Some problems in the theory of pseudo-random graphs

Andria Eleftheriou

November 7, 2019

Contents

Summary

Krivelevich, Lee and Sudakov proved the following theorem. Let k be either 3 or an even integer satisfying $k \geq 4$, and let $G = (V, E)$ be a (n, d, λ)-graph satisfying $d^{k-1}/n \gg \lambda^{k-2}$, then G has local resilience $(1/2 + o(1))\, d$ with respect to containing cycles of length t for all $k \leq t \leq n$. They conjectured that this can be extended to cycles of an arbitrary but fixed odd length. All notation will be defined in detail later. Aigner-Horev et al proved a substantial step towards this result, namely the following theorem:

For every integer $k \geq 1$ and every $\delta > 0$ there exists a $\gamma > 0$ such that for every sequence of densities $p = p(n)$ there exists an n_0 such that for any $n \geq n_0$ the following holds. If Γ is an n-vertex, (p, β)-jumbled graph satisfying

$$\beta = \beta(n) \leq \gamma p^{1+1/(2k-1)} n \log^{-2(k-1)} n \tag{1}$$

then

$$\mathrm{ex}(\Gamma, C_{2k+1}) < (\frac{1}{2} + \delta) p \binom{n}{2}. \tag{2}$$

They also speculated that the logarithmic 'fudge'factor $\log^{-2(k-1)} n$ could be removed. In this MPhil thesis we give an orderly and critical examination of the result of Krivelevich, Lee and Sudakov, the progress made on it by Aigner-Horev, Han and Schacht, and we also critically examine, to the extent that time allows, a very recent result by Berger, Lee and Sudakov which does indeed succeed in removing the logarithmic factor (`https : //arxiv.org/pdf/1906.05100.pdf`).

1 Introduction

A random graph $G(n, p)$ as described in [9], consists of a set V of n vertices and a set E of unordered pairs of vertices, called edges. We are interested in what happens when we start from a graph $G(n, p)$ and each edge in it, independently of all other edges, is either retained (with probability p) or discarded with probability $1-p$. Here p is a probability between 0 and 1, and all edges make this decision as to whether they are retained independently. Equivalently, the probability of a graph $G(V, E)$ with $V = \{1, \dots, n\}$ in $G(n, p)$ is $Pr[G] = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}$.

A pseudo-random graph $G = (V, E)$ as also described in [9], is a graph that behaves like a truly random graph $G(|V|, p)$ of the same edge density $p = |E|/\binom{|V|}{2}$. The most important characteristic of a truly random graph is its edge distribution. A pseudo-random graph is a graph with edge distribution resembling the one of a truly random graph with the same edge density. The pseudo-random graphs we will study are (n, d, λ)-graphs with $\lambda = o(d)$, where an (n, d, λ)-graph is a d-regular graph on n vertices whose second largest eigenvalue of the adjacency matrix is bounded by λ.

A graph property $\mathcal{P}$ is called monotone increasing (decreasing) if it is preserved under edge addition (deletion). The resilience of G with respect to $\mathcal{P}$ is a measure of how much one should change G in order to destroy $\mathcal{P}$. There are two kinds of resilience: global and local. Formally, following [10], we define the following.

Definition 1.1 *Let $\mathcal{P}$ be an monotone increasing (decreasing) property. The global resilience of G with respect to $\mathcal{P}$ is the minimum number r such that by deleting (adding) r edges from G one can obtain a graph not having $\mathcal{P}$.*

Definition 1.2 *Given a monotone increasing (decreasing) property $\mathcal{P}$. The* *local resilience of a graph G with respect to $\mathcal{P}$ is the minimum number r such that by deleting (adding) at each vertex of G at most r edges one can obtain a graph not having $\mathcal{P}$.*

Krivelevich, Lee and Sudakov in [7] stated an open problem. They believe that the theorem: Let k be either 3 or an even integer satisfying $k \geq 4$, and let $G = (V, E)$ be a (n, d, λ)-graph satisfying $d^{k-1}/n \gg \lambda^{k-2}$, then G has local resilience $(1/2 + o(1))\, d$ with respect to containing cycles of length t for all $k \leq t \leq n$ can be extended to cycles of an arbitrary but fixed odd length. More specifically, it is plausible that for an odd $k \geq 5$, if G is an (n, d, λ)-graph and $d^{k-1}/n \gg \lambda^{k-2}$, then the local resilience of G with respect to containing a cycle of length k is $(1/2 - o(1))d$. The validity of the conjecture below would allow us this extension.

Conjecture 1.3 *Let $k \geq 5$ be an odd integer and G be a (n, d, λ)-graph satisfying $d^{k-1} \gg \lambda^{k-2}$. Then G has global resilience $(1/4 + o(1))nd$ with respect to being C_k-free.*

The result of Aigner-Horev et al in [2], is the first step towards proving an open problem in section 7.3 in [7]. Aigner-Horev et al in [2] stated the following theorem.

Theorem 1.4 *Let $k \geq 1$ be an integer. If Γ is an (n, d, λ)-graph satisfying*

$$\lambda^{2k-1} \ll \frac{d^{2k}}{n} (\log n)^{-2(k-1)(2k-1)} \tag{3}$$

then

$$ex(\Gamma, C_{2k+1}) = (\frac{1}{2} + o(1)) \frac{dn}{2}. \tag{4}$$

The main part of this thesis will be an overview of their result. We shall also describe, to the extent which time allows, a more recent paper by Berger et al. which proves a stronger form of the result in which the 'fudge factor' $(\log n)^{-2(k-1)(2k-1)}$ is removed.

Theorem 1.4 is a consequence of Theorem 1.5 below for the so called jumbled graphs. Aigner-Horev et al proved the following result:

Theorem 1.5 *For every integer $k \geq 1$ and every $\delta > 0$ there exists a $\gamma > 0$ such that for every sequence of densities $p = p(n)$ there exists an n_0 such that for any $n \geq n_0$ the following holds. If Γ is an n-vertex, (p, β)-jumbled graph satisfying*

$$\beta = \beta(n) \leq \gamma p^{1+1/(2k-1)} n \log^{-2(k-1)} n,$$

then

$$ex(\Gamma, C_{2k+1}) < (\frac{1}{2} + \delta)p\binom{n}{2}.$$

Berger et al. eventually succeeded in removing the fudge factors. The precise statement they prove is the following - they use slightly different language from Aigner-Horev et al, but their result leads to the desired conclusion.

Theorem 1.6 *Let $k \geq 1$ be an integer and $\delta > 0$. Then there exist $\eta > 0$ and n_0 such that the following hold. If Γ is an (n, d, λ)-graph satisfying*

$$\lambda^{2k-1} \leq \eta \frac{d^{2k}}{n} \tag{5}$$

then, if G is a subgraph of Γ with $\geq (1/2 + \delta)\frac{d}{n}\binom{n}{2}$ edges, then there is a copy of C_{2k+1} in G.

We shall say more about the proof of Berger et al's result later. Summarising very briefly and crudely, the additional information used is a 'spectral' estimate - that is to say, one dependent on knowledge about the eigenvalues of relevant adjacency matrices - for the number of even cycles in (n, d, λ) graphs.

1.1 The structure of this thesis

We now explain more the structure of this thesis. In Chapter 2 we give basic background on graph theory and resilience. Chapter 3 collects basic information on pseudorandom graphs and in Chapter 4 we restate the result of Aigner-Horev et al and turn it from a problem about (n, d, λ) graphs into one about (p, β)-jumbled graphs. Chapter 5 outlines the structure of their proof, reducing to two substantial Lemmas which need to be proven. In Chapter 6 we provide an overview of the structure of the argument for one of these in the technically special case $k = 3$: in Chapter 7 we make these arguments more precise. Chapter 8 deals with counting shortest paths and Chapter 9 with counting cycles. These two chapters are perhaps the weightiest of the thesis.

Chapter 10 deals with the complete answer provided by Berger et al. We do not have time in this thesis to give all details of their argument, as this work appeared very late in our investigation, but we attempt to outline some of its main features and to convey some flavour of the result.

2 Basic graph theory definitions and Local Resilience of graphs

2.1 Basic graph theory definitions

Definition 2.1 *A graph G consists of a set V of vertices and a set E of unordered pairs of vertices, called edges.*

This definition implies that our graphs have no loops, no multiple edges, and no directions. We shall deal exclusively with finite graphs, where V (and hence E) are finite sets.

So, G has n vertices and m edges usually. We often informally write edges as vw rather than $\{v, w\}$.

As there are n vertices, there are $\binom{n}{2} = n(n-1)/2$ possible edges.

Definition 2.2 *A complete graph on n vertices is the graph on n vertices where all $\binom{n}{2} = n(n-1)/2$ possible edges are present. We denote it by K_n.*

We now give some other definitions related to graph theory.

Definition 2.3 *A vertex w is said to be adjacent to a vertex v if $vw \in E(G)$.*

Definition 2.4 *The degree of a vertex v is the number of vertices adjacent to it. The degree of a vertex v is denoted by $d(v)$.*

Definition 2.5 *The average degree d of a graph G with n vertices is $\sum_{i=1}^{n} d(i)/n$.*

This is a measure of how rich in edges the graph is.

The next few notions are to do with the idea of being able to move, along edges, from any vertex of the graph to any other.

Definition 2.6 *(i) A path between two vertices x and y in a graph G is a sequence of distinct vertices $x_0 = x, x_1, x_2, \dots x_r = y$ with the property that, for each $1 \le i \le r-1$, we have that $x_i x_{i+1}$ is an edge of G.*
(ii) A graph G is connected if and only if for every pair of vertices $x, y \in V(G)$ there is (at least one) path from x to y.

Definition 2.7 *A cycle in a graph is a path that starts at the same vertex at which it ends. Also a cycle can be defined as a closed path in which $n \ge 3$. The graph C_k is a k length cycle, consisting of k vertices and k edges that form a cycle.*

Definition 2.8 *A Hamilton cycle is a cycle of the graph which contains all the vertices.*

This thesis is not much about connectivity: however we make occasional passing references to it and as such we record two standard definitions.

Definition 2.9 *The edge-connectivity $\lambda(G)$ of a graph $G = (V, E)$ is the smallest size of a set of edges S such that, if we remove all the edges in S from the set E we get a disconnected graph. When $\lambda(G) \ge \lambda$ we can simply say that G is λ-edge-connected.*

There is a very similar notion for vertex-connectivity, where instead of removing edges we remove vertices. The next definition will be important when we come to talk about pseudo-random graphs.

Definition 2.10 *The adjacency matrix of a d-regular graph G on n vertices labeled be $1, 2, ..., n$, is the $n \times n$ binary matrix, $A = A(G)$, where $A_{ij} = 1$ iff $(i, j) \in E(G)$.*

2.2 Local and Global Resilience of graphs

2.2.1 Introduction

The rough idea is that the resilience should be a measure of how strongly the graph possesses some property $\mathcal{P}$ it does, in the sense of how many edges have to be removed to stop it having $\mathcal{P}$. There are however at least two ways in which one could interpret this: the first is in terms of just removing edges overall, without any restrictions on how many are removed at any particular vertex. Since this depends only on the number of edges removed, not local information about which vertices they are removed at, let us call this the global resilience. Here is a formal definition.

Definition 2.11 *(Global resilience in general). Let $\mathcal{P}$ be a property of graphs. Then the global resilience of $G = (V, E)$ with respect to $\mathcal{P}$ is the number of edges m in the graph H on vertex set V with the smallest number of edges such that $G \triangle H$ does not have $\mathcal{P}$.*

For example, the next result, which is just a way of stating Turán's theorem, answer the question how many edges have to removed from the complete graph K_n to make it K_k-free.

Theorem 2.12 *(Turán). The largest K_k-free subgraph of a graph on n vertices is a complete $(k-1)$-partite graph with all vertex classes as equal as possible in order. In particular, it has (for k fixed and $n \to \infty$)*

$$\leq (1 - \frac{1}{k-1})\frac{n(n-1)}{2}(1 + o(1))$$

edges. Putting it another way, one has to remove $\frac{1}{k-1}\frac{n(n-1)}{2}$ edges from K_n to make it K_k-free: the global resilience of K_n with respect to having a copy of K_k is $n(n-1)/(2(k-1))$.

Definition 2.13 *For two graphs G and H, the Turan number* $\mathrm{ex}(G, H)$ *of H in G is the largest integer e, such that there is an H-free subgraph of G with e edges.*

The other definition we shall need is local resilience.

Definition 2.14 *Let G and H be two graphs on the same vertex set V. $G \triangle H$ is a graph on vertex set $V(G)$ where we say $xy \in E(G \triangle H)$ if and only if either (a) $xy \in E(G)$ but $xy \notin E(H)$ or $xy \notin E(G)$ but $xy \in E(H)$.*

That is, H represents those edges whose status (present or not present) is reversed as compared with G. We fix a property $\mathcal{P}$ and consider all graphs on the same large vertex set. Local resilience is a graph function $f_{\mathcal{P}}(G)$. The value of this function depends on the graph and the property, and for some graphs it will be very difficult to determine.

Definition 2.15 *Consider a random graph $G(n, p)$ and a fixed property $\mathcal{P}$. The local resilience of the graph with respect to $\mathcal{P}$ is the minimum number r so that there is a graph H on $[n]$ with maximum degree at most r such that the graph $G(n, p) \triangle H$ does not have $\mathcal{P}$.*

We emphasise that the graph G is chosen at random first, and then an adversary is allowed to find a graph H with the smallest possible maximum degree in order to destroy the property P.

