Accepted Manuscript

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Please cite this article as: A.V. Mikhailov, G. Papamikos, J.P. Wang, Dressing method for the vector sine-Gordon equation and its soliton interactions, *Physica D* (2016), http://dx.doi.org/10.1016/j.physd.2016.01.010

Dressing method for the vector sine-Gordon equation and its soliton interactions

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Abstract

In this paper, we develop the dressing method to study the exact solutions for the vector sine-Gordon equation. The explicit formulas for one kink and one breather are derived. The method can be used to construct multi-soliton solutions. Two soliton interactions are also studied. The formulas for position shift of the kink and position and phase shifts of the breather are given. These quantities only depend on the pole positions of the dressing matrices.

1 Introduction

This paper is devoted to the study of an $O(n)$-invariant generalisation of the sine-Gordon equation

$$D_t \left(\frac{\vec{\alpha}_x}{\beta} \right) = \vec{\alpha}, \quad \beta^2 + \langle \vec{\alpha}, \vec{\alpha} \rangle = 1, \tag{1}$$

where the dependent variable $\vec{\alpha} = (\alpha^1, \cdots, \alpha^n)^T$ is n-dimensional real vector field and $\beta \in \mathbb{R}$. Here and in what follows the upper index T denotes the transpose of a vector or a matrix. We use the notation $\langle \cdot, \cdot \rangle$ for the Euclidean dot product of two vectors.

Equation (1) was first appeared in [1] viewed as a reduction of the two-dimensional $O(n)$ nonlinear σ-model [2]. Its integrability properties were further studied afterwards. The Lax pairs was given in [3] and its Lagrangian formulation in [4]. Later, this equation reappeared in the study of connection between finite dimensional geometry, infinite dimensional geometry and integrable systems [5]. It was derived as the inverse flow of the vector modified Korteweg-de Vries equation

$$\vec{u}_\tau = \vec{u}_{xxx} + \frac{3}{2} \langle \vec{u}, \vec{u} \rangle \vec{u}_x, \quad \vec{u} = \frac{\vec{\alpha}_x}{\beta}, \tag{2}$$

whose Hamiltonian, symplectic and hereditary recursion operators were naturally derived using the structure equation for the evolution of a curve embedded in an n-dimensional Riemannian manifold with constant curvature [6]. These have been recently re-derived in [7]. Besides, a partial classification of vector sine-Gordon equations using symmetry tests was done in [8].

Equation (1) is a higher-dimensional generalisation of the well-known scalar sine-Gordon equation

$$\theta_{xt} = \sin \theta. \tag{3}$$

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Indeed, it can be obtained by taking the dimension $n = 1$ and letting $\beta = \cos\theta$ and $\alpha^1 = \sin\theta$. The scalar sine-Gordon equation originates in differential geometry and has profound applications in physics and in life sciences (see recent review [9]). Vector generalisations of integrable equations have proved to be useful in applications [10]. They can be associated with symmetric spaces [11].

The rational dressing method was originally proposed in [2, 12] and developed in [13]. This method enables one to construct multi-soliton solutions and analyse soliton interactions in detail using basic knowledge of Linear Algebra. In this paper, we develop the dressing method for the vector sine-Gordon equation (1) and show that similar to the scalar sine-Gordon equation (3) there are two distinct types of solitons, namely kinks and breathers. One kink solution is a stationary wave propagating with a constant velocity. We show that a kink solution of the vector sine-Gordon equation can be obtained from a kink solution of (3) by setting $\vec{\alpha} = \mathbf{a}\sin\theta$, $\beta = \cos\theta$, where $\mathbf{a}$ is a constant unit length vector in $\mathbb{R}^n$. A general two kink solution of (1) cannot be obtained from solutions of (3), but it can be seen as a two kink solution of a vector sine-Gordon equation (1) with $n = 2$. One breather solution is a localised and periodically oscillating wave moving with a constant velocity. One breather solution of the general $O(n)$ invariant equation (1) can be obtained from a breather solution of (1) with $n = 2$ by an appropriate $O(n)$ rotation. Two breathers solution can be obtained from the corresponding solution of (1) with $n = 4$, etc. Surprisingly, the effects of interaction, such as the displacement and a phase shift (for breathers) are exactly the same as in the case of the scalar sine-Gordon equation (3) [14]. Such interaction properties are naturally valid for the vector modified Korteweg-de Vries equation (2). The detailed study of soliton interactions for (2) when $n = 2$ can be found in [15].

2 Dressing method for the vector sine-Gordon equation

In this section, we begin with the Lax representation of the vector sine-Gordon equation (1) given in [5], which is invariant under the reduction group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We then study the conditions for the dressing matrix (assumed to be rational in spectral parameter) with the same symmetries. The 1-soliton solutions of (1) correspond the dressing matrix with only simple poles belonging to a single orbit of the reduction group. For one kink, it has two pure imaginary simple poles and for one breather, it has four complex simple poles. Using the dressing method, we explicitly derive one kink and one breather solutions starting with a trivial solution.

The vector sine-Gordon equation (1) is equivalent to the compatibility condition [5] $[\mathcal{L}, \mathcal{A}] = 0$ for two linear problems

$$\mathcal{L}\Psi = 0, \qquad \mathcal{A}\Psi = 0, \tag{4}$$

where

$$\mathcal{L} = D_x - \lambda J - U \quad \text{and} \quad \mathcal{A} = D_t + \lambda^{-1} V, \tag{5}$$

and

$$J = \begin{pmatrix} 0 & 1 & \mathbf{0}^T \\ -1 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & 0_n \end{pmatrix}, \; U = \begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & -\vec{\alpha}_x^T/\beta \\ \mathbf{0} & \vec{\alpha}_x/\beta & 0_n \end{pmatrix}, \; V = \begin{pmatrix} 0 & \beta & \vec{\alpha}^T \\ -\beta & 0 & \mathbf{0}^T \\ -\vec{\alpha} & \mathbf{0} & 0_n \end{pmatrix}, \tag{6}$$

where $\mathbf{0}$ is n-dimensional zero column vector and 0_n is the $n \times n$ zero matrix. Without causing confusion, we sometimes simply write 0 instead.

The Lax operators $\mathcal{L}$ and $\mathcal{A}$ are invariant under the (reduction) group of automorphisms generated by the following three transformations: the first one is

$$\iota : \mathcal{L}(\lambda) \to -\mathcal{L}^\dagger(\lambda), \tag{7}$$

where $\mathcal{L}^\dagger(\lambda)$ is the adjoint operator defined by $\mathcal{L}^\dagger(\lambda) = -D_x - \lambda J^T - U^T$. The invariance under this transformation implies the matrices J and U are skew-symmetric. The second one is

$$r : \mathcal{L}(\lambda) \to \overline{\mathcal{L}(\bar{\lambda})}, \tag{8}$$

where $\overline{\mathcal{L}(\lambda)}$ means its complex conjugate. The invariance under this transformation reflects that the entries of matrices U and V are real. The last one is called Cartan involution

$$s : \mathcal{L}(\lambda) \to Q\mathcal{L}(-\lambda)Q, \tag{9}$$

where $Q = \mathrm{diag}(-1, 1, \ldots, 1)$, which leads to the reduction to the symmetric space.

These three commuting transformations satisfy

$$\iota^2 = r^2 = s^2 = \mathrm{id}$$

and therefore generate the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Indeed, the operator $\mathcal{A}$ is also invariant under it, that is,

$$\iota(\mathcal{A}(\lambda)) = \mathcal{A}(\lambda), \qquad r(\mathcal{A}(\lambda)) = \mathcal{A}(\lambda), \quad s(\mathcal{A}(\lambda)) = \mathcal{A}(\lambda). \tag{10}$$

Thus we say the Lax representation of (1) is invariant under the reduction group [13, 16, 17] $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

In what follows, we use the method of rational dressing [2, 12, 13] to construct new exact solutions of (1) starting from an exact solution $\vec{\alpha}_0$, β_0. Let us denote by U_0, V_0 the matrices U, V in which $\vec{\alpha}$, β are replaced by the exact solution $\vec{\alpha}_0$, β_0 of (1). The corresponding overdetermined linear system

$$\mathcal{L}_0\Psi_0 = (D_x - \lambda J - U_0)\Psi_0 = 0 \qquad \mathcal{A}_0\Psi_0 = (D_t + \lambda^{-1}V_0)\Psi_0 = 0 \tag{11}$$

has a fundamental solution $\Psi_0(\lambda, x, t)$ invariant under transformations (7)–(9). Following [2, 12] we shall assume that the fundamental solution $\Psi(\lambda, x, t)$ for the new ("dressed") linear problems

$$\mathcal{L}\Psi = (D_x - \lambda J - U)\Psi = 0 \qquad \mathcal{A}\Psi = (D_t + \lambda^{-1}V)\Psi = 0 \tag{12}$$

is of the form

$$\Psi = \Phi(\lambda)\Psi_0, \quad \det \Phi \neq 0, \tag{13}$$

where the dressing matrix $\Phi(\lambda)$ is assumed to be rational in the spectral parameter λ and to be invariant with respect to symmetries

$$\Phi(\lambda)^{-1} = \Phi^T(\lambda), \qquad \overline{\Phi(\bar{\lambda})} = \Phi(\lambda), \qquad Q\Phi(-\lambda)Q = \Phi(\lambda). \tag{14}$$

Conditions (14) guarantee that the corresponding Lax operators $\mathcal{L}$ and $\mathcal{A}$ are invariant with respect to transformations (7)–(9).

