [Jer96] *and vice-versa.*

To prove (5.22) we shall require the following lemma.

Lemma 5.6. *Let w, E, and $\tilde{r}$ be as in* (5.1). *There exists $C \geq 1$, depending only on the data and $\tilde{e}$, such that if $0 < r \leq \tilde{r}/C$ then*

$$r^{1-n} \int_{\Delta(w,r)} |\nabla \hat{U}|^p d\mathcal{H}^{n-1} \approx \min_{\Delta(w,r)} |\nabla \hat{U}|^p \approx r^{-p}\, \hat{U}(a_r(w))^p. \tag{5.23}$$

Proof. Let H be an open half-space with $H \cap E = \emptyset$ and ∂H a support plane for ∂E at w. Let ξ denote a unit normal pointing into H and let v be the $\hat{\mathcal{A}}$-harmonic function in $\mathbb{G} = H \cap B(w,r) \setminus \bar{B}(w + r\xi/2, r/8)$ with continuous boundary values, $v \equiv 0$ on $\partial(H \cap B(w,r))$ while $v \equiv \hat{U}(w + r\xi/2)$ on $\partial B(w + r\xi/2, r/8)$. Comparing boundary values and using Harnack's inequality for $\hat{\mathcal{A}}$-harmonic functions we see that $v \leq c\hat{U}$ in $\mathbb{G}$. Also using the boundary Harnack inequality in Lemma 3.2 and comparing v to a linear function, say l, which vanishes on ∂H with $l(w + r\xi/2) = \hat{U}(w + r\xi/2)$ we arrive at

$$\hat{U}(w + r\xi/2)/r \leq C' \hat{U}(w + s\xi)/s$$

whenever $0 < s \leq r/C'$ where $C' \geq 1$ depends only on the data. Letting $s \to 0$ in this display we get from Lemma 3.1 for $\mathcal{H}^{n-1}$-almost every $w \in \partial E$ that

$$\hat{U}(w + r\xi/2)/r \leq C' |\nabla \hat{U}(w)|. \tag{5.24}$$

Next observe from (3.9) with $q = (p-1)/p$, (3.8) of Lemma 3.1, and (3.5)(b) that there exists $C \geq 1$ depending only on the data and $\tilde{e}$ such that for $0 < r \leq \tilde{r}/C$,

$$r^{1-n} \int_{\Delta(w,r)} |\nabla \hat{U}|^p d\mathcal{H}^{n-1} \approx C r^{-p}\, \hat{U}(w + r\xi/2)^p. \tag{5.25}$$

Combining (5.25), (5.24), and using arbitrariness of w, Harnack' s inequality for $\hat{\mathcal{A}}$-harmonic functions, we conclude the validity of Lemma 5.6. □

Note from Lemma 5.6 that

$$\int_{\partial E} |\nabla \hat{U}|^p \leq C \quad \text{and} \quad \min_{\partial E} |\nabla \hat{U}| \geq C^{-1}. \tag{5.26}$$

Following [Jer96, Lemma 6.7] we first note from (5.1) that if $\zeta \in F$ and b denotes the radius of the largest $n-1$ dimensional ball contained in $\tilde{S}$ (the so called inradius of $\tilde{S}$) then

$$|\zeta - \pi(\zeta)| \leq C_+ b \tag{5.27}$$

for some $C_+ \geq 1$, depending only on the data and $\tilde{e}$. Second we state

Lemma 5.7. *If $y, z \in F$ and $\delta(y, F) \approx 1$, then*

$$\min_{\Delta(y,b)} |\nabla \hat{U}| \leq C \min_{\Delta(z,b)} |\nabla \hat{U}|. \tag{5.28}$$

Proof. The analogue of Lemma 5.7 in [Jer96] is Lemma 6.8. Given Lemma 5.6 and (5.27) we can essentially copy the clever geometric argument in [Jer96], so we refer to this paper for details. □

Lemma 5.8. *There exists $\epsilon_1 \in (0, 1]$ and $C \geq 2$ depending only on the data, $\tilde{e}$, such that $\epsilon_1 \geq C^{-1}$ when $1 < p \leq n-1$ while $\epsilon_1 \geq 1 + (1-n)/p + C^{-1}$ when $p > n-1$. Moreover, if $\hat{x} \in F$, then*

$$b^{1-n} \int_{\Delta(\hat{x},b)} |\nabla \hat{U}|^p d\mathcal{H}^{n-1} \leq C \delta(\hat{x}, F)^{-p+p\,\epsilon_1} \min_F |\nabla \hat{U}|^p. \tag{5.29}$$

