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Drinfel'd-Sokolov construction and exact solutions of vector modified KdV hierarchy

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Abstract

We construct the hierarchy of a multi-component generalisation of modified KdV equation and find exact solutions to its associated members. The construction of the hierarchy and its conservation laws is based on the Drinfel'd-Sokolov scheme, however, in our case the Lax operator contains a constant non-regular element of the underlying Lie algebra. We also derive the associated recursion operator of the hierarchy using the symmetry structure of the Lax operators. Finally, using the rational dressing method we obtain the one-soliton solution and we find the one-breather solution of general rank in terms of determinants.

1. Introduction

In this paper we explore certain integrability aspects of the vector modified Korteweg-de Vries equation (vmKdV)

$$\mathbf{u}_t + \mathbf{u}_{xxx} + \tfrac{3}{2}\|\mathbf{u}\|^2\mathbf{u}_x = 0\,, \quad \text{with} \quad \mathbf{u} = \mathbf{u}(x,t) \in \mathbb{R}^N\,, \tag{1}$$

such as the algebraic structure of its Lax representation, its integrable hierarchy, as well as soliton-type solutions constructed using Darboux transformations. Here and in what follows, we denote vectors by boldface, subscripts x and t mean partial differentiation with respect to

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the corresponding variable, and the upper index T denotes transposition, so that the standard Euclidean norm is $\|\mathbf{u}\| = \sqrt{\mathbf{u}^T\mathbf{u}}$. When $N = 1$, equation (1) reduces to the well-known modified KdV equation (mKdV)

$$u_t + u_{xxx} + \tfrac{3}{2}u^2 u_x = 0\,,$$

while for $N = 2$ one obtains the complex mKdV (also known as Hirota equation [18])

$$v_t + v_{xxx} + \tfrac{3}{2}|v|^2 v_x = 0$$

for the complex field $v = u_1 + iu_2$.

Equation (1) appeared in [21,33] in the study of the evolution of a curve in a $(N + 1)$-dimensional Riemannian manifold, while several generalisations of the mKdV equation have been introduced by various authors such as, for example, Iwao and Hirota [19], Fordy and Athorne [4] in association with symmetric spaces, Sokolov and Wolf [35] using the symmetry approach, Svinolupov and Sokolov [36,37] in relation to Jordan algebras, and non-associative algebras in general [34]. Moreover, the study of soliton solutions and their interactions for several multi-component generalisations of the scalar mKdV equation have been studied using the inverse scattering transform, or the Hirota, dressing, or other methods, see for example [3,11,19, 31,38,39]. For the case of the vmKdV equation (1) its Lax representation, recursion operator and Hamiltonian structure were presented by Sanders and Wang in [33], while the recursion operator and Hamiltonian structure were further investigated by Anco in [2]. Moreover, in [20], efficient numerical integration schemes of (1) were considered. We note here that the vmKdV equation which is the third member of the vector nonlinear Schrödinger (NLS) hierarchy (see for example [1]) is of the form

$$\mathbf{u}_t + \mathbf{u}_{xxx} + \tfrac{3}{4}\mathbf{u}^T\mathbf{u}_x\mathbf{u} + \tfrac{3}{4}\|\mathbf{u}\|^2\mathbf{u}_x = 0\,,$$

thus, different from the vmKdV equation that we consider in this paper. A relation between (1) and the NLS equation in the case when $N = 2$ is discussed in Section 3.3.