It follows from (11), (12) and (13) that

$$\Phi(D_x - \lambda J - U_0)\Phi^{-1} = -\lambda J - U; \tag{15}$$

$$\Phi(D_t + \lambda^{-1}V_0)\Phi^{-1} = \lambda^{-1}V. \tag{16}$$

These equations enable us to specify the form of the dressing matrix Φ and construct the corresponding "dressed" solution $\vec{\alpha}, \beta$ of the vector sine-Gordon equation (1).

Let us consider the most trivial case when the dressing matrix Φ does not depend on the spectral parameter λ. In this case the dressing results in a point transformation ($O(n)$ rotation) of the initial solution $\vec{\alpha}_0$.

Proposition 1. *Assume that Φ is a λ independent dressing matrix for the vector sine-Gordon equation (1). If it is invariant with respect to symmetries (14), then*

$$\Phi = \pm \begin{pmatrix} 1 & 0 & \mathbf{0}^T \\ 0 & 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \Omega \end{pmatrix}, \tag{17}$$

where $\Omega \in O(n, \mathbb{R})$ is a constant ($x, t-$ independent) matrix. The corresponding "dressed" solution is $\vec{\alpha} = \Omega \vec{\alpha}_0, \ \beta = \beta_0$.

Proof. Under the assumption that the dressing matrix Φ is independent of the spectral parameter λ, it follows from (15) and (16) that

$$[J, \Phi] = 0; \quad \Phi_x = U\Phi - \Phi U_0; \quad \Phi_t = 0; \quad V\Phi - \Phi V_0 = 0. \tag{18}$$

It follows from the first condition $[J, \Phi] = 0$ that the matrix Φ is of the form

$$\Phi = \begin{pmatrix} m_{11} & m_{12} & \mathbf{0}^T \\ -m_{12} & m_{11} & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \Omega \end{pmatrix},$$

where Ω is an $n \times n$ matrix. Substituting it into the last condition in (18), we get $m_{12} = 0$ and

$$(\beta - \beta_0)m_{11} = 0; \quad \vec{\alpha}^T \Omega = m_{11}\vec{\alpha}_0^T; \quad m_{11}\vec{\alpha} = \Omega \vec{\alpha}_0.$$

This implies that $\beta = \beta_0$ since $\det \Phi \neq 0$ and $\Omega \Omega^T = m_{11}^2 I_n$, where I_n is the $n \times n$ identity matrix. The rest two conditions in (18) imply that matrix Φ is independent of x and t. Finally, since Φ satisfies (14), we have that matrix Ω is real and $m_{11}^2 = 1$, i.e. $m_{11} = \pm 1$. □

A λ–dependent dressing matrix $\Phi(\lambda)$, which is invariant with respect to symmetries (14) has poles at the orbits of the reduction group. Simplest "one soliton" dressing correspond to the cases when matrix $\Phi(\lambda)$ has only simple poles belonging to a single orbit. There are two non-trivial cases:

(i) Matrix $\Phi(\lambda)$ has two pure imaginary poles at $\pm i\nu, \ \nu \neq 0, \ \nu \in \mathbb{R}$.

(ii) Matrix $\Phi(\lambda)$ has four complex poles at $\pm\mu, \ \pm\bar{\mu}, \ \mu = \gamma + i\omega, \ \gamma\omega \neq 0, \ \gamma, \omega \in \mathbb{R}$.

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These cases correspond to the "kink" and "breather" solutions respectively. We will show that the case of two real poles at $\pm\mu \in \mathbb{R}$ leads to a trivial dressing.

A more general case in which the poles belong to a finite union of orbits corresponds to a multi-soliton solution. Using this approach we can construct explicitly multi-kink-breather solutions and analyse the result of kink-kink, breather-breather and kink-breather collisions.

The explicit forms of matrix $\Phi(\lambda)$ corresponding the above two cases and satisfying the last two conditions in (14) are

$$(\text{i}) \qquad \Phi(\lambda) = C + \frac{A}{\lambda - \mu} - \frac{QAQ}{\lambda + \mu}, \quad A = -Q\bar{A}Q, \quad \mu = i\nu;$$

$$(\text{ii}) \qquad \Phi(\lambda) = C + \frac{A}{\lambda - \mu} - \frac{QAQ}{\lambda + \mu} + \frac{\bar{A}}{\lambda - \bar{\mu}} - \frac{Q\bar{A}Q}{\lambda + \bar{\mu}}, \quad \mu = \gamma + i\omega,$$

where $C = QCQ$ is a real matrix of size $(n+2) \times (n+2)$ independent of λ.

We now derive the conditions on matrices A and C such that $\Phi(\lambda)$ is an invariant dressing matrix, i.e. satisfying conditions (14). First we have

Proposition 2. *If $\Phi(\lambda)$ is a dressing matrix satisfying $\Phi(\lambda)\Phi^T(\lambda) = I_{n+2}$, then matrix C is a constant matrix of the same form as (17).*

Proof. All identities must be satisfied for all values of λ. For (15), the linear terms in λ as $\lambda \to \infty$ leads to $[J, C] = 0$. So matrix C is of the form

$$C = \begin{pmatrix} c_{11} & c_{12} & \mathbf{0}^T \\ -c_{12} & c_{11} & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \Omega \end{pmatrix},$$

where Ω is an $n \times n$ matrix. Substituting it into the last condition $C = QCQ$, we get $c_{12} = 0$.

The constant terms of (15) in λ as $\lambda \to \infty$ leads to

$$C_x = UC - CU_0 - [B, J], \tag{19}$$

where B is the coefficient at λ^{-1} in the expansion $\Phi(\lambda) = C + B\lambda^{-1} + \cdots$ as $\lambda \to \infty$. Substituting the form of C into it, we get $C_x = 0$ implying that matrix C is independent of x. For (16) and $\Phi(\lambda)\Phi^T(\lambda) = I_{n+2}$, taking the limit $\lambda \to \infty$, we get $C_t = 0$ and $CC^T = I_{n+2}$. Thus we proved the statement. $\square$

Note that a composition of two dressing matrices is also a dressing matrix. Therefore, without loss of generality, from now on, we take $C = I_{n+2}$ in the dressing matrix $\Phi(\lambda)$.

Proposition 3. *The dressing matrix satisfies $\Phi(\lambda)\Phi^T(\lambda) = I_{n+2}$ if and only if*

$$AA^T = 0; \qquad PA^T + AP^T = 0, \tag{20}$$

where

$$P = \left(\Phi(\lambda) - \frac{A}{\lambda - \mu}\right)\Big|_{\lambda=\mu}. \tag{21}$$

Proof. The product $\Phi(\lambda)\Phi^T(\lambda)$ is a rational matrix function of λ and at $\lambda = \infty$ it is equal to the unit matrix. Now it is sufficient to show that the rest of conditions in (20) are equivalent to vanishing all other poles of the product. The first condition of (20) is equivalent to vanishing of the second order pole at $\lambda = \mu$. All other second order poles of the product vanishes due to the reduction group symmetry. The residue at $\lambda = \mu$ vanishes if and only if A satisfies the second condition of (20). □

Therefore, the matrix A is degenerate. Here we shall study solution of rank one, that is, $\text{rank}(A) = 1$ and thus represent the matrix A by a bi-vector

$$A = b\rangle\langle a, \qquad \langle a = (p, q, a^1, \ldots, a^n), \qquad b\rangle = (b^1, \ldots, b^{n+2})^T, \tag{22}$$

where $p, q, a^i, b^i \in \mathbb{C}$. We use different notations for the first two component of the vector $\langle a$ in order to emphasise that they play a particular role in the solutions.

Under the assumption that A is a bi-vector, we rewrite conditions (20) for A into conditions for vectors $a\rangle$ and $b\rangle$.

Proposition 4. *Let $A = b\rangle\langle a \neq 0$ be a bi-vector. The dressing matrix satisfies $\Phi(\lambda)\Phi^T(\lambda) = I_{n+2}$ if and only if*

$$\langle aa \rangle = 0; \qquad Pa\rangle = 0, \tag{23}$$

where P is given by (21).

Proof. The first equation of (23) immediately follows from the first equation of (20). For matrix $A = b\rangle\langle a$ the second condition of (20) is equivalent to

$$PA^T = 0.$$

Indeed, the first term of the sum in the second equation of (20) is a bi-vector with image and co-image spaces spanned by $b\rangle$ and $\langle b$ respectively and thus it is equal to $\delta b\rangle\langle b$, that is,

$$PA^T = \delta b\rangle\langle b$$

for some $\delta \in \mathbb{C}$. The second term of the sum in the second equation of (20) is just the transposition of the first one and thus we get

$$PA^T + AP^T = 2\delta b\rangle\langle b = 0.$$

Thus $\delta = 0$, that is, $PA^T = 0$ implying $Pa\rangle = 0$. □

We now investigate the conditions (15) and (16) that $\Phi(\lambda)$ must satisfy in order to be a dressing matrix. Notice that they are rational matrix functions of λ. We compare the residues of all poles of both sides. For (15), we know that the linear terms in λ vanishes as $\lambda \to \infty$ from Proposition 2 and its constant terms in λ as $\lambda \to \infty$ (19) becomes

$$U = U_0 + [B, J], \tag{24}$$

where B is defined by $\Phi(\lambda) = I_{n+2} + B\lambda^{-1} + \cdots$ as $\lambda \to \infty$. We now compute residue of (15) at $\lambda = \mu$. This leads to

$$-AJA^T + A(D_x - \mu J - U_0)P^T + P(D_x - \mu J - U_0)A^T = 0, \tag{25}$$

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where $A = b\rangle\langle a \neq 0$ satisfying (23). In (25) the first term vanishes since A is a bi-vector and J is a skew-symmetric matrix. It follows from vanishing of the second order pole at $\lambda = \mu$ that

$$A(D_x - \mu J - U_0)A^T = 0. \tag{26}$$

Thus we can see that

$$\mathcal{L}_0 a\rangle = (D_x - \mu J - U_0)a\rangle = 0 \tag{27}$$

is a solution of (25) and (26) due to (23). In fact, the general solution should be $\mathcal{L}_0 a\rangle = \gamma a\rangle$, where γ is an arbitrary scalar function of x and t. Without any loss of generality we set $\gamma = 0$. Indeed, the γ can be removed by the rescaling $a\rangle$ to $\exp(\int \gamma \mathrm{d}x)a\rangle$ and it will be shown that the matrix A is not affected by the scaling of $a\rangle$.

