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Existence of an equilibrium in incomplete markets with discrete choices and many markets

Jonathan Halket

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Jonathan Halket†

Abstract

We define and prove the existence of an equilibrium for Bewley-style models of heterogeneous agents in incomplete markets with discrete and continuous choices. Our sample model also features endogenous price volatility across many markets (locations) but still has a steady state equilibrium with a finite-dimensional state space. Our proof of existence uses Kakutani's Fixed Point Theorem and does not require the set of households that are indifferent between two discrete choices to be measure zero.

Keywords: Incomplete Markets, Discrete Choices, Heterogeneous Agents

JEL Classification: C62, D58, R13.

1 Introduction

Many models study the effects of incomplete insurance in the tradition of Aiyagari [1994], Bewley [1984], Imrohoruglu [1989], Huggett [1993] by adding discrete choices. For instance, Chambers et al. [2009b] looks at the decision to own or rent a home, Chang and Kim [2006] looks at labor-force participation, Athreya and Simpson [2006] looks at defaulting, while Kitao [2008] looks at the decision to become an entrepreneur. Generically, proving existence has in the past in part reduced to guaranteeing that sets of households that are indifferent between two discrete choices are at most of measure zero. Otherwise aggregate demand and supply functions may not be continuous, a point discussed at length in elegant work by Chatterjee et al. [2007]. Their paper also offers a path for proving existence when the choice space is entirely discrete.

In this paper, we provide a path for proving existence when the choice space has both discrete and continuous choices and where sets of indifferent households can be arbitrarily large and thus

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†University of Essex and Institute for Fiscal Studies, *jonathan@halket.com, http://halket.com/*

aggregate demand and supply are correspondences rather than functions. We solve the problem created by the possibility of a large number of indifferent households by showing that there always exists a way of allocating these indifferent households such that there is an equilibrium. No additional nuisance costs such as the ones used in Chatterjee et al. [2007] are necessary[1]. The proof, which uses Kakutani's Fixed Point Theorem (FPT) rather than the usual Brouwer FPT, is adaptable to other Bewley-type models with discrete and continuous choices.

We illustrate the approach we use within a model of housing in local labor markets. We situate a Bewley-type model of earnings shocks in incomplete markets in a Lucas and Prescott [1971]-like island model of housing and labor markets. Exogenous stochastic variation in the quality of the local labor market will create endogenous household mobility and movements in house prices and rents. The model (a simplified version of Halket and Vasudev [2011]) has a continuum of discrete choices and markets - in this case locations. We show that there exists a stationary general equilibrium where the price of market-specific goods - in this case housing - is an exact, finite-dimensional function, even if the characteristics of a particular market (e.g. the distribution of wealth within the market) are stochastic. Heterogeneous agent, incomplete-market models with stochastic prices typically feature infinite dimensional state variables in the agents' decision problems, and thus afford only approximate solutions (as in, for instance, Krusell and Smith Jr. [1998]). We build an economy for which there is an exact stationary equilibrium[2] - in this case, the price of housing in a location will only depend upon a location-specific productivity factor. This allows us to characterize prices and allocations without having the distributions over households within or across locations entering the households' state space. The proof that locations with the same productivity have the same house prices relies on the upper hemi-continuity of the aggregate demand and supply correspondences and Kakutani's FPT in exactly the same way that the remedy for indifference over discrete choices does.

The rest of the paper is organized as follows: section 2 presents the model. Section 3 and 4 respectively defines and proves the existence of a stationary competitive equilibrium for the economy.

[1]Though since households are never indifferent in approximating computer simulations due to machine accuracy, both Chatterjee et al. [2007] and our solutions to indifference are mainly of theoretical importance.

[2]The equilibrium is theoretically exact. There is no analytic solution, so the computational solution is still approximate.

2 Model

We consider an OLG island model of household consumption choice. There is a continuum of measure 1 of agents and islands each in the economy[3]. Agents are indexed by $\iota \in [0,1]$ and islands are indexed by $\varepsilon \in \mathscr{E} \in [0,1]$.

Time is discrete and each period in the economy corresponds to one year in the data. Households are born at age $a = 1$ and live at most to age T. In every period, the household survives to next period with probability, $\lambda : \mathrm{A} \to [0,1]$, which is a function of the age of the head, $a \in \mathrm{A} = \{1,...,T\}$. We assume that $\lambda(a)$ is not only the probability for a particular individual of survival, but also the deterministic fraction of agents that survive until age $a+1$ having already survived until age a. Each period, a measure $\mu_1 = (1 + \sum_{\kappa=A}^{T} \prod_{a=A}^{\kappa} \lambda(a))^{-1}$ is born; so the population of agents in the economy is stationary.

2.1 Technology

There are two goods in the economy: a non-durable, globally available, consumption good and a durable housing good. The housing good is island specific and in fixed supply, $H(\varepsilon) = \overline{H} \quad \forall \varepsilon \in \mathscr{E}$, on each island. Housing is "putty" within an island; households choose housing $h \in \mathrm{H}$.

The consumption good is produced according to a production function $F(K,L)$, where K is the aggregate capital stock and L is the stock of available efficiency labor units in the entire economy[4] and F is strictly increasing and concave in both arguments. A consumption good produced can be consumed in that period, converted into capital next period, K', or used to maintain the housing stock or used up in transaction costs. Capital depreciates at rate δ. The aggregate resource constraint for the consumption good is:

$$C + K' - (1-\delta)K = F(K,L)$$

where C refers to consumption used for all purposes except investment in capital goods.