Proof. As in Lemma 6.13 of [Jer96] we note that if $\delta(\hat{x}, F) \approx 1$ then (5.29) follows from Lemmas 5.6 and 5.7. Thus we assume that $\delta(\hat{x}, F) << 1$ and choose $y, z \in F$ so that $\delta(y, F) \approx 1$ and $\pi(z) = z \in F \cap \tilde{S}$ with $\pi(\hat{x})$ lying on the line segment from $\pi(z)$ to $\pi(y)$. Let $\rho = |\pi(z) - \pi(y)|$. We note that if $\rho < 100\,b$, then (5.29) follows from Lemmas 5.7, 5.6 with $w = \hat{x}, y$, (5.27), and Harnack's inequality for $\hat{\mathcal{A}}$-harmonic functions with $\epsilon_1 = 1$. Thus we assume $\rho \geq 100\,b$. Then from the John ellipsoid theorem mentioned below (5.10) we deduce

$$\bar{\delta} := |\pi(\hat{x}) - \pi(z)|/\rho \approx \delta(\hat{x}, F)$$

so we assume, as we may, that $|\pi(\hat{x}) - \pi(z)| < \rho/100$. Next we define the cone:

$$\Gamma = \{\pi(z) + s(\zeta - \pi(z)) : \zeta \in E \cap \bar{B}(y, \rho/2) \text{ and } 0 < s < \infty\}.$$

Let X denote a point that lies ρ distance from $\Gamma \cup E$ and at most 2ρ from y. As in the proof of Theorem A we first construct V a positive $\hat{\mathcal{A}}$-harmonic function in $\mathbb{R}^n \setminus \Gamma$ which is continuous in $\mathbb{R}^n$ with $V \equiv 0$ on Γ and $V(X) = \hat{U}(X)$. Second we use the boundary Harnack inequality in Lemma 3.2 as in the proof of Theorem A to deduce that V is unique and satisfies

$$V(z + s(\hat{w} - z)) = s^{\epsilon_1} V(\hat{w}) \text{ for some } \epsilon_1 > 0 \text{ whenever } \hat{w} \in \mathbb{R}^n \setminus \{z\}, s > 0. \tag{5.30}$$

We observe that $\mathbb{R}^n \setminus \Gamma$ is contained in a translation and rotation of $K(\alpha)$ for some $\alpha \in (0, \pi)$. Using this fact, Lemma 3.2, and Theorem A we see that if $p > n-1$, $\epsilon_1 - 1 + (n-1)/p \geq C^{-1}$. If $1 < p \leq n-1$, one can use a compactness argument or an argument as in [KM72] to show that $\epsilon_1 \geq C^{-1}$. Let Γ_1 be the convex hull of $E \cap \bar{B}(y, \rho/2)$ and z. Also let V^* be the $\mathcal{A}$-harmonic capacitary function for $\mathbb{R}^n \setminus \Gamma_1$ when $1 < p < n$ while V^* is the $\mathcal{A}$-harmonic Green's function for $\mathbb{R}^n \setminus \Gamma_1$ when $p \geq n$.

Define $\hat{f}$ and $\hat{\mathcal{A}}$ as above (5.20) and observe that $\hat{V}^* = 1 - V^*$ is $\hat{\mathcal{A}}$-harmonic when $1 < p < n$ while $\hat{V}^* = V^*$ is $\hat{\mathcal{A}}$-harmonic when $p \geq n$ with continuous boundary value 0 on Γ_1. We first let

$$V' = \frac{\hat{U}(X)}{\hat{V}^*(X)}\, \hat{V}^*$$

and claim that

$$\begin{aligned} &(a) \quad \hat{U} \leq C V' \quad \text{in } B(y, 4\rho) \setminus E. \\ &(b) \quad V \approx V' \quad \text{in } B(z, \rho/4) \setminus \Gamma. \\ &(c) \quad \hat{U} \approx V' \approx V \quad \text{in } B(y, \rho/8) \setminus E\,. \end{aligned} \tag{5.31}$$

To prove (5.31) (a) observe from (2.8) that

$$\max_{\bar{B}(y,4\rho)} \hat{U} \leq c\, \hat{U}(X) = c\, V'(X) \leq c^2 \max_{\bar{B}(y,4\rho)} V'.$$