We carry out the similar analysis for (16) and obtain that

$$V = \Phi(0)V_0\Phi^T(0); \quad \text{(residue at } \lambda = 0) \tag{28}$$
$$A(D_t + \mu^{-1}V_0)A^T = 0; \quad \text{(double pole at } \lambda = \mu) \tag{29}$$
$$-\mu^{-2}AV_0A^T + A(D_t + \mu^{-1}V_0)P^T + P(D_t + \mu^{-1}V_0)A^T = 0. \quad \text{(residue at } \lambda = \mu) \tag{30}$$

For $A = b\rangle\langle a$, it is obviously $AV_0A^T = 0$ since V_0 is a skew-symmetric matrix. In the same way as we get (27) we can see that

$$(D_t + \mu^{-1}V_0)a\rangle = 0 \tag{31}$$

is a solution of (30) and (29). Thus for given constant vector $a_1\rangle$ satisfying $\langle a_1 a_1\rangle = 0$, we obtain that

$$a\rangle = \Psi_0(x,t,\mu)a_1\rangle. \tag{32}$$

To construct the exact solutions for the vector sine-Gordon equation (1) using (28) or (24), it is required to determine the vector $b\rangle$ in the bi-vector A using (23). The latter depends on the choice of the forms for $\Phi(\lambda)$. We will determine the vector $b\rangle$ for the kink and breather solutions in the following sections.

In the following two sections 2.1 and 2.2, we construct the exact solutions starting with the trivial solution $\beta_0 = 1$ and $\vec{\alpha}_0 = 0$ for the equation (1). Then $U_0 = 0$ and $V_0 = J$. It is easy to see that in this case the fundamental solution for (11) is

$$\Psi_0(x,t,\lambda) = \exp(\lambda Jx - \lambda^{-1}Jt) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \theta = \lambda x - \lambda^{-1}t. \tag{33}$$

Matrix $\Psi_0(x,t,\lambda)$ obviously satisfies the reduction group symmetry conditions (14).

2.1 A kink solution

Let us assume that the dressing matrix is regular at $\lambda = \infty$ and has only two simple poles at points $\lambda = \pm\mu \neq 0$. The set of poles is an orbit of the reduction group if μ is either real or pure imaginary. Then, without any loss of generality, it can be written in the form

$$\Phi(\lambda) = I_{n+2} + \frac{A}{\lambda - \mu} - \frac{QAQ}{\lambda + \mu}. \tag{34}$$

For the bi-vector $A = b\rangle\langle a \neq 0$, we now determine the vector $b\rangle$ using Proposition 4.

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Proposition 5. *Let $A = b\rangle\langle a \neq 0$ be a bi-vector . The dressing matrix (34) satisfies $\Phi(\lambda)\Phi^T(\lambda) = I_{n+2}$ if and only if*

$$\langle aa\rangle = 0; \qquad b\rangle = \frac{2\mu Qa\rangle}{\langle aQa\rangle}. \tag{35}$$

Proof. The first equation of (35) has been proved in Proposition 4. For matrix $A = b\rangle\langle a$ the second condition of (23) is

$$(I_{n+2} - \frac{Qb\rangle\langle aQ}{2\mu})a\rangle = 0, \quad \text{that is,} \quad a\rangle = \frac{Qb\rangle\langle aQa\rangle}{2\mu}.$$

This leads to the second equation of (35) since $Q^2 = I_{n+2}$. □

Remark 1. *It follows from this proposition that the bi-vector $A = b\rangle\langle a$ is parametrised by a subspace spanned by vector $a\rangle$ rather than vector $a\rangle$ itself. Indeed, scaling $a\rangle \mapsto \hat{\gamma}a\rangle$ does not change matrix A.*

Remark 2. *As we mention above, there are two possibilities for a two points orbit, namely $\mu = \pm\nu$ and $\mu = \pm i\nu$, $\nu \in \mathbb{R}$. In the first case to ensure that the solutions are real we require that the matrix A is real and consequently the vector $a\rangle$ is proportional to a real vector $a\rangle = \hat{\gamma}\hat{a}\rangle$, where $\hat{a}\rangle \in \mathbb{R}^{n+2}$. For real $\hat{a}\rangle$ it follows from the condition $\langle aa\rangle = 0$ that $\hat{a}\rangle = 0$ and so $A = 0$. Thus the case $\mu = \pm\nu$, $\nu \in \mathbb{R}$ leads to a trivial result.*

We shall see that the case $\mu = \pm i\nu$, $\nu \in \mathbb{R}$ and $\bar{A} = -QAQ$ yields a nontrivial solution.

To get the kink solution similar to the one for the classical sine-Gordon equation, we take μ to be pure imaginary, that is, $\mu = i\nu$, where $\nu \in \mathbb{R}$ as we stated in Remark 2.

Proposition 6. *A kink solution of the vector sine-Gordon system (1) on a trivial background ($\beta_0 = 1$, $\alpha_0^k = 0$) is given by*

$$\beta = 1 - 2(p_1^2 - q_1^2)(p_1\cosh\rho + q_1\sinh\rho)^{-2}, \tag{36}$$

$$\vec{\alpha} = 2\mathbf{a_1}(p_1\sinh\rho + q_1\cosh\rho)(p_1\cosh\rho + q_1\sinh\rho)^{-2}, \qquad i = 1, \ldots, n. \tag{37}$$

Here $\rho = \nu x + \nu^{-1}t$ with $\nu \in \mathbb{R}$ and $\langle a_1 = (ip_1, q_1, \mathbf{a_1}^T)$, where $p_1, q_1 \in \mathbb{R}$ and $\mathbf{a_1} \in \mathbb{R}^n$ is a constant vector satisfying $\langle a_1 a_1 \rangle = 0$.

Proof. When $\mu = i\nu$, the fundamental solution Ψ_0 given by (33) becomes

$$\Psi_0(x, t, i\nu) = \begin{pmatrix} \cosh\rho & i\sinh\rho & 0 \\ -i\sinh\rho & \cosh\rho & 0 \\ 0 & 0 & I \end{pmatrix}, \quad \rho = \nu x + \nu^{-1}t. \tag{38}$$

It follows from (32) that

$$a\rangle = \Psi_0(x, t, i\nu)a_1\rangle = (i(p_1\cosh\rho + q_1\sinh\rho), p_1\sinh\rho + q_1\cosh\rho, \mathbf{a_1}^T)^T.$$

Thus

$$\langle aQa\rangle = 2(p_1\cosh\rho + q_1\sinh\rho)^2.$$

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It follows from Proposition 5 that

$$b\rangle = \frac{i\nu}{(p_1\cosh\rho + q_1\sinh\rho)^2}(-i(p_1\cosh\rho + q_1\sinh\rho), p_1\sinh\rho + q_1\cosh\rho, \mathbf{a}_1^T)^T$$

We can now write down the matrix A. Further we use formula (28) with $V_0 = J$ and get (37) and (36) for $\vec\alpha$ and β as in the statement. $\qquad\square$

One kink solution (37) belongs to a one dimensional subspace (it is proportional to the vector $\mathbf{a}_1$) and thus it can be easily reconstructed from the kink solution of the scalar sine-Gordon equation. Indeed, if θ is a kink solution of equation $\theta_{xt} = \sin\theta$, then

$$\beta = \cos\theta, \qquad \vec\alpha = \frac{\mathbf{a}_1}{|\mathbf{a}_1|}\sin\theta$$

is the corresponding solution of the vector sine-Gordon equation.

Later we shall show that a general two kink solution belongs to two dimensional subspace $\mathrm{Span}_{\mathbb{R}}(\mathbf{a}_1, \mathbf{a}_2)$ and it cannot be obtained from solutions of the scalar sine-Gordon equation. In particular, for two-kink solutions we compute the position shifts after collision.

For one kink solution in Proposition 6, we rewrite their denominator in the following form:

$$(p_1\cosh\rho + q_1\sinh\rho)^2 = \left(\frac{p_1^2 - q_1^2}{2}\right)\left(\cosh\left(2\nu(x + \nu^{-2}t - x_0)\right) + 1\right), \ x_0 = \frac{1}{2\nu}\ln\left|\frac{p_1 - q_1}{p_1 + q_1}\right| . \quad (39)$$

Thus the kink is moving on the (x, t)-plane along the straight line and its position at time t is

$$x = -\nu^{-2}t + x_0,$$

which is only dependent on the pole $\mu = i\nu$ and the first two components of vector $a_1\rangle$. The width of the kink is of the order ν^{-1}.

2.2 A breather solution

A breather solution corresponds to the only simple poles at points of a generic orbit of the reduction group. Assuming that the dressing matrix has a simple pole at a point $\mu = \gamma + i\omega$, $\mu \neq \pm\bar\mu$, where we use $\bar\mu$ to denote the complex conjugate of μ, we have

$$\Phi(\lambda) = I_{n+2} + \frac{A}{\lambda - \mu} - \frac{QAQ}{\lambda + \mu} + \frac{\bar A}{\lambda - \bar\mu} - \frac{Q\bar AQ}{\lambda + \bar\mu} . \quad (40)$$

Assume that $A = b\rangle\langle a \neq 0$. We determine the vector $b\rangle$ using Proposition 4 and construct the solution on a trivial background ($\beta_0 = 1$, $\vec\alpha_0 = 0$).

