Starting small to communicate ☆

Alp Atakan [a,b], Levent Koçkesen [a], Elif Kubilay [c,*]

[a] *Koç University, Turkey*
[b] *Queen Mary Univ. of London, United Kingdom of Great Britain and Northern Ireland*
[c] *University of Essex, United Kingdom of Great Britain and Northern Ireland*

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ABSTRACT

We analyze a repeated cheap-talk game in which the receiver is privately informed about the conflict of interest between herself and the sender and either the sender or the receiver controls the stakes involved in their relationship. We focus on payoff-dominant equilibria that satisfy a Markovian property and show that if the potential conflict of interest is large, then the stakes increase over time, i.e., "starting small" is the unique equilibrium arrangement. In each period, the receiver plays the sender's ideal action with positive probability and the sender provides full information as long as he has always observed his ideal actions in the past. We also show that as the potential conflict of interest increases, the extent to which the stakes are back-loaded increases, i.e., stakes are initially smaller but grow faster.

1. Introduction

We analyze a repeated cheap-talk game in which the receiver is privately informed about the conflict of interest between herself and the sender and either the sender or the receiver controls the stakes involved in their relationship. For example, the sender could be a manufacturer who is better informed about demand and makes retail-price recommendations to a retailer, who in turn decides on the actual retail price and may have different objectives than the manufacturer.[1] The retailer might decide on how much to buy from the manufacturer, in which case we would have a situation where the receiver controls the stakes, or the manufacturer might decide on how much to deliver to that particular retailer, in which case it is the sender who controls the stakes.

Or the receiver could be an investor who first decides how much to save and then how to allocate these funds after receiving advice from an informed financial advisor. The investor is privately informed about her preferences, which might include risk preferences or some behavioral bias. This would be a situation in which the receiver chooses the stakes involved in the relationship. Alternatively, it could be the advisor (sender) who chooses the stakes by controlling which investment opportunities to present to the investor in each period. For example, he may present more important ones (or a big bunch of them) at the beginning and less important ones later, or vice versa.

In situations where the sender is a subordinate, e.g., an advisor, in a hierarchical relationship, we could also think of the problem faced by the receiver as the optimal design of the importance of the decisions about which the sender is consulted over time, i.e., the sender's career path. At what level of the hierarchy should the receiver start the sender and how should she go about promoting him? Is it best to start him at a very low rank and keep him there for a long time, or should the career of the sender progress at a steady pace? What is the role of potential conflict of interest between the sender and the receiver in the optimal design of the career path of the sender?

In our model, each period a sender observes a payoff relevant state of the world and communicates this information to the receiver. The receiver observes the message sent by the sender and makes a decision. State of the world and the decision jointly determine the payoffs, which are revealed at the end of the period. Overall payoff of each player is equal to the weighted sum of period payoffs, where the weight of each period is determined by the size of the stakes (or the importance of the decision) in that period.[2] The sender would like the decision to match the state of the world while the receiver might be biased. More crucially, the sender's preferences are common knowledge while that of the receiver is her private information.

We assume that the information on the state of the world is "soft", i.e., it cannot be verified, and that the messages are costless. This makes the communication phase in each period a "cheap-talk" game, i.e., the sender may lie and this has no direct costs for him. We also assume that the decisions of the receiver are not contractible. This could be due to legal reasons or because the decisions are impossible to reproduce before courts.[3] Our third crucial assumption is that states of the world are independently distributed across periods. This implies that the sender decides how much information to reveal each period without having to worry about its informational implications for the future states. Finally, we assume that the receiver's preferences are similar for each decision, i.e., she either shares the preferences of the sender or is biased in the same manner for all the decisions. Therefore, our model is more suitable for situations in which the decisions are related, such as a series of investment decisions, or budgetary decisions for different departments, etc. An alternative, and perhaps more plausible, way to interpret the model is that the receiver makes the same decision in each period and the state of the world observed by the sender corresponds to the optimal decision for that period.[4]

We assume that the receiver is either an unbiased type, who myopically chooses the decision best suited to the state given her beliefs in each period, or a biased type who acts strategically. The unbiased type resembles a commitment type that is common in the reputation literature. But, unlike a standard commitment type who always plays the same action, the unbiased receiver plays a best response to her beliefs in any given period. Furthermore, unlike in standard models of repeated games and reputation, where the discount rates and the stage game payoffs are fixed, in our model the size of the stakes (or the importance of the decision) in each period is chosen strategically by either the sender or the receiver.[5] Our aim is to characterize the perfect Bayesian equilibria of the resulting extensive form game with incomplete information.

In order to gain some intuition about the major forces at work in the model, note that the sender would like to receive his favorite decision, i.e., the unbiased decision which matches the state, in each period. Therefore, if he believes that the receiver is going to make the unbiased decision with high enough probability, then he has an incentive to reveal the state of the world truthfully. The biased receiver, on the other hand, would like to make a decision that is best for her, i.e., the biased decision, in any period and for that reason she would like to receive accurate information. However, if she makes a decision that is different from the decision that would be made by the unbiased commitment type, she would be revealed as biased and receive no information in the future. This introduces reputation concerns in the sense that she may masquerade as the unbiased receiver and act against her own interest today, in order to receive better information in the future.

It is clear that the receiver benefits from truthful communication. It turns out that, ex-ante, the sender also benefits from truthful communication, irrespective of whether the receiver is biased or not. In fact, if he could commit to a communication strategy before learning the state in the stage game, he would commit to full revelation.[6] Therefore, the stage game exhibits both conflict of interest, because of the possible bias, and common interest, because of the common preference for truthful communication.

These considerations imply that the sender (or the receiver) may choose the stakes in a strategic manner in order to utilize the reputational incentives and facilitate communication. In particular, if relatively larger stakes are left for the

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[2] From now on we will use the terms "importance of the decision" and "size of the stakes" interchangeably.

[3] The assumption that decisions are observable but not contractible follows the "incomplete contracts" perspective (e.g., Grossman and Hart (1986) and Hart and Moore (1990)).

[4] We are grateful to a referee for this interpretation of our model.

[5] This could also be thought of as choosing a time varying discount rate in a strategic manner.

[6] As in the standard cheap-talk games, we assume that the sender cannot commit to a communication strategy. Even if he could, however, commitment to truth-telling may not be in his interest in the repeated game, because that would eliminate the ability to punish the receiver for playing the biased action.

future, then the biased receiver may choose to play the unbiased action early on in the game and this may enable truthful communication.

As is usual in cheap-talk games, our model exhibits multiple equilibria. In order to circumvent this problem, we focus on equilibria that satisfy a Markovian property and yield the highest equilibrium payoffs for all the players, which we call the *payoff-dominant equilibria*. We show that if the potential bias is large and the initial reputation of the receiver bad enough, then there is a unique payoff-dominant equilibrium outcome in which the size of the stakes increases over time. In other words, the time path of the stakes exhibits "gradualism" or "starting small", which has been a recurring theme in economics in different contexts.[7] Our paper contributes to this literature by providing an alternative rationale for these phenomena, which is based on utilizing reputational incentives to maintain a healthy communication in a relationship.

We show that the time path of the stakes is chosen in such a way that the biased receiver is indifferent between the biased and the unbiased actions in each period. She mixes between these actions in a way that builds her reputation (after each successive unbiased action) at just the right speed in order to facilitate communication in every period. If her reputation were to evolve faster, then truthful communication would fail in the current period, while if it were to evolve slower, it would fail in the future. Sender reveals the state truthfully in every period as long as he has always observed the unbiased action in the past.[8]

Interestingly, the sender does not prefer a screening equilibrium in which the biased receiver reveals herself at the beginning of the game. This is because once the biased receiver reveals herself, the sender can no longer communicate with her and, as we have mentioned before, ex-ante the sender prefers to communicate even with the biased receiver. The biased receiver does not prefer to be screened either because, if her initial reputation is bad, she will receive information neither in the period she reveals herself nor afterwards. The unbiased receiver could potentially benefit from being revealed but only if that helps her receive more information. In the payoff-dominant equilibrium outcome, she receives complete information in every period except perhaps the first period in which the biased receiver plays a mixed strategy. However, if the biased receiver reveals herself by playing the biased action with probability one in the first period, rather than mixing, then the sender cannot communicate in that period. Thus, screening does not increase the amount of information provided to the unbiased receiver either.

Our results further imply that as the potential conflict of interest between the sender and the receiver increases, initial stakes become smaller but they grow faster. This is due the fact that as the bias becomes larger, the future must become relatively more important in order to provide sufficient incentives to the biased receiver to play the unbiased action in the current period. We also show that, both the sender and the receiver prefer to spread the total stake in their relationship over as many periods as possible. If the potential bias is large, this would mean that the stakes remain very small for a long period of time and then increase quickly towards the end of the relationship.

Finally, we should note that since the sender fully reveals the state in every period as long as he observes the unbiased action, a payoff-dominant equilibrium is also the most informative one on the equilibrium path.[9] Therefore, payoff-dominance is potentially a reasonable equilibrium selection in our context.[10]

2. The model

A sender and a potentially biased receiver play a repeated cheap-talk game for N periods. Since it is more convenient to do so, we will count the periods in reverse, so that the first period is labeled N, the second $N - 1$, and so on. In each period i, the following stage-game is played:

1. The sender (or the receiver) chooses the parameter $\delta_i \in [0, 1]$ for period i. The parameter δ_i represents the proportion of the total stakes deferred to subsequent periods and $1 - \delta_i$ represents the proportion in period i. Since there are no subsequent periods in the last period, we set $\delta_1 = 0$.
2. Nature chooses the state of the world $\theta_i \in \{0, 1\}$. We assume that each state is equally likely and that states are independent across periods.
3. The sender observes the state θ_i and chooses a message $m_i \in \{0, 1\}$.[11]
4. The receiver observes the sender's message and chooses an action $a_i \in \mathbb{R}$ without observing the state of the world.

We define the parameter γ_i as the importance of the decision or the proportion of the total stakes made in period i. More precisely, $\gamma_N = 1 - \delta_N$, $\gamma_i = \delta_N \cdots \delta_{i+1}(1 - \delta_i)$, and $\gamma_1 = \delta_N \cdots \delta_2(1 - \delta_1)$. The sender's payoff for period i is $v(a_i, \theta_i, \gamma_i) =$

$-\gamma_i(a_i - \theta_i)^2$ and the receiver's payoff is $u(a_i, \theta_i, \beta, \gamma_i) = -\gamma_i(a_i - (\theta_i + \beta))^2$, where $\beta \in \{0, b\}$ and $b > 0$. The parameter b measures the divergence of the preferences of the sender and the receiver, or simply the "bias" of the receiver. We assume that nature chooses $\beta = b$ with probability $p \in (0, 1)$ before the game begins and privately informs the receiver. The payoff of each player over the N periods is simply the sum of the payoffs from each period.

The state of the world, the messages, and the decisions of the receiver are unverifiable and hence cannot be contracted upon. Furthermore, as the payoff functions imply, the messages have no direct payoff consequence. This implies that the communication between the sender and the receiver is "cheap-talk" and that outcome contingent contracts cannot be written. After a period is over, the sender and receiver observe their payoffs and therefore the receiver learns the state in that period and the sender learns the receiver's action.

We assume that the unbiased (type $\beta = 0$) receiver is a commitment type who, in each period, plays the action that is perfectly aligned with the sender's preferences. More precisely, fix a period i and let $\lambda \in [0, 1]$ be the probability assigned by the receiver to the event that $\theta_i = 1$. Define the *best period action for type* $\beta \in \{0, b\}$ as follows:

$$a^\beta(\lambda) = \underset{a_i \in \mathbb{R}}{\operatorname{argmax}} \, \mathbb{E}\left[-(a_i - (\theta_i + \beta))^2\right] = \lambda + \beta. \tag{2.1}$$

We refer to $a^0(\lambda)$ as the *unbiased action* and to $a^b(\lambda)$ as the *biased action*. The unbiased receiver is a commitment type (or an automaton) who plays action $a^0(\lambda)$ in each period, i.e., she picks the myopic best response of a receiver who has zero bias and therefore chooses an action that is perfectly aligned with the sender's preferences. The biased receiver, in contrast, is rational and chooses her period action strategically.

The main question analyzed in the paper is the equilibrium allocation of the stakes or the importance of decisions over time. We assume that any $\delta_i \in [0, 1]$ can be chosen at the beginning of each period i and that at least one period has a positive weight. In other words, the importance of period i decision can be fine-tuned in any desired way. This could be motivated in three different ways: (1) The parameter γ_i is the proportion of the total stakes in the relationship that is assigned to period i; (2) There is a large set of decisions and each period a subset of these decisions is chosen; (3) There is a large set of decisions with varying importance and each period one decision from this set is chosen.

We will analyze two different versions of the model: (1) Sender chooses the parameter δ_i (*sender-game*); (2) Receiver chooses the parameter δ_i (*receiver-game*). For the sake of clarity, we will describe and present our results in detail for only the sender-game. In Section 5 we will describe the receiver-game and argue that our results go through for that model as well.

Let $o_i = (\delta_i, \theta_i, m_i, a_i)$ denote a period i outcome and O_i the set of all possible period i outcomes. For any $i < N$, let H_i be the set of all histories before decision i is made, i.e., sequences of the type $(o_N, \ldots, o_{i+1})$. Define $H_N = \{\emptyset\}$. The sender's belief in period i is a mapping $p_i : H_i \to [0, 1]$ where $p_i(h) = \text{prob}(\beta = b | h)$ for each $h \in H_i$ and $p_N(\emptyset) = p$. A period i strategy for the sender is comprised of two components: $\tau_i : H_i \to [0, 1]$, where $\tau_i(h)$ is the sender's choice of $\delta_i \in [0, 1]$ in period i after history $h \in H_i$, and $\mu_i : H_i \times [0, 1] \times \{0, 1\} \to [0, 1]$, where $\mu_i(h, \delta_i, \theta_i)$, denotes the probability of sending message 1 after history h, δ_i, and θ_i.[12] The receiver moves after histories of the type $(h, \delta_i, \theta_i, m_i)$ where $h \in H_i$. For any history $h \in H_i$, a period i information set for the receiver is given by $I_i = \{(h, \delta_i, \theta_i, m_i) : \theta_i \in \{0, 1\}\}$. In other words, before making a decision in period i, the only thing that is not known by the receiver is θ_i. Let the set of all period i information sets be $\mathscr{I}_i$. Receiver's belief that $\theta_i = 1$ is given by $\lambda_i : \mathscr{I}_i \to [0, 1]$. Since the unbiased receiver is a commitment type, we will only describe strategies for the biased receiver. Biased receiver's (mixed) strategy is given by $\alpha_i : \mathscr{I}_i \to \Delta(\mathbb{R})$, where $\Delta(\mathbb{R})$ denotes the set of all probability distributions with support in $\mathbb{R}$. For ease of exposition we will sometimes write $\lambda_i(h, \delta_i, m_i)$ and $\alpha_i(h, \delta_i, m_i)$ for any $h \in H_i$, $\delta_i \in [0, 1]$, and $m_i \in \{0, 1\}$. A collection $\sigma = (\tau_i, \mu_i, \alpha_i, p_i, \lambda_i)_{i=1}^N$ constitutes an *assessment*.

We are interested in the perfect Bayesian equilibria (PBE) of the game. For expositional reasons, we will restrict attention to equilibria in which for any $i \in \{N, N-1, \ldots, 1\}$, $h \in H_i$, and $\delta_i \in [0, 1]$ (1) The sender sends message $m = 0$ after observing $\theta_i = 0$, i.e., $\mu_i(h, \delta_i, 0) = 0$ and (2) The receiver puts positive probability only on the biased and unbiased actions, i.e.,

$$\alpha_i(h, \delta_i, m) \in \Delta\left(\left\{a^0(\lambda_i(h, \delta_i, m)), a^b(\lambda_i(h, \delta_i, m))\right\}\right).$$

We should note that restriction (1) is without loss of generality in terms of equilibrium outcomes and restriction (2) is satisfied in any equilibrium.[13]

Furthermore, we focus our attention on *equilibria* that satisfy the following two properties:

Property 1. Fix an assessment σ, a period i, a history $h \in H_i$, and an outcome $o_i = (\delta_i, \theta_i, m_i, a_i)$. If $a_i \neq a^0(\lambda_i(h, \delta_i, m_i))$, then σ is a PBE only if $p_j(\hat{h}) = 1$ in any period $j = i-1, \ldots, 1$ and history $\hat{h} \in H_j$ that follows o_i.

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[12] Although we define only the pure strategies in the choice of δ_i, in our analysis we allow for mixed strategies in both the sender and the receiver-game.

[13] As we will show in the sequel (see Lemma 2), any equilibrium where type 0 mixes between the two messages is completely uninformative and hence can be replicated with an equilibrium in which both types send message 0. Furthermore, playing any action other than the unbiased action reveals the receiver as biased, i.e., leads to the worst beliefs about her. This implies that if she plays any action other than the unbiased action with positive probability, it has to be her best period action, i.e., the biased action.

This property implies that any action other than the unbiased action is believed to be played by the biased type and once beliefs put probability one on the biased type, they are never altered.[14]

Property 2. For any $i = N, \ldots, 1$ and $h, h' \in H_i$: (1) $p_i(h) = p_i(h')$ implies $\tau_i(h) = \tau_i(h')$, $\mu_i(h, \delta_i, \theta_i) = \mu_i(h', \delta_i, \theta_i)$, $\alpha_i(h, \delta_i, m_i) = \alpha_i(h', \delta_i, m_i)$; (2) $\alpha_i(h, \delta_i, 0) = \alpha_i(h, \delta_i, 1)$.

The first part of this property states that past history matters only to the extent that it changes the reputation of the receiver, while the second part states that the receiver plays a symmetric strategy. Together, they imply that strategies do not depend on the past communication behavior of the sender.[15] Since the sender's past communication behavior has no effect on current and future payoffs or the states of the world, this is a *Markovian property* in the sense that strategies are independent of payoff irrelevant histories.[16] In particular, this restriction eliminates punishments in the form of "no information revelation" or "playing the biased action" after histories in which the sender did not communicate truthfully. In addition to its Markovian nature, this property is also implied by "renegotiation-proofness", because even after histories in which the sender has lied, both parties have an incentive to choose a continuation equilibrium in which there is full communication. In section 3.2 and 4 we will comment on how our results change when this restriction is removed.

From this point on, we restrict attention to PBE that satisfy Properties 1 and 2. Hence, when we say that an assessment σ constitutes an *equilibrium* we mean that the assessment is a PBE and the assessment satisfies Properties 1 and 2.

Given the above restrictions, we simplify notation and describe period i strategies by functions $\tau_i : H_i \to [0, 1]$, $\mu_i : H_i \times [0, 1] \to [0, 1]$, and $q_i : H_i \times [0, 1] \to [0, 1]$, where $\tau_i(h)$ determines the choice of δ_i, $\mu_i(h, \delta_i)$ determines the probability that the sender sends message 1 after δ_i and $\theta_i = 1$, and the function $q_i(h, \delta_i)$ is a distributional strategy (see Milgrom and Weber (1985)) for the receiver that determines the total probability with which the receiver plays the biased action.

As is usual in cheap-talk games, there are many equilibria of the game defined above, even under the Markovian restriction introduced in Property 2. In this paper, we focus on the *payoff-dominant equilibria*, i.e., equilibria that yield the highest equilibrium payoffs for the players.

