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Evolutionary Game Theory: Infinite and Finite Dynamics

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Contents

CHAPTER 1

Preliminaries

1.1 Summary

In this dissertation, I describe the philosophy of and techniques employed by evolutionary game theorists. I have considered various models used by evolutionary game theorists, and the dynamics of each model. I have compared the similarities and differences of each one, which are more viable in certain contexts, which fall apart under certain conditions and so on. Solution concepts for evolutionary games and their dynamical implications are discussed. The continuous replicator equation for two strategies is analysed rigorously using an argument based on bifurcations. The geometric point of view regarding ordinary differential equations is given a extensive coverage, including fixed points, periodic orbits, heteroclinic orbits, Lyapunov functions, manifolds, linearisation techniques and so on. Phase portraits and cobweb-staircase diagrams are produced using Matlab, the code for some of which are provided in an appendix at the end of this dissertation. There were some directions I wished to consider, but did not have the time nor scope: stochastic models, such as the Moran process [23] and Wright-Fisher model [15]; extensive form games, as covered in [6].

1.2 Dynamical Systems

The field of dynamical systems has wide application in the sciences to determine the state of some mathematical system across time. Ideas which are fundamental to dynamical systems are covered in this section, though it is recommended that a text such as [1], [10], or [31] is sought for a broader, more expansive coverage.

1.2.1 Fundamental ideas

In order to describe the phenomenon in question, an *initial value problem* is considered.[1] This comprises a system of *ordinary differential equations*, along with a set of *initial conditions* which specify the configuration of the system at some specific time t_0. There are two key features of a dynamical system: its *state variables* and its *rule for time evolution*. Using these two concepts, one can furnish all dynamical possibilities for the system under consideration.

We denote the *dependent state variables* as x_i: these are functions of the *independent* variable *time*, t. The vector $\mathbf{x} = (x_1, ..., x_n) : \mathbb{R} \mapsto \mathbb{R}^n$ represents a point in what is known as *phase space*: a space in which all possible states of a system are represented, with each possible state corresponding to one unique point in the phase space. We describe the rule for time evolution by a set of n functions, all of which have the state variables as their independent variables, as well as time. Denote them by $f_i : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}$. The initial value problem can then be written:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, \cdots, x_n, t), \qquad x_1(t_0) = X_1 \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
\dot{x}_n &= f_n(x_1, \cdots, x_n, t), \qquad x_n(t_0) = X_n.
\end{aligned}$$

Or, in a more compact form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{X}. \tag{1.2.1}$$

The standard dot notation is used to denote differentiation with respect to time.

Throughout this dissertation, we restrict our attention to *autonomous* systems of differential equations [10]. Such a system is a special case of (1.2.1), wherein the function $\mathbf{f}$ does not depend explicitly on time: that is, autonomous systems are of the form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{X}. \tag{1.2.2}$$

Note that in the case of autonomous systems, we can pick $t_0 = 0$ without loss of generality [10]. Solutions for this system are represented in phase space as integral curves.

[1]Other formulations can be used, for example PDEs: these shall not be considered here.

Definition 1.2.1 (Integral curves)
The curve $(x_1(t), ..., x_n(t)) \in \mathbb{R}^n$ *is an integral curve of equation* (1.2.2) *iff*

$$(\dot{x}_1(t), ..., \dot{x}_n(t)) = \mathbf{f}(x_1(t), ..., x_n(t))$$

for all t. *Thus the tangent to the integral curve at* $(X_1, ..., X_n)$ *is* $\mathbf{f}(X_1, ..., X_n)$.

Therefore, the vector field generated by $\mathbf{f}(\mathbf{x})$ specifies all 'velocity vectors' $\dot{\mathbf{x}}$ at all initial points. Using the notion of integral curves, we can define the *flow* as being all possible integral curves in phase space, each uniquely specified by an initial condition:

Definition 1.2.2 (Flow)
Consider the autonomous system of ordinary differential equations (1.2.2). *The solutions of this system define a flow,* $\varphi(\mathbf{X}, t)$ *such that* $\varphi(\mathbf{X}, t)$ *is the solution of the system at time* t *with initial value* $\mathbf{X}$. *Hence*

$$\frac{d}{dt}\varphi(\mathbf{X}, t) = f(\varphi(\mathbf{X}, t))$$

for all t *such that the solution through* $\mathbf{X}$ *exists and* $\varphi(\mathbf{X}, 0) = \mathbf{X}$.

As a result of this definition, the flow $\varphi : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ has the following properties:

Lemma 1.2.3 (Properties of the flow)

i) $\varphi(\mathbf{X}, 0) = \mathbf{X}$

ii) $\varphi(\mathbf{X}, t + s) = \varphi(\varphi(\mathbf{X}, t), s) = \varphi(\varphi(\mathbf{X}, s), t) = \varphi(\mathbf{X}, s + t)$

Due to the *existence and uniqueness theorem*, under sufficient continuity conditions for $\mathbf{f}(\mathbf{x})$, it is guaranteed that the solution curves never cross on some maximal time interval of existence.

Theorem 1.2.4 (Local existence and uniqueness)
Consider the initial value problem

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$$

with $\mathbf{f} \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ *with intial condition* $\mathbf{x}(t_0) = \mathbf{X}$. *There exists a* $t_+ > t_0$ *and a* $t_- < t_0$ *such that there is a unique solution* $\mathbf{x}(t) \in C^0((t_-, t_+), \mathbb{R}^n)$.

The stronger condition of Lipschitz continuity on $\mathbf{f}$ results in this maximal interval being $(t_-, t_+) = (-\infty, \infty)$ [6]. See Appendix A.1 for a proof of Theorem 1.2.4 in the infinite case.

Finding a system's *fixed points* is crucial in order to analyse its underlying dynamics.

Definition 1.2.5 (Fixed points)
A point $\mathbf{X}$ is a fixed point of the flow iff $\varphi(\mathbf{X}, t) = \mathbf{X}$ for all t.

A fixed point is an example of an *invariant set*.

Definition 1.2.6 (Invariant set)
A set M is invariant iff for all $\mathbf{X} \in M$, $\varphi(\mathbf{X}, t) \in M$ for all t. A set is forward (resp. backward) invariant if for all $\mathbf{X} \in M$, $\varphi(\mathbf{X}, t) \in M$ for all $t > 0$ (resp. $t < 0$).

As part of a fixed point's classification, one must be familiar with the concept of stability. Stability comes in several forms, though the notions of *Liapounov* ('start near, stay near'), *quasi-asymptotic* ('tends to eventually') and *asymptotic* ('tends to directly') stability are crucial.

Definition 1.2.7 (Liapounov stability)
A point $\mathbf{X}$ is Liapounov stable iff for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\mathbf{X} - \mathbf{Y}| < \delta$ then

$$|\varphi(\mathbf{X}, t) - \varphi(\mathbf{Y}, t)| < \varepsilon$$

for all $t \geq 0$.

Definition 1.2.8 (Quasi-asymptotic stability)
A point $\mathbf{X}$ is quasi-asymptotically stable iff there exists $\delta > 0$ such that if $|\mathbf{X} - \mathbf{Y}| < \delta$ then $|\varphi(\mathbf{X}, t) - \varphi(\mathbf{Y}, t)| \to 0$ as $t \to \infty$.

Note that this only describes the long term behaviour: in general, the solution can go all over the place before tending to the point.

Definition 1.2.9 (Asymptotic stability)
A point $\mathbf{X}$ is asymptotically stable iff it is both Liapounov stable and quasi-asymptotically stable.

Note that any point in the state space can have properties (1.2.7), (1.2.8) or (1.2.9), not just fixed points: however, they often go hand in hand with one another.

For expositional purposes, we shall now define the types of fixed point. We deliberately limit ourselves to the case of *planar*[2] *simple*[3] *linear* systems:

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{1.2.3}$$

where $A \in \mathbb{R}^{2\times 2}$. Systems of this nature are fundamental, and are extremely useful when dealing with the trickier – and more interesting – nonlinear cases.

Fixed points can be classified in many ways, however they broadly fall into two categories: *hyperbolic* and *non-hyperbolic*. For systems of the type (1.2.3), the only fixed point is the origin, $\mathbf{x} = (0, 0)$. In the hyperbolic case, it can be:

(i) a *regular node* (both stable or unstable)

(ii) a *singular node* (both stable or unstable)

(iii) a *degenerate node* (both stable or unstable)

(iv) a *focus* (both stable or unstable)

(v) a *saddle point*.

All other cases are considered *degenerate*; one example is a *centre*. The definitions of each fixed point shall be given in due course. We deal with nodes and saddle points immediately.

Definition 1.2.10 (Nodes)

The fixed point $\mathbf{x}^* = \mathbf{0}$ for system (1.2.3) *is a stable resp. unstable node if it is asymptotically stable for forward resp. backward time. Regular, singular, and degenerate nodes are all examples of nodes: as are foci.*

Definition 1.2.11 (Saddle point)

The fixed point $\mathbf{x}^* = \mathbf{0}$ for system (1.2.3) *is a saddle point if there exists some $\mathbf{X} \neq \mathbf{x}^*$ such that*

$$\varphi(\mathbf{X}, t) \to \mathbf{x}^*$$

as $t \to \infty$, and some $\mathbf{Y} \neq \mathbf{x}^*$ such that

$$\varphi(\mathbf{Y}, t) \to \mathbf{x}^*$$

as $t \to -\infty$.

[2] A two-dimensional system: the phase space is the x_1-x_2 plane.
[3] This means the defining matrix A has a non-zero determinant.

We can classify points in a more rigourous fashion by inspecting the matrix A of system (1.2.3). The eigenvalues and eigenvectors paint a complete picture when classifying fixed points in this case.

Therefore, if we denote the eigenvalues of A as λ_1 and λ_2, we have the following results:

- If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}^+$ then the fixed point is an unstable node.
- If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}^-$ then the fixed point is a stable node.
- If $\lambda_1 \neq \lambda_2$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \cdot \lambda_2 < 0$ then the fixed point is a saddle point.

If the eigenvalues $\lambda = a \pm ib$ are complex, the phase portrait depends on the values of a and b:

- If $a > 0$, then the fixed point is an unstable focus.
- If $a = 0$, then the fixed point is a centre.
- If $a < 0$, then the fixed point is a stable focus.
- If $b < 0$, then the trajectories spiral anti-clockwise around the origin.
- If $b > 0$, then the trajectories spiral clockwise around the origin.

In the case where the eigenvalues are not distinct, i.e. $\lambda_1 = \lambda_2$, there are two possibilities:

- If there are two linearly independent eigenvectors, then the fixed point is a singular node.
- If there is one linearly independent eigenvector, then the fixed point is a degenerate node.

It is a standard result in linear algebra that the sum of a square matrix's eigenvalues is equal to its trace and the product of said eigenvalues is its determinant [3]: in our case, we have that $T = trace(A) = \lambda_1 + \lambda_2$ and $D = det(A) = \lambda_1 \cdot \lambda_2$. Using this fact, we can summarise our results in the extremely useful diagram depicted in Figure 1.1.

To understand what follows, we define the solution of the general linear system $\dot{\mathbf{x}} = A\mathbf{x}$ as:

$$\mathbf{x}(t) = \varphi(\mathbf{X}, t) = \mathbf{X}e^{At},$$

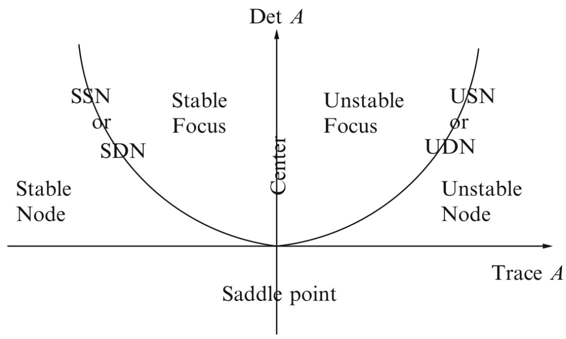


Figure 1.1: *A summary of fixed point classification for simple linear two dimensional systems [19]. The parabola has equation $T^2 - 4D = 0$. The abbreviations are as follows: SSN - Stable Singular Node; SDN - Stable Degenerate Node; USN - Unstable Singular Node; UDN - Unstable Degenerate Node.*

where we define the *matrix exponential*

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}.$$

This is a power series that converges for all real t.

Now consider a general nonlinear autonomous system. When classifying fixed points for such a system, the *Hartman-Grobman theorem* is invaluable:

Theorem 1.2.12 (Hartman-Grobman theorem)
If $\mathbf{x} = \mathbf{0}$ is a hyperbolic fixed point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ then there is a continuous invertible map, h, defined on some neighbourhood of $\mathbf{x} = \mathbf{0}$ which takes trajectories of the nonlinear flow to those of the linear flow $e^{tD\mathbf{f}(0)}$. This map can be chosen so that the parametrisation of trajectories by time is preserved.

The matrix $\mathbf{J} = D\mathbf{f}$ is the *Jacobian matrix*, defined element-wise as:

$$(\mathbf{J})_{ij} = (D\mathbf{f})_{ij} = \frac{\partial f_i}{\partial x_j}. \tag{1.2.4}$$

In Theorem 1.2.12, "hyperbolic" is the same as saying that none of the eigenvalues of the Jacobian matrix – when evaluated at the fixed point in question – are equal to zero. We use the Taylor expansion of **f** up to the linear term (often called 'linearisation') by computing the Jacobian matrix. After this, using Theorem (1.2.12) as justification, we can classify the fixed points of a system using similar eigenvalue techniques as before: this time, however, we inspect **J**.

Theorem 1.2.13 (Classification of hyperbolic fixed points)
Suppose we have a fixed point $\mathbf{x}^*$. Then, upon inspecting the eigenvalues λ_i ($i = 1, ..., n$) of the Jacobian matrix $D\mathbf{f}|_{\mathbf{x}=\mathbf{x}^*}$, we have the following possibilities[4]:

[4] "Sink" is another term for a stable node, while "source" is another term for an unstable node

- *If* $\mathrm{Re}(\lambda_i) < 0$ *for all* i, *then* $\mathbf{x}^*$ *is a sink: it is asymptotically stable in forwards time*
- *If* $\mathrm{Re}(\lambda_i) > 0$ *for all* i, *then* $\mathbf{x}^*$ *is a source: it is asymptotically stable in backwards time*
- *If* $\mathrm{Re}(\lambda_i) > 0$ *for some* i *and* $\mathrm{Re}(\lambda_i) < 0$ *for the remaining* $n-i$, *then* $\mathbf{x}^*$ *is a saddle: it is stable in some directions and unstable in others.*

We have that $\mathbb{R}^n$ can be represented as the direct sum of three subspaces denoted E^s, E^u, and E^c, which are defined as follows [31]:

$$\begin{aligned} E^s &= \text{span}\{e_1, ..., e_s\}, \\ E^u &= \text{span}\{e_{s+1}, ..., e_{s+u}\}, \qquad s+u+c=n, \\ E^c &= \text{span}\{e_{s+u+1}, ..., e_{s+u+e}\}, \end{aligned} \tag{1.2.5}$$

where $\{e_1, ..., e_s\}$ are the (generalised) eigenvetors of A (see (1.2.3), but for n dimensions) corresponding to the eigenvalues of A having *negative real part*, $\{e_{s+1}, ..., e_{s+u}\}$ are the (generalised) eigenvetors of A corresponding to the eigenvalues of A having *positive real part*, and $\{e_{s+u+1}, ..., e_{s+u+e}\}$ are the (generalised) eigenvetors of A corresponding to the eigenvalues of A having *zero real part*. E^s, E^u, and E^c are referred to as the linear stable, unstable, and centre invariant manifolds, respectively. Solutions starting in E^s approach the origin asymptotically as $t \to +\infty$ and solutions starting in E^u approach the origin asymptotically as $t \to -\infty$.

Furthermore, at a fixed point for a nonlinear flow, there exist nonlinear manifolds which are tangent to their linear counterparts. More precisely, we have the following.

Theorem 1.2.14 (Stable Manifold Theorem)
Suppose that the origin is a hyperbolic stationary point for $\dot{\mathbf{x}} = \mathbf{f}(x)$ *and* E^s *and* E^u *are the stable and unstable manifolds of the linear system* $\dot{x} = D\mathbf{f}(0)x$. *Then there exist local stable and unstable manifolds* $W^s_{loc}(0)$ *and* $W^u_{loc}(0)$ *of the same dimension as* E^s *and* E^u *respectively. These manifolds are (respectively) tangential to* E^s *and* E^u *at the origin and as smooth as the original function* $\mathbf{f}(x)$.

Similarly, there exists a local *centre manifold*, $W^c_{loc}(0)$, tangential to E^c at the origin, along which behaviour is neither asymptotically stable or unstable. The behaviour is determined by other means via normal forms, transformations and approximations: unlike the stable and unstable cases, the centre manifold is not unique.

So far, we have been dealing with continuous time, with generations blending continuously into one another. A deterministic evolution rule with discrete time and a continuous state space is called a map [22], $\mathbf{F} : \mathbb{R}^n \mapsto \mathbb{R}^n$, defined by iteration:

$$\mathbf{x}(t+1) = \mathbf{F}(\mathbf{x}(t)). \tag{1.2.6}$$

Many of the results regarding continuous flows extend to the theory of discrete maps, for instance, fixed points.

Definition 1.2.15 (Fixed points: maps)
A point $\mathbf{x}^*$ is a fixed point of the map $\mathbf{F}$ iff $\mathbf{F}(\mathbf{x}^*) = \mathbf{x}^*$ for all $t = 0, 1, ...$

One can also classify these fixed points in a similar fashion to the continuous case, with the necessary modifications.