Proposition 7. *Let $A = b\rangle\langle a \neq 0$ be a bi-vector, where $\langle a$ and $\langle b$ are two complex vectors given by $\langle a = (p, q, a^1, \cdots, a^n)$ and $\langle b = (b^1, b^2, \cdots, b^{n+2})$ respectively. The dressing matrix (40) satisfies $\Phi(\lambda)\Phi^T(\lambda) = I_{n+2}$ if and only if*

$$\langle aa\rangle = 0, \quad (41)$$

$$b^1 = \frac{2i\gamma\omega\mu\bar p}{\Delta}, \quad (42)$$

$$b^2 = \frac{2i\gamma\omega\mu}{\Delta^2}\left(\bar q(\bar\mu^2\langle\bar aa\rangle + 2\omega^2|p|^2) - 4\gamma\omega\bar p\,\mathrm{Im}(p\bar q)\right), \quad (43)$$

$$b^{k+2} = \frac{2i\gamma\omega\mu}{\Delta^2}\left(\bar a^k(\bar\mu^2\langle\bar aa\rangle + 2\omega^2|p|^2) - 4\gamma\omega\bar p\,\mathrm{Im}(p\bar a^k)\right), \quad k = 1, \ldots, n, \quad (44)$$

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where

$$\Delta = |\mu|^2 \langle \bar{a}a \rangle - 2\omega^2 |p|^2 \,. \tag{45}$$

Proof. Equation (41) has been proved in Proposition 4. For matrix $A = b\rangle\langle a$ the second condition of (23) is

$$P a\rangle = (I_{n+2} - \frac{QAQ}{2\mu} + \frac{\bar{A}}{\mu - \bar{\mu}} - \frac{Q\bar{A}Q}{\mu + \bar{\mu}})a\rangle = 0 \,,$$

that is,

$$a\rangle - \frac{Qb\rangle\langle aQa\rangle}{2\mu} + \frac{\bar{b}\rangle\langle \bar{a}a\rangle}{\mu - \bar{\mu}} - \frac{Q\bar{b}\rangle\langle \bar{a}Qa\rangle}{\mu + \bar{\mu}} = 0. \tag{46}$$

We solve this linear system of equations (46) for unknowns $b^1, \ldots, b^{n+2}$ by taking into account (41) and obtain the formulas in the statement. $\qquad\square$

Proposition 8. *A breather solution of the vector sine-Gordon system (1) on a trivial background* ($\beta_0 = 1,\ \vec{\alpha}_0 = 0$) *is given by*

$$\beta = 1 - \frac{16\gamma^2\omega^2}{\Delta^2} \left(\langle \bar{a}a \rangle |q|^2 - 2(\text{Im}(p\bar{q}))^2 \right) , \tag{47}$$

$$\alpha^k = \frac{8\gamma\omega}{\Delta^2} \left(\text{Im}((\bar{\mu}^2\langle \bar{a}a \rangle + 2\omega^2|p|^2)q\bar{a}^k) - 4\gamma\omega\,\text{Im}(q\bar{p})\,\text{Im}(p\bar{a}^k) \right) , \qquad k = 1, \ldots, n, \tag{48}$$

where Δ is given by (45) and the components of the vectors $\langle a = (p, q, a^1, \ldots, a^n)$ are the following functions of x, t given by

$$
\begin{aligned}
p &= \cos(\phi + i\eta)p_1 + \sin(\phi + i\eta)q_1, \\
q &= \cos(\phi + i\eta)q_1 - \sin(\phi + i\eta)p_1, \\
a^k &= a_1^k, \qquad k = 1, \ldots, n,
\end{aligned}
\tag{49}
$$

where $\phi = \gamma x - \frac{\gamma}{(\gamma^2+\omega^2)}t$, $\eta = \omega x + \frac{\omega}{(\gamma^2+\omega^2)}t$, vector $\langle a_1 = (p_1, q_1, a_1^1, \ldots, a_1^n)$ is an arbitrary complex constant vector satisfying the condition $\langle a_1 a_1 \rangle = 0$.

Proof. When $\mu = \gamma + i\omega$, the fundamental solution Ψ_0 given by (33) becomes

$$\Psi_0(\mu, x, t) = \begin{pmatrix} \cos(\phi + i\eta) & i\sin(\phi + i\eta) & 0 \\ -i\sin(\phi + i\eta) & \cos(\phi + i\eta) & 0 \\ 0 & 0 & I \end{pmatrix}, \tag{50}$$

where $\phi = \gamma\left(x - (\gamma^2 + \omega^2)^{-1}t\right)$, $\eta = \omega\left(x + (\gamma^2 + \omega^2)^{-1}t\right)$. Using (32) we get (49) for $a\rangle$.

It follows from (28) that

$$V = \Phi(0)J\Phi^T(0),$$

where $\Phi(\lambda)$ is defined in (40). Making substitution of $A = b\rangle\langle a$ in (40) and taking into account (6) we obtain

$$\beta = V_{1,2} = \frac{1}{|\mu|^4}(|\mu|^2 - 4\,\text{Re}(\bar{\mu}pb^1))(|\mu|^2 - 4\,\text{Re}(\bar{\mu}qb^2)); \tag{51}$$

$$\alpha_k = V_{1,k+2} = -\frac{4}{|\mu|^4}(|\mu|^2 - 4\,\text{Re}(\bar{\mu}pb^1))\,\text{Re}(\bar{\mu}qb^{k+2}), \quad k = 1, \cdots, n. \tag{52}$$

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From (42) in Proposition 7 it follows that

$$\mathrm{Re}(\bar{\mu} p b^1) = \mathrm{Re}\left(\bar{\mu} p \frac{2i\gamma\omega\mu\bar{p}}{\Delta}\right) = 0.$$

Thus $(|\mu|^2 - 4\,\mathrm{Re}(\bar{\mu} p b^1)) = |\mu|^2$, and therefore

$$\beta = 1 - \frac{4}{|\mu|^2}\,\mathrm{Re}(\bar{\mu} q b^2)\,, \tag{53}$$

$$\alpha_k = -\frac{4}{|\mu|^2}\,\mathrm{Re}(\bar{\mu} q b^{k+2})\,, \qquad k = 1, \ldots, n\,. \tag{54}$$

The substitution of (43) and (44) in the above equations leads to (47) and (48). $\qquad\square$

The breather solution (47), (48) represents a periodically oscillating localized wave of a characteristic width $\sim \omega^{-1}$ moving with the constant velocity $(\gamma^2 + \omega^2)^{-1}$. Thus the width of the wave, its speed and frequency of oscillations depend on the position of complex pole μ only. The location of the wave and the phase of its oscillations are determined by p_1, q_1, i.e. the first two entries of the vector $a_1\rangle$. These can be found using the denominator Δ defined by (45) as follows:

$$
\begin{aligned}
\Delta &= |\mu|^2(\langle\bar{a}_1 a_1\rangle - |p_1|^2 - |q_1|^2) + \gamma^2(|p_1|^2 + |q_1|^2)\cosh(2\eta) - 2\gamma^2\,\mathrm{Im}(\bar{p}_1 q_1)\sinh(2\eta) + \\
&\quad + \omega^2(|q_1|^2 - |p_1|^2)\cos(2\phi) - 2\omega^2\,\mathrm{Re}(p_1\bar{q}_1)\sin(2\phi) \\
&= (\gamma^2 + \omega^2)\sum_{k=1}^{n}|a_1^k|^2 + |p_1^2 + q_1^2|\left(\gamma^2\cosh(2\eta - 2\eta_0) - \omega^2\cos(2\phi - \phi_0)\right),
\end{aligned}
$$

where

$$\eta_0 = \frac{1}{2}\ln\left|\frac{p_1 - q_1 i}{p_1 + q_1 i}\right| \tag{55}$$

and ϕ_0 satisfies $\cos\phi_0 = \frac{|p_1|^2 - |q_1|^2}{|p_1^2 + q_1^2|}$ and $\sin\phi_0 = \frac{2\,\mathrm{Re}(p_1\bar{q}_1)}{|p_1^2 + q_1^2|}$. Thus we have

$$\phi_0 = \arg\left(\frac{p_1 + q_1 i}{p_1 - q_1 i}\right)\,. \tag{56}$$

Thus the breather is moving on the (x, t)-plane along the straight line and its position at time t is given by

$$x = -(\gamma^2 + \omega^2)^{-1}t + x_0, \quad x_0 = \omega^{-1}\eta_0,$$

which is only dependent on the position of the pole μ and the first two components of vector $a_1\rangle$. We define the phase of this breather as ϕ_0. In Figure 1, we plot one breather solution when $n = 2$ for α_1 and α_2 with $\mu = 2 + \frac{1}{4}i$ and $\langle a_1 = (i, 2, 3i, \sqrt{6})$ at time $t = 3$. The position of this breather is at $x \simeq -2.936$ and its phase $\phi_0 = 0$.