This inequality, $\Gamma_1 \subset E$, and the boundary maximum principle for $\hat{\mathcal{A}}$-harmonic functions give (5.31) (a). On the other hand, (5.31) (b) follows from (2.8), Harnack's inequality for $\hat{\mathcal{A}}$-harmonic functions, Lemma 3.2, and the fact that $\Gamma_1 \cap \bar{B}(z, \rho/2) = \Gamma \cap \bar{B}(z, \rho/2)$. Finally, (5.31) (c) follows from these inequalities and the fact that

$$\Gamma \cap \bar{B}(y, \rho/2) = \Gamma_1 \cap \bar{B}(y, \rho/2) = E \cap \bar{B}(y, \rho/2).$$

We conclude from (5.31) that

$$\hat{U} \leq CV \text{ in } B(z, \rho/4) \setminus E \quad \text{while} \quad \hat{U} \approx V \text{ in } B(y, \rho/8) \setminus E. \tag{5.32}$$

If $\tilde{C} \geq 1$ is large enough depending on $\tilde{e}$ and the data, then from (5.32), the fact that $\rho \geq 100\, b$, (5.27), Harnack's inequality for $\hat{\mathcal{A}}$-harmonic functions, and Lemma 5.6 we deduce that

$$b^{-1}\, V(\pi(y) - \tilde{C}be_n) \approx b^{-1}\, \hat{U}(\pi(y) - \tilde{C}be_n) \approx \min_F |\nabla \hat{U}| \tag{5.33}$$

and

$$b^{-1}\, \hat{U}(\pi(\hat{x}) - \tilde{C}be_n) \;\leq\; C\, b^{-1} V(\pi(\hat{x}) - \tilde{C}be_n). \tag{5.34}$$

Next we draw the line segment $\hat{l}$ from z to $\pi(y) - \tilde{C}be_n$. From similar triangles and the definition of $\bar{\delta}$ below (5.29), we see that $\pi(\hat{x}) - \bar{\delta}\tilde{C}be_n$ lies on $\hat{l}$. From this observation and homogeneity of V we get

$$V(\pi(\hat{x}) - \bar{\delta}\tilde{C}be_n) = \bar{\delta}^{\epsilon_1} V(\pi(y) - \tilde{C}b\, e_n). \tag{5.35}$$

Now since Γ is convex we can repeat the argument given in Lemma 5.6 with $\hat{U}$ replaced by V to get (5.23) with $\hat{U}$ replaced by V, w by $\pi(\hat{x})$, and r by $\bar{\delta}\tilde{C}b, \tilde{C}b$, provided $\tilde{C}$ is large enough. We obtain

$$\begin{aligned} &\min_{B(\pi(\hat{x}), \tilde{C}b) \cap \partial\Gamma} |\nabla V|^p \approx b^{-p} V(\pi(\hat{x}) - \tilde{C}be_n)^p, \\ &\min_{B(\pi(\hat{x}), \bar{\delta}\tilde{C}b) \cap \partial\Gamma} |\nabla V|^p \approx (\bar{\delta}b)^{-p} V(\pi(\hat{x}) - \bar{\delta}\tilde{C}be_n)^p\,. \end{aligned} \tag{5.36}$$

From Lemma 5.6, (5.33)-(5.36), Harnack's inequality for $\hat{\mathcal{A}}$-harmonic functions, we see that

$$
\begin{aligned}
b^{1-n}\int_{\Delta(\hat{x},b)}|\nabla\hat{U}|^p d\mathcal{H}^{n-1} &\approx \min_{\Delta(\hat{x},b)}|\nabla\hat{U}|^p \approx b^{-p}\hat{U}(\pi(\hat{x})-\tilde{C}be_n)^p \\
&\le C\,(\bar{\delta}b)^{-p}\,V(\pi(\hat{x})-\bar{\delta}\tilde{C}be_n)^p \\
&\le C'\,\bar{\delta}^{(\epsilon_1-1)p}\ \min_F|\nabla\hat{U}|^p
\end{aligned}
\tag{5.37}
$$

where C, C' depend only on the data and $\tilde{e}$. Thus Lemma 5.8 is valid. □

To complete the proof of Theorem B we need Lemma 6.17 from [Jer96] which in our situation can be stated as following lemma.