In this work we construct the hierarchy associated to equation (1) and derive the one-soliton and the general rank one-breather solution for the whole hierarchy. A method for constructing integrable hierarchies based on a given Lie algebra $\mathfrak{g}$ was introduced by Drinfel'd and Sokolov in [8,9], and later explored further in e.g. [7,10,26,40], and relies on the idea of dressing or formal (in the spectral parameter) Darboux transformations. A basic ingredient in the construction presented in these works is a Lax operator containing a constant regular element of $\mathfrak{g}$. Here, the construction of the hierarchy and its corresponding conservation laws are derived based on the ideas of the seminal works [8,9], considering, however, a Lax operator with a constant non-regular element of the underlying Lie algebra. The scattering theory of Lax operators containing a non-regular element of the Lie algebra has appeared in [13]. An additional property of the Lax operator related to the vmKdV equation (1) is its invariance under the Cartan involution introduced in Section 2. In particular, in Section 2 we present the Lax structure for the vmKdV equation (1) and its associated symmetries that will be extensively used in the following sections. Then, Section 3 is dedicated to the construction of the vmKdV hierarchy and its conservation laws, while the commutativity of the members of the hierarchy is also proved. The recursion operator for the hierarchy is also derived using an alternative method than those used in [2,33]. As mentioned previously, a relation to the NLS equation is discussed in Section 3.3. Finally, in Section 4, we employ the method of rational dressing [41,42], which is based on the concept of Darboux transformations, in order to construct solutions for the hierarchy. We derive the one-soliton solution, as well as the one-breather solution of general rank in terms of determinants.

2. Lax structure

We consider the following differential operator

$$\mathcal{L}(\lambda) = D_x - \mathcal{U}(\lambda) \quad \text{with} \quad \mathcal{U}(\lambda) = \lambda J + U\,, \tag{2}$$

which constitutes the spatial part of the Lax pair associated to the vmKdV equation (1), see [33]. We denote by D_x the operator of partial differentiation with respect to $x \in \mathbb{R}$, while J and U take values in the Lie algebra $\mathfrak{g} := \mathfrak{so}_{N+2}(\mathbb{R})$. In particular, J is constant while U depends on x through the potential $\mathbf{u}$ and are of the form

$$J = \begin{pmatrix} 0 & 1 & \mathbf{0}^T \\ -1 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \mathbb{O} \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{u}^T \\ \mathbf{0} & -\mathbf{u} & \mathbb{O} \end{pmatrix}, \tag{3}$$

where $\mathbb{O}$ stands for the $N \times N$ zero matrix. When there is no ambiguity we will use 0 to denote a square zero matrix of any dimension. Also, λ is a complex spectral parameter of the linear spectral problem $\mathcal{L}(\lambda)\Psi = 0$, $\mathcal{U}(\lambda)$ is a loop algebra $\mathfrak{g}[\lambda]$-valued function of x, and $\mathcal{L}(\lambda)$ is a function of x taking values in the ring of differential operators $\mathfrak{g}[\lambda][D_x]$.

The Lax operator (2) is invariant under the action of a group of automorphisms of the ring $\mathfrak{g}[\lambda][D_x]$ generated by the following transformations

$$\mathfrak{t}: \ \mathcal{X}(\lambda) \mapsto -\mathcal{X}(\lambda)^\dagger\,, \tag{4}$$

$$\mathfrak{q}: \ \mathcal{X}(\lambda) \mapsto Q\mathcal{X}(-\lambda)Q^{-1}\,, \tag{5}$$

$$\mathfrak{r}: \ \mathcal{X}(\lambda) \mapsto \mathcal{X}(\lambda^*)^*\,, \tag{6}$$

for all $\mathcal{X}(\lambda) \in \mathfrak{g}[\lambda][D_x]$. Here, $\mathcal{X}(\lambda)^\dagger$ denotes the formal adjoint of $\mathcal{X}(\lambda)$ defined by taking $D_x \mapsto -D_x$ and transposing elements in $\mathfrak{g}[\lambda]$, the $*$ denotes complex conjugation, and $Q = \text{diag}(-1, 1, \dots, 1)$. The reality requirement for the entries of U is equivalent to the invariance of $\mathcal{L}$ under the action of $\mathfrak{r}$. Invariance of $\mathcal{L}$ under $\mathfrak{t}$ implies that the matrices J, U are skew symmetric. Since the transformations $\mathfrak{t}, \mathfrak{q}, \mathfrak{r}$ are involutions and commute with each other they generate the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Such symmetry groups are known as reduction groups [24,25] and have been extensively used in the theory of integrable systems, see for example [5,23].