3. Preliminaries

Fix an assessment σ, a period i, a history $h \in H_i$, and $\delta_i \in [0, 1]$. Let $\Pr(m)$ be the total probability that the sender sends message $m \in \{0, 1\}$ in this assessment in period i after (h, δ_i).[17] Let $q = q_i(h, \delta_i)$ and $\lambda(m) = \lambda_i(h, \delta_i, m)$. We can then write the sender's and the biased receiver's ex-ante costs in that period as follows:

$$C_i^S(\sigma | h, \delta_i) = \underbrace{qb^2}_{\text{Cost of bias}} + \underbrace{\sum_{m \in \{0,1\}} \Pr(m) \lambda(m)(1 - \lambda(m))}_{\text{Cost of miscommunication}}$$

$$C_i^{BR}(\sigma | h, \delta_i) = \underbrace{\left(1 - \frac{q}{p}\right) b^2}_{\text{Cost of bias}} + \underbrace{\sum_{m \in \{0,1\}} \Pr(m) \lambda(m)(1 - \lambda(m))}_{\text{Cost of miscommunication}}$$

These costs are composed of two components: The first component (*cost of bias*) comes from the fact that the receiver plays the biased action with total probability q after both messages. The second component (*cost of miscommunication*) comes from the fact that the sender may not provide full information and is equal to the expected conditional variance of the state of the world. If, for example, the sender's message provides no information on θ_i, then $\lambda(m) = 1/2$ for $m \in \{0, 1\}$ and the cost of miscommunication is equal $1/4$. If it is perfectly informative, then $\lambda(1) = 1$ and $\lambda(0) = 0$ and the cost of miscommunication is equal to zero. The cost of the unbiased receiver is simply the cost of miscommunication.

3.1. One-period model

To fix ideas, we begin our analysis with the simple case of $N = 1$. Let q be the total probability with which the biased action is played and note that sequential rationality implies that the biased receiver plays the biased action with probability one after both messages, i.e., $q = p$. As it is the case in cheap-talk games, there is always an equilibrium in which the

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[14] The first part is quite reasonable because the unbiased action is the unique action that is available to the unbiased type. The second part is the same as the "never dissuaded once convinced" assumption in Rubinstein (1985).

[15] Note that the second part, i.e., symmetry, is necessary for this to be true. If the receiver plays differently after different messages, then the sender's choice of the message may induce different reputation levels, and hence different behavior, in the future. However, symmetry does not imply that the receiver plays the same mixed strategy over the same set of actions after each message. It only implies that the same probabilities are assigned after each message to the biased and unbiased actions, which themselves depend on beliefs and thus on the message.

[16] See Maskin and Tirole (2001) for the notion of payoff irrelevant histories and some arguments in favor of focusing on Markov perfect equilibria.

[17] More precisely, $\Pr(1) = 0.5 \mu_i(h, \delta_i)$ and $\Pr(0) = 0.5 + 0.5(1 - \mu_i(h, \delta_i))$.

sender's message provides no information about the state, the so called "babbling equilibrium". Instead, suppose that the sender communicates truthfully in equilibrium and, without loss of generality, assume that type $\theta = 0$ sends message $m = 0$ and type $\theta = 1$ sends message $m = 1$. Therefore, after message $m \in \{0, 1\}$, the unbiased action is m and the biased action is $m + b$. Optimality of the strategy of type $\theta = 1$ sender implies that the cost of sending message $m = 1$ is less than the cost of sending message $m = 0$, i.e.,

$$q(1 + b - 1)^2 + (1 - q)(1 - 1)^2 \leq q(0 + b - 1)^2 + (1 - q)(0 - 1)^2.$$

This is equivalent to $qb \leq 1/2$ or $q \leq \bar{q}$, where

$$\bar{q} \equiv \frac{1}{2b}.$$

Optimality of the strategy of type $\theta = 0$ implies that

$$q(b - 0)^2 + (1 - q)(0 - 0)^2 \leq q(1 + b - 0)^2 + (1 - p)(1 - 0)^2,$$

which is always satisfied. Conversely we can show that if $q \leq \bar{q}$, then there is an equilibrium with full revelation. If $\bar{q}/2 < q < \bar{q}$, then there is also an equilibrium in which type $\theta = 1$ completely mixes, but since this plays no major role in our analysis we relegate its proof to the Section 8 (see the proof of Lemma 2). Therefore, if $q > \bar{q}$ there is no information revelation in equilibrium. Note that truthful communication becomes an issue only when $\bar{q} < 1$, or equivalently $b > 1/2$. We summarize this discussion in the following lemma for easy reference.

Lemma 1. *Let $N = 1$ and $p \in [0, 1]$ be the probability that the receiver is biased. In any equilibrium, the receiver plays the biased action with probability one, i.e., $q = p$. If $p > \bar{q}$, then the sender's message provides no information. If $p \leq \bar{q}$, then there is an equilibrium where the sender truthfully reports the state and if $\bar{q}/2 < p < \bar{q}$, then there is an equilibrium where the sender's report is partially truthful (i.e., the sender sends the same message with positive probability in both states). These are the only equilibria.*

Note that the equilibria described in Lemma 1 are Pareto ranked: the more informative equilibrium yields a strictly higher expected payoff to both the sender and the receiver. In fact, in the truthful equilibrium, the receiver's cost is equal to zero whereas the (ex-ante) cost of the sender is pb^2. In the partially informative equilibrium, receiver's cost is $(1/2 - pb) \in (0, 1/4)$, whereas the sender's is equal to $pb^2 + (1/2 - pb)$. In the babbling equilibrium, expected costs of the receiver and the sender are $1/4$ and $pb^2 + 1/4$, respectively.

Remark 1. We should note that the observation in Lemma 1 is true for any period in which the communication incentives of the sender depends only on the total probability with which the biased action is played in that period. Property 2 ensures that this is indeed the case and hence we have Lemma 2, whose proof is in Section 8.

Lemma 2. *Fix a perfect Bayesian equilibrium, a period $i \in \{1, \ldots, N\}$, history $h \in H_i$, and $\delta_i \in [0, 1]$. Sender's equilibrium communication strategy can be of three types: (1) fully informative; (2) partially informative; or (3) non-informative. It is completely informative only if $q(h, \delta_i) \leq \bar{q}$ and partially informative only if $\bar{q}/2 < q(h, \delta_i) < \bar{q}$.*

3.2. Two-period model

In this subsection, we provide some intuition for our general result by analyzing the two-period version of the model. We will assume that the sender chooses the stakes and analyze the more interesting and challenging case of $p > \bar{q}$. We will comment later on what happens when $p \leq \bar{q}$ and in Section 5 we will argue that our results go through for the receiver-game. We will first establish two important facts that are true in any equilibrium, including those that do not satisfy the Markovian property (Property 2). After that we will construct an equilibrium and show that it yields the unique payoff dominant outcome among all equilibria that satisfy Property 2.

As we stated in Lemma 2, the Markovian property implies that truth-telling incentives of the sender in each period depends only on the total probability with which the biased action is played in that period: if the biased action is played with total probability q_i in period i, then full revelation is incentive compatible if $q_i \leq \bar{q}$ while no information can be revealed if $q_i > \bar{q}$. Note that we are using two aspects of Property 2 here: (1) communication behavior in the next period is independent of the communication behavior in the current period; and (2) symmetry, i.e., total probability of playing the biased action is the same and equal to some q_i after each message.[18]

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[18] We should note that the definitions of the biased and unbiased action depend on the receiver's beliefs (see Equation (2.1)) and hence potentially on the message. Therefore, symmetry does not mean that the receiver plays the same mixed strategy over the same set of actions after each message. For example, if the sender fully reveals the state, then after message $m = 0$, the biased and unbiased actions are b and 0, respectively, while after message $m = 1$, they are $1 + b$ and 1. Symmetry only means that probability assigned to b (resp. 0) after $m = 0$ is the same as the probability assigned to $1 + b$ (resp. 1) after $m = 1$.

This allows us to write the total cost of the sender in the following simple form:

$$C^S(\delta_2, q_2, c_2, c_1 | p) = (1 - \delta_2)\underbrace{\left[q_2 b^2 + c_2\right]}_{\text{Period 2 cost}} + \delta_2[q_2 \underbrace{(b^2 + \frac{1}{4})}_{\text{Cost after biased action}} + (1 - q_2)\underbrace{(p_1 b^2 + c_1)}_{\text{Cost after unbiased action}}]$$

$$= (1 - \delta_2)\left[q_2 b^2 + c_2\right] + \delta_2\left[pb^2 + q_2\frac{1}{4} + (1 - q_2)c_1\right], \tag{3.1}$$

where p is the prior probability assigned to the biased receiver, the posterior belief in period 1 is $p_1 = (p - q_2)/(1 - q_2)$ (by Bayes' rule), c_2 is the cost of miscommunication period 2, and the cost of miscommunication in period 1 is equal to $\frac{1}{4}$ and c_1 after the biased and unbiased actions, respectively. Similarly, we can write the total cost of the biased and unbiased receivers as follows:

$$C^{BR}(\delta_2, q_2, c_2, c_1 | p) = (1 - \delta_2)\left[\left(1 - \frac{q_2}{p}\right)b^2 + c_2\right] + \delta_2\left[\frac{q_2}{p}\frac{1}{4} + \left(1 - \frac{q_2}{p}\right)c_1\right] \tag{3.2}$$

$$C^{UR}(\delta_2, q_2, c_2, c_1 | p) = (1 - \delta_2)c_2 + \delta_2 c_1 \tag{3.3}$$

Note that, as we have remarked before, the costs of miscommunication are the same for the sender and the receiver and symmetry allows us to use one q_2 after both messages.

We now establish two equilibrium facts that will be useful later on. First define,

$$\delta_2^* = \frac{4b^2}{1 + 4b^2}$$

and

$$q_2^*(p) = \frac{p - \bar{q}}{1 - \bar{q}}$$

In the equilibrium that we will construct, the sender chooses δ_2^* and the receiver plays the biased action with total probability $q_2^*(p)$ in period 2. The first fact states that if the future is less important than δ_2^*, then the biased receiver plays the biased action with probability one in period 2.

Fact 1. *If $\delta_2 < \delta_2^*$, then in any equilibrium the biased receiver plays the biased action with probability one in period 2, i.e., $q_2 = p$.*

The proof of this fact is simple. If the biased receiver plays the biased action in period 2, i.e., $q_2 = p$, then her cost is $(1 - \delta_2)c_2 + \delta_2/4$. If, on the other hand, she plays the unbiased action her cost is at least $(1 - \delta_2)(c_2 + b^2)$. $\delta_2 < \delta_2^*$ implies $(1 - \delta_2)c_2 + \delta_2/4 < (1 - \delta_2)(c_2 + b^2)$, i.e., the cost of playing the biased action is smaller than the cost of playing the unbiased action. Intuitively, when the future is not important enough, the future reputational benefit of playing the unbiased action is smaller than its current cost. Therefore, in the current period the biased receiver plays the biased action.

In order to state our second fact, first note that if the total probability of playing the biased action in period 2 is $q_2^*(p)$, then Bayes' rule implies that beliefs must assign the following probability to the biased receiver in the next period (after the unbiased action):

$$p_1 = \frac{p - q_2^*(p)}{1 - q_2^*(p)}.$$

Definition of q_2^* implies that if the total probability of the biased action is $q_2 = q_2^*(p)$ in period 2, then $p_1 = \bar{q}$ and if $q_2 < q_2^*(p)$, then $p_1 > \bar{q}$. In other words, $q_2^*(p)$ is the smallest probability of the biased action in period 2 that makes $p_1 \leq \bar{q}$ and hence full revelation in period 1 possible. This leads to the following fact:

Fact 2. *In any equilibrium, the total probability of the biased action is at least $q_2^*(p)$ in period 2 and the probability assigned to the biased receiver in period 1 after the unbiased action is at most $\bar{q}$. In other words, $q_2 \geq q_2^*(p) > 0$ and $p_1 \leq \bar{q}$ in any equilibrium.*

The proof of this fact is also simple. As we have argued above, if $q_2 < q_2^*(p)$, then $p_1 > \bar{q}$. But $p_1 > \bar{q}$ implies that the sender will not provide any information in period 1 after the unbiased action, i.e., there is no benefit of playing the unbiased action for the biased receiver. Therefore, she should play the biased action with probability one, i.e., $q_2 = p \geq q_2^*(p)$, contradicting the hypothesis $q_2 < q_2^*(p)$. Finally, Bayes' rule and $q_2 \geq q_2^*(p)$ imply that $p_1 \leq \bar{q}$ in any equilibrium. The intuition behind Fact 2 is simple: the biased receiver must play the biased action with high enough probability, because, otherwise, her reputation next period after the unbiased action will be so bad that she will not receive any information. We should also note that, $q_2 \geq q_2^*(p)$ implies that the (minimum possible) equilibrium cost of miscommunication in period 2 is at least as high as the one in our equilibrium.

We are now ready to construct our equilibrium, which we denote by σ^*. We say that a period 2 outcome is cooperative if δ_2^* is chosen and the unbiased action is played. In period 2, the sender chooses δ_2^* and the unbiased receiver always plays the unbiased action given her beliefs. The biased receiver mixes after δ_2^* so that total probability of the biased action is $q_2^*(p)$, and plays the biased action with probability one (total probability p) after any other δ_2. Type $\theta_2 = 0$ sender always sends message $m_2 = 0$. Type $\theta_2 = 1$ sends message $m_2 = 1$, i.e., the sender fully reveals the state, if and only if δ_2^* has been chosen and it is incentive compatible to reveal the state, i.e., $q_2^*(p) \leq \bar{q}$. If $\delta_2 \neq \delta_2^*$ or $q_2^*(p) > \bar{q}$, type $\theta_2 = 1$ sender sends message $m_2 = 0$, i.e., the sender reveals no information. In period 1, the biased receiver plays the biased action with probability one and the sender reveals the state truthfully if and only if period 2 outcome is cooperative and it is incentive compatible to reveal truthfully, i.e., posterior probability that the receiver is biased after the unbiased action is smaller than $\bar{q}$, i.e., $p_1 \leq \bar{q}$. If period 2 outcome is not cooperative or $p_1 > \bar{q}$, then both senders send message $m_1 = 0$, i.e., the sender reveals no information. Beliefs are derived from Bayes' rule whenever possible.

Under σ^*, the sender's cost is equal to

$$C^S\left(\sigma^*|p\right) = C^S\left(\delta_2^*, q_2^*(p), c_2^*, 0|p\right) = (1 - \delta_2^*)\left(q_2^*(p)b^2 + c_2^*\right) + \delta_2^*\left(pb^2 + q_2^*(p)\frac{1}{4}\right), \tag{3.4}$$

the biased receiver's total cost is

$$C^{BR}\left(\sigma^*|p\right) = \left(1 - \delta_2^*\right)c_2^* + \delta_2^*\frac{1}{4}, \tag{3.5}$$

and the unbiased receiver's cost is

$$C^{UR}\left(\sigma^*|p\right) = \left(1 - \delta_2^*\right)c_2^*, \tag{3.6}$$

where $c_2^* = 0$ if $q_2^*(p) \leq \bar{q}$, and $c_2^* = 1/4$ otherwise.

We first show that σ^* is an equilibrium. Let us start with period 1. The biased receiver plays the biased action with probability one in period 1, which is the unique sequentially rational behavior. The sender reports truthfully if and only if the previous period's outcome is cooperative and $p_1 \leq \bar{q}$, which is also sequentially rational.

Now let us consider period 2 behavior. After histories that start with δ_2^*, the receiver plays the biased action with total probability $q_2^*(p)$, which implies that communicating truthfully is sequentially rational if $q_2^*(p) \leq \bar{q}$. If the receiver plays the unbiased action, then she induces a cooperative history and posterior belief is $p_1 = \bar{q}$ (As we have shown before, this follows from the definition of $q_2^*(p)$). Therefore, she learns the state perfectly in the next period and plays her best period action. This implies that the total cost of playing the unbiased action is $(1 - \delta_2^*)\left(b^2 + c_2\right)$, where $c_2 \in [0, 1/4]$ is the cost of miscommunication. If the receiver plays the biased action, then her cost is equal to the same cost of miscommunication c_2 in the current period, but she induces a history that is not cooperative and receives no information in the next period, which costs her $1/4$. Definition of δ_2^* implies that $(1 - \delta_2^*)\left(b^2 + c_2\right) = (1 - \delta_2^*)c_2 + \delta_2^*/4$, i.e., the receiver is indifferent between the biased and the unbiased actions and hence playing the biased action with total probability $q_2^*(p)$ is sequentially rational. Now consider histories that start with $\delta_2 \neq \delta_2^*$. In this case, the sender provides no information in both periods irrespective of what the receiver does. Therefore, it is sequentially rational for the biased receiver to play the biased action with probability one after any $\delta_2 \neq \delta_2^*$. If the sender chooses $\delta_2 \neq \delta_2^*$, there is no information revelation and the biased receiver plays the biased action with probability one in both periods. Therefore, the cost of the sender is equal to $pb^2 + 1/4$, which is the highest possible cost for him, and hence higher than the cost of choosing δ_2^*, which is at most $(1 - \delta_2^*)\left[q_2^*(p)b^2 + 1/4\right] + \delta_2^*\left[pb^2 + q_2^*(p)/4\right]$.

We will now show that σ^* yields the unique payoff dominant equilibrium outcome. Let us start with the *biased receiver*. As we have established in Fact 2, $q_2 > 0$ in any equilibrium, i.e., she plays the biased action in period 2 with positive probability. This implies that, in any equilibrium, her cost is equal to the cost of playing the biased action, i.e., $(1 - \delta_2)c_2 + \delta_2/4$. Since $c_2 \leq 1/4$, she prefers smaller δ_2. But if δ_2 is smaller than δ_2^*, then Fact 1 implies that $q_2 = p > \bar{q}$ and hence she receives no information in period 2. Note that here we are using the Markovian property, which, as we have argued before, implies that if the total probability of the biased action is greater than $\bar{q}$ then the sender reveals no information. Therefore, if $\delta_2 < \delta_2^*$, then the cost of miscommunication in period two is $c_2 = 1/4$ and her total cost is the highest possible, i.e., $1/4$. Therefore, the best for her is the minimum δ_2 that may possibly allow information revelation in period 2, i.e., δ_2^*.

Now let us consider the *unbiased receiver*. If $q_2^*(p) \leq \bar{q}$, then σ^* is optimal for the unbiased receiver because her cost is zero and hence no other equilibrium can yield a lower cost. If, on the other hand, $q_2^*(p) > \bar{q}$, then her cost in our equilibrium is equal to $(1 - \delta_2^*)/4$. Fact 2, i.e., $q_2 \geq q_2^*(p) > \bar{q}$, implies that her cost is at least $(1 - \delta_2)/4$ in any equilibrium (again an implication of the Markovian property). If $\delta_2 \leq \delta_2^*$, then $(1 - \delta_2)/4 \geq \left(1 - \delta_2^*\right)/4$, that is, the unbiased receiver's cost is at least as large as her cost in our equilibrium. If $\delta_2 > \delta_2^*$ and the unbiased receiver's cost is smaller than her cost in σ^*, then the *biased* receiver would be better off with playing the unbiased action herself, which contradicts $q_2 \geq q_2^*(p) > 0$ (Fact 2).[19]

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[19] More precisely, if $\delta_2 > \delta_2^*$ and $(1 - \delta_2)/4 + \delta_2 c_1 < \left(1 - \delta_2^*\right)/4$, or equivalently $\delta_2/4 > \delta_2^*/4 + \delta_2 c_1$, then $\delta_2^*/4 = (1 - \delta_2^*)b^2$ implies that $\delta_2/4 > (1 - \delta_2^*)b^2 + \delta_2 c_1 > (1 - \delta_2)b^2 + \delta_2 c_1$. In other words, the cost of playing the biased action is larger than the cost of playing the unbiased action for the biased receiver, which contradicts $q_2 \geq q_2^*(p) > 0$, i.e., she plays the biased action with positive probability in period 2 in any equilibrium.