3 Basic properties of pseudorandom graphs

Following [7] and [9] we define some useful properties below.

Definition 3.1 *A graph on n vertices is called pancyclic if it contains cycles of length t for all $3 \leq t \leq n$. A graph on n vertices with girth g is called weakly pancyclic if it has cycles of every length t with $g \leq t \leq n$.*

In this section we study the resilience of random and pseudorandom graphs with respect to this property.

Definition 3.2 *A random graph $G(n, p)$ is a probability space of all labeled graphs on n vertices $1, 2, ..., n$, where for each pair $1 \leq i < j \leq n$, (i, j) is an edge of $G(n, p)$ with probability $p = p(n)$, independently of any other edges.*

A pseudo-random graph $G = (V, E)$ is a graph that behaves like a truly random graph $G(|V|, p)$ of the same edge density $p = |E|/\binom{|V|}{2}$.

Definition 3.3 *A graph $G(V, E)$ is said to be (p, β)-jumbled if p, β are real numbers satisfying $0 < p < 1 \leq \beta$ if every subset of vertices $U \subset V$ satisfies:*

$$\left| e(U) - p\binom{|U|}{2} \right| \leq \beta\, |U|$$

We will now give another, somewhat more restrictive definition of a pseudo-random graph below.

Definition 3.4 *A (n, d, λ) graph is a d-regular graph on n vertices for which, writing the eigenvalues of G (i.e. the eigenvalues of the adjacency matrix $A(G)$) as $\lambda_1 = d$, $\lambda_2, \ldots, \lambda_n$ we have $\lambda = \max_{2 \leq i \leq n} |\lambda_i|$.*

Lemma 3.5 *An adjacency matrix A is real and symmetric. It has an orthogonal basis of real eigenvectors and all its eigenvalues are real. The eigenvalues of A are denoted in descending order by $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, where $\lambda_1 = d$ and its corresponding eigenvector is 1_n (the $n \times 1$ all ones vector).*

Theorem 3.6 *Let G be a d-regular graph on n vertices. Let $d = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the eigenvalues of G. Denote*

$$\lambda = \underbrace{max}_{2 \leq i \leq n} |\lambda_i| \, .$$

Then for every two subsets $U, W \subset V$,

$$|e(U,W) - \frac{d|U||W|}{n}| \leq \lambda \sqrt{|U||W|\left(1 - \frac{|U|}{n}\right)\left(1 - \frac{|W|}{n}\right)} < \lambda\sqrt{|U||W|}.$$

Theorem 3.7 *A (n, d, λ)-graph is (p, β) jumbled with $p = d/n$ and $\beta = \lambda$.*

The next proposition given by Krivelevich and Sudakov in [9].

Proposition 3.8 *Let $k \geq 1$ be an integer and let G be an (n, d, λ)-graph such that $d^{2k}/n \gg \lambda^{2k-1}$. Then G contains a cycle of length $2k + 1$.*

This proposition implies that when $d \gg n^{\frac{2}{2k+1}}$ and $\lambda \leq O(\sqrt{d})$ then any (n, d, λ)-graph contains a cycle of length $2k + 1$.

The start point for this dissertation is the following theorem given by Krivelevich, Lee and Sudakov in [7].

Theorem 3.9 *Let k be either 3 or an even integer satisfying $k \geq 4$, and let $G = (V, E)$ be a (n, d, λ)-graph satisfying $d^{k-1}/n >> \lambda^{k-2}$. Then G has local resilience $(1/2 + o(1))\,d$ with respect to containing cycles of length t for all $k \leq t \leq n$.*

Aigner-Horev et al proved the following result:

Theorem 3.10 *For every integer $k \geq 1$ and every $\delta > 0$ there exists a $\gamma > 0$ such that for every sequence of densities $p = p(n)$ there exists an n_0 such that* *for any $n \geq n_0$ the following holds. If Γ is an n-vertex, (p, β)-jumbled graph satisfying*

$$\beta = \beta(n) \leq \gamma p^{1+1/(2k-1)} n \log^{-2(k-1)} n,$$

then

$$ex(\Gamma, C_{2k+1}) < (\frac{1}{2} + \delta) p \binom{n}{2}.$$

This is clearly a resilience type result. In the rest of this chapter, we will try to show that the local resilience must be at most $(1/2 + o(1))\, d$. In subsequent chapters we will try to prove that this is the right asymptotic.

Theorem 3.11 *Any d-regular graph H can be made bipartite (which implies it has no (2k+1)-cycle) by removing about $\frac{d(1+o(1))}{2}$ edges from each vertex.*

Proof. Colour each vertex of the graph H red or blue equiprobably, each vertex being coloured independently of all other vertices. Our aim is to show that the graph G with only the red-blue edges retained has the required property. Since each vertex has Binomial $(d, 1/2)$ neighbours of the opposite colour to itself, the probability that any particular vertex has degree at least $d/2 - d^{2/3}$ and at most $d/2 + d^{2/3}$ is at most $2e^{-d^{1/3}/4}$. This comes true by Chernoff's inequality below:

Theorem 3.12 *If $X \sim Bin(n, p)$ and $\epsilon > 0$, then*

$$P(|X - \mathbf{E}[X]| \geq \epsilon \mathbf{E}[X]) \leq e^{-\Omega_\epsilon(\mathbf{E}[X])}$$

Consequently the probability that any vertex has degree not in the range $(d/2 - d^{2/3}, d/2 + d^{2/3})$ is (using the union bound) at most $2e^{-d^{1/3}/4}$. Provided that d is at least (say) n^α for some $\alpha > 0$ this will result in the relevant probability going to 0 as d (or equivalently n) goes to infinity. The fact that d is of this form follows from the condition $d^{k-1}/n \geq \lambda^{k-2}$.

Corollary 3.13 *The local resilience is $\leq \frac{d(1+o(1))}{2}$ by this partition argument.*

4 The result of Aigner-Horev et al

Aigner-Horev et al consider extremal problems for subgraphs of pseudorandom graphs. For graphs F and Γ the generalized Turán density $\pi_F(\Gamma)$ denotes the relative density of a maximum subgraph of Γ, which contains no copy of F. Extending classical Turán type results for odd cycles, they showed that $\pi_F(\Gamma) = 1/2$ provided F is an odd cycle and Γ is a sufficiently pseudorandom graph.

In particular, Aigner-Horev et al showed that for (n, d, λ)-graphs Γ, i.e. an n-vertex, d-regular graph with all non-trivial eigenvalues (i.e. those $\neq d$) in the interval $[-\lambda, \lambda]$ their result holds for odd cycles of length ℓ, provided

$$\lambda^{\ell-2} \ll \frac{d^{\ell-1}}{n} \log(n)^{-(\ell-2)(\ell-3)}$$

Upon to the polylog-factor this verifies the conjecture of Krivelevich, Lee and Sudakov. It is natural to ask if the polylogarithmic factor $\log(n)^{-(\ell-2)(\ell-3)}$ can be removed and this is what we discuss in the rest of this dissertation. In Chapter 10 we summarise some very recent work of Berger et al. which gives a full answer to this question.

We mention in passing a global resilience conjectures similar to ours. However it is not the main focus of what follows. Here is an open problem from [7], which is stated as a conjecture.

Conjecture 4.1 *Let $k \geq 5$ be an odd integer and G be a (n, d, λ)-graph satisfying $d^{k-1} \gg \lambda^{k-2}$. Then G has global resilience $(1/4 + o(1))nd$ with respect to being C_k-free.*

The Theorem 1.4 of Aigner-Horev et al will be a consequence of Theorem 5.4 which is stated for jumbled graphs below. The fact that Theorem 5.4 implies Theorem 4.2 is for the same reason as in Aigner-Horev et al's paper, namely that a (n, d, λ)-graph is (p, β) jumbled with $p = d/n$ and $\beta = \lambda$. We noted this fact at Theorem 3.7. We explain briefly how the relevant conditions will hold.

Theorem 4.2 *For every integer $k \geq 1$ and every $\delta > 0$ there exists a $\gamma > 0$ such that for every sequence of densities $p = p(n)$ there exist an n_0 such that for any $n \geq n_0$ the following holds. If Γ is an n-vertex (p, β)-jumbled graph satisfying*

$$\beta = \beta(n) \leq \gamma p^{1+1/(2k-1)} n \log^{-2(k-1)} n \tag{6}$$

then

$$ex(\Gamma, C_{2k+1}) < (\frac{1}{2} + \delta) p \binom{n}{2}. \tag{7}$$

For $k = 1$, the same problem was studied in [11]. In this case, we obtain the same result which is known to be best possible due to a construction of Alon in [1]. For $k \geq 2$, Alon's construction can be extended as to fit for general odd cycles, implying that for any $k \geq 2$ the equation 3 is best possible up to the polylog-factor. We also remark that Theorem 1.4 was essentially (up to the polylog-factor) conjectured by Krivelevich, Lee and Sudakov with Conjecture 7.1 in [7].

Proof of Theorem If we have $\lambda = \beta$ and $p = d/n$ as given by Theorem 3.7, then we get

$$\begin{aligned} \beta = \beta(n) \leq \gamma p^{1+1/(2k-1)} n \log^{-2(k-1)} n \\ \Rightarrow \lambda \leq \gamma \left(d/n\right)^{\frac{2k}{2k-1}} n \\ \Rightarrow \lambda^{2k-1} \leq \gamma^{2k-1} \frac{d^{2k}}{n^{2k}} n^{2k-1} \\ \Rightarrow \lambda^{2k-1} \ll \frac{d^{2k}}{n} \end{aligned}$$

and

$$\begin{aligned}
\mathrm{ex}(\Gamma, C_{2k+1}) &< (\frac{1}{2} + \delta)p\binom{n}{2} \\
&= (\frac{1}{2} + \delta)\frac{d}{n}\frac{n(n-1)}{2} \\
&= (\frac{1}{2} + \delta)\frac{d(n-1)}{2} \\
&= (\frac{1}{2} + o(1))\frac{d}{n}
\end{aligned}$$

Theorem 5.4 in turn with follow from Lemma 5.8 (which is unchanged from the paper by Aigner-Horev et al) and from Lemma 5.13, which will be more complicated.

5 First reduction of the problem

We start this article by giving some useful definitions following [2].

Definition 5.1 *Let Γ be a graph and $X, Y \subset V(\Gamma)$. By $vol(X, Y)$ we denote the number of all pairs with one element from X and the other element being from Y.*

Definition 5.2 *By $e_\Gamma(X, Y)$ we denote the number of actual edges $xy \in E(\Gamma)$ satisfying $x \in X$ and $y \in Y$. As usual, let $e_\Gamma(X) = e_\Gamma(X, X)$.*

Consider the following as an interpretation, in terms of volume, of the earlier Theorem 3.7.

Definition 5.3 *Let $p = p(n)$ be a sequence of densities, $0 \leq p \leq 1$ and $\beta = \beta(n)$. An n-vertex graph Γ is called (p, β)-jumbled graph if*

$$|e_\Gamma(X, Y) - pvol(X, Y)| \leq \beta vol(X, Y)^{1/2} \tag{8}$$

for all $X, Y \subseteq V(\Gamma)$

For disjoint sets X, Y

$$|e_\Gamma(X, Y) - p|X||Y|| \leq \beta(|X||Y|)^{1/2} \tag{9}$$

For $X = Y$,

$$|e_\Gamma(X) - p\binom{|X|}{2}| \leq \beta|X|. \tag{10}$$

Theorem 5.4 *For every integer $k \geq 1$ and every $\delta > 0$ there exists a $\gamma > 0$ such that for every sequence of densities $p = p(n)$ there exist an n_0 such that for any $n \geq n_0$ the following holds. If Γ is an n-vertex (p, β)-jumbled graph satisfying*

$$\beta = \beta(n) \leq \gamma p^{1+1/(2k-1)} n \log^{-2(k-1)} n \tag{11}$$

then

$$ex(\Gamma, C_{2k+1}) < (\frac{1}{2} + \delta)p\binom{n}{2}. \tag{12}$$

To prove Theorem 5.4 we need the help of two important Lemmas and some other definitions.

Definition 5.5 *For a graph G and disjoint sets $X, Y \subseteq V(G)$, $G[X, Y]$ is the bipartite subgraph of G whose vertex set is $X \cup Y$ and $E_G(X, Y)$ is the edge set which consists all the edges of G with one in X and the other in Y. Also, we write $E_G(X)$ to denote the edge set of $G[X]$.*

Definition 5.6 *For a graph R and a positive integer m, $R(m)$ is the graph obtained by replacing every vertex $i \in V(R)$ with a set of vertices V_i of size m and adding the complete bipartite graph between V_i and V_j whenever $ij \in E(R)$. A spanning subgraph of $R(m)$ is called a $R(m)$-graph.*

Definition 5.7 *A $G \subseteq R(m)$ is called an (α, p, ϵ)-degree-regular graph if*

$$deg_{G[V_i,V_j]}(v) \in [(\alpha - \epsilon)pm, (\alpha + \epsilon)pm] \tag{13}$$

holds whenever $ij \in E(R)$ and $v \in V_i \cup V_j$. The notation $R(m')$, for m' a positive real number, is a shorthand for $R(\lceil m' \rceil)$.

The following Lemma asserts under a certain assumption of jumbledness, a relatively dense subgraph of a sufficiently large (p, β)-jumbled graph contains a degree-regular $\mathcal{C}_l(m)$-graph with large m.