Theorem 1.2.16 (Classification of hyperbolic fixed points for maps)
Suppose we have a fixed point $\mathbf{x}^*$. Then, upon inspecting the eigenvalues λ_i ($i = 1, ..., n$) of the Jacobian matrix $D\mathbf{F}|_{\mathbf{x}=\mathbf{x}^*}$, we have the following possibilities:

- *If $|\lambda_i| < 1$ for all i, then $\mathbf{x}^*$ is a sink: it is asymptotically stable in forwards time*
- *If $|\lambda_i| > 1$ for all i, then $\mathbf{x}^*$ is a source: it is asymptotically stable in backwards time*
- *If $|\lambda_i| > 1$ for some i and for the rest $|\lambda_i| < 1$, then $\mathbf{x}^*$ is a saddle: it is stable in some directions and unstable in others.*

We therefore see that the condition for hyperbolicity in the case of maps is that $|\lambda_i| \neq 1$ for all i. In other words, no eigenvalues of the Jacobian matrix when evaluated at the fixed point lie on the unit circle in the complex plane. This gives rise to non-hyperbolic fixed points.

It is important to note here that the field of dynamical systems is not so much concerned with finding closed form solutions to initial value problems. Instead, it is concerned with the qualitative behaviour of the system under consideration: in particular, its long-term behaviour. This point is stressed by virtually any relevant textbook, though a few examples are [1, 10, 19]. It is a much more fruitful procedure to consider the nature of the fixed points of the system and its phase portrait.

1.2.2 The simple pendulum

For illustrative purposes, we consider the example of an undamped pendulum in the case of small oscillations[5]. This is an example of *simple harmonic motion*. That is to say, it is governed

[5] This simply means that we can use the small angle approximation $\sin\theta \approx \theta$.

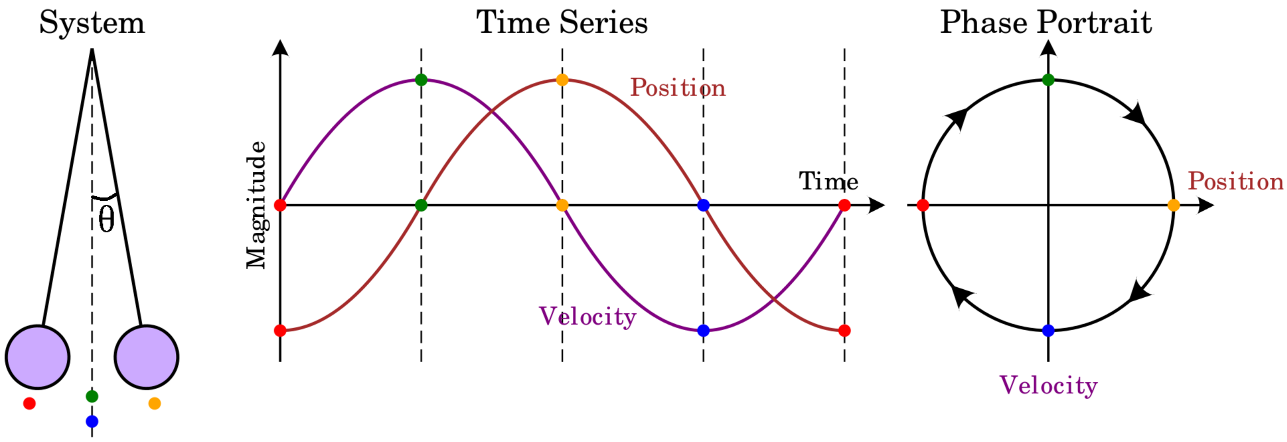


Figure 1.2: *Simple pendulum: depiction of the physical system (left); evolution of trajectory across time (middle); phase portrait (right).*

by the second-order differential equation:

$$\ddot{\theta} = -\theta, \tag{1.2.7}$$

where θ is the angle formed between the rod and the downward vertical (see Figure 1.2). We consider this to be our first state variable: call it $x_1 = \theta$. We can reduce (1.2.7) to a coupled system of first-order ODEs by introducing a second co-ordinate, $x_2 = \dot{x}_1$, i.e. the bob's angular velocity. It then becomes the system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1. \end{aligned} \tag{1.2.8}$$

The system (1.2.8) is a dynamical system. The state variables x_1 and x_2 specify its phase space – the (x_1, x_2)-plane – which consists of infinitely many closed integral curves, representing periodic orbits: one such orbit can be seen in Figure 1.2.

We can find the fixed point(s) of the dynamical system (1.2.8) by solving the (in this case, rather trivial) equation $\mathbf{f} = \mathbf{0}$. This yields only the origin, $\mathbf{x}^* = (x_1^*, x_2^*) = (0, 0)$, describing the setup where the pendulum hangs directly downward with no velocity: it remains as such for all time. To determine the stability of $\mathbf{x}^*$, we inspect the corresponding Jacobian matrix of (1.2.8). This is given by:

$$D\mathbf{f} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.2.9}$$

Since (1.2.9) is a matrix with constant entries, it is the same for all points in the (x_1, x_2)-plane: in particular, $\mathbf{x}^*$. Its eigenvalues are the complex conjugate pair $\lambda_\pm = \pm i$. This makes the origin an unstable *centre* (see subsection 1.2.1) with infinitely many periodic orbits organised in concentric circles travelling clockwise around it. Each periodic orbit can be specified by a unique initial condition: they "fill up" the (x_1, x_2)-plane.

1.3 Evolutionary Game Theory (EGT)

Evolutionary Game Theory (EGT) is a cross-fertilisation of the theories of classic evolutionary theory and classic game theory [7].

1.3.1 Fundamental ideas

Throughout this dissertation, the majority of games considered are in *normal form*[6], and are between two players; that is, each interaction is between two individuals. A normal form game is defined by a *pure payoff matrix*, as described below.

We consider an infinite population comprised of n types of individuals $E_1, E_2, ..., E_n$. In a biological context, this is often thought of as a single species exhibiting various behavioural traits, or '*phenotypes*'. These types have their behaviour represented by a *strategy*. There are two types of strategy; *pure* and *mixed*. The pure strategies are the extremes. There are N pure strategies $R_1, R_2, ..., R_N$ in a normal form game. The interactions among these pure strategies are ultimately what defines the game in question.

For each evolutionary game, there is a pure payoff matrix which describes the payoff given to each individual. It is defined element-wise by:

$$U = [u_{ij}], \tag{1.3.1}$$

where u_{ij} is the (constant) payoff received by a player using the pure strategy R_i versus a player using the pure strategy R_j. As such, $U \in \mathbb{R}^{N \times N}$.

Mixed strategies are the general case of a strategy; they are defined by an individual playing each pure strategy R_i with some fixed probability p_i. This therefore means that the set of all possible mixed strategies $\mathbf{p} \in \mathbb{R}^N$ is defined on the N-dimensional *simplex*:

$$S_N = \left\{ \mathbf{p} = (p_1, ..., p_N) \in \mathbb{R}^N \, : \, p_i \geq 0 \text{ and } \sum_{i=1}^{N} p_i = 1 \right\}. \tag{1.3.2}$$

[6]Note here that we are not talking of the "normal forms" associated with dynamical systems (see https://en.wikipedia.org/wiki/Normal_form_(dynamical_systems)). The reader should take care with this distinction going forward.

The simplex S_N is an $N-1$ dimensional structure embedded in an N-dimensional Euclidean space [24]. It is the generalisation of a triangle. Namely, S_3 is a planar triangle contained in $\mathbb{R}^3_+$; S_4 is a tetrahedron contained in $\mathbb{R}^4_+$, and so on.

It should be noted here that a pure strategy R_i is a special case of a mixed strategy: that is, when the i-th probability is equal to one while all remaining probabilities are zero. These are the standard unit vectors $\mathbf{e}_i$ and correspond to the corners of the simplex S_N.

In contrast, the interior of the simplex contains the *completely mixed* strategies, where all probabilities are strictly positive. Each type $E_1, E_2, ..., E_n$ has a corresponding strategy $\mathbf{p}^1, \mathbf{p}^2, ..., \mathbf{p}^n \in S_N$.

It is important to point out here that n and N need not be equal, as they represent different phenomena. In a normal form game, there is always a finite amount N of predetermined pure strategies. However, using the pure strategies, an infinite amount of mixed strategies can be produced: we only pick a select amount n to analyse. Therefore, the reader is advised to take care regarding this subtlety.

Using the pure payoff matrix U, one can assemble the general *mixed* payoff matrix:

$$A = [a_{ij}] = \left[\mathbf{p}^i \cdot \left(U \cdot \mathbf{p}^j\right)\right]. \tag{1.3.3}$$

Each element of this matrix represents the expected (since we are dealing with probabilities) payoff received by a type E_i individual versus a type E_j individual.

As before, the pure representation is a special case of the general mixed representation: one arrives back at the pure payoff matrix when considering pure strategies.

Suppose type E_i is playing the pure strategy $\mathbf{p}^i = \mathbf{e}_i$ against an individual E_j who is playing the pure strategy $\mathbf{p}^j = \mathbf{e}_j$. The payoff received by type E_i is therefore:

$$\mathbf{p}^i \cdot \left(U \cdot \mathbf{p}^j\right) = \mathbf{e}_i \cdot (U \cdot \mathbf{e}_j) = u_{ij},$$

as expected.

In the case of a symmetric normal form game involving two pure strategies, for the sake of brevity we often denote the pure payoff matrix by:

$$U = \begin{array}{c c} & \begin{array}{cc} \mathbf{e}_1 & \mathbf{e}_2 \end{array} \\ \begin{array}{c} \mathbf{e}_1 \\ \mathbf{e}_2 \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}. \tag{1.3.4}$$

We can represent the state of the population using the vector $\mathbf{x} = (x_1, ..., x_n)$, where each x_i represents the relative frequency of type E_i within the population. Since these are *relative* frequencies, we have that:

$$\sum_{i}^{n} x_i = 1.$$

This therefore means that the set of possible compositions of the population is defined on the n-dimensional simplex:

$$S_n = \left\{ \mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n \;:\; x_i \geq 0 \,\text{and}\, \sum_{i=1}^{n} x_i = 1 \right\}.$$

Note the non-negative frequencies: it does not make sense to have a negative frequency. The evolution of the composition of the population across time is central to evolutionary game theory.

When considering an evolutionary game, it is fundamental to consider the notion of *fitness*. The fitness f_i of a strategy is considered frequency-dependent, meaning we have that $f_i = f_i(\mathbf{x})$. In the case of normal form games, payoff is equated with fitness.

As part of the analysis, the notion of *average fitness* is considered, denoted by:

$$\bar{f}(\mathbf{x}) = \sum_{i} x_i f_i.$$

This is simply a weighted average based on the relative frequencies of each strategy within the population. As such, the average fitness also varies across time. The strategies which have fitness higher than that of the average fitness within the population ultimately come out on top. 'On top' can mean a few different things: this shall be covered in due course.

The key idea behind evolutionary game theory is to see which strategies rise to the top and outcompete their rivals; that is, which ones are the fittest from a reproductive standpoint.

1.3.2 The game of "Chicken"

For illustrative purposes, we consider the symmetric, normal form game of "Chicken". This example is taken from [24].

In the Chicken game, two cars head for each other at high speed. The loser is whoever chickens out first. The winner stays on the track. If neither driver chickens out, there is the substantial cost of a collision. Consider two pure strategies: R_1 means you go for it. R_2 means you chicken out after some time. The reward for winning is b, the cost of a collision is c. If both players decide to chicken out, then the chance that you will win is $1/2$. Hence the pure payoff matrix is:

$$\begin{array}{cc} & \begin{array}{cc} \mathbf{e}_1 & \mathbf{e}_2 \end{array} \\ U = \begin{array}{c} \mathbf{e}_1 \\ \mathbf{e}_2 \end{array} & \begin{pmatrix} -c & b \\ 0 & \frac{b}{2} \end{pmatrix}. \end{array} \tag{1.3.5}$$

Since there are two pure strategies, the dynamics of the normal form game are represented on the two-dimensional simplex S_2.

We could further our analysis by considering the mixed strategy[7] $\mathbf{p}^3 = (\frac{b}{b+2c}, \frac{2c}{b+2c})$. That is, we "go for it" with probability $\frac{b}{b+2c}$, and "chicken out" with probability $\frac{2c}{b+2c}$. Correspondingly, denote the two pure strategies mentioned above as $\mathbf{p}^1 = \mathbf{e}_1 = (1, 0)$ and $\mathbf{p}^2 = \mathbf{e}_2 = (0, 1)$. By using the definition described in equation (1.3.3), we can derive the mixed payoff matrix:

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{p}^1 & \mathbf{p}^2 & \mathbf{p}^3 \end{array} \\ A = \begin{array}{c} \mathbf{p}^1 \\ \mathbf{p}^2 \\ \mathbf{p}^3 \end{array} & \begin{pmatrix} -c & b & \frac{bc}{b+2c} \\ 0 & \frac{b}{2} & \frac{bc}{b+2c} \\ \frac{-bc}{b+2c} & \frac{b(b+c)}{b+2c} & \frac{bc}{b+2c} \end{pmatrix}. \end{array} \tag{1.3.6}$$

Using the game defined by (1.3.6), we could model how a population comprised of individuals playing one of the three available strategies evolves across time on the three-dimensional simplex S_3 as they compete with one another.

1.4 What EGT sets out to achieve

According to [7], "evolution is the central [organising] principle of biology." It governs the change of species over several thousands of generations in terms of their genotype and phenotype. The remarkable fact is that the underlying process of natural selection is almost entirely mathematical, relying on chance and the most effective phenotypes ("strategies" in this context) rising to the top and outcompeting their rivals by being passed down through generations by way of reproduction. Evolution rests upon three principles: replication, selection and mutation [24]. Replication appears in organisms which make copies of themselves via reproduction, both sexual and asexual, such as cells, viruses and multicellular organisms. The DNA and RNA produced by these individuals is replicated and passed on to the offspring. Selection occurs when these individuals compete with one another. According to

[7] As it turns out, this strategy is an ESS: more on this later.

[20], "fitness of particular phenotypes depend on the frequencies in the population." That is, fitness is frequency dependent. Certain phenotypes will be replicated more than others [14]: "strategies with high payoff will spread within the population." No organism is perfect, and so genetic "mistakes" known as mutations can occur.

Within this framework, the frequency-dependent selection among phenotypes which arises as a consequence of natural selection can be evaluated with the mathematical formulation of evolutionary game theory. As humans, it may be tempting to view animals competing for mates, territory, breeding grounds, or resources as a rational affair, wherein each individual has greedy motives and wishes to exploit one's opponent and reap the highest possible reward by predicting an opponent's move (much like humans do in chess: each strategy is a sequence of moves, planned long into the future). This is in fact indicative of the classical game theory, as formulated by John von Neumann [17]. In evolutionary game theory, this premise of a "rational agent" is disregarded: it is antithetical to the underlying methodology. Each phenotype exhibited by an individual is seen as a strategy. Individuals in a population interact with one another in the form of an evolutionary game. Different combinations of interactions result in each player receiving a fixed payoff. In the evolutionary setup, payoff is equated with fitness, where fitness is simply taken to mean reproductive success [5].

The key idea which evolutionary game theory seeks to flesh out is the constitution of a population of a species comprised of different phenotypes and how it evolves over time, based on intrinsic interactions among these strategies in the form of games. Evolutionary games can therefore be fully described by the following objects [4]: "the set of strategies, the possible states of the population, and the payoffs corresponding to different states of the population." With this setup, a meaningful analysis can be performed.

CHAPTER 2

Models

2.1 The Discrete Replicator Equation

The map $\mathbf{F} : S_n \mapsto S_n$ for discrete replicator dynamics is written:

$$F_i(\mathbf{x}) = \frac{\mathbf{e}_i \cdot A\mathbf{x}}{\mathbf{x} \cdot A\mathbf{x}} \cdot x_i. \tag{2.1.1}$$

Denoting the fitness of strategy $\mathbf{e}_i$ by $f_i = \mathbf{e}_i \cdot A\mathbf{x}$ and the average fitness of the population by $\bar{\mathbf{f}} = \mathbf{x} \cdot A\mathbf{x}$, the interpretation of the iterative process is clear. At each time step, the strategy's share within the population evolves according to its fitness compared to the average fitness of the population. If the ratio $f_i/\bar{\mathbf{f}} > 1$, the strategy increases in frequency: if the ratio $f_i/\bar{\mathbf{f}} < 1$, the strategy decreases in frequency. In general, there exist fixed points for the map (2.1.1). These occur at points for which the fitness of each strategy is equal, thus making all of the fitnesses equal to the average fitness of the population at some point $\mathbf{x}^*$, which ensure that $\mathbf{F}(\mathbf{x}^*) = \mathbf{x}^*$ across all time [31]. If, for each column of the payoff matrix (1.3.3), all entries are equal, this means that $f_i/\bar{\mathbf{f}} = 1$. In this case, the strategies are neutral variants to selection, making this – mathematically speaking – a degenerate case, since all points $\mathbf{x} \in S_n$ are fixed points in this case. This makes the dynamics uninteresting.

In the context of the discrete replicator equation, the payoff matrix must have strictly positive part: this is because the payoff is interpreted as the amount of offspring the individual produces in the next generation [6].Clearly, it is nonsense to assume to the contrary, since one cannot give birth to a negative amount of progeny! Furthermore, this assumption ensure that divisions by zero do not occur for (2.1.1).

In the simplest case of two pure strategies, since we have the constraint $x_1 + x_2 = 1$, it suffices to consider the dynamics of the variable $x = x_1 = 1 - x_2$. With this reduction, (2.1.1) becomes:

$$F(x) = \frac{(a-b)x^2 + bx}{(a-b-c+d)x^2 + (b+c-2d)x + d}, \tag{2.1.2}$$

where payoffs are given by the matrix (1.3.4). The equation (2.1.2) is an example of a one-dimensional map [10, 31] on the real interval $[0, 1]$.

2.2 The Continuous Replicator Equation

The most prolific method (see [1, 7, 13, 24, 30]) for analysing evolutionary games is performed by considering a population comprised of several different types. In the context of EGT, we typically think of this as a population of a species exhibiting various behavioural traits or *phenotypes*. These individuals are modelled as *replicators*; that is, they reproduce asexually and pass down a single copy of their genes to their progeny, e.g. bacteria [24]. It is worth mentioning that the assumption of asexual reproduction simplifies the analysis considerably, though there is a loss of realism [13]. Also, there is no mutation from generation to generation [30]: "strategies breed true." Each type is represented by a strategy, be it pure or mixed, where mixed means the type plays each pure strategy with some fixed probability for each interaction. Each game has its corresponding payoff (fitness) matrices, which describe the outcome of interactions of strategies within that game: with the pure payoff matrix, one can generate the mixed payoff matrix. This setup is formally known as a *normal form* game [13].