The breather solution (47), (48) also depends on the constant complex n dimensional vector $\mathbf{a}_1 = (a_1^1, \ldots, a_1^n)^T \in \mathbb{C}^n$, which is the last n components of the vector $a_1\rangle = (p_1, q_1, a_1^1, \ldots, a_1^n)^T$. In general, the vector $\vec{\alpha}$ of the solution (48) belongs to the two dimensional subspace $W(\mathbf{a}_1) = \mathrm{Span}_{\mathbb{R}}(\mathrm{Re}\,\mathbf{a}_1, \mathrm{Im}\,\mathbf{a}_1) \subseteq \mathbb{R}^n$. If the subspace $W(\mathbf{a}_1)$ is one dimensional ($\mathrm{Re}\,\mathbf{a}_1 \sim \mathrm{Im}\,\mathbf{a}_1$) then the

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obtained one breather solution represents the well known breather solution of the scalar sine-Gordon equation (with $\theta_{xt} = \sin\theta : \ \beta = \cos\theta, \ \vec{\alpha} = \frac{\mathbf{a_1}}{|\mathbf{a_1}|}\sin\theta$), which is a subsystem of the vector sine-Gordon equation. Solution (48) corresponding to a two-dimensional subspace $W(\mathbf{a})$ cannot be obtained from a solution of the scalar sine-Gordon equation. However, it can be obtained from the breather solution of the two-dimensional vector sine-Gordon equation.

Similar to the kink case (see Remark 1), the breather solution is parametrised by a point on the complex Grassmanian $\mathrm{Gr}_{1,2+n} = \mathbb{CP}^{n+1}$ rather than the complex vector $a\rangle$ itself.

3 Multi-soliton solutions and soliton interactions

The vector sine-Gordon equation has two types of one soliton solutions, namely the kink and breather solutions discussed in sections 2.1 and 2.2. The set of poles of the dressing matrix $\Phi(\lambda)$ for multi-soliton solutions is a union of a finite number of the kink and breather type orbits. We can construct the dressing matrix recursively by presenting it as a composition of elementary dressing factors of the kink and breather type. Another approach is based on a representation of the dressing matrix in the form of partial fractions with respect to the spectral parameter λ. The latter approach is more conventional, but leads to a big system of linear algebraic equations, the size of which is determined by the number of solitons in the multi-soliton solution.

In order to construct a general multi-soliton dressing of the vector sine-Gordon equation (1) with m_1 kinks and m_2 breathers we need to choose the following data:

- a set of m_1 distinct positive real numbers $\{\nu_s\}_{s=1}^{m_1}$,

- a set of m_1 vectors $a_s\rangle, \ s = 1, \ldots, m_1$ of the form $a_s\rangle = (ip_s, q_s, a_s^1, \ldots, a_s^n)^T$, such that

$$\langle a_s a_s\rangle = -p_s^2 + q_s^2 + (a_s^1)^2 + \cdots + (a_s^n)^2 = 0, \qquad p_s, q_s, a_s^1, \ldots, a_s^n \in \mathbb{R}$$

  and the real "sub-vector" $(a_s^1, \ldots, a_s^n)$ is non-zero.

- a set of m_2 distinct complex numbers $\{\mu_s = \gamma_s + i\omega_s \,|\, \gamma_s > 0, \ \omega_s > 0\}_{s=m_1+1}^{m_1+m_2}$,

- a set of m_2 complex vectors $a_s\rangle = (p_s, q_s, a_s^1, \ldots, a_s^n)^T, \ s = m_1 + 1, \ldots, m_1 + m_2$ and $p_s, q_s, a_s^1, \ldots, a_s^n \in \mathbb{C}$ such that $\langle a_s a_s\rangle = 0$ and "sub-vectors" (p_s, q_s) and $(a_s^1, \ldots, a_s^n)$ are both non-zero.

Let us first construct the m_1-kink solution. We start from a trivial solution of the vector sine-Gordon equation $\beta_0 = 1, \vec{\alpha}_0 = 0$, so that

$$U_0 = 0, \quad V_0 = J, \quad \Psi_0(\lambda, x, t) = \exp(\lambda x J - \lambda^{-1} t J).$$

We denote the kink dressing matrix constructed in the Section 2.1 as

$$\widehat{\Phi}(\lambda, \mu, a\rangle) = I_{n+2} + \frac{1}{(\lambda - \mu)}\frac{Qa\rangle\langle a}{\langle aQa\rangle} - \frac{1}{(\lambda + \mu)}\frac{a\rangle\langle aQ}{\langle aQa\rangle}.$$

Now the multi-kink solution of the vector sine-Gordon equation and the corresponding the fundamental solution for the associated linear problems can be found recursively

$$
\begin{aligned}
&V_s = \widehat{\Phi}(0, i\nu_s, \Psi_{s-1}(i\nu_s, x, t)a_s\rangle)V_{s-1}\widehat{\Phi}^T(0, i\nu_s, \Psi_{s-1}(i\nu_s, x, t)a_s\rangle); \\
&\Psi_s(\lambda, x, t) = \widehat{\Phi}(\lambda, i\nu_s, \Psi_{s-1}(i\nu_s, x, t)a_s\rangle)\Psi_{s-1}(\lambda, x, t);
\end{aligned}
\qquad s = 1, 2, \ldots, m_1.
$$

Having constructed the m_1-kink solution V_{m_1} and the corresponding fundamental solution of the linear problem $\Psi_{m_1}(\lambda, x, t)$ we can add m_2 breathers using the breather dressing matrix constructed in Section 2.2 (see Proposition 7) denoted as

$$\widetilde{\Phi}(\lambda, \mu, a\rangle) = I_{n+2} + \frac{A}{\lambda - \mu} - \frac{QAQ}{\lambda + \mu} + \frac{\bar{A}}{\lambda - \bar{\mu}} - \frac{Q\bar{A}Q}{\lambda + \bar{\mu}}$$

in a similar recursive way

$$V_s = \widetilde{\Phi}(0, \mu_s, \Psi_{s-1}(\mu_s, x, t)a_s\rangle)V_{s-1}\widetilde{\Phi}^T(0, \mu_s, \Psi_{s-1}(\mu_s, x, t)a_s\rangle);$$
$$\Psi_s(\lambda, x, t) = \widetilde{\Phi}(\lambda, \mu_s, \Psi_{s-1}(\mu_s, x, t)a_s\rangle)\Psi_{s-1}(\lambda, x, t); \qquad s = m_1 + 1, ..., m_1 + m_2.$$

Moreover, we could change the order of dressings by creating m_2-breather solutions and then adding m_1 kinks, or even make dressings in an arbitrary order. Such recursive approach is useful for the study of the effects of soliton interactions as shown in the following sections.

3.1   Interaction of two kinks

In this section, we study the interaction of two kinks. A two-kink solution corresponding to the sets

$$\{\nu_1, a_1\rangle = (ip_1, q_1, a_1^1, \ldots, a_1^n)^T\}, \quad \{\nu_2, a_2\rangle = (ip_2, q_2, a_2^1, \ldots, a_2^n)^T\}, \quad 0 < \nu_1 < \nu_2$$

represents two kinks moving to the left with speeds ν_1^{-2} and ν_2^{-2} respectively. Their trajectories on the (x, t)–plane intersect near the point (X, T), which can be found from the system of equations (c.f. (39))

$$\nu_1 X + \nu_1^{-1}T = \frac{1}{2}\ln\left|\frac{p_1 - q_1}{p_1 + q_1}\right|, \qquad \nu_2 X + \nu_2^{-1}T = \frac{1}{2}\ln\left|\frac{p_2 - q_2}{p_2 + q_2}\right|.$$

Far from this point as $t \to \pm\infty$ the solution tends to the sum of two simple kinks moving along the trajectories

$$\nu_1(x - x_1^\pm) + \nu_1^{-1}t = 0, \qquad \nu_2(x - x_2^\pm) + \nu_2^{-1}t = 0$$

respectively. Thus, the effect of the interaction is a shift of a straight line trajectory of each kink. In Figure 2, we plot a two-kink solutions when $n = 2$ for α_1 and α_2 at time $t = 6$ with

$$\nu_1 = 0.9, \qquad \langle a_1 = (i, 0, 0, 1); \qquad \nu_2 = 1.1, \qquad \langle a_2 = (i, 0, 1, 0).$$

These two kinks intersect at the point $x = 0$. At $t = 6$, the position of the kink corresponding to ν_1 is at $x = \simeq -7.41$ and the other kink is at $x = -4.96$.

In order to determine the shift $x_2 = x_2^+ - x_2^-$ of the trajectory of the second kink we consider the limits $t \to \pm\infty$ assuming $\rho_2 = \nu_2 x + \nu_2^{-1}t$ to be finite. In this case

$$\rho_1 = \nu_1 x + \nu_1^{-1}t = \frac{\nu_1}{\nu_2}\rho_2 + \frac{t}{\nu_1}\left(1 - \frac{\nu_1^2}{\nu_2^2}\right) \to \pm\infty, \quad \text{as} \ \ t \to \pm\infty.$$

Therefore

$$\Psi_1(\lambda, x, t)a_2\rangle = \widehat{\Phi}(\lambda, i\nu_1, \Psi_0(i\nu_1, x, t)a_1\rangle)\Psi_0(\lambda, x, t)a_2\rangle \to \widehat{\Phi}_1^\pm(\lambda)\Psi_0(\lambda, x, t)a_2, \quad \text{as} \ \ t \to \pm\infty, \quad (57)$$

where

$$\widehat{\Phi}_1^\pm(\lambda) = \lim_{\rho_1 \to \pm\infty} \widehat{\Phi}(\lambda, i\nu_1, \Psi_0(i\nu_1, x, t)a_1\rangle).$$

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The latter limit can be easily computed since $\Psi_0(i\nu_1, x, t)$ is a function of ρ_1 only (38). In particular we have

$$\lim_{\rho_1 \to \pm\infty} \frac{Q\Psi_0(i\nu_1, x, t)a_1\rangle\langle a_1\Psi_0(i\nu_1, x, t)^T}{\langle a_1\Psi_0(i\nu_1, x, t)^T Q\Psi_0(i\nu_1, x, t)a_1\rangle} = \frac{1}{2}\begin{pmatrix} 1 & \mp i & 0 \\ \pm i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and therefore

$$\widehat{\Phi}_1^{\pm}(\lambda) = \frac{1}{\lambda^2 + \nu_1^2}\begin{pmatrix} \lambda^2 - \nu_1^2 & \pm 2\lambda\nu_1 & 0 \\ \mp 2\lambda\nu_1 & \lambda^2 - \nu_1^2 & 0 \\ 0 & 0 & (\lambda^2 + \nu_1^2)I \end{pmatrix}. \tag{58}$$