Lemma 5.8 *For any integer $\ell \geq 3$, all $\rho > 0$, $\alpha_0 > 0$ and $0 < \epsilon < \alpha_0$ there exist a $\nu > 0$ and a $\gamma > 0$ such that for every sequence of densities $p = p(n) \gg logn/n$ there exists an n_0 such that for every $n \geq n_0$ the following holds.*

Let Γ be an n-vertex (p, β)-jumbled graph with $\beta = \beta(n) \leq \gamma p^{1+\rho} n$ and let $G \subset \Gamma$ be a subgraph of Γ satisfying $e(G) \geq \alpha_0 p\binom{n}{2}$. Then, there exist an $\alpha \geq \alpha_0$ such that G contains an (α, p, ϵ)-degree-regular $C_\ell(\nu n)$-graph as a subgraph.

Using the above Lemma we try to prove Theorem 5.4.

When we have $C_l(m)$ large degree-regular graphs which are hosted in a sufficiently large jumbled graph Γ we should, in our search for odd cycles, concentrate on those cycles which have all but one of their edges in the hosted graph $C_l(m)$ graph, with the remaining one being in Γ.

Let us fix one vertex w of the cycle C_{2k+1} and the other $2k$ vertices will be

$$(u_k, u_{k-1}, \ldots, u_1, v_1, v_2, \ldots, v_k).$$

Definition 5.9 *For a given graph Γ let $H \subseteq \Gamma$ be a $C_{2k+1}(m)$ with the corresponding vertex partition $(U_k, \ldots, U_1, W, V_1, \ldots, V_k)$. U_i is the set of m vertices corresponding to u_i and V_i is the set corresponding to v_i.*

Definition 5.10 *Using H and Γ as in the previous definition by $\mathcal{C}(H, \Gamma)$ we denote the set of all cycles of length $(2k+1)$ of the form $(u'_k, \ldots, u'_1, w', v'_1, \ldots, v'_k)$ such that $w' \in W, u'_i \in U_i, v'_i \in V_i, v'_k u'_k \in E(\Gamma)$, and all edges other than $v'_k u'_k$ are in $E(H)$. In other words, a member of $\mathcal{C}(H, \Gamma)$ is a cycle of Γ of length $2k+1$ which respects the vertex partition of H and which has all edges but possibly $v'_k u'_k$ being in H.*

Definition 5.11 *For a real number $\mu > 0$, an edge of $\Gamma[V_k, U_k]$ is called μ-saturated if it is contained in at least $p(\mu p m)^{2k-1}$ members of $\mathcal{C}(H, \Gamma)$.*

Definition 5.12 *A cycle in $\mathcal{C}(H, \Gamma)$ containing a μ-saturated edge is called a μ-saturated cycle. We write $\mathcal{S}(\mu, H, \Gamma)$ to denote the set of μ-saturated cycles in $\mathcal{C}(H, \Gamma)$.*

We expect that an edge of $\Gamma[U_k, V_k]$ extends to $(\alpha p)^{2k} m^{2k-1}$ members of $\mathcal{C}(H,\Gamma)$. For $\mu \approx \alpha$, a μ-saturated edge overshoots this expectation by a factor $1/\alpha$.

Lemma 5.13 *For any integer $k \geq 1$ and all reals $0 < \nu, \alpha_0 \leq 1$ and $0 < \epsilon \leq \alpha_0/3$ there exists a $\gamma > 0$ such that for every sequence of densities $p = p(n)$ there exists an n_0 such that for any $n \geq n_0$ the following holds.*

If Γ is an n-vertex (p,β)-jumbled graph with

$$\beta = \beta(n) \leq \gamma p^{1+1/(2k-1)} n \log^{-2(k-1)} n \tag{14}$$

then for any $m \geq \nu n$ and any $\alpha \geq \alpha_0$ an (α, p, ϵ)-degree-regular $C_{2k+1}(m)$-graph $H \subseteq \Gamma$ satisfies

$$|\mathcal{C}(H,\Gamma)| \geq (\alpha - 2\epsilon)^{2k} (pm)^{2k+1} \tag{15}$$

and

$$|\mathcal{S}(\alpha + 2\epsilon, H, \Gamma)| \leq (3\epsilon)^{2k} (pm)^{2k+1}. \tag{16}$$

With the lemmas 5.8 and 5.13 above we will try to show how they imply Theorem 5.4.

Proof of Theorem 5.4

These parameters are all we need. Let $k \geq 1$, m a positive integer, $\delta > 0$ and $p = p(n)$ be given. We assume that $\delta \leq \frac{1}{2}$. We set that

$$\ell = 2k+1, \rho = \ell^{-1}, \epsilon = \frac{\delta}{4 + 32k + 6^{2k+1}}, \alpha_0 = 1/2 + \delta \tag{17}$$

Also, $\nu > 0$ and $\gamma_1 > 0$ will be the values obtained by applying Lemma 5.8. With these choices of l, ρ, ϵ and α_0 and also $\gamma_2 > 0$, $k \geq 1$, $0 < \nu$,$\alpha_0 \leq 1$ $0 < \epsilon \leq \alpha_0/3$ as obtained in Lemma 5.13. We set

$$\gamma = \min\{\gamma_1, \gamma_2, \frac{\delta\nu}{4}\}. \tag{18}$$

For the remainder of the proof, we shall let n_0 be large enough that both Lemma 5.8 and 5.13 are operational.

Continuing with the proof of Theorem 5.4, let Γ be a (p, β)-jumbled graph with β satisfying

$$\beta = \beta(n) \leq \gamma p^{1+1/(2k-1)} n \log^{-2(k-1)} n.$$

We need to show that every subgraph G of Γ satisfying $e(G) \geq \alpha_0 p \binom{n}{2}$ contains a C_{2k+1}. Let G be such a subgraph of Γ and let $H \subseteq G$ be an (α, p, ϵ)-degree-regular $C_{2k+1}(m)$-graph - this of course exists by Lemma 5.8, where $\alpha \geq \alpha_0$ and $m \geq vn$. Recall that a $C_{2k+1}(m)$-graph has vertex classes V_i of order m for each $1 \leq i \leq k$, U_i of order m for each $1 \leq i \leq k$, and a further class W.

Definition 5.14 *Let $F = F(U_k, V_k) \subseteq E_\Gamma(U_k, V_k)$ denote those edges of $\Gamma[U_k, V_k]$ met by a member of $\mathcal{C}(H, \Gamma)$. Every edge in F completes a path of length $2k$ in H into a cycle of length $2k + 1$.*

In what follows, we prove that $F \cap E_H(U_k, V_k) \neq \emptyset$ which then implies that $C_{2k+1} \subseteq H \subseteq G$ completing the proof of Theorem 5.4. To this end it is sufficient to show that

$$|F| \geq (\alpha - \frac{\delta}{2})pm^2. \tag{19}$$

As H is an (α, p, ϵ)-degree-regular and $\epsilon < \frac{\delta}{2}$ we can see that

$$e_H(U_k, V_k) \geq (\alpha - \frac{\delta}{2})pm^2. \tag{20}$$

We now consider $\beta = \beta(n)$. By our condition on β in the statement of the Theorem 5.4, we have

$$\beta m \leq \gamma p^{1+\frac{1}{2k-1}} nm \log^{-2(k-1)} n \leq \gamma p^{1+\rho} nm \tag{21}$$

as $\rho = \ell^{-1}$ and $\ell = 2k+1$. Thus we conclude that $\beta m \leq \gamma p^{1+\rho} nm$.

We next need to prove that $\gamma p^{1+\rho} nm \leq \frac{\delta}{2} pm^2$. To see this, divide both sides by pm, getting $\gamma p^{\rho} n \leq \frac{\delta}{2} m$. But $\gamma \leq \frac{\delta\nu}{4}$ as $\gamma = \min\{\gamma_1, \gamma_2, \frac{\delta\nu}{4}\}$, so it is enough to show $\frac{\delta\nu}{4} p^{\rho} n \leq \frac{\delta}{2} m$. Simplifying, this is equivalent to $\nu p^{\rho} n \leq 2m$. As $m \geq \nu n$, it is enough to show $\nu p^{\rho} n \leq 2\nu n$, and this is just the simple fact that $p^{\rho} \leq 2$ as $p \in (0,1)$.

We now use our upper bound on β to finish off. We have, by jumbledness of Γ with $m \geq \nu n$ and $\gamma \leq \frac{\delta\nu}{4}$ and using the definition 5.3 of jumbled graphs at the beginning $|e_{\Gamma}(U_k, V_k) - p|U_k||V_k|| \leq \beta(|U_k||V_k|)^{1/2}$. Since $|U_k| = |V_k| = m$, $|e_{\Gamma}(U_k, V_k)| \leq pm^2 + \beta m \leq pm^2 + \gamma p^{1+\rho} nm$. We have already shown that $\gamma p^{1+\rho} nm \leq \frac{\delta\nu}{2} pm^2$ so

$$pm^2 + \beta m \leq pm^2 + \frac{\delta}{2} pm^2 = (1 + \frac{\delta}{2}) pm^2. \tag{22}$$

Suppose we could show that $|F| \geq (\alpha - \frac{\delta}{2}) pm^2$. Then we would, using the fact we have just proved that $e_{\Gamma}(U_k, V_k) \leq (1 + \frac{\delta}{2}) pm^2$ and splitting the above in two factors we have that

$$(1 + \frac{\delta}{2}) pm^2 = (\alpha - \frac{\delta}{2}) pm^2 + (1 - \alpha + \delta) pm^2$$

and

$$\begin{aligned} e_{\Gamma}(U_k, V_k) \leq (\alpha - \frac{\delta}{2}) pm^2 &+ (1 - \alpha + \delta) pm^2 \\ &\leq |F| + (1 - \alpha + \delta) pm^2 \\ &< |F| + (\alpha - \frac{\delta}{2}) pm^2 \\ &\leq |F| + e_H(U_k, V_k). \end{aligned} \tag{23}$$

To get this sequence of inequalities to work, we need to show that $1 - \alpha + \delta < \alpha - \frac{\delta}{2}$. This claim is equivalent to $1 + \frac{3\delta}{2} < 2\alpha$. But $\alpha \geq \alpha_0 = \frac{1}{2} + \delta$, so

$$2\alpha \geq 1 + 2\delta > 1 + \frac{3\delta}{2}$$

as required.

This now gives us that $e_\Gamma(U_k, V_k) < |F| + e_H(U_k, V_k)$ and this will indeed imply that F and $E_H(U_k V_k)$ have non-empty intersection. It remains to show that $|F| \geq (\alpha - \frac{\delta}{2})pm^2$. We need some additional notations first. By definitions 5.1, 5.2 and 5.5 recall that $F = F(U_k, V_k) \subseteq E_\Gamma(U_k, V_k)$ denote those edges of $\Gamma[U_k, Y_k]$ met by a member of $\mathcal{C}(H, \Gamma)$. Each member of F is contained in at most $p(\alpha + 2\epsilon)^{2k-1}(pm)^{2k-1}$ members of $\mathcal{C}' = \mathcal{C}(H, \Gamma) \backslash \mathcal{S}(\alpha + 2\epsilon, H, \Gamma)$. To see this, note that $\mathcal{C}'$ is those cycles in $\mathcal{C}(H, \Gamma)$ which are not $(\alpha + 2\epsilon)$-saturated, and thus are contained in at most $p(\alpha + 2\epsilon)pm^{2k-1}$ members of $\mathcal{C}(H, \Gamma)$ so that

$$|F| \geq \frac{|\mathcal{C}'|}{p(\alpha + 2\epsilon)^{2k-1}(pm)^{2k-1}}. \tag{24}$$

However we also have that, using the estimates from Lemma 5.13.

$$|\mathcal{C}(H, \Gamma)| \geq (\alpha - 2\epsilon)^{2k}(pm)^{2k+1} \tag{25}$$

and

$$|\mathcal{S}(\alpha + 2\epsilon, H, \Gamma)| \leq (3\epsilon)^{2k}(pm)^{2k+1}. \tag{26}$$

Combining the above three inequalities (29), (30) and (31) we have that

$$|F| \geq \frac{(\alpha - 2\epsilon)^{2k}(pm)^{2k+1} - (3\epsilon)^{2k}(pm)^{2k+1}}{p(\alpha + 2\epsilon)^{2k-1}(pm)^{2k-1}}.$$

Factorising using the common factor and simplifying that fraction we obtain

$$|F| \geq (\frac{(\alpha - 2\epsilon)^{2k}}{(\alpha + 2\epsilon)^{2k-1}} - \frac{(3\epsilon)^{2k}}{(\alpha + 2\epsilon)^{2k-1}})pm^2.$$

Separate the fractions above and work independently. Let

$$\mathcal{T} = \frac{(\alpha - 2\epsilon)^{2k}}{(\alpha + 2\epsilon)^{2k-1}} = (\alpha - 2\epsilon)(\frac{\alpha - 2\epsilon}{\alpha + 2\epsilon})^{2k-1}.$$

We now claim that for $a, b > 0$ that $\frac{a-b}{a+b} \geq 1 - \frac{2}{a}b$. To see why this is true, note that

$$(a-b)a \geq a(a+b) - 2b(a+b)$$

which is the same as

$$a^2 - ab \geq a^2 + ab - 2ab - 2b^2$$

making the inequality obvious. Simplifying, $b^2 \geq 0$ as required as any $a, b > 0$. As $\alpha \geq 1/2$ we get that

$$\mathcal{T} \geq (\alpha - 2\epsilon)(1 - \frac{2}{\alpha}2\epsilon)^{2k-1} \geq (\alpha - 2\epsilon)(1 - 8\epsilon)^{2k-1}.$$

Definition 5.15 *Bernoulli's inequality states that*

$$(1+x)^r \geq 1 + rx$$

for every integer $r \geq 0$ and every real number $x \geq -1$

By Bernoulli inequality $(1-8\epsilon)^{2k-1} \geq (1 - 8\epsilon(2k-1)) \geq (1 - 16k\epsilon)$. In detail,

$$\mathcal{T} \geq (\alpha - 2\epsilon)(1-8\epsilon)^{2k-1} \geq (\alpha - 2\epsilon)(1 - 16k\epsilon) = \alpha - 16\alpha k\epsilon - 2\epsilon + 32k\epsilon^2.$$

We need to show that

$$\alpha - 16\alpha k\epsilon - 2\epsilon + 32k\epsilon^2 \geq \alpha - 2\epsilon - 16k\epsilon.$$

So, $-16\alpha k\epsilon + 32k\epsilon^2 \geq -16k\epsilon$.