2.2 Preferences

The household derives utility from housing and from the consumption good, which is the numeraire good. Preferences are time-separable where β is the time discount factor. The instantaneous utility function $u(c,h)$ is strictly increasing and concave.

[3] Judd [1985] notes two technical issues with continuums of independent random numbers relating to the Law of Large Numbers and measurability. We follow the approach of Miao [2006], Karatzas et al. [1994], Feldman and Gilles [1985] in dealing with these problems.

[4] i.e. capital is fully mobile across islands

2.3 Labor productivity and job offers

Each island has a productivity, indexed by j, which follows a finite state Markov chain with state space $j \in \mathrm{J} \equiv \{1,...,J\}$ and transition probabilities given by the matrix $\pi_J(j'|j)$. Let Π_J denote the unique invariant measure associated with π_J.

Similarly each household has an ability, indexed by i, which follows a Markov chain with state space $i \in \mathrm{I} \subset [-I,I]$ and transition probabilities given by the matrix $\pi_I(i'|i)$. The initial realization of a newborn household's ability is assumed to be drawn from the distribution Π_I for all households.

Households are endowed with one unit of time per period supplied inelastically in the labor market. A household's effective labor supply in any period is the product of two elements that depend on the household's ability, and the productivity of the island on which it chooses to work:

$$l(i,j) = l_i(i)l_j(j)$$

where $l_i : \mathrm{I} \to \mathbb{R}_{++}$, $l_j : \mathrm{J} \to \mathbb{R}_{++}$ are known functions. $l_j(\cdot)$ is assumed to be an increasing function.

2.4 Assets and Prices

There exists a one-period, risk-free asset b with price $\rho = \frac{1}{1+r}$ (where r is the net interest rate). All firms and households may borrow or lend at this rate however households can only lend. Households choose asset holdings from the set $b' \in \mathrm{B} \equiv [0,\overline{b}]$[5]. Wages per unit of effective labor supply are w.

2.5 Institutional structure of the housing market

A household can only consume housing on the island on which it works. Housing is rented from a risk-neutral, competitive agency in the real estate industry[6]. The household pays $q(j)$ per unit of housing h on island ε of productivity j. Real estate agencies can buy and sell housing on island ε of productivity j at the price $p(j)$.

We assume that all houses require upkeep in consumption goods in an amount proportional to the house value as maintenance: $\delta_h p(j)h$. In addition, we assume that this keeps the house at a

[5]Proofs of boundedness of the choice space in the Bewley-class of models tend to be idiosyncratic . Often times, boundedness is just computationally verified. Of course, boundedness is an important hurdle for any proof of existence. However since the main focus here is on providing a generic (even within a specific example) general equilibrium definition and proof of existence for Bewley models with discrete choices, we assume that the choice space is bounded rather than rig our particular example.

[6]For models with both home ownership and renting see e.g. Chambers et al. [2009b,a], Halket and Vasudev [2011], Iacoviello and Pavan [2009], Rios-Rull and Sanchez-Marcos [2008]

constant quality over time.

Real estate firms are risk-neutral and can borrow at the interest rate, r. The real estate industry is competitive, so the size and number of individual firms is indeterminate.

Real estate firms pay maintenance, and interest and potentially earn capital gains/losses from housing (which we assume that they carry forward). Since real estate is competitive, firms make zero profit on average. The zero-profit condition is:[7]

$$q(j) = (\delta_h + \frac{r}{1+r})p(j) - \frac{1}{1+r}E(p(j') - p(j)|j),$$

where $E(p(j') - p(j)|j)$ is the expected capital gain for a unit of housing on an island of productivity j.

Housing is immovable and, for the households, indivisible; i.e. moving is costly. Any household that moves pays a transaction cost that is a proportion θ_h of the rental cost of the new house. Newborn agents are born with no housing and therefore their initial location is unimportant (since they will pay the moving costs regardless).

2.6 Timing

The timing within a period is as follows:

1. Some households die. A household of age a enters the period, observes its ability i, and its island's productivity j. All of the dead households' housing stock is sold to the real estate agency.

2. The household chooses to locate/work on island ε' which is of productivity $\tilde{j}$. If the household moves, the household chooses home size h' and how many consumption goods c to consume this period and next period's financial assets b'. If it chooses to stay in its current housing, the household only chooses c and b'.

3. Efficiency labor units and capital are supplied.

4. Factor payments are made, and consumption and housing services are consumed.

All information is commonly known and all decisions are publicly observable.

[7] We allow households to move frictionlessly between rental houses of the same size on the same island, so ruling out any loss-leader, teaser-rate type rental prices.

3 Definition of Equilibrium

Section 3 and 4 respectively defines and proves the existence of a stationary competitive equilibrium for the economy. Since our model has both discrete and continuous state variables the proof of existence of an equilibrium correspondingly differs from the one in Aiyagari [1994]. Our proof involves a selection of state-contingent action plans in areas of indifference. In order to formalize this, we introduce mixed allocations which will serve as tie-breaking criteria[8]. Since our economy is populated by a continuum of agents, there is no aggregate uncertainty using a mixed allocation.