Lemma 5.9. *With the same notation as in Lemma 5.8 choose a coordinate system with axes parallel to the axes of an optimal inscribed ellipsoid contained in $\tilde{S}$. Let $\mathcal{Q}$ be a tiling of $\tilde{S}$ by closed cubes $\subset \sum_1$ and of side-length $s\le b$ with sides parallel to the coordinate axes. If $Q\in\mathcal{Q}$, let Q^* be the cube concentric to Q with side-length $10((n-1)!)^2 s$ and let*

$$\delta^*(Q)=\max_{y\in Q^*\cap\tilde{S}}\delta(y,\tilde{S}).$$

There exists $c(n)\ge 1$ such that

$$\sum_{\{Q:\,\delta^*(Q)<\sigma\}}\mathcal{H}^{n-1}(Q)\le c(n)\sigma\mathcal{H}^{n-1}(\tilde{S}) \tag{5.38}$$

where C depends only on the data and $\tilde{e}$.

Let ϵ_1 be as in Lemma 5.8 and put $\epsilon_0=\epsilon_1$ if $1<p\le n-1$ while $\epsilon_0=\epsilon_1-1+(n-1)/p$ if $p>n-1$. To prove (5.22) and thus complete the proof of Theorem B we first note from Lemma 5.9 that if $\epsilon\in(0,1)$,

$$\sum_{\{Q\in\mathcal{Q}\}}\delta^*(Q)^{-1+\epsilon}\mathcal{H}^{n-1}(Q)\le C(\epsilon)\mathcal{H}^{n-1}(\tilde{S}), \tag{5.39}$$

as follows from summing separately over cubes $Q\in\mathcal{Q}$ with $\delta^*(Q)\le 2^{-k}s, k=0,1,2,\dots$ Second from Lemmas 5.6, 5.8, we deduce that if $\hat{y},\hat{z}\in F$ and $\pi(\hat{y}),\pi(\hat{z})\in Q^*$, then

$$\max_{\hat{\zeta}\in\{\hat{x},\hat{y}\}} b^{1-n}\int_{\Delta(\hat{\zeta},b)}|\nabla\hat{U}|^p d\mathcal{H}^{n-1}\le C\min_{\hat{\zeta}\in\{\hat{y},\hat{z}\}}\delta^{-p+\epsilon_1 p}(\hat{\zeta},F)\min_F|\nabla\hat{U}|^p.$$

Hence,

$$b^{1-n}\int_{\pi^{-1}(Q)\cap F}|\nabla\hat{U}|^p d\mathcal{H}^{n-1}\le C\delta^*(Q)^{-p+p\epsilon_1}\min_F|\nabla\hat{U}|^p.$$

Using (5.39) with $s = b/2$, and the above inequality we conclude that

$$\begin{aligned}\int_F \delta^{1-\epsilon_0}(\cdot, F)\, |\nabla \hat{U}|^p \, d\mathcal{H}^{n-1} &\leq C \sum_{Q\in\mathcal{Q}} \delta^*(Q)^{1-\epsilon_0} \int_{\pi^{-1}(Q)\cap F} |\nabla \hat{U}|^p d\mathcal{H}^{n-1} \\ &\leq C^2 \sum_{Q\in\mathcal{Q}} \delta^*(Q)^{1-\epsilon_0-p+p\epsilon_1} \mathcal{H}^{n-1}(Q) \min_F |\nabla \hat{U}|^p \\ &\leq C^3 \mathcal{H}^{n-1}(F) \min_F |\nabla \hat{U}|^p\end{aligned} \tag{5.40}$$

as we obtain from (5.39) if $1 < p \leq 2$ or $n = 2, 3$, and $p > 2$. Indeed , $\delta^*(Q)^{1-\epsilon_0-p+p\epsilon_1} \leq \delta^*(Q)^{-1+(p-1)\epsilon_0}$ if $1 < p \leq 2$ while $\delta^*(Q)^{1-\epsilon_0-p+p\epsilon_1} = \delta^*(Q)^{2-n+(p-1)\epsilon_0} \leq \delta^*(Q)^{-1+(p-1)\epsilon_0}$ if $n = 2, 3$ and $p > 2$. Thus (5.22) is valid and the proof of Theorem B is now complete.

Acknowledgement

Part of this research was done while the second author was visiting TIFR in Bangalore India. The second author thanks TIFR for their gracious hospitality.