Under the basic tenet of Darwinism, the proliferation of a strategy depends on the difference between its fitness and the average fitness of the population [13]. When regarding infinite populations with generations that blend continuously into one another, we can then arrive at a system of nonlinear ordinary differential equations; the celebrated *continuous replicator equation* (seen in its infancy in [28]):

$$\dot{x}_i = x_i(f_i(\mathbf{x}) - \bar{f}(\mathbf{x})). \tag{2.2.1}$$

It is key to note here that the fitnesses are dependent on the composition of the population. This is a key element in the EGT formulation; fitness is *frequency-dependent* [7]. The fitness functions f_i are linear, and in fact are produced using the matrix (1.3.3) such that:

$$f_i(\mathbf{x}) = (A\mathbf{x})_i. \tag{2.2.2}$$

The fitness of a strategy depends on both the nature of the payoff matrix A and the composition of the population $\mathbf{x}$. We assume a population is sufficiently *well-mixed* such that interactions among types are random. That is to say, the probability of meeting a type E_i individual is equal to its relative frequency x_i in the population. This accounts for the nature of the fitness of a strategy; it is the payoff recieved by an individual for playing against other phenotypes multiplied by the probability of meeting that phenotype, then summed over all possibilities. It follows the structure of an expected value [29].

Under these assumptions, (2.2.1) becomes:

$$\dot{x}_i = x_i((A\mathbf{x})_i - \mathbf{x} \cdot A\mathbf{x}). \tag{2.2.3}$$

This is an autonomous system of nonlinear ordinary differential equations (see section 1.2), forming a dynamical system. A dynamical system [22] is “a rule for time evolution on a state space.” In this case, the state variables are the relative frequencies of the phenotypes. These form the state space; the simplex S_n. The rule for time evolution is the difference between the fitness of the phenotype and the average fitness of the population.

In the case of two strategies, since we have the constraint $x_1 + x_2 = 1$, we can make use of a new variable $x = x_1 = 1 - x_2$ and carry out the matrix products in (2.2.3), leaving us with a single ODE [6, 24]:

$$\dot{x} = x(1-x)((a-b-c+d)x + b - d). \tag{2.2.4}$$

For a formal derivation of (2.2.1), see [30]. One can also derive the continuous model from the discrete one: see Appendix A.2. Similar to the set of all possible mixed strategies, this dynamical system is defined on a simplex. However, the dynamical system is defined on the simplex S_n, **not necessarily** S_N; n and N need not be equal, as they represent different phenomena. N represents the number of possible pure strategies within the framework of the game under consideration, whereas n represents the number of types considered using these pure strategies as a basis to form mixed strategies. All trajectories starting on the simplex remain on the simplex, since under the flow induced by (2.2.3), S_n is a *forward invariant set* [10, 13].

The following lemma [30] is useful when it comes to analysing games under continuous replicator dynamics.

Lemma 2.2.1 *If some constant $v \in \mathbb{R}$ is added to all entries in some column of the payoff matrix, the replicator dynamics are unaffected.*

CHAPTER 3

Solution concepts

In game theory as a whole, the concept of a solution is a tricky one and comes in various hues and persuasions. There are a few, however, which have strong applications in evolutionary game theory. They often also have links with the dynamics of the model being considered, giving information regarding long term behaviour. This is why it is often more important to take a qualitative view of differential equations as opposed to a quantitative one [10]. We will cover the notions of Nash equilibria and Evolutionarily Stable strategies, along with their biological significance and relation to the underpinning dynamics. As said in [24] "Bioligists are interested in the concepts of strict Nash equilibrium or an evolutionarily stable strategy, because natural selection protects populations of such strategies against invasion by mutants."

3.1 Nash equilibrium

The concept of a Nash equilibrium represents a strategy which is a best reply to itself. That is to say [24], in a game between two people, "if both play a strategy that happens to be a Nash equilibrium, then neither person can deviate from this strategy and increase his payoff." Having been described [13] as "the most important notion in game theory", it makes sense to give it some exposition. We start by expressing the idea mathematically.

Definition 3.1.1 ((Strict) Nash equilibrium)

Consider a normal form game between two players with pure payoff matrix U. A strategy $\mathbf{q}$ is said to *be a strict Nash equilibrium if it is the unique best reply to itself, i.e. if*

$$\mathbf{q} \cdot U\mathbf{q} > \mathbf{p} \cdot U\mathbf{q}$$

holds for all strategies $\mathbf{p} \neq \mathbf{q}$.

Such a strategy is necessarily pure in this case [13].

We can relax the strict inequality given in Definition (3.1.1) to yield the standard (weaker) Nash equilibrium. These strategies $\hat{\mathbf{p}}$ are completely mixed, i.e. $\hat{\mathbf{p}} \in \text{int}S_N$. The completely mixed strategy $\hat{\mathbf{p}}$ is a Nash equilibrium if and only if its coordinates satisfy the system of linear equations [13]:

$$(U\mathbf{p})_1 = ... = (U\mathbf{p})_N \tag{3.1.1}$$

$$p_1 + \cdots + p_N = 1. \tag{3.1.2}$$

Every normal form game admits at least one Nash equilibrium [13]. What about a particular mixture of a population of individuals, as opposed to one individual adopting some specific strategy? For this idea, we have the notion of a Nash equilibrium for a state [13].

Definition 3.1.2 (Nash equilibrium (state))
Consider a normal form game between two players with mixed payoff matrix A*. The state of the population* $\hat{\mathbf{x}}$ *is a Nash equilibrium if*

$$\hat{\mathbf{x}} \cdot A\hat{\mathbf{x}} \geq \mathbf{x} \cdot A\hat{\mathbf{x}} \qquad \forall \mathbf{x} \in S_n$$

Recall that a state $\mathbf{x}$ refers to the composition of the population – comprised of n types – at a given point in time.

3.2 Evolutionarily Stable Strategy (ESS)

An evolutionarily stable strategy has its roots in biological thinking. It is concerned with the idea of a mutant invading a homogeneous population. In other words, imagine some population consisting purely of one phenotype. Suppose an infinitesimal quantity of a mutant phenotype is added to the population. Can this mutant not only invade this population, but also completely absorb it, i.e. replace all of the original resident population? A strategy which resists both the former condition (*invasion*) and the latter condition (*replacement*) is known as an *evolutionarily stable strategy*, or ESS for short. Originally developed by John Maynard Smith [20], it has close ties with the notion of a Nash equilibrium, but is in fact a stronger condition: stronger in the sense that a strategy being an ESS is a sufficient condition for it being a Nash equilibrium [24].

The rigorous definition in terms of payoffs and strategies is as follows [13].

Definition 3.2.1 (Evolutionarily Stable Strategy)

The strategy $\hat{\mathbf{p}} \in S_N$ will be said to be evolutionarily stable if for all $\mathbf{p} \in S_N$ with $\mathbf{p} \neq \hat{\mathbf{p}}$, the inequality

$$\mathbf{p} \cdot U(\varepsilon \mathbf{p} + (1-\varepsilon)\hat{\mathbf{p}}) < \hat{\mathbf{p}} \cdot U(\varepsilon \mathbf{p} + (1-\varepsilon)\hat{\mathbf{p}}) \tag{3.2.1}$$

holds for all $\varepsilon > 0$ that are sufficiently small, i.e. smaller than some appropriate invasion barrier $\bar{\varepsilon}(\mathbf{p}) > 0$.

The inequality (3.2.1) can be written as:

$$(1-\varepsilon)(\hat{\mathbf{p}} \cdot U\hat{\mathbf{p}} - \mathbf{p} \cdot U\hat{\mathbf{p}}) + \varepsilon(\hat{\mathbf{p}} \cdot U\mathbf{p} - \mathbf{p} \cdot U\mathbf{p}) > 0. \tag{3.2.2}$$

This therefore means $\hat{\mathbf{p}}$ is an ESS if and only if the following conditions are satisfied:

(a) *equilibrium condition*

$$\mathbf{p} \cdot U\hat{\mathbf{p}} \leq \hat{\mathbf{p}} \cdot U\hat{\mathbf{p}} \quad \text{for all} \quad \mathbf{p} \in S_N \tag{3.2.3}$$

(b) *stability condition*

$$\text{if} \quad \mathbf{p} \neq \hat{\mathbf{p}} \quad \text{and} \quad \mathbf{p} \cdot U\hat{\mathbf{p}} = \hat{\mathbf{p}} \cdot U\hat{\mathbf{p}}, \quad \text{then} \quad \mathbf{p} \cdot U\mathbf{p} < \hat{\mathbf{p}} \cdot U\mathbf{p}. \tag{3.2.4}$$

Therefore, we have the following useful criterion to check for an ESS.

Theorem 3.2.2

The strategy $\hat{\mathbf{p}} \in S_N$ is an ESS if and only if

$$\hat{\mathbf{p}} \cdot U\mathbf{q} > \mathbf{q} \cdot U\mathbf{q} \tag{3.2.5}$$

for all $\mathbf{q} \neq \hat{\mathbf{p}}$ in some neighbourhood of $\hat{\mathbf{p}}$ in S_N.

We also have the concept of an *evolutionarily stable state*:

Definition 3.2.3 (Evolutionarily Stable State)

Consider a game and state which satisfies Definition 3.1.2. Then, if $\hat{\mathbf{x}}$ satisfies the further condition

$$\hat{\mathbf{x}} \cdot A\mathbf{x} > \mathbf{x} \cdot A\mathbf{x}$$

for all $\mathbf{x} \neq \hat{\mathbf{x}}$ in a neighbourhood of $\hat{\mathbf{x}}$, we say that $\hat{\mathbf{x}}$ is an evolutionarily stable state.

CHAPTER 4

Two Strategy Games

Symmetric, normal form games with two strategies can be divided into three main categories: Prisoner's Dilemma, Hawk-Dove and Co-ordination Class. By inspecting the difference in payoff for all strategies, one can determine the existence of Nash Equilibria and Evolutionarily Stable strategies. The corresponding replicator equations, in both their continuous and discrete forms, give an intuitive geometrical picture of the dynamics of each game. It is also shown why they are such good choices by means of example.

4.1 The Prisoner's Dilemma

The Prisoner's Dilemma game describes a scenario involving cooperation and defection. If both individuals cooperate, they both end up winners. If both defect, they both end up losers. One would then think that players would opt to cooperate: however, this is often not the case, as the case where one party cooperates and the other defects yields the defector a payoff, while the cooperator gets nothing. This is the dilemma: despite mutual cooperation being the best strategy in the long run, defection always yields the higher payoff, no matter what one's opponent chooses. With this in mind, both players assume that their opponent will defect, and so do the same: had they both chosen to cooperate, they would come out with a higher overall payoff [24].

The name "The Prisoner's Dilemma" was coined in 1950 by Albert Tucker [25], who used the analogy of two partners in crime facing prison sentences to illustrate the underlying philosophy. A summation of the scenario can be found in [25]:

> *"Two individuals are being held in a prison in separate, isolated cells and they have each been told that, if they defect (from their partner in crime) by confessing, they will go free while the other will get five years in prison. However, if they cooperate (with their partner in crime) by not confessing, there is enough evidence to send them both to prison for two years. If they both defect, by confessing, they will each get four years. The potential exists to go free. This occurs if the focal individual defects (confesses) while its partner cooperates (remains mum)."*

If we generalise the payoffs, they can be classified by four distinct outcomes, namely: Reward for mutual cooperation (R); Temptation to defect (T); Punishment for mutual defection (P); Sucker's payoff (S). These satisfy the conditions $T > R > P > S$ and, when considering the iterated version of the game [24] (where several bouts are undertaken between the same two players over a finite period of time), $R > (T + P)/2$. This latter condition ensures the feasibility of a purely cooperative strategy.

The payoff matrix is then given by:

$$U = \begin{array}{c c} & \begin{array}{cc} \mathbf{e}_1 & \mathbf{e}_2 \end{array} \\ \begin{array}{c} \mathbf{e}_1 \\ \mathbf{e}_2 \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \equiv \begin{array}{c c} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} R & S \\ T & P \end{pmatrix} \end{array}. \tag{4.1.1}$$

We have that:

$$\mathbf{e}_2 \cdot U\mathbf{e}_2 = P > S = \mathbf{e}_1 \cdot U\mathbf{e}_2,$$

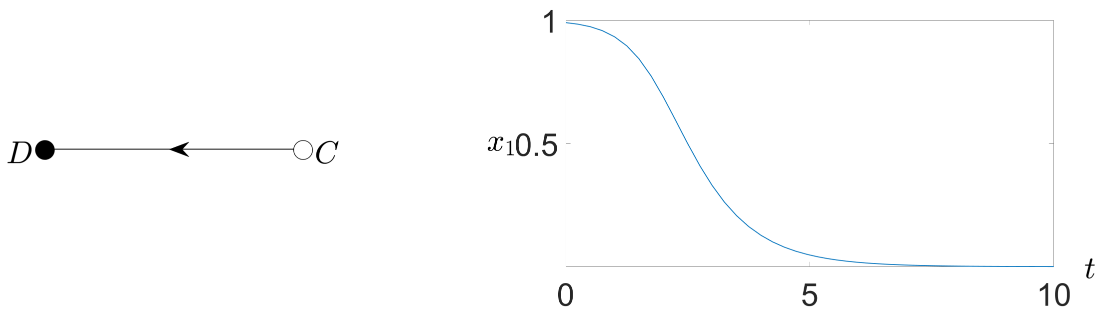
meaning pure defection is a (unique) strict Nash equilibrium, and hence an ESS. The Prisoner's Dilemma game is therefore typified by the relation $(a - c)(d - b) < 0$.

4.1.1 The Continuous Replicator Equation

For the simple Prisoner's Dilemma game, we consider the replicator equation (2.2.3) where A is given by (4.1.1).

Since we are considering the dynamics on the two dimensional simplex, we can use (2.2.4) which, in terms of the Prisoner's Dilemma game, can be expressed as:

$$\dot{x}_1 = x_1(1 - x_1)((R - S - T + P)x_1 + S - P). \tag{4.1.2}$$



(a) *Phase portrait* (b) *Typical trajectory*

Figure 4.1: *Dynamics (generated in MATLAB) of the* 2×2 *PD game in symmetric normal form, with* $T = 5$, $R = 3$, $P = 1$, $S = 0$, *as in Axelrod (1984). The blue trajectory corresponds to the initial condition* $x_1 = 0.99$.

There are two fixed points for (4.2.2): the vertices $x_1 = 0$ and $x_1 = 1$. In order for the unique interior fixed point to exist, we must find values which satisfy the equation:

$$(R - S - T + P)x_1 + S - P = 0.$$

However, we have that $T > R > P > S$, meaning:

$$(P - S + R - T)x_1 + S - P < 0,$$

for all $0 < x_1 < 1$. Therefore, the Prisoner's Dilemma has no interior fixed point under replicator dynamics.

We can inspect the Jacobian matrix of (4.1.2) to determine the stability of the fixed points rigorously.

When evaluated at $x_1 = 0$, the Jacobian matrix has a single eigenvalue given by $\lambda = S - P$. Since we have that $P > S$, this means that $\lambda < 0$, making $x_1 = 0$ a sink: all interior initial conditions converge to it asymptotically in forwards time. More formally, we say that $x_1 = 0$ has a *basin of attraction* [10] given by $x_1 \in (0, 1)$.

When evaluated at $x_1 = 1$, the Jacobian matrix has a single eigenvalue given by $\lambda = T - R$. Since we have that $T > R$, this means that $\lambda > 0$, making $x_1 = 1$ a source: all interior initial conditions converge to it asymptotically in backwards time. The dynamics are depicted in Figure 4.1.

A more intuitive justification can be gleaned by simply inspecting the sign of $\dot{x}_1$ for varying x_1. According to (4.1.2), we have that $\dot{x}_1 < 0$ for all $x_1 \in (0, 1)$: there is motion away from "Co-operate" and hence towards "Defect", since $\dot{x}_1 = -\dot{x}_2$. This therefore means that the temptation to defect is too great for any interior initial population, and thus they are all doomed to become defectors.

4.1.2 The Discrete Replicator Equation

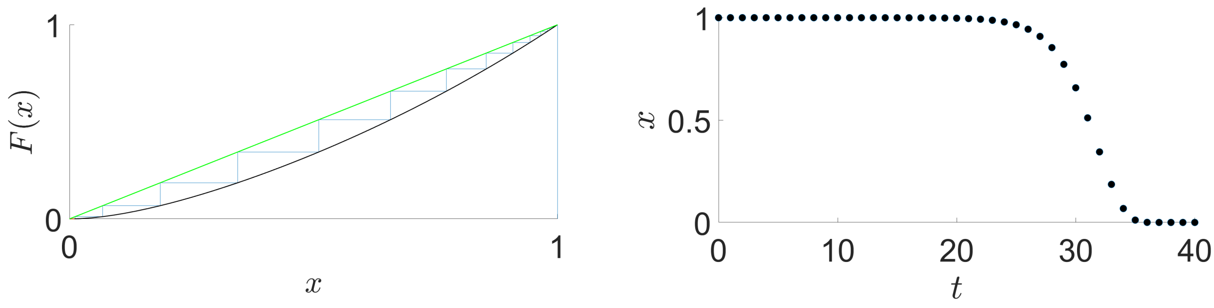
We perform a similar analysis using the reduced discrete replicator equation given by (2.1.2). In terms of the Prisoner's Dilemma, this is given by:

$$F(x) = \frac{(R-S)x^2 + Sx}{(R-S-T+P)x^2 + (S+T-2P)x + P}. \tag{4.1.3}$$

The fixed points which lie on the simplex S_2 are given by the vertices $x = 1$ and $x = 0$, since $F(0) = 0$ and $F(1) = 1$. To determine the stability of the endpoints, we proceed thusly.