Now let us consider the *sender*. First note that since the cost of miscommunication in period 1 is at most 1/4, i.e., $c_1 \leq 1/4$, the sender's cost is strictly increasing in the total probability of the biased action in period 2, i.e., q_2 (see equation (3.1)). Intuitively, this is because a higher q_2 leads to a higher cost of the biased action today and a higher probability of miscommunication tomorrow. Assume first that $\delta_2 < \delta_2^*$ in equilibrium. In this case, Fact 1 implies that the biased receiver plays the biased action with probability one in period 2, i.e., $q_2 = p > q_2^*(p)$. This implies that, for a fixed δ_2, the sender's cost is strictly higher than his cost if q_2 were to equal to $q_2^*(p)$: (1) cost of miscommunication in period 2 is the highest possible, i.e., 1/4; (2) cost of biased action is strictly higher, i.e., $pb^2 > [(1-\delta_2)q_2^*(p) + \delta_2 p]b^2$. Furthermore, $(1-\delta_2) > (1-\delta_2^*)$ implies that more weight is assigned to today. Since the cost of miscommunication today $(= 1/4)$ is higher than the (expected) cost of miscommunication tomorrow $(= p/4)$, smaller δ_2 is costlier. Therefore, sender's cost is higher than his cost in our equilibrium if $\delta_2 < \delta_2^*$.

Assume now that $\delta_2 > \delta_2^*$ and remember that $q_2 \geq q_2^*(p) > 0$ in any equilibrium. Since the sender's cost is increasing in q_2, the cost in any equilibrium where $q_2 > q_2^*(p)$ is higher than the cost if we replace q_2 with $q_2^*(p)$. Furthermore, when $q_2 = q_2^*(p)$ sender's cost is strictly increasing in δ_2: As δ_2 increases the cost of the biased action increases because in period 1 the biased receiver plays the biased action with a higher probability than in period 2. However, the cost of miscommunication decreases because it is equal to 1/4 today while its expectation is $q_2^*/4$ tomorrow. A straightforward computation shows that when $b > 1/2$, the increase in the cost of biased action outweighs the decrease in cost of miscommunication. Therefore, starting with an arbitrary equilibrium with $q_2 > q_2^*(p)$ and $\delta_2 > \delta_2^*$ and decreasing q_2 to $q_2^*(p)$ and δ_2 to δ_2^*, decreases the sender's cost. This shows that the sender's cost is higher than his cost in our equilibrium if $\delta_2 > \delta_2^*$. We conclude that $\delta_2 = \delta_2^*$ and $q_2 = q_2^*(p)$ uniquely minimizes the sender's cost.

The important ingredients in our result are the following facts that are true in any equilibrium: (1) $\delta_2 < \delta_2^*$ implies that the biased receiver plays the biased action with probability one in period 2; (2) $q_2 \geq q_2^*(p) > 0$ in any equilibrium; (3) $q_2^*(p)$ is the smallest probability of biased action that makes information revelation possible in the next period. The two other properties that we have used come from the Markovian property: (4) the receiver plays the biased action with the same probabilities after each message; (5) if the total probability of the biased action in period 2 is greater than $\bar{q}$, then there is no information revelation. The rest of the intuition can be summarized as follows: $\delta_2 < \delta_2^*$ is bad for the sender because it leads to a higher probability of the biased action and no information today and higher probability of no information tomorrow. Similarly, it is bad for both types of the receiver because it leads to no information today. Sender prefers the smallest possible probability of the biased action today that allows information revelation tomorrow, which is $q_2^*(p)$. If this probability is indeed equal to $q_2^*(p)$, then he prefers smaller δ_2 because the biased action is played with higher probability tomorrow, i.e., $p > q_2^*(p)$. The biased receiver also likes smaller δ_2 because she may receive information today but will not receive it tomorrow. So, it is best for her that δ_2 is the smallest that allows information revelation today, i.e., δ_2^*. Large δ_2 may benefit the unbiased receiver, but if it does then the biased receiver would also prefer playing the unbiased action, contradicting that she must play the biased action with positive probability, i.e., ingredient (2) above.

To summarize, if the initial reputation of the receiver is bad, i.e., $p > \bar{q}$, then in the unique payoff-dominant equilibrium outcome: (1) The relative size of the stakes in period 2 is chosen so that the receiver is indifferent between the biased and the unbiased actions for that period, given that the sender will communicate truthfully after the unbiased action and will provide no information otherwise; (2) The receiver mixes in such a way that her reputation next period is just good enough to make truthful communication possible; (3) A larger share of the stakes is left to the future $(\delta_2^* > 1/2)$, i.e., starting small is the unique payoff-dominant equilibrium outcome.

Remark 2. In order to understand the role of reputational concerns, it might be instructive to compare the equilibrium we have presented above with the scenario in which the sender learns the receiver's type. Assuming that $b > 1/2$, once the sender learns that the receiver is biased, he will provide no information and the receiver will play the biased action in each period. Therefore, the cost of the sender will be $b^2 + 1/4$. If, on the other hand, the sender learns that the receiver is unbiased, then the sender's cost will be zero in every period. Therefore, if the sender knows that he will learn the type of the receiver, his ex ante cost is $p(b^2 + 1/4)$. This cost is higher than his cost in our equilibrium. Intuitively, in our equilibrium, the probability of the biased action is smaller in each period, which directly benefits the sender, and also leads to higher probability of truthful communication in the future. This is the main reason why the sender does not prefer to screen the receiver early on in the game.

Remark 3. We should also note that, for our results, it is not necessary that the sender and the receiver share the same δ_2. The main role of δ_2 is to provide incentives to the biased receiver to play the unbiased action with positive probability. It is easy to show that if the sender has a fixed δ_2 and the choice variable is the size of the stakes for the receiver, then our main result still goes through, i.e., σ^* is still the payoff-dominant equilibrium.

Remark 4. In the analysis above we have restricted the search for payoff-dominant equilibrium to those in which the Markov property, i.e., Property 2, holds.

If, however, we allow the sender to coordinate her communication strategy across periods, then the incentive compatibility constraint for telling the truth is relaxed. When the Markov property holds, the binding constraint for telling the truth

in the first period is given by $q_2 b^2 \leq q_2 b^2 + 1 - 2q_2 b$, or $q_2 \leq \bar{q}$, where q_2 is the total probability with which the agent plays the biased action. Without the Markov restriction, this constraint is given by

$$(1 - \delta_2) q_2 b^2 + \delta_2 \left(p_2 b^2 + \frac{1}{4} q_2 \right) \leq (1 - \delta_2) \left(q_2 b^2 + 1 - 2q_2 b \right) + \delta_2 \left(p_2 b^2 + \frac{1}{4} \right),$$

which is equivalent to

$$q_2 \leq \hat{q}(\delta) \equiv \frac{\delta_2/4 + (1 - \delta_2)}{\delta_2/4 + (1 - \delta_2)2b}.$$

Otherwise, the non-Markov payoff-dominant equilibrium is exactly the same as the Markov equilibrium. In particular, $q_2 = 1 - (1 - p) / (1 - \bar{q})$ in any payoff-dominant equilibrium, both Markov and non-Markov. Since, $\hat{q}(\delta) < \bar{q}$ for any $b > 1/2$, truth-telling is optimal for a larger set of prior probabilities in the non-Markov payoff-dominant equilibrium.

Remark 5. It is easy to adapt the arguments in the text and show that, in any payoff-dominant equilibrium, the receiver plays the biased action with the same probability after any message sent with positive probability. In other words, the symmetry part of the property is not binding for our results in the two-period model. However, the analysis rapidly becomes intractable for $N > 2$ without the symmetry assumption. Beyond tractability, there is also an economic justification for the symmetry assumption: the choice of stakes is independent of whether the receiver plays symmetrically because stakes are chosen to leave the receiver indifferent between playing the biased action in the current period and playing the unbiased action in the current period and then switching to the biased action in the subsequent period. Importantly, the probability of playing the biased action in the current period and therefore the communication strategy of the sender does not enter this calculation of the stakes.

Remark 6. If $p \leq \bar{q}$, then the sender's cost in our equilibrium is $\delta_2^* p b^2$. The sender's cost in any equilibrium with $\delta_2 > \delta_2^*$ is at least $\delta_2 p b^2$, which is strictly greater than $\delta_2^* p b^2$. If $\delta_2 < \delta_2^*$, then $q_2 = p$ and the sender's cost is at least $p b^2$, which is also strictly greater than $\delta_2^* p b^2$. Therefore, our equilibrium uniquely minimizes the sender's cost when $p \leq \bar{q}$ as well.

However, when $p \leq \bar{q}$, equilibrium σ^* is not optimal for the receiver. Still, there is an equilibrium which is receiver-optimal and has the same property of starting small. In this equilibrium $\delta_2 = 1$, the sender provides full information in both periods and the biased receiver plays the unbiased action in the first and the biased action in the last period. In other words, the players effectively play a one-period game with the same prior. The costs of both the biased and the unbiased receivers are equal to zero in this equilibrium, while the cost of the biased receiver is at least $\delta_2^*/4$ in equilibrium σ^*.

4. The main result

In this section, we will show that the main results we have obtained in the two-period version of the model go through in the general model with N periods: There is a unique sender-optimal equilibrium outcome and this outcome is also optimal for the receiver as long as the receiver has a sufficiently bad initial reputation. In other words, for sufficiently bad initial reputation levels, there is a unique equilibrium outcome that Pareto dominates all other equilibrium outcomes. If the potential conflict of interest is large enough, i.e., $b > 1/2$, this equilibrium outcome is characterized by "starting small", i.e., increasing the stakes over time as long as the receiver does the "right thing". We also show that as the receiver's potential bias b increases, the initial stakes become smaller but they grow faster.

In order to facilitate the definition of the payoff-dominant equilibrium we first need some preliminary definitions. Let $\delta_1^* = 0$ and define δ_i^* recursively as

$$\delta_2^* = \frac{4b^2}{1 + 4b^2} \tag{4.1}$$

$$\delta_i^* = \frac{4b^2}{1 + 4b^2 \prod_{j=2}^{i-1} \delta_j^*}, \quad i = 3, \ldots, N \tag{4.2}$$

For any $p \in [0, 1]$, let[20]

$$q_i^*(p) = \begin{cases} \left(1 - \frac{1-p}{(1-\bar{q})^{i-1}} \right)^+, & \bar{q} < 1 \text{ or } i = 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.3}$$

Define the set of period i histories H_i^* as follows. $H_N^* = \{\emptyset\}$ and, for any $i = N - 1, \ldots, 1$, a history $h = (o_N, o_{N-1} \ldots, o_{i+1}) \in H_i^*$ if (1) period N outcome, $o_N = (\delta_N, \theta_N, m_N, a_N)$, is such that $\delta_N = \delta_N^*$ and

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[20] $x^+ = \max\{0, x\}$ for any $x \in \mathbb{R}$.

$$a_N = \begin{cases} m_N, & \text{if } q_N^*(p) \leq \bar{q} \\ 1/2, & \text{if } q_N^*(p) > \bar{q} \end{cases}$$

(2) for all $j = N - 1, \ldots, i + 1$, period j outcome, $o_j = (\delta_j, \theta_j, m_j, a_j)$, is such that $\delta_j = \delta_j^*$ and $a_j = m_j$. In other words, a history belongs to H_i^* if in each previous period j, δ_j^* has been chosen and the receiver played the unbiased action after each message believing that the sender was telling the truth (except in period N, where she believes the sender is telling the truth if and only if doing so is sequentially rational for the sender).

We will define the payoff-dominant equilibrium assessment $\sigma^* = (\tau_i, \mu_i, q_i, p_i, \lambda_i)$ for the sender-game and point out how it differs for the receiver-game in Section 5. After each history in H_i^*, the sender chooses δ_i^* and after any other history he chooses $\delta_i = 0$, i.e.,

$$\tau_i(h) = \begin{cases} \delta_i^*, & h \in H_i^* \\ 0, & \text{otherwise} \end{cases} \tag{4.4}$$

If $h \in H_i^*$ and δ_i^* has been chosen in period i, then the receiver's total probability of playing the biased action is equal to $q_i^*(p_i(h))$, where q_i^* is defined in (4.3); otherwise, the biased receiver plays the biased action with probability one, i.e.,[21]

$$q_i(h, \delta_i) = \begin{cases} q_i^*(p_i(h)), & \text{if } h \in H_i^*, \ \delta_i = \delta_i^*, \ p_i(h) < 1 \\ p_i(h), & \text{otherwise} \end{cases} \tag{4.5}$$

The sender communicates truthfully in period i as long as the total probability of the biased action in that period is less than or equal to $\bar{q}$, the history belongs to H_i^*, and δ_i^* has been chosen. Since we assumed type $\theta_i = 0$ sender sends message $m_i = 0$, this implies that the probability with which type $\theta_i = 1$ sends message $m_i = 1$ is given by

$$\mu_i(h, \delta_i) = \begin{cases} 1, & \text{if } h \in H_i^*, \ \delta_i = \delta_i^*, \ q_i(h, \delta_i) \leq \bar{q}, \ p_i(h) < 1 \\ 0, & \text{otherwise} \end{cases} \tag{4.6}$$

which implies that

$$\mu_N(\delta_N) = \begin{cases} 1, & \text{if } \delta_N = \delta_N^*, \ q_N^*(p) \leq \bar{q} \\ 0, & \text{otherwise} \end{cases}$$

In any period i, the unbiased action is given by $a^0(\lambda_i(h, \delta_i, m))$, where

$$\lambda_i(h, \delta_i, m) = \begin{cases} \frac{1 - \mu_i(h, \delta_i)}{2 - \mu_i(h, \delta_i)}, & m = 0 \\ 1, & m = 1 \end{cases} \tag{4.7}$$

Beliefs on the receiver's type are defined as follows: $p_N(\emptyset) = p$ and

$$p_{i-1}(h, o_i) = 1 - \frac{1 - p_i(h)}{1 - q_i(h, \delta_i)}, \quad \text{for all } h \in H_i, \ o_i \in O_i \tag{4.8}$$

Note that if the players play according to σ^* up to and including period $i + 1$, which implies that $h_i \in H_i^*$, then $h_{i-1} \notin H_{i-1}^*$ if $\delta_i \neq \delta_i^*$ is chosen or if in period i the receiver plays an action that is different from the unbiased action, i.e., plays $a_i \neq m_i$. In that case, the sender assigns probability one to the event that the receiver is biased, provides no information, and terminates the game by choosing $\delta_{i-1} = 0$.

Theorem 1. *The assessment σ^* is a perfect Bayesian equilibrium and induces the unique sender-optimal equilibrium outcome. If $p > 1 - (1 - \bar{q})^{N-1}$, then σ^* is also a receiver-optimal equilibrium and therefore payoff-dominant.*

In the equilibrium σ^* described above, the sender leaves a proportion δ_i^* of the total stakes in the relationship to subsequent periods on the equilibrium path. This proportion leaves the receiver exactly indifferent between the biased and the unbiased actions in period i, given that in each subsequent period, the sender communicates truthfully after having observed the unbiased action in all prior periods and provides no information otherwise.

In order to further describe the equilibrium, let us first focus on the case in which σ^* is payoff-dominant, i.e., $p > 1 - (1 - \bar{q})^{N-1}$. In this case the receiver plays the biased action with total probability equal to $1 - (1 - p)/(1 - \bar{q})^{N-1}$ in period N and plays the biased action with total probability $\bar{q}$ thereafter. The sender reports the state truthfully in every

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[21] Note that in the last period $\delta_1 = \delta_1^* = 0$ by construction.
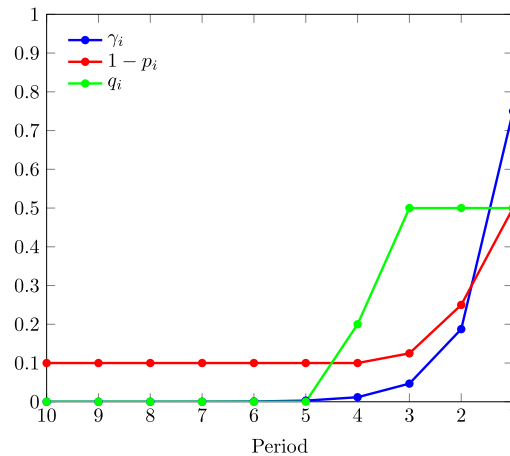
Fig. 1. Equilibrium behavior for $b = 1$, $p = 0.9$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

period except possibly the first period, i.e., period N in our notation. In period N, total probability of the biased action may exceed $\bar{q}$ and if this is the case the sender communicates no information. In other words, if the receiver has a sufficiently bad initial reputation, then informative communication may fail in the first period, i.e., period N, but communication is fully informative thereafter.

If $p \leq 1 - (1 - \bar{q})^{N-1}$, then the sender-optimal equilibrium in not receiver-optimal anymore. In the sender-optimal equilibrium σ^*, the receiver plays the unbiased action with probability one until the game reaches period k, where k is the first period (largest integer) such that $p > 1 - (1 - \bar{q})^{k-1}$. The receiver plays the biased action with total probability equal to $1 - (1 - p)/(1 - \bar{q})^{k-1} \leq \bar{q}$ in period k and plays the biased action with total probability $\bar{q}$ in all subsequent periods. The receiver's reputation remains constant and equal to $1 - p$ until the game reaches period k and then monotonically increases in each period to reach exactly $1 - \bar{q}$ in the last period of the game. The sender reports the state truthfully in every period after observing the unbiased action.

Fig. 1 plots the importance parameter (γ_i), reputation of the receiver ($1 - p_i$), and the total probability with which the receiver plays the biased action (q_i) for each period i, when the bias is equal to 1, the prior on b is 0.9, and total number of periods is 10. Note that $\bar{q} = 1/2$ and hence $k = 4$.

Remark 7. If $p \leq 1 - (1 - \bar{q})^{N-1}$, then σ^* is not receiver-optimal. Let $i^*(p)$ denote the largest i such that $p > 1 - (1 - \bar{q})^{i-1}$. Consider an equilibrium σ^{**} where in all periods N through $i^*(p) + 1$, the stakes are chosen to be zero, i.e., $\delta_i = 1$, the sender communicates truthfully, and the receiver plays the unbiased action with probability one. Starting in period $i^*(p)$, the equilibrium σ^{**} then follows σ^*. Lemma 6 in the Appendix proves that this equilibrium is receiver-optimal irrespective of the identity of the player who chooses the stakes. In this equilibrium, players act as if the game effectively begins in period $i^*(p)$. Notice that this equilibrium also entails starting small. The unbiased receiver is indifferent between this equilibrium and σ^* because the sender communicates truthfully in every period. On the other hand, the biased receiver's payoff in equilibrium σ^{**} strictly exceeds her payoff in equilibrium σ^*. This is because the stakes are chosen to be zero in exactly those periods in which the unbiased action is played with probability one in equilibrium σ^*. That is, the receiver maintains her reputation at zero cost in periods N through $i^*(p) + 1$.

Remark 8. In Remark 4, we have commented on how the results would change in the two-period model if we remove restrictions imposed by the Markov property, i.e., Property 2. For $N > 2$, we conjecture that any payoff-dominant non-Markov equilibrium involves similar behavior to that in the Markov equilibrium, except that truth-telling constraints are relaxed, i.e., in (4.6) $\bar{q}$ is replaced with a properly defined $\hat{q}_i(\delta_i, \delta_{i-1}, \ldots, \delta_2) > \bar{q}$ in a similar way as in the two-period model.

We should also note that, when the receiver chooses the stakes and faces a series of senders each of whom interacts with the receiver for only one period, then our main results go through in the general model without imposing this Markovian restriction. We will comment more on the receiver-game in Section 5.

