

# Dynamic Competitive Economies with Complete Markets and Collateral Constraints\*

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## Abstract

In this paper we examine the competitive equilibria of a dynamic stochastic economy with complete markets and collateral constraints. We show that, provided the sets of asset payoffs and of collateral levels are sufficiently rich, the equilibrium allocations with sequential trades and collateral constraints are equivalent to those obtained in Arrow–Debreu markets subject to a series of limited pledgeability constraints.

We provide both necessary and sufficient conditions for equilibria to be Pareto efficient and show that when collateral is scarce equilibria are not only Pareto inefficient but also often constrained inefficient, in the sense that imposing tighter borrowing restrictions can make everybody in the economy better off.

We derive sufficient conditions for the existence of Markov equilibria and, for the case of two agents, for the existence of equilibria that have finite support. These equilibria can be computed with arbitrary accuracy and the model is very tractable.

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# 1 Introduction

We examine the competitive equilibria of an infinite-horizon exchange economy where the only limit to risk sharing comes from the presence of a collateral constraint. Consumers face a borrowing limit, determined by the fact that all loans must be collateralized, as for example in Kiyotaki and Moore (1997) or Geanakoplos (1997), but otherwise financial markets are complete. Only part of the consumers' future endowment can be pledged as collateral, hence the borrowing constraint may be binding and limit risk sharing opportunities in the economy. More specifically, we consider an environment where consumers are unable to commit to repaying their debt obligations and the seizure of the collateral by lenders is the only loss an agent faces for his default. There is no additional punishment, for instance in the form of exclusion from trade in financial markets as in the model considered by Kehoe and Levine (1993), (2001). However, like in that model, and in contrast to Bewley (1977) and the literature which followed it<sup>1</sup>, the borrowing (and collateral) constraint is endogenously determined in equilibrium by the agents' limited commitment problem. The analysis is carried out in a heterogeneous agents version of Lucas's (1978) asset pricing. The part of a consumer's endowment that can be pledged as collateral can be naturally interpreted as the agent's initial share of the Lucas tree — a long-lived asset in positive supply that pays dividends at each date event.

We show in this paper that this is a tractable model of dynamic economies under uncertainty, analyze the welfare properties of competitive equilibria, and establish the existence of simple dynamic equilibria. More specifically, we first show the equivalence between the competitive equilibria when trade occurs in a complete set of contingent commodity markets at the initial date, as in Arrow–Debreu, subject to a series of appropriate limited-pledgeability constraints, and the equilibria when trade is sequential, in a sufficiently rich set of financial markets where short positions must be backed by collateral. This allows us to clearly identify market structures, and in particular the specification of asset payoffs and of the associated collateral requirements, such that the only financial friction is the limited commitment requiring all loans to be collateralized. Second, we provide necessary and sufficient conditions for competitive equilibria to be fully Pareto-efficient – that is for the amount of available collateral to be sufficiently large that the collateral constraint never binds. We then show that, whenever the constraint binds, competitive equilibria in this model are not only Pareto inefficient but are also often constrained inefficient, in the sense that introducing tighter restrictions on borrowing from some date  $t > 0$  makes all agents better off. Third, we derive sufficient conditions for the existence of a Markov equilibrium in this model and show that Markov equilibria often have “finite support” in the sense that individuals' consumption only takes finitely many values. A unique Markov equilibrium exists whenever each agent's coefficient of relative risk aversion is bounded above by one. Under the same assumption, or alternatively when all agents have identical, constant rel-

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<sup>1</sup>See Heathcote, Storesletten, and Violante (2009) for a survey.

ative risk aversion utility functions, or when there is no aggregate uncertainty, equilibria have finite support if there are only two types of agent.

Several papers (the quoted work of Kiyotaki and Moore (1997), Geanakoplos (1997), and various others) have formalized the idea that borrowing on collateral might give rise to cyclical fluctuations in real activity and enhance the volatility of prices. They typically assume that financial markets are incomplete, and/or that the collateral requirements are exogenously specified, so that it is not clear if the source of the inefficiency is the missing markets or the limited ability of the agents to use the existing collateral for their borrowing needs. Furthermore, dynamic models with collateral constraints and incomplete markets turn out to be very difficult to analyze (see Kubler and Schmedders (2003) for a discussion), no conditions are known that ensure the existence of recursive equilibria, and there are therefore few quantitative results about welfare losses due to collateral.

We show here that considering an environment where financial markets are complete and there are no restrictions on how the existing collateral can be used to back short positions allows matters to be simplified considerably. In our model, equilibria can often be characterized as the solution of a finite system of equations. We show that a numerical approximation of equilibria is fairly simple and a rigorous error analysis is possible. Moreover, we can use the implicit function theorem to conduct local comparative statics.

As mentioned above, there is also a large literature that assumes that agents can trade in complete financial markets, default is punished with permanent exclusion from future trades, and loans are not collateralized. We refer for convenience in what follows to these models as “limited enforcement models”. As shown in Kehoe and Levine (2001), Ligon et al. (2002), and Alvarez and Jermann (2000), these models are quite tractable since competitive equilibria can be written as the solution to a planner’s problem subject to appropriate constraints. Even though this is not true in the environment considered here — the limited commitment constraint has a different nature and we show that competitive equilibria may be constrained inefficient — tractability is still obtained.

Chien and Lustig (2010) (also Lustig (2000) in an earlier, similar work) examine a version of the model in this paper with a continuum of agents and growth. The main focus of their analysis is on a quantitative assessment of the asset pricing implications of the model and their similarities with Alvarez and Jermann (2000). Their notion of recursive equilibrium also uses instantaneous weights (Chien and Lustig call them “stochastic Pareto-Negishi weights”) as an endogenous state variable and is essentially identical to ours. However, our results on the existence of such recursive equilibria and of finite support equilibria are rather different, as explained in more in detail in Section 4. Also, they do not examine how the allocation can be decentralized in asset markets with collateral constraints, nor they discuss the constrained inefficiency of competitive equilibria.

Cordoba (2008) considers an economy with production, no aggregate uncertainty, and a continuum of ex ante identical agents and derives sufficient conditions for Pareto-efficiency that are similar to ours.

Lorenzoni (2008), Kilenthong and Townsend (2011), and Gromb and Vayanos (2002) also show that collateral constraints can lead to constrained inefficient equilibrium allocations. However, the analyses by Lorenzoni and by Kilenthong and Townsend are different as they consider a production economy where capital accumulation links different periods and the reallocation is induced by a change in the level of investment that modifies available resources. In our pure exchange setup resources are fixed, only their distribution can vary, and the reallocation is induced by tightening the borrowing constraints with respect to their level endogenously determined in equilibrium. Gromb and Vayanos consider a model with segmented markets and competitive arbitrageurs who need to collateralize separately their positions in each asset, giving conditions under which reducing the arbitrageurs' short positions in the initial period leads to a Pareto improvement. They also consider a pure exchange economy but the segmentation of markets is a key ingredient in their analysis.

Geanakoplos and Zame (2002, and, in a later version, 2009) are the first to formally introduce collateral constraints and default into general equilibrium models. They consider a two-period model with incomplete markets where a durable good needs to be used as collateral. They are the first to point out that, even if markets are complete and the amount of collateral in the economy is large, the Pareto-efficient Arrow–Debreu allocation may not be obtained unless one allows for collateralized financial securities to be used as collateral in addition to the durable good (they refer to this as “pyramiding”). Our equivalence result in Section 2 below makes crucial use of this insight.

The remainder of this paper is organized as follows. In Section 2 we describe the environment, and define an Arrow–Debreu equilibrium with limited pledgeability and a financial markets equilibrium with collateral constraints. We establish the equivalence of equilibrium allocations in these two concepts when there are sufficiently many assets available for trade. In Section 3 we analyze the welfare properties of equilibria. We derive conditions on the level of collateral under which equilibria are Pareto efficient and show that if these conditions are not satisfied they may be constrained inefficient. In Section 4 we study the existence of Markov equilibria and derive conditions under which they can be described by a finite system of equations. Proofs are collected in the Appendix.

## 2 The model

In this section we describe the physical economy, define a notion of Arrow–Debreu equilibrium with limited pledgeability and of a financial markets equilibrium with collateral constraints, and give conditions for these two concepts to be equivalent.

### 2.1 The physical economy

We examine an infinite-horizon stochastic exchange economy with a single perishable consumption good available at each date  $t = 0, 1, \dots$ . We represent the resolution of uncer-

tainty by an event tree. At each period  $t = 1, \dots$  one of  $S$  possible exogenous shocks  $s \in \mathcal{S} = \{1, \dots, S\}$  occurs, with a fixed initial state  $s_0 \in \mathcal{S}$ . Each node of the tree is characterized by a history of shocks  $\sigma = s^t = (s_0, \dots, s_t)$ . The exogenous shocks follow a Markov process with transition matrix  $\pi$ , where  $\pi(s, s')$  denotes the probability of shock  $s'$  given  $s$ . We assume that  $\pi(s, s') > 0$  for all  $s, s' \in \mathcal{S}$ . With a slight abuse of notation we also write  $\pi(s^t)$  to denote the unconditional probability of node  $s^t$ . We collect all nodes of the infinite tree in a set  $\Sigma$  and we write  $\sigma' \succeq \sigma$  if node  $\sigma'$  is either the same as node  $\sigma$  or a (not necessarily immediate) successor.

There are  $H$  infinitely lived agents which we collect in a set  $\mathcal{H}$ . Agent  $h \in \mathcal{H}$  maximizes a time-separable expected utility function

$$U^h(c) = u^h(c_0, s_0) + E \left( \sum_{t=1}^{\infty} \beta^t u^h(c_t, s_t) \middle| s_0 \right),$$

where (conditional) expectations are formed with respect to the Markov transition matrix  $\pi$ , and the discount factor satisfies  $\beta \in (0, 1)$ . We assume that the possibly state dependent Bernoulli function  $u^h(\cdot, s) : \mathbb{R}_{++} \rightarrow \mathbb{R}$  is strictly monotone,  $C^2$ , strictly concave, and satisfies the Inada-condition  $u^{h'}(c, s) = \frac{\partial u^h(c, s)}{\partial c} \rightarrow \infty$  as  $c \rightarrow 0$ , for all  $s \in \mathcal{S}$ .

Each agent  $h$ 's endowment over his lifetime consists of two parts. The first part is given by an amount of the consumption good that the agent receives at any date event,  $e^h(s^t) = e^h(s_t)$  where  $e^h : \mathcal{S} \rightarrow \mathbb{R}_{++}$  is a time-invariant function of the shock. In addition, the agent is endowed at period 0 with an exogenously given share  $\theta^h(s^{-1}) \geq 0$  of a Lucas tree. The tree is an infinitely lived physical asset that pays each period strictly positive dividends  $d : \mathcal{S} \rightarrow \mathbb{R}_{++}$ , which depend solely on the current shock realization  $s \in \mathcal{S}$ . The tree exists in unit net supply,  $\sum_{h \in \mathcal{H}} \theta^h(s^{-1}) = 1$ , and its shares can be traded at any node  $\sigma$  for a unit price  $q(\sigma)$ . The total endowment of the consumer is therefore  $\omega^h(s_t) = e^h(s_t) + \theta^h(s^{-1})d(s_t)$ .

## 2.2 Arrow–Debreu equilibrium with limited pledgeability

We assume that  $e^h(s_t)$  cannot be sold in advance in order to finance consumption or savings at any date before the endowment is received, it thus constitutes the non-pledgeable component of the agent's total endowments  $\omega^h(s_t)$ . To formalize the notion of an equilibrium with non-pledgeable endowments, we define an **Arrow–Debreu equilibrium with limited pledgeability** as a collection of prices  $(\rho(\sigma))_{\sigma \in \Sigma}$  and a consumption allocation  $(c^h(\sigma))_{\sigma \in \Sigma}^{h \in \mathcal{H}}$  such that

$$\sum_{h \in \mathcal{H}} (c^h(\sigma) - \omega^h(\sigma)) = 0, \quad \text{for all } \sigma \in \Sigma \tag{1}$$

and for all agents  $h$

$$(c^h(\sigma))_{\sigma \in \Sigma} \in \arg \max_{c \geq 0} U^h(c) \text{ s.t.} \quad (2)$$

$$\sum_{\sigma \in \Sigma} \rho(\sigma) c(\sigma) \leq \sum_{\sigma \in \Sigma} \rho(\sigma) \omega^h(\sigma) < \infty \quad (3)$$

$$\sum_{\sigma \succeq s^t} \rho(\sigma) c(\sigma) \geq \sum_{\sigma \succeq s^t} \rho(\sigma) e^h(\sigma) \text{ for all } s^t. \quad (4)$$

The definition is the same as that of an Arrow–Debreu competitive equilibrium, where agents are able to trade at the initial date  $t = 0$  in a complete set of contingent commodity markets, except for the additional constraints (4). These constraints express precisely the condition that  $e^h(\sigma)$  is unalienable – that is to say this component of the endowment can only be used to finance consumption in the node  $\sigma$  in which it is received or in any successor node. Note that these additional constraints are likely to be binding whenever the  $e^h$ -part of the agent’s endowments is large relative to the part given by the tree’s dividends – that is when there is only a small amount of future endowments that can be traded at earlier nodes of the event tree.

### 2.3 Financial markets with collateral constraints

We will show that the abstract equilibrium notion proposed above allows us to capture the allocations attained as competitive equilibria in a standard setting where agents trade sequentially in financial markets and short positions must be backed by collateral, provided markets are “complete” in a sense made precise below.

We consider an environment where at each node  $s^t$  any agent  $h$  can trade the tree as well as  $J$  financial assets (in zero net supply), collected in a set  $\mathcal{J}$ . These assets are one-period securities: asset  $j$  traded at node  $s^t$  promises a payoff  $b_j(s^{t+1}) = b_j(s_{t+1}) \geq 0$  at the  $S$  successor nodes ( $s^{t+1}$ ). The agent can hold any amount  $\theta(s^t) \geq 0$  of shares of the tree, which trade at the price  $q(s^t)$ . In addition, for each security  $j \in \mathcal{J}$ , with price  $p_j(s^t)$ , the agent can hold any *long* position  $\phi_{j+}(s^t) \geq 0$  as well as a *short* position  $\phi_{j-}(s^t) \leq 0$ . The net position in security  $j$  is denoted by  $\phi_j(s^t) = \phi_{j+}(s^t) + \phi_{j-}(s^t)$ . We assume that all loans are non-recourse – that is consumers can default at no cost on the prescribed payments. To ensure that some payments are made, each short position in a security must be backed by an appropriate amount of the tree or of long positions in other financial securities that are admissible as collateral. The specification of a financial security  $j \in \mathcal{J}$  is then given not only by its promised payoff  $b_j(\cdot)$  but also by its collateral requirement, described by the vector<sup>2</sup>  $k^j \in \mathbb{R}_+^{J+1}$ . For each unit of security  $j$  sold short by a consumer, he is required to hold  $k_{J+1}^j$  units of the tree as well as  $k_i^j$  units of each security  $i \in \mathcal{J}$  as collateral.

Since all loans are non-recourse, the consumer will find it optimal to default on his promise to deliver  $b_j(s_{t+1})$  per unit sold whenever  $b_j(s_{t+1})$  is higher than the value of the

<sup>2</sup>In principle this collateral requirement could vary with the exogenous shock, but for our purposes it suffices to assume that it is fixed.

collateral associated with the short position. In this case the buyer of the financial security gets the collateral associated with the promise. Hence, the actual payoff of any security  $j \in \mathcal{J}$  at any node  $s^{t+1}$  is endogenously determined by the agents' incentives to default and the collateral requirements, as in Geanakoplos and Zame (2002) and Kubler and Schmedders (2003). It is given by the values  $f_j(s^{t+1})$  satisfying the following system of equations, for all  $j \in \mathcal{J}$ :

$$f_j(s^{t+1}) = \min \left\{ b_j(s_{t+1}), \sum_{i=1}^J k_i^j f_i(s^{t+1}) + k_{J+1}^j (q(s^{t+1}) + d(s_{t+1})) \right\}. \quad (5)$$

For this equation to have a nontrivial solution, we assume that the tree is used as collateral for each security  $j$ , either directly or indirectly. If the tree is not used as collateral for security  $j$  — that is  $k_{J+1}^j = 0$  — it must be used as collateral for some other security, in turn used as collateral for another security and so on until we reach one of the securities used as collateral for  $j$ . In this way, the tree backs, indirectly, the claims of all securities along the chain. This construction will be made precise in the proof of Theorem 1.

A **collateral constrained financial market equilibrium** is defined as a collection of choices  $(c^h(\sigma), \theta^h(\sigma), (\phi_+^h(\sigma), \phi_-^h(\sigma)))_{\sigma \in \Sigma}$  for all agents  $h \in \mathcal{H}$ , prices,  $(p(\sigma), q(\sigma))_{\sigma \in \Sigma}$ , and payoffs  $(f(\sigma))_{\sigma \in \Sigma}$  satisfying (5) such that the following conditions hold.

(CC1) Market clearing:

$$\sum_{h \in \mathcal{H}} \theta^h(\sigma) = 1 \text{ and } \sum_{h \in \mathcal{H}} \phi_+^h(\sigma) + \sum_{h \in \mathcal{H}} \phi_-^h(\sigma) = 0 \text{ for all } \sigma \in \Sigma.$$

(CC2) Individual optimization: for each agent  $h$

$$\begin{aligned} & (\theta^h(\sigma), \phi_+^h(\sigma), \phi_-^h(\sigma), c^h(\sigma))_{\sigma \in \Sigma} \in \arg \max_{\theta \geq 0, \phi_+ \geq 0, \phi_- \leq 0, c \geq 0} U^h(c) \text{ s.t.} \\ & c(s^t) = e^h(s_t) + \phi(s^{t-1}) \cdot f(s^t) + \theta(s^{t-1})(q(s^t) + d(s_t)) - \theta(s^t)q(s^t) - \phi(s^t) \cdot p(s^t), \quad \forall s^t \\ & \theta(s^t) + \sum_{j \in \mathcal{J}} k_{J+1}^j(s^t) \phi_{j-}(s^t) \geq 0, \quad \forall s^t \\ & \phi_{j+}(s^t) + \sum_{i \in \mathcal{J}} k_j^i(s^t) \phi_{i-}(s^t) \geq 0, \quad \forall s^t, \forall j \in \mathcal{J}. \end{aligned}$$

Condition (CC1) is the standard market clearing condition for the tree and the financial assets, where long ( $\phi_+$ ) and short ( $\phi_-$ ) positions of securities are separated. Condition (CC2) requires that each agent chooses asset holdings and consumption at each node to maximize utility subject to a standard budget constraint and additional constraints requiring the agent to hold sufficient amounts of the tree (the first inequality constraint) and of long positions in financial securities (the second inequality constraint) so as to satisfy the collateral requirements for these assets. The existence of a collateral constrained financial markets equilibrium follows by the argument in Kubler and Schmedders (2003).