The eigenvalue associated with $DF|_{x=0}$ is $S/P < 1$, since $S < P$. Therefore, $x = 0$ is asymptotically stable. The eigenvalue associated with $DF|_{x=1}$ is $T/R > 1$, since $T > R$. Therefore, $x = 0$ is linearly unstable. See Figure 4.2 for the dynamics.

One can see that the continuous and discrete replicator equation are in agreement in the case of the Prisoner's Dilemma.



(a) *Cobweb-staircase diagram* (b) *Typical orbit*

Figure 4.2: *Discrete dynamics (generated in MATLAB) of the* 2×2 *PD game in symmetric normal form, with* $T = 5$, $R = 3$, $P = 1$, $S = 0$, *as in Axelrod (1984).*

4.2 The Hawk-Dove game

The Hawk-Dove game describes dynamics where those adopting a rarer strategy can take over the population: that is to say, neither pure strategy is an ESS [13]. Originally discussed by John Maynard Smith [20], the setup refers to a pairwise contest between animals for a resource of finite fitness value, thus leading to greater reproductive success: this can be thought of as a piece of food, mating rights or territory, for instance. The two pure strategies are Hawk and Dove. Hawk types will always fight, no matter what. In this way, they run the risk of injury. If they win, they gain the resource of value V. If they lose, they are penalised with a cost of C. Doves will display: if they are confronted by a Hawk, they will flee, yielding them a payoff of 0. But, if confronted by another Dove type, they will split the resource. If we make the assumption that likewise interaction among Hawks yields a chance of victory of $1/2$, the outcome of the game is described by the following payoff matrix:

$$U = \begin{array}{c c} & \begin{array}{cc} \mathbf{e}_1 & \mathbf{e}_2 \end{array} \\ \begin{array}{c} \mathbf{e}_1 \\ \mathbf{e}_2 \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \equiv \begin{array}{c c} & \begin{array}{cc} H & D \end{array} \\ \begin{array}{c} H \\ D \end{array} & \begin{pmatrix} \frac{1}{2}(V-C) & V \\ 0 & V/2 \end{pmatrix} \end{array}. \tag{4.2.1}$$

H is an ESS if $\frac{1}{2}(V-C) > 0$, or $V > C$. In other words, if it is worth risking injury to obtain the resource, H is the only sensible strategy [20]. This structure, with pure Hawk as the ESS, is identical to that of the Prisoner's Dilemma.

In contrast, the Hawk-Dove game is typified by the relations $a < c$ and $d < b$ (see (4.2.1)): it makes sense to do the opposite to one's opponent, since the best response to $\mathbf{e}_1$ is $\mathbf{e}_2$ and vice versa. Therefore, we restrict our attention to the case where $0 < V < C$, ensuring that neither pure strategy is a strict Nash equilibrium.

We can consider the set of equations given by (3.1.1), (3.1.2) when applied to (4.2.1). This has the unique solution $\hat{p}_1 = V/C$, $\hat{p}_2 = 1 - V/C$, making the strategy $\hat{\mathbf{p}} = (\frac{V}{C}, 1 - \frac{V}{C})$ the unique Nash equilibrium.

Using Theorem (3.2.2) and the fact that $p_2 = 1 - p_1$ as a basis, we have that:

$$\hat{\mathbf{p}} \cdot U\mathbf{p} - \mathbf{p} \cdot U\mathbf{p} = \frac{V(C + V - 2Cp_1)}{2C} - \frac{V - Cp_1^2}{2} = \frac{1}{2C}(V - Cp_1)^2 > 0,$$

which holds for all $p_1 \neq V/C$. This therefore means that the strategy $\hat{\mathbf{p}}$ is an ESS as well.

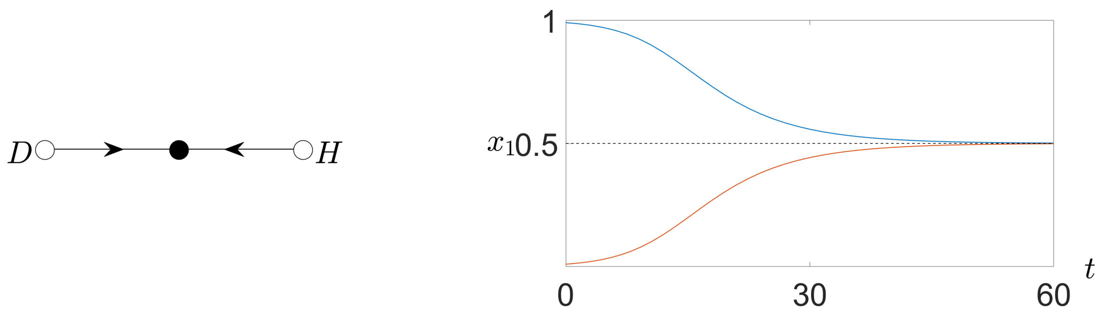
It is interesting to note that the names of the strategies are not technically apt in biological terms. Maynard Smith was using the vernacular employed during the Vietnamese war [21, 26]. In reality, doves are not so peaceable. In fact, they will fight to the death when confined in a cage [13, 24].

4.2.1 The Continuous Replicator Equation

For the simple Hawk-Dove game, we consider the replicator equation (2.2.3) where A is given by (4.2.1).

Since we are considering the dynamics on the two dimensional simplex, we can use (2.2.4) which, in terms of the Hawk-Dove game, can be expressed as:

$$\dot{x}_1 = \frac{1}{2}x_1(1 - x_1)(V - Cx_1). \tag{4.2.2}$$



(a) *Phase portrait* (b) *Typical trajectories*

Figure 4.3: *Dynamics (generated in MATLAB: see Appendices B.1 and B.2) of the* 2×2 *HD game in symmetric normal form, with* $V = 0.5$ *and* $C = 1$*. The dashed line indicates the evolutionarily stable state, given by* $x_1 = x_1^* = V/C$*. The blue and orange trajectories correspond to the initial conditions* $x_1 = 0.99$ *and* $x_1 = 0.01$*, respectively.*

There are three fixed points for (4.2.2): the interior fixed point $x_1^* = V/C$ and the vertices of the simplex, $x_1 = 0$ and $x_1 = 1$. The latter is to be expected, since the vertices of the simplex are always fixed points under continuous replicator dynamics [24]. We can inspect the Jacobian matrix of (4.2.2) to determine the stability of the fixed points rigorously.

When evaluated at $x_1 = 0$, the Jacobian matrix has a single eigenvalue given by $\lambda = V/2$. Since we have that $V > 0$, this means that $\lambda > 0$, making $x_1 = 0$ a source: initial conditions $x_1 \in (0, x_1^*)$ converge to it asymptotically in backwards time.

When evaluated at $x_1 = 1$, the Jacobian matrix has a single eigenvalue given by $\lambda = \frac{1}{2}(C - V)$. Since we have that $V > C > 0$, this means that $\lambda > 0$, meaning $x_1 = 1$ is also a source: initial conditions $x_1 \in (x_1^*, 1)$ converge to it asymptotically in backwards time.

When evaluated at $x_1 = x_1^*$, the Jacobian matrix has a single eigenvalue given by $\lambda = -\frac{1}{2C}V(C - V)$. In this case, $\lambda < 0$, making $x_1 = x_1^*$ a sink: all interior initial conditions (other than $x_1 = x_1^*$ itself) converge to it asymptotically in forwards time. This is in agreement with the result that any asymptotically stable fixed point for a two player game is also an ESS under (2.2.3), and vice versa [6]. The dynamics are depicted in Figure 4.3.

A more intuitive justification can be gleaned by simply inspecting the sign of $\dot{x}_1$ for varying x_1. According to (4.2.2), we have that $\dot{x}_1 > 0$ for $x_1 < x_1^*$ and also that $\dot{x}_1 < 0$ for $x_1 > x_1^*$. In plain English, this means that initial conditions starting nearer to "all Dove" evolve towards "all Hawk" (and vice versa), converging to the fixed point $x_1 = x_1^*$.

4.2.2 The Discrete Replicator Equation

We perform a similar analysis using the reduced discrete replicator equation given by (2.1.2). This is given by:

$$F(x) = \frac{(a-b)x^2 + bx}{(a-b-c+d)x^2 + (b+c-2d)x + d}. \tag{4.2.3}$$

The fixed points which lie on the simplex S_2 are given by the vertices $x = 1$ and $x = 0$, since $F(0) = 0$ and $F(1) = 1$. Furthermore, using similar logic, there is an interior fixed point given by $x = x^* = (d-b)/(a-c+d-b)$. To determine the stability of the endpoints, we proceed thusly.

The eigenvalue associated with $DF|_{x=0}$ is $b/d > 1$, since $d < b$. Therefore, $x = 0$ is linearly unstable. The eigenvalue associated with $DF|_{x=1}$ is $c/a > 1$, since $c > a$. Therefore, $x = 0$ is linearly unstable. The eigenvalue associated with $DF|_{x=x^*}$ is $-(ab - 2ad + cd)/(ad - bc) < 1$ for similar reasons. Hence, $x = x^*$ is asymptotically stable. See Figure 4.4 for the dynamics.

Once again, both the continuous and discrete case show stark similarities.

4.2.3 Discrete Monotone Selection Dynamics

There is a slight subtlety regarding the monotonicity of the convergence to the interior fixed point x^* in the Hawk-Dove game. As it turns out, there are a whole class of *monotone selection dynamics* (of which the discrete replicator equation is a special case) which are defined by[6]:

$$\mathbf{F}(x_i) = x_i + f_i(\mathbf{x}), \tag{4.2.4}$$

where the vector field $\mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), ..., f_n(\mathbf{x}))$ satisfies the relevant conditions:

i. $\sum_{i=1}^{n} f_i(\mathbf{x}) = 0$ for all $\mathbf{x} \in S_n$.

i. $f_i(\mathbf{x})/x_i$ extends to a continuous real-valued function on S_n.

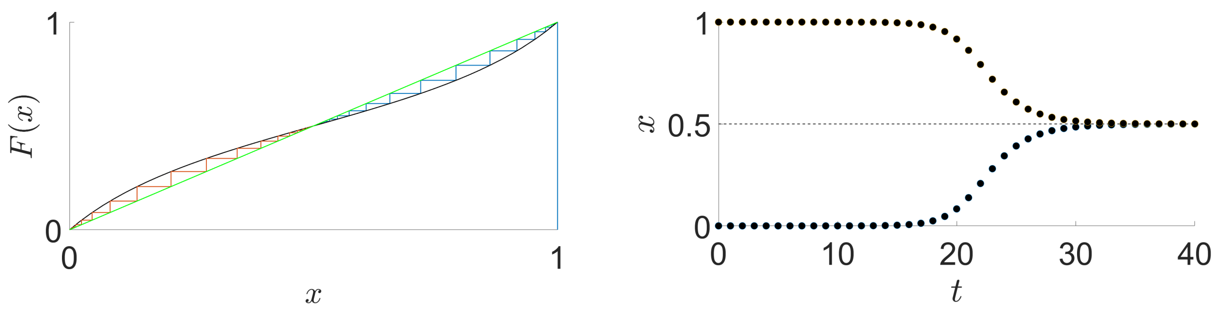
i. If we have the further condition that the entirety of S_n is forward invariant, the dynamic considered is a regular selection dynamic.

One could then, for example, use the discrete dynamic defined by:

$$F(x_1) = x_1 + rx_1(1-x_1)(\frac{1}{2} - x_1), \tag{4.2.5}$$

to describe a Hawk-Dove game with $a = 1$, $b = 2$, $c = 2$, $d = 1$ such that $x_1^* = \frac{1}{2}$. We have that $0 < r < 16$ is a parameter which describes the rate at which evolution acts. In fact, (4.2.5) is a discrete-time monotone selection dynamic which leaves the simplex forward invariant.

However, the qualitative behaviour observed in the discrete replicator equation only occurs for $0 < r \leq 4$, wherein interior trajectories converge monotonically to x_1^*. Things change when $4 < r \leq 8$: convergence to x_1^* is not necessarily monotone, and orbits are often attracted to a nearby orbit of period 2. Then, finally, for $8 < r < 16$, we have wildly unpredictable results which are not in line with the Hawk-Dove game's behaviour. This is because at $r = 8$ there occurs a period-doubling bifurcation [6]. See Figure 4.5 for an initial condition in this region.



(a) *Cobweb-staircase diagram* (b) *Typical orbits*

Figure 4.4: *Discrete dynamics (generated in MATLAB) of the 2×2 HD game in symmetric normal form.*

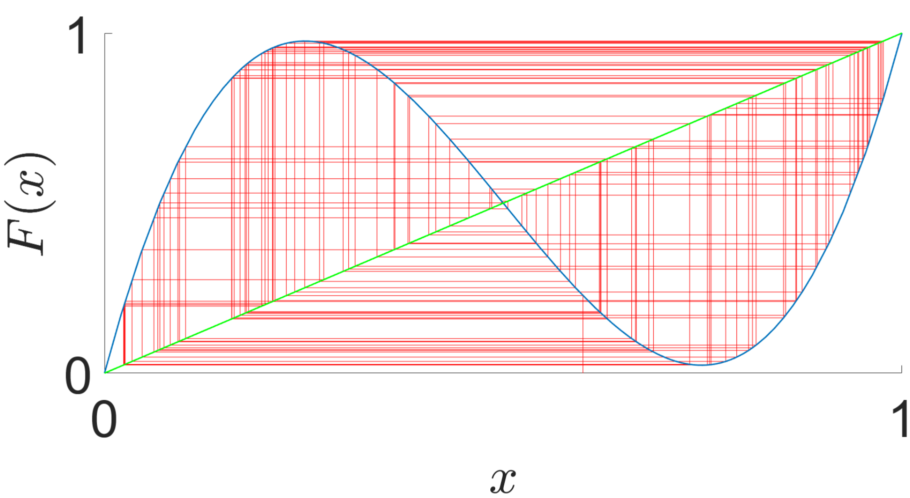


Figure 4.5: *A typical initial condition for the discrete dynamic defined by* (4.2.5). *See Appendix* B.3.

4.3 The Stag Hunt (co-ordination class)

In contrast to the Hawk-Dove game, the Stag Hunt game represents the case where both pure strategies are strict Nash equilibria: it is always best to do as your opponent does. In this way, it is often known as the "co-ordination class" of two-strategy normal form games.

The idea was originally documented by philosopher Jean-Jacques Rosseau [27], as a prototype for his *social contract*. Context is given by imagining two people who are starving and go on a hunt. They are aware of the presence of both deer and hare. Deer is of course much larger and thus more valuable: as a corollary, it requires the effort of both hunters to bring one down. Hare, on the other hand, is smaller: it has less meat and can easily be brought down by one hunter.

This defines the game. Should the hunters co-operate, they both end up better off. The worse outcome occurs when one hunter opts for the deer, while the other betrays him and goes for the small fry in a rash display of selfishness.

The key is that choosing the opposite strategy to one's opponent results in a lower yield than if one had co-operated, hence we have the payoff matrix:

$$U = \begin{array}{c c} & \begin{array}{cc} \mathbf{e}_1 & \mathbf{e}_2 \end{array} \\ \begin{array}{c} \mathbf{e}_1 \\ \mathbf{e}_2 \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \equiv \begin{array}{c c} & \begin{array}{cc} S & H \end{array} \\ \begin{array}{c} S \\ H \end{array} & \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \end{array}, \tag{4.3.1}$$

where $A > 0$ and $D > 0$. If $A > D$, $\mathbf{e}_1$ payoff dominates $\mathbf{e}_2$. If $D > A$, $\mathbf{e}_2$ risk dominates $\mathbf{e}_1$[1]. While both pure strategies are strict Nash equilibria, we can also consider the set of equations given by (3.1.1), (3.1.2) when applied to (4.3.1). This has the unique solution $\hat{p}_1 = D/(A+D)$, $\hat{p}_2 = 1 - D/(A+D)$, making the strategy $\hat{\mathbf{p}} = (D/(A+D), 1-D/(A+D))$ a Nash equilibrium.

However, since we have that:

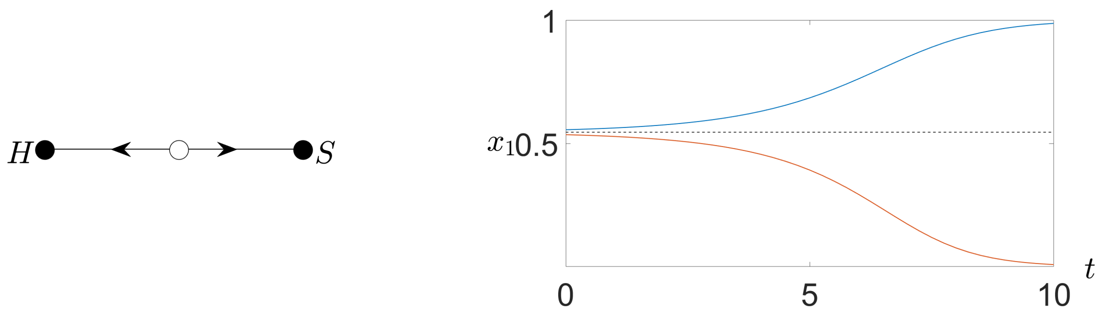
$$\hat{\mathbf{p}} \cdot U\mathbf{p} - \mathbf{p} \cdot U\mathbf{p} = -\frac{((A+D)p_1 - D^2)}{A+D} \not> 0,$$

this strategy is not an ESS.

4.3.1 The Continuous Replicator Equation

For the simple Stag Hunt game, we consider the replicator equation (2.2.3) where A is given by (4.3.1).

[1]See [9] for more information.



(a) *Phase portrait* (b) *Typical trajectories*

Figure 4.6: *Dynamics (generated in MATLAB) of the* 2×2 *CO game in symmetric normal form, with* $A = 1$ *and* $D = 1.2$. *The dashed line indicates the fixed point, given by* $x_1 = x_1^* = D/(A + D)$. *The blue and orange trajectories correspond to the initial conditions* $x_1 = x_1^* + 0.01$ *and* $x_1 = x_1^* - 0.01$, *respectively.*

Since we are considering the dynamics on the two dimensional simplex, we can use (2.2.4) which, in terms of the Stag Hunt game, can be expressed as:

$$\dot{x}_1 = x_1(1 - x_1)((A + D)x_1 - D). \tag{4.3.2}$$

There are three fixed points for (4.3.2): the interior fixed point $x_1^* = D/(A + D)$ and the vertices of the simplex, $x_1 = 0$ and $x_1 = 1$. The latter is to be expected, since the vertices of the simplex are always fixed points under continuous replicator dynamics [24]. We can inspect the Jacobian matrix of (4.3.2) to determine the stability of the fixed points rigorously.