3.1 Mixed allocations and the distribution of households

Definition. The state space $S = \mathrm{A} \times \mathrm{I} \times \mathrm{J} \times \mathrm{H} \times \mathrm{B} \times \mathscr{E}$. A state can be written as $s = (a, i, j, h, b, \varepsilon) \in \mathrm{S}$. Let $\overline{j} : \mathscr{E} \to J$ be the (stochastic) function that maps an island to its productivity. The vector of house prices is $(p)_J = (p_1, ...p_J)$. The vector of rents is $(q)_J = (q_1, ...q_J)$. The price vector, $\overrightarrow{p} = ((p)_J, (q)_J, r, \omega) \in \mathrm{P} \subset \mathbb{R}^{2J+2}$. $\mathrm{Y} = \mathrm{H} \times \mathrm{B} \times \mathrm{J} \times \mathscr{E} \times \mathrm{C}$ is the choice space. $y = (h', b', \tilde{j}, \varepsilon', c) \in \mathrm{Y}$ is a particular choice vector.

Since $\mathrm{Y} \in \mathbb{R}^n$, $(\mathrm{Y}, \mathscr{B}(\mathrm{Y}))$ is a measure space[9] where $\mathscr{B}(\mathrm{Y})$ is the standard Borel space on Y. We can then define the probability space, and a mixed allocation as an element of the probability space.

Definition. Let $\tilde{\Lambda}$ be the set of probability measures on Y, with elements $\lambda^y : \mathscr{B}(\mathrm{Y}) \to [0,1]$. Let Δ be the space of functions $f : \mathrm{S} \to \tilde{\Lambda}$.

Now we can define a mixed allocation as a state-contingent distribution over optimal choices.

Definition. A *mixed allocation*, $\alpha : \mathrm{S} \times \mathrm{P} \to \tilde{\Lambda}$ is map that specifies the probability distribution over the optimal choice set given by $Y(s, \overrightarrow{p})$.

$$\alpha(s, \overrightarrow{p}) \in \{\tilde{\alpha} \in \tilde{\Lambda} : supp(\tilde{\alpha}) \subseteq Y(s, \overrightarrow{p})\}$$

Let Λ be the space of mixed allocations. We will at times find it convenient to abuse notation by referring to the value of the c.d.f given by $\alpha(s, \overrightarrow{p})$ at a point $\tilde{y}$, $\lambda^y(\tilde{y})$, as $\alpha(s, \overrightarrow{p}, \tilde{y})$.

[8] The mixed allocations do not affect preferences. Rather they will just allocate indifferent households to one or another of the items in the set they are indifferent over.

[9] For a set $X \subseteq \mathbb{R}^n$, we assume that the standard Borel space is used in constructing measure and probability measure spaces. That is, the statement "μ is a probability measure on X" implies that $(X, \mathscr{B}(X), \mu)$ is a probability measure space.

3.2 Stationary competitive equilibrium

In this paper we consider only stationary competitive equilibria. Before we define the equilibrium we set out the notion of the distribution of households over the state space. Our stationary equilibrium requires that this distribution does not change over time.

Definition. The *household distribution over states*, $\mu : \mathscr{B}(\mathrm{S}) \to [0,1]$ is a probability measure on S. Let $\mathscr{M}$ be the space of probability distributions on S.

Definition. A stationary competitive equilibrium is a vector of strictly positive prices, $\vec{p}^*$, a set of correspondences

$$Y^*(s;\overrightarrow{p}^*) = (H^*(s,\overrightarrow{p}^*), B^*(s,\overrightarrow{p}^*), J^*(s,\overrightarrow{p}^*), \varepsilon^*(s,\overrightarrow{p}^*), C^*(s,\overrightarrow{p})),$$

a mixed allocation $\alpha^* \in \Lambda$, a probability measure μ^*, firm capital and labor holdings K^* and L^*, and a $J^*(s;\overrightarrow{p}^*)$ such that:

(i) y^* solves the household's problem 1 for each $y^* \in Y^*(s,\overrightarrow{p})$

(ii) K^* and L^* solve the firm's optimization problem:

$$r^* + \delta = F_K(K^*, L^*)$$

$$w^* = F_L(K^*, L^*)$$

(iii) Goods market clears:

$$F(K^*, L^*) \;=\; \delta K^* + \int_{\mathrm{S}} \int_{Y^*(s,\vec{p}^*)} (c^* + 1^*_m hq^*(j^*)\theta_h + p^*(j^*)h^*\delta_h) d\alpha^*(s,\overrightarrow{p}^*, y^*) d\mu^*$$

(iv) Capital market clears:

$$K^* = \int_{\mathrm{S}} \int_{Y^*(s,\overrightarrow{p}^*)} (b^* + (q(j^*) - (1+\delta_h)p^*(j^*))h^*) d\alpha^*(s,\overrightarrow{p}^*, y^*) d\mu^*$$

(v) Labor market clears:

$$L^* \;=\; \int_{\mathrm{S}} \int_{Y^*(s,\vec{p}^*)} l(i, j^*) d\alpha^*(s,\overrightarrow{p}^*, y^*) d\mu^*(s)$$

(vi) Housing market clears:

$$\overline{H} = H(\varepsilon) = \int_{S} \int_{Y^*(s,\vec{p}^*)} h^* \cdot 1\{\varepsilon^* = \varepsilon\} d\alpha^*(s, \overrightarrow{p}^*, y^*) d\mu^*(s) \qquad \forall \varepsilon \in \mathscr{E}$$

(viii) Zero profits in the real estate sector:

$$q(j) = (\delta_h + \frac{r^*}{1+r^*})p^*(j) - \frac{1}{1+r^*}E(p^*(j') - p^*(j)|j), \qquad \forall j \in J$$

(ix) Steady-state distribution:

$$\mu^* = \Upsilon_{p,\alpha}\mu^*$$

where $\Upsilon_{p,\alpha}$ is the transition function generated by the optimal choice correspondence of the household and the mixed allocation, α.