References

[AGH+17] M. Akman, J. Gong, J. Hineman, J. Lewis, and A. Vogel, *The Brunn-Minkowski inequality and A Minkowski problem for nonlinear capacity*, To appear in Memoirs of the AMS, arXiv:1709.00447 (2017). (Cited on pages 4, 5, 6, 7, 11, 13, 14, 15, and 20).

[ALSV18] Murat Akman, John Lewis, Olli Saari, and Andrew Vogel, *The Brunn-Minkowski inequality and A Minkowski problem for $\mathcal{A}$-harmonic Green's function*, To appear in Advances in Calculus of Variations, arXiv:1810.03752 (2018). (Cited on pages 4, 5, 6, 7, 11, 12, 13, 14, and 15).

[ALV19] Murat Akman, John Lewis, and Andrew Vogel, *Note on an eigenvalue problem for an ode originating from a homogeneous p-harmonic function*, Algebra i Analiz **31** (2019), no. 2, 75–87. (Cited on pages 5, 7, and 20).

[Aro86] Gunnar Aronsson, *Construction of singular solutions to the p-harmonic equation and its limit equation for $p = \infty$*, Manuscripta Math. **56** (1986), no. 2, 135–158. MR 850366 (Cited on page 2).

[AS05] Hiroaki Aikawa and Nageswari Shanmugalingam, *Carleson-type estimates for p-harmonic functions and the conformal Martin boundary of John domains in metric measure spaces*, Michigan Math. J. **53** (2005), no. 1, 165–188. MR 2125540 (Cited on page 10).

[BL05] Björn Bennewitz and John Lewis, *On the dimension of p-harmonic measure*, Ann. Acad. Sci. Fenn. Math. **30** (2005), no. 2, 459–505. MR 2173375 (Cited on page 23).

[Caf89] Luis A. Caffarelli, *Interior a priori estimates for solutions of fully nonlinear equations*, Ann. of Math. (2) **130** (1989), no. 1, 189–213. MR 1005611 (Cited on pages 7 and 29).

[Caf90a] L. A. Caffarelli, *A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity*, Ann. of Math. (2) **131** (1990), no. 1, 129–134. MR 1038359 (Cited on pages 7 and 29).

[Caf90b] Luis A. Caffarelli, *Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation*, Ann. of Math. (2) **131** (1990), no. 1, 135–150. MR 1038360 (Cited on pages 7 and 29).

[Caf91] Luis A. Caffarelli, *Some regularity properties of solutions of Monge Ampère equation*, Comm. Pure Appl. Math. **44** (1991), no. 8-9, 965–969. MR 1127042 (Cited on pages 7 and 27).

[CNS+15] A. Colesanti, K. Nyström, P. Salani, J. Xiao, D. Yang, and G. Zhang, *The Hadamard variational formula and the Minkowski problem for p-capacity*, Advances in Mathematics **285** (2015), 1511 – 1588. (Cited on pages 6, 7, and 25).

[DS16] Dante DeBlassie and Robert G. Smits, *The p-harmonic measure of a small spherical cap*, Matematiche (Catania) **71** (2016), no. 1, 149–171. MR 3528055 (Cited on page 3).

[DS18] Dante DeBlassie and Robert G. Smits, *The p-harmonic measure of small axially symmetric sets*, Potential Anal. **49** (2018), no. 4, 583–608. MR 3859537 (Cited on page 3).

[EL91] Alexandre Eremenko and John Lewis, *Uniform limits of certain A-harmonic functions with applications to quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I Math. **16** (1991), no. 2, 361–375. MR 1139803 (Cited on page 10).

[Fig17] Alessio Figalli, *The Monge-Ampère equation and its applications*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2017. MR 3617963 (Cited on page 26).

[GH00] Cristian E. Gutiérrez and Qingbo Huang, *Geometric properties of the sections of solutions to the Monge-Ampère equation*, Trans. Amer. Math. Soc. **352** (2000), no. 9, 4381–4396. MR 1665332 (Cited on pages 7 and 27).

[Gut01] Cristian E. Gutiérrez, *The Monge-Ampère equation*, Progress in Nonlinear Differential Equations and their Applications, vol. 44, Birkhäuser Boston, Inc., Boston, MA, 2001. MR 1829162 (Cited on page 28).