The automorphism $a \mapsto QaQ^{-1}$ of the Lie algebra $\mathfrak{g}$ is an involution and hence its eigenvalues are ± 1. We define the corresponding eigenspaces

$$\mathfrak{g}^{(j)} = \{a \in \mathfrak{g} \mid QaQ^{-1} = (-1)^j a\}, \quad j \in \mathbb{Z}_2, \tag{7}$$

which satisfy the following commutation relations

$$\left[\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}\right] \subseteq \mathfrak{g}^{(i+j)}, \tag{8}$$

and thus define a $\mathbb{Z}_2$-grading $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ known as Cartan decomposition of a Lie algebra [17]. Such reductions to symmetric spaces have been used in [4,12] to construct multi-component integrable equations of KdV, mKdV and NLS type, and for example in [14] in relation to Heisenberg ferromagnet models. They have also appeared in [27,28] in relation to a vectorial generalisation of the sine-Gordon equation. Elements that belong to either $\mathfrak{g}^{(0)}$ or $\mathfrak{g}^{(1)}$ are called homogeneous, and in particular elements of $\mathfrak{g}^{(0)}$ are called even while those of $\mathfrak{g}^{(1)}$ are called odd, and they are of the form

$$\begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & *_1^T \\ \mathbf{0} & -*_1 & * \end{pmatrix}, \quad \begin{pmatrix} 0 & *_1 & *_2^T \\ -*_1 & 0 & \mathbf{0}^T \\ -*_2 & \mathbf{0} & \mathbb{0} \end{pmatrix}, \tag{9}$$

respectively, with the bottom right block belonging in $\mathfrak{so}_N(\mathbb{R})$. In particular, in (3) U is even while J is odd.

We define the map $\mathrm{ad}_J(a) := [J, a]\,, \forall a \in \mathfrak{gl}_{N+2}(\mathbb{R})$. The restriction of ad_J to the Im ad_J is invertible. The inverse ad_J^{-1} can be written in terms of sums of powers of ad_J by representing ad_J as a matrix and then using the Cayley-Hamilton theorem together with the fact that the minimal polynomial of a matrix divides its characteristic polynomial. We find that

$$\mathrm{ad}_J^{-1} = -\frac{5}{4}\mathrm{ad}_J - \frac{1}{4}\mathrm{ad}_J^3\,. \tag{10}$$

Moreover, it follows that the projectors to Im ad_J and Ker ad_J are

$$P_{\mathrm{Im}} := \mathrm{ad}_J^{-1} \circ \mathrm{ad}_J = -\frac{5}{4}\mathrm{ad}_J^2 - \frac{1}{4}\mathrm{ad}_J^4 \quad \text{and} \quad P_{\mathrm{Ker}} := \mathrm{id} - P_{\mathrm{Im}}\,, \tag{11}$$

respectively. We will use the above in the proof of Propositions 1 and 2 in Section 3. Considering the restriction ad_J in $\mathfrak{g}$ we can define the spaces $\mathfrak{K} = \mathrm{Ker}\,\mathrm{ad}_J$ and $\mathfrak{I} = \mathrm{Im}\,\mathrm{ad}_J$, such that

$$\mathfrak{g} = \mathfrak{K} \oplus \mathfrak{I}\,, \tag{12}$$

which follows from the fact that $\mathfrak{K} \cap \mathfrak{I} = \{0\}$. The subspace $\mathfrak{K}$ consists of skew-symmetric matrices of the form

$$\begin{pmatrix} 0 & *_1 & \mathbf{0}^T \\ -*_1 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & * \end{pmatrix}.$$

Using the $\mathbb{Z}_2$ gradation given above, we further define the spaces $\mathfrak{K}^{(i)} = \mathfrak{K} \cap \mathfrak{g}^{(i)}$ and $\mathfrak{I}^{(i)} = \mathfrak{I} \cap \mathfrak{g}^{(i)}$, with $i \in \mathbb{Z}_2$. Comparing with (9) we have that $\mathfrak{K}^{(1)} = \mathrm{span}_{\mathbb{R}}(J)$, hence it is abelian, and further $\mathfrak{K}^{(0)}$ consists of the bottom right $\mathfrak{so}_N$ block in $\mathfrak{so}_{N+2}$. Hence, the following commutation relations hold for $\mathfrak{K}^{(0)}$ and $\mathfrak{K}^{(1)}$