4 Existence of Equilibrium

We use Kakutani's FPT in order to establish the existence of a stationary CE. The proof can be broadly divided into four steps:

1. Show that the household's problem is well-defined and has a solution.
2. The optimal policy function generates a transition function for the household distribution over states. We show that there is a household distribution over states that is invariant with respect to the transition function.
3. Show that the set of stationary household distributions over states is upper hemi-continuous in the price vector.
4. Construct a price transition operator and show this map has a fixed point using Kakutani's theorem.

Our innovation is to add as an equilibrium object mixed allocations over the optimal choice set, which act as tie-breaking criteria. This gives us a convex (probability) space of optimal choices and a convex set of macroeconomic variables. We show that this is sufficient to satisfy the conditions necessary for Kakutani's theorem to derive a stationary competitive equilibrium[10].

[10] We use $\exists!$ to denote "there exists a unique". We will be dealing with convergent sequences in $\mathbb{R}^n$ throughout this proof. We follow the convention that if the space in question, $S \subseteq \mathbb{R}^n$, the metric is the standard metric on $\mathbb{R}^n$. In addition, if the space in consideration is a probability space then the corresponding metric is the sup-norm. Any non-standard metrics will be indicated in the proof. We simplify the feasible correspondence by eliminating c from the budget constraint, using the non-satiation of preferences.

Definition. The period utility function $u : \mathrm{S} \times \mathrm{Y} \to \mathbb{R}$ is defined implicitly by the utility function $u(\cdot,\cdot)$. A *state-contingent action plan* is a function $\tilde{y} : \mathrm{S} \times \mathrm{P} \to \mathrm{Y}$ that specifies the household's choice at every state given the price vector.

4.1 The household's problem

The value function, $V : \mathrm{S} \times \mathrm{P} \to \mathbb{R}$ and the optimal policy correspondence, $Y : \mathrm{S} \times \mathrm{P} \rightrightarrows \mathrm{Y}$ are given by the following:

$$V(s;\overrightarrow{p}) \quad = \quad \sup_{y \in \Gamma(s;\overrightarrow{p})} u(s,y) + \beta \lambda(a) E[V(s')|s,y]$$

where $\Gamma : \mathrm{S} \times \mathrm{P} \rightrightarrows \mathrm{Y}$ is given by

$$\begin{gathered} c + \rho b' + h'(1 + 1_m \theta_h) q(\tilde{j}) + \leq wl(i,\tilde{j}) + b \\ b' \in \mathrm{B} \\ c \geq 0 \\ h' \geq 0 \\ \bar{j}(\varepsilon') = \tilde{j} \end{gathered}$$

$$1_m = \begin{cases} 0 & \text{if } h' = h, \varepsilon' = \varepsilon \\ 1 & \text{else} \end{cases}$$

Remark. Given the assumptions above, S, Y are compact.

Lemma 1. Γ *is continuous.*

Proof. Omitted □

Theorem 2. *(Theorem of the Maximum). There is a solution to the household's problem such that*

1. $\exists! V : \mathrm{S} \times \mathrm{P} \to \mathbb{R}$ that solves the household problem.

2. The optimal policy correspondence, $Y : \mathrm{S} \times \mathrm{P} \rightrightarrows \mathrm{Y}$ is non-empty, compact-valued and upper hemi-continuous.

Proof. $u(\cdot,\cdot)$ is continuous and Γ is compact-valued and upper hemi-continuous. So Berge's maximum theorem applies and the result follows (see Stokey et al. [1989], Thm. 3.6). □

When a household moves to an island of productivity j, we can see from the household problem that it is indifferent between all islands of that productivity. We formalize this below.

Fact. *For* $\forall s, \vec{p}$, *let* $y(s;\overrightarrow{p}) = (h'(s;\overrightarrow{p}), b'(s;\overrightarrow{p}), \tilde{j}(s;\overrightarrow{p}), \varepsilon'(s;\overrightarrow{p})) \in Y(s;\overrightarrow{p})$. If $h'(s;\overrightarrow{p}) \neq h$ or $\varepsilon'(s;\overrightarrow{p}) \neq \varepsilon$ then

$$\hat{y}(s;\overrightarrow{p}) = (h'(s;\overrightarrow{p}), b'(s;\overrightarrow{p}), \tilde{j}(s;\overrightarrow{p}), \hat{\varepsilon}(s;\overrightarrow{p})) \in Y(s;\overrightarrow{p}) \ \forall \hat{\varepsilon} : \bar{j}(\hat{\varepsilon}) = \tilde{j}$$

Definition. Given a price $\overrightarrow{p}$ and mixed allocation α, the household transition function for survivors, $GS_{\overrightarrow{p},\alpha} : \mathrm{S} \times \mathscr{B}(\mathrm{S}) \to [0,1]$ is defined as

$$GS_{\overrightarrow{p},\alpha}(s,S') = \int_{\tilde{y} \in Y(s;\overrightarrow{p})} \int_{s' \in S'} 1\{\tilde{h} = h', \tilde{b} = b', \tilde{\varepsilon} = \varepsilon', a' = a+1\} \pi_i(i'|i) \pi_j(j'|\tilde{j}) ds' d\alpha(s, \overrightarrow{p}, \tilde{y})$$

where, e.g., $\tilde{h}$ refers to the element of $\tilde{y}$

The transition function for newborns, $GN : \mathscr{B}(\mathrm{S}) \to [0,1]$ is defined as

$$GN(S') = \int_{s' \in S'} 1\{h' = 0, a' = 1, j' = \bar{j}(\varepsilon')\} \pi_i(i'|i) f_{\varepsilon}(\varepsilon') ds'$$