When evaluated at $x_1 = 0$, the Jacobian matrix has a single eigenvalue given by $\lambda = -D$. Since we have that $D > 0$, this means that $\lambda < 0$, making $x_1 = 0$ a sink: initial conditions in $x_1 \in (0, x_1^*)$ converge to it asymptotically in forwards time.

When evaluated at $x_1 = 1$, the Jacobian matrix has a single eigenvalue given by $\lambda = -A$. Since we have that $A > 0$, this means that $\lambda < 0$, meaning $x_1 = 1$ is also a sink: initial conditions $x_1 \in (x_1^*, 1)$ converge to it asymptotically in forwards time.

When evaluated at $x_1 = x_1^*$, the Jacobian matrix has a single eigenvalue given by $\lambda = \frac{AD}{A+D}$. In this case, $\lambda > 0$, making $x_1 = x_1^*$ a source: all interior initial conditions (other than $x_1 = x_1^*$ itself) converge to it asymptotically in backwards time. The dynamics are depicted in Figure 4.6.

Note also that for $x_1 \in (0, x_1^*)$, $\dot{x}_1 < 0$ and for $x_1 \in (x_1^*, 1)$, $\dot{x}_1 > 0$. This shows that initial conditions lying either side of x_1^* converge directly to their respective extreme.

4.3.2 The Discrete Replicator Equation

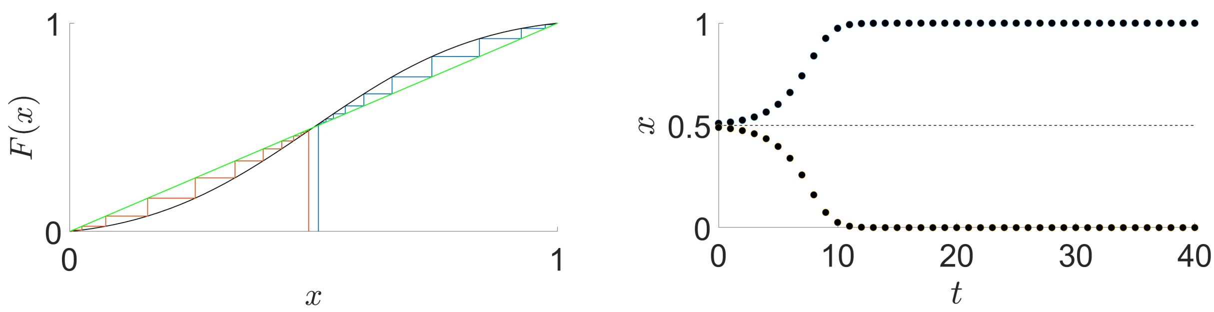
We perform a similar analysis using the reduced discrete replicator equation given by (2.1.2). This is given by:

$$F(x) = \frac{(a-b)x^2 + bx}{(a-b-c+d)x^2 + (b+c-2d)x + d}. \tag{4.3.3}$$

The fixed points which lie on the simplex S_2 are given by the vertices $x = 1$ and $x = 0$, since $F(0) = 0$ and $F(1) = 1$. Furthermore, using similar logic, there is an interior fixed point given by $x = x^* = (d-b)/(a-c+d-b)$. To determine the stability of the endpoints, we proceed thusly.

The eigenvalue associated with $DF|_{x=0}$ is $b/d < 1$, since $d > b$. Therefore, $x = 0$ is asymptotically stable. The eigenvalue associated with $DF|_{x=1}$ is $c/a < 1$, since $c < a$. Therefore, $x = 0$ is also asymptotically stable. The eigenvalue associated with $DF|_{x=x^*}$ is $-(ab-2ad+cd)/(ad-bc) > 1$ for similar reasons. Hence, $x = x^*$ is linearly unstable. See Figure 4.7 for the dynamics.

Both infinite and countable dynamics agree with one another.



(a) *Cobweb-staircase diagram*

(b) *Typical orbits*

Figure 4.7: *Discrete dynamics (generated in MATLAB) of the 2×2 CO game in symmetric normal form.*

CHAPTER 5

Bifurcations of the Reduced Continuous 2-D Replicator Equation

In what follows, we consider the reduction of the two-dimensional replicator equation, given by (2.2.4). Using Lemma 2.2.1 as a basis, we can define new parameters $A = a - c$ and $D = d - b$. This gives an easy criterion for strict Nash equilibria (and hence ESS): if $A > 0$ (resp. $D > 0$), then $\mathbf{e}_1$ (resp. $\mathbf{e}_2$) is a strict NE; if $A < 0$ (resp. $D < 0$), then $\mathbf{e}_1$ (resp. $\mathbf{e}_2$) is not a strict NE.

Then, (2.2.4) becomes:

$$\dot{x} = x(1-x)((A-D)x + D) = f^*(x, A, D). \tag{5.0.1}$$

With this reduction, the analysis is greatly simplified. The term $(A-D)x + D$ represents the difference in fitness between each strategy, $f_1 - f_2 = \Delta\mathbf{f}$. This gives a useful interpretation (see [7]). At a given moment in time: if $\Delta\mathbf{f} > 0$, strategy $\mathbf{e}_1$ is fitter than $\mathbf{e}_2$; if $\Delta\mathbf{f} < 0$, strategy $\mathbf{e}_2$ is fitter than $\mathbf{e}_1$; if $\Delta\mathbf{f} = 0$, the strategies are equally as fit. This last case gives the criterion for finding the interior fixed point of (5.0.1) (should it be defined on S_2, the conditions for which we shall find out presently).

In what follows, we see how the qualitative nature of (5.0.1) changes as we vary the parameters A and D. We start by finding the fixed points, and inspecting their stability for various A and D. This is well worth doing, as [1] "the phase portrait of any autonomous equation is determined completely by the nature of its fixed points."

The fixed points occur when the rate of change of each strategy's relative frequency –

with respect to time – is zero. Namely, when $\dot{x} = h(x, A, D) = 0$. This yields the set of points $x^* \in \{0, 1, \frac{D}{A+D}\}$. The Jacobian matrix (which in this case is simply a scalar, since we are dealing with a one-dimensional system) is given by:

$$Df^*(x, A, D) = (-3A - 3D)x^2 + (2A + 4D)x - D. \tag{5.0.2}$$

We now inspect this quantity (which is the eigenvalue, once again due to the one-dimensionality of the system) for each fixed point, thus revealing its stability.

For $x = 0$, we have that:

$$Df^*(0, A, D) = -D. \tag{5.0.3}$$

For $x = 1$, we have that:

$$Df^*(1, A, D) = -A. \tag{5.0.4}$$

For $x = x_3^* = \frac{D}{A+D}$, we have that:

$$Df^*(x_3^*, A, D) = \frac{AD}{A + D}. \tag{5.0.5}$$

5.1 A fixed

We start by considering what happens to (5.0.1) if we fix A and vary D, using equations (5.0.3), (5.0.4) and (5.0.5) for reference.

5.1.1 A positive

When both A and D are positive, the game under consideration has that both pure strategies are strict Nash equilibria, i.e. it is of the Co-ordination Class type. Both fixed points $x = 0$ and $x = 1$ at the extremes of the simplex S_2 are asymptotically stable, representing the fact that they are both an ESS. There is a branch of unstable fixed points given by $x = x^* = D/(A + D)$ in this region, representing the polymorphic state which all interior initial conditions (besides $x_0 = x^*$ itself) will diverge away from.

When A is positive and D is negative, the game under consideration has $\mathbf{e}_1$ as a strict Nash equilibrium, while $\mathbf{e}_2$ is not: thus, it is akin to the Prisoner's Dilemma. The fixed point $x = 1$ remains asymptotically stable, however $x = 0$ is now a source. The branch of fixed points is stable in the region $|D| < |A|$ and unstable for $|D| > |A|$. It no longer lies in the simplex S_2 however, and so it is not important from an evolutionary game theorist's perspective: be that as it may, without context, it still remains interesting to the ODE enthusiast.

We thus conclude that there is a bifurcation in the (D, x)-plane at $(0, 0)$, wherein the fixed points $x = 0$ and $x = D/(A + D)$ exchange stability. Furthermore, there is an infinite amount of points, represented by the line $D = -A$, along which the stability of the fixed point $x = D/(A + D)$ changes. See Figure 5.1.

5.1.2 A negative

When both A and D are negative, the game under consideration has that neither strategy is a strict Nash equilibrium, i.e. it is of the Hawk-Dove type. Both fixed points $x = 0$ and $x = 1$ at the extremes of the simplex S_2 are sources: initial conditions converge asymptotically towards them in backwards time. There is a branch of sinks given by $x = x^* = D/(A + D)$ in this region, representing the polymorphic state which all interior initial conditions will converge towards across time.

When A is negative and D is positive, we have a qualitatively identical setup for when A is positive and D is negative, but with the strategies swapped. That is to say, $\mathbf{e}_2$ is a strict Nash equilibrium, whilst $\mathbf{e}_1$ is not, making this configuration ($A < 0$ and $D > 0$) dynamically identical to the other ($A > 0$ and $D < 0$): it is of the Prisoner's Dilemma type. The fixed point $x = 1$ remains unstable, however $x = 0$ is now a sink. The situation regarding the branch of fixed points is directly opposite to the setup in subsection 5.1.1: the bifurcation is also congruent. See Figure 5.2.

5.1.3 A zero

When $A = 0$ and $D > 0$, based on the requirements (3.2.3) and (3.2.4), $\mathbf{e}_2$ is a strict Nash equilibrium (and hence an ESS) while $\mathbf{e}_1$ is not an ESS, but it is a non-strict Nash equilibrium. This setup [6] is known as the "Degenerate Prisoner's Dilemma".

The fixed point at $x = 0$ is asymptotically stable. The branch of fixed points given by $x = D/(A + D)$ collapses to $x = 1$, which is now a non-hyperbolic fixed point. Upon inspecting the reduced ODE (when $A = 0$):

$$\dot{x} = -Dx(1 - x)^2,$$

we see that the fixed point $x = 1$ is a non-hyperbolic saddle: trajectories starting in $(0, 1)$ yield $\dot{x} < 0$ and hence diverge away from it (not exponentially quickly for nearby initial conditions) towards $x = 0$. Similarly, for $x_0 \in (1, \infty)$, $\dot{x} < 0$, meaning convergence towards $x = 1$ in positive time. This is an artefact of the dynamics which helps to analyse the game from an EGT perspective: although the saddle behaviour occurs outside the simplex S_2, it helps to appreciate the rate of divergence from $x = 1$.

When $A = 0$ and $D < 0$, $\mathbf{e}_2$ is no longer an ESS, becoming a non-strict Nash equilibrium. On the other hand, $\mathbf{e}_1$ is now an ESS. Once again, we have the Degenerate Prisoner's Dilemma.

The fixed point given by $x = 0$ is now a source. The fixed point given by $x = 1$ is once again a non-hyperbolic saddle, however the stable and unstable directions swap over. All interior trajectories now converge towards $x = 1$.

When both $A = 0$ and $D = 0$, both pure strategies are neutral variants with respect to selection: in other words, the replicator equation does not predict domination of either strategy in the long run. Both strategies are as fit as one another across all time, since we have that:

$$f_1 = Ax = 0 = D(1 - x) = f_2.$$

This means the simplex comprises entirely of fixed points, due to the fact that $\dot{x} = 0$ for all $x \in S_2$. See Figure 5.3.

5.1.4 Classification of Bifurcations

When considering (5.0.1), we have that $f^*(0, A, 0) = 0 = Df^*(0, A, 0)$, making $(x, D) = (0, 0)$ a non-hyperbolic fixed point, and hence a bifurcation point.

We can classify the bifurcations of (5.0.1) by treating A as a constant, denoting $f^*(x, A, D) = g(x, D)$, and inspecting the partial derivatives of $g(x, D)$ when evaluated at $(x, D) = (0, 0)$.

We have that:

$$\begin{aligned}
g_{xx} &= -(6A + 6D)x + 2A + 4D \\
g_D &= -x^3 + 2x^2 - x \\
g_{DD} &= 0 \\
g_{Dx} &= -3x^2 + 4x - 1 \\
g_{Dx}^2 - g_{xx}g_{DD} &= (3x^2 - 4x + 1)^2,
\end{aligned}$$

making $g_D(0, 0) = 0$, $g_{xx}(0, 0) = 2A \neq 0$ and $[g_{Dx}^2 - g_{xx}g_{DD}]|_{(0,0)} = 1 > 0$. We therefore conclude that the origin is a *transcritical* bifurcation, with branches of varying stability depending on the sign of g_{xx} (see [10], Theorem 8.3). If $A > 0$, then $g_{xx} > 0$, making the upper branch unstable and the lower branch stable. If $A < 0$, then $g_{xx} < 0$, making the upper branch stable and the lower branch unstable. See Figure 5.1 and Figure 5.2.

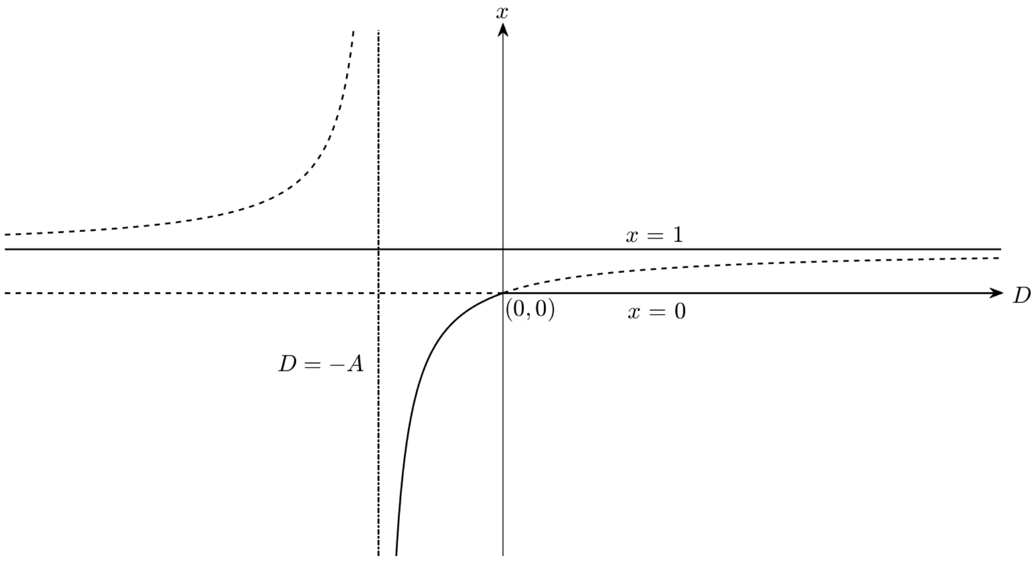


Figure 5.1: $A > 0$, *varying* D. *The line* $D = -A$ *represents the asymptote for the set of points given by* $x = D/(A+D)$. *Solid lines indicate stability, while dashed indicate instability. The point* $(0, 0)$ *is a transcritical bifurcation, with* $g_{xx} > 0$, *resulting in instability above and stability below* $(0, 0)$. *See Appendix B.4.*

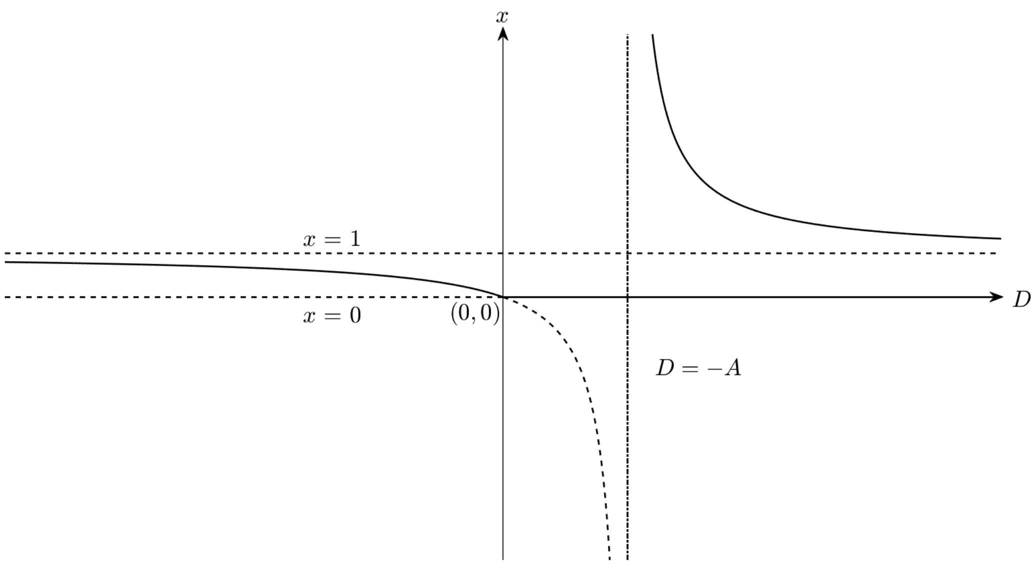


Figure 5.2: $A < 0$, *varying* D. *The line* $D = -A$ *represents the asymptote for the set of points given by* $x = D/(A+D)$. *Solid lines indicate stability, while dashed indicate instability. The point* $(0, 0)$ *is a transcritical bifurcation, with* $g_{xx} < 0$, *resulting in stability above and instability below* $(0, 0)$.