$$[\mathfrak{K}^{(1)}, \mathfrak{K}^{(1)}] = 0\,,\ [\mathfrak{K}^{(0)}, \mathfrak{K}^{(0)}] \subseteq \mathfrak{K}^{(0)},\ [\mathfrak{K}^{(0)}, \mathfrak{K}^{(1)}] = 0\,, \tag{13}$$

and thus provide the decomposition $\mathfrak{K} = \mathfrak{K}^{(0)} \oplus \mathfrak{K}^{(1)}$ into a direct sum of Lie algebras. We remark here that the construction in which $\mathfrak{K}$ is an abelian subalgebra of $\mathfrak{g}$ was presented in [9], as well as used in later works, e.g. [7,12,40]. In the current situation, although $\mathfrak{K}$ is not abelian, the decomposition of $\mathfrak{K}$ to odd and even parts is crucial for the construction of the conservation laws for the vmKdV hierarchy. In the case where $N = 2$, $\mathfrak{K}^{(0)}$ is abelian and thus from (13) it follows that $\mathfrak{K}$ is also abelian. We discuss this situation further in Section 3.3. We will use the above definitions together with the properties of the $\mathcal{L}(\lambda)$ operator (2) in the following section, where we construct the vmKdV hierarchy following the ideas of Drinfel'd and Sokolov.

3. The vector mKdV hierarchy

The vmKdV hierarchy is defined as the set of all compatibility conditions $[\mathcal{L}(\lambda), \mathcal{A}_k(\lambda)] = 0$, $k \in \mathbb{Z}_+$, of the systems of spectral problems $\mathcal{L}(\lambda)\Psi = 0$, $\mathcal{A}_k(\lambda)\Psi = 0$, where $\mathcal{L}(\lambda)$ is given in (2) and $\mathcal{A}_k(\lambda)$ are of the form

$$\mathcal{A}_k(\lambda) = D_{t_k} - \mathcal{V}_k(\lambda), \tag{14}$$

with D_{t_k} denoting partial differentiation with respect to t_k. Each $\mathcal{V}_k(\lambda)$ is an appropriately chosen matrix-valued polynomial in λ of degree k depending on $\mathbf{u}$ and its derivatives. In [27,28], the authors studied the vector sine-Gordon equation which arises as the compatibility condition of $\mathcal{L}(\lambda)$ with a Lax operator having a λ^{-1}-dependence.

For given k, the compatibility condition $[\mathcal{L}, \mathcal{A}_k] = 0$ is equivalent to the Lax equation

$$D_{t_k}\mathcal{L} = [\mathcal{V}_k, \mathcal{L}]\,. \tag{15}$$

The left-hand side of equation (15) is equal to

$$D_{t_k}\mathcal{L} = -D_{t_k}U = -\begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{u}_{t_k}^T \\ \mathbf{0} & -\mathbf{u}_{t_k} & \mathbb{0} \end{pmatrix}, \tag{16}$$

therefore, if the commutator $[\mathcal{V}_k, \mathcal{L}]$ is of the same form as (16) then we obtain consistent evolutionary equations, integrable in the Lax sense. Hence, the problem of determining the vmKdV hierarchy reduces to finding all those $\mathcal{V}_k$ such that $[\mathcal{V}_k, \mathcal{L}]$ is λ-independent and skew-symmetric matrix of the form (16). To this end, we define the set of formal series

$$\mathfrak{g}((\lambda^{-1}))^{\langle \mathfrak{q}\rangle} := \left\{ \sum_{i=-\infty}^{l} a_i\lambda^i \,, a_i \in \mathfrak{g}\,, l \in \mathbb{Z}_+\,, Q a_i Q^{-1} = (-1)^{i \bmod 2} a_i \right\}$$

and the projections of $a(\lambda) \in \mathfrak{g}((\lambda^{-1}))^{\langle \mathfrak{q}\rangle}$ to its polynomial in λ part $a(\lambda)_+$, and the part consisting of strictly negative powers in λ, $a(\lambda)_- = a(\lambda) - a(\lambda)_+$. The following lemma gives sufficient conditions for the characterisation of the $\mathcal{V}_k(\lambda)$ as described above.