While our assumptions on the rules governing collateral are obviously abstracting from many important issues arising in practice, we try to model two key aspects of collateral contracts. First, we assume that margins are asset specific in that an asset cannot be used for two different short positions at the same time even if these two positions require payment in mutually exclusive states. It is important to point out that in this case no information over all the trades carried out by an agent is required to enforce these collateral constraints — it suffices to post the required collateral for each short position; hence we can say that the financial contracts traded in the markets are non-exclusive. This is in contrast to other limited commitment models, such as Kehoe and Levine (1993, 2001) or Alvarez and Jermann (2000) where observability of all trades in financial markets is assumed. It is also in contrast to Chien and Lustig (2010) who assume that margins are portfolio-specific. They analyze a model with collateral requirements where, in addition to the tree, a complete set of  $S$  Arrow securities is available for trade at each node and the tree must be used as collateral for short positions in these Arrow securities. Chien and Lustig assume that each unit of the tree can be used to secure short positions in several Arrow securities at the same time, i.e. the collateral constraint only has to hold for the whole portfolio of securities held. These “portfolio margins” clearly allow economizing on the use of the tree as collateral but they generally also require a stronger enforcement and coordination ability among lenders, or the full observability of agents’ trades, not needed in the environment considered here. The specification adopted here, based on asset specific margins, is closer to trading practices used in financial markets (see, e.g., Appendix A in Brunnermeier and Pedersen (2009) for details).

Second, the fact that margins are asset specific requires other channels to economize on collateral. This is done via our assumption that not only the tree but also financial securities can be used as collateral. Geanakoplos and Zame (2002) refer to this assumption as “pyramiding”. In practice, financial securities are routinely used for collateralized borrowing (e.g., in repo agreements, see Bottazzi et al. (2012), but also in other transactions) — however, as Brunnermeier and Pedersen (2009) point out, in order to take short positions in more complicated securities such as derivatives brokers typically require cash-collateral.

Our assumption of pyramiding implies an implicit re-use of collateral that is somewhat similar to “rehypothecation”, but there are some important differences. Rehypothecation refers to the common practice in financial trades that allows a lender to use the collateral received on a loan as collateral he pledges to enter a short position with a third party. In many collateralized trades the borrower remains the owner of the asset used as collateral but the lender gains broad rights to use the collateral; in some trades the borrower loses ownership over the pledged asset altogether (see, e.g., Monnet (2011) for a description of institutional details). We assume instead that a lender can reuse the collateral that is backing his loan only indirectly by using the long position in the loan as collateral. Since agents can default on their debt obligations, at the cost only of losing the posted collateral, it is clear that the tree is ultimately backing all financial claims, directly or indirectly. But



the lender can never profit from a situation where the value of the collateral exceeds that of the borrower’s obligations, by not returning the collateral. One possible reason why in practice one sees rehypothecation rather than pyramiding is that financial securities are only good collateral if they are traded on liquid markets, which might make it difficult to build a large pyramid of financial securities with possibly different payoffs, backed by a single physical asset.

## 2.4 Equivalence of Arrow–Debreu and financial market equilibrium

We will show that any Arrow–Debreu equilibrium allocation with limited pledgeability can also be attained at a collateral constrained financial market equilibrium, provided financial markets are complete. As we will see, in this environment the notion of “complete markets” is a little subtler than usual, since it requires the presence of a sufficiently rich set of financial assets not only in terms of the specification of their payoff but also of their collateral requirement. To illustrate what sufficiently rich means here, it is useful to first consider a simple two-period example.

Suppose there are three agents with identical preferences trading in period 1 to insure against uncertainty in the second period. There are three equiprobable states in the second period (which in a slight abuse of notation we refer to as  $s = 1, 2, 3$ ) and the tree pays 1 unit in each of these states. Each agent has initial holdings of the tree equal to 4 units<sup>3</sup> while the non-pledgeable second period endowment of the three agents is

$$e^1 = (0, 6, 9), \quad e^2 = (6, 9, 0) \quad \text{and} \quad e^3 = (9, 0, 6).$$

In this environment it is easy to verify that the Arrow–Debreu equilibrium with limited pledgeability features a constant level of consumption in the three states:

$$c^1 = c^2 = c^3 = (9, 9, 9).$$

This equilibrium is Pareto-efficient and coincides with the standard Arrow–Debreu equilibrium.

It is also easy to see that a complete set of Arrow securities, each of them collateralized by the tree, does not suffice to complete the market. To implement the above allocation, in fact agent 1 would need to hold his endowment of the tree, buy 5 units of the Arrow security for state 1, and sell short, respectively, 1 and 4 units of the Arrow securities for states 2 and 3. However, this violates his collateral constraint since he needs a total of 5 units of the tree as collateral while he only holds four units.

More interestingly, if in addition to the Arrow securities there were also three assets paying zero in one state and 1 unit in the two others, each agent could achieve his Arrow–Debreu consumption level without violating his collateral constraints by selling 1 unit of

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<sup>3</sup>It simplifies the exposition to assume that there are 12 trees in the economy. Alternatively, we could take the tree to be in unit supply and assume that it pays 12 units of dividends.

the asset that pays in the states where he has positive endowments (in addition to selling 3 units of the Arrow security that pays in the state where his endowment is 9). However no other agent would buy this asset since any agent needs to buy only one Arrow security to achieve his Arrow-Debreu consumption. Market clearing would not be possible. The same argument applies to all other specifications of the asset payoffs<sup>4</sup>. In this example it is therefore not possible to achieve the Arrow-Debreu equilibrium outcome if all promises are only backed by the tree.

In contrast, once one allows for pyramiding – that is to say for the presence of promises backed by financial securities and not the tree, one can easily find asset trades that satisfy the collateral constraints and implement the Arrow-Debreu consumption allocation with limited pledgeability. Suppose there are two financial securities with promises  $b_1 = (0, 1, 1)$ ,  $b_2 = (0, 0, 1)$ . One unit of the tree needs to be used as collateral for each short position in security  $j = 1$ , while (only) one unit of financial security 1 is used as collateral for each short position in  $j = 2$ . Consider, then, the following portfolios. Agent 1 holds 9 units of the tree,  $\theta^1 = 9$ , shorts 9 units of security 1,  $\phi_{1-}^1 = -9$  using his holdings of the tree as collateral, and, at the same time, buys 3 units of this security back,  $\phi_{1+}^1 = 3$ . Finally, he shorts 3 units of security 2,  $\phi_{2-}^1 = -3$ , using the long position in security 1 as collateral. Agent 2 holds 3 units of the tree, shorts 3 units of security 1 and buys 9 units of security 2,  $\theta^2 = 3$ ,  $\phi_{1-}^2 = -3$ ,  $\phi_{2+}^2 = 9$ . Agent 3 holds no tree, buys 9 units of security 1 and shorts 6 units of security 2, backed by his holdings of security 1, i.e.  $\theta^3 = 0$ ,  $\phi_{1+}^3 = 9$ ,  $\phi_{2-}^3 = -6$ .

Note that given the above specification of asset payoffs and collateral requirements, it is obviously crucial that agent 3 can use a long position in security 1 as collateral to back his short sales of security 2. The need for agent 1 to go at the same time long and short in the same security on the other hand is not essential and depends on our assumption that only security 1 can be used as collateral for short positions in a security with payoff  $(0, 0, 1)$ . Alternatively, we could have assumed that there are two distinct securities with payoff  $(0, 0, 1)$ , one of which is collateralized by financial security  $j = 1$ , as above, the other by the tree, in which case agent 1 could just short 6 units of security 1 and 3 units of the second security with payoff  $(0, 0, 1)$ , both collateralized by the tree.

The previous example illustrates the basic intuition of how to construct a set of assets which allows one to attain the Arrow-Debreu equilibrium allocations with limited pledgeability in a setup with sequential trading of financial securities and collateral constraints. Our main result in this section generalizes this construction and the above argument to the infinite-horizon, stochastic economy under consideration with any number of states and consumers. To prove the result, it is convenient to introduce an alternative equilibrium notion with sequential trading, where each period intermediaries purchase the tree from consumers and issue a complete set of one period, state-contingent claims (options) on the tree, which are bought by consumers. This specification, although slightly artificial, allows

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<sup>4</sup>Kilenthong (2011) makes this point in a slightly different environment with capital.

us to simplify the proof and turns out to be useful for analyzing the properties of collateral constrained equilibria when markets are complete.

More precisely, at each node  $s^t$  intermediaries purchase the tree from the consumers and issue  $J = S$  “tree options”, where option  $j$  promises the delivery of one unit of the tree the subsequent period if, and only if, shock  $s = j$  realizes. At each date event  $s^t$  households can trade the tree and, in addition, can only take long positions  $\theta_s(s^t) \geq 0$ ,  $s = 1, \dots, S$  in these  $S$  tree options at the prices  $q_s(s^t) > 0$ ,  $s = 1, \dots, S$ . The intermediaries’ holdings of the tree ensure that all due dividend payments can be made.

An **equilibrium with intermediaries** is defined as a collection of individual consumption levels  $(c^h(\sigma))_{\sigma \in \Sigma}^{h \in \mathcal{H}}$ , portfolios  $(\theta_s^h(\sigma))_{\sigma \in \Sigma, s \in S}^{h \in \mathcal{H}}$ , and prices  $(q(\sigma), q_s(\sigma))_{\sigma \in \Sigma, s \in S}$ , such that markets clear and agents maximize their utility – that is

(IE1) at all nodes  $s^t$ ,

$$\sum_{h \in \mathcal{H}} \theta_s^h(s^t) = 1 \text{ for all } s \in \mathcal{S}.$$

(IE2) for all agents  $h \in \mathcal{H}$

$$(c^h, \theta^h) \in \arg \max_{\theta, c \geq 0} U^h(c) \text{ s.t.}$$

$$c(s^t) = e^h(s_t) + \theta_{s_t}(s^{t-1}) (q(s^t) + d(s_t)) - \sum_{s'=1}^S \theta_{s'}(s^t) q_{s'}(s^t) \text{ for all } s^t,$$

$$\theta_s(s^t) \geq 0, \quad \text{for all } s^t, s.$$

(IE3) at all nodes  $s^t$ ,

$$q(s^t) = \sum_{s=1}^S q_s(s^t).$$

Condition (IE3) ensures that intermediaries make zero profit in equilibrium, since the intermediation technology, with zero costs, exhibits constant returns to scale.

As mentioned above, the concept of equilibrium with intermediaries is used to show under which conditions Arrow–Debreu equilibria can be implemented as financial markets equilibria. The following theorem formalizes this.

**THEOREM 1** *For any Arrow–Debreu equilibrium with limited pledgeability there exists an equilibrium with intermediaries with the same consumption allocation. Moreover, one can construct the payoffs and collateral requirements of  $S - 1$  financial securities such that there exists a collateral constrained financial markets equilibrium with the same consumption allocation.*

It is relatively easy to show that any Arrow–Debreu equilibrium allocation with limited pledgeability can also be attained as an equilibrium with intermediaries. In order to show that this equilibrium can be attained as an equilibrium with collateral constraints, one needs to construct a rich enough asset structure that ensures that the payoffs achieved

with the tree options can be replicated by trading in the asset market, subject to collateral constraints. The basic construction is to set the payoff of each security  $j = 1, \dots, S-1$ , equal to zero in states  $s = 1, \dots, j$  and equal to 1 in states  $j+1, \dots, S$ . One unit of security  $j$  must be used as collateral for each unit short position in security  $j+1$ , for  $j = 2, \dots, S-1$ , while a unit of the tree is used as collateral for unit short positions in security 1. This specification generalizes that of the simple example above. It suffices to verify that we can always find portfolios that allow us to replicate the consumption allocation of the equilibrium with intermediaries. The details of the proof are in the Appendix.

Note that the reverse implication of that stated in Theorem 1 also holds for collateral constrained equilibria without bubbles.<sup>5</sup> Given the equivalence established in the theorem above, in most of the paper we will consider the notion that turns out to be more convenient, depending on the issue — that one of equilibrium with intermediaries or that of Arrow–Debreu equilibrium with limited pledgeability.

Our collateral constrained equilibrium concept with complete financial markets has some interesting similarities to both Kehoe and Levine (1993) and Golosov and Tsyvinski (2007).<sup>6</sup> Kehoe and Levine (1993) differs from most of the other papers in the literature on limited enforcement models by the fact that an environment with several physical commodities is considered. In the event of default only part of the agents' endowment can be seized and agents also face the punishment of permanent exclusion from trade in financial markets, but their trades in the spot commodity markets are not observable and cannot be prevented. In addition to the intertemporal budget constraint agents then face at each node a constraint on their continuation utility level, which in this case depends on (spot market) prices. Golosov and Tsyvinski (2007) consider an environment where insurance contracts are offered in the presence of moral hazard, but hidden trades by the agents in some markets cannot be prevented and hence prices (together with agents' utilities) again enter the agents' incentive constraints. Prices also enter the additional constraint given by (4) above, which has, however, the form of a budget constraint (agents' utilities do not appear), and reflects the fact that no exclusion from trade in any market is possible. Agents' incentives are captured by the specification of asset payoffs in (5) and trades at each node are always restricted by the collateral constraint. A possible interpretation of this is that agents can always hide all their trades: as argued in the previous section, no information on agents' trades is needed to enforce the collateral constraints.

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<sup>5</sup>Since the existence proof in Kubler and Schmedders (2003) shows that collateral constrained financial markets equilibria without bubbles exist, Theorem 1 implies the existence also of Arrow–Debreu equilibria with limited pledgeability.

<sup>6</sup>We thank an anonymous referee for pointing out this connection to us.

### 3 Welfare properties of equilibria

In this section we investigate the welfare properties of competitive equilibria with collateral constraints. We first examine the case where there are Pareto-efficient competitive equilibria since the amount of available collateral is “sufficiently large” to satisfy the collateral needs of the economy and the collateral constraints never bind. We derive both necessary and sufficient conditions for the existence of Pareto-efficient equilibria in general economies with no aggregate uncertainty as well as those with aggregate uncertainty when consumers have identical constant relative risk aversion (CRRA) utility.

We then consider the case where collateral is scarce and no competitive equilibrium is Pareto efficient. We study an example where there is an equilibrium that is inefficient and characterize its properties. The main result of this section shows that in this case equilibria are not only Pareto inefficient, but may also be constrained inefficient. That is, even by taking the borrowing restrictions imposed by the collateral constraints into account, a welfare improvement can still be obtained with respect to the competitive equilibrium.

#### 3.1 When do Pareto-efficient equilibria exist?

It is useful to begin the analysis by examining the issue in the framework of the following simple example, which will also be used in other parts of the paper. There are two types of agents, two possible realizations of the shocks each period, and no aggregate uncertainty. The shocks are i.i.d. with probabilities  $\pi(1, 1) = \pi(2, 1) = \pi(1, 2) = \pi(2, 2) = \frac{1}{2}$ . We assume the tree has a deterministic dividend  $\mathfrak{d}$  and the endowments of agent 1 are  $e^1(1) = \mathfrak{h}$ ,  $e^1(2) = 0$ , the endowments of agent 2 are  $e^2(1) = 0$ ,  $e^2(2) = \mathfrak{h}$ , where  $0 < \mathfrak{h}$ , and the agents’ Bernoulli utility function is state invariant,  $u^h(c, s) = u^h(c)$  for  $h = 1, 2$ . While we assumed above that endowments are strictly positive, it is useful to consider the example with  $e^1(2) = e^2(1) = 0$  since this simplifies computations considerably. Our results carry over to an example where endowments are strictly positive.

Since there is no aggregate uncertainty, at a Pareto-efficient allocation agents’ consumption is constant, i.e.  $c^h(s^t) = c^h$  for all  $s^t$ , for  $h = 1, 2$ , and the same is true at an Arrow–Debreu equilibrium, with supporting prices given by  $\rho(s^t) = (\beta/2)^t$ . For this allocation to be also an equilibrium in the present environment, the collateral constraints, or equivalently the limited pledgeability constraints (4), must all be satisfied. The latter reduce to

$$c^h - \mathfrak{h} + c^h \frac{\beta}{1 - \beta} - \mathfrak{h} \frac{\beta/2}{1 - \beta} \geq 0, \text{ for } h = 1, 2. \quad (6)$$

In addition, by feasibility we have  $c^1 + c^2 = \mathfrak{h} + \mathfrak{d}$ . If the initial distribution of the tree among agents is such that  $c^1 = c^2$  at the Arrow–Debreu equilibrium, it is easy to see that (6) is satisfied if, and only if,

$$\frac{\mathfrak{d}}{1 - \beta} \geq \mathfrak{h} \quad (7)$$

that is, if the total discounted flow of dividends paid by the tree are larger than the variability of agents' non-pledgeable endowments.

To generalize the simple example, first note that in the stationary environment considered in this paper Pareto-efficient allocations are always such that agents' consumption only depends (at most) on the current realization of the shock (see, e.g., Judd et al. (2003)). Consider a Pareto-efficient allocation  $\{c^h(s)\}_{s \in \mathcal{S}}^{h \in \mathcal{H}}$ . For this allocation to be supported as an Arrow–Debreu equilibrium with limited pledgeability the supporting prices, given by  $\rho(s^t) = u^h(c^h(s_t), s_t) \beta^t \pi(s^t)$  for all  $s^t$  and any  $h$ , must be such that the limited pledgeability constraints are satisfied for all agents  $h \in \mathcal{H}$  and all shocks  $s \in \mathcal{S}$ :

$$u^h(c^h(s), s)(c^h(s) - e^h(s)) + E \left( \sum_{t=1}^{\infty} \beta^t u^h(c^h(s_t), s_t)(c^h(s_t) - e^h(s_t)) \middle| s_0 = s \right) \geq 0. \quad (8)$$

Moreover, the initial distribution of the tree at  $t = 0$  (i.e., the initial conditions) must ensure that the intertemporal budget constraint (3) holds. In what follows we will say that *Pareto-efficient equilibria exist* for an economy if there are initial distributions for which the competitive equilibrium is Pareto-efficient. We discuss in Subsection 4.2.2 below what happens if Pareto efficient equilibria exist but initial conditions are such that collateral constraints bind initially. We provide conditions that guarantee that in the long run the equilibrium allocation will converge to that of a Pareto-efficient equilibrium.

For general utility functions and endowments, there are no simple conditions on fundamentals that ensure (8) since equilibrium allocations cannot be derived analytically. However, for the case of no aggregate uncertainty and for the case of identical CRRA utility equilibrium allocations can be determined easily.