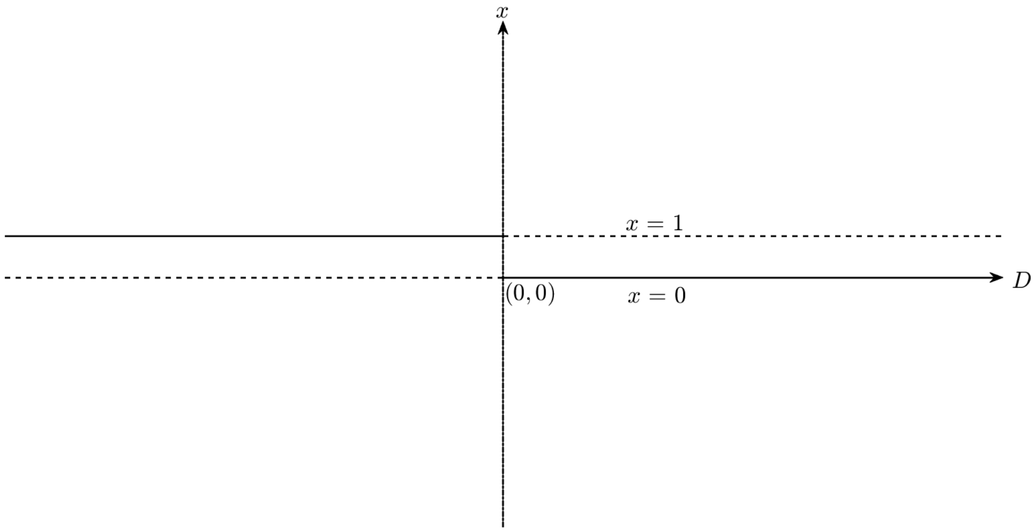


Figure 5.3: $A = 0$, *varying* D. *Solid lines indicate stability, while dashed indicate instability. The entire* x*-axis comprises fixed points, hence the alternation of dots and dashes.*

5.2 Summary

It is worth noting at this point that the process of fixing A and varying D is not a unique setup: one could just as easily fix D and vary A. However, the analysis yields the same results. If we consider the (A, D)-plane, we can categorise each possible configuration [30].

In the preceding analysis, we considered the changes which (5.0.1) underwent when A was fixed and D varied. This ruled out the possibility of switching from a Co-ordination class game to a Hawk-Dove game (and vice-versa), since this transformation requires the variation of both A and D simultaneously.

In fact, this is impossible. We can determine the bifurcations in terms of D and A by finding the values of x for which $Df^*(x, A, D) = 0$, then using this to eliminate x from the relation $f^*(x, A, D) = 0$, since these are the exact points for which the bifurcations occur: the non-hyperbolic fixed points [10].

Solving the equation $Df^*(x, A, D) = 0$ yields:

$$x^* = \frac{A + 2D \pm \sqrt{A^2 + AD + D^2}}{3(A + D)}. \tag{5.2.1}$$

Note that this excludes the origin $(A, D) = (0, 0)$ and points for which $D = -A$ right from the outset, since both of these result in division by zero and are hence undefined.

We now use (5.2.1) to solve $f^*(x^*, A, D) = 0$ for both A and D. The solutions are $D = 0$

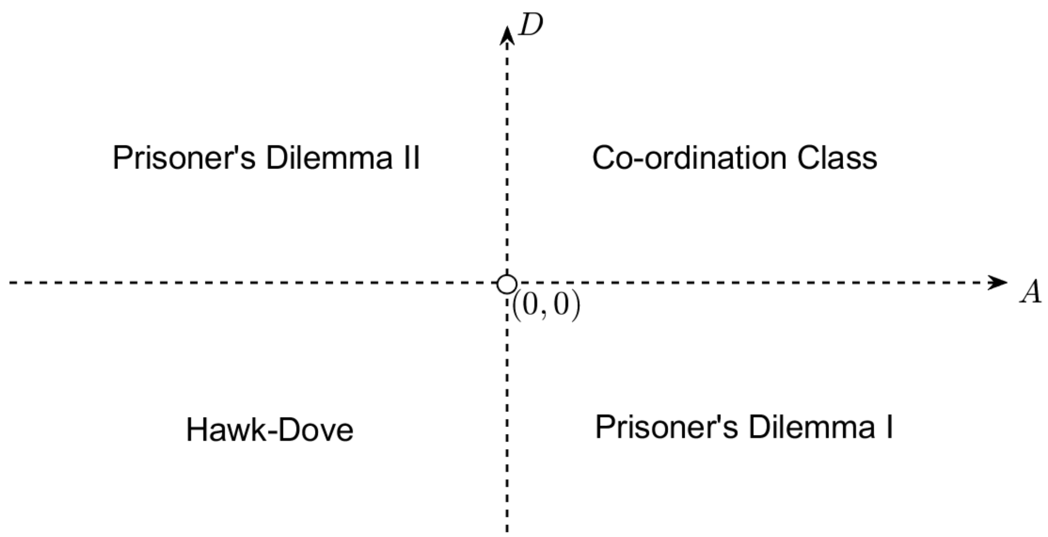


Figure 5.4: *Classification of each symmetric normal form two strategy game under continuous replicator dynamics. Transcritical bifurcations occur along both axes (hence the dashed lines), except for at the origin.*

for $|A| > 0$ and $A = 0$ for $|D| > 0$. In other words, the bifurcations occur along both the A and D axes, except at their intersection $(0, 0)$. See Figure 5.4.

This is the exact reason why one cannot shift directly from a Co-ordination class to a Hawk-Dove game with a continuous variation of both A and D: the only possible way for this to occur is if there was a bifurcation at the origin. Since there is no such bifurcation, this behaviour is impossible. It is only possible to shift from Hawk-Dove to Prisoner's Dilemma, then to Co-ordination class.

CHAPTER 6

Three Strategy Games

More interesting dynamical behaviour occurs for planar systems, as the preliminaries suggest. This chapter aims to give an outline of some of these behaviours, as well as the underlying concepts, as concepts such as manifolds are trivial in one dimension. Lyapunov functions are touched on. More rich ideas from evolutionary game theory are also covered, such as evolutionarily stable sets. An example shows how the dynamics of the continuous and discrete replicator equation dot always line up.

6.1 Rock-Scissors-Paper

Known by most as a playground game, *Rock-Scissors-Paper*[1] typifies games with a cyclical structure. The setup is easy to grasp: in a bout between two players, each individual faces their opponent with their hand behind their back. They then chant "Rock, Scissors, Paper!". Upon the utterance of "Paper!", each player then reveals their hand in the shape of a rock (balled fist), a pair of scissors (all fingers closed except for middle and index), or a sheet of paper (completely open hand).

The strategies are cyclical due to the possible outcomes: Rock beats Scissors, Scissors beats Paper and Paper beats Rock. In the traditional game, we assume that the gain of one player is equal in magnitude to the loss of the other, i.e. it is a *zero sum game* [13]. In terms of

[1] I always knew it as "Rock, Paper, Scissors", but what do I know?

the payoff matrix (1.3.1), we say that $u_{ij} = -u_{ji}$. The payoff matrix is then defined by

$$U = \begin{array}{c c} & \begin{array}{ccc} R & S & P \end{array} \\ \begin{array}{c} R \\ S \\ P \end{array} & \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array}. \tag{6.1.1}$$

No pure strategy is better than the other, in the sense that none of them are a strict NE, ESS or even standard NE.

However, the mixed strategy $\hat{\mathbf{p}} = (1/3, 1/3, 1/3)$ is a NE, since it solves the system of equations:

$$\begin{aligned} &p_2 - p_3 = -p_1 + p_3 = p_1 - p_2 \\ &p_1 + p_2 + p_3 = 1. \end{aligned}$$

Upon checking condition (3.2.2) for $\hat{\mathbf{p}}$ to be an ESS, we see that:

$$\hat{\mathbf{p}} \cdot U\mathbf{p} - \mathbf{p} \cdot U\mathbf{p} = 0 - 0 = 0 \not> 0,$$

since in zero sum games, we have that $\mathbf{p} \cdot U\mathbf{p} = 0$ for all $\mathbf{p} \in S_N$.[2]

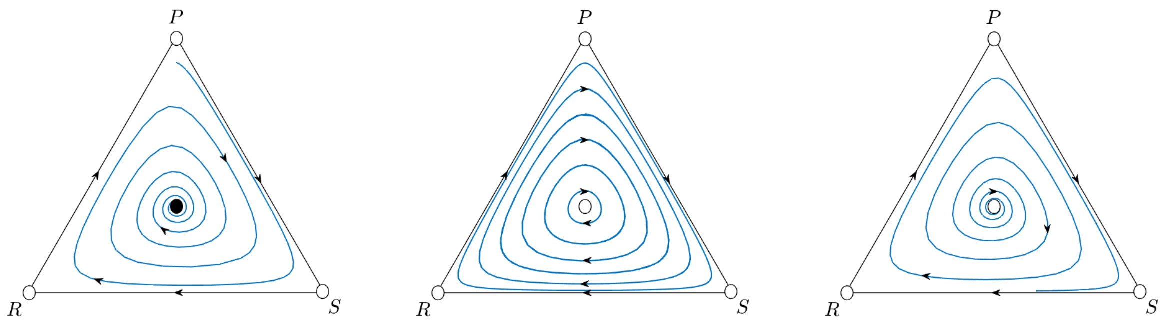
6.1.1 The Continuous Replicator Equation

When considering the Rock-Scissors-Paper game under continuous replicator dynamics, we use the payoff matrix (6.1.1) with a slight adjustment. Rather than mutual interaction resulting in a gain of zero for both individuals, we allow this payoff to vary by using the single real parameter $|\varepsilon|$, usually considered close to zero [6]. The payoff matrix is then given by:

$$A = \begin{array}{c c} & \begin{array}{ccc} R & S & P \end{array} \\ \begin{array}{c} R \\ S \\ P \end{array} & \begin{pmatrix} \varepsilon & 1 & -1 \\ -1 & \varepsilon & 1 \\ 1 & -1 & \varepsilon \end{pmatrix} \end{array}. \tag{6.1.2}$$

When we set $\varepsilon = 0$, we recover the original game as described by (6.1.1). Interesting behaviour occurs for all considered values of ε, which we shall see presently.

[2]When viewed in terms of $\mathbf{x}$ rather than $\mathbf{p}$, this is qualitatively identical to saying that the average fitness of the population is zero. See [13].



(a) $-1 < \varepsilon < 0$ (b) $\varepsilon = 0$ (c) $0 < \varepsilon < 1$

Figure 6.1: *The continuous replicator dynamics (generated in MATLAB) of the RSP game for various values of* ε.

Firstly, consider the mixed strategy $\hat{\mathbf{p}} = (1/3, 1/3, 1/3)$ under the modified game described by (6.1.2). For $\mathbf{p} \neq \hat{\mathbf{p}}$, we now have that:

$$\hat{\mathbf{p}} \cdot A\mathbf{p} - \mathbf{p} \cdot A\mathbf{p} = -\varepsilon(p_1^2 + p_2^2 + p_3^2 - 1). \tag{6.1.3}$$

For $-1 < \varepsilon < 0$, we have that (6.1.3) is strictly positive, making $\hat{\mathbf{p}}$ an ESS in this case. For $0 \leq \varepsilon < 1$, (6.1.3) is non-positive, meaning $\hat{\mathbf{p}}$ is not an ESS.

Inserting the payoff matrix (6.1.2) into the continuous replicator equation (2.2.1) yields:

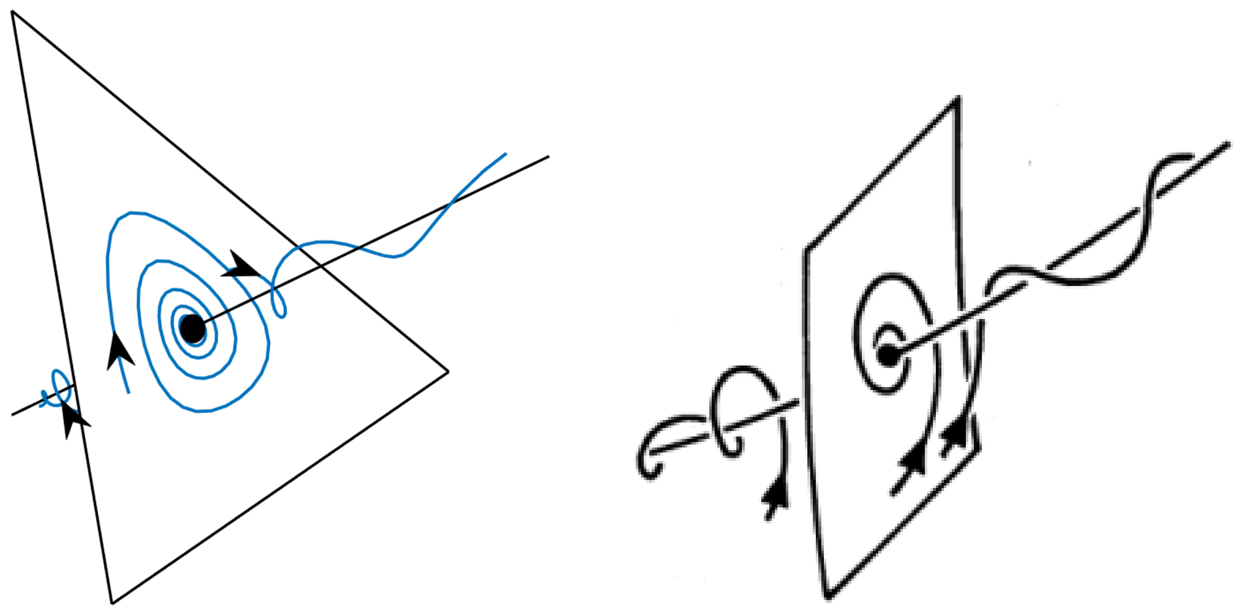
$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_1(\varepsilon x_1^2 - \varepsilon x_1 + \varepsilon x_2^2 - x_2 + \varepsilon x_3^2 + x_3) \\ -x_2(\varepsilon x_1^2 + x_1 + \varepsilon x_2^2 - \varepsilon x_2 + \varepsilon x_3^2 - x_3) \\ -x_3(\varepsilon x_1^2 - x_1 + \varepsilon x_2^2 + x_2 + \varepsilon x_3^2 - \varepsilon x_3) \end{bmatrix} = \mathbf{f}. \tag{6.1.4}$$

This system of ODEs has fixed points given by the set of points $\mathbf{x}^* \in \mathbb{R}^3$ which satisfy $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$, i.e. the set:

$$\left\{ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{\varepsilon - 1}{2\varepsilon}, \frac{\varepsilon + 1}{2\varepsilon}, 0\right), \left(0, \frac{\varepsilon - 1}{2\varepsilon}, \frac{\varepsilon + 1}{2\varepsilon}\right), \left(\frac{\varepsilon + 1}{2\varepsilon}, 0, \frac{\varepsilon - 1}{2\varepsilon}\right) \right\}. \tag{6.1.5}$$

We consider the case where $|\varepsilon| < 1$. This then excludes the last three elements of (6.1.5), since they do not lie on the simplex S_3 for these values of ε. To analyse the remaining fixed points, we inspect the Jacobian matrix $D\mathbf{f}$ when evaluated at the fixed point.

For $\mathbf{e}_i$ – where $i = 1, 2, 3$ – the eigenvalues of $D\mathbf{f}|_{\mathbf{x}=\mathbf{e}_i}$ are given by $\lambda_1 = -1 - \varepsilon$, $\lambda_2 = 1 - \varepsilon$ and $\lambda_3 = -\varepsilon$. We now write the eigenvectors in the respective order of the corresponding eigenvalues.



(a) (b)

Figure 6.2: *Figure 6.2a shows the behaviour (generated in MATLAB: see Appendix B.5) along the invariant manifolds of* $\hat{\mathbf{x}}$ *for the dynamical system* (6.1.4) *when* $-1 < \varepsilon < 0$*: Figure 6.2b shows similar behaviour, taken from [10].*

The eigenvectors for $\mathbf{e}_1$ are given by:

$$\Xi^{(1)} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

For $\mathbf{e}_2$, they are:

$$\Xi^{(2)} = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

For $\mathbf{e}_3$, they are:

$$\Xi^{(3)} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

For $0 < \varepsilon < 1$, $\lambda_1 < 0$, $\lambda_2 > 0$ and $\lambda_3 < 0$. Hence, the states corresponding to the pure strategies are saddle points in this case. For $\varepsilon = 0$, $\lambda_1 < 0$, $\lambda_2 > 0$ and $\lambda_3 = 0$, making them non-hyperbolic saddles. For $-1 < \varepsilon < 0$, $\lambda_1 < 0$, $\lambda_2 > 0$ and $\lambda_3 > 0$: they are now hyperbolic saddles, as when $0 < \varepsilon < 1$. See Figure 6.1. There is therefore a bifurcation when $\varepsilon = 0$.

For each value of ε, one can show that the eigenvectors of each vertex of the simplex form a *heteroclinic orbit* from $\mathbf{e}_i$ to $\mathbf{e}_{i+1}$, and thus the vertices form a *heteroclinic cycle* [13].

For the fixed point given by $\mathbf{x} = \hat{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we have that the eigenvalues of $D\mathbf{f}|_{\mathbf{x}=\hat{\mathbf{x}}}$ are $\lambda_1 = -\frac{\varepsilon}{3}$, $\lambda_{2,3} = \frac{\varepsilon}{3} \pm \frac{\sqrt{3}}{3}i$. The corresponding eigenvectors are given by:

$$\mathbf{\Xi}^{(4)} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{3}i}{2} \\ -\frac{1}{2} + \frac{\sqrt{3}i}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} + \frac{\sqrt{3}i}{2} \\ -\frac{1}{2} - \frac{\sqrt{3}i}{2} \\ 1 \end{bmatrix} \right\}.$$

Straight away we can make conclusions about the linearised flow near $\hat{\mathbf{x}}$. We start by considering $-1 < \varepsilon < 0$. Then, we have that $\lambda_1 > 0$ and $\mathrm{Re}(\lambda_{2,3}) < 0$. Up to a linear change of co-ordinates through rotations, we have that the normal form for $D\mathbf{f}|_{\mathbf{x}=\hat{\mathbf{x}}}$ shows the following behaviour [10]. The point $\hat{\mathbf{x}}$ has an unstable linear manifold spanned by $(1,1,1)^T$, while the linear stable manifold is coplanar to the simplex S_3: that is to say, $S_3 \subset E^s$. Thus, the local behaviour for all $\mathbf{x} \in \mathrm{int}S_3$ is asymptotic convergence towards $\hat{\mathbf{x}}$, since the simplex S_3 is contained within the stable manifold: see Figure 6.2.