As, $\alpha \leq 1$ so

$$16\alpha k\epsilon \leq 16k\epsilon$$

$$-16\alpha k\epsilon \geq -16k\epsilon$$

and $32k\epsilon^2 \geq 0$. Since, $\delta \leq 1/2$ it follows that

$$\mathcal{T} \geq \alpha - 2\epsilon - 16k\epsilon. \tag{27}$$

Combining the fact that the second fraction is, using that $\alpha \geq 1/2$, at most

$$\frac{(3\epsilon)^{2k}}{(\alpha+2\epsilon)^{2k-1}} \leq 2^{2k-1}3^{2k}\epsilon \leq 6^{2k}\epsilon$$

so

$$|F| \geq (\alpha - 2\epsilon - 16k\epsilon - 6^{2k}\epsilon)pm^2.$$

Before we set that $\epsilon = \frac{\delta}{4+32k+6^{2k+1}}$ and it is obvious that $\epsilon = \frac{\delta}{4+32k+6^{2k+1}} \leq \frac{\delta}{4}$ so that $2\epsilon + 16k\epsilon + 6^{2k}\epsilon \leq \frac{\delta}{2}$. At the end we get what we want:

$$|F| \geq (\alpha - \frac{\delta}{2})pm^2. \tag{28}$$

6 Proofs of the two Lemmas: initial steps

The form of Lemma 5.8 we are going to use is exactly the same as in Aigner-Horev at al's paper and so no modification of it is required. However, we need a better form of Lemma 5.13 that is in that paper, and that is what we shall now try to prove.

Recall that Γ denotes an n-vertex (p, β)-jumbled graph and H denotes an (α, p, ϵ)-degree-regular $C_{2k+1}(m)$-graph which is a subgraph of Γ. H has a partition $(U_k, \ldots, U_1, W, V_1, \ldots, V_k)$ of its vertex set, as given by Definition 5.6.

The proof has two parts.

The first is the one that concerns $\mathcal{C}(H, \Gamma)$. By definition 5.9 this set consists of all $(2k+1)$-cycles of the form $(u_k, \ldots, u_1, w, v_1, \ldots, v_k)$, where $u_i \in U_i, w \in W, v_i \in V_i$, the edge $u_k v_k$ is an edge of $\Gamma[U_k, V_k]$ and the remaining edges are that of H. So what we want to prove first is that $|\mathcal{C}(H, \Gamma)| \geq (\alpha - 2\epsilon)^{2k}(pm)^{2k+1}$.

To see this, the fact that H is an almost degree regular graph means, using the definition 5.7 that all degrees in $H[V_i, V_j]$ are in the range $((\alpha - \epsilon)pm, (\alpha + \epsilon)pm)$ whenever $ij \in E(\Gamma(U_k, V_k))$, $1 \leq i, j \leq k$ and $v \in V_i \cup V_j$. This implies that the number of paths of the form $(u_k, \ldots, u_1, w, v_1, \ldots, v_k)$ in H is in the set$((\alpha - \epsilon)pm, (\alpha + \epsilon)pm))^{2k}$. As H is an arbitrary subgraph of Γ it may occur that this set of paths "clusters" on a small number of pairs of vertices $(u_k, v_k) \in U_k$ x V_k. (We should be a little careful here: of course one of the edges is, for most vertices, ruled out as it is the edge by which one entered that vertex, but this will be not be significant, e.g. replacing ϵ by a slightly smaller value). The lower bound stated above asserts that these many paths are NOT clustered on a small number of pairs (u_k, v_k) of vertices: they are spread out in a 'random-like', way. In fact, approximately a p proportion of these paths extend to cycles in $\mathcal{C}(H, \Gamma)$ as one would expect in a purely random setting.

The second part of Lemma 5.13 concerns the set $\mathcal{S}(\alpha + 2\epsilon, H, \Gamma)$ of $(\alpha + 2\epsilon)$-saturated cycles. This consists of all cycles $(u_k, \dots, u_1, w, v_1, \dots, v_k)$ in $\mathcal{C}(H, \Gamma)$ for which the edge $u_k v_k \in E(\Gamma[U_k, V_k])$ is $(\alpha+2\epsilon)$-saturated, meaning that it is contained in at least $p((\alpha + 2\epsilon)pm)^{2k-1}$ members of $\mathcal{C}(H, \Gamma)$. The aim is to show that this set is not too large: indeed that it is negligible by comparison with $|\mathcal{C}(H, \Gamma)|$ by bounding it above appropriately. Informally, $|\mathcal{C}(H, \Gamma)| \geq (\alpha - 2\epsilon)^{2k}(pm)^{2k+1}$ and $e_\Gamma[U_k, V_k]$ is approximately pm^2, so we expect an edge of $\Gamma[U_k, V_k]$ to be contained in at least approximately $p(\alpha - 2\epsilon)(pm)^{2k-1}$ members of $\mathcal{C}(H, \Gamma)$. In short, for a small ϵ, the number of cycles in $\mathcal{C}(H, \Gamma)$ an $(\alpha + 2\epsilon)$-saturated edge is contained in overshoots this lower bound by a factor $1/\alpha$.

We include a brief and rough sketch of our approach for $k = 3$ in which H has the partition $(U_3, U_2, U_1, W, V_1, V_2, V_3)$. For a vertex $w \in W$, we write $\mathcal{C}(H, \Gamma, w)$ to denote the cycles in $\mathcal{C}(H, \Gamma)$ that contains w. To estimate the size of $\mathcal{C}(H, \Gamma, w)$ we shall repeatedly apply the jumbledness condition as described in the definition 5.3 $|e_\Gamma(X, Y) - pvol(X, Y)| \leq \beta vol(X, Y)^{1/2}$ to pairs of subsets (L, R) where $L \subseteq U_3$ and $R \subseteq V_3$ consist of certain vertices connected to w by a path of length 3. The way we shall do this - details below - will be such that the edges in $\Gamma[L, R] \subseteq \Gamma[U_k, V_k]$ will be contained in roughly the same number of cycles in $\mathcal{C}(H, \Gamma, w)$.

To achieve this, consider the neighborhood $X_w = N_H(w) \cap U_1$ of w in U_1. We partition the set U_2 with regard to the "backwards" degrees of its vertices into X_w. For example, the ith partition class will consist of all vertices $u \in U_2$ satisfying $(1 + \eta)^{i-1} \leq |N_H(u) \cap X_w| < (1 + \eta)^i$ for some small η. Some of these classes may be empty: certainly they together cover all vertices in U_2

which are connected to X_w.

We proceed to U_3 in a similar way. Each partition class of U_2 will define a partition of U_3. U_3 is defined in a similar manner to the partition just defined for U_2 using X_w. Given the ith partition class of U_2, say $Z_\eta(i, X_w)$, we assign a vertex $u \in U_3$ to the j-th partition class of U_3 if it satisfies $(1+\eta)^{j-1} \leq |N_H(u) \cap Z_\eta(i, X_w)| < (1+\eta)^j$. The resulting partition class is denoted $Z_\eta(i, j, X_w)$.

We partition the sets V_2 and V_3 in a similar manner where we use $Y_w = N_H(w) \cap V_1$, the neighbourhood of w in V_1, instead of X_w.

The number of paths between w and any vertex $u \in Z_\eta(i, j, X_w) \subset U_3$, confined to the set $Z_\eta(i, X_w)$ - i.e., paths of the form (w, u', u) where $u' \in Z_\eta(i, X_w)$- is at least $(1+\eta)^{i+j-2}$ and at most $(1+\eta)^{i+j}$. This number is known up to a factor of $(1+\eta)^2$. The same bounds hold for w and any vertex $v \in Z_\eta(i', j', Y_w) \subset V_3$, where $Z_\eta(i', j', Y_w)$ is the set obtained by partitioning with respect to Y_w, V_2 and V_3.

If we take a path from w to $u \in Z_\eta(i, j, X_w)$, confined to $Z_\eta(i, X_w)$, and another path from w to $v \in Z_\eta(i', j', Y_w)$, confined to $Z_\eta(i', X_w)$, then these two paths yield a path from u to v. The number of such (u, v)-paths is at least $(1+\eta)^{i+i'+j+j'-4}$ and at most $(1+\eta)^{i+i'+j+j'}$.

Since u and v were arbitrary vertices from $Z_\eta(i, j, X_w)$ and $Z_\eta(i', j', Y_w)$, respectively, we conclude that if uv is an edge of Γ, then the number of cycles in $\mathcal{C}(H, \Gamma, w)$ is in the interval $((1+\eta)^{i+i'+j+j'-4}, (1+\eta)^{i+i'+j+j'+4})$ that contain the edge uv and are confined to $Z_\eta(i, X_w)$ and $Z_\eta(i', X_w)$.

On the other hand, jumbledness of Γ yields that the relevant number of edges in Γ between $Z_\eta(i, j, X_w)$ and $Z_\eta(i', j', Y_w)$ is in the interval

$$(p|Z_\eta(i, j, X_w)||Z_\eta(i', j', Y_w)| - \beta\sqrt{|Z_\eta(i, j, X_w)||Z_\eta(i', j', Y_w)|},$$

$$p|Z_\eta(i, j, X_w)||Z_\eta(i', j', Y_w)| + \beta\sqrt{|Z_\eta(i, j, X_w)||Z_\eta(i', j', Y_w)|}).$$

Summing over all $1 \leq i, j, i', j' \leq \lceil \log_{1+\eta} n \rceil + 1$, we obtain a good estimate for $|\mathcal{C}(H, \Gamma, w)|$. The main term of the contribution of w is in the range

$$(\sum_{i,j,i',j'} p|Z_\eta(i,j,X_w)||Z_\eta(i',j',Y_w)|(1+\eta)^{i+i'+j+j'-4},$$

$$\sum_{i,j,i',j'} p|Z_\eta(i,j,X_w)||Z_\eta(i',j',Y_w)|(1+\eta)^{i+i'+j+j'+4}).$$

This, we will show to be

$$(|X_w||Y_w|p((\alpha-\epsilon)pm)^4, |X_w||Y_w|p((\alpha+\epsilon)pm)^4).$$

The main obstacle will be to show that the error term is negligible compared to the main term, i.e., that

$$\sum_{i,j,i',j'} \beta\sqrt{|Z_\eta(i,j,X_w)||Z_\eta(i',j',Y_w)|}(1+\eta)^{i+i'+j+j'+4} = o(p(pm)^6),$$

where $1 \leq i, j, i', j' \leq \lceil \log_{1+\eta} n \rceil + 1$, provided Γ is sufficiently jumbled. This was the approach for establishing the first part of Lemma 5.13.

For the second part, we shall estimate $|\mathcal{S}(\alpha+2\epsilon, H, \Gamma)|$. This will be done by employing similar arguments to those above, but applied to a rearrangement of the partition $(U_3, U_2, U_1, W, V_1, V_2, V_3)$. We will use the rearrangement

$$(\tilde{U}_3, \tilde{U}_2, \tilde{U}_1, \tilde{W}, \tilde{V}_1, \tilde{V}_2, \tilde{V}_3) = (W, U_1, U_2, U_3, V_3, V_2, V_1).$$

This is a valid partition of the $C_7(m)$-graph H.

The interest is to estimate the number of $(\alpha+2\epsilon)$-saturated cycles. The $(\alpha+2\epsilon)$-saturated cycles that these cycles contain now lie between the sets $\tilde{W} = U_3$ and $\tilde{V}_1 = V_3$. For a given vertex $\tilde{w} \in \tilde{W}$, we set $\tilde{X}_{\tilde{w}} = N_H(\tilde{w}) \cap \tilde{U}_1$. Unlike before, we shall define the set $\tilde{Y}_{\tilde{w}} \subset \tilde{V}_1$ to consist of those vertices of $\tilde{V}_1$ that are incident to $\tilde{w}$ through $(\alpha+2\epsilon)$-saturated edges.

The same arguments as above, yield bounds corresponding to

$$|X_w||Y_w|p((\alpha \pm \epsilon)pm)^4$$

and

$$\sum_{i,j,i',j'} \beta\sqrt{|Z_\eta(i,j,X_w)||Z_\eta(i',j',Y_w)|}(1+\eta)^{i+i'+j+j'+4} = o(p(pm)^6)$$

will lead to an upper bound on the number of $(\alpha+2\epsilon)$-saturated cycles containing $\tilde{w}$. We shall have that since every $(\alpha+2\epsilon)$-saturated edge is contained at least $p((\alpha+2\epsilon)pm)^5$ cycles containing $\tilde{w}$, then

$$|\tilde{Y}_{\tilde{w}}|p((\alpha+2\epsilon)pm)^5 \leq |\tilde{X}_w||\tilde{Y}_w|p(((\alpha+\epsilon)pm)^4 + o(p(pm)^6).$$

As, $|\tilde{X}_w| \leq (\alpha+\epsilon)pm$, we conclude that this inequality can hold provided that $|\tilde{Y}_{\tilde{w}}| = o(pm)$, implying that the number of $(\alpha+2\epsilon)$-saturated cycles containing $\tilde{w}$ is bounded from above by $|\tilde{X}_w||\tilde{Y}_w|p(((\alpha+\epsilon)pm)^4 + o(p(pm)^6)$ which is $o(p(pm)^6)$. Summing over all $\tilde{w} \in \tilde{W}$ yields the desired bound.