Remark 9. We have assumed that the unbiased receiver is a commitment type who plays her myopic best response, i.e., the unbiased action given her beliefs, in each period, which is also the best period action for the sender. The fact that the unbiased receiver's preferences are perfectly aligned with that of the sender provides one justification for that assumption. Furthermore, the equilibrium we have specified remains an equilibrium even if the unbiased receiver is a fully strategic player. However, there may exist other Markov equilibria, such as a separating equilibrium where, in some period, the unbiased receiver plays an action different from the unbiased action while the biased receiver plays the biased action. Our assumption that the unbiased receiver is a commitment type who always plays the unbiased action rules out such equilibria.
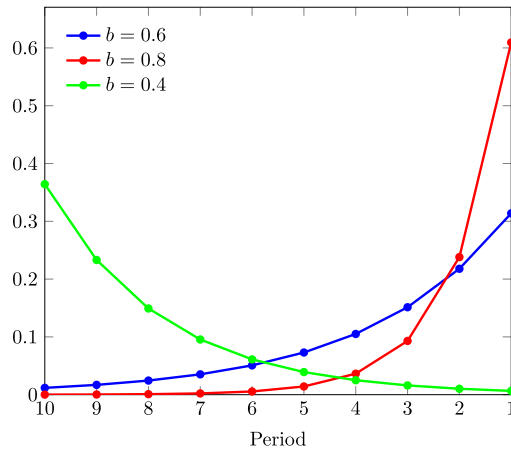
Fig. 2. Importance parameter for different biases.

4.1. Allocation of stakes

We are now ready to answer the main question that motivated our model: What is the equilibrium path of the stakes or the importance of the decisions?

Proposition 1. *Suppose that play unfolds according to equilibrium σ^* and denote by γ_i^* the proportion of the total stakes in period i in this equilibrium. If $b > 1/2$, then the equilibrium path is characterized by progressively larger stakes, i.e., $\gamma_N^* < \gamma_{N-1}^* < \cdots < \gamma_2^* < \gamma_1^*$, whereas if $b < 1/2$, then the size of the stakes decreases over time, i.e., $\gamma_N^* > \gamma_{N-1}^* > \cdots > \gamma_2^* > \gamma_1^*$. As b increases, the initial stakes become smaller but they grow faster, i.e., γ_j^*/γ_i^* is increasing in b for all $j < i$.*

Proof. See Section 8. □

Let $a = 4b^2$. The proof of Proposition 1 shows that the proportion of the stakes in the first period, i.e., period N, is given by

$$\gamma_N = \frac{1}{\sum_{j=0}^{N-1} a^j},$$

and then each subsequent proportion is just a times the previous one. If $a > 1$, i.e., $b > 1/2$, this implies that each period receives more weight than the previous one. More precisely, the growth rate of γ_i is equal to $a - 1 > 0$, i.e., the greater the potential bias of the receiver, the higher the growth rate of the stakes. If, on the other hand $a < 1$, i.e., $b < 1/2$, then the size of the stakes decreases over time.

In order to gain some intuition for this result let us consider the two-period model where the first period's proportion is γ_2 and the last period's γ_1. If the biased receiver plays the biased action in the first period, then her cost is $\gamma_1/4$. If she plays the unbiased action, then her cost is $\gamma_2 b^2$, assuming that after playing the unbiased action she receives full information as she does in our equilibrium. Therefore, the receiver is indifferent between the biased and the unbiased actions if and only if $\gamma_1/4 = \gamma_2 b^2$ or $\gamma_1 = a\gamma_2$.

Fig. 2 plots the evolution of the stakes over time for three different bias parameters, two of which are greater than 1/2 and one is smaller than 1/2. Observe that, when the potential bias is larger, the stakes are initially smaller but they grow faster. This is because the larger the bias the more important the future must become in order for the biased receiver to play the unbiased action. Since the total size of the stakes is normalized to one, this implies smaller stakes at the beginning and a higher growth rate.

Finally, we can show that the equilibrium costs of the players are decreasing in the number of periods N.

Proposition 2. *Cost of the sender strictly decreases in N in any sender-optimal equilibrium. Cost of (both types of) the receiver is decreasing in N in any receiver-optimal equilibrium.*

Proof. See Section 8. □

This result implies that, if the sender or the receiver had a choice over the number of periods over which to spread the total stakes available in their relationship, then they would choose as many periods as possible. Of course, this neglects any cost of time, which would act as a countervailing force.

5. The receiver-game

In the receiver-game, it is the receiver who chooses δ_i at the beginning of each period i rather than the sender. Therefore, τ_i is the receiver's strategy and specified as $\tau_i : H_i \times \{0, b\} \to [0, 1]$. The sender's belief in period i is a mapping $p_i : H_i \times [0, 1] \to [0, 1]$, and hence different δ_i may lead to different beliefs. Finally, Property 2 changes as follows: For any $i = N, \ldots, 1$ and $h, h' \in H_i$: (1) $p_i(h) = p_i(h')$ implies $\tau_i(h, \beta) = \tau_i(h', \beta)$ for any $\beta \in \{0, b\}$, $\mu_i(h, \delta_i, \theta_i) = \mu_i(h', \delta_i, \theta_i)$, $\alpha_i(h, \delta_i, m_i) = \alpha_i(h', \delta_i, m_i)$; (2) $\alpha_i(h, \delta_i, 0) = \alpha_i(h, \delta_i, 1)$.

Adapt the assessment σ^* defined in Section 4 by replacing the definition of τ_i given in (4.4) with

$$\tau_i(h, 0) = \tau_i(h, b) = \begin{cases} \delta_i^*, & h \in H_i^* \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

and adding

$$p_i(h, \delta_i) = \begin{cases} p_i(h), & \delta_i = \delta_i^* \\ 1, & \text{otherwise} \end{cases} \tag{5.2}$$

In other words, both types of the receivers choose δ_i^* as long as the history is cooperative and the sender's belief does not change after observing δ_i^*, while it assigns probability one to the biased receiver after any other δ_i. We prove in Section 8 that Theorem 1 as well as Propositions 1 and 2 are true verbatim for the receiver-game. In what follows, we provide some intuition for the two-period version of the model.

It is easy to show that σ^* is a perfect Bayesian equilibrium of the two-period receiver-game. Furthermore, Fact 1 is true verbatim and Fact 2 is true as long as the prior on the biased receiver is greater than or equal to p. We now argue that σ^* yields the payoff-dominant equilibrium outcome if $p > \bar{q}$. Let us consider the biased receiver first. The key observation in this case is that the posterior on the biased receiver, say p', must be greater than or equal to the prior p after some δ_2 chosen with positive probability by the biased receiver. (This follows from Bayes' rule.) But then the arguments we have made in the sender-game go through. Again, intuitively, $p' \geq p > \bar{q}$ implies that the biased receiver must play the biased action with positive probability in period 2, because otherwise her reputation in period 1 will be $p' > \bar{q}$ and she will receive no information from the sender, which makes playing the unbiased action suboptimal in period 2. Therefore, her total cost is $(1 - \delta_2)c_2 + \delta_2/4$, i.e., she accrues the highest information cost in period 1 and hence she would like to lower δ_2. But if δ_2 goes below δ_2^*, Fact 1 and $p' > \bar{q}$ imply that she receives no information in period 2. Therefore, the best for her is the minimum δ_2 that may possibly allow information revelation in period 2, i.e., δ_2^*.

Now consider the unbiased receiver. If $q_2^*(p) \leq \bar{q}$, then our equilibrium is optimal for the unbiased receiver because her cost under that equilibrium is zero. Suppose now that $q_2^*(p) > \bar{q}$. The key observation is that, as we argued above, the posterior after some δ_2 chosen with positive probability by the biased receiver must be greater than or equal to the prior and Fact 2 implies that total probability of the biased action $q_2 \geq q_2^*(p) > \bar{q}$ after such δ_2. Therefore, the information cost must be $1/4$ after δ_2. But then the information cost must be $1/4$ after any δ_2' chosen by the unbiased receiver since, otherwise, the biased receiver would rather choose δ_2', receive some information and then play the biased action. This would lead to a smaller cost for her. Therefore, the cost of the unbiased receiver must be at least $(1 - \delta_2')/4$. The rest of the arguments for why this must be greater than equal to her cost in our equilibrium is exactly like the one we provided in the sender-game.

Now let us consider the sender. First suppose that both types pool and choose the same δ_2. After observing δ_2, the posterior belief of the sender is the same as the prior and the arguments we have made for the sender-game are still valid. Second, suppose that the sender and receiver choose different δ_2s in period 2, i.e., let σ be a separating equilibrium. The sender's payoff after any δ_2 in the support of the biased receiver's strategy is equal to $b^2 + \frac{1}{4}$, because the sender believes that the receiver is biased with probability one after observing any such δ_2, communicates no information in either period, and the biased receiver plays the biased action in both periods. Therefore, the sender's cost under σ is at least $p(b^2 + \frac{1}{4})$, which strictly exceeds the sender's cost under σ^*.

Below, we sketch the argument for the more general case. Consider any equilibrium σ where the biased and unbiased receivers randomize over a range of δ_2s. Bayesian consistency (i.e., the martingale property of beliefs) implies that the expectation of the sender's posterior belief, formed after observing the receiver's choice of δ_2, is equal to her prior belief. More precisely, $\mathbb{E}[p(\delta_2)] = p$, where $p(\delta_2)$ is the sender's posterior after observing δ_2 and the expectation is taken over the choice of δ_2 using the probability distribution in equilibrium σ. Define the function $f(p') = C_2^S(\sigma^*|p')$ if $q_2^*(p') \leq \bar{q}$ and $f(p') = C_2^S(\sigma^*|p') - \frac{(1-\delta_2^*)}{4}$, otherwise. In words, $f(\cdot)$ is equal to the sender's cost under σ^*, excluding the information costs in period 2. The arguments we have made for the sender-game are still valid after each realization of δ_2 and therefore the sender's cost after observing any δ_2 in the support of σ exceeds the cost under σ^* when the initial belief is equal to $p(\delta_2)$. Therefore, $C_2^S(\sigma|p(\delta_2)) \geq C_2^S(\sigma^*|p(\delta_2)) \geq f(p(\delta_2))$. A straightforward computation shows that $f(\cdot)$ is a convex function. Therefore, $\mathbb{E}[C_2^S(\sigma|p(\delta_2))] \geq \mathbb{E}[f(p(\delta_2))] \geq f(\mathbb{E}[p(\delta_2)]) = f(p)$.

If $q_2^*(p) \leq \bar{q}$, then the information cost in period 2 is equal to zero and thus $\mathbb{E}[C_2^S(\sigma|p(\delta_2))] \geq f(p) = C_2^S(\sigma^*|p)$. On the other hand, if $q_2^*(p) > \bar{q}$, a key step in our argument shows that $C_2^S(\sigma|p(\delta_2)) \geq (1 - \delta_2^*)\frac{1}{4} + f(p(\delta_2))$ for each δ_2 in

the support of σ.[22] Therefore, using the convexity of the function f establishes the result through the following chain of inequalities

$$\mathbb{E}\left[C_2^S\left(\sigma|p(\delta_2)\right)\right] \geq \mathbb{E}\left[(1-\delta_2^*)\frac{1}{4} + f(p(\delta_2))\right] \geq (1-\delta_2^*)\frac{1}{4} + f(p) = C_2^S\left(\sigma^*|p\right).$$

Intuitively, information revelation through the choice of stakes only hurts the sender because it makes communication more difficult and the sender prefers to communicate even with the biased receiver.

6. Related literature

The main issue addressed in this paper, i.e., dynamic allocation stakes, has been analyzed in prisoners' dilemma type environments before. Watson (1999) and Watson (2002) analyze an infinitely repeated prisoners' dilemma type game with incomplete information and variable stakes over time. In the stage game, the "low" type player prefers to "betray", which benefits herself, injures the other player, and ends the game, while the "high" type prefers cooperation as long as the other player also cooperates. He shows that starting with small stakes supports perpetual cooperation between the high types as an equilibrium outcome. Watson (2002) assumes that players commit to the way stakes change over time while Watson (1999) characterizes an equilibrium in which the stakes satisfy a renegotiation condition. Andreoni and Samuelson (2006) study a twice repeated prisoners' dilemma game with incomplete information and variable stakes. Players are conditional cooperators in the sense that, in the stage game, a type-α player prefers to cooperate if she believes that the other player cooperates with at least α probability.[23] They characterize the equilibria of this game with exogenously given stakes and show that "starting small" leads to the best payoffs for the players.[24] The main point of departure of our model from these papers is that, in contrast to a prisoners' dilemma game, our stage game is a game of strategic communication that exhibits common interest as well as conflict of interest. Furthermore, we assume that one of the players has the authority to determine the stakes involved in their relationship and analyze how they are determined in equilibrium.[25]

Our paper is also related to Aghion et al. (2004), which shows that, if control rights are not contractible, then transferring them unconditionally to the agent and learning her willingness to cooperate could be the optimal arrangement for the principal. In their model, the principal would like to learn the agent's type and the "bad" agent has no incentive for reputation, while in our case the "bad" receiver has an incentive to maintain a "good" reputation and the sender does not prefer to screen the agent types because doing so hinders communication.[26]

In each period of our model, the sender and the receiver are involved in a cheap-talk game, which has been introduced by Crawford and Sobel (1982). They analyze the equilibrium communication behavior between an informed but biased sender and an uninformed receiver and show that the informativeness of equilibrium decreases in the degree of the sender's bias. There are two main differences between Crawford and Sobel (1982) and our model: (1) The degree of preference divergence between the sender and receiver is the private information of the receiver; (2) The game is repeated, where in each period a new state of the world is realized but preferences remain the same.

Morris (2001) also differs from Crawford and Sobel along those two dimensions. The main difference is that in Morris (2001) the bias is the private information of the sender whereas in our model it is the private information of the receiver. Morris (2001) finds that the unbiased sender, who prefers to inform the receiver about the state of the world, may choose not to do so in the first period in order to be regarded as unbiased and hence better inform the receiver in the future. In contrast, in our model, the biased receiver may mimic the unbiased receiver in order to maintain a good reputation and receive better information in the future. Furthermore, we analyze the equilibrium allocation of the stakes over time whereas it is exogenously given in Morris (2001).

Morgan and Stocken (2003) analyzes a one period cheap-talk game with a sender with uncertain preferences, whereas Sobel (1985) and Benabou and Laroque (1992) are earlier papers that analyze repeated cheap-talk games, except that they assume that the unbiased (or good) sender always tells the truth. Li and Madarász (2008) extend Morgan and Stocken (2003) so that the bias can be in either direction and compare equilibria under known and unknown biases, while Dimitrakas and Sarafidis (2005) allow the bias to have an arbitrary distribution. Our model differs from these papers in that we assume the bias is receiver's private information and that the cheap-talk game is repeated.

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[22] This is because the information cost in period 2 must be the same for any δ_2 in the support σ since the biased agent must be indifferent between these choices of stakes. Moreover, there is a δ_2 in the support of σ where the information cost in period 2 is equal to 1/4. This is because there must be a δ_2 in the support of σ such that $p(\delta_2) \geq p$ and therefore $q_2(p(\delta_2)) \geq q_2^*(p) > \bar{q}$.

[23] They allow α to be negative or greater than one, which corresponds to unconditional cooperators or defectors, respectively.

[24] Andreoni and Samuelson (2006) also test their theory experimentally and find empirical support for their predictions. Andreoni et al. (2016) extend this paper so that players choose the stakes themselves in the experiment. They show that the subjects indeed choose the payoff maximizing strategy of starting small. Other papers that feature gradualism as an optimal or equilibrium outcome include Marx and Matthews (2000), Blonski and Probst (2004), and the loan model in Section 6 of Sobel (1985).

[25] We also show that gradualism is not always the best equilibrium arrangement for the sender. Indeed, if the potential conflict of interest between the receiver and the sender is small enough, then the opposite arrangement of "starting big" turns out to be optimal for the sender.

[26] Also related is Halonen (1997), which shows that a joint ownership structure and a concern for reputation may help solve the hold-up problem and implement the first best in a (twice) repeated game with incomplete information.

There is also a recent literature on repeated cheap-talk games, which mostly assumes infinite horizon and explores the set of equilibria (in the limit) as the players' discount rates converge to one. (See Renault et al. (2013), Best and Quigley (2017), Margaria and Smolin (2018).)[27] Our model is outside of this framework because we assume that the game is finitely repeated and that the size of the stakes is determined in equilibrium. Furthermore, this literature assumes that only the sender has private information, while in our model the receiver has private information as well, which is the main factor that sustains truthful communication.

Another related paper is Ottaviani and Sørensen (2001) in which a sequence of privately informed experts, who are exclusively concerned about their reputation for being well-informed, offer public advice to an uninformed receiver. They show that reputational concerns may lead to herding by the experts.[28] Our model can also be framed as a model of sequential cheap-talk with multiple experts (senders) but we have a receiver who is privately informed about the preference divergence between herself and the experts, and it is the receiver who is concerned about reputation.[29]

Our work is also related to the literature on pandering. Maskin and Tirole (2004) analyze a two-period model where in the first period an official chooses a policy, which determines whether she stays in office in the second period. They show that if the official's desire to stay in office is sufficiently strong, then in the first period she could choose a popular action, i.e., she could pander to public opinion even if she does not think that the public opinion is the optimal policy. In our model, incentives to pander come from the desire to receive better information rather than the desire to stay in office.[30]

Another related strand of literature is the one on career concerns pioneered by Holmström (1999), in which an employee's concern about her reputation for talent leads her to exert costly effort even without explicit incentives provided by a contract.[31] In our model, concern for reputation for being unbiased arises from the receiver's incentives to obtain accurate information and leads her to act in the interest of the sender.

7. Concluding remarks

We have analyzed a model of repeated cheap-talk in which the conflict of interest is the private information of the receiver and either the sender or the receiver can determine how the stakes involved in their relationship evolve over time. We find that, if the potential conflict of interest is large, then the stakes increase over time, i.e., "starting small" or "gradualism" is the unique (payoff-dominant) equilibrium arrangement.

Basically, the stakes are designed in order to utilize the receiver's incentives to build reputation for being unbiased and facilitate communication. In equilibrium, the receiver mixes between her own and the sender's favorite action and the sender communicates truthfully with the receiver throughout their relationship as long as the receiver always does the "right thing".

We also showed that if the potential bias is small, i.e., smaller than $1/2$, then this pattern is reversed and the size of the stakes decreases over time. Note that if $b < 1/2$, or more generally the initial reputation of the receiver is good enough to make truthful communication possible even in the one-shot game, i.e., $pb < 1/2$, the sender-optimal equilibrium is not receiver-optimal. In fact, in this case, the sender-optimal equilibrium is sustained by the sender's off-the-equilibrium threat to communicate no information if the receiver plays the biased action. This threat is perfectly credible when $b > 1/2$, because the only equilibrium behavior once the receiver is revealed to be biased is to reveal no information. The same is not true when $b < 1/2$. Since the sender prefers truthful communication ex-ante, such a threat may be regarded as non-credible. If that is the position one takes, then our results should be deemed most convincing and interesting for those cases in which the potential bias of the receiver is large enough.

The current work raises many other questions and could be extended in a number of ways. For example, what would be the equilibrium path of stakes in a situation where reputational concerns create perverse incentives as in Morris (2001), Ely and Välimäki (2003), Maskin and Tirole (2004), or Kartik and Van Weelden (2015)? As opposed to what happens in our model, would it be optimal to front-load the decisions in order to avoid such perverse incentives? More technical extensions include richer type spaces for the players, but our preliminary analyses of such models have so far proved non-trivial.

8. Proofs

Proof of Theorem 1. We start with the proof of Lemma 2.

Proof of Lemma 2. Let $\lambda_m = \lambda_i(h, \delta_i, m)$, $q = q(h, \delta_i)$, and note that the period cost of sending message m for type $\theta \in \{0, 1\}$ is

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[27] In Golosov et al. (2014), the game is finitely repeated but, unlike in ours, the state is fixed throughout. Also related is Hörner et al. (2015), which provides a more general analysis of dynamic Bayesian games.

[28] Also see Ottaviani and Sørensen (2006a,b) in which an expert with reputational concerns (but no bias) fails to provide full information to the receiver.