The complete transition function, $G : \mathrm{S} \times \mathscr{B}(\mathrm{S}) \to [0,1]$ is defined as

$$G_{\overrightarrow{p},\alpha}(s,S') = \lambda(a) GS_{\overrightarrow{p},\alpha}(s,S') + (1-\lambda(a)) GN(S')$$

Given a price vector $\overrightarrow{p}$ and a mixed allocation α, the operator $\Upsilon_{\overrightarrow{p},\alpha} : \mathscr{M} \to \mathscr{M}$ is defined by the transition function G and gives the household distribution over the next period's states

$$\Upsilon_{\overrightarrow{p},\alpha}(\mu_{\overrightarrow{p},\alpha})(S') = \int_{s \in S} G_{\overrightarrow{p},\alpha}(s,S') d\mu_{\overrightarrow{p},\alpha}$$

Theorem 3. *[Existence of a unique invariant household distribution] For each* $\overrightarrow{p} \in \mathrm{P}$ *and* $\alpha \in \Lambda$, $\exists! \ \mu_{\overrightarrow{p},\alpha} \in \mathscr{M}$ *s.t.* $\Upsilon_{\overrightarrow{p},\alpha}(\mu_{\overrightarrow{p},\alpha}) = \mu_{\overrightarrow{p},\alpha}$

Proof. We use Theorem 11.10 of Stokey et al. [1989]. First, we show that $G_{\overrightarrow{p},\alpha}$ satisfies Doeblin's condition. From exercise 11.4g of Stokey et al. [1989], it is sufficient to show that GN satisfies Doeblin's condition. We must show that there exists a finite measure η on $(\mathrm{S}, \mathscr{B}(\mathrm{S}))$, an integer $N \geq 1$ and a number $n > 0$ such that if $\eta(z) \leq n$ then $GN^N(s,S') \leq 1-n \ \forall s \in S$. Set $\eta(S') = GN(S')$. Then we can see that GN satisfies Doeblin's condition for $N = 1$ and $n < 1/2$. This guarantees the existence of an invariant distribution.

Observe also that if $\eta(S') > 0$, then $G_{\overrightarrow{p},\alpha}(s,S') \geq (1-\lambda(a)) GN(S') > 0$. This implies that the invariant distribution is unique. □

Lemma 4. *If $\{(s_n,\vec{p}_n)\}$ is a sequence in $S\times \mathrm{P}$ converging to $(s_0,\vec{p}_0)$ then there exists a sequence $\{\alpha_n\}$ that converges to α_0 such that $G_{\vec{p}_n,\alpha_n}(s_n,\cdot)$ converges weakly to $G_{\vec{p}_0,\alpha_0}(s_0,\cdot)$.*

Proof. GN is independent of $\vec{p}$, so

$$\lim_{n\to\infty} GN(S') = GN(S')\ \forall S' \in \mathscr{B}(\mathrm{S})$$

WLOG we can focus on sequences where the discrete states remain the same. Since Y is upper hemi-continuous in s and $\vec{p}$, $\exists\,(y_n(s_n,\vec{p}_n)) \to y(s_0,\vec{p}_0)$. Pick $\alpha(s_n,\vec{p}_n) = 1\{y_n(s_n,\vec{p}_n)\}$. Then,

$$\lim_{n\to\infty} GS_{\vec{p}_n,\alpha_n}(s_n,S') = GS_{\vec{p}_0,\alpha_0}(s_0,S')\ \forall S' \in \mathscr{B}(\mathrm{S})$$

□

Lemma 5. *Take a sequence of $\{\vec{p}_n\} \in \vec{P} \to \vec{p}_0$. Then, $\exists\{\alpha_n\} \to \alpha_0$ such that*

$$\mu_{\vec{p}_n,\alpha_n} \to \mu_{\vec{p}_0,\alpha_0}$$

where $\mu_{\vec{p}_n,\alpha_n} = \Upsilon_{\vec{p}_n,\alpha_n}\mu_{\vec{p}_n,\alpha_n}$ and $\mu_{\vec{p}_0,\alpha_0} = \Upsilon_{\vec{p}_0,\alpha_0}\mu_{\vec{p}_0,\alpha_0}$.

Proof. Theorem 3 and Lemma 4 are sufficient to use Theorem 12.13 of Stokey et al. [1989] which gives us the result. □

So far we have shown that a solution to the household problem exists, and that given any price vector, we can find a unique stationary distribution of households over the state space. Significantly, we have shown that the set of transition functions and the set of household distributions as a function of price are upper hemi-continuous. The next step is to define the aggregate variables and show that they are bounded as well. In what follows, we use $\mu_{\overrightarrow{p},\alpha}$ to represent the invariant household distribution given $\overrightarrow{p}$ and α.

Remark. Equilibrium prices, $\overrightarrow{p}^*$, lie within the following bounded set.

The bounds on labor supply are set as follows: $\overline{L} = \overline{l} = \max(l(i,j))$ and $\underline{L} = \underline{l} = \min(l(i,j))$. A lower and upper bounds on wages $\underline{w} = 0$, $\overline{w} = F_l(\overline{b},\underline{l})$ and on the price of savings $\underline{\rho} = 0$, $\overline{\rho} = \frac{1}{1-\delta}$ follow from the firm's problem. A lower bound on rents is $\underline{q} = 0$. The upper bound on rental price, $\overline{q}$, is set subject to the following condition: $\overline{q}\overline{H}\min_j \Pi_j > \overline{b} + \overline{w}\overline{l}$. This condition states that there is some rental price, $\overline{q}$, at which all the housing cannot be bought with the maximum wealth in hand. This ensures that there is a price vector where the optimal housing demand is less than the housing supply. A lower bound on the price of housing is $\underline{p} = 0$.