### 3.1.1 No aggregate uncertainty

When there is no aggregate uncertainty – that is to say  $\sum_{h \in \mathcal{H}} \omega^h(s)$  is equal to a constant  $\omega$  for all shock realizations  $s \in \mathcal{S}$ , and agents' Bernoulli functions are state independent, all Pareto-efficient allocations must satisfy  $c^h(s) = c^h$  for all  $s, h$ . Hence condition (8) simplifies to

$$\max_{s \in \mathcal{S}} \left[ e^h(s) + E \left( \sum_{t=1}^{\infty} \beta^t e^h(s_t) \middle| s_0 = s \right) \right] \leq \frac{c^h}{1 - \beta} \text{ for all } h \in \mathcal{H} \quad (9)$$

and using the feasibility of the allocation we obtain the following<sup>7</sup>:

**THEOREM 2** *A necessary and sufficient condition for the existence of a Pareto-efficient equilibrium with no aggregate uncertainty is*

$$(1 - \beta) \sum_{h \in \mathcal{H}} \max_{s \in \mathcal{S}} \left[ e^h(s) + E \left( \sum_{t=1}^{\infty} \beta^t e^h(s_t) \middle| s_0 = s \right) \right] \leq \omega. \quad (10)$$

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<sup>7</sup>Necessity is obvious given (9). Sufficiency follows from the observation that under (10) it is always possible to find a Pareto-efficient allocation  $\{c^h\}_{h \in \mathcal{H}}$  that satisfies (9).

Recalling that  $\omega - \sum_{h \in \mathcal{H}} e^h(s) = d(s)$ , condition (10) requires the amount of collateral in every state, measured by  $d(s)$ , to be sufficiently large relative to the variability of the present discounted value of the agents' non-pledgeable endowment, captured by the term on the left-hand side of (10).

If in addition shocks are i.i.d. and  $d(s) = d$  for all  $s$ , condition (10) simplifies to<sup>8</sup>

$$\sum_{h \in \mathcal{H}} \max_{s \in \mathcal{S}} e^h(s) \leq \frac{\omega - \beta e}{1 - \beta} = \frac{d}{1 - \beta} + e, \quad (11)$$

where  $e = \sum_{h \in \mathcal{H}} e^h(s)$  for any  $s$ .

### 3.1.2 Identical CRRA preferences

Consider next the case where all agents have identical CRRA preferences with coefficient of relative risk aversion  $\tau$ . In this case, all Pareto-efficient allocations satisfy the property that, for all  $h, s$ ,  $c^h(s) = \lambda^h \omega(s)$  for some  $\lambda^h \geq 0$  and  $\sum_{h \in \mathcal{H}} \lambda^h = 1$ , where  $\omega(s) = \sum_{h \in \mathcal{H}} \omega^h(s)$ . We can therefore write condition (8) as

$$\frac{\lambda^h}{\omega(s)^{\tau-1}} + E \left( \sum_{t=1}^{\infty} \beta^t \frac{\lambda^h}{\omega(s_t)^{\tau-1}} \middle| s_0 = s \right) \geq \frac{e^h(s)}{\omega(s)^\tau} + E \left( \sum_{t=1}^{\infty} \beta^t \frac{e^h(s_t)}{\omega(s_t)^\tau} \middle| s_0 = s \right) \text{ for all } s \in \mathcal{S}, h \in \mathcal{H}.$$

As in the previous section, feasibility allows us to obtain from the above inequality the following necessary and sufficient condition for the existence of an efficient equilibrium:

$$1 \geq \sum_{h \in \mathcal{H}} \max_{s \in \mathcal{S}} \frac{\frac{e^h(s)}{\omega(s)^\tau} + E \left( \sum_{t=1}^{\infty} \beta^t \frac{e^h(s_t)}{\omega(s_t)^\tau} \middle| s_0 = s \right)}{\frac{1}{\omega(s)^{\tau-1}} + E \left( \sum_{t=1}^{\infty} \beta^t \frac{1}{\omega(s_t)^{\tau-1}} \middle| s_0 = s \right)} \quad (12)$$

When all agents have log-utility, i.e.  $\tau = 1$ , condition (12) greatly simplifies and reduces to

$$\frac{1}{1 - \beta} \geq \sum_{h \in \mathcal{H}} \max_{s \in \mathcal{S}} \left[ \frac{e^h(s)}{\omega(s)} + E \left( \sum_{t=1}^{\infty} \beta^t \frac{e^h(s_t)}{\omega(s_t)} \middle| s_0 = s \right) \right],$$

analogous to (10).

Condition (12), while not very intuitive, can obviously be verified numerically for given processes of individual endowments and dividends. It is obviously beyond the scope of this paper to take a stand on which values should be considered as realistic for the level of persistence and the size of the idiosyncratic shocks as well as for the amount of available collateral. It might be interesting, however, to consider an example of a calibrated economy from the applied literature. Heaton and Lucas (1996) calibrate a Lucas-style economy with two types of agents to match key facts about the US economy. They take the dividend-share to be earnings to stock-market capital and estimate this number to be around 15 percent of total income. They assume that aggregate growth rates follow an 8-state Markov chain and calibrate their model using the PSID (Panel Study of Income Dynamics) and NIPA (National

<sup>8</sup>This condition is a clear generalization of (7) obtained for the example.

Income and Product Accounts). We consider their calibration for the “Cyclical Distribution Case” but de-trend the economy to ensure we remain in our stationary environment. We find that for their specification of the economy the competitive equilibrium is Pareto-efficient (i.e., condition (12) holds). This shows that, if one considers the specification of idiosyncratic risks in Heaton and Lucas (1996) to be somewhat realistic, Pareto-efficient equilibria exist (in the sense that there are initial conditions for which equilibria are Pareto efficient) for all realistic levels of collateral.

### 3.2 Constrained inefficiency of equilibria

If the collateral in the economy is too scarce to support a Pareto-efficient allocation, it could still be the case that the equilibrium allocation is constrained Pareto efficient in the sense that no reallocation of the resources that is feasible and satisfies the collateral constraints can make everybody better off. We show here that this may not be true by presenting a robust example for which a welfare improvement can indeed be found subject to these constraints. We consider in particular the reallocation obtained when agents are subject to constraints on trades that are tighter than the collateral constraints.

#### 3.2.1 Pareto Inefficient equilibria

Consider again the simple environment of the example of Section 3.1. When

$$\mathfrak{h} > \frac{\mathfrak{d}}{1 - \beta}, \tag{13}$$

as we showed, a Pareto-efficient competitive equilibrium does not exist, so the only possible equilibrium is one where the collateral constraints bind (at least in some state). We show next that when (13) holds, an inefficient equilibrium exists where agents’ consumption and prices are time invariant functions of the shock  $s$  alone. We will refer to equilibria satisfying this property as *steady-state equilibria*. We will show below that this steady-state equilibrium might be constrained inefficient.

It turns out to be simpler to carry out this analysis in terms of the notion of equilibrium with intermediaries. Given the symmetry of the environment, it is natural to conjecture that the steady-state equilibrium is symmetric with  $c^1(1) = c^2(2)$  and  $\theta^1 = (0, 1)$ ,  $\theta^2 = (1, 0)$ . In the following we will verify this conjecture. Letting  $q_1(s)$  and  $q_2(s)$  denote the equilibrium prices of the tree options (with the price of the tree satisfying the zero profit condition  $q(s) = q_1(s) + q_2(s)$  stated above), the consumption values of agent 1 supported by the above portfolios readily obtain from the budget constraints:

$$\begin{aligned} c^1(1) &= \mathfrak{h} - q_2(1), \\ c^1(2) &= \mathfrak{d} + q_1(2) + q_2(2) - q_2(2) = \mathfrak{d} + q_1(2) \end{aligned}$$

The values of the equilibrium prices must satisfy the first-order conditions of agent 1 for the security paying in state 2 (since agent 1 is always unconstrained in his holdings of this



asset)

$$\begin{aligned} q_2(1)u'(c^1(1)) &= \beta \frac{1}{2}(q_1(2) + q_2(2) + \mathfrak{d})u'(c^1(2)) \\ q_2(2)u'(c^1(2)) &= \beta \frac{1}{2}(q_1(2) + q_2(2) + \mathfrak{d})u'(c^1(2)) \end{aligned}$$

and the corresponding conditions of agent 2 for the security paying in state 1. From the second condition above we obtain:

$$q_2(2) = \frac{\beta}{2-\beta}(q_1(2) + \mathfrak{d}).$$

By symmetry we also have  $q_1(1) = q_2(2)$  and  $q_1(2) = q_2(1)$ .

To prove the existence of a steady-state it suffices to show that the first-order conditions have a solution in prices that support an allocation with  $c^1(1) > c^1(2)$ , i.e. that satisfy  $\mathfrak{h} - q_2(1) > \mathfrak{d} + q_2(1)$ .

To do so we can reduce the system to a single equation in  $q_2(1)$  and obtain the following.

$$q_2(1)u'(\mathfrak{h} - q_2(1)) = \frac{\beta}{2-\beta}(q_2(1) + \mathfrak{d})u'(\mathfrak{d} + q_2(1))$$

The existence of a positive solution  $q_2(1) > 0$  follows directly from the intermediate value theorem. To see that the solution must satisfy  $\mathfrak{h} - q_2(1) > \mathfrak{d} + q_2(1)$ , suppose to the contrary that  $\mathfrak{h} - q_2(1) \leq \mathfrak{d} + q_2(1)$ . By concavity of  $u(\cdot)$  this must imply  $q_2(1) \leq \frac{\beta}{2-\beta}(q_2(1) + \mathfrak{d})$  or  $2q_2(1) \leq \beta \frac{\mathfrak{d}}{1-\beta}$ . But then we would obtain  $\mathfrak{h} \leq \mathfrak{d} + 2q_2(1) \leq \frac{\mathfrak{d}}{1-\beta}$ , which contradicts (13) above.

We have thus shown that under (13) a symmetric steady-state equilibrium exists. If the initial conditions of the economy are such that  $s_0 = 1$  and  $\theta^1(s_-) = 0$ , there is a competitive equilibrium that is identical to this steady state at each date  $t$ . Since  $c^1(1) > c^1(2)$  the equilibrium is clearly inefficient. In the rest of this section we will focus on this equilibrium<sup>9</sup>. When the initial conditions are different from those stated above — similarly to what has already been stated in Section 3.1 — the analysis in Subsection 4.2.2 shows that there always exists a competitive equilibrium that converges to this steady state.

### 3.2.2 Pareto-improving intervention

Suppose the economy is at a steady-state equilibrium as described in the previous subsection, where  $\theta^1 = (0, 1)$ ,  $\theta^2 = (1, 0)$  at each date  $t$ , and consider the welfare effect of tightening the portfolio restriction to  $\theta_s^h(s^t) \geq \varepsilon$ , for  $\varepsilon > 0$  and all  $s \in \mathcal{S}$ . This tighter restriction is assumed to be introduced at  $t = 1$  and to hold for all  $t \geq 1$ . The intervention is announced at  $t = 0$  after all trades have taken place. In the light of the equivalence established in Section 2.4 this

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<sup>9</sup>For the case of general preferences we cannot rule out the possibility that other equilibria exist. In the following Subsection 3.2.3 we assume that all agents have log utility, in which case it follows from Theorem 5 in Section 4 that the equilibrium is unique (and hence our argument shows that all equilibria are constrained inefficient).

is equivalent to increasing the collateral requirements in a collateral constrained financial market equilibrium. We consider the equilibrium obtained as a result of this intervention, where agents optimize subject to these tighter constraints and markets clear, and evaluate agents' welfare ex ante, at date 0, at the new equilibrium allocation. This allocation clearly satisfies the collateral constraints. At the same time, since the tighter constraints modify agents' trades and hence securities' prices, it may not be budget feasible at the original prices. We show the intervention is Pareto improving, for an open set of the parameter values describing the economy. Thus, inefficient steady-state equilibria (as characterized in the previous subsection) are also constrained inefficient: making the collateral constraint tighter in some date events improves welfare.

Given the nature of the intervention and the fact that the economy is initially in a steady state, there is a transition phase of one period before the economy settles to a new steady state (at  $t = 2$ ): the new equilibrium prices and consumption levels depend on time (whether it is  $t = 1$  or  $t > 1$ ), and on the realization of the current shock. It is thus convenient to use the notation  $q_{s'}(s; t)$  to indicate the price at time  $t$  and state  $s$  of the tree option that pays in state  $s'$ . The new equilibrium portfolios are, at all dates  $t \geq 1$ ,  $\theta^1 = (\varepsilon, 1 - \varepsilon)$ ,  $\theta^2 = (1 - \varepsilon, \varepsilon)$  – that is, the short-sale constraint always binds. The consumption level of type 1 consumers at the date of the intervention,  $t = 1$ , is

$$\begin{aligned} c^1(s_1 = 1) &= \mathfrak{h} - q_1(1; 1)\varepsilon - q_2(1; 1)(1 - \varepsilon) \\ c^1(s_1 = 2) &= \mathfrak{d} + q_1(2; 1) + q_2(2; 1) - q_1(2; 1)\varepsilon - q_2(2; 1)(1 - \varepsilon) \\ &= \mathfrak{d} + q_1(2; 1)(1 - \varepsilon) + q_2(2; 1)\varepsilon \end{aligned}$$

and, at all subsequent dates  $t > 1$ ,

$$\begin{aligned} c^1(s_t = 1) &= \mathfrak{h} + \varepsilon(q_1(1) + q_2(1) + \mathfrak{d}) - q_1(1)\varepsilon - q_2(1)(1 - \varepsilon) \\ &= \mathfrak{h} + \varepsilon\mathfrak{d} - q_2(1)(1 - 2\varepsilon) \\ c^1(s_t = 2) &= (\mathfrak{d} + q_1(2) + q_2(2))(1 - \varepsilon) - q_1(2)\varepsilon - q_2(2)(1 - \varepsilon) \\ &= \mathfrak{d}(1 - \varepsilon) + q_1(2)(1 - 2\varepsilon) \end{aligned}$$

where  $q_{s'}(s) = q_{s'}(s; t)$  for all  $t > 1$ ,  $s, s'$ .

The above expressions allow us to gain some intuition for the effects of the intervention considered. Consider first the direct effect, ignoring the price changes: we see that the intervention unambiguously increases the variability of consumption across states, not only at all dates  $t > 1$  but also at  $t = 1$ .<sup>10</sup> Next, turning our attention to the price changes, we show in what follows that the equilibrium price of the tree options unambiguously increases, as a result of the intervention, since their effective supply (the amount that can be traded in the market) decreases, from 1 to  $1 - 2\varepsilon$ . From the above expressions we see that an increase in prices reduces the variability of consumption across states, since consumers are

<sup>10</sup>This last property follows from the fact that the consumers' optimality conditions imply that, at an initial steady-state equilibrium, we have  $q_2(1) > q_1(1)$  and  $q_1(2) > q_2(2)$ .

net buyers of assets when they are rich and net sellers when they are poor. Hence, the price effect improves risk sharing, in contrast to the direct effect. We also show that for an open set of parameter values the price effect prevails over the direct effect.

We have eight new equilibrium prices to determine. By symmetry (of consumers' preferences, endowments, and shocks), however, these reduce to four, since  $q_1(1; 1) = q_2(2; 1)$ ,  $q_2(1; 1) = q_1(2; 1)$ , as well as  $q_1(1) = q_2(2)$  and  $q_2(1) = q_1(2)$  for all  $t = 2, \dots$ . Using the above expressions of the budget constraints, the equilibrium prices can be obtained from the first-order conditions for the consumers' optimal choices. After some substitutions, we obtain<sup>11</sup> the following equation that can be solved for  $q_2(1) = q_1(2)$ :

$$q_1(2)u'(\mathfrak{h} + \varepsilon\mathfrak{d} - q_1(2)(1 - 2\varepsilon)) - \frac{\beta(q_1(2) + \mathfrak{d})}{2 - \beta}u'(\mathfrak{d}(1 - \varepsilon) + q_1(2)(1 - 2\varepsilon)) = 0. \quad (14)$$

It is useful to denote by  $q_1^0(2)$  the solution of this equation when  $\varepsilon = 0$  (that is, at the initial steady state).

Differentiating (14) with respect to  $\varepsilon$  and evaluating it at  $\varepsilon = 0$  yields the following expression for the change in equilibrium prices in the new steady state:

$$\left. \frac{dq_1(2)}{d\varepsilon} \right|_{\varepsilon=0} = \frac{- \left[ \beta \frac{\mathfrak{d} + q_1^0(2)}{2 - \beta} u''_{\mathfrak{d}} + q_1^0(2) u''_{\mathfrak{h}} \right] (\mathfrak{d} + 2q_1^0(2))}{u'_{\mathfrak{h}} - \frac{\beta}{2 - \beta} u'_{\mathfrak{d}} - \beta \frac{\mathfrak{d} + q_1^0(2)}{2 - \beta} u''_{\mathfrak{d}} - q_1^0(2) u''_{\mathfrak{h}}} \quad (15)$$

where  $u'_{\mathfrak{h}} = u'(\mathfrak{h} - q_1^0(2))$  and  $u'_{\mathfrak{d}} = u'(\mathfrak{d} + q_1^0(2))$  with  $u''_{\mathfrak{h}}$  and  $u''_{\mathfrak{d}}$  defined analogously. In the above expression the numerator is clearly positive, and so is the denominator, since equation (14) evaluated at  $\varepsilon = 0$  yields  $u'_{\mathfrak{h}} = \frac{\mathfrak{d} + q_1^0(2)}{q_1^0(2)} \frac{\beta}{2 - \beta} u'_{\mathfrak{d}} > \frac{\beta}{2 - \beta} u'_{\mathfrak{d}}$ . Turning our attention to the effect on equilibrium consumption, from the above expressions of the budget constraints and the symmetry of equilibrium prices we obtain

$$\left. \frac{dc^1(s_t = 1)}{d\varepsilon} \right|_{\varepsilon=0} = - \left. \frac{dc^1(s_t = 2)}{d\varepsilon} \right|_{\varepsilon=0} = 2q_1^0(2) + \mathfrak{d} - \left. \frac{dq_1(2)}{d\varepsilon} \right|_{\varepsilon=0}. \quad (16)$$

From (15) we immediately see that

$$0 < \left. \frac{dq_1(2)}{d\varepsilon} \right|_{\varepsilon=0} < \mathfrak{d} + 2q_1^0(2),$$

so that  $\left. \frac{dc^1(s_t=1)}{d\varepsilon} \right|_{\varepsilon=0} > 0$ . Hence, in the new steady-state for all  $t > 1$ , the equilibrium price of the tree options unambiguously increases, as claimed, as a result of the intervention, but the change in prices is not enough to overturn the direct effect of the intervention, and so the variability in consumption across states increases too.