[29] There are other models in which multiple experts with known biases are involved in simultaneous or sequential cheap-talk, among which are Gilligan and Krehbiel (1989), Austen-Smith (1990), and Krishna and Morgan (2001).

[30] Brandenburger and Polak (1996), Vidal and Möller (2007), Acemoglu et al. (2013), Che et al. (2013), and Morelli and Van Weelden (2013) are some of the other papers in the pandering literature.

[31] Holmstrom's model was originally developed in a paper published in 1982 in an edited book. See also Holmström and Costa (1986).

$$(\lambda_m - \theta)^2 + 2(\lambda_m - \theta)qb + qb^2.$$

Property 2 implies that the continuation payoff does not depend on the message and hence only the period payoff matters for sequential rationality of the sender. As it is always the case in cheap-talk models, there is always an equilibrium in which the sender's strategy is completely uninformative irrespective of his beliefs, the so called "babbling equilibrium". Suppose that in equilibrium the sender provides full information to the receiver. Sequential rationality of type $\theta_i = 0$ is always satisfied, whereas sequential rationality of type $\theta_i = 1$ implies that $qb^2 \leq 1 - 2qb + qb^2$ or $qb \leq 1/2$. If type $\theta_i = 0$ plays a completely mixed strategy, then $\lambda_1^2 + 2\lambda_1 qb = \lambda_0^2 + 2\lambda_0 qb$, which implies $\lambda_0 = \lambda_1$. This implies that both types mix with equal probabilities and hence the sender's strategy is non-informative. Therefore, in any other type of equilibrium behavior, type 0 must be playing a pure strategy while type 1 completely mixes. Suppose, without loss of generality, that type 0 sends message 0. This implies that $\lambda_1 = 1$ and $\lambda_0 \in (0, 1/2)$. It is easy to show that type 0's sequential rationality is satisfied while type 1's sequential rationality implies that $\lambda_0 = 1 - 2qb$, which, in turn, implies that $1/4 < qb < 1/2$. $\quad\square$

Lemma 3. *Suppose that the sender chooses the allocation δ, then the assessment σ^* is a perfect Bayesian equilibrium.*

Proof. Fix a history $h \in H_1^*$ and note that under σ^* the biased receiver plays the biased action with probability one and the sender provides information if and only if $p_1(h) \leq \bar{q}$. The receiver's strategy is sequentially rational since period 1 is the last period and the sender's strategy is sequentially rational by Lemma 2.

Let $i > 1$ and fix a history $h_i \in H_i^*$ such that $p_i(h_i) < 1$. If $\delta_i = \delta_i^*$, then under σ^* the receiver is indifferent between the biased and unbiased actions after any message m_i. In order to see this, note that it is true for $i = 2$, as we previously showed in section 3.2. Suppose that it is true in period $i - 1$. If, in period i, the receiver chooses the biased action, then she induces a history that is not in H_{i-1}^*, which implies that in period $i - 1$ the sender chooses $\delta_{i-1} = 0$, provides no information, and the receiver plays the biased action. Therefore, the cost of playing the biased action is $\delta_i^*/4$. If she plays the unbiased action instead, then she suffers a cost equal to b^2 in period i but induces a history in H_{i-1}^*. In the next period, the sender choses δ_{i-1}^* and provides full information. Under the induction hypothesis, her expected cost starting from period $i - 1$ is equal to $\delta_{i-1}^*/4$, i.e., the cost of playing the biased action in period $i - 1$. Therefore, the cost of playing the unbiased action in period i is equal to $(1 - \delta_i^*)b^2 + \delta_i^*\delta_{i-1}^*/4$. Definition of δ_i^* (see (4.2)) implies that

$$\delta_i^* \frac{1}{4} = (1 - \delta_i^*)b^2 + \delta_i^* \delta_{i-1}^* \frac{1}{4}$$

which, in turn, implies that she is indifferent between the biased and unbiased actions in period i. Therefore, playing the biased action with total probability $q_i^*(p_i(h))$ is optimal after such histories.

Lemma 2 implies that the communication strategy of the sender (see (4.6)) is sequentially rational. If the sender chooses $\delta_i \neq \delta_i^*$, then in period i he provides no information and the biased receiver plays the biased action. In period $i - 1$, he chooses $\delta_{i-1} = 0$, provides no information, and the biased receiver again plays the biased action. Therefore, his expected cost of choosing $\delta_i \neq \delta_i^*$ is equal to

$$p_i(h) b^2 + \frac{1}{4}.$$

If he chooses δ_i^*, then the receiver plays the biased action with total probability $q_i^*(p_i(h)) \leq p_i(h)$, and this cannot lead to a higher expected cost. Therefore, it is sequentially rational for the sender to choose δ_i^*.

If $h_i \notin H_i^*$ or $p_i(h) = 1$, then the biased receiver plays the biased action with probability one. This is sequentially rational because the sender provides no information in any subsequent period. The sender is willing to provide no information because babbling is always an equilibrium of the cheap-talk game. Moreover, it is sequentially rational for the sender to choose $\delta_i = 0$ because his continuation payoff is equal to $p_i(h_i)b^2 + \frac{1}{4}$ and independent of his choice of δ_i.

Finally, it is straightforward to check that the beliefs defined in (4.7) and (4.8) satisfy the Bayes' rule whenever it can be applied conditional on reaching any $h \in H_i$. $\quad\square$

Lemma 4. *Suppose that the receiver chooses the allocation δ, then the assessment σ^* is a perfect Bayesian equilibrium.*

Proof. Fix a history $h \in H_1^*$ and note that under σ^* the biased receiver plays the biased action with probability one and the sender provides information if and only if $p_1(h) \leq \bar{q}$. The biased receiver's strategy is sequentially rational since period 1 is the last period and the sender's strategy is sequentially rational by Lemma 2.

Let $i > 1$ and fix a history $h_i \in H_i^*$ such that $p_i(h_i) < 1$. If $\delta_i = \delta_i^*$, then under σ^* the biased receiver is indifferent between the biased and unbiased actions after any message m_i. In order to see this, note that it is true for $i = 2$, as we previously showed in section 3.2. Suppose that it is true in period $i - 1$. If, in period i, the receiver chooses the biased action, then she induces a history that is not in H_{i-1}^*, which implies that in period $i - 1$ the receiver chooses $\delta_{i-1} = 0$, the sender provides no information, and the biased receiver plays the biased action. Therefore, the cost of playing the biased action is $\delta_i^*/4$. If she plays the unbiased action instead, then she suffers a cost equal to b^2 in period i but induces a history in

H_{i-1}^*. In the next period, the receiver choses δ_{i-1}^* and the sender provides full information. Under the induction hypothesis, her expected cost starting from period $i-1$ is equal to $\delta_{i-1}^*/4$, i.e., the cost of playing the biased action in period $i-1$. Therefore, the cost of playing the unbiased action in period i is equal to $\left(1-\delta_i^*\right)b^2 + \delta_i^*\delta_{i-1}^*/4$. Definition of δ_i^* (see (4.2)) implies that

$$\delta_i^* \frac{1}{4} = \left(1-\delta_i^*\right)b^2 + \delta_i^*\delta_{i-1}^* \frac{1}{4}$$

which, in turn, implies that she is indifferent between the biased and unbiased actions in period i. Therefore, playing the biased action with total probability $q_i^*(p_i(h))$ is sequentially rational after such histories. Moreover, Lemma 2 implies that the communication strategy of the sender (see (4.6)) is sequentially rational because $q_i^*(p_i(h)) \le \bar{q}$.

If the receiver chooses $\delta_i \ne \delta_i^*$, then in period i the sender provides no information and the biased receiver plays the biased action. In period $i-1$, the receiver chooses $\delta_{i-1} = 0$, the sender provides no information, and the biased receiver again plays the biased action. Therefore, the expected cost of choosing $\delta_i \ne \delta_i^*$ is equal to $1/4$ for both the biased and unbiased receivers. The expected cost of choosing δ_i^* is equal to zero for the unbiased receiver which is clearly lower than the cost $1/4$ which results from choosing δ_i. The expected cost of choosing δ_i^* is equal to $\delta_i^* \frac{1}{4}$ for the biased receiver because this receiver is indifferent between the biased and unbiased actions following δ_i^*. However, this cost is clearly lower than $1/4$ which results from choosing $\delta_i \ne \delta_i^*$.

If $h_i \notin H_i^*$ or $p_i(h) = 1$, then the biased receiver plays the biased action with probability one. This is sequentially rational because the sender provides no information in any subsequent period. The sender is willing to provide no information because babbling is always an equilibrium of the cheap-talk game. Moreover, it is sequentially rational for the receiver to choose $\delta_i = 0$ because his continuation payoff is equal to $1/4$ and independent of her choice of δ_i.

Finally, it is straightforward to check that the beliefs defined in (4.7) and (4.8) satisfy the Bayes' rule whenever it can be applied conditional on reaching any $h \in H_i$. $\quad\square$

Lemma 5. *If $p > 1 - (1-\bar{q})^{N-1}$, then the assessment σ^* is receiver optimal.*

Proof. We first argue that σ^* is receiver optimal for the biased receiver.

Claim 1. *Suppose that the sender chooses the parameter δ. We argue that the assessment σ^* is biased receiver optimal.*

Proof. Assume that the assessment σ^* is receiver optimal for all reputation levels $p > 1 - (1-\bar{q})^{i-1}$ in the i stage communication game. Under this induction hypothesis, we show that σ^* is receiver optimal in the $i+1$ stage communication game for all reputation levels $p > 1 - (1-\bar{q})^i$, i.e., the cost under assessment σ^* is smaller than the cost under any other assessment σ given the induction hypothesis. Fix an assessment $\sigma \ne \sigma^*$ and assume, on the way to a contradiction, that $C_{i+1}^R(\sigma) < C_{i+1}^R(\sigma^*)$.

Step 1. Suppose that the receiver plays the biased action with total probability $q_{i+1} > \bar{q}$ in period $i+1$. This implies that the receiver's cost is $1/4$ since the sender cannot communicate in period $i+1$ given that $q_{i+1} > \bar{q}$. However, the receiver's cost under σ^* is at most $1/4$.

Step 2. On the other hand if $q_{i+1} \le \bar{q}$, then $p > 1 - (1-\bar{q})^i$ and Bayes' rule together imply that the receiver's reputation in period i, p_i, is strictly greater than $1 - (1-\bar{q})^{i-1}$. There are two cases to consider:

Case 1. $q_{i+1} < q_{i+1}^*$. If $q_{i+1} < q_{i+1}^*$, then $q_i^*(p_i) > \bar{q}$ by construction and therefore $C^R(i|\sigma^*) = \frac{1}{4}$. Note that if $\delta_{i+1} = 1$, then, by the induction hypothesis, $C_{i+1}^R(\sigma) = C_i^R(\sigma|h_i) \ge C_i^R(\sigma^*) = \frac{1}{4}$ where h_i is the history under which the unbiased action is played in period i. However, this would contradict our initial hypothesis that $\frac{1}{4} = C_{i+1}^R(\sigma) < C_{i+1}^R(\sigma^*)$ because $C_{i+1}^R(\sigma^*) \le \frac{1}{4}$. Therefore, $\delta_{i+1} < 1$.

In order for the biased receiver to play the unbiased action in period i we must have, $\frac{1}{4}\delta_{i+1} \ge (1-\delta_{i+1})b^2 + \delta_{i+1}C_i^R(\sigma|h_i)$. However, since $C_i^R(\sigma^*) \le C_i^R(\sigma|h_i)$ by the induction hypothesis we have

$$\frac{1}{4}\delta_{i+1} \ge (1-\delta_{i+1})b^2 + \delta_{i+1}C_i^R(\sigma|h_i)$$
$$\ge (1-\delta_{i+1})b^2 + \delta_{i+1}C_i^R(\sigma^*)$$
$$= (1-\delta_{i+1})b^2 + \frac{1}{4}\delta_{i+1}$$

leading to a contradiction.

Case 2. $q_{i+1} \ge q_{i+1}^*$. The facts that $q_{i+1} \ge q_{i+1}^*$ and $q_{i+1} \le \bar{q}$ together imply that $q_{i+1}^* \le \bar{q}$. If $q_{i+1}^* \le \bar{q}$, then the cost under σ^* is equal to $\delta_{i+1}^*/4$. The cost under σ is at least $\delta_{i+1}/4$ because the receiver must play the biased action with positive probability in period $i+1$. The fact that the receiver is indifferent between the biased and the unbiased action implies the following equalities:

$$\frac{1}{4}\delta_{i+1} = (1 - \delta_{i+1})b^2 + \delta_{i+1}C_i^R(\sigma)$$

$$\frac{1}{4}\delta_{i+1}^* = (1 - \delta_{i+1}^*)b^2 + \delta_{i+1}^* C_i^R(\sigma^*)$$

where $C_i^R(\sigma)$ is the biased receiver's cost in the continuation game under the strategy σ. The induction hypothesis and $p_i > 1 - (1 - \bar{q})^{i-1}$ together imply that $C_i^R(\sigma^*) \leq C_i^R(\sigma)$. The fact that $C_i^R(\sigma^*) \leq C_i^R(\sigma)$ implies that

$$\delta_{i+1}\frac{1}{4} \geq (1 - \delta_{i+1})b^2 + \delta_{i+1}C_i^R(\sigma^*).$$

Therefore

$$(\delta_{i+1} - \delta_{i+1}^*)(\frac{1}{4} + b^2 - C_i^R(\sigma^*)) \geq 0$$

However, because $C_i^S(\sigma^*) < \frac{1}{4} + b^2$ we have $\delta_{i+1} \geq \delta_{i+1}^*$ and hence $\delta_{i+1}^*/4 \leq \delta_{i+1}/4$ showing that the cost under σ exceeds the cost under σ^*.

We now complete the inductive argument by showing that σ^* is optimal for the biased receiver for $i = 2$ if $p > 1 - (1 - \bar{q})^{2-1} = \bar{q}$. This conclusion follows immediately from the argument above because for any two strategy profiles σ and σ^* we have $C_1^R(\sigma) = C_1^R(\sigma^*) = 1/4$ if the reputation $p_1 > \bar{q}$ in period $i = 1$ and $C_1^R(\sigma) \geq C_1^R(\sigma^*) = 0$, otherwise.   □

Claim 2. *Suppose that the receiver chooses the parameter δ. We argue that the assessment σ^* is biased receiver optimal.*

Proof. Assume that the assessment σ^* is receiver optimal for all reputation levels $p > 1 - (1 - \bar{q})^{i-1}$ in the i stage communication game. Under this induction hypothesis, we show that σ^* is receiver optimal in the $i + 1$ stage communication game for all reputation levels $p > 1 - (1 - \bar{q})^i$. Fix an assessment $\sigma \neq \sigma^*$ and assume, on the way to a contradiction, that $C_{i+1}^R(\sigma) < C_{i+1}^R(\sigma^*)$.

Let δ_{i+1} be a δ in the support of σ such that $p_{i+1} \geq p$. Bayes' rule implies that there must be such a δ_{i+1} in the support of σ. Let q_{i+1} denote the receiver's strategy after the history where δ_{i+1} has been chosen.

Suppose that the receiver plays the biased action with total probability $q_{i+1} > \bar{q}$ in period $i + 1$. This implies that the receiver's cost is $1/4$ since the sender cannot communicate in period $i + 1$ given that $q_{i+1} > \bar{q}$. However, the receiver's cost under σ^* is at most $1/4$.

On the other hand if $q_{i+1} \leq \bar{q}$, then $p_{i+1} > 1 - (1 - \bar{q})^i$ and Bayes' rule together imply that the receiver's reputation in period i, p_i, is strictly greater than $1 - (1 - \bar{q})^{i-1}$. There are two cases to consider:

Case 1. $q_{i+1} < q_{i+1}^*(p_{i+1})$ where $q_{i+1}^*(p_{i+1})$ is the probability of playing the biased action under σ^* if the initial reputation is equal to p_{i+1} and given that δ_{i+1}^* is chosen by the receiver. If $q_{i+1} < q_{i+1}^*(p_{i+1})$, then $q_i^*(p_i) > \bar{q}$ by construction and therefore $C_i^R(\sigma^*) = \frac{1}{4}$. Note that if $\delta_{i+1} = 1$, then, by the induction hypothesis, $C_{i+1}^R(\sigma) = C_i^R(\sigma|h_i) \geq C_i^R(\sigma^*) = \frac{1}{4}$ where h_i is the history under which the unbiased action is played in period i. However, this would contradict our initial hypothesis that $\frac{1}{4} = C_{i+1}^R(\sigma) < C_{i+1}^R(\sigma^*)$ because $C_{i+1}^R(\sigma^*) \leq \frac{1}{4}$. Therefore, $\delta_{i+1} < 1$.

In order for the biased receiver to play the unbiased action in period i we must have, $\frac{1}{4}\delta_{i+1} \geq (1 - \delta_{i+1})b^2 + \delta_{i+1}C_i^R(\sigma|h_i)$. However, since $C_i^R(\sigma^*) \leq C_i^R(\sigma|h_i)$ by the induction hypothesis we have

$$\frac{1}{4}\delta_{i+1} \geq (1 - \delta_{i+1})b^2 + \delta_{i+1}C_i^R(\sigma|h_i)$$

$$\geq (1 - \delta_{i+1})b^2 + \delta_{i+1}C_i^R(\sigma^*)$$

$$= (1 - \delta_{i+1})b^2 + \frac{1}{4}\delta_{i+1}$$

leading to a contradiction.

Case 2. $q_{i+1} \geq q_{i+1}^*(p_{i+1})$ where $q_{i+1}^*(p_{i+1})$ is the probability of playing the biased action under σ^* if the initial reputation is equal to p_{i+1} and given that δ_{i+1}^* is chosen by the receiver. Note that $q_{i+1}^*(p_{i+1}) \geq q_{i+1}^*(p)$ because $p_{i+1} \geq p$. The facts that $q_{i+1} \geq q_{i+1}^*(p_{i+1}) \geq q_{i+1}^*(p) > 0$ and $q_{i+1} \leq \bar{q}$ together imply that $q_{i+1}^*(p) \leq \bar{q}$. If $q_{i+1}^*(p) \leq \bar{q}$, then the cost under σ^* is equal to $\delta_{i+1}^*/4$. The cost under σ is $\delta_{i+1}/4$ because the receiver plays the biased action with positive probability in period $i + 1$. The fact that the receiver is indifferent between the biased and the unbiased action implies the following equalities:

$$\frac{1}{4}\delta_{i+1} = (1 - \delta_{i+1})b^2 + \delta_{i+1}C_i^R(\sigma)$$

$$\frac{1}{4}\delta_{i+1}^* = (1 - \delta_{i+1}^*)b^2 + \delta_{i+1}^* C_i^R(\sigma^*)$$

where $C_i^R(\sigma)$ is the biased receiver's cost in the continuation game under the strategy σ. The induction hypothesis and $p_i > 1 - (1 - \bar{q})^{i-1}$ together imply that $C_i^R(\sigma^*) \leq C_i^R(\sigma)$. The fact that $C_i^R(\sigma^*) \leq C_i^R(\sigma)$ implies that

$$\delta_{i+1}\frac{1}{4} \geq (1 - \delta_{i+1})b^2 + \delta_{i+1}C_i^R(\sigma^*).$$

Therefore

$$(\delta_{i+1} - \delta_{i+1}^*)(\frac{1}{4} + b^2 - C_i^R(\sigma^*)) \geq 0$$

However, because $C_i^S(\sigma^*) < \frac{1}{4} + b^2$ we have $\delta_{i+1} \geq \delta_{i+1}^*$ and hence $\delta_{i+1}^*/4 \leq \delta_{i+1}/4$ showing that the cost under σ exceeds the cost under σ^*. $\square$

Claim 3. *Suppose that the sender chooses the parameter δ. We argue that σ^* is receiver optimal for the unbiased receiver.*

Proof. If $q_N^* \leq \bar{q}$, then σ^* is optimal for the unbiased receiver because her cost under σ^* is equal to zero since the sender communicates truthfully in each period.