From a real estate firm's zero profit condition, we get:

$$q(j) + E(p(j') - p(j)|j) = (\delta_h + \frac{r}{1+r})p(j)$$

Now consider $j = J$. On the highest island productivity the expected capital gain on housing is negative, so that

$$p(J) \quad \leq \quad \frac{q(J)}{\delta_h}$$

So we set $\overline{p}(J) = \frac{\overline{q}}{\delta_h}$. But since $p(j) \leq p(J)$, we set the upper bound on the house price space $\overline{p} = \frac{\overline{q}}{\delta_h}$. So the upper bound on rent gives us the upper bound on the housing price as well.

Definition. Aggregate capital supply, labor supply and housing demand are given by

$$K(\vec{p},\alpha) \quad = \quad \int_S \int_{y^* \in Y(s,\vec{p})} (b^* + (q(j^*) - (1+\delta_h)p(j^*))h^*) d\alpha(s,\overrightarrow{p},y^*) d\mu_{\vec{p},\alpha}(s)$$

$$L(\vec{p},\alpha) = \int_S \int_{Y^*(s,\vec{p})} l(i,\tilde{j}) d\alpha(s,\overrightarrow{p},y^*) d\mu_{\vec{p},\alpha}(s)$$

$$H(j;\vec{p},\alpha) = \int_S \int_{Y^*(s,\vec{p})} h^* 1\{\tilde{j}^* = j\} d\alpha(s,\overrightarrow{p},y^*) d\mu_{\vec{p},\alpha}(s)$$

Remark. $K(\overrightarrow{p},\alpha), L(\overrightarrow{p},\alpha), H(j;\overrightarrow{p},\alpha)$ are continuous in α given $\overrightarrow{p}$.

Lemma 6. *There exist* $\overline{K}, \overline{L}, \overline{H} < \infty$ *and* $\underline{L} > 0$ *such that* $K(\vec{p},\alpha) \leq \overline{K}$, $\underline{L} \leq L(\vec{p},\alpha) \leq \overline{L}$, $H(j;\vec{p},\alpha) \leq \overline{H}$ *for all* $(\vec{p},\alpha)$.

Proof. Since the state space is bounded,

$$\begin{aligned}
K(\vec{p},\alpha) &\leq \int_S \int_{Y^*(s,\vec{p})} (\overline{b} + \overline{q}\overline{h}) d\alpha(s,\overrightarrow{p},y^*) d\mu_{\vec{p},\alpha}(s) \leq \overline{b} + \overline{q}\overline{h} \\
K(\vec{p},\alpha) &\geq \int_S \int_{Y^*(s,\vec{p})} \underline{b} - (1+\delta_h)\overline{h} d\alpha(s,\overrightarrow{p},y^*) d\mu_{\vec{p},\alpha} \geq \underline{b} - (1+\delta_h)\overline{h} \\
L(\vec{p},\alpha) &\leq \int_S \int_{Y^*(s,\vec{p})} \overline{l} d\alpha(s,\overrightarrow{p},y^*) d\mu_{\vec{p},\alpha}(s) = \overline{l} \\
L(\vec{p},\alpha) &\geq \int_S \int_{Y^*(s,\vec{p})} \underline{l} d\alpha(s,\overrightarrow{p},y^*) d\mu_{\vec{p},\alpha}(s) = \underline{l} \\
H(j;\vec{p},\alpha) &\leq \int_S \int_{Y^*(s,\vec{p})} \overline{h} d\alpha(s,\overrightarrow{p},y^*) d\mu_{\vec{p},\alpha}(s) = \overline{h} \\
H(j;\vec{p},\alpha) &\geq 0
\end{aligned}$$

□

We are now in a position to define the price transition function and examine its properties.

Definition. We define the components of the price transition correspondence as follows:

$$\Omega^w : \vec{P} \times \Lambda \to \mathbb{R}, \text{ where } \Omega^w(\vec{p},\alpha) = \begin{cases} F_L(K(\vec{p},\alpha), L(\vec{p},\alpha)) & \text{if } K(\vec{p},\alpha) > 0 \\ F_L(0, L(\vec{p},\alpha)) & \text{if } K(\vec{p},\alpha) \leq 0 \end{cases}$$

$$\Omega^\rho : \vec{P} \times \Lambda \to \mathbb{R}, \text{ where } \Omega^\rho(\vec{p},\alpha) = \begin{cases} (1 + F_K(K(\vec{p},\alpha), L(\vec{p},\alpha)) - \delta)^{-1} & \text{if } K(\vec{p},\alpha) > 0 \\ 0 & \text{if } K(\vec{p},\alpha) \leq 0 \end{cases}$$

It is convenient to define $\Omega^h : \mathrm{P} \to \mathbb{R}$ iteratively. Set $p_0 = \underline{p}$, $p'_0 = \underline{p}$, $p_{J+1} = \overline{p}$, $p'_{J+1} = \overline{p}$. Then, to get p', we start from $j = 1$, and proceed to $j = J$.

$$\begin{aligned} p'_j \;\; \in \;\; & [p'_{j-1}, p_{j+1}] = p_j + 1\{H_j > \overline{H}_j\}(p_{j+1} - p_j) \cdot \frac{H_j - \overline{H}_j}{H_j} \\ & -1\{H_j \leq \overline{H}_j\}(p_j - p'_{j-1}) \cdot \frac{\overline{H}_j - H_j}{\overline{H}_j};\; j = \{1, J\} \end{aligned}$$

where $\overline{H}_j = \overline{H}\Pi_j(j)$ and $H_j = \int h^* 1\{j^* = j\} d\mu^*$.