Noting that $[\widehat{\Phi}_1^{\pm}(\lambda), \Psi_0(\lambda, x, t)] = 0$ we can represent the limits (57) in the form

$$\widehat{\Phi}_1^{\pm}(i\nu_2)\Psi_0(i\nu_2, x, t)a_2\rangle = \Psi_0(i\nu_2, x, t)a_2^{\pm}\rangle$$

where

$$a_2^{\pm}\rangle = \begin{pmatrix} i(\gamma_1 p_2 \mp \gamma_2 q_2) \\ \gamma_1 q_2 \mp \gamma_2 p_2 \\ \mathbf{a}_2 \end{pmatrix}, \qquad \gamma_1 = \frac{\nu_2^2 + \nu_1^2}{\nu_2^2 - \nu_1^2}, \qquad \gamma_2 = \frac{2\nu_1\nu_2}{\nu_2^2 - \nu_1^2}.$$

Using formula (39) we obtain

$$x_2^+ = \lim_{\rho \to +\infty} x_0 = \frac{1}{2\nu_2}\ln\Big|\frac{(\gamma_1 + \gamma_2)(p_2 - q_2)}{(\gamma_1 - \gamma_2)(p_2 + q_2)}\Big|;$$

$$x_2^- = \lim_{\rho \to -\infty} x_0 = \frac{1}{2\nu_2}\ln\Big|\frac{(\gamma_1 - \gamma_2)(p_2 - q_2)}{(\gamma_1 + \gamma_2)(p_2 + q_2)}\Big|.$$

Therefore, the position shift for the kink with speed $1/\nu_2^2$ is

$$\Delta x_2 = x_2^+ - x_2^- = \frac{1}{2\nu_2}\ln\left(\frac{\gamma_1 + \gamma_2}{\gamma_1 - \gamma_2}\right)^2 = \frac{1}{\nu_2}\ln\left(\frac{\nu_1 + \nu_2}{\nu_1 - \nu_2}\right)^2 = \frac{2}{\nu_2}\ln\frac{\nu_2 + \nu_1}{\nu_2 - \nu_1}.$$

In a similar way one can obtain that the position shift for the kink with speed $1/\nu_1^2$ is

$$\Delta x_1 = x_1^+ - x_1^- = -\frac{1}{\nu_1}\ln\left(\frac{\nu_1 + \nu_2}{\nu_2 - \nu_1}\right)^2 = \frac{2}{\nu_1}\ln\frac{\nu_2 - \nu_1}{\nu_2 + \nu_1}.$$

Notice that these position shifts Δx_1 and Δx_2 for kinks after collision only depend on the pole position ν_1 and ν_2. They are exactly the same as that of the scalar sine-Gordon equation (3) given in [14].

3.2 Interaction of two breathers

A two-breather solution corresponding to the sets

$$\{\mu_1, a_1\rangle = (p_1, q_1, a_1^1, \ldots, a_1^n)^T\}, \quad \{\mu_2, a_2\rangle = (p_2, q_2, a_2^1, \ldots, a_2^n)^T\}, \quad 0 < |\mu_1| < |\mu_2|$$

represents two breathers moving to the left with speeds $|\mu_1|^{-2}$ and $|\mu_2|^{-2}$ respectively. Their trajectories on the (x, t)–plane intersect near the point (X, T), which can be found from the system of equations (c.f. (55))

$$\omega_l X + \omega_l(\gamma_l^2 + \omega_l^2)^{-1}T = \frac{1}{2}\ln\Big|\frac{p_l - q_l i}{p_l + q_l i}\Big|, \quad l = 1, 2.$$

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Far from this point as $t \to \pm\infty$ the solution tends to the sum of two simple breathers moving along the trajectories

$$x - x_l^\pm + (\gamma_l^2 + \omega_l^2)^{-1}t = 0, \qquad l = 1, 2.$$

with phase $\phi_l^\pm$. Thus, the effect of the interaction is a shift of a straight line trajectory and the change of the phase for each breather.

In order to determine the shift $\eta_2 = \eta_2^+ - \eta_2^-$ of the trajectory and the change of the phase for the second breather we consider the limits $t \to \pm\infty$ assuming $\rho_2 = x + (\gamma_2^2 + \omega_2^2)^{-1}t$ to be finite. In this case

$$\rho_1 = x + (\gamma_1^2 + \omega_1^2)^{-1}t = \rho_2 + \left(\frac{1}{\gamma_1^2 + \omega_1^2} - \frac{1}{\gamma_2^2 + \omega_2^2} \right) t \to \pm\infty, \quad \text{as} \ \ t \to \pm\infty.$$

Therefore

$$\Psi_1(\lambda, x, t)a_2\rangle = \widetilde{\Phi}(\lambda, \mu_1, \Psi_0(\mu_1, x, t)a_1\rangle)\Psi_0(\lambda, x, t)a_2\rangle \to \widetilde{\Phi}_1^\pm(\lambda)\Psi_0(\lambda, x, t)a_2, \quad \text{as} \ \ t \to \pm\infty, \quad (59)$$

where

$$\widetilde{\Phi}_1^\pm(\lambda) = \lim_{\rho_1 \to \pm\infty} \widetilde{\Phi}(\lambda, \mu_1, \Psi_0(\mu_1, x, t)a_1\rangle).$$

The latter limit can be easily computed using $\Psi_0(\mu_1, x, t)$ (50). In particular we have

$$\lim_{\rho_1 \to \pm\infty} A = \frac{\omega_1 \mu_1}{\gamma_1} \begin{pmatrix} i & \pm 1 & 0 \\ \mp 1 & i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and therefore

$$\widetilde{\Phi}_1^\pm(\lambda) = \begin{pmatrix} \frac{(\lambda^2 - |\mu_1|^2)^2 - 4\lambda^2\omega_1^2}{(\lambda^2 - \mu_1^2)(\lambda^2 - \bar{\mu}_1^2)} & \pm\frac{4\lambda\omega_1(\lambda^2 - |\mu_1|^2)}{(\lambda^2 - \mu_1^2)(\lambda^2 - \bar{\mu}_1^2)} & 0 \\ \mp\frac{4\lambda\omega_1(\lambda^2 - |\mu_1|^2)}{(\lambda^2 - \mu_1^2)(\lambda^2 - \bar{\mu}_1^2)} & \frac{(\lambda^2 - |\mu_1|^2)^2 - 4\lambda^2\omega_1^2}{(\lambda^2 - \mu_1^2)(\lambda^2 - \bar{\mu}_1^2)} & 0 \\ 0 & 0 & I \end{pmatrix}. \quad (60)$$

Noting that $[\widetilde{\Phi}_1^\pm(\lambda), \Psi_0(\lambda, x, t)] = 0$ we can represent the limits (59) in the form

$$\widetilde{\Phi}_1^\pm(\mu_2)\Psi_0(\mu_2, x, t)a_2\rangle = \Psi_0(\mu_2, x, t)a_2^\pm\rangle.$$

We denote $(1, 1)$ and $(1, 2)$ entries of matrix $\widetilde{\Phi}_1^+(\mu_2)$ by κ_1 and κ_2, respectively. Notice that

$$\kappa_1 - \kappa_2 i = \frac{(\mu_2^2 - |\mu_1|^2 - 2\mu_2\omega_1 i)^2}{(\mu_2^2 - \mu_1^2)(\mu_2^2 - \bar{\mu}_1^2)} = \frac{(\mu_2 - \mu_1)^2(\mu_2 + \bar{\mu}_1)^2}{(\mu_2^2 - \mu_1^2)(\mu_2^2 - \bar{\mu}_1^2)};$$

$$\kappa_1 + \kappa_2 i = \frac{(\mu_2^2 - |\mu_1|^2 + 2\mu_2\omega_1 i)^2}{(\mu_2^2 - \mu_1^2)(\mu_2^2 - \bar{\mu}_1^2)} = \frac{(\mu_2 + \mu_1)^2(\mu_2 - \bar{\mu}_1)^2}{(\mu_2^2 - \mu_1^2)(\mu_2^2 - \bar{\mu}_1^2)}$$

Using formula for η_0 in (55) we obtain

$$x_2^+ = \frac{1}{\omega_2} \lim_{\rho_1 \to +\infty} \eta_0 = \frac{1}{2\omega_2} \ln \left| \frac{(\kappa_1 + \kappa_2 i)(p_2 - q_2 i)}{\kappa_1 - \kappa_2 i)(p_2 + q_2 i)} \right| = \frac{1}{2\omega_2} \ln \left| \frac{(\mu_1 + \mu_2)^2(\bar{\mu}_1 - \mu_2)^2(p_2 - q_2 i)}{(\mu_1 - \mu_2)^2(\bar{\mu}_1 + \mu_2)^2(p_2 + q_2 i)} \right|;$$

$$x_2^- = \frac{1}{\omega_2} \lim_{\rho_1 \to -\infty} \eta_0 = \frac{1}{2\omega_2} \ln \left| \frac{(\kappa_1 - \kappa_2 i)(p_2 - q_2 i)}{\kappa_1 + \kappa_2 i)(p_2 + q_2 i)} \right| = \frac{1}{2\omega_2} \ln \left| \frac{(\mu_1 - \mu_2)^2(\bar{\mu}_1 + \mu_2)^2(p_2 - q_2 i)}{(\mu_1 + \mu_2)^2(\bar{\mu}_1 - \mu_2)^2(p_2 + q_2 i)} \right|.$$