7 Starting to make the above more precise

This section will start the process of making the above overview precise not only for $k = 3$ but for all k.

Partitioning the neighborhoods For a real $\eta > 0$, we set $L_\eta = \lceil \log_{1+\eta} 2pm \rceil + 1$ and let $\mathcal{I}_\eta = \{0\} \times [L_\eta]^{k-1}$ where $[n]$ denotes the set $\{1, 2, \dots n\}$.

Definition 7.1 *By* $\mathbf{s}$ *we mean a tuple of integers* $(s_1, s_2, \dots, s_k) \in \mathcal{I}_\eta$ *(so that* $s_1 = 0$*), and we write* $\mathbf{s}_j$ *to denote the prefix* $(s_1, s_2, \dots, s_j)$*, where* $j \in [k]$*, and write* $\mathbf{s}$ *instead of* $\mathbf{s}_k$*.*

Be careful not to confuse s_j and $\mathbf{s}_j$.

Definition 7.2 *For a set* $X \subset U$*, we put* $Z_\eta(\mathbf{s}_1, X) = Z_\eta(0, X) = X \subset U_1$*, and for* $j = 2, \dots, k$ *we define*

$$Z_\eta(\mathbf{s}_j, X) = \{x \in U_j : (1+\eta)^{s_j - 1} \leq |N_H(x) \cap Z_\eta(\mathbf{s}_{j-1}, X)| < (1+\eta)^{s_j}\}$$

so that $Z_\eta(\mathbf{s}_j, X) \subseteq U_j$ *for each* $j \in [k]$*.*

For $\eta \in (0, 1]$, the value $(1+\eta)^{s_j}$ is essentially bounded by the maximum degree of H for any $j \in [k]$: more precisely, we have for such η that $(1+\eta)^{s_j} \leq 8pm$, for any $s_j \in [L_\eta]$, since

$$L_\eta \leq \log_{1+\eta}(2pm) + 2 \leq \log_{1+\eta}(2pm) + \log_{1+\eta}(4) = \log_{1+\eta} 8pm$$

whereas in the other direction $(1+\eta)^{L_\eta - 1} \geq 2pm \geq (\alpha + \epsilon)pm$, and that the maximum degree of a vertex in H is $(\alpha+\epsilon)pm$. This means that $Z_\eta(\mathbf{s}_j, X) = \{x \in U_j : (1+\eta)^{s_j-1} \leq |N_H(x) \cap Z_\eta(\mathbf{s}_{j-1}, X)| < (1+\eta)^{s_j}\}$ defines a partition of the neighborhood of $Z_\eta(\mathbf{s}_{j-1}, X)$ in U_j, i.e.

$$\dot{\bigcup_{1 \leq i \leq L_\eta}} Z_\eta((s_1, \dots, s_{j-1}, i), X) = N_H(Z_\eta(\mathbf{s}_{j-1}, X)) \cap U_j$$

where some of these sets may possibly be empty.

8 Counting shortest paths

We count shortest (X, U_k)-paths in H.

Definition 8.1 *If L and R are two subsets of vertices, we write (L, R)-path to denote a shortest path with one end in L and the other in R.*

With this convention, and recalling that the edges in H which connect two of the U_i are between U_j and U_{j+1}, we see that, if $X \subset U_1$, an (X, U_k)-path in H has a single vertex in each set $U_i, i \in [k]$. (Only one by virtue of being a shortest path). Instead of $(X, \{y\})$-path we write (X, y)-path.

Definition 8.2 *For $j \in \{2, \ldots, k\}$ and a tuple $\mathbf{s} \in \mathcal{I}_\eta$, we write $\sum \mathbf{s}_j$ to denote the sum $\sum_{i=1}^{j} s_i = \sum_{i=2}^{j} s_i$ (using that $s_1 = 0$). We write $\sum \mathbf{s}$ instead of $\sum \mathbf{s}_k$.*

Definition 8.3 *The subgraph of H induced by the vertex sets*

$$\{Z_\eta(\mathbf{s}_1, X), Z_\eta(\mathbf{s}_2, X), \ldots, Z_\eta(\mathbf{s}_j, X)\}$$

is denoted by $H(\mathbf{s}_j)$.

Note that we do not advertise the fact that this depends on η as well.

For a vertex $z \in Z_\eta(\mathbf{s}_j, X)$ the number $\pi_H(X, \mathbf{s}_j, z)$ of (X, z)-paths confined to $H(\mathbf{s}_j)$ clearly satisfies

$$\prod_{i=2}^{j}(1+\eta)^{s_i-1} \leq \pi_H(X, \mathbf{s}_j, z) \leq \prod_{i=2}^{j}(1+\eta)^{s_i}.$$

We may write the above as

$$(1+\eta)^{-(j-1)}(1+\eta)^{\sum \mathbf{s}_j} \leq \pi_H(X, \mathbf{s}_j, z) \leq (1+\eta)^{\sum \mathbf{s}_j}. \tag{29}$$

We observe that for any two vertices $z, z' \in Z_\eta(\mathbf{s}_j, X)$, the variation between $\pi_H(X, \mathbf{s}_j, z)$ and $\pi_H(X, \mathbf{s}_j, z')$ is bounded by a factor of $(1+\eta)^{j-1}$. Consequently the number

$$\pi_H(X, \mathbf{s}_j) = \sum_{z \in Z_\eta(\mathbf{s}_j, X)} \pi_H(X, \mathbf{s}_j, z)$$

of $(X, Z_\eta(\mathbf{s}_j, X))$-paths confined to $H(\mathbf{s}_j)$satisfies

$$\begin{aligned} &(1+\eta)^{-(j-1)} |Z_\eta(\mathbf{s}_j, X)| (1+\eta)^{\sum \mathbf{s}_j} \\ &\leq \pi_H(X, \mathbf{s}_j) \leq |Z_\eta(\mathbf{s}_j, X)| (1+\eta)^{\sum \mathbf{s}_j}. \end{aligned} \tag{30}$$

However, using the previously noted on page 32 fact that

$$\dot{\bigcup_{1 \leq i \leq L_\eta}} Z_\eta((s_1, \ldots, s_{j-1}, i), X) = N_H(Z_\eta(\mathbf{s}_{j-1}, X)) \cap U_j \tag{31}$$

every (X, U_j)-path is contained in $H(\mathbf{s}_j)$ for exactly one $\mathbf{s}_j$. Summing over all $\mathbf{s}_j$, we obtain the following inequality for the number $\pi_H(X, U_j)$ of (X, U_j)-paths:

$$\begin{aligned} &(1+\eta)^{-(j-1)} \sum_{\mathbf{s}_j} |Z_\eta(\mathbf{s}_j, X)| (1+\eta)^{\sum \mathbf{s}_j} \\ &\leq \pi_H(U_j, X) \leq \sum_{\mathbf{s}_j} |Z_\eta(\mathbf{s}_j, X)| (1+\eta)^{\sum \mathbf{s}_j}. \end{aligned} \tag{32}$$

We now approach the estimation of the numbers $\pi_H(U_j, X)$ from another viewpoint. Using the degree-regularity of H, we obviously have, for all $j \in [k]$

$$|X|((\alpha - \epsilon)pm)^{j-1} \leq \pi_H(U_j, X) \leq |X|((\alpha + \epsilon)pm)^{j-1}. \tag{33}$$

Finally, for $j \in \{1, ..., k\}$, a tuple $\mathbf{t} \in \mathcal{I}_\eta$ and a set $Y \subseteq V_1$ we define the sets $\{Z_\eta(\mathbf{t}_j, Y)\}_{j=1}^k$. The subgraph H induced by the vertex sets

$$\{Z_\eta(t_1, Y), Z_\eta(t_2, Y), \ldots, Z_\eta(t_j, Y)\}$$

The numbers $\pi_H(Y, \mathbf{t}_j, u), \pi_H(Y, \mathbf{t}_j), \pi_H(V_j, Y)$ are defined in an analogous manner to the sets and numbers just defined. These hold also,

$$\dot{\bigcup_{1 \leq i \leq L_\eta}} Z_\eta((t_1, \ldots, t_{j-1}, i), Y) = N_H(Z_\eta(\mathbf{t}_{j-1}, Y)) \cap V_j \tag{34}$$

and

$$|Y|((\alpha - \epsilon)pm)^{j-1} \leq \pi_H(V_j, Y) \leq |Y|((\alpha + \epsilon)pm)^{j-1}. \tag{35}$$

We will use the estimates implied by these facts later on.

9 Counting cycles

Given $u \in U_k, v \in V_k$ we let, as before, $\pi_H(X,u)$ and $\pi_H(Y,v)$ be the number of (X,u)-paths and (Y,v)-paths respectively. We define

$$\mathcal{O}(X,Y) = \sum_{uv \in E_\Gamma(U_k,V_k)} \pi_H(X,u)\pi_H(Y,v) \tag{36}$$

to be the number of composed paths each of which comprises a (X,u)-path and a (Y,v)-path in H connected by the edge $uv \in \Gamma[U_k,V_k]$.

Definition 9.1 *Let $\mathcal{C}(H,\Gamma,w)$ denote the set of cycles in $\mathcal{C}(H,\Gamma)$ containing the vertex $w \in W$ and observe that*

$$|\mathcal{C}(H,\Gamma,w)| = \mathcal{O}(N_H(w) \cap U_1, N_H(w) \cap V_1). \tag{37}$$

Our next paragraph will be concerned with estimating the size of $\mathcal{S}(\mu,H,\Gamma)$. Recall the definition of $\mathcal{S}$.

Definition 9.2 *A cycle in $\mathcal{C}(H,\Gamma)$ containing a μ-saturated edge is called a μ-saturated cycle. We write $\mathcal{S}(\mu,H,\Gamma)$ to denote the set of μ-saturated cycles in $\mathcal{C}(H,\Gamma)$.*

We rearrange the partition $(U_k,\dots,U_1,W,V_1,\dots,V_k)$ to yield a partition $(\tilde{U_k},\dots,\tilde{U_1},\tilde{W},\tilde{V_1},\dots,\tilde{V_k})$ obtained by renaming the partition classes as follows:

$$\tilde{W} = U_k, \tilde{U_1} = U_{k-1},\dots,\tilde{U}_{k-1} = U_1, \tilde{U_k} = W, \tilde{V_1} = V_k,\dots,\tilde{V_k} = V_1. \tag{38}$$

The new partition is still a valid partition of the $C_{2k+1}(m)$-graph H, and the μ-saturated edges now lie between $\tilde{W}$ and $\tilde{V_1}$.

Definition 9.3 *For a vertex $\tilde{w} \in \tilde{W}$ let $D_\mu(\tilde{w})$ denote the set of vertices in $\tilde{V_1}$ adjacent to $\tilde{w}$ in Γ through a μ-saturated edge and let $\mathcal{S}(\mu,H,\Gamma,\tilde{w})$ denote the set of μ-saturated cycles containing $\tilde{w}$.*

Then,

$$|\mathcal{S}(\mu, H, \Gamma, \tilde{w})| \leq \tilde{\mathcal{O}}(N_H(\tilde{w}) \cap \tilde{U}_1, D_\mu(\tilde{w})) \tag{39}$$

where $\tilde{\mathcal{O}}(\tilde{X}, \tilde{Y})$ is defined in the same way as $\mathcal{O}(\tilde{X}, \tilde{Y})$ only with respect to the partition $(\tilde{U}_k, \ldots, \tilde{U}_1, \tilde{W}, \tilde{V}_1, \ldots, \tilde{V}_k)$ and where $\tilde{X} \subset \tilde{U}_1$ and $\tilde{Y} \subset \tilde{V}_1$.

In $|\mathcal{S}(\mu, H, \Gamma, \tilde{w})| \leq \tilde{\mathcal{O}}(N_H(\tilde{w}) \cap \tilde{U}_1, D_\mu(\tilde{w}))$ we have an upper bound only as cycles in $\tilde{O}(N_H(\tilde{w}) \cap \tilde{U}_1, D_\mu(\tilde{w}))$ may involve edges in Γ between $\tilde{U}_k$ and $\tilde{V}_k$ which might not belong to H.