Definition. $\Omega^h : \mathrm{P} \times \Lambda \to \mathbb{R}^J$ is defined as $\Omega^h_j(\overrightarrow{p}, \alpha) = p'_j$

$\Omega^p : \mathrm{P} \times \Lambda \to \mathrm{P}$ is defined as $\Omega^p(\overrightarrow{p}, \alpha) = (\Omega^w(\overrightarrow{p}, \alpha), \Omega^\rho(\overrightarrow{p}, \alpha), \Omega^h(\overrightarrow{p}, \alpha))$

The *price transition correspondence*, $\Omega : \mathrm{P} \to \mathbb{R}^{J+2}$ is defined as

$$\Omega(\vec{p}) = \{\Omega^p(\overrightarrow{p}, \alpha) : \alpha \in \Lambda\}$$

Define $\tilde{\Omega}^x(\overrightarrow{p}) = \{\Omega^x(\overrightarrow{p}, \alpha) : \alpha \in \Lambda\}$ for $x \in \{w, \rho\}$ and $\tilde{\Omega}^h_j(\overrightarrow{p}) = \{\Omega^h_j(\overrightarrow{p}, \alpha) : \alpha \in \Lambda\}$ $\forall j \in \mathrm{J}$. These are corresponding components of $\Omega(\overrightarrow{p})$.

Lemma 7. *Under the conditions above, the price transition function, Ω, is a self-map: range$(\Omega) \subseteq$ P.*

Proof. First we look at Ω^w. Since $\underline{K} \leq K(\overrightarrow{p}, \alpha) \leq \overline{K}$ and $\underline{L} \leq L(\overrightarrow{p}, \alpha) \leq \overline{L}$,

$$\underline{w} = F_L(0, \overline{L}) \leq F_L(K(\vec{p},\alpha), L(\vec{p},\alpha)) \leq F_L(\overline{K}, \underline{L}) = \overline{w}$$

Hence $range(\Omega^w) \subseteq [\underline{w}, \overline{w}]$.

By construction, $\Omega^\rho(\overrightarrow{p}, \alpha) \geq 0 \, \forall \overrightarrow{p}, \alpha$. Further,

$$\Omega^\rho(\overrightarrow{p}, \alpha) \leq (1 + F_K(\overline{K}, \underline{L}) - \delta)^{-1} = \overline{\rho} \; \forall \overrightarrow{p}, \alpha$$

Therefore, $range(\Omega^\rho) \subseteq [0, \overline{\rho}]$.

By construction, $\Omega^h_{j-1} \leq \Omega^h_j \leq \Omega^h_{j+1} \ \forall j \in \{2,...J-1\}$ and $\underline{p} \leq \Omega^h_1 \leq \Omega^h_2$ and $\Omega^h_{J-1} \leq \Omega^h_J \leq \overline{p}$. Hence $range(\Omega^h) \subseteq [\underline{p}, \overline{p}]$.

From the definition of Ω^p, we get that $range(\Omega^p) \subseteq \mathrm{P}$. This implies that $\Omega(\overrightarrow{p}) \subseteq \mathrm{P} \ \forall \overrightarrow{p} \in \mathrm{P}$. □

Lemma 8. *Let Γ be a correspondence- $\Gamma : X \Rightarrow Y$, $f : X \times Y \to Z$, f continuous in y, and $\Gamma' : X \Rightarrow Z$, where $\Gamma'(x) = \{z : \exists y : y \in \Gamma(x)$ and $z = f(x,y)\}$. Then the following holds:*

1. If Γ is compact-valued, then Γ' is compact-valued also.

2. If Γ is upper hemi-continuous, then Γ' is upper hemi-continuous also.

Proof. Using Ok [2004], Prop. 3, Ch. D3, Pg 222, f takes compact sets to compact sets. Hence Γ' is compact-valued.

Pick $(x_m) \to x$ and $(z_m) \in \Gamma'(x_m) \forall m$. We want to show that there is a subsequence $(z_{m_k}) \to z \in \Gamma'(x)$. Since f is a function, for every $z_m \ \exists y_m$ such that $z_m = f(x_m, y_m)$, which implies that $y_m \in \Gamma(x_m) \ \forall m$. Since Γ is uhc, $\exists$ a subsequence $(y_{m_k}) \to y \in \Gamma(x)$. From continuity of f, the subsequence $z_{m_k} = f(x_{m_k}, y_{m_k}) \to z = f(x,y)$. So $z \in \Gamma'(x)$, hence Γ' is uhc. □

Definition. $K^p : \mathrm{P} \rightrightarrows \mathbb{R}$ is defined as $K^p(\overrightarrow{p}) = \{k : k = K(\overrightarrow{p}, \alpha) \ \alpha \in \Lambda\}$. H^p_j and L^p are defined analogously.

Lemma 9. *$K^p, \{H^p_j\}^J_{j=1}, L^p$ are upper hemi-continuous and close-valued.*

Proof. We show the result for $H_j(\vec{p})$. The proof for the remaining is analogous. From Theorem (2), Y changes continuously in $\vec{p}$. Now let $\lambda_p : \mathrm{P} \rightrightarrows \Delta$, where $\lambda_p(\overrightarrow{p}) = \Lambda_{\overrightarrow{p}}$.