Not only this, the eigenvalues tell the whole story: when the motion is restricted to the planar system on S_3 with $-1 < \varepsilon < 0$, $\hat{\mathbf{x}}$ is a stable focus with trajectories spiralling clockwise (see section 1.2).

For $0 < \varepsilon < 1$, the situation is much the same but with the stabilities reversed. The fixed point is now an unstable focus with interior trajectories spiralling out to the boundary of S_3.

For $\varepsilon = 0$, the fixed point is now a centre, surrounded by an infinite number of periodic orbits contained within S_3.

One can also deduce similar results using the Lyapunov function given by:

$$V(\mathbf{x}) = x_1 x_2 x_3. \tag{6.1.6}$$

We consider the neighbourhood U of $\hat{\mathbf{x}}$ to be the simplex S_3 without the boundary $\mathrm{bd}S_3$, i.e. the interior $U = \mathrm{int}S_3 \backslash \hat{\mathbf{x}}$. Indeed, we have that $V(\hat{\mathbf{x}}) < V(\mathbf{x})$ for all $\mathbf{x} \in U$. Furthermore:

$$\dot{V} = \nabla V \cdot \mathbf{f} = -3\varepsilon V(x_1^2 + x_2^2 + x_3^2 - \frac{1}{3}).$$

In the case where $-1 < \varepsilon < 0$, $\dot{V} > 0$ for all $\mathbf{x} \in U$. This, in conjuction with the previous properties, makes (6.1.6) a *strict* Lyapunov function [6]. As such, we conclude that $\hat{\mathbf{x}}$ is asymptotically stable on the simplex S_3.

When $\varepsilon = 0$, $\dot{V} = 0$ for all $\mathbf{x} \in U$. Hence, S_3 is filled with closed level curves, signifying periodic orbits.

For $0 < \varepsilon < 1$, we inspect the behaviour of V on the set $U = S_3$. We have that $\dot{V} \leq 0$ for all $\mathbf{x} \in U$. Furthermore, since $\dot{V} = 0$ for $\mathbf{x} \in \mathrm{bd}S_3$ and $\hat{\mathbf{x}}$, we know that [13] the ω and α limit sets for all interior trajectories must be contained within the set $\mathbf{x} \in \mathrm{bd}S_3 \cup \hat{\mathbf{x}}$. Since $\hat{\mathbf{x}}$ is an unstable focus, we have that $\alpha(\mathbf{x}) = \hat{\mathbf{x}}$. Hence, by deduction, it must be that $\omega(\mathbf{x}) = \mathrm{bd}S_3$. That is to say, for $0 < \varepsilon < 1$, all interior initial conditions (save $\hat{\mathbf{x}}$) are attracted to the boundary of the simplex $\mathrm{bd}S_3$ (the heteroclinic cycle mentioned before) [24].

6.1.2 The Discrete Replicator Equation

This section serves to show the qualitative differences in dynamics which both the continuous and discrete versions of the replicator equation produce, specifically for the RSP game. Since we cannot have negative entries in our payoff matrix (N.B. section 2.1) we must adjust (6.1.2) in the following way. Continuous replicator dynamics are unaffected by the addition of a constant to each entry of a column of the underlying payoff matrix, meaning we can add 1 to each entry, hence producing:

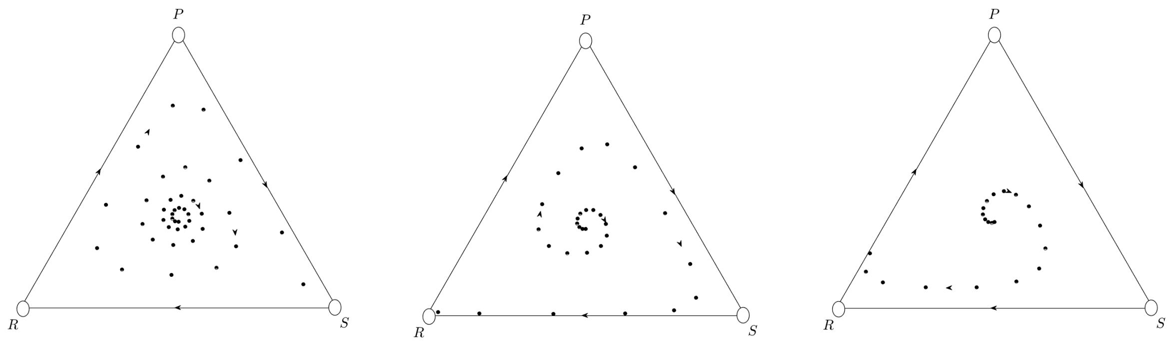
$$A = \begin{array}{c} \\ {\scriptstyle R} \\ {\scriptstyle S} \\ {\scriptstyle P} \end{array} \begin{array}{c} \begin{array}{ccc} {\scriptstyle R} & {\scriptstyle S} & {\scriptstyle P} \end{array} \\ \begin{pmatrix} 1+\varepsilon & 2 & 0 \\ 0 & 1+\varepsilon & 2 \\ 2 & 0 & 1+\varepsilon \end{pmatrix} \end{array}. \tag{6.1.7}$$

By manipulating the payoff matrix in this way, we ensure that our comparisons with the continuous case are legitimate. The discrete replicator equation (2.1.1) with this adjusted payoff matrix (6.1.7) is then:

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} F_1(\mathbf{x}) \\ F_2(\mathbf{x}) \\ F_3(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \frac{x_1(2x_2+x_1(\varepsilon+1))}{x_1(2x_2+x_1(\varepsilon+1))+x_2(2x_3+x_2(\varepsilon+1))+x_3(2x_1+x_3(\varepsilon+1))} \\ \frac{x_2(2x_3+x_2(\varepsilon+1))}{x_1(2x_2+x_1(\varepsilon+1))+x_2(2x_3+x_2(\varepsilon+1))+x_3(2x_1+x_3(\varepsilon+1))} \\ \frac{x_3(2x_1+x_3(\varepsilon+1))}{x_1(2x_2+x_1(\varepsilon+1))+x_2(2x_3+x_2(\varepsilon+1))+x_3(2x_1+x_3(\varepsilon+1))} \end{bmatrix}. \tag{6.1.8}$$

The fixed points are now given by seeking values $\mathbf{x}^*$ which satisfy the relation $\mathbf{F}(\mathbf{x}^*) = \mathbf{x}^*$. Namely, the set:

$$\left\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(\frac{\varepsilon-1}{2\varepsilon}, \frac{\varepsilon+1}{2\varepsilon}, 0\right), \left(0, \frac{\varepsilon-1}{2\varepsilon}, \frac{\varepsilon+1}{2\varepsilon}\right), \left(\frac{\varepsilon+1}{2\varepsilon}, 0, \frac{\varepsilon-1}{2\varepsilon}\right)\right\}. \tag{6.1.9}$$



(a) $\varepsilon = -\frac{1}{3}$ (b) $\varepsilon = 0$ (c) $\varepsilon = \frac{1}{3}$

Figure 6.3: *The discrete replicator dynamics (generated in MATLAB) of the RSP game for various values of* ε*, indicating the instability of* $\hat{\mathbf{x}} = (1/3, 1/3, 1/3)$.

Note the equivalence with the continuous case. Once again, we consider the case where $|\varepsilon| < 1$ to restrict our attention to the vertices of the simplex and the barycentre[3]. As in the continuous case, all eigenvalues of $D\mathbf{F}|_{\mathbf{x}=\mathbf{e}_i}$ are the same: that is to say, for each $\mathbf{e}_i$, we have $\lambda_{1,2} = 0$ and $\lambda_3 = 2/(\varepsilon + 1)$. Therefore $|\lambda_1| = |\lambda_2| = 0 < 1$, and since $|\varepsilon| < 1$, $|\lambda_3| > 1$. We thus conclude that all vertices of the simplex are saddles, once again in agreement with the continuous case.

The differences start to occur when considering $\mathbf{x} = \hat{\mathbf{x}} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The eigenvalues of $D\mathbf{F}|_{\mathbf{x}=\hat{\mathbf{x}}}$ are given by $\lambda_1 = 0$ and $\lambda_{2,3} = 1 + \frac{\varepsilon \pm \sqrt{3}i}{3+\varepsilon}$. We have that $|\lambda_1| = 0 < 1$ and $|\lambda_{2,3}| = 2\sqrt{\varepsilon^2 + 3\varepsilon + 3}/(3 + \varepsilon) > 1$ for all $|\varepsilon| < 1$ (and, in fact, all $\varepsilon > -1$) [6]. The motion on the nonlinear stable manifold of $\hat{\mathbf{x}}$ – tangent to the span of the first eigenvector, $(1, 1, 1)^T$ – plays no part in the motion on the simplex S_3, since all trajectories remain on the simplex for all forward time, never leaving this invariant set.

The previous points made have important implications, namely that the strategy $\hat{\mathbf{p}}$ does not give rise to an evolutionarily stable state under discrete replicator dynamics, as it does for the continuous case: specifically, $-1 < \varepsilon < 0$ does not give rise to asymptotic stability for the discrete case, whereas for the continuous case it does. See Figure 6.3.

6.2 An Extended Hawk-Dove Model

The following example highlights the fact that a *mixed strategy* can be an ESS, however its corresponding *state* may not be ES [13].

[3]The co-ordinates we use when specifying points on the simplex S_n are known as *Barycentric*. See [7, 13].

Consider an extended version of the original hawk-dove game (see section 4.2), whereby we have three distinct types: E_1, corresponding to pure Hawk; E_2, corresponding to pure Dove; E_3, corresponding to the mixed strategy $\hat{\mathbf{p}} = (\frac{V}{C}, 1 - \frac{V}{C})$. The payoff matrix is then given by:

$$A = \begin{array}{c c} & \begin{array}{ccc} H & D & \hat{\mathbf{p}} \end{array} \\ \begin{array}{c} H \\ D \\ \hat{\mathbf{p}} \end{array} & \begin{pmatrix} \frac{1}{2}(V-C) & V & \frac{V(C-V)}{2C} \\ 0 & V/2 & \frac{V(C-V)}{2C} \\ \frac{V(V-C)}{2C} & \frac{V(V+C)}{2C} & \frac{V(C-V)}{2C} \end{pmatrix} \end{array}. \tag{6.2.1}$$

As we already know, the type E_3 cannot be invaded by E_1 or E_2, since the corresponding *strategy* is ESS. However the state $\mathbf{e}_3 = (0, 0, 1)$ (where *all* individuals play $\hat{\mathbf{p}}$) is not evolutionarily stable: neither is the mixture $\mathbf{x}^* = (\frac{V}{C}, 1 - \frac{V}{C}, 0)$.

We can see this by inspecting conditions set out by definitions 3.1.2 and 3.2.3. For both $\hat{\mathbf{x}} = \mathbf{e}_3$ and $\hat{\mathbf{x}} = (V/C, 1 - V/C, 0)$, with $\mathbf{x} = (x_1, x_2, x_3)$, we have:

$$\hat{\mathbf{x}} \cdot A\hat{\mathbf{x}} - \mathbf{x} \cdot A\hat{\mathbf{x}} = -\frac{V}{2C}(C - V)(x_1 + x_2 + x_3 - 1) = 0 \geq 0,$$

making both states Nash equilibria. They cannot be ES states, however, since they lie on an ES *set* [13]. The only way for them to be ES states is if the ES set was a singleton; however it is actually the line connecting the points $\mathbf{e}_3$ and $(V/C, 1 - V/C, 0)$: see Figure 6.4.

6.2.1 The Continuous Replicator Equation

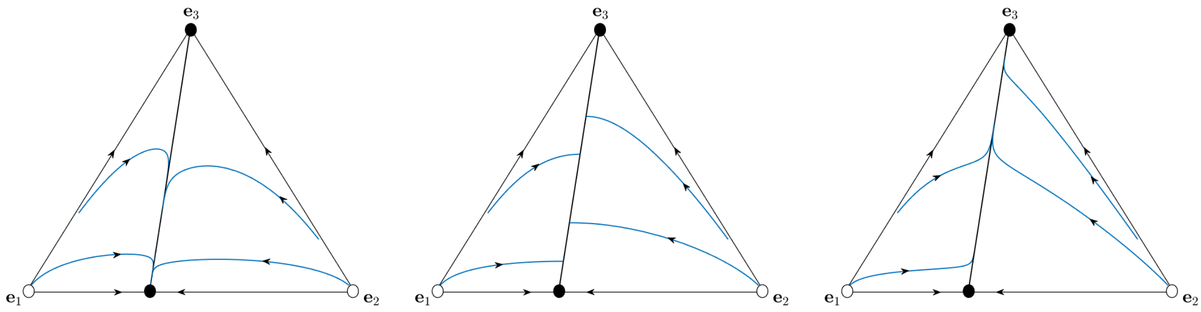
In the following analysis, we consider the continuous replicator equation (2.2.3) where A is given by (6.2.1). That is to say:

$$\begin{aligned}
\dot{x}_1 &= -\frac{x_1(-C^2x_1^2+C^2x_1+CVx_1^2+2CVx_1x_2-CVx_1+CVx_2^2+2CVx_2x_3-2CVx_2+CVx_3^2-CVx_3-V^2x_3^2+V^2x_3)}{2C}\\
\dot{x}_2 &= -\frac{x_2(-C^2x_1^2+CVx_1^2+2CVx_1x_2+CVx_2^2+2CVx_2x_3-CVx_2+CVx_3^2-CVx_3-V^2x_3^2+V^2x_3)}{2C}\\
\dot{x}_3 &= -\frac{x_3(-C^2x_1^2+CVx_1^2+2CVx_1x_2+CVx_1+CVx_2^2+2CVx_2x_3-CVx_2+CVx_3^2-CVx_3-V^2x_1-V^2x_2-V^2x_3^2+V^2x_3)}{2C}.
\end{aligned} \tag{6.2.2}$$

The fixed points are given by: the vertices $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ of the simplex S_3; the boundary fixed point, given by $\mathbf{x}^* = (V/C, 1 - V/C, 0)$.

The eigenvalues of $D\mathbf{f}|_{\mathbf{x}=\mathbf{e}_1}$ are $\lambda_{1,2} = \frac{1}{2}(C - V) > 0$ and $\lambda_3 = \frac{1}{2C}(C - V)^2 > 0$, since we have that $0 < V < C$. All eigenvalues are positive, making $\mathbf{e}_1$ a source.

The eigenvalues of $D\mathbf{f}|_{\mathbf{x}=\mathbf{e}_2}$ are $\lambda_1 = -\frac{V}{2}$, $\lambda_2 = \frac{V}{2} > 0$ and $\lambda_3 = \frac{V^2}{2C}$. Thus, $\mathbf{e}_2$ is a saddle, with unstable manifold (based on the eigenvectors) spanned by its adjacent edges on the



(a) $0 < C < 1$ (b) $C = 1$ (c) $C > 1$

Figure 6.4: *The continuous replicator dynamics (generated in MATLAB) of the extended Hawk-Dove game for various values of* C*. The line joining* $\mathbf{e}_3$ *and* $\mathbf{x}^*$*, along which* $\dot{x}_1 = \dot{x}_2$*, is given by* $\mathbf{x} = (X, \frac{C}{V}X - X, 1 - \frac{C}{V}X)$*.*

simplex. Both of these fixed points are in agreement with the original two dimensional Hawk-Dove game.

The eigenvalues of $D\mathbf{f}|_{\mathbf{x}=\mathbf{e}_3}$ are $\lambda_{1,2} = 0$ and $\lambda_3 = \frac{V}{2C}(C - V) < 0$. Therefore, $\mathbf{e}_3$ is a stable non-hyperbolic fixed point. Since the unstable manifold for $\mathbf{e}_3$ is empty, the local motion is unequivocally towards the centre manifold tangent to the span of the adjacent edges of the simplex [31].

The eigenvalues of $D\mathbf{f}|_{\mathbf{x}=\mathbf{x}^*}$ are $\lambda_1 = 0$ and $\lambda_{2,3} = -\frac{V}{2C}(C - V)^2 < 0$. Once again, the unstable manifold for $\mathbf{x}^*$ is empty, meaning the local motion is unequivocally towards the centre manifold tangent to the span of the vector $(V/C, 1 - V/C, 1)$. This eigenspace is precisely the points along the line which joins $\mathbf{x}^*$ and $\mathbf{e}_3$.

At $C = 1$, all of the *isoclines*[4] "collide", turning the line connecting $\mathbf{x}^*$ and $\mathbf{e}_3$ entirely into fixed points: see Figure 6.4b. This is because the value $C = 1$ satisfies the condition that the fitnesses of the strategies are equal.

[4] These are the lines along which the growth rates of $n - 1$ strategies are equal.

APPENDIX A

Important results

A.1 Proof of the Picard-Lindelöf theorem

In this section, we prove the existence and uniqueness theorem for initial value problems of the form described in Theorem 1.2.4. We need only prove it for a single scalar component of **f** (say, f).

Let

$$C_{a,b} = \overline{I_a(t_0)} \times \overline{B_b(y_0)}$$

where:

$$\overline{I_a(t_0)} = [t_0 - a, t_0 + a]$$
$$\overline{B_b(y_0)} = [y_0 - b, y_0 + b].$$

This is the compact cylinder where f is defined. Let

$$M = \sup_{C_{a,b}} ||f||,$$

that is, the maximum slope of the function in modulus. Finally, let L be the Lipschitz constant of f with respect to the second variable.

We will proceed to apply the Banach fixed point theorem using the metric on $\mathcal{C}(I_a(t_0), B_b(y_0))$ induced by the uniform norm

$$||\varphi||_\infty = \sup_{t \in I_a} |\varphi(t)|.$$

We define an operator between two functional spaces of continuous functions, Picard's operator, as follows:

$$\Gamma : \mathcal{C}(I_a(t_0), B_b(y_0)) \to \mathcal{C}(I_a(t_0), B_b(y_0))$$

defined by:

$$\Gamma\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))ds.$$

We must show that this operator maps a complete non-empty metric space $\mathcal{X}$ onto itself and also is a contraction mapping.