We can similarly proceed to determine the effect on consumption at the transition date  $t = 1$ :

$$\left. \frac{dc^1(s_1 = 1)}{d\varepsilon} \right|_{\varepsilon=0} = - \left. \frac{dc^1(s_1 = 2)}{d\varepsilon} \right|_{\varepsilon=0} = q_1^0(2) - \frac{\beta(q_1^0(2) + \mathfrak{d})}{2 - \beta} - \left. \frac{dq_2(1; 1)}{d\varepsilon} \right|_{\varepsilon=0},$$

<sup>11</sup>The details for this as well as the similar derivation of (19) below can be found in the Appendix.

where we used the fact that  $q_1(1; 1)$ , evaluated at  $\varepsilon = 0$ , is equal to  $q_1(1)$  and the steady-state value before the intervention,  $\frac{\beta(q_1^0(2)+\vartheta)}{2-\beta}$ .

The effect on the discounted expected utility of consumer 1 of an infinitesimal tightening of the portfolio restriction – that is from  $\varepsilon = 0$  to  $d\varepsilon > 0$  is given by

$$\left. \frac{dU}{d\varepsilon} \right|_{\varepsilon=0} = \frac{1}{2} (u'_h - u'_\vartheta) \left. \frac{dc^1(s_1 = 1)}{d\varepsilon} \right|_{\varepsilon=0} + \frac{\beta}{2(1-\beta)} (u'_h - u'_\vartheta) \left. \frac{dc^1(s_t = 1)}{d\varepsilon} \right|_{\varepsilon=0}. \quad (17)$$

By symmetry, the expression for the change in consumer 2's expected utility has the same value. Hence the welfare effect of the intervention considered is determined by the sign of the expression in (17).

Since  $u'_h < u'_\vartheta$ , our finding on the sign of (16) implies that the effect of the intervention considered on agents' steady state welfare, given by the second term in (17), is always negative. For the intervention to be welfare improving we therefore need to have a welfare improvement in the initial period that is sufficiently large to compensate for the negative effect after that period. More precisely, from (17) it follows that  $\left. \frac{dU}{d\varepsilon} \right|_{\varepsilon=0} > 0$  if, and only if,

$$\left. \frac{dc^1(s_1 = 1)}{d\varepsilon} \right|_{\varepsilon=0} < -\frac{\beta}{1-\beta} \left. \frac{dc^1(s_t = 1)}{d\varepsilon} \right|_{\varepsilon=0},$$

or equivalently, substituting the expressions obtained above for the consumption changes and rearranging terms,

$$\left. \frac{dq_2(1;1)}{d\varepsilon} \right|_{\varepsilon=0} > \frac{2q_1^0(2) + \vartheta\beta}{(2-\beta)(1-\beta)} - \frac{\beta}{1-\beta} \left. \frac{dq_2(1)}{d\varepsilon} \right|_{\varepsilon=0} \quad (18)$$

That is, for an improvement to obtain the price change in the first period,  $\left. \frac{dq_2(1;1)}{d\varepsilon} \right|_{\varepsilon=0}$ , has to be sufficiently large that  $c^1(s_1 = 1)$  decreases, increasing risk sharing in this intermediate period, and by a sufficiently large amount.

Again by differentiating the consumers' first-order conditions with respect to  $\varepsilon$  we obtain the following expression for the price effect at the intermediate date:

$$\frac{\left. \frac{dq_2(1;1)}{d\varepsilon} \right|_{\varepsilon=0}}{u'_h - q_1^0(2)u''_h} = \frac{q_1^0(2) \left( \frac{\beta(q_1^0(2)+\vartheta)}{2-\beta} - q_1^0(2) \right) u''_h - \frac{\beta(q_1^0(2)+\vartheta)}{2-\beta} u''_\vartheta \left( \vartheta + 2q_1^0(2) - \left. \frac{dq_1(2)}{d\varepsilon} \right|_{\varepsilon=0} \right) + \frac{\beta}{2-\beta} u'_\vartheta \left. \frac{dq_1(2)}{d\varepsilon} \right|_{\varepsilon=0}}{u'_h - q_1^0(2)u''_h} \quad (19)$$

Substituting this expression into the condition obtained above for the intervention to be improving, (18), and rearranging terms we get:

$$\begin{aligned} & q_1^0(2) (1-\beta) [\beta\vartheta - 2(1-\beta)q_1^0(2)] u''_h - (1-\beta)\beta(q_1^0(2) + \vartheta)u''_\vartheta (\vartheta + 2q_1^0(2)) \\ & \quad - (2q_1^0(2) + \vartheta\beta) \left( u'_h - q_1^0(2)u''_h \right) \\ & + \left[ \beta(2-\beta) \left( u'_h - q_1^0(2)u''_h \right) + \beta(1-\beta)u'_\vartheta + (1-\beta)\beta(q_1^0(2) + \vartheta)u''_\vartheta \right] \left. \frac{dq_1(2)}{d\varepsilon} \right|_{\varepsilon=0} > 0 \end{aligned} \quad (20)$$

Condition (20) is stated in terms of endogenous variables, which obviously raises the question if there are economies for which the equilibrium values satisfy it. We have the following result.

**THEOREM 3** *There are specifications of economies in the environment under consideration that are robust with respect to perturbations in  $(\mathfrak{h}, \mathfrak{d}, \beta)$ , and to perturbations of preferences, for which condition (20) holds and hence there exists a steady-state equilibrium that is constrained inefficient.*

To prove the theorem, we show (in the Appendix) that for sufficiently small  $\beta$  condition (20) is satisfied if

$$1 + \mathfrak{d} \frac{u''(\mathfrak{d} + q_1^0(2))}{u'(\mathfrak{d} + q_1^0(2))} + \frac{u'(\mathfrak{h} - q_1^0(2))}{u'(\mathfrak{d} + q_1^0(2))} < 0. \quad (21)$$

As shown in the previous Subsection, 3.2.1, when  $\mathfrak{h}(1 - \beta) > \mathfrak{d}$  a Pareto-inefficient steady-state equilibrium exists with  $\frac{u'(\mathfrak{h} - q_1^0(2))}{u'(\mathfrak{d} + q_1^0(2))} < 1$ . It, then, follows that the inequality  $-\frac{u''(\mathfrak{d} + q_1^0(2))}{u'(\mathfrak{d} + q_1^0(2))} > \left(1 + \frac{u'(\mathfrak{h} - q_1^0(2))}{u'(\mathfrak{d} + q_1^0(2))}\right) / \mathfrak{d}$  is satisfied when the absolute risk-aversion is sufficiently high. Therefore, condition (20) holds and the steady-state equilibrium is constrained inefficient whenever the agents' absolute risk aversion is uniformly above  $2/\mathfrak{d}$  and  $\beta$  is sufficiently small. It is clear that this is true for an open set of parameters and utility functions.

### 3.2.3 Logarithmic preferences

While Theorem 3 above is all one can say in general, it is useful to illustrate, for a given specification of the agents' utility function, how large the set of parameter values is for which one obtains constrained inefficient equilibria. We consider here the case where  $u(c) = \log(c)$ . It can be verified that in this case an explicit solution of (14) for the equilibrium price can be found, given by

$$q_1^0(2) = \beta \frac{\mathfrak{h}}{2}.$$

Since the utility is homothetic it is without loss of generality to normalize  $\mathfrak{d} = 1$ . Direct computations show that

$$\left. \frac{dq_2(1; 1)}{d\varepsilon} \right|_{\varepsilon=0} = \frac{\beta(1 + \mathfrak{h})(1 + \beta\mathfrak{h})}{2 + \beta\mathfrak{h}}$$

and

$$\left. \frac{dq_1(2)}{d\varepsilon} \right|_{\varepsilon=0} = \frac{\beta(-4\mathfrak{h} + \beta^2\mathfrak{h}(2 + 3\mathfrak{h}) + 2\beta(1 + \mathfrak{h} - 2\mathfrak{h}^2))}{2(\beta - 2)(2 + \beta\mathfrak{h})}.$$

According to equation (18) an improvement is possible if

$$\beta \left. \frac{dq_2(1)}{d\varepsilon} \right|_{\varepsilon=0} + (1 - \beta) \left. \frac{dq_2(1; 1)}{d\varepsilon} \right|_{\varepsilon=0} - \frac{2q_2(1) + \mathfrak{d}\beta}{(2 - \beta)} > 0$$

Substituting these expressions into (18) we find that, in the case of logarithmic preferences, the intervention considered is welfare improving if, and only if,

$$2 - \beta(\mathfrak{h} - 2)\mathfrak{h} + \beta^2\mathfrak{h}^2 < 0.$$

Figure 1 shows, in the space  $\mathfrak{h}, \beta$ , the region of values of these parameters for which competitive equilibria are constrained inefficient as well as the region where equilibria are

Pareto efficient. We see that the region where constrained inefficiency holds is quite large, while the region where full Pareto efficiency cannot be attained but still the intervention considered is not welfare improving is very small.

## 4 Stationarity properties of equilibria

In this section we give conditions for the existence of a (stationary) Markov equilibrium and, for the case of two agents, we give conditions that ensure that this Markov equilibrium has finite support.

The example in Section 3.2.1 demonstrates that even when the constraints bind there can exist equilibria where consumption and prices only depend on the current realization of the exogenous shock. Since the stochastic process of the exogenous variables has finite support, these equilibria have finite support. While equilibria of this type generally exist for pure exchange economies with Pareto-efficient equilibria, in models with incomplete markets, or with overlapping generations, equilibrium prices and consumption levels typically take infinitely many values along an equilibrium path. It is obviously an important question whether along the equilibrium path the endogenous variables take finitely many or infinitely many values. If they take finitely many values, the equilibrium can be characterized by a finite system of equations, it can typically be computed easily and, one can conduct local comparative statics using the implicit function theorem. Ligon et al. (2002) show that in limited enforcement models finite support equilibria always exist if there are two agents. However, in those models equilibrium allocations are constrained efficient and can be obtained as the solution of a convex programming problem. As we have demonstrated, competitive equilibrium in our model may be constrained inefficient and it is not possible to derive equilibrium allocations as the solution to a planner's problem — the argument in Ligon et al. (2002) crucially depends on this property. In this respect our model is closer to models with incomplete financial markets and in these models finite support equilibria typically do not exist.

Even if equilibria have infinite support, they might still be tractable if they are Markov for some simple, endogenous state variable (as, for example, is the case in the stochastic growth model). In many models with heterogeneous agents and market imperfections, however, it is an open problem under which conditions Markov equilibria exist (see, e.g., Kubler and Schmedders (2002) or Santos (2002)).

In this section we investigate the conditions under which there exist Markov equilibria, and equilibria where individuals' consumption follow a Markov process with finite support. As argued above, the existence of equilibria with these properties is important, as such equilibria are simpler to study and to compute. We begin by providing some sufficient conditions for the existence of Markov equilibria. This analysis becomes simpler if we consider the Arrow–Debreu equilibrium notion with limited pledgeability and use as endogenous state variable the *instantaneous Negishi weights*, yielding current consumption levels (as

in Chien and Lustig (2010)), rather than the beginning-of-period distribution of the tree. The intuition for why this simplifies the analysis is that, as we will see below, if at some node an agent's limited pledgeability constraint does not bind, the agent's instantaneous Negishi weight remains constant — hence one only needs to analyze the evolution of the weight for nodes where the constraints are binding. In the remainder of the section we show that, for economies with only two types of agents, finite support equilibria exist under more general conditions, and then conclude with a brief discussion of the existence of finite support equilibria with more than two agents.

#### 4.1 Existence of Markov equilibria

We take the endogenous state at some node  $s^t$  to be the Negishi consumption weights  $\lambda(s^t) \in \mathbb{R}_{++}^H$  where

$$(c^1(s^t), \dots, c^H(s^t)) \in \arg \max_{c \in \mathbb{R}_+^H} \sum_{h \in \mathcal{H}} \lambda^h(s^t) u^h(c^h, s_t) \text{ s.t. } \sum_{h \in \mathcal{H}} (c^h - \omega^h(s_t)) = 0.$$

Negishi's (1960) approach to proving the existence of a competitive equilibrium, instead of solving for consumption values that clear markets, solves for weights that enforce budget balance (see also Dana (1993)). Judd et al. (2003) show how to use this approach to compute equilibria in Lucas-style models with complete markets (and without collateral constraints). Chien and Lustig (2010) (see also Chien et al. (2011)) consider a Markov equilibrium notion that features individual multipliers — interpretable as the inverse of our consumption weights — as the endogenous state variable in a model analogous to ours, though for a slightly different economy with a continuum of agents.

The state, then, consists of the current shock and all agents' current Negishi weights,  $(s, \lambda)$ . To define a competitive equilibrium satisfying the Markov property (in short, a Markov equilibrium), we need to rewrite the equilibrium conditions in a recursive form, specifying a policy function that determines how the endogenous variables depend on the state and a transition map that associates to the current state a probability distribution over next period's states. The consumption policy function,  $C : \mathcal{S} \times \mathbb{R}_{++}^H \rightarrow \mathbb{R}_+^H$  is obviously given by

$$C(s, \lambda) = \arg \max_{c \in \mathbb{R}_+^H} \sum_{h \in \mathcal{H}} \lambda^h u^h(c^h, s) \text{ s.t. } \sum_{h \in \mathcal{H}} (c^h - \omega^h(s)) = 0. \quad (22)$$

To understand how  $\lambda$  evolves across time periods and shock realizations, consider an Arrow–Debreu equilibrium with limited pledgeability, with prices  $(\rho(\sigma))_{\sigma \in \Sigma}$ , and a consumption allocation  $(c^h(\sigma))_{\sigma \in \Sigma}^{h \in \mathcal{H}}$ . If for an agent  $h$  and a node  $s^t$  the limited pledgeability constraint does not bind, i.e.

$$\sum_{\sigma \succeq s^t} \rho(\sigma) c^h(\sigma) > \sum_{\sigma \succeq s^t} \rho(\sigma) e^h(\sigma),$$

from the agent's first-order conditions it follows that his marginal rate of substitution between  $s^{t-1}$  and  $s^t$  must equal the price ratio,  $\frac{\rho(s^t)}{\rho(s^{t-1})}$  and, as we show formally below, we

have  $\lambda^h(s^t) = \lambda^h(s^{t-1})$ . If, on the other hand, the constraint binds

$$\sum_{\sigma \succeq s^t} \rho(\sigma) c^h(\sigma) = \sum_{\sigma \succeq s^t} \rho(\sigma) e^h(\sigma),$$

his marginal rate of substitution must be higher than  $\frac{\rho(s^t)}{\rho(s^{t-1})}$  and we have  $\lambda^h(s^t) > \lambda^h(s^{t-1})$ . A key determinant for the transition of  $\lambda$  is thus the value of an agent's future lifetime consumption in excess of his non-pledgeable endowments at any date event. We write this recursively as a function of the state  $(s, \lambda)$  denoting it agent  $h$ 's "excess expenditure function"  $V^h(s, \lambda)$ , for each  $h$ . In what follows, it is useful to write (in a slight abuse of notation)

$$u^{h'}(s, \lambda) = u^{h'}(C^h(s, \lambda), s).$$

A **Markov equilibrium** consists of a policy function  $C : \mathcal{S} \times \mathbb{R}_{++}^H \rightarrow \mathbb{R}_{++}^H$ , together with a transition function  $L : \mathcal{S} \times \mathbb{R}_{++}^H \rightarrow \mathbb{R}_{++}^H$ , and excess expenditure functions  $V^h : \mathcal{S} \times \mathbb{R}_{++}^H \rightarrow \mathbb{R}$  for all agents  $h \in \mathcal{H}$ , such that for all  $h \in \mathcal{H}$ , all  $s \in \mathcal{S}$ , and all  $\lambda \in \mathbb{R}_{++}^H$ :

$$V^h(s, \lambda) = u^{h'}(s, \lambda) (C^h(s, \lambda) - e^h(s)) + \beta \sum_{s'} \pi(s, s') V^h(s', L(s', \lambda)) \quad (23)$$

and for all  $s' \in \mathcal{S}$ ,

$$V(s', L(s', \lambda)) \geq 0 \quad (24)$$

$$L(s', \lambda) - \lambda \geq 0 \quad (25)$$

$$V(s', L(s', \lambda)) (L(s', \lambda) - \lambda) = 0 \quad (26)$$

Note that since consumption is homogenous of degree zero in the Negishi weights we could normalize  $\lambda$  to lie in the  $H - 1$  dimensional unit simplex,  $\Delta^{H-1} = \{\lambda \in \mathbb{R}_+^H : \sum_{h=1}^H \lambda^h = 1\}$ . At this point it simplifies the exposition and the notation not to do so, since without this normalization (25) implies that we always have  $L(s', \lambda) \geq \lambda$  while the relative Negishi weight of an agent actually decreases if he is unconstrained and some other agents are constrained.

With  $\lambda$  normalized to lie in the simplex, the conditions defining a Markov equilibrium become a little messier. Since in parts of the argument in Section 4.2 it will turn out to be convenient to adopt this normalization, it is useful to briefly illustrate here how the definition of a Markov equilibrium changes with such a normalization. The functions  $C(\cdot), V(\cdot)$  and  $L(\cdot)$  become maps from  $\mathcal{S} \times \Delta^{H-1}$  and we need to introduce an auxiliary function  $\gamma : \mathcal{S} \times \Delta^{H-1} \rightarrow \mathbb{R}_+^H$ . While the definition of  $V^h$  is as in equation (23) and equation (24) is unchanged, equations (25)-(26) become:

$$L_h(s', \lambda) = \frac{\lambda_h + \gamma_h(s', \lambda)}{\sum_{i=1}^H (\lambda_i + \gamma_i(s', \lambda))} \text{ and } \gamma_h(s', \lambda) V^h(s', L(s', \lambda)) = 0 \text{ for all } h \in \mathcal{H} \quad (27)$$

To show that a Markov equilibrium as defined above indeed satisfies all the properties of an Arrow-Debreu equilibrium with limited pledgeability, we compare the first-order conditions for the latter equilibrium with conditions (23) – (26) above. To get some understanding



of the expression of the excess expenditure function in (23), note that we proceeded as in the Negishi approach, by taking an agent's marginal utility to value his net consumption. Intuitively this is possible because, as argued above, whenever the agent is unconstrained at some state  $s$ , his marginal rate of substitution between  $s$  and the predecessor state  $s'$  equals the prices while, when he is constrained, we have  $V^h(s', \lambda) = 0$  and it is irrelevant whether this term is multiplied by the agents' marginal rate of substitution between  $s$  and  $s'$  or by actual market prices since the product is always zero. With regard to the remaining conditions, note that the first-order conditions for the optimality of consumption of agent  $h$  at some node  $s^t$  can be written as follows:

$$\begin{aligned} \beta^t \pi(s^t) u^h(c^h(s^t), s_t) - \eta^h \rho(s^t) + \sum_{\sigma: s^t \succeq \sigma} \mu^h(\sigma) \rho(s^t) &= 0 \\ \mu^h(s^t) \sum_{\sigma \succeq s^t} \rho(\sigma) (c^h(\sigma) - e^h(\sigma)) &= 0, \end{aligned}$$

for multipliers  $\eta^h \geq 0$  (associated with the intertemporal budget constraint (3)) and  $\mu^h(\sigma) \geq 0$  (associated with the collateral constraint (4) at node  $\sigma$ ). Taking  $\frac{1}{\lambda^h(\sigma)} = \eta^h - \sum_{\sigma': s^t \succeq \sigma'} \mu^h(\sigma')$  for all  $h, \sigma$ , we see that conditions (24)–(26) follow from the above first-order conditions and that the evolution of  $\lambda^h(\sigma)$  is determined by that of the Lagrange multipliers  $\mu^h(\sigma)$ .