[GV18] Konstantinos T. Gkikas and Laurent Véron, *The spherical p-harmonic eigenvalue problem in non-smooth domains*, J. Funct. Anal. **274** (2018), no. 4, 1155–1176. MR 3743193 (Cited on page 2).

[HKM06] Juha Heinonen, Tero Kilpeläinen, and Olli Martio, *Nonlinear potential theory of degenerate elliptic equations*, Dover Publications Inc., 2006. (Cited on pages 4, 9, and 10).

[Jer96] David Jerison, *A Minkowski problem for electrostatic capacity*, Acta Math. **176** (1996), no. 1, 1–47. MR 1395668 (Cited on pages 6, 7, 25, 27, 29, 30, 31, and 33).

[JK82] David S Jerison and Carlos E Kenig, *Boundary behavior of harmonic functions in non-tangentially accessible domains*, Advances in Mathematics **46** (1982), no. 1, 80 – 147. (Cited on page 10).

[KM72] I. N. Krol' and V. G. Maz'ja, *The absence of the continuity and Hölder continuity of the solutions of quasilinear elliptic equations near a nonregular boundary*, Trudy Moskov. Mat. Obšč. **26** (1972), 75–94. MR 0377265 (Cited on pages 2 and 31).

[Kro73] I. N. Krol', *The behavior of the solutions of a certain quasilinear equation near zero cusps of the boundary*, Trudy Mat. Inst. Steklov. **125** (1973), 140–146, 233, Boundary value problems of mathematical physics, 8. MR 0344671 (Cited on page 2).

[KZ03] Tero Kilpeläinen and Xiao Zhong, *Growth of entire $\mathcal{A}$-subharmonic functions*, Ann. Acad. Sci. Fenn. Math. **28** (2003), no. 1, 181–192. MR 1976839 (Cited on page 10).

[Lie88] Gary M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219. MR 969499 (Cited on page 28).

[LLN08] John Lewis, Niklas Lundström, and Kaj Nyström, *Boundary Harnack inequalities for operators of p-Laplace type in Reifenberg flat domains*, Perspectives in partial differential equations, harmonic analysis and applications, Proc. Sympos. Pure Math., vol. 79, Amer. Math. Soc., Providence, RI, 2008, pp. 229–266. MR 2500495 (Cited on pages 6 and 19).

[LMTW19] José G. Llorente, Juan J. Manfredi, William C. Troy, and Jang-Mei Wu, *On p-harmonic measures in half-spaces*, Annali di Matematica Pura ed Applicata (1923 -) (2019). (Cited on page 3).

[LN07] John Lewis and Kaj Nyström, *Boundary behaviour for p harmonic functions in Lipschitz and starlike Lipschitz ring domains*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 5, 765–813. MR 2382861 (Cited on pages 6 and 15).

[LN10] John Lewis and Kaj Nyström, *Boundary behavior and the Martin boundary problem for p harmonic functions in Lipschitz domains*, Ann. of Math. (2) **172** (2010), no. 3, 1907–1948. MR 2726103 (Cited on pages 6, 15, and 17).

[LN18] John Lewis and Kaj Nyström, *Quasi-linear PDEs and low-dimensional sets*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 7, 1689–1746. MR 3807311 (Cited on pages 6, 7, 18, 19, 20, and 23).

[LV13] Niklas L. P. Lundström and Jonatan Vasilis, *Decay of a p-harmonic measure in the plane*, Ann. Acad. Sci. Fenn. Math. **38** (2013), no. 1, 351–366. MR 3076814 (Cited on page 3).

[PV09] Alessio Porretta and Laurent Véron, *Separable p-harmonic functions in a cone and related quasilinear equations on manifolds*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 6, 1285–1305. MR 2557136 (Cited on pages 2 and 5).

[Ser64] James Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247–302. MR 0170096 (Cited on page 8).

[Tol83] Peter Tolksdorf, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Partial Differential Equations **8** (1983), no. 7, 773–817. MR 700735 (Cited on page 2).

[Tol84] Peter Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), no. 1, 126–150. MR 727034 (Cited on page 8).

Murat Akman, Department of Mathematics, University of Connecticut, Storrs, CT 06269-1009

Email address: murat.akman@uconn.edu

John Lewis, Department of Mathematics, University of Kentucky, Lexington, Kentucky, 40506

Email address: johnl@uky.edu

Andrew Vogel, Department of Mathematics, Syracuse University, Syracuse, New York 13244

Email address: alvogel@syr.edu