If, on the other hand, $q_N^* > \bar{q}$, then her cost under σ^* is equal to $(1 - \delta_N^*)/4$ because the sender communicates truthfully in each period except period N. Choose any other assessment σ.

Case 1. Suppose that the receiver plays the biased action with total probability $q_N \leq \bar{q}$. However, then $q_N^* > \bar{q}$, and $q_N \leq \bar{q}$ together imply that $q_j > \bar{q}$ for some $j < N$. However, then the sender will communicate no information in period j. This implies that, the biased receiver will play the biased action in period $j + 1$ with probability one. However, this further implies that the sender will communicate no information in period $j + 1$ either. Working backwards in this way, we find that biased receiver will play the biased action in each period $i > j$ contradicting our initial hypothesis that $q_N \leq \bar{q}$.

Case 2. Suppose that the receiver plays the biased action with total probability $p > q_N > \bar{q}$. In this case, the sender communicates no information in period N. Therefore, the unbiased receiver's cost under σ is at least $(1 - \delta_N)/4$. If $\delta_N \leq \delta_N^*$, then $(1 - \delta_N)/4 \leq (1 - \delta_N^*)/4$, that is, the unbiased receivers cost under σ^* is at most the cost under σ.

Suppose instead that $\delta_N > \delta_N^*$. Let $\hat{i}$ denote the largest $i < N$ such that $\delta_i \leq \delta_i^*$. Note that $\hat{i}$ is possibly equal to one where $\delta_1 = \delta_1^* = 0$. Also, note that we have $\delta_i > \delta_i^*$ for $i > \hat{i}$ and $\delta_{\hat{i}} \leq \delta_{\hat{i}}^*$ by construction.

The biased receiver is indifferent between the biased action and the unbiased action in period N. Therefore, playing the biased action in period N is a best response for the receiver given the sender's strategy under σ and must result in a cost that is less than or equal to any other strategy that the receiver may use. In particular, the cost must be less than or equal to the cost of playing the unbiased action in each period $i \in \{N, \hat{i} + 1\}$ and then playing the biased action in period $\hat{i}$. Therefore,

$$\delta_N\frac{1}{4} \leq (1 - \delta_N)b^2 + \sum_{i=\hat{i}+1}^{N-1} \gamma_i \left(c_i + b^2\right) + \gamma_{\hat{i}}c_i + \frac{1}{4}\delta_{\hat{i}}\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i\right)$$

$$\leq b^2 \sum_{i=\hat{i}+1}^{N} \gamma_i + \sum_{i=1}^{N-1} \gamma_i c_i + \frac{1}{4}\delta_{\hat{i}}\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i\right)$$

where the second inequality follows because $c_i \geq 0$ for all i.

The fact that the biased receiver is indifferent between the biased and unbiased action in each period $i > 1$ under strategy σ^* implies that

$$\delta_N^*\frac{1}{4} = b^2 \sum_{i=\hat{i}+1}^{N} \gamma_i^* + \frac{1}{4}\delta_{\hat{i}}^*\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i^*\right)$$

Moreover,

$$(1 - \delta_N)\frac{1}{4} + \sum_{i=1}^{N-1} \gamma_i c_i < \left(1 - \delta_N^*\right)\frac{1}{4}$$

$$\delta_N^*\frac{1}{4} + \sum_{i=1}^{N-1} \gamma_i c_i < \delta_N\frac{1}{4}$$

by assumption. Therefore, $\delta_N^*\frac{1}{4} + \sum_{i=1}^{N-1} \gamma_i c_i < \delta_4\frac{1}{4}$ and the equality displayed above together imply that

$$\delta_N \frac{1}{4} > b^2 \left(\sum_{i=\hat{i}+1}^{N} \gamma_i^* \right) + \frac{1}{4}\delta_{\hat{i}}^* \left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i^* \right) + \sum_{i=1}^{N-1} \gamma_i c_i.$$

The facts that $\delta_i > \delta_i^*$ for $i > \hat{i}$ and $\delta_{\hat{i}} \leq \delta_{\hat{i}}^*$ together imply that

$$b^2 \left(\sum_{i=\hat{i}+1}^{N} \gamma_i^* \right) + \frac{1}{4}\delta_{\hat{i}}^* \left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i^* \right) \geq b^2 \left(\sum_{i=\hat{i}+1}^{N} \gamma_i^* \right) + \frac{1}{4}\delta_{\hat{i}} \left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i^* \right)$$

$$\geq b^2 \sum_{i=\hat{i}+1}^{N} \gamma_i + \frac{1}{4}\delta_{\hat{i}} \left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i \right)$$

The first inequality above follows because $\delta_{\hat{i}} \leq \delta_{\hat{i}}^*$. The second inequality above follows because $\delta_i > \delta_i^*$ for each $i > \hat{i}$ implies that $\sum_{i=\hat{i}+1}^{N} \gamma_i^* > \sum_{i=\hat{i}+1}^{N} \gamma_i$ and because $\frac{1}{4}\delta_{\hat{i}} \leq b^2$ (since $b^2 \geq \frac{1}{4}$ and $\delta_{\hat{i}} \leq 1$). However, the above inequality implies that

$$\sum_{i=\hat{i}+1}^{N} \gamma_i b^2 + \sum_{i=1}^{N-1} \gamma_i c_i + \frac{1}{4}\delta_{\hat{i}} \left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i \right) \geq \delta_4 \frac{1}{4} > \sum_{i=\hat{i}+1}^{N} \gamma_i b^2 + \sum_{i=1}^{N-1} \gamma_i c_i + \frac{1}{4}\delta_{\hat{i}} \left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i \right)$$

which leads to a contradiction. $\quad\square$

Claim 4. *Suppose that the receiver chooses the parameter δ. We argue that σ^* is receiver optimal for the unbiased receiver.*

Proof. If $q_N^* \leq \bar{q}$, then σ^* is optimal for the unbiased receiver because her cost under σ^* is equal to zero since the sender communicates truthfully in each period.

If, on the other hand, $q_N^* > \bar{q}$, then her cost under σ^* is equal to $(1 - \delta_N^*)/4$ because the sender communicates truthfully in each period except period N. Choose any other assessment σ. Let δ_N be a δ in the support of σ such that $1 > p_N \geq p$. Bayes' rule implies that there must be such a δ_N in the support of σ. Let q_N denote the receiver's strategy after the history where δ_N has been chosen.

Case 1. Suppose that the receiver plays the biased action with total probability $q_N \leq \bar{q}$ after each δ_N in the support of σ. However, then $q_N^* > \bar{q}$, and $q_N \leq \bar{q}$ together imply that $q_j > \bar{q}$ after some δ_j for $j < N$ in the support of σ. However, then the unraveling argument detailed further above implies that $q_N = p > \bar{q}$ leading to a contradiction.

Case 2. Suppose that the biased receiver chooses δ_N and plays the biased action with total probability $p_N > q_N > \bar{q}$. If δ_N is in the support of the unbiased receiver's strategy, then the unbiased receiver's cost under σ is at least $(1 - \delta_N)/4$. Alternatively, suppose that the biased receiver chooses a δ_N' under σ that is not in the support of the unbiased receiver. In this case, the sender communicates no information in period N. Let δ_N denote any choice of δ in the support of the unbiased receiver. Then

$$\left(1 - \delta_N' \right) \frac{1}{4} + \delta_N' \frac{1}{4} = (1 - \delta_N) \frac{1}{4} + \delta_N \frac{1}{4}$$

$$\leq (1 - \delta_N) c_N + \delta_N \frac{1}{4}$$

and therefore $c_N = \frac{1}{4}$. Hence, the unbiased receiver's cost under σ is at least $(1 - \delta_N)/4$.

If $\delta_N \leq \delta_N^*$, then $(1 - \delta_N)/4 \leq (1 - \delta_N^*)/4$, that is, the unbiased receivers cost under σ^* is at most the cost under σ. Suppose instead that $\delta_N > \delta_N^*$. Let $\hat{i}$ denote the largest $i < N$ such that $\delta_i \leq \delta_i^*$ for some δ_i in the support of the unbiased receiver. Note that $\hat{i}$ is possibly equal to one where $\delta_1 = \delta_1^* = 0$. Also, note that we have $\delta_i > \delta_i^*$ for $i > \hat{i}$ and $\delta_{\hat{i}} \leq \delta_{\hat{i}}^*$ by construction.

Suppose that δ_N is in the support of the unbiased receiver. Therefore, choosing δ_N and playing the biased action in period N is a best response for the biased receiver given the senders strategy under σ and must result in a cost that is less than or equal to any other strategy that the biased receiver may use. In particular, the cost must be less than or equal to the cost of choosing δ_N in the support of the unbiased receiver and playing the unbiased action in each period $i \in \left\{ N, \hat{i}+1 \right\}$ and then playing the biased action in period $\hat{i}$. Therefore,

$$\delta_N \frac{1}{4} \leq (1 - \delta_N) b^2 + \sum_{i=\hat{i}+1}^{N-1} \gamma_i \left(c_i + b^2 \right) + \gamma_{\hat{i}} c_i + \frac{1}{4}\delta_{\hat{i}} \left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i \right)$$

$$\leq b^2 \sum_{i=\hat{i}+1}^{N} \gamma_i + \sum_{i=1}^{N-1} \gamma_i c_i + \frac{1}{4}\delta_{\hat{i}}\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i\right)$$

where the second inequality follows because $c_i \geq 0$ for all i.

Alternatively, suppose that the biased receiver chooses a δ'_N under σ that is not in the support of the unbiased receiver. In this case, the sender communicates no information in period N. Let δ_N denote any choice of δ in the support of the unbiased receiver. Choosing δ'_N is a best response for the biased receiver given the senders strategy under σ and must result in a cost that is less than or equal to any other strategy that the biased receiver may use. In particular, the cost must be less than or equal to the cost of choosing δ_N in the support of the unbiased receiver and playing the unbiased action in each period $i \in \{N, \hat{i}+1\}$ and then playing the biased action in period $\hat{i}$. Therefore,

$$\left(1 - \delta'_N\right)\frac{1}{4} + \delta'_N\frac{1}{4} = \left(1 - \delta_N\right)\frac{1}{4} + \delta_N\frac{1}{4} \leq \left(1 - \delta_N\right)\left(b^2 + \frac{1}{4}\right) + \sum_{i=\hat{i}+1}^{N-1} \gamma_i\left(c_i + b^2\right) + \gamma_{\hat{i}}c_i + \frac{1}{4}\delta_{\hat{i}}\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i\right)$$

$$\delta_N\frac{1}{4} \leq \left(1 - \delta_N\right)b^2 + \sum_{i=\hat{i}+1}^{N-1} \gamma_i\left(c_i + b^2\right) + \gamma_{\hat{i}}c_i + \frac{1}{4}\delta_{\hat{i}}\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i\right)$$

The fact that the biased receiver is indifferent between the biased and unbiased action in each period $i > 1$ under strategy σ^* implies that

$$\delta_N^*\frac{1}{4} = b^2 \sum_{i=\hat{i}+1}^{N} \gamma_i^* + \frac{1}{4}\delta_{\hat{i}}^*\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i^*\right)$$

Moreover,

$$\left(1 - \delta_N\right)\frac{1}{4} + \sum_{i=1}^{N-1} \gamma_i c_i < \left(1 - \delta_N^*\right)\frac{1}{4}$$

$$\delta_N^*\frac{1}{4} + \sum_{i=1}^{N-1} \gamma_i c_i < \delta_N\frac{1}{4}$$

by assumption. Therefore, $\delta_N^*\frac{1}{4} + \sum_{i=1}^{N-1} \gamma_i c_i < \delta_4\frac{1}{4}$ and the equality displayed above together imply that

$$\delta_N\frac{1}{4} > b^2 \left(\sum_{i=\hat{i}+1}^{N} \gamma_i^*\right) + \frac{1}{4}\delta_{\hat{i}}^*\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i^*\right) + \sum_{i=1}^{N-1} \gamma_i c_i.$$

The facts that $\delta_i > \delta_i^*$ for $i > \hat{i}$ and $\delta_{\hat{i}} \leq \delta_{\hat{i}}^*$ together imply that

$$b^2 \left(\sum_{i=\hat{i}+1}^{N} \gamma_i^*\right) + \frac{1}{4}\delta_{\hat{i}}^*\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i^*\right) \geq b^2 \left(\sum_{i=\hat{i}+1}^{N} \gamma_i^*\right) + \frac{1}{4}\delta_{\hat{i}}\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i^*\right)$$

$$\geq b^2 \sum_{i=\hat{i}+1}^{N} \gamma_i + \frac{1}{4}\delta_{\hat{i}}\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i\right)$$

The first inequality above follows because $\delta_{\hat{i}} \leq \delta_{\hat{i}}^*$. The second inequality above follows because $\delta_i > \delta_i^*$ for each $i > \hat{i}$ implies that $\sum_{i=\hat{i}+1}^{N} \gamma_i^* > \sum_{i=\hat{i}+1}^{N} \gamma_i$ and because $\frac{1}{4}\delta_{\hat{i}} \leq b^2$ (since $b^2 \geq \frac{1}{4}$ and $\delta_{\hat{i}} \leq 1$). However, the above inequality implies that

$$\sum_{i=\hat{i}+1}^{N} \gamma_i b^2 + \sum_{i=1}^{N-1} \gamma_i c_i + \frac{1}{4}\delta_{\hat{i}}\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i\right) \geq \delta_4\frac{1}{4} > \sum_{i=\hat{i}+1}^{N} \gamma_i b^2 + \sum_{i=1}^{N-1} \gamma_i c_i + \frac{1}{4}\delta_{\hat{i}}\left(1 - \sum_{i=\hat{i}+1}^{N} \gamma_i\right)$$

which leads to a contradiction. $\square$

Claims 1-4 complete the proof of the lemma. $\square$

Lemma 6. *Suppose $p \leq 1 - (1 - \bar{q})^{N-1}$. Let $i^*(p)$ denote the largest i such that $p > 1 - (1 - \bar{q})^{i^*-1}$. The assessment that chooses $\tau_i = \delta_i = 1$ in all periods N through $i^*(p)$ and then follows σ^* is receiver optimal irrespective of the identity of the player that chooses δ.*

Proof. First note that the above specified assessment is optimal for the unbiased receiver because the sender communicates truthfully in each period where $\gamma_i > 0$.

Choose an assessment σ. Let σ^{**} denote the above specified assessment. Assume that the above described assessment is receiver optimal for all $N - 1$ period games. Note that

$$C_N^R\left(\sigma^{**}|p\right) = \left(1 - \delta_{i^*(p)}^* \cdots \delta_2^*\right) b^2$$

because the receiver is indifferent between the unbiased and biased action in each period $1 < i \leq i^*(p)$ and the receiver plays the biased action in period 1.

If the receiver plays the biased action with probability $q_N > \bar{q}$ in period N, then the receiver's cost under σ is equal to $1/4$ which is clearly worse than her cost under the above specified assessment.

If $q_N = 0$, then the cost under σ is clearly higher than the cost under σ^{**}. This is because

$$\begin{aligned}
C_N^R(\sigma) = (1 - \delta_N)b^2 + \delta_N C_{N-1}^R\left(\sigma|\frac{p - q_N}{1 - q_N}\right) &\geq (1 - \delta_N)b^2 + \delta_N C_{N-1}^R\left(\sigma^{**}|\frac{p - q_N}{1 - q_N},\right). \\
&= (1 - \delta_N)b^2 + \delta_N C_{N-1}^R\left(\sigma^{**}|p\right) \\
&\geq C_{N-1}^R\left(\sigma^{**}|p\right) = C_N^R\left(\sigma^{**}|p\right)
\end{aligned}$$

Suppose that $0 < q_N < q_{i*}^* \leq \bar{q}$, then

$$C_{N-1}^R\left(\sigma^{**}|\frac{p - q_N}{1 - q_N}\right) = C_{N-1}^R\left(\sigma^{**}|p\right) = C_N^R\left(\sigma^{**}|p\right).$$

This is because $q_N < q_{i*}^*$ implies that

$$\frac{p - q_N}{1 - q_N} > 1 - (1 - \bar{q})^{i^*-1}$$

Under σ^{**}, we have $\gamma_i^{**} = 0$ for each period N through i^* because $\frac{p - q_N}{1 - q_N} > 1 - (1 - \bar{q})^{i^*-1}$. Moreover, the receiver is indifferent between the biased and unbiased actions in each period $i \leq i^*$. Therefore, $C_{N-1}^R\left(\sigma^{**}|\frac{p - q_N}{1 - q_N}\right) = C_{N-1}^R\left(\sigma^{**}|p\right) = C_N^R(\sigma^{**}|p)$. Hence, the logic further above implies that $C_N^R(\sigma|p) \geq C_N^R(\sigma^{**}|p)$.

Suppose that $q_{i*}^* \leq q_N \leq \bar{q}$. The fact that $q_N \leq \bar{q}$ implies that $i^*\left(\frac{p - q_N}{1 - q_N}\right) = i^*(p) - 1$. Therefore,

$$C_{N-1}^R\left(\sigma^{**}|\frac{p - q_N}{1 - q_N}\right) = \left(1 - \delta_{i^*(p)-1}^* \cdots \delta_2^*\right) b^2.$$

The fact that the receiver is indifferent between the biased and unbiased action in period N implies that

$$(1 - \delta_N)b^2 + \delta_N C_{N-1}^R\left(\sigma|\frac{p - q_N}{1 - q_N}\right) - \frac{1}{4}\delta_N = 0$$

Moreover,

$$(1 - \delta_N)b^2 + \delta_N C_{N-1}^R\left(\sigma^{**}|\frac{p - q_N}{1 - q_N}\right) - \frac{1}{4}\delta_N \leq 0$$

because $C_{N-1}^R\left(\sigma|\frac{p-q_N}{1-q_N}\right) \geq C_{N-1}^R\left(\sigma^{**}|\frac{p-q_N}{1-q_N},\right)$. Hence,

$$(1 - \delta_N)b^2 + \delta_N \left(\left(1 - \delta_{i^*(p)-1}^* \cdots \delta_2^*\right) b^2 - \frac{1}{4}\right) \leq 0.$$

Moreover,

$$\left(1 - \delta_{i^*(p)}^*\right) b^2 + \delta_{i^*(p)}^* \left(\left(1 - \delta_{i^*(p)-1}^* \cdots \delta_2^*\right) b^2 - \frac{1}{4}\right) = 0$$

by construction. Thus, we conclude that $\delta_N \geq \delta_{i^*(p)}^*$ because $\left(1 - \delta_{i^*(p)-1}^* \cdots \delta_2^*\right) b^2 - \frac{1}{4} \leq 0$. However,

$$C_N^R(\sigma) = \delta_N \frac{1}{4} \geq \frac{1}{4}\delta_{i^*(p)}^* = C_N^R\left(\sigma^{**}|p\right)$$

concluding the proof. $\square$

Lemma 7. *For an N period game suppose that $\sigma = \{\tau_i, \mu_i, q_i, p_i\}$ is such*

1. *The belief $p_j = 1$ for some $j \leq N$ and $p_i < 1$ for all $i > j$.*
2. *The choices q_i and μ_i are sequentially rational under σ.*
3. *The probability $q_i(h) \leq \bar{q}$ for all $i \geq j$ where only the unbiased action has been played in history h and h is a history which has positive probability under σ.*
4. *The communication strategy μ_i entails truthful communication in each period $N > i \geq j$.*
5. *The total probability of playing the biased action in periods N through j is equal to p.*
6. *The γ_i implied by τ_i is equal to zero for all periods $i < j$.*

Then $C_{N-j+1}^S(\sigma) \geq C_N^S(\sigma^*)$ where the initial reputation is equal to p.