**THEOREM 4** *Given a Markov equilibrium  $(C, V, L)$  and any  $\lambda_0 \in R_{++}^H$  with  $V^h(s_0, \lambda_0) \geq 0$  for all  $h$ , there exist initial tree-holdings  $(\theta^h(s^{-1}))_{h \in \mathcal{H}}$  and an Arrow–Debreu equilibrium with limited pledgeability with  $c^h(s^t) = C^h(s_t, \lambda(s^t))$  and  $\lambda(s^t) = L(s_t, \lambda(s^{t-1}))$  for all  $s^t, t > 0$ .*

Note that if there is a competitive equilibrium with  $\lambda(s^t) = \lambda^*$  for all  $s^t$ , this must be an unconstrained Arrow–Debreu equilibrium. The fact that  $\lambda(s^t)$  does not change over time implies that the additional constraint (4) is never binding in equilibrium and the allocation is identical to the unconstrained Arrow–Debreu equilibrium allocation and is Pareto efficient. Therefore, if for a given Markov equilibrium a vector of weights  $\lambda^*$  exists with  $V^h(s, \lambda^*) \geq 0$  for all  $s \in \mathcal{S}$  and all  $h \in \mathcal{H}$ , there exist initial conditions (corresponding to the weights  $\lambda^*$ ) for which the Markov equilibrium is identical to an unconstrained Arrow–Debreu equilibrium.

It is well known that in models where the equilibrium may be constrained inefficient Markov equilibria might not always exist. For the model with collateral constraints, when financial markets are incomplete no sufficient conditions are known that ensure the existence of a Markov equilibrium (see Kubler and Schmedders (2003)). In contrast, in the environment considered here, with complete markets, as shown in the next theorem, the assumption that all agents' preferences satisfy the gross substitute property implies that Markov equilibria always exist. Dana (1993) shows that this assumption guarantees the uniqueness of Arrow–Debreu equilibria in infinite horizon exchange economies without constraints. We show that her argument extends to our model and guarantees the existence of a Markov equilibrium through the uniqueness of the “continuation-equilibrium”. As pointed out by Dana (1993), in our context the assumption of gross substitutes is equivalent to assuming

that for all agents  $h$  and all shocks  $s$ , the term  $c u^{h'}(c, s)$  is increasing in  $c$ ; or equivalently, that the coefficient of relative risk aversion  $-c \frac{u^{h''}(c, s)}{u^{h'}(c, s)}$  is always less than or equal to one. While in applied work it is often assumed that relative risk aversion is significantly above one it is also sometimes argued (see, e.g., Boldrin and Levine (2001)) that a value below one might be the empirically more relevant case.

We have the following result.

**THEOREM 5** *Suppose that for all agents  $h$  and all shocks  $s$ ,  $c u^{h'}(c, s)$  is increasing in  $c$  for all  $c > 0$ . Then a Markov equilibrium exists and it is unique.*

To establish the result we prove first that for each initial value of  $\lambda$  a unique equilibrium exists. The proof proceeds by contradiction and is similar to the standard proof of uniqueness of equilibrium when demand functions exhibit the gross substitute property. Suppose two equilibrium allocations existed, with two distinct processes for the associated Negishi weights. Define a new process as the (pointwise) minimum of these two. At the allocation implied by this process all agents violate their budget constraints, which must violate feasibility since in our economy prices are summable. The uniqueness of the equilibrium for each  $\lambda$ , then, directly implies the existence — and uniqueness — of a Markov equilibrium.

## 4.2 Markov equilibria with finite support

The main difficulty in determining whether Markov equilibria with finite support exist lies in specifying the support. We show that for the case of two agent types,  $H = 2$ , there is a natural characterization of the support. These equilibria constitute a generalization of the steady-state equilibrium obtained in the example considered in Section 3.2.1. We also demonstrate how these finite support equilibria are always reached at an equilibrium when the economy starts with arbitrary initial conditions.

Finally, we discuss briefly the case of more than two agents and show that in this case there are examples of equilibria with finite support but there are also examples where all equilibria have infinite support.

### 4.2.1 Finite support Markov equilibria in economies with two types of agents

When there are only two types of agents finite support equilibria exist under rather general conditions and are very easy to characterize. It is convenient here to normalize  $(\lambda^1, \lambda^2)$  to always lie in the unit interval. This allows us to denote by  $\lambda = \lambda^1$  the value of the consumption weight for agent 1 and to take this as a state variable. In a slight abuse of notation, we write  $C^h(s, \lambda) = C^h(s, (\lambda, 1 - \lambda))$ ,  $V^h(s, \lambda) = V^h(s, (\lambda, 1 - \lambda))$  etc.

In order to fix ideas, suppose that a Markov equilibrium exists and that, for all agents  $h = 1, 2$  and all  $s \in \mathcal{S}$ , the excess expenditure function  $V^h(s, \cdot)$ , has a unique zero. Denote

by  $\underline{\lambda}^*(s)$  the zero of  $V^1(s, \cdot)$  and by  $\bar{\lambda}^*(s) > \underline{\lambda}^*(s)$  the zero of  $V^2(s, \cdot)$ .<sup>12</sup> Figure 2 illustrates a possible form of these functions and their zeros in the simple case where there are two possible shocks,  $s = 1, 2$ . By equations (24) and (27), for each  $(s, \lambda) \in \mathcal{S} \times (0, 1)$  we must have  $L(s, \lambda) = \lambda$  or  $L(s, \lambda) \in \{\underline{\lambda}^*(s), \bar{\lambda}^*(s)\}$ . To see this note that if  $V^h(s, \lambda) > 0$  for  $h = 1, 2$  then  $L(s, \lambda) = \lambda$  while if  $V^1(s, \lambda) < 0$ , Equation (27) implies that  $\lambda$  must “jump” to  $\underline{\lambda}^*(s)$ , and if  $V^2(s, \lambda) < 0$  it implies that  $\lambda$  must jump to  $\bar{\lambda}^*(s)$ . If  $V^h(s, \lambda) = 0$  for some  $h = 1, 2$  we must have  $\lambda \in \{\underline{\lambda}^*(s), \bar{\lambda}^*(s)\}$  and since we assumed that there is a unique zero of each  $V^h(s, \cdot)$  we have  $L(s, \lambda) = \lambda$ . Therefore, if each  $V^h(s, \cdot)$  has a unique zero the entire equilibrium transition can be described by the  $2S$  numbers  $\underline{\lambda}^*(1), \bar{\lambda}^*(1), \dots, \underline{\lambda}^*(S), \bar{\lambda}^*(S)$ : either  $\lambda$  stays constant or it jumps to one of these  $2S$  values. In Figure 2 if we start with  $\lambda = \bar{\lambda}^*(1)$  when the current state is  $s = 1$ , the endogenous state has to move to  $\underline{\lambda}^*(2)$  when state 2 occurs and will alternate (as in Ligon et al. (2002)) between the values  $\bar{\lambda}^*(1)$  and  $\underline{\lambda}^*(2)$ .

Unfortunately, it is not straightforward to identify conditions under which, when a Markov equilibrium exists, each excess expenditure function  $V^h(s, \cdot)$  has a unique zero. Our main result of this section gives sufficient conditions<sup>13</sup>.

**THEOREM 6** *Suppose there are two types of agents. A finite support Markov equilibrium exists for all initial conditions if any one of the following three conditions is satisfied*

1. *The coefficient of relative risk aversion satisfies  $-cu^{h''}(c, s)/u^{h'}(c, s) \leq 1$  for all  $c, s, h$ .*
2. *All agents have identical, constant relative risk aversion (CRRA) Bernoulli utility functions.*
3.  *$u^h(c, s)$  is state independent for all  $h$  and there is no aggregate uncertainty.*

To prove the result we adopt a constructive approach and conjecture that a competitive equilibrium with finite support exists. We then derive a finite system of equations and inequalities that characterize the competitive equilibrium in this case and find conditions under which this system has a solution and one can construct monotone functions  $V^h(s, \cdot)$  for all  $h = 1, 2$ .

As in the simple case of Figure 2, we want to prove that a Markov equilibrium exists where at most  $2S$  points are visited in the endogenous state space. In order to characterise these points as the solution to a system of inequalities, we define for any  $\underline{\lambda} \in (0, 1)^S$  and

<sup>12</sup>The Inada condition on agents’ utility functions together with the fact that  $V^h(s, \cdot)$  must be bounded above ensure that  $V^1(s, \lambda) < 0$  for  $\lambda$  sufficiently small and  $V^2(s, \lambda) < 0$  for  $\lambda$  sufficiently close to 1. Moreover, since  $d(s) > 0$ ,  $\bar{\lambda}^*(s)$  must always be larger than  $\underline{\lambda}^*(s)$ .

<sup>13</sup>In an earlier working paper version of their published paper, Chien and Lustig also characterize equilibria with finite support for the case of two shocks and two agents with identical CRRA utility. Our result holds for any number of shocks under more general conditions.

$\bar{\lambda} \in (0, 1)^S$  a function  $L : \mathcal{S} \times [0, 1] \rightarrow [0, 1]$  by

$$L_{(\underline{\lambda}, \bar{\lambda})}(s, \lambda) = \begin{cases} \lambda & \text{if } \underline{\lambda}(s) \leq \lambda \leq \bar{\lambda}(s) \\ \underline{\lambda}(s) & \text{if } \lambda < \underline{\lambda}(s) \\ \bar{\lambda}(s) & \text{if } \lambda > \bar{\lambda}(s). \end{cases}$$

This function  $L_{(\underline{\lambda}, \bar{\lambda})}(\cdot, \cdot)$  describes a law of motion for  $\lambda$ .

To characterize when the values  $(\underline{\lambda}, \bar{\lambda})$  induce the transition function of a finite support Markov equilibrium we define, for each  $h = 1, 2$ ,  $2S^2$  numbers  $V^h(s, \underline{\lambda}(\tilde{s}))$ ,  $V^h(s, \bar{\lambda}(\tilde{s}))$  for  $s, \tilde{s} \in \mathcal{S}$  to be the solution of the following linear system of  $2S^2$  equations:

$$V^h(s, \underline{\lambda}(\tilde{s})) = u^h(s, \underline{\lambda}(\tilde{s})) \left( C^h(s, \underline{\lambda}(\tilde{s})) - e^h(s) \right) + \beta \sum_{s'} \pi(s, s') V^h(s', L_{(\underline{\lambda}, \bar{\lambda})}(s', \underline{\lambda}(\tilde{s}))), \quad (28)$$

$$V^h(s, \bar{\lambda}(\tilde{s})) = u^h(s, \bar{\lambda}(\tilde{s})) \left( C^h(s, \bar{\lambda}(\tilde{s})) - e^h(s) \right) + \beta \sum_{s'} \pi(s, s') V^h(s', L_{(\underline{\lambda}, \bar{\lambda})}(s', \bar{\lambda}(\tilde{s}))). \quad (29)$$

The values  $(\underline{\lambda}(s), \bar{\lambda}(s))_{s \in \mathcal{S}}$  induce an equilibrium, and are denoted as  $(\underline{\lambda}^*(s), \bar{\lambda}^*(s))_{s \in \mathcal{S}}$  if they satisfy the following conditions:

$$V^1(s, \underline{\lambda}^*(s)) = V^2(s, \bar{\lambda}^*(s)) = 0 \text{ for all } s \in \mathcal{S}, \quad (30)$$

and, for all  $s, s'$

$$L_{(\underline{\lambda}^*, \bar{\lambda}^*)}(s', \underline{\lambda}^*(s)) = \underline{\lambda}^*(s) \Rightarrow V^h(s', \underline{\lambda}^*(s)) \geq 0 \text{ for } h = 1, 2 \quad (31)$$

$$L_{(\underline{\lambda}^*, \bar{\lambda}^*)}(s', \bar{\lambda}^*(s)) = \bar{\lambda}^*(s) \Rightarrow V^h(s', \bar{\lambda}^*(s)) \geq 0 \text{ for } h = 1, 2 \quad (32)$$

When the above conditions are satisfied, by construction,  $L_{(\underline{\lambda}^*, \bar{\lambda}^*)}(\cdot)$  describes a transition function that ensures that  $V^h(s, L(s, \lambda)) \geq 0$  for all  $\lambda \in \{\underline{\lambda}^*(s), \bar{\lambda}^*(s), s = 1, \dots, S\}$  so that (24) holds and (27) holds. This constitutes a competitive equilibrium for economies with initial conditions in  $\{\underline{\lambda}^*(s), \bar{\lambda}^*(s), s = 1, \dots, S\}$ . The advantage of this formulation is that the computation of an equilibrium reduces to solving a non-linear system of equations and verifying finitely many inequalities. In the above theorem, however, we claim a stronger result since we want the finite support equilibrium to exist for all initial conditions. The stated conditions on preferences and endowments guarantee that each  $V^h(s, \cdot)$  function has a unique zero.

In the proof we first show that there always exist  $S$  pairs  $(\underline{\lambda}^*(s), \bar{\lambda}^*(s))$  such that the solutions to (28) and (29) satisfy (30). This turns out to be always true under the general conditions on endowments and preferences considered in this paper and follows from a standard, fixed-point argument. However, we need rather restrictive conditions as those stated in the proposition to make sure that (31) and (32) also hold. These conditions turn out to ensure that the functions  $V^h(s, \cdot)$  have unique zeros.

Note that in the above construction the intervals  $([\underline{\lambda}^*(s), \bar{\lambda}^*(s)])_{s \in \mathcal{S}}$  uniquely define the values  $(V^h(s, \underline{\lambda}(\tilde{s})), V^h(s, \bar{\lambda}(\tilde{s})))_{s, \tilde{s} \in \mathcal{S}}^{h \in \mathcal{H}}$  and characterize the equilibrium. The equilibrium

dynamics of these equilibria are straightforward. If one starts at an initial condition that corresponds to a welfare weight on the boundary of the interval  $[\underline{\lambda}^*(s), \bar{\lambda}^*(s)]$  for the initial state  $s = s_0$ , only finitely many different welfare weights are visited along the equilibrium. If the initial condition corresponds to a different value,  $\lambda$  stays constant until we reach a state  $s'$  where one of the constraint binds in which case  $\lambda$  jumps to  $\underline{\lambda}^*(s)$  or  $\bar{\lambda}^*(s)$  and the dynamics proceeds as above.

Ligon et al. (2002) establish an analogous result for two-agent economies with limited enforcement, where equilibria are solutions of a constrained planner's problem. In that model, because the planner's problem can be formulated as a stationary programming problem, Markov equilibria always exist and to establish the finite support result it suffices to ensure the monotonicity of the agents' indirect utility function. In our model the existence of a Markov equilibrium is not guaranteed and even if Markov equilibria exist we need to ensure the monotonicity of the expenditure function. Hence the conditions in Ligon et al. (2002) are much weaker than those in Theorem 6.

#### 4.2.2 An example

To illustrate the construction of Theorem 6 it is useful to consider again the example considered in Sections 3.1 and 3.2.1. In that example we showed that when (7) holds, a Pareto-efficient equilibrium exists with constant consumption for some initial conditions. However, we did not discuss if and how that steady state could be reached for arbitrary initial conditions. Applying the results of the previous section, we can show that there always exists an equilibrium where the steady state is reached, but that it might take arbitrarily long to reach it. This equilibrium is clearly Pareto inefficient since, along the transition to the steady state, risk-sharing is imperfect as the collateral constraints will frequently bind. Nevertheless, the competitive equilibrium always has finite support. The endogenous state variable,  $\lambda$ , never takes more than three different values.

Suppose for simplicity that  $\mathfrak{h} = \frac{\mathfrak{d}}{1-\beta}$ ,  $u^1(c) = u^2(c) = \log(c)$ . Denote aggregate endowments by  $\omega = \mathfrak{h} + \mathfrak{d}$ . It is easy to check that with log utility we have  $C^1(s, \lambda) = \lambda\omega$  for both  $s = 1, 2$ . Also, it is easy to see that under these parameter values there exists a unique efficient steady-state where each agent's consumption is given by  $\frac{\omega}{2}$ .<sup>14</sup> As pointed out after Theorem 4, a Pareto-efficient Markov equilibrium exists if, for some  $\lambda^*$ , we have  $V^h(s, \lambda^*) \geq 0$  for all  $h$  and all  $s$ . If we start with this initial condition, the economy will be immediately at the efficient steady state.