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Therefore, the position shift for the breather with speed $1/|\mu_2|^2$ is

$$\Delta x_2 = x_2^+ - x_2^- = \frac{2}{\omega_2} \ln \Big| \frac{(\mu_1 + \mu_2)(\bar{\mu}_1 - \mu_2)}{(\mu_1 - \mu_2)(\bar{\mu}_1 + \mu_2)} \Big|.$$

Using the formula for ϕ_0 in (56), we compute the phase shift for this breather. Indeed, we have

$$\phi_2^+ = \lim_{\rho_1 \to +\infty} \phi_0 = \arg \left(\frac{(\mu_1 - \mu_2)^2 (\bar{\mu}_1 + \mu_2)^2 (p_2 + q_2 i)}{(\mu_1 + \mu_2)^2 (\bar{\mu}_1 - \mu_2)^2 (p_2 - q_2 i)} \right);$$
$$\phi_2^- = \lim_{\rho_1 \to +\infty} \phi_0 = \arg \left(\frac{(\mu_1 + \mu_2)^2 (\bar{\mu}_1 - \mu_2)^2 (p_2 + q_2 i)}{(\mu_1 - \mu_2)^2 (\bar{\mu}_1 + \mu_2)^2 (p_2 - q_2 i)} \right).$$

Hence the phase shift is

$$\Delta \phi_2 = \phi_2^+ - \phi_2^- = 2 \arg \left(\frac{(\mu_1 - \mu_2)^2 (\bar{\mu}_1 + \mu_2)^2}{(\mu_1 + \mu_2)^2 (\bar{\mu}_1 - \mu_2)^2} \right) = 4 \arg \left(\frac{(\mu_1 - \mu_2)(\bar{\mu}_1 + \mu_2)}{(\mu_1 + \mu_2)(\bar{\mu}_1 - \mu_2)} \right).$$

In a similar way we obtain that the position and phase shifts for the breather with speed $1/|\mu_1|^2$ are

$$\Delta x_1 = x_1^+ - x_1^- = \frac{2}{\omega_1} \ln \Big| \frac{(\mu_1 - \mu_2)(\bar{\mu}_1 + \mu_2)}{(\mu_1 + \mu_2)(\bar{\mu}_1 - \mu_2)} \Big|;$$
$$\Delta \phi_1 = \phi_1^+ - \phi_1^- = 4 \arg \left(\frac{(\mu_1 + \mu_2)(\bar{\mu}_1 - \mu_2)}{(\mu_1 - \mu_2)(\bar{\mu}_1 + \mu_2)} \right).$$

Similar to the case of interaction of two kinks, the position and phase shifts for breathers after collision only depend on the pole position μ_1 and μ_2, which are exactly the same as that of the scalar sine-Gordon equation (3) given in [14].

3.3   Interaction of one kink and one breather

A kink-breather or breather-kink solution corresponding to the sets

$$\{i\nu, a\rangle = (i\hat{p}, \hat{q}, \hat{a}^1, \dots, \hat{a}^n)^T\}, \quad \{\mu, \tilde{a}\rangle = (\tilde{p}, \tilde{q}, \tilde{a}^1, \dots, \tilde{a}^n)^T\}, \quad 0 < \nu < |\mu|,$$

where $\nu, \hat{p}, \hat{q}, \hat{a}^1, \dots, \hat{a}^n \in \mathbb{R}$ and $\mu = \gamma + i\omega, \tilde{p}, \tilde{q}, \tilde{a}^1, \dots, \tilde{a}^n \in \mathbb{C}$, represents one kink and one breather moving to the left with speeds ν^{-2} and $|\mu|^{-2}$ respectively. Their trajectories on the (x, t)–plane intersect near the point (X, T), which can be found from the system of equations (cf. (39))

$$\nu X + \nu^{-1} T = \frac{1}{2} \ln \Big| \frac{\hat{p} - \hat{q}}{\hat{p} + \hat{q}} \Big|, \qquad \omega X + \omega (\gamma^2 + \omega^2)^{-1} T = \frac{1}{2} \ln \Big| \frac{\tilde{p} - \tilde{q}i}{\tilde{p} + \tilde{q}i} \Big|.$$

Far from this point as $t \to \pm\infty$ the solution tends to the sum of one kink and one breather moving along the trajectories

$$\nu(x - \hat{x}^\pm) + \nu^{-1} t = 0, \qquad (x - \tilde{x}^\pm) + (\gamma^2 + \omega^2)^{-1} t = 0$$

respectively, and the phase for the breather is $\phi^\pm$. Thus, the effect of the interaction is a shift of a straight line trajectory of each of them.

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In order to determine the shift $\Delta\tilde{x} = \tilde{x}^+ - \tilde{x}^-$ of the trajectory and phase shift $\Delta\phi = \phi^+ - \phi^-$ of the breather we consider the limits $t \to \pm\infty$ assuming $\tilde{\rho} = x + (\gamma^2 + \omega^2)^{-1}t$ to be finite. In this case

$$\hat{\rho} = \nu x + \nu^{-1}t = \nu\left(\tilde{\rho} + \left(\frac{1}{\nu^2} - \frac{1}{\gamma^2 + \omega^2}\right)t\right) \to \pm\infty, \quad \text{as} \ \ t \to \pm\infty.$$

Therefore

$$\Psi_1(\lambda, x, t)\tilde{a}\rangle = \widehat{\Phi}(\lambda, i\nu, \Psi_0(i\nu, x, t)\hat{a}))\Psi_0(\lambda, x, t)\tilde{a}\rangle \to \widehat{\Phi}_1^\pm(\lambda)\Psi_0(\lambda, x, t)\tilde{a}, \quad \text{as} \ \ t \to \pm\infty, \qquad (61)$$

where

$$\widehat{\Phi}_1^\pm(\lambda) = \lim_{\hat{\rho}\to\pm\infty} \widehat{\Phi}(\lambda, i\nu, \Psi_0(i\nu, x, t)\hat{a})) = \frac{1}{\lambda^2 + \nu^2}\begin{pmatrix} \lambda^2 - \nu^2 & \pm 2\lambda\nu & 0 \\ \mp 2\lambda\nu & \lambda^2 - \nu^2 & 0 \\ 0 & 0 & (\lambda^2 + \nu^2)I \end{pmatrix}.$$

according to (58). Noting that $[\widehat{\Phi}_1^\pm(\lambda), \Psi_0(\lambda, x, t)] = 0$ we represent the limits (61) in the form

$$\widehat{\Phi}_1^\pm(\mu)\Psi_0(\mu, x, t)\tilde{a}\rangle = \Psi_0(\mu, x, t)\tilde{a}^\pm\rangle$$

where

$$\langle\tilde{a}^\pm = \left(\frac{\mu^2 - \nu^2}{\mu^2 + \nu^2}\tilde{p} \pm \frac{2\mu\nu}{\mu^2 + \nu^2}\tilde{q}, \mp\frac{2\mu\nu}{\mu^2 + \nu^2}\tilde{p} + \frac{\mu^2 - \nu^2}{\mu^2 + \nu^2}\tilde{q}, \tilde{a}^1, \ldots, \tilde{a}^n\right)$$

Using formula (55) we obtain

$$\tilde{x}^+ = \frac{1}{\omega}\lim_{\hat{\rho}\to+\infty}\eta_0 = \frac{1}{2\omega}\ln\frac{((\nu+\omega)^2 + \gamma^2)|\tilde{p} - \tilde{q}i|}{((\nu-\omega)^2 + \gamma^2)|\tilde{p} + \tilde{q}i|};$$
$$\tilde{x}^- = \frac{1}{\omega}\lim_{\hat{\rho}\to-\infty}\eta_0 = \frac{1}{2\omega}\ln\frac{((\nu-\omega)^2 + \gamma^2)|\tilde{p} - \tilde{q}i|}{((\nu+\omega)^2 + \gamma^2)|\tilde{p} + \tilde{q}i|}.$$

It follows that the position shift for the breather is

$$\Delta\tilde{x} = \tilde{x}^+ - \tilde{x}^- = \frac{2}{\omega}\ln\left|\frac{\mu + \nu i}{\mu - \nu i}\right| = \frac{1}{\omega}\ln\frac{(\nu+\omega)^2 + \gamma^2}{(\nu-\omega)^2 + \gamma^2}.$$

Using formula (56) we obtain

$$\phi^+ = \lim_{\hat{\rho}\to+\infty}\phi_0 = \arg\left(\frac{(\mu - \nu i)^2(\tilde{p} + \tilde{q}i)}{(\mu + \nu i)^2(\tilde{p} - \tilde{q}i)}\right); \quad \phi^- = \lim_{\hat{\rho}\to-\infty}\phi_0 = \arg\left(\frac{(\mu + \nu i)^2(\tilde{p} + \tilde{q}i)}{(\mu - \nu i)^2(\tilde{p} - \tilde{q}i)}\right).$$

Hence the phase shift for the breather is

$$\Delta\phi = \phi^+ - \phi^- = 4\arg\left(\frac{\mu - \nu i}{\mu + \nu i}\right)$$

which are only dependent on the positions of poles, the values of ν and μ.