Proof. We will prove this for the case where the period where $p_j = 1$ is taken as period 1. This is without loss because we can take $N^* = N - j + 1$. Moreover, a simple calculation shows $C_{N-j+1}^S(\sigma^*) \geq C_N^S(\sigma^*)$.

The assessment σ that satisfies 1-6 is feasible for the following minimization problem.

$$\min_{q, \gamma} \gamma_1 b^2 q_1 \prod_{j=2}^{N}(1 - q_j) + \sum_{i=2}^{N}(q_i \prod_{j=i+1}^{N}(1 - q_j))(b^2(\sum_{j=1}^{i} \gamma_j) + \frac{1}{4}(\sum_{j=1}^{i-1} \gamma_j)) + 1_{\{q_N > \bar{q}\}} \gamma_N \frac{1}{4}$$

Subject to

$$4b^2 \sum_{j=2}^{i} \gamma_j \leq \sum_{j=1}^{i-1} \gamma_j \text{ for all } i > 1 \tag{8.1}$$

$$(4b^2 \sum_{j=2}^{i} \gamma_j - \sum_{j=1}^{i-1} \gamma_j)q_i = 0 \text{ for all } i > 1 \tag{8.2}$$

$$q_i \leq \bar{q} \text{ for all } i < N \tag{8.3}$$

$$\sum_{i=1}^{N}(q_i \prod_{j=i+1}^{N}(1 - q_j)) = p \tag{8.4}$$

$$\sum_{j=1}^{N} \gamma_j = 1 \tag{8.5}$$

$$\gamma_j \geq 0 \tag{8.6}$$

$$q_j \geq 0 \tag{8.7}$$

In the optimization above $\gamma_i = \prod_{j=i+1}^{N} \delta_{j+1}(1 - \delta_i)$ and q_i is the total probability of playing the biased action in periods i and $1_{\{q_N > \bar{q}\}}$ is the indicator function which is equal to one if $q_N > \bar{q}$ and zero otherwise. The objective function is the total cost of the sender under the assumption that the sender communicates truthfully in every period except possibly period N.

Constraint (8.1) states that playing the biased action is at least as costly as the unbiased action in every period except the last, i.e.,

$$\frac{\sum_{j=2}^{i} \gamma_j}{\sum_{j=1}^{i} \gamma_j} b^2 \leq \frac{\sum_{j=1}^{i-1} \gamma_j}{\sum_{j=1}^{i} \gamma_j} \frac{1}{4}$$

canceling $\sum_{j=1}^{i} \gamma_j$ from both sides we obtain the constraint. This constraint must hold, because if it did not, then the receiver would play the biased action with probability one in that period. However, then the probability of playing the unbiased action would exceed $\bar{q}$ in that period. This would however contradict step 2.

Constraint (8.2) says that Constraint (8.1) can only hold strictly in periods where the receiver plays the biased action with probability zero. Constraint (8.3) says that the receiver must play the biased action with probability at most $\bar{q}$. This follows from step 1. Constraint (8.4) says that the biased type eventually plays the biased action. Therefore, the total probability of the biased action is at least equal to the prior probability that the sender faces a biased receiver.

The optimization problem above is feasible because the assessment that we constructed satisfies all of the constraints. Moreover, the constraint set is compact. Therefore, the optimization problem admits a solution. Below we argue that our assessment solves the problem and therefore is sender optimal.

We argue that Constraint (8.1) holds with equality for all $i > 1$ in any solution to the optimization problem. Suppose that $4b^2 \sum_{j=2}^{i} \gamma_j < \sum_{j=1}^{i-1} \gamma_j$ for some $i < N$ and suppose that i is the largest index where this constraint does not hold with equality. This implies that $q_i = 0$ because of the second constraint.

We show that if we increase γ_i by Δ to $\hat{\gamma}_i$ so that the ith constraint binds, decrease γ_{i+1} by Δ to $\hat{\gamma}_{i+1}$, set $\hat{q}_i = q_{i+1}$, and $\hat{q}_{i+1} = 0$ and leave all other variables unchanged, then all the constraints continue to hold. However, we show that this new feasible choice has strictly lower cost and dominates the old plan.

Let

$$\Delta = \left(\sum_{j=1}^{i-1} \gamma_j\right) \frac{1}{4b^2} - \left(\sum_{j=2}^{i} \gamma_j\right).$$

This choice of Δ ensures that the ith constraint binds with equality. Note that

$$\gamma_{i+1} = \frac{\gamma_i}{4b^2} + \left(\sum_{j=1}^{i-1} \gamma_j\right) \frac{1}{4b^2} - \left(\sum_{j=2}^{i} \gamma_j\right)$$

$$= \frac{\gamma_i}{4b^2} + \Delta > 0.$$

Hence, $\hat{\gamma}_{i+1} > 0$. Also, note that the $i+1$st constraint now holds strictly. This is because

$$(\hat{\gamma}_{i+1} + \hat{\gamma}_i)b^2 + b^2 \sum_{j=2}^{i-1} \gamma_j = b^2 \sum_{j=2}^{i+1} \gamma_j$$

$$\leq \frac{1}{4} \sum_{j=1}^{i} \gamma_j$$

$$= \frac{1}{4}\gamma_i + \frac{1}{4} \sum_{j=1}^{i-1} \gamma_j$$

$$< \frac{1}{4}\gamma_i + \frac{1}{4}\Delta + \frac{1}{4} \sum_{j=1}^{i-1} \gamma_j$$

$$= \hat{\gamma}_i \frac{1}{4} + \left(\sum_{j=1}^{i-1} \gamma_j\right) \frac{1}{4}.$$

Also, all constraints $j > i+1$ continue to hold with equality:

$$b^2 \sum_{k=i+2}^{j} \gamma_k + (\hat{\gamma}_{i+1} + \hat{\gamma}_i)b^2 + b^2 \sum_{k=2}^{i-1} \gamma_k = b^2 \sum_{k=2}^{j} \gamma_k$$

$$= \frac{1}{4} \sum_{k=1}^{j-1} \gamma_k$$

$$= \frac{1}{4} \sum_{k=i+2}^{j-1} \gamma_k + \frac{1}{4}(\hat{\gamma}_{i+1} + \hat{\gamma}_i) + \frac{1}{4} \sum_{j=1}^{i-1} \gamma_j.$$

Note that this new strategy entails strictly less cost for the sender. This is because we have decreased δ_i and the sender's cost is decreasing in δ_i. See the proof of Step 2 for an argument. However, this contradicts the assertion that the initial plan solved the optimization problem. Therefore this line of reasoning establishes that all the such constraints must hold with equality.

In period N if the Constraint (8.1) holds as an inequality, then $q_N = 0$ and therefore the miscommunication cost in period N is equal to zero. However, in this case the sender's cost is again decreasing δ_N and therefore it is possible to decrease δ_N to decrease cost. Moreover, decreasing δ_N does not affect any of the subsequent such constraints.

If all such constraints hold with equality, then we have $\gamma_i = \prod_{j=i+1}^{N} \delta_{j+1}^*(1 - \delta_i^*)$ in the optimal solution. This follows immediately from the construction of the sequence $\{\delta_i^*\}$.

Given that we have $\gamma = \prod_{j=i+1}^{N} \delta_{j+1}^*(1 - \delta_i^*)$, we now show that $q_i = q_i^*$. Note that we can move mass from any period i to any other period j because the Constraints (8.1) hold with equality. First note that $\sum_{i=1}^{N}(q_i \prod_{j=i+1}^{N}(1 - q_j)) = p = \sum_{i=1}^{N}(q_i^* \prod_{j=i+1}^{N}(1 - q_j^*))$, i.e., constraint (8.4) holds with equality, because otherwise the total cost can be reduced by decreasing the probability q_i in some period i.

Suppose that there exists a period i such that $q_i > q_i^*$ and let i be the largest such period. Then, by the construction of the sequence of q_i^*, there must exist a period $j < i$ where $q_j < q_j^* = \bar{q}$. However, then the sender's cost can be reduced by decreasing q_i and increasing q_j by $\epsilon > 0$ sufficiently small.

Suppose that there is a period i such that $0 \leq q_i < q_i^*$ and let i be the smallest such period. Note that $i < N$ by construction because otherwise we would have $\sum_{i=1}^{N}(q_i \prod_{j=i+1}^{N}(1 - q_j)) < p$ which would violate constraint 4. This implies by construction that 1) $q_i^* > 0$ and 2) $q_j^* = \bar{q}$ for all $j < i$. However, $q_j \leq \bar{q}$ for all $j < N$ because of Constraint (8.3) of the optimization problem. Therefore, $q_j = q_j^*$ for all $j < i$. Also we know that $\sum_{i=1}^{N}(q_i \prod_{j=i+1}^{N}(1 - q_j)) = p$. This implies that, there must be a period $k > i$ such that $q_k > q_k^*$. However, then the sender's cost can be reduced by decreasing q_k and increasing q_i by $\epsilon > 0$ sufficiently small. $\quad \square$

Lemma 8. *If σ is sender-optimal, then the total probability that the receiver plays the biased action is at most $\bar{q}$ for any $i < N$ after any history where only the unbiased action has been observed. Therefore, if σ is sender-optimal, then $q_N(p) \geq q_N^*(p)$.*

Proof. The assertion is automatically true if $b \leq 1/2$. Assume that $b > 1/2$. On the way to a contradiction, suppose that there is a period $i < N$ where the receiver plays the biased action with strictly higher probability than $\bar{q}$ under σ. We argue below that the receiver plays the biased action with probability $q_j = p > \bar{q}$ in period N under σ.

To see this, note that in period $i + 1$ the receiver's cost from playing the biased action is $\frac{\delta_i}{4} + (1 - \delta_i)x$. The receiver's cost from playing the unbiased action is $(1 - \delta_i)(x + b^2) + \frac{\delta_i}{4}$ where the continuation payoff is $\frac{\delta_i}{4}$ because the sender's communication is uninformative in period i and the receiver plays the biased action with positive probability. Therefore, the biased receiver will play the biased action with probability one in period $i + 1$. Working in this manner recursively, we find that the receiver will play the biased action with probability one in each period $j > i$, i.e., $q_j = p_j$.

We now argue that assessment σ^* entails strictly lower cost than σ. The cost under σ is at least

$$C_N^S(\sigma) \geq p(\frac{1}{4} + b^2).$$

In contrast, the cost in assessment σ^* is at most

$$C_N^S(\sigma^*) \leq q(b^2 + \frac{1}{4}) + (1 - q)((1 - \delta)\frac{1}{4} + \delta\frac{p - q}{1 - q}b^2),$$

where $\frac{p-q}{1-q} = \bar{q}$. Hence,

$$p(\frac{1}{4} + b^2) - C^S(\sigma^*) \geq (p - q)(b^2 + \frac{1}{4}) - (1 - q)((1 - \delta)\frac{1}{4} + \delta\frac{p - q}{1 - q}b^2)$$

$$= (p - q)\frac{1}{4} + (p - q)b^2(1 - \delta) - (1 - q)(1 - \delta)\frac{1}{4}$$

$$= (p - q)\frac{1}{4} + \bar{q}(1 - q)b^2(1 - \delta) - (1 - q)(1 - \delta)\frac{1}{4}$$

$$\geq (p - q)\frac{1}{4} + (\frac{b}{2} - \frac{1}{4})(1 - q)(1 - \delta) > 0.$$

To complete the proof, note that $q_N(p) \geq q_N^*(p)$ is automatically true if $q_N^* = 0$. If $q_N^*(p) > 0$, then $q_i^*(h_i) = \bar{q}$ in any subsequent period where h_i is any history on the equilibrium path where only the unbiased action has been observed. If $q_N(p) < q_N^*(p)$, then there must exist a history on the equilibrium path h_i where only the unbiased action has been played such that $q_i(h_i) > \bar{q}$. However, then the sender communicates no information in period $i < N$ leading to a contradiction. $\quad \square$

Lemma 9. *Fix an assessment σ. Let period $m^* \geq 1$ be such that $p_{m^*} = 1$ under σ. The assessment σ is sender optimal only if the miscommunication costs are equal to zero in every period $m^* \leq i < N$ and if $\gamma_{m^*} = 0$ for all $i < m^*$ under σ.*

Proof. Suppose $m^* = 1$. We come back to the case where $m^* > 1$ at the end of the proof.

On the way to a contradiction, suppose that there are information costs $x_j > 0$ in some period $j < N$ in a sender-optimal assessment σ which satisfies Property 1 and 2 after an history where only the unbiased action has been observed. Under this hypothesis, we show that there is an assessment σ', which satisfies Property 1 and 2 and has strictly lower costs for both the sender and the receiver than assessment σ. In this new assessment the sender communicates truthfully in each period except possibly period N after observing the unbiased action. This is incentive compatible because we have $q_i \leq \bar{q}$ in every period $i < N$ by assumption.

Case 1. Suppose that $q_N \leq \bar{q}$. Then the sender communicating truthfully in period N is incentive compatible

Let $N > j^* \geq 1$ be a period such that there are miscommunication costs in this period and no information costs in any period $i \in \{j^* - 1, ..., 1\}$. Period j^* is well defined since we allow for j^* to equal one. In any period $i > j^*$ the following inequality holds

$$\delta_i \frac{1}{4} \geq (1 - \delta_i \cdots \delta_{z+1})b^2 + \sum_{k=z}^{i-1}(\prod_{l=k+1}^{i}\delta_l)(1 - \delta_k)x_k + \frac{1}{4}\delta_z,$$

where $z < i$ is any period in which the receiver plays the biased action with positive probability. Such a period z must exist because the receiver will play the biased action with probability one in the last period of the game. The sum $\sum_{k=z}^{i-1}(\prod_{l=k+1}^{i}\delta_l)(1 - \delta_k)x_k \geq 0$ is the total miscommunication costs that the receiver incurs in the posited equilibrium in the periods $\{i - 1, ..., z\}$. The inequality holds because the receiver must prefer to play the unbiased action until period z and then switch to the biased action in period z. Moreover, note that in any period $i > j$ where the receiver plays a mixed strategy the inequality holds with equality.

In the assessment that we construct, we leave the δ_i's unchanged in any period $i \leq j$ and in any period in which the receiver plays the unbiased action with probability one. In this new assessment, the sender communicates truthfully in each period $i < N$.

Let $i_1 > j^*$ be the first period where the receiver plays a mixed strategy and choose $\hat{\delta}_{i_1}$ such that

$$\hat{\delta_{i_1}} \frac{1}{4} = (1 - \hat{\delta_{i_1}} \delta_{i_1-1} \cdots \delta_2)b^2.$$

Note that $\hat{\delta_{i_1}} < \delta_{i_1}$. We now show that for any period $k > i_1$, the receiver strictly prefers to play the unbiased action. For any period $k > i_1$, the receiver would weakly prefer to play the unbiased action with probability 1 until period i_1, then play the biased action in period i under the posited equilibrium σ. More precisely,

$$\delta_k \frac{1}{4} \geq (1 - \delta_k \cdots \delta_{k-1}\delta_{i_1+1})b^2 + X + \delta_{i_1}\frac{1}{4}$$

where X denotes the total miscommunication costs in periods $\{k - 1, ..., i_1\}$. Therefore, in the equilibrium without miscommunication costs we find that,

$$\delta_k \frac{1}{4} > (1 - \delta_k \cdots \delta_{k-1}\delta_{i+1})b^2 + \hat{\delta}_{i_1}\frac{1}{4}$$

because $\hat{\delta_{i_1}} < \delta_{i_1}$. Suppose that k is a period where the receiver plays a mixed strategy. Recall that we leave δ_i's are unchanged in any period $i < k$ where the receiver plays the unbiased action with probability one, further suppose that $\hat{\delta}_i < \delta_i$ in any period $i < k$ where the receiver plays a mixed strategy which leaves the receiver is indifferent in period i given the new choice of δ's. Note that the above argument implies that

$$\delta_k \frac{1}{4} > (1 - \delta_k \cdots \delta_{k-1}\delta_{i+1})b^2 + \hat{\delta}_i\frac{1}{4}.$$

Pick $\hat{\delta}_k$ such that

$$\hat{\delta}_k \frac{1}{4} = (1 - \hat{\delta}_k \cdots \delta_{k-1}\delta_{i+1})b^2 + \hat{\delta}_i\frac{1}{4}$$

and note that $\hat{\delta}_k < \delta_k$. Choosing δ's in this manner ensures that the receiver is indifferent in any period that he plays a mixed strategy and prefers the unbiased action in any period where he plays the unbiased action with probability one. Also note that $\hat{\delta}_i \leq \delta_i$ in any period i.

If we leave everything else in the assessment σ unchanged, decreasing the δ's in the way described above, and have the sender communicate truthfully in each period, then we obtain a new equilibrium where all miscommunication costs have been eliminated and which satisfies Property 1 and 2. In this new assessment, the sender's costs have strictly decreased. To see this first note that eliminating miscommunication costs without changing the δ_i's strictly decreases the sender's cost. Moreover, in any period where there are no miscommunication costs, the sender's continuation cost is strictly increasing in δ. This is because the sender's continuation cost in any period $i < N$ is given by $q_i\left[(1 - \delta_i)b^2 + \delta_i(b^2 + 1/4)\right] + (1 - q_i)\delta_i C_{i-1}^S(\sigma|h_{i-1})$ and the fact that $C_{i-1}^S(\sigma|h_{i-1}) \geq 0$ implies that this expression is strictly increasing in δ. If the sender communicates truthfully in period N, then this argument is also true for period N. Therefore, the original σ could not have been a sender-optimal equilibrium.

Case 2. Suppose that $q_N > \bar{q}$ and the sender does not communicate truthfully under σ.

If $N - 1$ is a period where the receiver plays a mixed strategy and $q_{N-1} = \bar{q}$, then $C_{N-1}^S(\sigma|h_{N-1}) \geq 1/4$ and the sender's cost in period N is again strictly increasing in δ. This is because

$$C_{N-1}^S(\sigma|h_{N-1}) \geq \bar{q}b^2 = \frac{b}{2} \geq \frac{1}{4}$$

If the sender's costs are increasing in δ under the new assessment σ', then we are done by the reasoning above. Assume that the sender's costs are decreasing in δ under the new assessment σ' and alter σ' in the following way:

Assume either that Case 2a) $N - 1$, is a period where the receiver plays a mixed strategy and $q_{N-1} < \bar{q}$, or Case 2b) If $N - 1$ is a period where the receiver plays the unbiased action with probability 1.

Case 2a) If $N - 1$, is a period where the receiver plays a mixed strategy and $q_{N-1} < \bar{q}$, then increase q_{N-1} and decrease q_N until either $q_{N-1} = \bar{q}$ or $q_N \leq \bar{q}$. Note that we can move mass from q_N to q_{N-1} until q_{N-1} reaches $\bar{q}$. If the mass at q_N dips below $\bar{q}$ before q_{N-1} increases to $\bar{q}$ we can stop and allow the sender to communicate truthfully also in period N. However, if the sender communicates truthfully in period N, then case 1 implies that σ cannot be sender optimal. If on the other hand $q_{N-1} = \bar{q}$, then the reasoning in the previous paragraph implies that σ cannot be sender optimal.