In the environment considered here, since agent 1 has a high endowment in shock 1, we must have  $\underline{\lambda}^*(1) > \underline{\lambda}^*(2)$  and, for an efficient steady state to exist, we must also have  $\underline{\lambda}^*(1) \leq \bar{\lambda}^*(2)$ . In fact, we show now that for  $\mathfrak{h} = \frac{\mathfrak{d}}{1-\beta}$  we have  $\underline{\lambda}^*(1) = \bar{\lambda}^*(2) = \frac{1}{2}$ . To do so, let us conjecture this property holds,  $\underline{\lambda}^*(1) = \bar{\lambda}^*(2) = \frac{1}{2}$ , and that  $L(s, \underline{\lambda}^*(1)) = \underline{\lambda}^*(1)$

<sup>14</sup>When  $\mathfrak{h} < \frac{\mathfrak{d}}{1-\beta}$  there is a continuum of efficient steady-states. The same logic of the argument applies in that case though calculations are a little more complex.

for  $s = 1, 2$ . With log utility, equations (28) and (29) become

$$V^1(1, \underline{\lambda}^*(1)) = 1 - \frac{\mathfrak{h}}{\underline{\lambda}^*(1)\omega} + \frac{\beta}{2} (V^1(1, \underline{\lambda}^*(1)) + V^1(2, \underline{\lambda}^*(1)))$$

and

$$V^1(2, \underline{\lambda}^*(1)) = 1 + \frac{\beta}{2} (V^1(1, \underline{\lambda}^*(1)) + V^1(2, \underline{\lambda}^*(1))).$$

To verify that our conjecture is correct we need to show that these two equations have a the solution for  $V^1(1, \underline{\lambda}^*(1))$  and  $V^1(2, \underline{\lambda}^*(1))$  that satisfies  $V^1(1, \underline{\lambda}^*(1)) = 0$  and  $V^1(2, \underline{\lambda}^*(1)) > 0$ . It is easy to see that if  $V^1(1, 0.5) = 0$  a solution of the second equation is given by  $V^1(2, 0.5) = \frac{1}{1-\beta/2}$ , which is always positive. Substituting this value into the first equation yields

$$V^1(1, 0.5) = 1 - \frac{\mathfrak{h}}{0.5\omega} + \frac{\beta}{2} V^1(2, 0.5) = \frac{\mathfrak{d} - \mathfrak{h}}{\omega} + \frac{\beta(\mathfrak{h} + \mathfrak{d})}{(2 - \beta)\omega} = 0,$$

which holds whenever  $(2-\beta)(\mathfrak{d}-\mathfrak{h})+\beta(\mathfrak{h}+\mathfrak{d}) = 0$ , equivalent to our assumption that  $\mathfrak{h} = \frac{\mathfrak{d}}{1-\beta}$ . By symmetry,  $\bar{\lambda}^*(2) = \frac{1}{2}$  solves the corresponding system for  $V^2(s, \bar{\lambda}^*(2))$ ,  $s = 1, 2$ .

Since endowments in the low state are zero, it is easy to verify that  $V^1(2, \lambda) \geq 0$  and  $V^2(1, \lambda) \geq 0$  for all  $\lambda \in (0, 1)$ . Therefore, in this example the values  $\underline{\lambda}^*(1), \bar{\lambda}^*(2)$  completely characterize the Markov equilibrium. If the initial conditions are such that the initial Negishi weight  $\lambda(s_0) = 1/2$ , the Markov equilibrium coincides with the efficient steady state. On the other hand, if — for example —  $\lambda(s_0) < \frac{1}{2}$ , the state variable remains unchanged at the initial value  $\lambda(s_0)$  as long as only shock 2 occurs, since  $V^h(2, \lambda(s_0)) \geq 0$  for both  $h = 1, 2$ . Agent 1 will consume an amount less than 1/2. When shock 1 occurs, we have  $V^1(1, \lambda(s_0)) < 0$  since  $\lambda(s_0) < \underline{\lambda}^*(1)$  and  $V^1(1, \cdot)$  is increasing. Therefore  $\lambda$  must jump to  $\underline{\lambda}^*(1)$  where it will stay from there on. Hence the steady state will be reached after each shock has realized at least once.

The same argument can also be used to analyze the case where the steady state is inefficient. It is easy to see that when  $\mathfrak{h} > \frac{\mathfrak{d}}{1-\beta}$  we have  $\underline{\lambda}^*(1) > \bar{\lambda}^*(2)$ . There is no efficient steady state and along the equilibrium path, after an initial transition similar to the one above, the instantaneous Negishi weight  $\lambda$  oscillates between the two values  $\bar{\lambda}^*(2)$  and  $\underline{\lambda}^*(1)$ .

### 4.2.3 Existence and nonexistence of finite support equilibria when $H > 2$

Unfortunately, for the general case with more than two agent types we do not know of general conditions that ensure the existence of finite support Markov equilibria. The problem is that the dynamics of the Negishi weights, which as we saw have a simple pattern when  $H = 2$ , can be much more complex when  $H > 2$ . In fact, even for limited enforcement models no existence results of finite support equilibria are available when  $H > 2$ . Nevertheless, it is useful to show by example that finite support equilibria might exist and to give an example where finite support equilibria do not exist.

Suppose there are 3 types of agents and three equiprobable i.i.d. shocks. Assume again the agents have identical log-utility functions,  $u^h(c) = \log(c)$  for all  $h$ , endowments are

$e^1 = (0, \mathfrak{h}, \mathfrak{h})$ ,  $e^2 = (\mathfrak{h}, 0, \mathfrak{h})$ , and  $e^3 = (\mathfrak{h}, \mathfrak{h}, 0)$  for some  $\mathfrak{h} > 0$ , while the tree pays constant dividends  $\mathfrak{d} > 0$ . The aggregate endowment is deterministic and equal to  $\omega = 2\mathfrak{h} + \mathfrak{d}$ . As in the case of two agents, we again normalize instantaneous Negishi weights to lie in the unit simplex – that is to say, we have  $\lambda^3 = 1 - \lambda^2 - \lambda^1$ . This allows us to write  $C^h(s, \lambda) = \lambda^h \omega$  for all  $h$  and  $s$ .

Using symmetry, we show in what follows that under the condition

$$\mathfrak{h} > \frac{\mathfrak{d}}{1 - \beta}$$

there exists a finite support equilibrium where agents 2 and 3 are constrained in state 1, agents 1 and 3 in state 2, and agents 1 and 2 in state 3. Denoting by  $\lambda(s, h)$  the value of the Negishi weights in state  $s$  where only type  $h$  is unconstrained, we need to find the values of the vectors  $\lambda(1, 1)$ ,  $\lambda(2, 2)$ , and  $\lambda(3, 3)$  constituting the support of the equilibrium. By symmetry, the weights of all agents when constrained are identical across all states, i.e.  $\lambda^1(2, 2) = \lambda^1(3, 3) = \lambda^2(1, 1) = \lambda^2(3, 3) = \lambda^3(1, 1) = \lambda^3(2, 2) = \lambda_{\mathfrak{h}}$  for some  $\lambda_{\mathfrak{h}}$ . Similarly,  $\lambda^1(1, 1) = \lambda^2(2, 2) = \lambda^3(3, 3) = \lambda_{\mathfrak{l}} = 1 - 2\lambda_{\mathfrak{h}}$ . In this situation, the transition function must therefore satisfy the following property

$$L(s, \lambda) = \lambda(s, s) \text{ whenever } \lambda \in \{\lambda(1, 1), \lambda(2, 2), \lambda(3, 3)\}.$$

Given this property of the transition function and the above specification of the states where each agent is constrained, proceeding analogously to the previous section we obtain

$$\begin{aligned} V^1(1, \lambda(1, 1)) &= 1 + \frac{\beta}{3} V^1(1, \lambda(1, 1)) = \frac{1}{1 - \frac{\beta}{3}} \\ V^1(s, \lambda(s, s)) &= 0 = 1 - \frac{\mathfrak{h}}{\lambda^1(s, s)\omega} + \frac{\beta}{3} V^1(1, \lambda(1, 1)), \quad s = 2, 3 \end{aligned}$$

where the equality  $V^1(s, \lambda(s, s)) = 0$  holds in the states where agent 1 is constrained. Hence we must have

$$1 - \frac{\mathfrak{h}}{\lambda_{\mathfrak{h}}\omega} + \frac{\beta}{3 - \beta} = 0,$$

or

$$\lambda_{\mathfrak{h}} = \frac{(3 - \beta)\mathfrak{h}}{3(\mathfrak{d} + 2\mathfrak{h})},$$

and  $1 > \lambda_{\mathfrak{h}} > 1/3 > \lambda_{\mathfrak{l}}$  given the assumption  $\mathfrak{h} > \frac{\mathfrak{d}}{1 - \beta}$ . A finite support equilibrium exists if we start with initial conditions  $s_0 = 1$  and  $\lambda(s_0) = \lambda(1, 1)$  — along the equilibrium path the instantaneous Negishi weights will only take values in  $\{\lambda(1, 1), \lambda(2, 2), \lambda(3, 3)\}$ .

To illustrate why it is difficult to find general conditions that ensure the existence of a finite support equilibrium when  $H > 2$ , consider the following small modification of the example above. Instead of assuming that each agent's individual endowments are high in two out of the three states, suppose they are high only in one out of the three states. That is  $e^1 = (\mathfrak{h}, 0, 0)$ ,  $e^2 = (0, \mathfrak{h}, 0)$ , and  $e^3 = (0, 0, \mathfrak{h})$  for some  $\mathfrak{h} > 0$ . Under the maintained assumption of logarithmic utility, Theorem 5 ensures the existence of a unique Markov

equilibrium. However, we will show that under the following assumption on fundamentals the unique Markov equilibrium has infinite support (and hence there exists no finite support equilibrium).

$$\beta < \frac{3}{4}, \quad \mathfrak{h} > \frac{3\mathfrak{d}}{3-4\beta} \quad (33)$$

To do so, note that whenever agents have zero endowments they must be unconstrained. We conjecture that in equilibrium each agent  $h$  is always constrained in state  $s = h$  and denote his relative instantaneous Negishi weight in this case by  $\lambda_{\mathfrak{h}}$ . We conjecture that in any equilibrium we have

$$\lambda_{\mathfrak{h}} = \frac{\mathfrak{h}}{\omega} \frac{3-2\beta}{3},$$

that the instantaneous Negishi weights  $\lambda$  lie in the set

$$\Delta_{\epsilon} = \left\{ \lambda \in \mathbb{R}_+^3 : \sum_h \lambda^h = 1, \lambda^h \leq \lambda_{\mathfrak{h}} \text{ for all } h = 1, 2, 3 \right\},$$

and that the transition  $L(s, \cdot)$  takes the following form for any  $s \in \mathcal{S}$  and any  $\lambda \in \Delta_{\epsilon}$

$$L(s, \lambda) = \begin{cases} \left( \lambda^{1'}, \lambda^{2'}, \lambda^{3'} \right) : & \lambda^{j'} = \lambda_{\mathfrak{h}} & \text{if } j = s \\ & \lambda^{j'} = \lambda^j \frac{1-\lambda_{\mathfrak{h}}}{\sum_{h \neq s} \lambda^h} & \text{otherwise.} \end{cases}$$

To verify this conjecture and confirm that this transition describes a competitive equilibrium by symmetry, it suffices to verify (24) and (27) only for agent 1. In state  $s = 1$ , for any  $\lambda \in \Delta_{\epsilon}$  with  $\lambda = (\lambda_{\mathfrak{h}}, \lambda_2, \lambda_3)$  we obtain

$$V^1(1, \lambda) = 0 = 1 - \frac{\mathfrak{h}}{\lambda_{\mathfrak{h}}\omega} + \frac{\beta}{3} \sum_{s'=2}^3 V^1(s', L(s', \lambda)),$$

while in the other two states for any  $\lambda \in \Delta_{\epsilon}$  we have

$$\begin{aligned} V^1(2, \lambda) &= 1 + \frac{\beta}{3} \sum_{s'=2}^3 V^1(s', L(s', \lambda)), \\ V^1(3, \lambda) &= 1 + \frac{\beta}{3} \sum_{s'=2}^3 V^1(s', L(s', \lambda)). \end{aligned}$$

From the last two equations we see that  $V^1(2, \lambda) = V^1(3, \lambda) = \frac{1}{1-\frac{2\beta}{3}}$  for all  $\lambda \in \Delta_{\epsilon}$ . By substituting this value into the first equation we obtain that  $V^1(1, \lambda) = 0$  is equivalent to the following.

$$1 - \frac{\mathfrak{h}}{\lambda_{\mathfrak{h}}\omega} + \frac{2\beta}{3-2\beta} = 0 \Leftrightarrow \lambda_{\mathfrak{h}} = \frac{\mathfrak{h}}{\omega} \frac{3-2\beta}{3}.$$

Under assumption (33) we have  $1 > \lambda_{\mathfrak{h}} > \frac{1}{2}$  and therefore Equation (27) is satisfied. In equilibrium, for all admissible initial conditions, we have  $\lambda(s^t) \in \Delta_{\epsilon}$  for all  $s^t$ .

To illustrate the construction suppose initial conditions are  $s_0 = 1$  and  $\theta^2(s_{-1}) = \theta^3(s_{-1}) = 1/2$ . The initial value of the welfare weights is given by  $\lambda(s_0) = (\lambda_{\mathfrak{h}}, \frac{1-\lambda_{\mathfrak{h}}}{2}, \frac{1-\lambda_{\mathfrak{h}}}{2})$ .



Since  $V^h(s, \lambda) > 0$  whenever  $s \neq h$  and, by construction, for all  $\lambda(s^t)$  along the equilibrium path  $V^h(h, \lambda(s^t)) = 0$ , the constructed transition function describes a Markov equilibrium. It is easy to check that this equilibrium does not have finite support. To see this, consider for instance a sequence of shocks for  $t = 1, 2, \dots$  with  $s_t = 1$  if  $t$  is odd and  $s_t = 2$  if  $t$  is even. It is easy to see that we must have  $\lambda^3(s^{t+1}) = \lambda^3(s^t) \frac{1-\lambda_h}{\lambda_h + \lambda^3(s^t)}$  and hence

$$\frac{1}{\lambda^3(s^{t+1})} = \frac{1}{1-\lambda_h} + \frac{\lambda_h}{1-\lambda_h} \frac{1}{\lambda^3(s^t)},$$

which always diverges since  $\lambda_h > \frac{1}{2}$ . In the process, as  $\lambda^3 \rightarrow 0$  it takes infinitely many values.

The economic reason for the non-existence of finite support equilibria in this example is as follows. Consider a sequence of shocks where shocks 1 and 2 alternate but shock 3 never occurs. Agent 3 has positive endowments only in shock 3 but he will obviously have savings. However along that sequence of shocks agent 3's consumption will converge asymptotically to zero. Since in each period a new "rich" agent is present, the returns to savings will be so low that agent 3 cannot guarantee a non-decreasing consumption stream. Along this sequence, his individual consumption will take infinitely many values. Note that the fact that shocks alternate is crucial for this argument. Instantaneous Negishi weights remain constant along a path where shock 2 is occurs every period.

## 5 Appendix

### 5.1 Proofs

#### Proof of Theorem 1

We first show that each Arrow–Debreu equilibrium allocation with limited pledgeability is also an equilibrium allocation with intermediaries. Given the equilibrium Arrow–Debreu prices  $(\rho(\sigma))_{\sigma \in \Sigma}$ , set the prices of the tree equal to  $q(s^t) = \frac{1}{\rho(s^t)} \sum_{\sigma \succ s^t} \rho(\sigma) d(\sigma)$  and the prices of the tree options as

$$q_{s_{t+1}}(s^t) = \frac{1}{\rho(s^t)} \rho(s^{t+1}) (q(s^{t+1}) + d(s^{t+1})) \quad (34)$$

for every  $s^t, s_{t+1}$ . It is then easy to see that the set of budget-feasible consumption levels is the same for the budget set in (IE2) and for the budget set defined by (3) and (4). For any  $h \in \mathcal{H}$ , given an arbitrary consumption sequence  $(c(\sigma))_{\sigma \in \Sigma}$  that satisfies (IE2), using (34) we get

$$\rho(s^t) \theta_{s_t}(s^{t-1}) (q(s^t) + d(s^t)) = \rho(s^t) (c(s^t) - e^h(s^t)) + \rho(s^t) \sum_{s_{t+1} \in \mathcal{S}} \theta_{s_{t+1}}(s^t) \frac{\rho(s^{t+1})}{\rho(s^t)} (q(s^{t+1}) + d(s_{t+1}))$$

for each  $s^t$  with  $t \geq 1$ . Substituting recursively for the second term on the right-hand side we obtain

$$\rho(s^t) \theta_{s_t}(s^{t-1}) (q(s^t) + d(s^t)) = \sum_{\sigma \succeq s^t} \rho(\sigma) (c(\sigma) - e^h(\sigma)) \geq 0,$$

that is, (4) holds. At the root node  $s_0$  we have

$$\theta^h(s_-)(q(s_0) + d(s_0)) = \sum_{\sigma \succeq s_0} \rho(\sigma)(c(\sigma) - e^h(\sigma)),$$

which is equivalent to (3). The reverse implication can be similarly shown.

We show next that an equilibrium allocation with intermediaries is a collateral constrained financial markets equilibrium allocation for a sufficiently rich asset structure  $\mathcal{J}$ , constructed as follows. In addition to the tree, at each node there are  $S - 1$  financial securities. Security  $j = 1, \dots, S - 1$  promises a zero payment in all states  $s = 1, \dots, j$  and a payment equal to 1 in the other states  $j + 1, \dots, S$ .