In view of the equation $|\mathcal{C}(H, \Gamma, w)| = \mathcal{O}(N_H(w) \cap U_1, N_H(w) \cap V_1)$ and the inequality $|\mathcal{S}(\mu, H, \Gamma, \tilde{w})| \leq \tilde{\mathcal{O}}(N_H(\tilde{w}) \cap \tilde{U}_1, D_\mu(\tilde{w}))$, we focus on $\mathcal{O}(X, Y)$ in order to estimate $|\mathcal{C}(H, \Gamma, w)|$ and $|\mathcal{S}(\mu, H, \Gamma, \tilde{w})|$. For tuples $\mathbf{s}, \mathbf{t} \in \mathcal{I}_\eta$, we write $e_\Gamma(\mathbf{s}, \mathbf{t})$ for $e_\Gamma(Z_\eta(\mathbf{s}, X), Z_\eta(\mathbf{t}, Y))$ and observe that due to the inequality (37) we may write

$$\frac{\sum_{\mathbf{s}} \sum_{\mathbf{t}} e_\Gamma(\mathbf{s}, \mathbf{t})(1+\eta)^{\Sigma \mathbf{s} + \Sigma \mathbf{t}}}{(1+\eta)^{2(k-1)}} \leq \mathcal{O}(X, Y) \leq \sum_{\mathbf{s}} \sum_{\mathbf{t}} e_\Gamma(\mathbf{s}, \mathbf{t})(1+\eta)^{\Sigma \mathbf{s} + \Sigma \mathbf{t}}. \tag{40}$$

Next, we use the jumbledness of Γ to get the estimates

$$\begin{aligned} e_\Gamma(\mathbf{s}, \mathbf{t}) \in (p|Z_\eta(\mathbf{s}, X)||Z_\eta(\mathbf{t}, Y)| - \beta\sqrt{|Z_\eta(\mathbf{s}, X)||Z_\eta(\mathbf{t}, Y)|}, \\ p|Z_\eta(\mathbf{s}, X)||Z_\eta(\mathbf{t}, Y)| + \beta\sqrt{|Z_\eta(\mathbf{s}, X)||Z_\eta(\mathbf{t}, Y)|}). \end{aligned} \tag{41}$$

Substituting this estimate for $e_\Gamma(\mathbf{s}, \mathbf{t})$ in the inequality above we find the two bounds for $\mathcal{O}(X, Y)$.

$$(1+\eta)^{2(k-1)} \mathcal{O}(X, Y) \geq pP_\eta(X)P_\eta(Y) - \beta Q_\eta(X)Q_\eta(Y), \tag{42}$$

and

$$\mathcal{O}(X, Y) \leq pP_\eta(X)P_\eta(Y) + \beta Q_\eta(X)Q_\eta(Y), \tag{43}$$

where

$$P_\eta(X) = \sum_{\mathbf{s}} |Z_\eta(\mathbf{s}, X)|(1+\eta)^{\Sigma \mathbf{s}}, \tag{44}$$

$$P_\eta(Y) = \sum_{\mathbf{t}} |Z_\eta(\mathbf{t}, Y)|(1+\eta)^{\sum \mathbf{t}}, \tag{45}$$

$$Q_\eta(X) = \sum_{\mathbf{s}} \sqrt{|Z_\eta(\mathbf{s}, X)|}(1+\eta)^{\sum \mathbf{s}}, \tag{46}$$

$$Q_\eta(Y) = \sum_{\mathbf{t}} \sqrt{|Z_\eta(\mathbf{t}, Y)|}(1+\eta)^{\sum \mathbf{t}}. \tag{47}$$

We will now argue that $\mathcal{O}(X, Y)$ is for a small η essentially equal to the main term $pP_\eta(X)P_\eta(Y)$ with the error term $\beta Q_\eta(X)Q_\eta(Y)$ being small by comparison. To see this we rewrite (30) for $j = k$ as to obtain

$$P_\eta(X) \leq \pi_H(U_k, X)(1+\eta)^{k-1} \leq P_\eta(X)(1+\eta)^{k-1}. \tag{48}$$

With (33) this yields

$$|X|((\alpha - \epsilon)pm)^{k-1} \leq P_\eta(X) \leq |X|((\alpha + \epsilon)(1+\eta)pm)^{k-1}. \tag{49}$$

A similar assertion holds for $P_\eta(Y)$. Clearly,

$$|Y|((\alpha - \epsilon)pm)^{k-1} \leq P_\eta(Y) \leq |Y|((\alpha + \epsilon)(1+\eta)pm)^{k-1}. \tag{50}$$

For $w \in W$, the sets $X_w = N_H(w) \cap U_1$ and $Y_w = N_H(w) \cap V_1$ both have size $((\alpha - \epsilon)pm, (\alpha + \epsilon)pm)$ due to the degree regularity of H. With the help of (37), (42) and (49) we have that

$$\begin{aligned}
(1+\eta)^{2k}|\mathcal{C}(H, \Gamma, w)| &\overset{(37)}{=} (1+\eta)^{2k}\mathcal{O}(N_H(w) \cap U_1, N_H(w) \cap V_1) \\
&= (1+\eta)^{2k}\mathcal{O}(X_w, Y_w)
\end{aligned}$$

where we used our definitions of X_w and Y_w. This in turn is

$$\begin{aligned}
&\overset{(42)}{\geq} pP_\eta(X_w)P_\eta(Y_w) - \beta Q_\eta(X_w)Q_\eta(Y_w) \\
&\overset{(49)}{\geq} p|X_w|((\alpha - \epsilon)pm)^{k-1}|Y_w|((\alpha - \epsilon)pm)^{k-1} - \beta Q_\eta(X_w)Q_\eta(Y_w) \\
&\geq p((\alpha - \epsilon)pm)((\alpha - \epsilon)pm)^{k-1}((\alpha - \epsilon)pm)((\alpha - \epsilon)pm)^{k-1} - \beta Q_\eta(X_w)Q_\eta(Y_w) \\
&\geq p((\alpha - \epsilon)pm)^{2k}
\end{aligned} \tag{51}$$

where in the second last line we used our bounds on the size of X_w and Y_w.

Let $\tilde{w} \in \tilde{W}$ and $X_{\tilde{w}} = N_H(\tilde{w}) \cap \tilde{U}_1$ and $Y_{\tilde{w}} = D_\mu(\tilde{w})$. Also, let $\tilde{P}_\eta(\tilde{X})$ and $\tilde{Q}_\eta(\tilde{X})$ be defined as

$$\tilde{P}_\eta(\tilde{X}) = \sum_s |Z_\eta(s, \tilde{X})|(1+\eta)^{\sum s}, \tag{52}$$

and

$$\tilde{Q}_\eta(\tilde{X}) = \sum_s \sqrt{|Z_\eta(s, \tilde{X})|}(1+\eta)^{\sum s}. \tag{53}$$

Using (39), (43) and (49) we have that

$$\begin{aligned} |\mathcal{S}(\mu, H, \Gamma, \tilde{w})| &\overset{(39)}{\leq} \tilde{\mathcal{O}}(N_H(\tilde{w}) \cap \tilde{U}_1, D_\mu(\tilde{w})) \\ &\overset{(43)}{\leq} p\tilde{P}_\eta(X_{\tilde{w}})\tilde{P}_\eta(D_\mu(\tilde{w})) + \beta\tilde{Q}_\eta(X_{\tilde{w}})\tilde{Q}_\eta(D_\mu(\tilde{w})) \end{aligned}$$

where we used the definition of X_w. This in turn is

$$\leq p|X_{\tilde{w}}|((\alpha+\epsilon)(1+\eta)pm)^{k-1}|D_\mu(\tilde{w})|((\alpha+\epsilon)(1+\eta)pm)^{k-1} + \beta\tilde{Q}_\eta(X_{\tilde{w}})\tilde{Q}_\eta(D_\mu(\tilde{w}))$$

this time using bounds on the size of $X_{\tilde{w}}$. Finally this in turn is

$$\leq p|D_\mu(\tilde{w})|((\alpha+\epsilon)pm)^{2k-1} + \beta\tilde{Q}_\eta(X_{\tilde{w}})\tilde{Q}_\eta(D_\mu(\tilde{w})). \tag{54}$$

We will now give a key claim which we will have to use in order to finish our proof of Lemma 5.13.

Claim. For $k \geq 1$ and real numbers $0 < \xi, \alpha, \eta, \nu \leq 1$ and $0 < \epsilon \leq \alpha/3$ there exists $\gamma > 0$ such that for every sequence of densities $p = p(n) > 0$ there is an n_0 such that for all $n > n_0$ the following holds. If Γ is an n-vertex (p, β)-jumbled graph with $\beta < \gamma p^{1+1/(2k-1)} n \log^{-2(k-1)} n$ and let $H \subseteq \Gamma$ be an (α, p, ϵ)-degree-regular $C_{2k+1}(m)$-graph, $m \geq \nu n$ with partition

$(U_k, U_{k-1}, \dots U_1, W, V_1, \dots V_k)$. If $X \subset U_1$ and $Y \subset V_1$ both have size at most $(\alpha + \epsilon)pm$ then

$$\beta Q_\eta(X) Q_\eta(Y) < \xi p(pm)^{2k}. \tag{55}$$

We defer the proof of this claim for now, and use it to show how it implies Lemma 5.13.

Proof of Lemma 5.13

Given k, ν, α_0 and ϵ we put

$$\xi = (\epsilon/4)^{4k}, \eta = \min\{\frac{\epsilon}{4\alpha_0}, 1\}, \tag{56}$$

and note $\xi \in (0, 1]$ and $\eta \in (0, 1]$. Let γ' be the number obtained applying the Claim above with $k, \xi, \alpha = \alpha_0, \eta, \nu$ and ϵ. In the statement of Lemma 5.13 we now take γ to be this γ' and n_0 sufficiently large.

Using (51) we have

$$|\mathcal{C}(H, \Gamma, w)| \geq \left(\frac{\alpha - \epsilon}{1 + \eta}\right)^{2k} p(pm)^{2k} - \frac{1}{(1 + \eta)^{2k}} \beta Q_\eta(X_w) Q_\eta(Y_w), \tag{57}$$

for any $w \in W$, where $X_w = N_H(w) \cap U_1$ and $Y_w = N_H(w) \cap V_1$. Since, H is (α, p, ϵ)-degree-regular graph both X_w and Y_w have size at most $(\alpha + \epsilon)pm$. Then, using our choice $\xi = (\epsilon/4)^{4k}$ and $\eta = \min\{\frac{\epsilon}{4\alpha_0}, 1\}$ we have that

$$\begin{aligned} |\mathcal{C}(H, \Gamma, w)| &\geq \left(\frac{\alpha - \epsilon}{1 + \eta}\right)^{2k} p(pm)^{2k} - (1 + \eta)^{-2k} \xi p(pm)^{2k} \\ &\geq \frac{(\alpha - \epsilon)^{2k} - (\epsilon/4)^{4k}}{(1 + \eta)^{2k}} p(pm)^{2k} \\ &\geq \left(\frac{\alpha - (3/2)\epsilon}{1 + \eta}\right)^{2k} p(pm)^{2k} \\ &\geq (\alpha - 2\epsilon)^{2k} p(pm)^{2k} \end{aligned} \tag{58}$$

holds for any $w \in W$. We need to show that

$$\left(\frac{\alpha - (3/2)\epsilon}{1 + \eta}\right) \geq (\alpha - 2\epsilon).$$

Using that $\alpha = \alpha_0$ and the definition of η that $\eta = \min\{\frac{\epsilon}{4\alpha_0}, 1\}$ we have that

$$\begin{aligned}(\alpha - (3/2)\epsilon) &\geq (\alpha - 2\epsilon)(1+\eta)\\ \leftrightarrow (\alpha - (3/2)\epsilon) &\geq (\alpha - 2\epsilon)(1+\frac{\epsilon}{4\alpha})\\ \leftrightarrow 4\alpha(\alpha - (3/2)\epsilon) &\geq (\alpha - 2\epsilon)(4\alpha+\epsilon)\\ \leftrightarrow 4\alpha^2 - 6\alpha\epsilon &\geq 4\alpha^2 - 7\alpha\epsilon - 2\epsilon^2\end{aligned}$$

so this is true.

We used in this argument the fact that

$$(\alpha - 3\epsilon/2)^{2k} + (\epsilon^2/16)^{2k} \leq (\alpha - 3\epsilon/2 + \epsilon^2/16)^{2k} \leq (\alpha-\epsilon)^{2k}$$

the last inequality being equivalent to $\epsilon^2/16 < \epsilon/2$ which comfortably follows from $\epsilon \leq 1$.

Summing over all the vertices in W yields (15), the first assertion of Lemma 5.13.

We now have to prove the second assertion made in Lemma 5.13, namely that $|\mathcal{S}(\alpha+2\epsilon, H, \Gamma)| \leq (3\epsilon)^{2k}(pm)^{2k+1}$. We use again the partition $\tilde{W} = U_k$, $\tilde{U}_1 = U_{k-1}, \ldots, \tilde{U}_{k-1} = U_1$ and $\tilde{U}_k = W$, $\tilde{V}_1 = V_k, \tilde{V}_2 = V_{k-1}, \ldots \tilde{V}_k = V_1$.

Now, using the partition (38) it is sufficient to prove that for any $\tilde{w} \in \tilde{W}$ it holds that

$$|D_{\alpha+2\epsilon}(\tilde{w})| \leq \epsilon^{2k} pm. \tag{59}$$

Indeed, to confirm this, note that, if we assume the inequality (59) yields

$$\begin{aligned}|\mathcal{S}(\alpha+2\epsilon, H, \Gamma, \tilde{w})| &\overset{(54)}{\leq} |D_{\alpha+2\epsilon}(\tilde{w})| p((\alpha+\epsilon)pm)^{2k-1} + \beta \tilde{Q}_\eta(X_{\tilde{w}})\tilde{Q}_\eta(D_{\alpha+2\epsilon}(\tilde{w}))\\ &\leq \epsilon^{2k}(\alpha+\epsilon)^{2k-1}p(pm)^{2k} + \xi p(pm)^{2k}\\ &\overset{(56)}{\leq} \left(\epsilon^{2k}(\alpha+\epsilon)^{2k-1} + (\epsilon/4)^{4k}\right) p(pm)^{2k}\\ &\leq (3\epsilon)^{2k}p(pm)^{2k}.\end{aligned}$$

We need to show that $\epsilon^{2k}(\alpha+\epsilon)^{2k-1} + (\epsilon/4)^{4k} \leq (3\epsilon)^{2k}$.