First, we show that λ_p is upper hemi-continuous. Since λ_p is closed, it is sufficient to show that λ_p has a closed graph. Pick a sequence $\vec{p}_n \to \vec{p}$ and $\alpha_n \to \alpha$ with $\alpha_n \in \lambda_p(\vec{p}_n) \ \forall n$. We want to show that $\alpha \in \lambda_p(\vec{p}) \implies supp(\alpha) \subseteq Y(\vec{p})$. Suppose not. Then $\exists y \in supp(\alpha)$ such that $y \notin Y(\vec{p})$. Construct an open set O such that $Y(\vec{p}) \subseteq O$, $y \notin O$. Since Y is upper hemi-continuous, $\exists \delta$ such that $Y(N_\delta(\vec{p})) \subseteq O$. Since $\vec{p}_n \to \vec{p}$, $\exists N$ such that $\forall n \geq N, y \notin Y(\vec{p}_n) \implies d^s(\alpha_n, \alpha) \geq \alpha(y) > 0$, where $d^s(\cdot,\cdot)$ refers to the sup norm. This contradicts the fact that $\alpha_n \to \alpha$. Hence λ_p is upper hemi-continuous.

$\lambda_p : \vec{P} \rightrightarrows \Delta$ and $H_j(\overrightarrow{p}, \alpha) : \mathrm{P} \times \Delta \to \mathbb{R}$. From the definition of H_j, H_j is continuous. Hence, using Lemma (8), $H^p_j(\vec{p}) = H_j(\overrightarrow{p}, \Lambda(\vec{p}))$ is upper hemi-continuous. Since λ_p is close-valued and $H_j(\overrightarrow{p}, \alpha)$ is continuous, H^p_j is close-valued. □

Lemma 10. *Ω is upper hemi-continuous, convex-valued and close-valued.*

Proof. To show that Ω is convex-valued, we show that $\tilde{\Omega}^w, \tilde{\Omega}^\rho$, and $\tilde{\Omega}_j^h$ are convex-valued. We prove this for $\tilde{\Omega}_\alpha^w$; the proof for the rest is analogous.

$$\Omega^w(\overrightarrow{p}, \alpha) = \begin{cases} F_L(K^*(\overrightarrow{p}, \alpha), L^*(\overrightarrow{p}, \alpha)) & \text{if } K^*(\overrightarrow{p}, \alpha) > 0 \\ F_L(0, L^*(\overrightarrow{p}, \alpha)) & \text{if } K^*(\overrightarrow{p}, \alpha) \leq 0 \end{cases}$$

Now, K^* and L^* are continuous in α, and F_L is continuous in both its arguments. Therefore Ω^w is continuous in α. Since Λ is a convex set,

$$\Omega^w(\vec{p}) = range(\Omega^w(\alpha))$$

is a convex set. The same reasoning applies for the other components of $\Omega(\vec{p})$. Therefore, $\Omega(\vec{p})$ is convex-valued.

From the definition of $\Omega(\vec{p})$, it is a continuous transformation of Δ. Using Lemma (8), we get that $\Omega(\vec{p})$ is upper hemi-continuous. Since Δ is compact and Ω is a continuous transformation, the image is compact as well. □

Theorem 11. *A competitive equilibrium exists.*

Proof. P is a convex and compact space. Ω is convex-valued, and since it is upper hemi-continuous and compact-valued, it has a closed graph. Using Kakutani's FPT, $\exists \vec{p} \in$ P such that $\vec{p} \in \Omega(\vec{p})$. This implies that $\exists \alpha$ such that $\vec{p} = \Omega^p(\vec{p})$. The aggregate capital and labor supply equations are satisfied by $K^*(\vec{p}, \alpha)$ and $L^*(\vec{p}, \alpha)$. The aggregate housing demand on all islands of productivity j is $H_j(\vec{p}, \alpha)$. Households are indifferent between islands with the same productivity when moving (Fact (4.1)). Let $\hat{\alpha}$ be such that

$$\int_0^1 1\{\bar{j}(\varepsilon) \quad = \quad \bar{j}(\varepsilon')\} GS_{\vec{p},\hat{\alpha}}(s, S') d\varepsilon \quad = \quad \int_0^1 1\{\bar{j}(\varepsilon) \quad = \quad \bar{j}(\varepsilon')\} GS_{\vec{p},\alpha}(s, S') d\varepsilon$$

and

$$\int_0^1 1\{\bar{j}(\varepsilon) \quad = \quad \bar{j}(\varepsilon')\} GN_{\vec{p},\hat{\alpha}}(S') d\varepsilon \quad = \quad \int_0^1 1\{\bar{j}(\varepsilon) \quad = \quad \bar{j}(\varepsilon')\} GN_{\vec{p},\alpha}(s, S') d\varepsilon$$

for all $s \in S$ and $S' \in B(S)$. Then, if $\alpha :\ \vec{p} = \Omega^p(\vec{p})$, then $\hat{\alpha} :\ \vec{p} = \Omega_{\alpha'}(\vec{p})$. So $\exists\ \alpha :\ \vec{p} = \Omega_\alpha(\vec{p})$ and

$$\overline{H} = H(\varepsilon) = \int_{s \in S} \int_{y^* \in Y^*(s, \vec{p}^*)} h^* \cdot 1\{\varepsilon^* = \varepsilon\} d\alpha'(y^*) d\mu^*(s) \qquad \forall \varepsilon \in [0, 1]$$

By Walras' Law, the goods market clears. Hence all the conditions for a competitive equilibrium are satisfied. □

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