Given any equilibrium with intermediaries, with consumption allocation  $(\bar{c}^h(s^t))^{h \in \mathcal{H}}$ , prices  $\bar{q}_s(s^t)$ , and portfolios  $(\bar{\theta}_s^h(s^t))^{h \in \mathcal{H}}$  of the tree options, for all  $s, s^t$ , let

$$\bar{k} \equiv \sup_{s^t} \left( \sum_s \bar{q}_s(s^t) + d(s^t) \right) < \infty.$$

Consider the following specification of the collateral requirements of the  $J = S - 1$  financial securities:

$$\begin{aligned} k_{J+1}^1 &= \frac{1}{\bar{k}}, & k_j^1 &= 0 \text{ for all } j = 1, \dots, J \\ k_{j-1}^j &= 1, & k_i^j &= 0 \text{ for all } i \neq j - 1, \text{ for all } j = 2, \dots, J \end{aligned}$$

It suffices to show that a collateral constrained financial markets equilibrium exists with the same consumption allocation  $(\bar{c}^h(s^t))_{s^t \in \Sigma}^{h \in \mathcal{H}}$  and tree prices  $q(s^t) = \sum_s \bar{q}_s(s^t)$ . At this equilibrium, the payoffs of the financial securities are:

$$f_j(s^t) = \begin{cases} \frac{q(s^t) + d(s^t)}{\bar{k}} & \text{if } s^t > j \\ 0 & \text{otherwise.} \end{cases}$$

and the securities' prices

$$p_j(s^t) = \frac{1}{\bar{k}} \sum_{s=j+1}^S \bar{q}_s(s^t), \quad j = 1, \dots, S - 1.$$

Consider then the following portfolio holdings for each agent  $h$ , and each node  $s^t$ : set  $\theta^h(s^t) = \bar{\theta}_1^h(s^t)$ ,  $\phi_{1-}^h(s^t) = -\bar{k}\bar{\theta}_1^h(s^t)$ ,  $\phi_{1+}^h(s^t) = \bar{k}\bar{\theta}_2^h(s^t)$ ,  $\phi_{S-1+}^h(s^t) = \bar{k}\bar{\theta}_S^h(s^t)$  and for all other  $j = 2, \dots, J - 1$

$$\phi_{j+}^h(s^t) = \bar{k}\bar{\theta}_{j+1}^h(s^t), \quad \phi_{j+1-}^h(s^t) = -\bar{k}\bar{\theta}_{j+1}^h(s^t).$$

It is easy to verify that these portfolio holdings, together with the above prices of the tree and the securities, satisfy the collateral constraints, yield the consumption allocation  $(\bar{c}^h(s^t))_{s^t \in \Sigma}^{h \in \mathcal{H}}$  and are so the consumers' optimal choices.  $\square$

### Proof of Theorem 4

To construct equilibrium prices, set  $\rho(s_0) = 1$  and

$$\rho(s^t) = \rho(s^{t-1})\beta\pi(s_{t-1}, s_t) \max_{h \in \mathcal{H}} \frac{u^{h'}(s_t, \lambda(s^t))}{u^{h'}(s_{t-1}, \lambda(s^{t-1}))}.$$

Agent  $h$ 's first-order conditions for optimal consumption at some node  $s^t$  can be written as follows:

$$\begin{aligned} \beta^t \pi(s^t) u^{h'}(c^h(s^t), s_t) - \eta^h \rho(s^t) + \sum_{\sigma: s^t \succeq \sigma} \mu^h(\sigma) \rho(s^t) &= 0 \\ \mu^h(s^t) \sum_{\sigma \succeq s^t} \rho(\sigma) (c^h(\sigma) - e^h(\sigma)) &= 0, \end{aligned}$$

for multipliers  $\eta^h \geq 0$  (associated with the intertemporal budget constraint (3)) and  $\mu^h(\sigma) \geq 0$  (associated with the collateral constraint (4) at node  $\sigma$ ). It is standard to show that for summable and positive prices these conditions, together with the budget inequalities (3) and (4) are necessary and sufficient for a maximum (see, e.g., Dechert (1982)). But then at each  $s^t$  and for all agents  $h = 2, \dots, H$  we have

$$\frac{u^{1'}(c^1(s^t), s_t)}{u^{h'}(c^h(s^t), s_t)} = \frac{\eta^1 - \sum_{\sigma: s^t \succeq \sigma} \mu^1(\sigma)}{\eta^h - \sum_{\sigma: s^t \succeq \sigma} \mu^h(\sigma)},$$

which is equivalent to the first-order conditions of (22) if  $1/\lambda^h(\sigma) = \eta^h - \sum_{\sigma: s^t \succeq \sigma} \mu^h(\sigma)$  for all  $h, \sigma$ . It remains to be shown that the budget inequalities (4) as well as the market clearing conditions are satisfied. The latter is obvious, given (22). Regarding the budget inequalities we need to show that  $V^h(s^t, \lambda(s^t)) = 0$  if, and only if,  $\sum_{\sigma \succeq s^t} \rho(\sigma) (c^h(\sigma) - e^h(\sigma)) = 0$ . Since for any agent  $h \in \mathcal{H}$ ,  $\frac{\rho(s^{t+1})}{\rho(s^t)} = \frac{u^{h'}(s_{t+1}, \lambda(s^{t+1}))}{u^{h'}(s_t, \lambda(s^t))}$  whenever  $V^h(s_{t+1}, \lambda(s^{t+1})) \neq 0$ , this follows from the definition of  $V^h$ .  $\square$

### Proof of Theorem 5

To prove the result we need the following lemma.

LEMMA 1 *Suppose that for all  $s, h$ ,  $u^h(c, s)$  satisfies the property that  $cu^{h'}(c, s)$  is (weakly) increasing in  $c$ . For any  $\lambda_1, \lambda_2 \in \mathbb{R}_{++}^H$ , let  $\underline{\lambda}^h = \min[\lambda_1^h, \lambda_2^h]$ ,  $h = 1, \dots, H$ ; if  $\underline{\lambda} \neq \lambda_1$  and  $\underline{\lambda} \neq \lambda_2$ , for all  $h$  we have*

$$\begin{aligned} u^{h'}(s, \underline{\lambda})(C^h(s, \underline{\lambda}) - e^h(s)) &> \\ \min \left[ u^{h'}(s, \lambda^1)(C^h(s, \lambda^1) - e^h(s)), u^{h'}(s, \lambda^2)(C^h(s, \lambda^2) - e^h(s)) \right], \end{aligned} \tag{35}$$

and

$$\underline{\lambda}^h u^{h'}(s, \underline{\lambda}) < \min \left[ \lambda_1^h u^{h'}(s, \lambda^1), \lambda_2^h u^{h'}(s, \lambda^2) \right]. \tag{36}$$

**Proof.** Assume without loss of generality that  $\underline{\lambda}^h = \lambda_1^h \leq \lambda_2^h$ . Since  $\underline{\lambda} < \lambda_1$  we must have  $C^h(s, \underline{\lambda}) > C^h(s, \lambda_1)$  and so (35) follows from the assumption made on  $u^h(\cdot)$ . Concavity of  $u^h(\cdot)$  implies that  $\underline{\lambda}^h u^{h'}(s, \underline{\lambda}) < \lambda_1^h u^{h'}(s, \lambda_1)$ . To prove that  $\underline{\lambda}^h u^{h'}(s, \underline{\lambda}) \leq \lambda_2^h u^{h'}(s, \lambda_2)$  and therefore (36) holds, define  $\tilde{\lambda}$  by  $\tilde{\lambda}^h = \underline{\lambda}^h \leq \lambda_2^h$  and  $\tilde{\lambda}^i = \lambda_2^i$  for all  $i \neq h$ . Since  $\tilde{\lambda}^h u^{h'}(s, \tilde{\lambda}) = \tilde{\lambda}^i u^{i'}(s, \tilde{\lambda})$ , we must have  $\tilde{\lambda}^h u^{h'}(s, \tilde{\lambda}) \leq \lambda_2^h u^{h'}(s, \lambda_2)$ . Furthermore, we have  $\underline{\lambda}^h u^{h'}(s, \underline{\lambda}) \leq \tilde{\lambda}^h u^{h'}(s, \tilde{\lambda})$ . Also one of the last two inequalities must hold strictly, so that (36) follows.  $\square$

**Proof of the theorem.** Given an Arrow–Debreu equilibrium with limited pledgeability we can describe the equilibrium consumption allocation by the associated instantaneous weights  $\lambda(s^t)$ , which are uniquely determined if we normalize the initial weights  $\sum_{h \in \mathcal{H}} \lambda^h(s_0) = 1$  and for all  $t > 0$ , all  $s^t$ , require that  $\lambda(s^t) \geq \lambda(s^{t-1})$  and  $\lambda^h(s^t) = \lambda^h(s^{t-1})$  for at least one agent  $h \in \mathcal{H}$ .

It is a standard argument<sup>15</sup> to show that for each  $l \in \mathbb{R}_{++}^H$ ,  $\sum_h l^h = 1$  there exists an Arrow–Debreu equilibrium with limited pledgeability with  $\lambda(s_0) = l$  and some transfers at  $t = 0$ . To prove the existence of a Markov equilibrium it suffices to show that the equilibrium associated with any given, initial  $\lambda(s_0)$  is unique. Suppose to the contrary that there exist two equilibria with instantaneous weights  $\lambda_1, \lambda_2$  with  $\lambda_1(s_0) = \lambda_2(s_0)$  but  $\lambda_1(s^t) \neq \lambda_2(s^t)$  for some  $s^t$ . Define for each  $s^t$  and all  $h \in \mathcal{H}$ ,  $\underline{\lambda}^h(s^t) = \min(\lambda_1^h(s^t), \lambda_2^h(s^t))$ . Since both  $\lambda_1(s^t)$  and  $\lambda_2(s^t)$  describe equilibria we must have  $\underline{\lambda}(s^t) \neq \lambda_1(s^t)$  and  $\underline{\lambda}(s^t) \neq \lambda_2(s^t)$  for some  $s^t$ . Define recursively

$$v^h(s^t) = u^{h'}(s_t, \underline{\lambda})(C^h(s_t, \underline{\lambda}) - e^h(s^t)) + \beta \sum_{s'} \pi(s_t, s') v^h(s^{t+1}).$$

By Lemma 1 we have, for each  $s^t$ ,

$$u^{h'}(s_t, \underline{\lambda}(s^t))(C^h(s_t, \underline{\lambda}(s^t)) - e^h(s_t)) \geq \min \left[ u^{h'}(s_t, \lambda_1(s^t))(C^h(s_t, \lambda_1(s^t)) - e^h(s_t)), u^{h'}(s_t, \lambda_2(s^t))(C^h(s_t, \lambda_2(s^t)) - e^h(s_t)) \right]$$

and therefore  $v^h(s_0) \geq 0$ . By the first-order conditions of the Negishi maximization problem the terms  $\lambda^h u^{h'}(s, \lambda)$  are identical across all agents  $h$ , for all  $s$  and  $\lambda$ , hence the Arrow–Debreu prices in the two equilibria are given by  $\rho_i(s^t) = \beta^t \pi(s^t) \lambda_i^1(s^t) u^{1'}(s_t, \lambda_i(s^t))$  for  $i = 1, 2$ . Since the price of the tree is finite, prices are summable, each  $(\beta^t \pi(s^t) \lambda_i^h(s^t))$  is also summable and so is  $(\beta^t \pi(s^t) \underline{\lambda}^h(s^t))$ . Define  $\underline{\rho}(s^t) = \beta^t \pi(s^t) \underline{\lambda}^1(s^t) u^{1'}(s_t, \underline{\lambda}(s^t))$  for all  $s^t$ .

Lemma 1 also implies  $\underline{\lambda}^h(s^t) u^{h'}(\underline{\lambda}(s^t), s_t) \leq \min [\lambda_1^h(s^t) u^{h'}(s_t, \lambda_1(s^t)), \lambda_2^h(s^t) u^{h'}(s_t, \lambda_2(s^t))]$  for all  $s^t$ , with the inequality holding strict for some  $s^t$ . Therefore the allocation  $\underline{c}(s^t) =$

<sup>15</sup>Kubler and Schmedders (2003) show existence for all initial levels of tree-holdings, the same technique can then be applied to all initial Negishi-weights.

$C(s_t, \underline{\lambda}(s^t))$  would have to satisfy

$$\sum_{s^t} \underline{\rho}(s^t) \left( \sum_{h \in \mathcal{H}} (\underline{c}^h(s^t) - e^h(s_t)) - d(s_t) \right) > 0.$$

Since  $\underline{\rho}$  is summable this contradicts feasibility and the equilibrium must be unique.  $\square$

### Proof of Theorem 6

To prove existence we first show that there always exist  $S$  pairs  $(\underline{\lambda}^*(s), \bar{\lambda}^*(s))$  such that the solution to (28) and (29) satisfy (30).

LEMMA 2 *There always exists  $\underline{\lambda}^*(1), \dots, \bar{\lambda}^*(S) \in (0, 1)^{2S}$  solving (28), (29) and (30).*

To prove the lemma we need the following version of Brouwer's fixed-point theorem (see e.g. Zeidler (1985), Proposition 2.8).

LEMMA 3 *Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous function such that*

$$\inf_{\|x\|=r} \sum_{i=1}^n x_i f_i(x) \geq 0, \text{ for some } r > 0.$$

*Then  $f$  has at least one zero, i.e. there is a  $x$  with  $\|x\| \leq r$  and  $f(x) = 0$ .*

**Proof of Lemma 2** To show the existence of a solution of (28)-(30), we can substitute out all  $V^1(s, \bar{\lambda}(\tilde{s}))$  and  $V^2(s, \underline{\lambda}(\tilde{s}))$  as well as all  $V^1(s, \underline{\lambda}(\tilde{s}))$  and  $V^2(s, \bar{\lambda}(\tilde{s}))$  for  $s \neq \tilde{s}$ . We obtain a function  $f : (0, 1)^{2S} \rightarrow \mathbb{R}^{2S}$ , where each  $f_i$ ,  $i = 1, \dots, 2S$  is the weighted sum of terms of the form

$$u^{1'}(s, \bar{\lambda}(\tilde{s})) (C^1(s, \bar{\lambda}(\tilde{s})) - e^1(s)) \text{ and } u^{1'}(s, \underline{\lambda}(\tilde{s})) (C^1(s, \underline{\lambda}(\tilde{s})) - e^1(s)), \quad (37)$$

where the weights on the terms involving  $\bar{\lambda}(s)$  are positive (bounded away from zero) if, and only if, there is an  $s'$  with  $\underline{\lambda}(s') > \bar{\lambda}(s)$  (recall that  $\pi(s, s') > 0$  for all  $s, s'$ ). Similarly each  $f_i$  with  $i = S + 1, \dots, 2S$  is a weighted sum of terms

$$u^{2'}(s, \bar{\lambda}(\tilde{s})) (C^2(s, \bar{\lambda}(\tilde{s})) - e^2(s)) \text{ and } u^{2'}(s, \underline{\lambda}(\tilde{s})) (C^2(s, \underline{\lambda}(\tilde{s})) - e^2(s)), \quad (38)$$

where the weights on the terms involving  $\underline{\lambda}(s)$  are positive if, and only if, there is an  $s'$  with  $\bar{\lambda}(s') < \underline{\lambda}(s)$ . We obtain that  $f(\underline{\lambda}(1), \bar{\lambda}(1), \dots, \underline{\lambda}(S), \bar{\lambda}(S)) = 0$  precisely when there exists a solution to (28) and (29) with

$$V^1(s, \underline{\lambda}(s)) = V^2(s, \bar{\lambda}(s)) = 0 \text{ for all } s \in \mathcal{S}.$$

To prove the lemma it therefore suffices to show that for sufficiently small  $\epsilon > 0$ , there exist  $x \in [\epsilon, 1 - \epsilon]^{2S}$  with  $f(x) = 0$ . This result follows directly by applying Lemma 3 above to a slight modification of the function  $f(\cdot)$ . For  $x \in [\epsilon, 1 - \epsilon]^{2S}$ , set  $g_i(x) = f_i(x)$

for  $i = 1, \dots, S$  and  $g_i(x) = -f_i(x)$  for  $i = S + 1, \dots, 2S$  and Extend the function  $g$  to the whole domain  $\mathbb{R}^{2S}$  by setting it to a constant outside of  $[\epsilon, 1 - \epsilon]^{2S}$  which is chosen to ensure continuity. All one needs to prove is the appropriate boundary behavior. Clearly, as some  $\bar{\lambda}(s)$  is sufficiently large or some  $\underline{\lambda}(s)$  is sufficiently small, we have that  $\sum_i x_i g_i(x) < 0$  since each  $f_i(x)$  is bounded above. The key is to show that if  $\bar{\lambda}(s)$  is sufficiently small, or if  $\underline{\lambda}(s)$  is sufficiently large, we also have that some  $|g_i(x)|$  becomes arbitrarily large. To show this, note that in (38) the terms involving  $\underline{\lambda}(s)$  have positive (and bounded away from zero) weight whenever there is an  $s'$  with  $\bar{\lambda}(s') < \underline{\lambda}(s)$ . If this is the case, clearly some  $f_i(x)$ ,  $i = 1, \dots, S$  can be made arbitrarily small; if it is not the case, some  $\bar{\lambda}(s')$  becomes arbitrarily close to 1 and we are in the case above. The argument for  $\bar{\lambda}(s)$  is analogous.  $\square$

**Proof of the theorem.**

To prove the theorem it suffices to show that given  $\underline{\lambda}^*, \bar{\lambda}^*$  we can construct functions  $V^h(s, \cdot)$  that have a unique zero.

If all agents' relative risk aversion is below or equal to 1, the utility satisfies the gross substitute property and the result follows from the proof of Theorem 5, since we showed uniqueness of Markov equilibria. In order to prove the sufficiency of conditions 2. and 3., it is useful to define the following functions  $\tilde{V}^h(s, \lambda) = \frac{1}{u^{h'}(s, \lambda)} V^h(s, \lambda)$  for  $h = 1, 2$ . Clearly,  $V^h(s, \cdot)$  has a unique zero if, and only if,  $\tilde{V}^h(s, \cdot)$  does. We have

$$\begin{aligned} \tilde{V}^1(s, \lambda) &= C^1(s, \lambda) - e^1(s) + \beta \sum_{s': \lambda \in [\underline{\lambda}^*(s'), \bar{\lambda}^*(s')]} \pi(s, s') \frac{u^{1'}(s', \lambda)}{u^{1'}(s, \lambda)} \tilde{V}^1(s', \lambda) \\ &\quad + \beta \sum_{s': \lambda > \bar{\lambda}^*(s')} \pi(s, s') \frac{u^{1'}(s', \bar{\lambda}^*(s'))}{u^{1'}(s, \lambda)} \tilde{V}^1(s', \bar{\lambda}^*(s')) \end{aligned} \quad (39)$$

$$\begin{aligned} \tilde{V}^2(s, \lambda) &= C^2(s, \lambda) - e^2(s) + \beta \sum_{s': \lambda \in [\underline{\lambda}^*(s'), \bar{\lambda}^*(s')]} \pi(s, s') \frac{u^{2'}(s', \lambda)}{u^{2'}(s, \lambda)} \tilde{V}^2(s', \lambda) \\ &\quad + \beta \sum_{s': \lambda < \underline{\lambda}^*(s')} \pi(s, s') \frac{u^{2'}(s', \underline{\lambda}^*(s'))}{u^{2'}(s, \lambda)} \tilde{V}^2(s', \underline{\lambda}^*(s')) \end{aligned} \quad (40)$$

for all  $s$  with  $\lambda \in [\underline{\lambda}^*(s'), \bar{\lambda}^*(s')]$ .