We first show that, given certain restrictions on a, Γ takes $\overline{B_b(y_0)}$ onto itself in the space of continuous functions with uniform norm. Here, $\overline{B_b(y_0)}$ is a closed ball in the space of continuous (and bounded) functions "centered" at the constant function y_0. Hence we need to show that

$$||\varphi_1||_\infty \leq b$$

implies

$$||\Gamma\varphi(t) - y_0|| = \left|\left|\int_{t_0}^{t} f(s, \varphi(s))ds\right|\right| \leq \int_{t_0}^{t'} ||f(s, \varphi(s))||ds \leq M|t' - t_0| \leq Ma \leq b$$

where t' is some number in $[t_0 - a, t_0 + a]$ where the maximum is achieved. The last step is true if we impose the requirement $a < \frac{b}{M}$.

Now let's try to prove that this operator is a contraction.

Given two functions $\varphi_1, \varphi_2 \in \mathcal{C}(I_a(t_0), B_b(y_0))$, in order to apply the Banach fixed point theorem we want

$$||\Gamma\varphi_1 - \Gamma\varphi_2||_\infty \leq q||\varphi_1 - \varphi_2||_\infty,$$

for some $q < 1$. So, let t be such that

$$||\Gamma\varphi_1 - \Gamma\varphi_2||_\infty = ||(\Gamma\varphi_1 - \Gamma\varphi_2)(t)||.$$

Then, using the definition of Γ

$$\begin{aligned}
||(\Gamma\varphi_1 - \Gamma\varphi_2)(t)|| &= \left|\left|\textstyle\int_{t_0}^{t}(f(s, \varphi_1(s)) - f(s, \varphi_2(s)))ds\right|\right| \\
&\leq \textstyle\int_{t_0}^{t} ||(f(s, \varphi_1(s)) - f(s, \varphi_2(s)))||ds \\
&\leq L\textstyle\int_{t_0}^{t} ||\varphi_1(s) - \varphi_2(s)||ds \qquad f \text{ is Lipschitz-continuous} \\
&\leq La||\varphi_1 - \varphi_2||_\infty
\end{aligned}$$

This is a contraction if $a < \frac{1}{L}$.

We have established that the Picard's operator is a contraction on the Banach spaces with the metric induced by the uniform norm. This allows us to apply the Banach fixed point theorem to conclude that the operator has a unique fixed point. In particular, there is a unique function

$$\varphi \in \mathcal{C}(I_a(t_0), B_b(y_0))$$

such that $\Gamma\varphi = \varphi$. This function is the unique solution of the initial value problem, valid on the interval I_a where a satisfies the condition

$$a < \min\left\{\frac{b}{M}, \frac{1}{L}\right\}.$$

■

A.2 Derivation of the continuous replicator equation from the discrete model

The following derivation is taken from chapter 1 of [7]. The discrete time replicator equation describes the population of competing individuals over time. At times $t = 0, 1, 2, ...$ the state is given by:

$$x_i(t+1) = \frac{f_i^*}{\overline{f^*}} x_i(t), \quad i = 1, 2, ..., n, \tag{A.2.1}$$

where $\overline{f^*} = \sum_{i=1}^{n} x_i f_i^*$ is the average fitness in the population. When dealing with shorter time intervals, we can generalise (A.2.1) by writing:

$$x_i(t+\Delta t) = \left[\frac{f_i^*}{\overline{f^*}}\right]^{\Delta t} x_i(t), \tag{A.2.2}$$

where Δt is the time interval of interest. This equation agrees with (A.2.1) when $\Delta t = 0$ or 1, but it also makes sense for intermediate values of Δt. Now take the natural lograithm of both sides to obtain:

$$\ln x_i(t+\Delta t) = \Delta t \left[\ln f_i^* - \ln \overline{f^*}\right] + \ln x_i(t). \tag{A.2.3}$$

Subtract $\ln x_i(t)$ from both sides and divide both sides by Δt to obtain:

$$\frac{\ln x_i(t+\Delta t) - \ln x_i(t)}{\Delta t} = \left[\ln f_i^* - \ln \overline{f^*}\right]. \tag{A.2.4}$$

To get the continuous time replicator equation, take the limit $\Delta t \to 0$ and the left-hand side of (A.2.4) becomes $\frac{d \ln x_i(t)}{dt} = \frac{1}{x_i}\frac{dx_i}{dt} = \frac{\dot{x}_i}{x_i}$. Using the notation $f_i = \ln f_i^*$ and $\bar{f} = \ln \overline{f^*}$, the right-hand side of (A.2.4) becomes $[f_i - \bar{f}]$. Multiply through by x_i and we have the elegant system of ordinary differential equations,

$$\dot{x}_i(t) = (f_i - \bar{f})x_i(t), \quad i = 1, ..., n, \tag{A.2.5}$$

known as **continuous replicator dynamics**. ∎

APPENDIX B

MATLAB code

This section contains some of the MATLAB code used to produce the figures in this dissertation. In an effort to remain concise, where figures are similar, I have picked one example which typifies the ideas considered.

B.1 Figure 4.3a

```
% Hawk-Dove phase portrait

clear;
clf;
clc;

% V > C > 0

x = 0:0.1:1;
y = 1 - x;
hold on
plot(x,y,'k','linewidth',2);
axis([0 1 0 1]);
xlabel('Hawk'); ylabel('Dove');
```

```
set(gcf,'Color','w');
set(gca,'visible','off');
set(gca,'view',[-45;90]);
set(gca,'DataAspectRatio',[1 1 1],'PlotBoxAspectRatio',[1 1 2]);
annotation(gcf,'textbox',...
    [0.25 0.507 0.0341390922401171 0.0552995391705069],...
    'String','$D$',...
    'Interpreter','latex',...
    'FontSize',72,...
    'FitBoxToText','off',...
    'EdgeColor','none');
annotation(gcf,'textbox',...
    [0.74 0.507 0.034139092240117 0.055299539170507],'String',{'$H$'},...
    'Interpreter','latex',...
    'FontSize',72,...
    'FitBoxToText','off',...
    'EdgeColor','none');
annotation(gcf,'ellipse',[0.71 0.485 0.03 0.06],'LineWidth',1.5,...
    'FaceColor',[1 1 1]);
annotation(gcf,'ellipse',[0.3 0.485 0.03 0.06],'LineWidth',1.5,...
    'FaceColor',[1 1 1]);
annotation(gcf,'ellipse',[0.513 0.485 0.03 0.06],'LineWidth',1.5,...
    'FaceColor',[0 0 0]);
annotation(gcf,'arrow',[0.62 0.6],...
    [0.516932270916335 0.516932270916335],'LineStyle','none',...
    'HeadLength',45,'HeadWidth',45);
annotation(gcf,'arrow',[0.42 0.44],...
    [0.516932270916335 0.516932270916335],'LineStyle','none',...
    'HeadLength',45,'HeadWidth',45);
hold off
```

B.2 Figure 4.3b

```
% Graph of typical ICs for the Hawk-Dove game

clear;
clf;
clc;

% C > V > 0

V = 0.5;
C = 1;
fp = V/C;

sys = @(t,x) [(1/2)*x(1)*(1 - x(1))*(V - C*x(1))];
tspan = [0 60];
x0 = [0.99;0.01];
n = size(x0);
tfinal = tspan(1,2);
x = linspace(0,tfinal);
m = size(x);
y = fp*ones(m);

hold on
for i = 1:n
    [ts,xs] = ode45(sys,tspan,x0(i));
plot(ts,xs,'LineWidth',2);
end
plot(x,y,'k','LineWidth',1.5,'LineStyle','--');
axis([0 tfinal 0 1]);
set(gcf,'color','w');
set(gca,'XTick',[0,tfinal/2,tfinal]);
```

```
set(gca,'YTick',[0.5,1]);
set(gca,'box','on');
set(gca,'FontSize',72);
annotation(gcf,'textbox',...
    [0.95 0.14 0.0208750000000003 0.0587649402390438],...
    'String',{'$t$'},...
    'LineStyle','none',...
    'Interpreter','latex',...
    'FontSize',72,...
    'FitBoxToText','off');
annotation(gcf,'textbox',...
    [0.001 0.475 0.0510833333333338 0.126315697907061],...
    'String','$x_1$',...
    'LineStyle','none',...
    'Interpreter','latex',...
    'FontSize',72,...
    'FitBoxToText','off');
hold off
```

B.3 Figure 4.5

```
% Discrete monotone selection dynamic: cobweb-staircase diagram

clear; % Clears workspace
clf;
clc; % Clears command window
fsize=72; % Used to set font size
nmax=200;halfm=nmax/2; % Define the amount of iterates
t=zeros(1,nmax);t1=zeros(1,nmax);t2=zeros(1,nmax); % Define empty vectors
% to then fill with numbers... not completely necessary, however saves a
% lot of time when coding.
```

```
t(1)= 0.6; % Set the initial condition
r = 15.5; % Set the value of mu
axis([0 1 0 1]); % Set the range of the axes
for n=1:nmax
    t(n+1) = t(n) + r*t(n)*(1 - t(n))*(1/2 - t(n));
end
% Now, the vector t has nmax entries, containing the value of the iterates
% Need two lines for each iterate... one vertically to meet F(x), one
% traveling horizontally to meet the line F(x) = x
for n=1:halfm
    t1(2*n-1)=t(n); % sets every odd entry to the iterates
    t1(2*n)=t(n); % sets every even entry to the iterates
end

t2(1)=0;t2(2)=t(2); % sets the first entry to 0, and the second to the
% second iterate
for n=2:halfm
    t2(2*n-1)=t(n); % starting from the third entry, sets all entries out
    % identically to t1, but shifts them over by one. This ensures the "up
    % and across" motion occurs
    t2(2*n)=t(n+1);
end
hold on
plot(t1,t2,'r'); % Plot the iterates, in red
fplot(@(x) x + r.*x.*(1-x).*(1/2 - x),[0 1],'LineWidth',2);
x=[0 1];y=[0 1];
plot(x,y,'g','LineWidth',2); % Plot the line y = x, in green
hold off
set(gca,'xtick',[0 1],'Fontsize',fsize)
set(gca,'ytick',[0 1],'Fontsize',fsize)
xlabel('$x$','Interpreter','latex','Fontsize',fsize)
ylabel('$F(x)$','Interpreter','latex','Fontsize',fsize)
```

B.4 Figure 5.1

```
% Bifurcation diagram for one dimensional replicator: A > 0

clear;
clf;
clc;

A = 0.5;
Dmin = -2;
Dmax = -Dmin;

% Int_1 = [Dmin -A];
% Int_2 = [-A 0];
% Int_3 = [0 inf];

D_1 = Dmin:0.001:-A;
D_2 = -A:0.001:0;
D_3 = 0:0.001:Dmax;
D_4 = Dmin:0.1:Dmax;
D_5 = Dmin:0.1:0;
D_6 = 0:0.1:Dmax;

dim_D_4 = size(D_4);
n = dim_D_4(1,2);

dim_D_5 = size(D_5);
m = dim_D_5(1,2);

x_1 = D_1./(A + D_1); % Branch of fixed points
```

```
x_2 = D_2./(A + D_2); % Branch of fixed points
x_3 = D_3./(A + D_3); % Branch of fixed points
x_4 = ones(n); % Fixed point x = 1
x_5 = zeros(m); % Fixed point x = 0 (unstable)
x_6 = x_5; % Fixed point x = 0 (stable)

asymp_x = -6:0.1:6; % Asymptote at D = -A
dim_asymp_x = size(asymp_x);
o = dim_asymp_x(1,2);
asymp_D = -A*ones(o); % Asymptote at D = -A

hold on
plot(D_1,x_1,'Color','k','LineWidth',2,'LineStyle','--');
plot(D_2,x_2,'Color','k','LineWidth',2);
plot(D_3,x_3,'Color','k','LineWidth',2,'LineStyle','--');
plot(asymp_D,asymp_x,'Color','k','LineWidth',2,'LineStyle','-.');
plot(D_4,x_4,'Color','k','LineWidth',2);
plot(D_5,x_5,'Color','k','LineWidth',2,'LineStyle','--');
plot(D_6,x_6,'Color','k','LineWidth',2);
plot(zeros(o),asymp_x,'Color','k','LineWidth',1); % x-axis (vertical axis)
axis([Dmin Dmax -6 6]);
set(gca,'Visible','off');
annotation(gcf,'arrow',[0.517708333333333 0.517708333333333],...
    [0.894418326693227 0.936254980079681],'LineStyle','none',...
    'HeadLength',15,'HeadWidth',15);
annotation(gcf,'arrow',[0.878125 0.906770833333333],...
    [0.517928286852589 0.517928286852589],'LineStyle','none',...
    'HeadLength',15,'HeadWidth',15);
annotation(gcf,'textbox',...
    [0.508291666666666 0.941235059760956 0.0245208333333333...
    0.0368525896414342],...
    'String','$x$',...
```

```
    'Interpreter','latex',...
    'FontSize',26,...
    'FitBoxToText','off',...
    'EdgeColor','none');
annotation(gcf,'textbox',...
    [0.909854166666667 0.5 0.0245208333333333 0.0368525896414342],...
    'String','$D$',...
    'Interpreter','latex',...
    'FontSize',26,...
    'FitBoxToText','off',...
    'EdgeColor','none');
annotation(gcf,'textbox',...
    [0.609333333333332 0.594621513944223 0.0573333333333334...
    0.0368525896414342],...
    'String','$x$ = 1',...
    'Interpreter','latex',...
    'FontSize',26,...
    'FitBoxToText','off',...
    'EdgeColor','none');
annotation(gcf,'textbox',...
    [0.610895833333332 0.47609561752988 0.0573333333333334...
    0.0368525896414345],...
    'String','$x$ = 0',...
    'Interpreter','latex',...
    'FontSize',26,...
    'FitBoxToText','off',...
    'EdgeColor','none');
annotation(gcf,'textbox',...
    [0.3385 0.394422310756972 0.0719166666666666 0.0368525896414343],...
    'String','$D = -A$',...
    'Interpreter','latex',...
    'FontSize',26,...
```

```
    'FitBoxToText','off',...
    'EdgeColor','none');
annotation(gcf,'textbox',...
    [0.515062499999999 0.4800796812749 0.0573333333333334...
    0.0368525896414346],...
    'String','$(0,0)$',...
    'Interpreter','latex',...
    'FontSize',26,...
    'FitBoxToText','off',...
    'EdgeColor','none');
hold off
```

B.5 Figure 6.2a

```
% Manifolds

% |epsilon| < 1... measures the rate at which the trajectories spiral
% Payoff matrix given by A =...
% [epsilon, 1, -1; -1, epsilon, 1; 1, -1, epsilon]
clear;
clf;
clc;
% Simplex

map = [1 1 1];
[X,Y] = meshgrid(0:0.1:1,0:0.1:1);
Z = 1 - X - Y;
hold on
s = surf(X,Y,Z,'EdgeColor','none'); % removes grid on surface
colormap(map); % sets the colourmap
X = 0:0.1:1;
Y = 1 - X;
```

```
Z = zeros(1,11);
% In what follows, the 3D line plots represent the outline of the simplex.
plot3(X,Y,Z,'k','LineWidth',2.5);
plot3(Z,X,Y,'k','LineWidth',2.5);
plot3(Y,Z,X,'k','LineWidth',2.5);
axis([0 1 0 1 0 1]);
set(gca,'visible','off');
set(gca,'PlotBoxAspectRatio',[1 1 1]);
set(gca,'Position',[0.1300 0.0486 0.7750 0.8150]);
hold off

% Trajectories

epsilon = -0.15;
phase_blue = [0, 0.4470, 0.7410];
sys = @(t,x) [-x(1)*(x(3) - x(2) - x(1)*epsilon + x(3)*(x(2) - x(1) +...
    x(3)*epsilon) + x(1)*(x(3) - x(2) + x(1)*epsilon) + x(2)*(x(1) -...
    x(3) + x(2)*epsilon));
    -x(2)*(x(1) - x(3) - x(2)*epsilon + x(3)*(x(2) - x(1) +...
    x(3)*epsilon) + x(1)*(x(3) - x(2) + x(1)*epsilon) + x(2)*(x(1) -...
    x(3) + x(2)*epsilon));
    -x(3)*(x(2) - x(1) - x(3)*epsilon + x(3)*(x(2) - x(1) +...
    x(3)*epsilon) + x(1)*(x(3) - x(2) + x(1)*epsilon) + x(2)*(x(1) -...
    x(3) + x(2)*epsilon))
    ];
tspan = [0 100];
x0 = [0.56,0.14,0.3;0.15,0.15,0.07;0.4,0.4,0.5];
dim = size(x0);
n = dim(1,1);
hold on
for i = 1:n
[ts,xs] = ode45(sys,tspan,x0(i,:));
```

```
plot3(xs(:,1),xs(:,2),xs(:,3),'color',phase_blue,'LineWidth',3);
end
view(77.0815860742184, 25.6137579617834);
xlabel('Rock'); ylabel('Scissors'); zlabel('Paper');
hold off

%% Linear unstable manifold for x*

x = -2:1:2;
y = -2:1:2;
z = -2:1:2;
hold on
plot3(x,y,z,'k','LineWidth',2.5)
annotation(gcf,'arrow',[0.415104166666667 0.4109375],...
    [0.355140186915888 0.398753894080997],'LineStyle','none',...
    'HeadSize',35);
annotation(gcf,'arrow',[0.388020833333333 0.377083333333333],...
    [0.268951194184839 0.31048805815161],'LineStyle','none',...
    'HeadSize',35);
annotation(gcf,'arrow',[0.4921875 0.505208333333333],...
    [0.47871235721703 0.469366562824507],'LineStyle','none',...
    'HeadSize',35);
annotation(gcf,'ellipse',...
    [0.453604166666667 0.383177570093458 0.0156666666666667...
    0.0363447559709242],'FaceColor',[0 0 0]);
hold off
```

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