In order to determine the shift $\Delta\hat{x} = \hat{x}^+ - \hat{x}^-$ of the trajectory of the kink we consider the limits $t \to \pm\infty$ assuming $\hat{\rho} = \nu x + \nu^{-1}t$ to be finite. In this case

$$\tilde{\rho} = x + (\gamma^2 + \omega^2)^{-1}t = \frac{\hat{\rho}}{\nu} - \left(\frac{1}{\nu^2} - \frac{1}{\gamma^2 + \omega^2}t\right) \to \mp\infty, \quad \text{as} \ \ t \to \pm\infty.$$

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Therefore

$$\Psi_1(\lambda, x, t)\hat{a}\rangle = \widetilde{\Phi}(\lambda, \mu, \Psi_0(\mu, x, t)\tilde{a}\rangle)\Psi_0(\lambda, x, t)\hat{a}\rangle \to \widetilde{\Phi}_1^{\pm}(\lambda)\Psi_0(\lambda, x, t)\hat{a}, \quad \text{as } t \to \pm\infty, \qquad (62)$$

where

$$\widetilde{\Phi}_1^{\pm}(\lambda) = \lim_{\tilde{\rho} \to \mp\infty} \widetilde{\Phi}(\lambda, \mu, \Psi_0(\mu, x, t)\tilde{a}\rangle) = \begin{pmatrix} \frac{(\lambda^2-|\mu|^2)^2-4\lambda^2\omega^2}{(\lambda^2-\mu^2)(\lambda^2-\bar{\mu}^2)} & \mp\frac{4\lambda\omega(\lambda^2-|\mu|^2)}{(\lambda^2-\mu^2)(\lambda^2-\bar{\mu}^2)} & 0 \\ \pm\frac{4\lambda\omega(\lambda^2-|\mu|^2)}{(\lambda^2-\mu^2)(\lambda^2-\bar{\mu}^2)} & \frac{(\lambda^2-|\mu|^2)^2-4\lambda^2\omega^2}{(\lambda^2-\mu^2)(\lambda^2-\bar{\mu}^2)} & 0 \\ 0 & 0 & I \end{pmatrix}.$$

according to (60). Noting that $[\widetilde{\Phi}_1^{\pm}(\lambda), \Psi_0(\lambda, x, t)] = 0$ we represent the limits (62) in the form

$$\widetilde{\Phi}_1^{\pm}(i\nu)\Psi_0(i\nu, x, t)\hat{a}\rangle = \Psi_0(i\nu, x, t)\hat{a}^{\pm}\rangle$$

where

$$\hat{a}^{\pm}\rangle = \begin{pmatrix} i\frac{\left((\gamma^2+\omega^2+\nu^2)^2+4\nu^2\omega^2\right)\hat{p} \pm 4\nu\omega(\gamma^2+\omega^2+\nu^2)\hat{q}}{(\gamma^2+\omega^2+\nu^2)^2-4\nu^2\omega^2} \\ \frac{\pm4\nu\omega(\gamma^2+\omega^2+\nu^2)\hat{p}+\left((\gamma^2+\omega^2+\nu^2)^2+4\nu^2\omega^2\right)\hat{q}}{(\gamma^2+\omega^2+\nu^2)^2-4\nu^2\omega^2} \\ \hat{\mathbf{a}} \end{pmatrix}$$

Using formula (39) we obtain

$$\hat{x}^+ = \lim_{\tilde{\rho} \to -\infty} x_0 = \frac{1}{2\nu} \ln \frac{((\nu-\omega)^2+\gamma^2)^2|\hat{p}-\hat{q}|}{((\nu+\omega)^2+\gamma^2)^2|\hat{p}+\hat{q}|};$$

$$\hat{x}^- = \lim_{\tilde{\rho} \to +\infty} x_0 = \frac{1}{2\nu} \ln \frac{((\nu+\omega)^2+\gamma^2)^2|\hat{p}-\hat{q}|}{((\nu-\omega)^2+\gamma^2)^2|\hat{p}+\hat{q}|}.$$

Therefore, the position shift for the kink with speed $1/\nu^2$ is

$$\Delta\hat{x} = \hat{x}^+ - \hat{x}^- = \frac{1}{2\nu} \ln \frac{((\nu-\omega)^2+\gamma^2)^4}{((\nu+\omega)^2+\gamma^2)^4} = \frac{2}{\nu} \ln \frac{(\nu-\omega)^2+\gamma^2}{(\nu+\omega)^2+\gamma^2}.$$

In Figure 3, we give the contour plot of α_1 to in (x, t)-plane to demonstrate the phase and position shifts of one kink and one breather interaction, where the kink and breather are corresponding to

$$\nu = 1, \ \langle a_{\text{kink}} = (10i, 7, 4, 5.916), \qquad \mu = 2 + 0.6i, \ \langle a_{\text{breather}} = (i, 2, 3i, 2.45).$$

They intersect approximately at the point $(-0.930, 0, 0626)$. After the collision, the position of kink gets shifted forward by 0.455 and the position of breather gets shifted backward by 0.759.

4 Conclusion

In this paper we have found and studied kink, breather and multi-soliton solutions of the vector sine-Gordon equation (1). The soliton solutions obtained in this paper are of rank one and are generic. Actually a kink solution cannot be of rank more than one. For example, in order to construct a kink solution of rank two, one needs to have two vectors

$$\langle a_1 = (ip_1, q_1, \mathbf{a}_1^T), \ \langle a_2 = (ip_2, q_2, \mathbf{a}_2^T), \ p_1, p_2, q_1, q_2 \in \mathbb{R}, \ \mathbf{a_1}, \mathbf{a_2} \in \mathbb{R}^n$$

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satisfying the conditions $\langle a_1 a_1 \rangle = 0$, $\langle a_2 a_2 \rangle = 0$, $\langle a_1 a_2 \rangle = 0$. This implies that vectors $\langle a_1$ and $\langle a_2$ are linearly dependent. However, in the case of breathers the corresponding vectors are complex and therefore there is no such obstacle. Thus higher rank breather solutions may exist.

Our results show that a k-kink solution of the n-component ($\vec{\alpha} \in \mathbb{R}^n$) vector sine-Gordon equation ($k \leq n$) is also a solution of k-component sine-Gordon equation. Similarly, a general k-breather solution of n-component equation (1) ($k \leq n/2$) is a solution of $2k$ component vector sine-Gordon equation. In particular, a general breather solution of the vector sine-Gordon equation cannot be obtained from any solution of the scalar sine-Gordon equation, but is a solution of a two-component vector sine-Gordon equation.

The result of interactions (positional shifts and phase shifts) due to collisions in the m_1-kink and m_2-breather solution of the vector sine-Gordon equation (1) is exactly the same as in the case of the scalar sine-Gordon equation (3), which has been studied in [14]. It only depends on the positions of poles in the dressing matrices. This is significantly different from the case of the vector generalisation of nonlinear Schrödinger equation, where the result of interactions depends on the initial polarisation of the colliding solitons [10]. This can be explained by the difference in the spectral and symmetry properties of their Lax operators. These properties have been essentially used in the construction of multi-soliton solutions.

Acknowledgements

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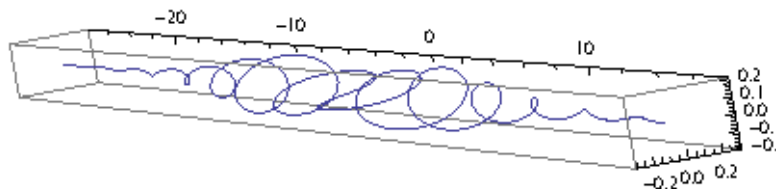[17] A.V. Mikhailov. Reduction in integrable systems. The reduction group. *JETP Lett.*, 32(2):187–192, 1980.

Figure 1: One breather solution. The curve (x, α_1, α_2) with $\mu = 2 + 0.25i$ is plotted at time $t = 3$.
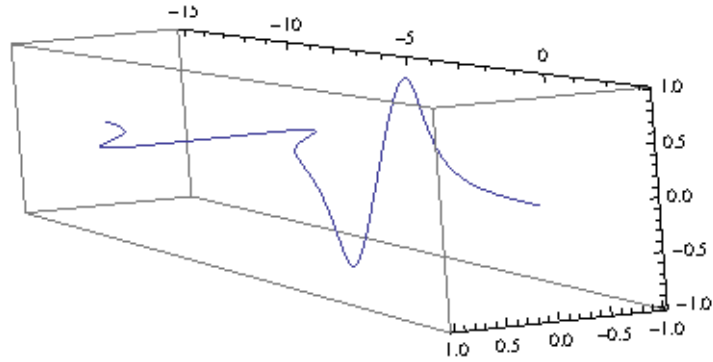
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Figure 2: Two kink solution. The curve (x, α_1, α_2) with $\nu_1 = .9$, $\nu_2 = 1.1$ is plotted at time $t = 6$.
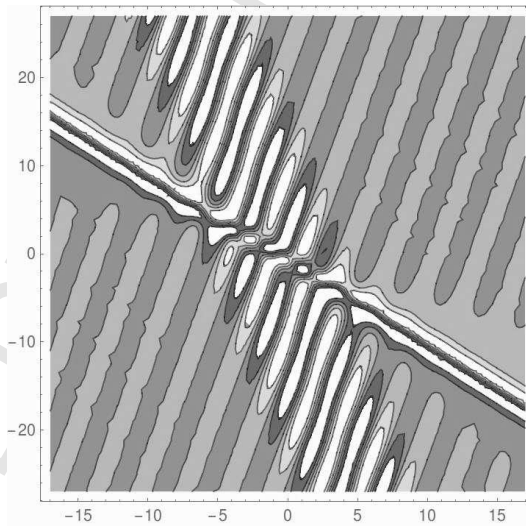

Figure 3: The contour plot of α_1 in (x, t)-plane: Interaction of a kink with a breather

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Highlights

- The explicit formulas for kink and breather solutions are derived
- The method can be used to construct multi-soliton solutions
- The soliton interactions are studied in details