Assume that agents have identical CRRA preferences. Then the term  $u^{h'}(s', \lambda)/u^{h'}(s, \lambda)$  is independent of  $\lambda$ . Therefore  $\lambda$  only enters  $\tilde{V}^1(s, \lambda)$  through the term  $C^1(s, \lambda)$ , which is clearly increasing in  $\lambda$ , and through the term  $\pi(s, s') \frac{u^{1'}(s', \bar{\lambda}^*(s'))}{u^{1'}(s, \lambda)} \tilde{V}^1(s', \bar{\lambda}^*(s'))$ , which is also increasing in  $\lambda$  since  $u^{1'}(s, \lambda)$  is decreasing in  $\lambda$ . Therefore the function  $\tilde{V}^1$  must be monotonically increasing and has a unique zero. Finally, if there is no aggregate uncertainty, the term  $u^{h'}(s', \lambda)/u^{h'}(s, \lambda)$  is simply equal to 1 and the same argument as for identical CRRA preferences shows the monotonicity of  $\tilde{V}^h(s, \cdot)$ .  $\square$

## 5.2 Further details on Section 3.3

**Derivation of Equation (14).** At each date  $t \geq 2$  in state 1 the price  $q_2(1)$  of the tree option paying in state 2 is determined by agent 1's first-order condition, since agent 2 is constrained in that state in his holdings of that asset.. On the other hand, in state 2 the consumption of both agents is the same as in the subsequent date in state 2, hence both agents are not constrained in their holdings of the tree options paying in state 2 and its price is determined by the first-order conditions of any of them. We obtain

$$q_2(1)u'(\mathfrak{h} + \varepsilon\mathfrak{d} - q_2(1)(1 - 2\varepsilon)) = \frac{\beta}{2}(q_1(2) + q_2(2) + \mathfrak{d})u'(\mathfrak{d}(1 - \varepsilon) + q_1(2)(1 - 2\varepsilon)) \quad (41)$$

$$q_2(2)u'(\mathfrak{d}(1 - \varepsilon) + q_1(2)(1 - 2\varepsilon)) = \frac{\beta}{2}(q_1(2) + q_2(2) + \mathfrak{d})u'(\mathfrak{d}(1 - \varepsilon) + q_1(2)(1 - 2\varepsilon)) \quad (42)$$

From (42) we obtain for  $t > 1$  that

$$q_1(1) = q_2(2) = \frac{\beta(q_1(2) + \mathfrak{d})}{2 - \beta} \quad (43)$$

and therefore

$$q_1(2) + q_2(2) + \mathfrak{d} = \frac{2(q_1(2) + \mathfrak{d})}{2 - \beta}.$$

Substituting this expression into equation (41) we obtain (14).

**Derivation of Equation (19).** At  $t = 1$  agent 1's first order conditions with respect to the tree option paying in state 2 still determine its price in state 1 since agent 2 is constrained in that state

$$q_2(1; 1)u'(\mathfrak{h} - q_1(1; 1)\varepsilon - q_2(1; 1)(1 - \varepsilon)) = \frac{\beta}{2}(q_1(2) + q_2(2) + \mathfrak{d})u'(\mathfrak{d}(1 - \varepsilon) + q_1(2)(1 - 2\varepsilon)). \quad (44)$$

The expression for the price change in (19) is obtained by differentiating (44) with respect to  $\varepsilon$ , evaluated at  $\varepsilon = 0$ , when  $q_2(1; 1)$ ,  $q_1(2)$  and  $q_1(1; 1)$ ,  $q_2(2)$  are at their steady state values before the intervention, given respectively by  $q_1^0(2)$  for the first two and by  $\frac{\beta(q_1^0(2) + \mathfrak{d})}{2 - \beta}$  for the last two. Noting that  $\left. \frac{dq_1(1; 1)}{d\varepsilon} \right|_{\varepsilon=0} = 0$ , since the price  $q_1(1; 1)$  also changes with  $\varepsilon$  but the expression is evaluated at  $\varepsilon = 0$ , we get (19).

**Derivation of Condition (21).** From equation (14) we find that  $q_1^0(2)$  can be written in terms of  $u'_{\mathfrak{h}}$  and  $u'_{\mathfrak{d}}$ ,

$$q_1^0(2) = \frac{\frac{\beta\mathfrak{d}}{2 - \beta}u'_{\mathfrak{d}}}{u'_{\mathfrak{h}} - \frac{\beta}{2 - \beta}u'_{\mathfrak{d}}}.$$

Substituting this expression and (15) into (20) we obtain that this condition is equivalent to  $\frac{A}{B} > 0$ , where

$$A = 4 \frac{u'_h}{u'_o} \left[ 1 + \mathfrak{d} \frac{u''_o}{u'_o} + \frac{u'_h}{u'_o} \right] - 2\beta \left[ 2 + (4 + 3\mathfrak{d} \frac{u''_o}{u'_o}) \frac{u'_h}{u'_o} + 2 \left( \frac{u'_h}{u'_o} \right)^2 + \mathfrak{d} \frac{u''_h}{u'_o} \right] + \beta^2 \left[ \frac{u'_o}{u'_h} + (3 + 2\mathfrak{d} \frac{u''_o}{u'_o}) \frac{u'_h}{u'_o} + \left( \frac{u'_h}{u'_o} \right)^2 + \mathfrak{d} \frac{\frac{u''_h}{u'_o}}{\left( \frac{u'_h}{u'_o} \right)^2} + (3 + \mathfrak{d} \frac{u''_h}{u'_o}) \right]$$

and

$$B = \left[ -2 \left( \frac{u'_h}{u'_o} \right)^2 + \beta \left( \frac{u'_h}{u'_o} + \left( \frac{u'_h}{u'_o} \right)^2 + \mathfrak{d} \frac{u''_h}{u'_o} \right) \right] \left[ 4 \left( \frac{u'_h}{u'_o} \right)^2 + \beta^2 (1 + (2 + \mathfrak{d} \frac{u''_o}{u'_o}) \frac{u'_h}{u'_o}) + \left( \frac{u'_h}{u'_o} \right)^2 + \mathfrak{d} \frac{u''_h}{u'_o} \right] - 2\beta \left[ (2 + \mathfrak{d} \frac{u''_o}{u'_o}) \frac{u'_h}{u'_o} + 2 \left( \frac{u'_h}{u'_o} \right)^2 + \mathfrak{d} \frac{u''_h}{u'_o} \right]$$

It can then be easily seen that, since all marginal utilities are evaluated at positive numbers, which remain bounded away from zero as  $\beta \rightarrow 0$ , for sufficiently small  $\beta$  we have  $\frac{A}{B} > 0$  if

$$1 + \mathfrak{d} \frac{u''(\mathfrak{d} + q_1^0(2))}{u'(\mathfrak{d} + q_1^0(2))} + \frac{u'(\mathfrak{h} - q_1^0(2))}{u'(\mathfrak{d} + q_1^0(2))} < 0.$$

## References

- [1] Alvarez, F. and U. J. Jermann (2000), “Efficiency, Equilibrium and Asset Pricing with Risk of Default,” *Econometrica* 68, 775–797.
- [2] Bewley, T. (1977): “The Permanent Income Hypothesis: A Theoretical Formulation,” *Journal of Economic Theory*, 16(2), 252-292.
- [3] Boldrin, M. and D. Levine (1991) “Growth Cycles and Market Crashes”, *Journal of Economic Theory*, 96, 13–39.
- [4] Bottazzi, J.-M., J. Luque and M. Pascoa (2012) “Securities Market Theory: Possession, Repo and Rehypothecation,” *Journal of Economic Theory*, 477-500.
- [5] Brunnermeier, M.K. and L.H. Pedersen (2009) “Market Liquidity and Funding Liquidity” *Review of Financial Studies*, 22, 2201-2238.
- [6] Chien, Y. and H. Lustig (2010), “The Market Price of Aggregate Risk and the Wealth Distribution,” *Review of Financial Studies*, 23, 1596-1650.
- [7] Chien, Y., H.L. Cole and H. Lustig (2011) “A Multiplier Approach to Understanding the Macro Implications of Household Finance,” *Review of Economic Studies*, 78, 199–234.
- [8] Cordoba, J.C. (2008) “U.S: Inequality: Debt Constraints or Incomplete Asset Markets”, *Journal of Monetary Economics*, 55, 350-364.



- [9] Dana, R.A. (1993), “Existence and Uniqueness of Equilibria when Preferences are Additively Separable”, *Econometrica* 61, 953–957.
- [10] Dechert, W.D., (1982), “Lagrange Multiplier in Infinite Horizon Discrete Time Optimal Control Models”, *Journal of Mathematical Economics*, 9, 285-302.
- [11] Geanakoplos, J. (1997): “Promises Promises,” in *The Economy as an Evolving Complex System II*, ed. by W. B. Arthur, S. N. Durlauf, and D. A. Lane. Addison Wesley Longman, Reading, MA, 285–320.
- [12] Geanakoplos, J., and W. Zame (2002): “Collateral and the Enforcement of Intertemporal Contracts,” discussion paper, Yale University.
- [13] Geanakoplos, J., and W. Zame (2009): “Collateralized Security Markets,” Discussion Paper, UCLA.
- [14] Golosov, M. and A. Tsyvinski (2007): “Optimal Taxation with Endogenous Insurance Markets” *Quarterly Journal of Economics* 122, 487-534.
- [15] Gromb, D. and D. Vayanos (2002): “Equilibrium and welfare in markets with financially constrained arbitrageurs,” *Journal of Financial Economics* 66, 361–407.
- [16] Heathcote, J., K. Storesletten and G. Violante (2009): “Quantitative Macroeconomics with Heterogeneous Households,” *Annual Review of Economics* 1, 319-354.
- [17] Heaton, J., and D.J. Lucas (1996): “Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing,” *Journal of Political Economy* 104, 443–487.
- [18] Judd, K.L., F. Kubler and K. Schmedders (2003): “Asset Trading Volume in Infinite-Horizon Economies with Dynamically Complete Markets and Heterogeneous Agents”, *Journal of Finance* , 63, 2203-2217.
- [19] Kehoe, T., and D. Levine (1993): “Debt Constrained Asset Markets,” *Review of Economic Studies* 60, 856–888.
- [20] Kehoe, T. and D. Levine (2001): “Liquidity Constrained vs. debt Constrained Markets,” *Econometrica* 69, 575-598.
- [21] Kilenthong, W.T. (2011): “Collateral premia and risk sharing under limited commitment”, *Economic Theory*, 46, 475–501.
- [22] Kilenthong, W.T. and R.M. Townsend (2011): “Market Based, Segregated Exchanges with Default Risk”, discussion paper, MIT.
- [23] Kiyotaki, N., and J. Moore (1997): “Credit Cycles,” *Journal of Political Economy* 105, 211–248.

- [24] Kubler, F. and K. Schmedders (2002): “Recursive Equilibria in Economies with Incomplete Markets”, *Macroeconomic Dynamics*, 6, 284-306.
- [25] Kubler, F. and K. Schmedders (2003): “Stationary Equilibria in Asset-Pricing Models with Incomplete Markets and Collateral,” *Econometrica* 71, 1767–1795.
- [26] Ligon, E., J.P. Thomas and T. Worrall (2002): “Mutual insurance and limited commitment: Theory and evidence from village economies”, *Review of Economic Studies*, 69(1), 209-244.
- [27] Lorenzoni, G. (2008), “Inefficient Credit Booms,” *Review of Economic Studies* 75, 809–833.
- [28] Lucas, R.E.Jr. (1978): “Asset Prices in an Exchange Economy,” *Econometrica* 46, 1429–1445.
- [29] Lustig, H. (2000): “Secured Lending and Asset Pricing,” mimeo, Stanford University.
- [30] Monnet, C. (2011): “Rehypothecation,” *Federal Reserve Bank of Philadelphia Business Review*
- [31] Negishi, T. (1960), “Welfare Economics and Existence of an Equilibrium for a Competitive Economy,” *Metroeconomica* 12, 92–97.
- [32] Santos, M.S. (2002), “On non-existence of Markov equilibria in competitive-market economies”, *Journal of Economic Theory*, 105:73-98.
- [33] Zeidler, E. (1986), *Nonlinear Functional Analysis and its Applications I*, Springer Verlag, New York.

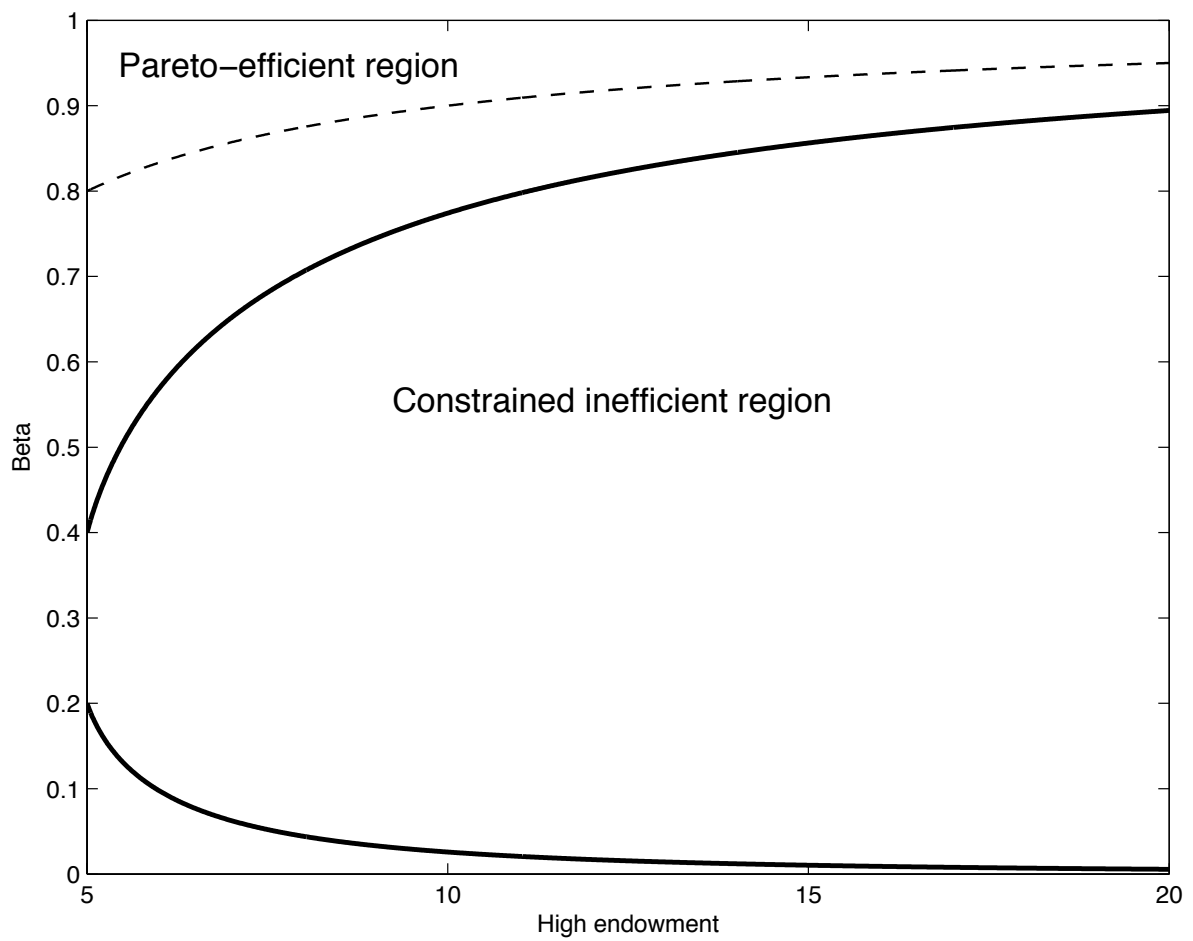


Figure 1: Constrained inefficient region

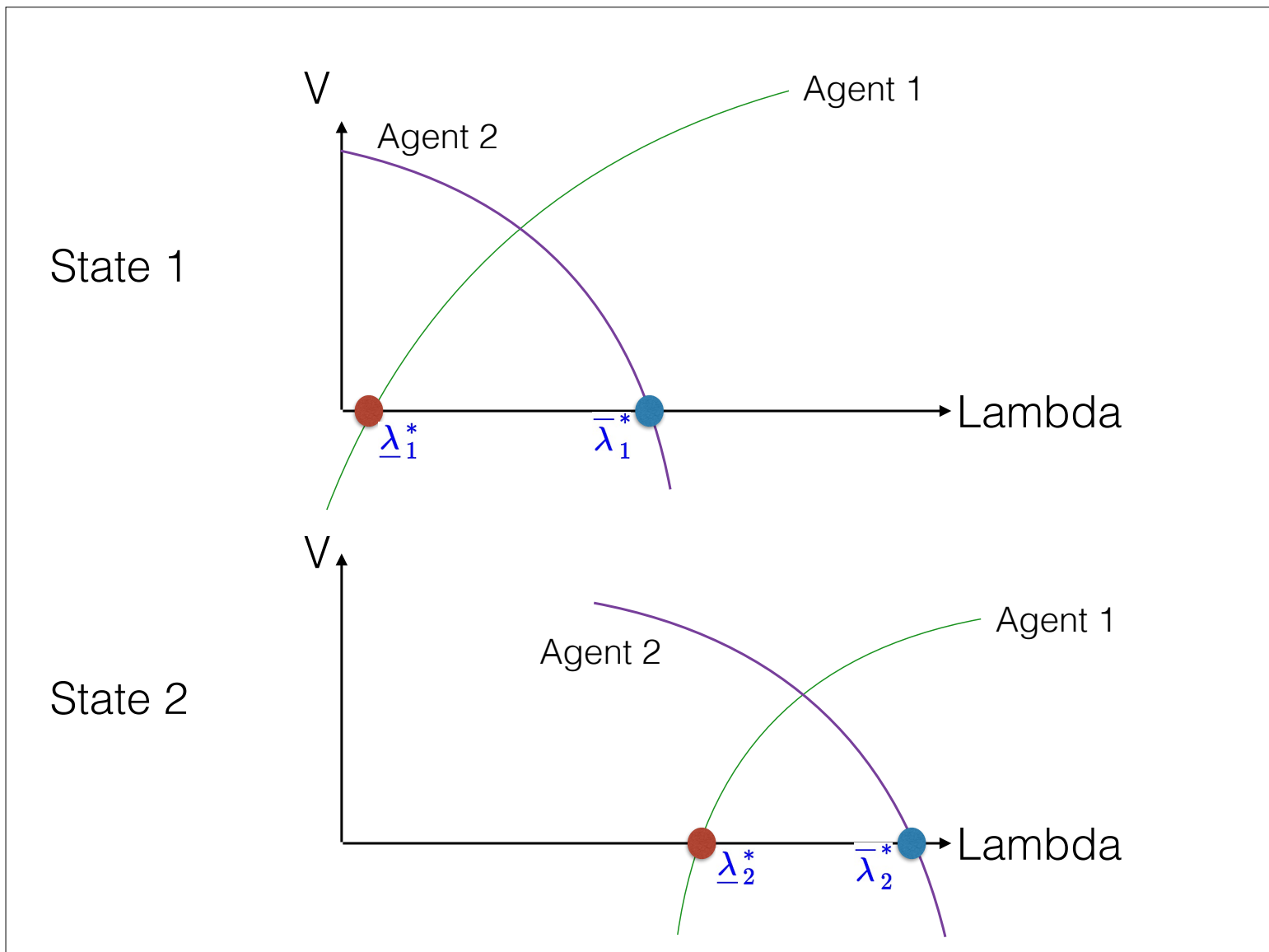


Figure 2: Finite support equilibrium