Dividing both sides by ϵ^{2k} we get $(\alpha+\epsilon)^{2k-1} + (\frac{\epsilon^{2k}}{16^{2k}}) \leq 3^{2k}$.

As $\epsilon \leq \alpha \leq 1$ then $(\alpha+\epsilon)^{2k-1} \leq 2^{2k}$. The other term $\epsilon^{2k}/16^{2k} \leq 1$ so we have that $2^{2k}+1 \leq 3^{2k}$. So, the inequality $\epsilon^{2k}(\alpha+\epsilon)^{2k-1} + (\epsilon/4)^{4k}) \leq (3\epsilon)^{2k}$ is true.

Summing over all the vertices in $\tilde{W}$ yields (16).

It remains to prove (59). Suppose instead that $|D_{\alpha+2\epsilon}(\tilde{z})| > \epsilon^{2k}pm$ for some vertex $\tilde{z} \in \tilde{W}$, and choose $B \subseteq D_{\alpha+2\epsilon}(\tilde{z}) \subseteq \tilde{V}_1$ of size $\lceil \epsilon^{2k}pm \rceil \leq (\alpha+\epsilon)pm$ as stated at the Claim on page 39. Count the number of members of $\mathcal{S}(\alpha+2\epsilon, H, \Gamma, \tilde{z})$ as in the definition 9.3 with the $(\alpha+2\epsilon)$-saturated edge of the form $\tilde{z}b$ where $b \in B$. We write $\mathcal{S}(B,\tilde{z})$ to denote this number. By the definition of an $(\alpha+2\epsilon)$-saturated edge, we get

$$|\mathcal{S}(B,\tilde{z})| \geq |B|p((\alpha+\epsilon)pm)^{2k-1} = \epsilon^{2k}(\alpha+2\epsilon)^{2k-1}p(pm)^{2k}. \tag{60}$$

On the other hand, using (54) with $X_{\tilde{z}} = N_H(\tilde{z}) \cap \tilde{U}_1$ and B instead of $D_\mu \tilde{z}$ with claim on page 39 with $X = X_{\tilde{z}}$ and $Y = B$ yield

$$\begin{aligned}|\mathcal{S}(B,\tilde{z})| &\leq |B|p((\alpha+\epsilon)pm)^{2k-1} + \xi p(pm)^{2k} \\ &\overset{(56)}{=} \left(\epsilon^{2k}(\alpha+\epsilon)^{2k-1} + (\epsilon/4)^{4k}\right) p(pm)^{2k}. \end{aligned} \tag{61}$$

The fact we need to prove here is that

$$\epsilon^{2k}(\alpha+\epsilon)^{2k-1} + (\epsilon/4)^{4k} < \epsilon^{2k}(\alpha+2\epsilon)^{2k-1}.$$

More clearly,

$$\begin{aligned}(\alpha+\epsilon)^{2k-1} + \frac{\epsilon^{2k}}{16^{2k}} \leq (\alpha+\epsilon)^{2k-1} + (\epsilon/16)^{2k-1} &\leq (\alpha+\epsilon+\epsilon/16)^{2k-1} \\ &< (\alpha+2\epsilon)^{2k-1}. \end{aligned} \tag{62}$$

This proves (59) and completes the proof of Lemma 5.13.

We will now give the proof of the key Claim mentioned in page 39.

Proof of Claim

Given $k, \xi, \alpha, \epsilon, \eta$ and ν, we set

$$\gamma = \frac{\xi(\log(1+\eta))^{2k}}{2^{8k}\nu} \tag{63}$$

and choose n_0 be sufficiently large, and let Γ be a (p,β)-jumbled graph, where β satisfies $\beta = \beta(n) \le \gamma p^{1+1/(2k-1)} n \log^{-2(k-1)} n$.

We need to show that $\beta Q_\eta(X) Q_\eta(Y) \le \xi p(pm)^{2k}$ where

$$Q_\eta(X) = \sum_{\mathbf{s}\in\mathcal{I}_\eta} \sqrt{|Z_\eta(\mathbf{s},X)|}(1+\eta)^{\Sigma\mathbf{s}}$$

and $Q_\eta(Y)$ is defined similarly as $\sum_{\mathbf{s}\in\mathcal{I}_\eta} \sqrt{|Z_\eta(\mathbf{s},Y)|}(1+\eta)^{\Sigma\mathbf{s}}$.

We consider the term

$$q_\eta(\mathbf{s},X) = \sqrt{|Z_\eta(\mathbf{s},X)|}(1+\eta)^{\Sigma\mathbf{s}} \tag{64}$$

where $\mathbf{s}\in\mathcal{I}_\eta$.

We must prove that for any $\mathbf{s}\in\mathcal{I}_\eta$

$$q_\eta(\mathbf{s},X) \le 2^{4k} p^{k-\frac{1}{2(2k-1)}} m^{\frac{2k-1}{2}} \tag{65}$$

This estimate holds for the term $q_\eta(\mathbf{t},Y)$ with X replaced by Y and $\mathbf{t}\in\mathcal{I}_\eta$.

Assuming (65), we prove that $\beta Q_\eta(X)Q_\eta(Y) \le \xi p(pm)^{2k}$ as follows:

$$\beta Q_\eta(X)Q_\eta(Y) = \beta \sum_{\mathbf{s}\in\mathcal{I}_\eta} \sqrt{|Z_\eta(\mathbf{s},X)|}(1+\eta)^{\Sigma\mathbf{s}} \sum_{\mathbf{s}\in\mathcal{I}_\eta} \sqrt{|Z_\eta(\mathbf{s},Y)|}(1+\eta)^{\Sigma\mathbf{s}} =$$

$$\overset{(64)}{=} \beta \sum_{\mathbf{s},\mathbf{t}} q_\eta(\mathbf{s},X) q_\eta(\mathbf{t},Y)$$

where $\mathbf{s}, \mathbf{t} \in \mathcal{I}_\eta$, and put $L = (2\log_{1+\eta} n)^{2(k-1)} \geq L_\eta^{2(k-1)}$, which is the upper bound on the number of summands in the equation above.

Then,

$$
\begin{aligned}
\beta Q_\eta(X)Q_\eta(Y) &\leq \beta L(2^{4k} p^{k-\frac{1}{2(2k-1)}} m^{\frac{2k-1}{2}})^2 \\
&= \beta L 2^{8k} p^{2k-\frac{1}{(2k-1)}} m^{2k-1} \\
&\overset{(14)}{\leq} \gamma p^{1+1/(2k-1)} n (\frac{1}{\log(1+\eta)})^{2k-1} 2^{8k} p^{2k-\frac{1}{(2k-1)}} m^{2k-1} \\
&\overset{(14)}{\leq} \gamma (\frac{1}{\log(1+\eta)})^{2k-1} 2^{8k} p^{2k+1} m^{2k-1} n \\
&\overset{(63)}{\leq} \xi p (pm)^{2k}.
\end{aligned}
$$

For a tuple $\mathbf{s} \in \mathcal{I}_\eta$ and using $\alpha + \epsilon \leq 2$ and $1 + \eta \leq 2$ we need to show (65).

There are two cases:

The first case is that we suppose that $|Z_\eta(\mathbf{s}_j, X)| < p^{1/(2k-1)} m$ for all $2 \leq j \leq k$. We should show that

$$
q_\eta(\mathbf{s}, X) = \sqrt{|Z_\eta(\mathbf{s}, X)|}(1+\eta)^{\sum \mathbf{s}} \leq (1+\eta)^k \sqrt{|X|} \prod_{j=2}^{k} M_j, \tag{66}
$$

where

$$
M_j = 2\max\{\beta, p\sqrt{|Z_\eta(\mathbf{s}_j, X)||Z_\eta(\mathbf{s}_{j-1}, X)|}\} < 2p^{1+1/(2k-1)} m.
$$

Then, (66) with the assumption that $\sqrt{|X|} \leq \sqrt{(\alpha+\epsilon)pm}$ gives

$$
\begin{aligned}
q_\eta(\mathbf{s}, X) &\leq (1+\eta)^k \sqrt{(\alpha+\epsilon)pm} (2p^{1+\frac{1}{2k-1}} m)^{k-1} \\
&\leq (1+\eta)^k \sqrt{(\alpha+\epsilon)} \sqrt{pm} 2^{k-1} p^{\frac{2k}{2k-1}\cdot(k-1)} m^{k-1} \\
&\leq 2^k \sqrt{2} \sqrt{pm} 2^{k-1} p^{\frac{2k^2-2k}{2k-1}} m^{k-1} \leq 2^{\frac{4k-1}{2}} \sqrt{pm} p^{\frac{2k^2-2k}{2k-1}} m^{k-1} \\
&\leq 2^{\frac{4k-1}{2}} p^{\frac{4k^2-2k-1}{2(2k-1)}} m^{\frac{2k-1}{2}} \leq 2^{2k} p^{\frac{4k^2-2k-1}{2(2k-1)}} m^{\frac{2k-1}{2}}.
\end{aligned}
$$

To verify (66), we first show for all $2 \leq j \leq k$

$$\sqrt{|Z_\eta(\mathbf{s}_j, X)|}(1+\eta)^{s_j-1} \leq M_j\sqrt{|Z_\eta(\mathbf{s}_{j-1}, X)|}. \tag{67}$$

Note that (67) holds if $(1+\eta)^{s_j-1} \leq 2p|Z_\eta(\mathbf{s}_{j-1}, X)|$.

On the other hand, if

$$(1+\eta)^{s_j} > 2p|Z_\eta(\mathbf{s}_{j-1}, X)| \tag{68}$$

holds then

$$\sqrt{|Z_\eta(\mathbf{s}_j, X)|} \leq \frac{\beta\sqrt{|Z_\eta(\mathbf{s}_{j-1}, X)}}{(1+\eta)^{s_j-1} - p|Z_\eta(\mathbf{s}_{j-1}, X)|} \leq \frac{\beta\sqrt{|Z_\eta(\mathbf{s}_{j-1}, X)|}}{\frac{1}{2}(1+\eta)^{s_j-1}}$$

It is important to mention here that the first inequality above is due to a fact.

Fact Let Γ be a (p,β)-jumbled graph. If $X, Y \subseteq V(\Gamma)$ are disjoint and satisfy $e_\Gamma(X,Y) \geq k|X| \neq p|Y||X|$, then

$$|X| \leq \frac{\beta^2|Y|}{(k-p|Y|)^2}$$

Proof We know that $\beta\sqrt{|X||Y|} \geq |e(X,Y) - p|X||Y|| \geq (k-p|Y|)|X|$

Using algebra this implies to $\beta^2|X||Y| \geq (k-p|Y|)^2|X|^2$ and so

$$\frac{\beta^2|Y|}{(k-p|Y|)^2} \geq |X|$$

More clearly, we take $X = \sqrt{Z_\eta(\mathbf{s}, X)}$ (the notation clash is unavoidable here!) and $Y = Z_\eta(\mathbf{s}_{j-1}, X)$. Also, we are using here that $k = (1+\eta)^{s_j-1}$ and this is implicit in Definition 7.2. Then, the inequality becomes

$$\begin{aligned}|Z_\eta(\mathbf{s}, X)| &\leq \sqrt{\frac{\beta^2|Z_\eta(\mathbf{s}_{j-1}, X)|}{((1+\eta)^{s_j-1} - p|Z_\eta(\mathbf{s}_{j-1}, X)|)^2}} \leq \frac{\beta\sqrt{|Z_\eta(\mathbf{s}_{j-1}, X)|}}{(1+\eta)^{s_j-1} - p|Z_\eta(\mathbf{s}_{j-1}, X)|} \\ &\leq \frac{\beta\sqrt{|Z_\eta(\mathbf{s}_{j-1}, X)|}}{(1+\eta)^{s_j-1} - \frac{(1+\eta)^{s_j}}{2}} = \frac{\beta\sqrt{|Z_\eta(\mathbf{s}_{j-1}, X)|}}{\frac{1}{2}(1+\eta)^{s_j-1}(1-\eta)} \leq \frac{\beta\sqrt{|Z_\eta(\mathbf{s}_{j-1}, X)|}}{\frac{1}{2}(1+\eta)^{s_j-1}}.\end{aligned}$$

Recall (67) for all $2 \leq j \leq k$

$$\sqrt{|Z_\eta(\mathbf{s}_j, X)|}(1+\eta)^{s_{j-1}} \leq M_j \sqrt{|Z_\eta(\mathbf{s}_{j-1}, X)|}. \tag{69}$$

The second case is that $|Z_\eta(\mathbf{s}_j, X)| \geq p^{1/(2k-1)}m$ for some $2 \leq j \leq k$. To prove (65) we make $q_\eta(\mathbf{s}, X)$ as the product of two numbers so we write it as below

$$q_\eta(\mathbf{s}, X) = \sqrt{|Z_\eta(\mathbf{s}, X)|}(1+\eta)^{\sum \mathbf{s}} = R_1 \times R_2, \tag{70}$$

where R_1 is

$$R_1 = \sqrt{|Z_\eta(\mathbf{s}_j, X)|}(1+\eta)^{\sum \mathbf{s}_j}, \tag{71}$$

and R_2 is

$$R_2 = \prod_{r=j}^{k-1} \sqrt{\frac{|Z_\eta(\mathbf{s}_{r+1}, X)|}{|Z_\eta(\mathbf{s}_r, X)|}}(1+\eta)^{s_r+1}. \tag{72}$$

The set $Z_\eta(\mathbf{s}, X)$ is not empty and this implies that every set $Z_\eta(\mathbf{s}_r, X)$ is not empty for each $r \in [k]$. To upper bound $q_\eta(\mathbf{s}, X)$ we prove that

$$R_1 \leq 4^j p^{j-\frac{1}{2(2k-1)}} m^{\frac{2j-1}{2}}, \tag{73}$$

and that

$$R_2 \leq (6pm)^{k-j}. \tag{74}$$

To see (73) we first check that

$$|Z_\eta(\mathbf{s}_j, X)|(1+\eta)^{\sum \mathbf{s}_j} \leq |X|((1+\eta)(\alpha+\epsilon)pm)^{j-1} \leq (4pm)^j, \tag{75}$$

where the first inequality is due to the degree-regularity of H recalling the equation (30) and (33).