Case 2b) If $N - 1$ is a period where the receiver does not play a mixed strategy, then

$$\delta_{N-1}\frac{1}{4} > (1 - \delta_{N-1}\cdots\delta_{i+1})b^2 + \hat{\delta}_i\frac{1}{4}$$

this is because under the assessment σ' playing the unbiased action in each period is sequentially rational for the receiver. Moreover,

$$\delta_N\frac{1}{4} > (1 - \delta_N\delta_{N-1}\cdots\delta_{i+1})b^2 + \hat{\delta}_i\frac{1}{4}$$

The cost is given by

$$C^S(\sigma') = q_N\left(\frac{1}{4} + b^2\right) + (1 - q_N)\left((1 - \delta_N)\frac{1}{4} + \delta_N\delta_{N-1}C^S_{N-2}(\sigma')\right).$$

The fact that the cost is decreasing in δ implies that $\delta_{N-1}C^S_{N-2}(\sigma') < \frac{1}{4}$, therefore the cost is decreasing in for all $\delta'_{N-1} \leq \delta_{N-1}$

Suppose that $\delta_N\frac{1}{4} < (1 - \delta_N\hat{\delta}_{N-1}\cdots\delta_{i+1})b^2 + \hat{\delta}_i\frac{1}{4}$. Then, there is $\hat{\delta}_{N-1} < \delta^*_{n-1} < \delta_{N-1}$ such that $\delta_N\frac{1}{4} < (1 - \delta_N\delta^*_{N-1}\cdots\delta_{i+1})b^2 + \hat{\delta}_i\frac{1}{4}$. Picking the period $N - 1$'s delta equal to δ^*_{N-1} leaves δ_{N-1} unchanged but lowers cost in all subsequent periods. Hence improves upon assessment σ.

Suppose on the other hand $\delta_N\frac{1}{4} \geq (1 - \delta_N\hat{\delta}_{N-1}\cdots\delta_{i+1})b^2 + \hat{\delta}_i\frac{1}{4}$. Then picking $\hat{\delta}_N$ and $\hat{\delta}_{N-1}$ and then increasing q_{N-1} and decrease q_N until either $q_{N-1} = \bar{q}$ or $q_N \leq \bar{q}$ reduces cost by the reasoning given in case 2a).

Suppose now that $m^* > 1$

Let i be the smallest period where $p_i < 1$. If $i = N$, then the cost under σ^* is lower than the cost under σ implying that σ is not sender optimal.

Suppose that $i < N$. Create a new assessment σ' by deleting all the miscommunication costs as in step 2. In the new assessment σ' set period i as the last period, i.e., set $\gamma_i = \sum_{j=1}^{i-1}\gamma_j$ and set $\gamma_j = 0$ for $j < i$. This reduces the cost for the sender by $\delta_i\frac{1}{4}$ because the biased receiver plays the biased action in all periods i through 1. In period any k we have the following:

$$\delta_k\frac{1}{4} > (1 - \delta_k\cdots\delta_{k-1}\delta_{i+1})b^2 + \delta_{i+1}C^R_i(\sigma') = (1 - \delta_k\cdots\delta_{k-1}\delta_{i+1})b^2,$$

because $C^R_i(\sigma') = 0$ under the new assessment. Therefore, the above logic can now be used to ensure all incentive constraints are satisfied while at the same time reducing the sender's cost. □

Lemma 10. *Suppose that the sender chooses δ, then the assessment σ^* is sender-optimal.*

Proof. If an assessment is sender optimal, then it satisfies the conditions 1-6 outlined in Lemma 7 by Lemmata 8 and 9. Therefore, Lemma 7 shows that σ^* is sender optimal. □

Lemma 11. *Suppose that the receiver chooses δ, then the assessment σ^* is sender-optimal.*

Proof. Let σ be an assessment where the sender and receiver separate and choose different δ_Ns in period N. The sender's payoff after any δ_N in the support of the biased receiver is equal to $\frac{1}{4} + b^2$ because the sender's belief that the receiver is equal to one after observing any such δ_N, therefore the sender is unable to communicate any information in any subsequent period and the biased receiver plays the biased action in every period. Hence, the sender's cost under σ is at least $p\left(\frac{1}{4} + b^2\right)$ which strictly exceeds the sender's cost under σ^*.

Let σ be an assessment where the biased and unbiased receiver pool and choose the same δ after any history on the equilibrium path where only the unbiased action has been observed. In this case the argument given for Lemmata 7, 8 and 9 together imply that the cost under σ^* is lower than the cost under σ.

Suppose that $supp\,[\tau] = \{\delta_N, \delta'_N\}$. Note $\Pr(\delta_N)\,p(\delta_N) + (1 - \Pr(\delta_N))\,p(\delta'_N) = p$.

Case 1. $q^*_N(p) \leq \bar{q}$. Lemmata 7, 8 and 9 together imply that

$$C^s_N(\sigma|\delta_N) \geq C^s_N(\sigma^*|p(\delta_N))$$
$$C^s_N(\sigma|\delta'_N) \geq C^s_N(\sigma^*|p(\delta'_N))$$

If $q_N^*(p) \le \bar{q}$, then

$$\Pr(\delta_N) C_N^S(\sigma^*|p(\delta_N)) + (1 - \Pr(\delta_N)) C_N^S(\sigma^*|p(\delta_N')) \ge C_N^S(\sigma^*|p).$$

Therefore,

$$C_N^s(\sigma) = \Pr(\delta_N) C_N^S(\sigma|\delta_N) + (1 - \Pr(\delta_N)) C_N^S(\sigma|\delta_N') \ge C_N^S(\sigma^*|p).$$

Case 2. $q_N^*(p) > \bar{q}$, $q_N^*(p(\delta_N)) > \bar{q}$ and $q_N^*(p(\delta_N')) > \bar{q}$. Lemmata 7, 8 and 9 together imply that

$$C_N^s(\sigma|\delta_N) \ge C_N^s(\sigma^*|p(\delta_N))$$
$$C_N^s(\sigma|\delta_N') \ge C_N^s(\sigma^*|p(\delta_N'))$$

If $q_N^*(p) > \bar{q}$, $q_N^*(p(\delta_N)) > \bar{q}$ and $q_N^*(p(\delta_N')) > \bar{q}$, then

$$\Pr(\delta_N) C_N^S(\sigma^*|p(\delta_N)) + (1 - \Pr(\delta_N)) C_N^S(\sigma^*|p(\delta_N)) \ge C_N^S(\sigma^*|p).$$

Therefore,

$$C_N^s(\sigma) = \Pr(\delta_N) C_N^S(\sigma|\delta_N) + (1 - \Pr(\delta_N)) C_N^S(\sigma|\delta_N) \ge C_N^S(\sigma^*|p).$$

Case 3. $q_N^*(p) > \bar{q}$, $q_N^*(p(\delta_N)) \le \bar{q}$ and $q_N^*(p(\delta_N')) > \bar{q}$.

$$C_N^s(\sigma|\delta_N) = (1 - \delta_N) c_N + q_N \left(\delta_N \frac{1}{4} + b^2\right) + \delta_N (1 - q_N) C_{N-1}^s(\sigma|p(h_N))$$

The fact that $q_N^*(p(\delta_N')) > \bar{q}$ implies that the biased receiver's cost after such a δ_N' is equal to 1/4. Hence, after any δ_N in the support of τ,

$$(1 - \delta_N) c_N(\delta_N) + \delta_N \frac{1}{4} \ge \frac{1}{4}.$$

Therefore, $c_N(\delta_N) \ge \frac{1}{4}$ and we have

$$C_N^s(\sigma|\delta_N) = (1 - \delta_N) \frac{1}{4} + q_N \left(\delta_N \frac{1}{4} + b^2\right) + \delta_N (1 - q_N) C_{N-1}^s(\sigma|p(h_N))$$

Case 3 a). Suppose that $\delta_N < \delta_N^*$.

The fact that $q_N^*(p(\delta_N)) \le \bar{q}$ and Lemmata 7, 8 and 9 together imply that

$$q_N \left(\delta_N \frac{1}{4} + b^2\right) + \delta_N (1 - q_N) C_{N-1}^s(\sigma|p(h_N)) \ge \delta_N^* C_{N-1}^s(\sigma^*|p(\delta_N)) = C_N^s(\sigma^*|p(\delta_N))$$

Hence,

$$C_N^s(\sigma|\delta_N) \ge (1 - \delta_N) \frac{1}{4} + \delta_N^* C_{N-1}^s(\sigma|p(\delta_N))$$

$$> \left(1 - \delta_N^*\right) \frac{1}{4} + C_N^s(\sigma^*|p(\delta_N))$$

Moreover, Lemmata 7, 8 and 9 together imply that

$$C_N^s(\sigma|\delta_N') \ge C_N^s(\sigma^*|p(\delta_N'))$$

If $q_N^*(p) > \bar{q}$, $q_N^*(p(\delta_N)) \le \bar{q}$ and $q_N^*(p(\delta_N')) > \bar{q}$, then

$$\Pr(\delta = \delta_N) \left(C_N^S(\sigma^*|p(\delta_N)) + \frac{1}{4}\left(1 - \delta_N^*\right)\right) + \left(1 - \Pr(\delta_N')\right) C_N^S(\sigma^*|p(\delta_N')) \ge C_N^S(\sigma^*|p).$$

Therefore,

$$\Pr(\delta_N) C_N^S(\sigma|\delta_N) + \left(1 - \Pr(\delta_N')\right) C_N^S(\sigma|\delta_N') \ge C_N^S(\sigma^*|p).$$

Case 3 b). Suppose that $\delta_N \geq \delta_N^*$. If $q_N^*(p(\delta_N)) > 0$, then $q_{N-1}^*(\delta_N) = \bar{q}$. Therefore,

$$
\begin{aligned}
C_N^s(\sigma|\delta_N) &= (1-\delta_N)\frac{1}{4} + q_N\left(\delta_N\frac{1}{4} + b^2\right) + \delta_N(1-q_N)C_{N-1}^s(\sigma|p(h_N)) \\
&\geq (1-\delta_N)\frac{1}{4} + \delta_N C_{N-1}^s(\sigma^*|p(\delta_N)) \\
&\geq \left(1-\delta_N^*\right)\frac{1}{4} + \delta_N^* C_{N-1}^s(\sigma^*|p(\delta_N))
\end{aligned}
$$

because $q_{N-1}^*(\delta_N) = \bar{q}$ implies that $C_{N-1}^s(\sigma^*|p(h_N)) \geq 1/4$. Moreover,

$$
C_N^s\left(\sigma|\delta_N'\right) \geq C_N^s\left(\sigma^*|p\left(\delta_N'\right)\right)
$$

Hence,

$$
\Pr(\delta_N)C_N^s(\sigma|\delta_N) + \left(1 - \Pr\left(\delta_N'\right)\right)C_N^s\left(\sigma|\delta_N'\right) \geq C_N^s(\sigma^*|p).
$$

Suppose that $q_N^*(p(\delta_N)) = 0$ and $q_{N-1}^*(p(\delta_N)) < \bar{q}$. Note that

$$
\begin{aligned}
C_N^s(\sigma) &\geq \Pr(\delta_N)\left((1-\delta_N)\frac{1}{4} + \delta_N C_{N-1}^s\left(\sigma^*|p(\delta_N')\right)\right) + \\
&(1 - \Pr(\delta_N))\left(\left(1-\delta_N^*\right)\frac{1}{4} + q_N^*\left(p(\delta_N')\right)\left(\delta^*\frac{1}{4} + b^2\right) + \delta_N^*\left(1 - q_N^*\left(p(\delta_N')\right)\right)C_{N-1}^s\left(\sigma^*|\frac{p(\delta_N') - q_N^*\left(p(\delta_N')\right)}{1 - q_N^*\left(p(\delta_N')\right)}\right)\right)
\end{aligned}
$$

Let p_1 be such that $q_{N-1}^*(p_1) = \bar{q}$. Note that $q_{N-1}^*(p(\delta_N)) < \bar{q}$ implies that $p_1 > p(\delta_N)$. Pick p_2 such that

$$
\Pr(\delta_N)p_1 + (1 - \Pr(\delta_N))p_2 = p.
$$

The fact that $q_{N-1}^*(p_1) = \bar{q}$ and $q_{N-1}^*(p_2) = \bar{q}$, together imply that

$$
(1 - \Pr(\delta_N))q_N^*(p_2) = q_N^*(p).
$$

Moreover,

$$
\begin{aligned}
&\Pr(\delta_N)\left((1-\delta_N)\frac{1}{4} + \delta_N C_{N-1}^s\left(\sigma^*|p(\delta_N')\right)\right) + \\
&(1 - \Pr(\delta_N))\left(\left(1-\delta_N^*\right)\frac{1}{4} + q_N^*\left(p(\delta_N')\right)\left(\delta^*\frac{1}{4} + b^2\right) + \delta_N^*\left(1 - q_N^*\left(p(\delta_N')\right)\right)C_{N-1}^s\left(\sigma^*|\frac{p(\delta_N') - q_N^*\left(p(\delta_N')\right)}{1 - q_N^*\left(p(\delta_N')\right)}\right)\right) \geq \\
&\Pr(\delta_N)\left((1-\delta_N)\frac{1}{4} + \delta_N C_{N-1}^s\left(\sigma^*|p_1\right)\right) + \\
&(1 - \Pr(\delta_N))\left(\left(1-\delta_N^*\right)\frac{1}{4} + q_N^*(p_2)\left(\delta^*\frac{1}{4} + b^2\right) + \delta_N^*\left(1 - q_N^*\left(p(\delta_N')\right)\right)C_{N-1}^s\left(\sigma^*|\frac{p_2 - q_N^*(p)}{1 - q_N^*(p)}\right)\right) \geq \\
&\Pr(\delta_N)\left((1-\delta_N^*)\frac{1}{4} + \delta_N^* C_{N-1}^s\left(\sigma^*|p_1\right)\right) + \\
&(1 - \Pr(\delta_N))\left(\left(1-\delta_N^*\right)\frac{1}{4} + q_N^*(p)\left(\delta^*\frac{1}{4} + b^2\right) + \delta_N^*\left(1 - q_N^*\left(p(\delta_N')\right)\right)C_{N-1}^s\left(\sigma^*|\frac{p_2 - q_N^*(p)}{1 - q_N^*(p)}\right)\right) = \\
&\hspace{10cm} C_N^s(\sigma^*)
\end{aligned}
$$

completing the argument. Although throughout the proof we have assumed that τ has a two point support, the argument follows without alteration for the case where $\tau's$ support is arbitrary by simply applying Jensen's inequality. $\quad\square$

$\hspace{14cm}\square$

Proof of Proposition 1. Define $D_1 = 1$, let $a = 4b^2$ and note that δ_i^*, $i = 2, \ldots, N$, is defined by the following system of equations:

$$
\delta_i^* = \frac{a}{1 + aD_{i-1}} \tag{8.8}
$$

$$
D_i = \delta_i^* D_{i-1} \tag{8.9}
$$

for all $i = 2, \ldots, N$. This, in turn, can be reduced to the following difference equation with initial condition $D_1 = 1$:

$$D_i = \frac{a D_{i-1}}{1 + a D_{i-1}}, \ i = 2, \ldots, N. \tag{8.10}$$

Claim 5. *Unique solution to the difference equation given in (8.10) is given by*

$$D_i = \frac{a^{i-1}}{\sum_{j=0}^{i-1} a^j} \tag{8.11}$$

Proof of Claim 5. Proof is by induction. $D_2 = a/(1 + a)$, so it is true for $i = 2$. Suppose now that it is true for $2 \leq k \leq N - 1$. Then

$$D_{k+1} = \frac{a D_k}{1 + a D_k} = \frac{a \frac{a^{k-1}}{\sum_{j=0}^{k-1} a^j}}{1 + a \frac{a^{k-1}}{\sum_{j=0}^{k-1} a^j}} = \frac{a^k}{\sum_{j=0}^{k} a^j}$$

which establishes the claim. □

Substituting (8.11) into (8.8), we obtain

$$\delta_i^* = \frac{a}{1 + a \frac{a^{i-2}}{\sum_{j=0}^{i-2} a^j}} = \frac{\sum_{j=1}^{i-1} a^j}{\sum_{j=0}^{i-1} a^j}. \tag{8.12}$$

Claim 6.

$$\gamma_{N-i} = \frac{a^i}{\sum_{j=0}^{N-1} a^j}, \ i = 0, 1, \ldots, N - 1. \tag{8.13}$$

Proof of Claim 6. First, note that

$$\gamma_N = 1 - \delta_N^* = 1 - \frac{\sum_{j=1}^{N-1} a^j}{\sum_{j=0}^{N-1} a^j} = \frac{1}{\sum_{j=0}^{N-1} a^j}$$

Second, by definition $\gamma_{N-i} = \delta_N^* \delta_{N-1}^* \ldots \delta_{N-i+1}^* (1 - \delta_{N-i}^*)$ for any $i = 1, \ldots, N - 1$. Again by definition $D_i = \delta_i^* \delta_{i-1}^* \ldots \delta_2^*$, which implies that

$$\delta_N^* \delta_{N-1}^* \ldots \delta_{N-i+1}^* = \frac{\delta_N^* \delta_{N-1}^* \ldots \delta_2^*}{\delta_{N-i}^* \delta_{N-i-1}^* \ldots \delta_2^*} = \frac{D_N}{D_{N-i}}$$

Therefore,

$$\gamma_{N-i} = \frac{D_N}{D_{N-i}} (1 - \delta_{N-i}^*) = \frac{\frac{a^{N-1}}{\sum_{j=0}^{N-1} a^j}}{\frac{a^{N-i-1}}{\sum_{j=0}^{N-i-1} a^j}} \left(1 - \frac{\sum_{j=1}^{N-i-1} a^j}{\sum_{j=0}^{N-i-1} a^j} \right) = \frac{a^i}{\sum_{j=0}^{N-1} a^j}$$

for any $i = 1, \ldots, N - 1$. This proves the claim. □

It is now easy to show that growth rate of the importance parameter γ is $a - 1$ and that γ_N decreases in a. □

Proof of Proposition 2. Let the prior be $p > \bar{q}$ and k the largest integer such that $p > 1 - (1 - \bar{q})^{k-1}$. Assume that $k \geq 2$. Theorem 1 and the discussion that follows it implies that, if $N \geq k$, the cost of the sender is equal to

$$C^S = (\gamma_1 + \cdots + \gamma_{k-1}) \bar{q} b^2 + \gamma_k q_k b^2 < \bar{q} b^2$$

since $q_k \leq \bar{q}$. Cost when $N < k$, on the other hand, is at least $\bar{q} b^2$, because the total probability of the biased action is greater than or equal to $\bar{q}$ in period N. This implies that it is strictly better to choose $N \geq k$ rather than $N < k$. Let $W_i = \gamma_1 + \cdots + \gamma_i$ and note that $W_k = D_N / D_k$ by definition. Since $\gamma_k = W_k - W_{k-1}$, cost can be written as

$$C^S = [W_{k-1}(\bar{q} - q_k) + W_k q_k]b^2$$
$$= [\delta_k^*(\bar{q} - q_k) + q_k]\frac{D_N}{D_k}b^2.$$

Note that $p \le 1 - (1 - \bar{q})^k$ by definition of k. Therefore, if $N \ge k$, then there is no cost of miscommunication in period N and hence the cost of the unbiased receiver is equal to zero while the cost of the biased receiver is $\delta_k/4$ in the receiver-optimal equilibrium. If $N < k$, however, there is a cost of miscommunication in period N, which implies that the cost of the unbiased receiver is $(1 - \delta_N)/4$ and the cost of the biased receiver is $1/4$. Therefore, any $N \ge k$ minimizes the cost for the receivers. This implies that the cost of the receivers is (weakly) decreasing in N.

If $k = 1$ or $p \le \bar{q}$, then the cost is equal to $\gamma_1 p b^2 = D_N p b^2$. Equation (8.11) implies that D_N is strictly decreasing in N, which implies that the cost of the sender is decreasing N. Furthermore, if $b > 1/2$, then $\lim_{N \to \infty} D_N = 1 - 1/a > 0$, which implies that the lower bound on the total cost is strictly positive. If $b < 1/2$, then $\lim_{N \to \infty} D_N = 0$.

When $k = 1$ or $p \le \bar{q}$, the cost of both types of the receivers in the receiver-optimal equilibrium is equal to zero, and hence it is constant in N.   □

Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.geb.2020.03.001.

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