Equation (30) says that

$$(1+\eta)^{-(j-1)}\sum_{\mathbf{s}_j}|Z_\eta(\mathbf{s}_j,X)|(1+\eta)^{\sum\mathbf{s}_j}$$
$$\le \pi_H(\mathbf{s}_j,X)\le \sum_{\mathbf{s_j}}|Z_\eta(\mathbf{s}_j,X)|(1+\eta)^{\sum\mathbf{s}_j}.$$

and equation (33) says that

$$|X|((\alpha-\epsilon)pm)^{j-1}\le \pi_H(\mathbf{s}_j,X)\le |X|((\alpha+\epsilon)pm)^{j-1}$$

Combining the two inequalities above (30) and (33), we get that

$$(1+\eta)^{-(j-1)}\sum_{\mathbf{s}_j}|Z_\eta(\mathbf{s}_j,X)|(1+\eta)^{\sum\mathbf{s}_j}\le \pi_H(\mathbf{s}_j,X)\le |X|((\alpha+\epsilon)pm)^{j-1}$$

Dividing both sides by the factor $(1+\eta)^{-(j-1)}$ we have

$$\sum_{\mathbf{s}_j}|Z_\eta(\mathbf{s}_j,X)|(1+\eta)^{\sum\mathbf{s}_j}\le \frac{\pi_H(\mathbf{s}_j,X)}{(1+\eta)^{-(j-1)}}\overset{(38)}{\le}\frac{|X|((\alpha+\epsilon)pm)^{j-1}}{(1+\eta)^{-(j-1)}}$$
$$=|X|((1+\eta)(\alpha+\epsilon)pm)^{j-1}$$

Due to the inequalities $1+\eta\le 2$ and $\alpha+\epsilon\le 2$ we conclude to

$$|Z_\eta(\mathbf{s}_j,X)|(1+\eta)^{\sum\mathbf{s}_j}\le |X|((1+\eta)(\alpha+\epsilon)pm)^{j-1}$$
$$\overset{|X|\le(\alpha+\epsilon)pm}{\le}(\alpha+\epsilon)pm((1+\eta)(\alpha+\epsilon)pm)^{j-1}\le 2pm(4pm)^{j-1}\le(4pm)^j \quad (76)$$

This together with the assumption that $|Z_\eta(\mathbf{s}_j,X)|\ge p^{1/(2k-1)}m$ yield that

$$(1+\eta)^{\sum\mathbf{s}_j}\le\frac{(4pm)^j}{|Z_\eta(\mathbf{s}_j,X)|}\le 4^jp^{j-\frac{1}{2k-1}}m^{j-1}. \quad (77)$$

We rewrite (71) as below

$$R_1=\sqrt{|Z_\eta(\mathbf{s}_j,X)|(1+\eta)^{\sum\mathbf{s}_j}}(1+\eta)^{\frac{1}{2}\sum\mathbf{s}_j}.$$

Using (76) and (77) we have that

$$R_1 \le (4pm)^{j/2}(4^j p^{j-\frac{1}{2k-1}} m^{j-1})^{1/2} = 4^{j/2}p^{j/2}m^{j/2}4^{j/2}p^{j/2-\frac{1}{2(2k-1)}}m^{j/2-1/2}$$
$$\le 4^j p^{j-\frac{1}{2(2k-1)}} m^{\frac{2j-1}{2}}$$

and so (73) follows.

To prove (74) we rewrite (72) to the form

$$R_2 = \prod_{r=j}^{k-1} \sqrt{\frac{|Z_\eta(\mathbf{s}_{r+1},X)|}{|Z_\eta(\mathbf{s}_r,X)|}}(1+\eta)^{s_r+1}$$
$$= \prod_{r=j}^{k-1} \sqrt{\frac{|Z_\eta(\mathbf{s}_{r+1},X)|}{|Z_\eta(\mathbf{s}_r,X)|}}((1+\eta)^{s_{r+1}})^{1/2}((1+\eta)^{s_{r+1}})^{1/2}$$
$$= \prod_{r=j}^{k-1} (\frac{|Z_\eta(\mathbf{s}_{r+1},X)|}{|Z_\eta(\mathbf{s}_r,X)|}(1+\eta)^{s_{r+1}})^{1/2}(1+\eta)^{\frac{s_{r+1}}{2}} \tag{78}$$

For $r \in [k-1]$ we observe that

$$|Z_\eta(\mathbf{s}_{r+1},X)|(1+\eta)^{s_{r+1}} \le (1+\eta)e_H(Z_\eta(\mathbf{s}_r,X), Z_\eta(\mathbf{s}_{r+1},X)),$$

so the term $|Z_\eta(\mathbf{s}_{r+1},X)|(1+\eta)^{s_{r+1}}/|Z_\eta(\mathbf{s}_r,X)|$ exceeds the average degree of a vertex in $Z_\eta(\mathbf{s}_r,X)$ in the graph $H\left[(Z_\eta(\mathbf{s}_r,X), Z_\eta(\mathbf{s}_{r+1},X)\right]$ by a factor at most $1+\eta$. Owing the degree-regularity of H this average degree is bounded by $2pm$. So,

$$\frac{|Z_\eta(\mathbf{s}_{r+1},X)|}{|Z_\eta(\mathbf{s}_r,X)|}(1+\eta)^{s_r+1} \le (1+\eta)2pm \le 4pm.$$

To see this,

$$\frac{|Z_\eta(\mathbf{s}_{r+1},X)|}{|Z_\eta(\mathbf{s}_r,X)|}(1+\eta)^{s_{r+1}} \le \frac{|X|((\alpha+\epsilon)pm)^r}{p^{\frac{1}{2k+1}}}(1+\eta)^{\mathbf{s}_{r+1}}$$
$$\overset{|X|\le(\alpha+\epsilon)pm}{=} \frac{(\alpha+\epsilon)pm((\alpha+\epsilon)pm)^r}{p^{\frac{1}{2k+1}}}(1+\eta)^{\mathbf{s}_{r+1}}$$
$$= (\alpha+\epsilon)^{r+1}p^{r+1-\frac{1}{2k+1}}m^{r+1}(1+\eta)^{\mathbf{s}_{r+1}}$$
$$\le (\alpha+\epsilon)pm(1+\eta)$$

Due to the inequalities $\alpha + \epsilon \leq 2$ and $1 + \eta \leq 2$

$$\frac{|Z_\eta(\mathbf{s}_{r+1}, X)|}{|Z_\eta(\mathbf{s}_r, X)|}(1+\eta)^{s_r+1} \leq 2pm(1+\eta) \leq 4pm \tag{79}$$

To see the upper bound of (79) we need to bound the factor $(1+\eta)^{\frac{s_{r+1}}{2}}$. So, using the same method as before

$$\frac{|Z_\eta(\mathbf{s}_{r+1}, X)|}{|Z_\eta(\mathbf{s}_r, X)|}(1+\eta)^{\frac{\mathbf{s}_{r+1}}{2}} \leq (1+\eta)^2 2pm \leq 8pm \tag{80}$$

The multiplication of the two upper bounds (79) and (80) gives us $\sqrt{4pm}\sqrt{8pm} \leq 6pm$.

10 The result of Berger,Lee and Schacht

In [3] by Berger, Lee and Schacht we have solutions to the following two problems. This work appeared very late in our period of research into these equations, we can only give a short summary of the paper here. Firstly, Berger, Lee and Schacht proved an analogue of the Erdős-Stone theorem.

Theorem 10.1 *Let $k \geq 1$ be an integer and let $\delta > 0$. Then there exist $\eta > 0$ and n_0 such that the following holds: let $n \geq n_0$ and let Γ be an (n, d, λ)-graph satisfying $\lambda^{2k-1} \leq \eta d^{2k}/n$. If $G \subset \Gamma$ is a subgraph such that $e(G) \geq (\frac{1}{2} + \delta)\frac{d}{n}\binom{n}{2}$, then there is a copy of C_{2k+1} in G.*

The proof of the theorem 10.1 is a consequence of the theorem below.

Theorem 10.2 *For $0 < \mu, \delta < 1$ and an integer $k \geq 1$, there exists $\eta = \eta(\delta, \mu, k) > 0$ such that the following holds: let Γ be an (n, d, λ)-graph satisfying $\lambda^{2k-1} \leq \eta d^{2k}/n$ and let X be a δ-almost-regular vertex subset of Γ with $|X| \geq \mu n$. Then, for every subgraph G of $\Gamma[X]$, we have*

$$N_{C_{2k+1}}(G) + N_{C_{2k+1}}(\Gamma[X] \setminus G) \geq \frac{1}{2^{2k}}(p\,|X|)^{2k+1}(1 - 2^{8k}\delta).$$

We need to explain some terminology here. A set X of vertices of Γ is *almost regular* if, letting $\Gamma[X]$ be the subgraph of Γ induced on X, we have $d_{\Gamma[X]}(x) \in (1-\delta)p|X|, (1+\delta)p|X|$ for all $x \in X$. Further $N_H(G)$, for two graphs H and a (larger) graph G, is the number of copies of H in G.

The Theorem 10.2 above can be viewed as an extension of the Erdős-Stone theorem in extremal graph theory. Recall that this theorem - see e.g. [4] - states that if H is a (fixed) graph, then

$$\lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{n(n-1)/2} = 1 - \frac{1}{\chi(H) - 1}$$

where $\chi(H)$ is the chromatic number of H. Thus if the number of edges is at least $(1 - 1/(\chi(H) - 1) + \epsilon)\binom{n}{2}$ there is a copy of H in a graph on n vertices for large enough n. The result in Theorem 10.2 is more general.

To prove Theorem 10.2, they used the spectral estimate for the number of even cycles in (n, d, λ)-graphs using the lemma below.

Lemma 10.3 *Let Γ be an (n, d, λ)-graph and let k be a positive integer. Then,*

$$h_{C_{2k}}(\Gamma) \leq d^{2k} + \lambda^{2k-2} dn.$$

where $h_{C_{2k}}(\Gamma)$ is the number of all homomorphisms from C^{2k+1} to Γ. Recall that a homomorphism from a graph H to a graph G is a function f from $V(H)$ to $V(G)$ such that if $xy \in E(H)$ then $f(x)f(y) \in E(G)$.

11 Possible directions for future work

A question which arises is what happens when $\lambda^{\ell-2}$ and $d^{\ell-1}/n$ are of the same order of magnitude. For example, at the moment the result of Berger et al. is that there is some η such that if $\lambda^{2k-1} \leq \eta d^{2k}/n$ then we get a copy of C_{2k+1} in any subgraph G of Γ with more than $(1/2+\delta)\frac{d}{n}\binom{n}{2}$ edges. The nature of their argument makes it difficult to see what exatly η is and it would be desirable to have better understanding of this.

References

[1] Alon, N.; Explicit Ramsey graphs and orthogonal labelings, Electron. J. Combin. 1 (1994).

[2] Aigner-Horev, E.; Han, H.; Schacht, M. Extremal results for odd cycles in sparse pseudorandom graphs. Combinatorica (2014), **34**, Issue 4, 379-406.

[3] Berger, S.; Joonkyung, L; Schacht, M; Odd cycles in subraphs of sparse pseudorandom graphs. Combinatorics (math.CO) (2019) 1-14.

[4] Bollobás, B. Modern Graph Theory.

[5] Dellamonica, D. Jr; Kohayakawa, Y.; Marciniszyn, M.: Steger, A. On the resilience of long cycles in random graphs. Electronic Journal Combinatorics **15** (2008), #R32.

[6] Donadelli, J: Haxell, P.: Kohayakawa, Y. A note on the size Ramsey number. RAIRO- Theoretical Informatics and Applications **39** (2005) no.1, 191-206.

[7] Krivelevich, M.; Lee, C.; Sudakov, B. Resilient pancyclicity of random and pseudorandom graphs, SIAM Journal on Discrete Mathematics 24(1), (2010) 1-16.

[8] Krivelevich, M.; Sudakov, B.; Vu, V. H. A sharp threshold for network reliability. Combinatorics, Probability and Computing (2002) **11**, 465-474.

[9] Krivelevich, M.; Sudakov, B. Pseudo-random graphs. Journal of Graph Theory 52 (2006), 123-134.

[10] Sudakov, B.; Vu, V. H. Local resilience of graphs. Journal Random Structures and Algorithms. Volume 33, Issue 4, (2008) 409-433.

[11] Sudakov, B.; Szabo, T.; Vu, V.H. A generalization of Turan's theorem. Journal of Graph Theory, **49**, Issue 3 (2005) 187-195.

[12] Thomason, A. Pseudo-random graphs. Annals of Discrete mathematics **33** (1987) 307-331.