Lemma 3.1. *Let $a(\lambda) = \sum_{i=-\infty}^{k} a_i\lambda^i \in \mathfrak{g}((\lambda^{-1}))^{\langle \mathfrak{q}\rangle}$, with a_k non-zero, and such that $[a(\lambda), \mathcal{L}(\lambda)] = 0$. Then, $[a(\lambda)_+, \mathcal{L}(\lambda)]$ is λ-independent and of the form* (16)*. Moreover, if k is an odd positive integer then $a_k = J$.*

Proof. The condition $[a(\lambda), \mathcal{L}(\lambda)] = 0$ implies that $[a(\lambda)_+, \mathcal{L}(\lambda)] = -[a(\lambda)_-, \mathcal{L}(\lambda)]$. The left-hand side of this relation is polynomial in λ, while the right-hand side contains only non-positive powers of λ. It follows that

$$[a(\lambda)_+, \mathcal{L}(\lambda)] = -[J, a_{-1}] \in \mathfrak{I}^{(0)} \tag{17}$$

and thus is of the same form as (16). Then, expanding $[a(\lambda), \mathcal{L}(\lambda)] = 0$ in λ we obtain the following equations for the coefficients a_i of $a(\lambda)$

$$[J, a_k] = 0\,, \quad D_x a_i = [U, a_i] + \big[J, a_{i-1}\big]\,, \quad i \le k\,. \tag{18}$$

If k is odd, it follows that $a_k \in \mathfrak{K}^{(1)} = \text{span}_{\mathbb{R}}(J)$, thus $a_k = c\,J$ with c a scalar function. From the second equation in (18), for $i = k$ we obtain that $D_x c = 0$. Hence, since c is x-independent it can be set to one without affecting the Lax equation. □

From the discussion so far it follows that the $\mathcal{V}_k(\lambda)$ which are of the form $a(\lambda)_+$, with $a(\lambda)$ as in Lemma 3.1, lead to consistent Lax equations (15). Moreover, above we show that if k is odd then a_k is determined uniquely. If k is even then $a_k \in \mathfrak{K}^{(0)}$ and by solving recursively equations

(18) we obtain non-local terms in the $\mathfrak{I}$ part of a_{-1}, i.e. terms that involve D_x^{-1}, the formal inverse operator of D_x such that $D_x D_x^{-1} = D_x^{-1} D_x = id$. Since from (17) the Lax equations are of the form $U_{t_k} = [J, a_{-1}]$, it follows that they will also contain non-local terms. Such types of evolutionary non-local equations were studied in, for example, [6] and have been shown to admit very interesting solutions (boomerons). Also, in [12] the existence of such non-local equations within the NLS hierarchy was remarked. In this work we will primarily consider the case of odd k where $a_k = J$.

From Lemma 3.1 it follows that in order to determine $a(\lambda)$ we have to study equation $[a(\lambda), \mathcal{L}(\lambda)] = 0$, which is equivalent to the Lax equation

$$D_x a(\lambda) = [\mathcal{U}(\lambda), a(\lambda)] \,. \tag{19}$$

Equation (19) implies that

$$a(\lambda) = P(\lambda)C(\lambda)P(\lambda)^{-1} \,, \quad \text{with} \quad D_x P(\lambda) = \mathcal{U}(\lambda)P(\lambda) \tag{20}$$

and $C(\lambda)$ a constant matrix with respect to x. Since we are interested in autonomous equations we also assume that $C(\lambda)$ is independent of t_k. Moreover, the fundamental solution $P(\lambda)$ of the differential equation (20) is a formal Laurent series in λ taking values in the group $SO_{N+2}(\mathbb{R})$ and is invariant under the action of $\mathfrak{q}$, namely belongs in the loop group

$$LG^{\langle \mathfrak{q} \rangle} := \left\{ A(\lambda) = \sum_{i=-\infty}^{\infty} A_i \lambda^i \in SO_{N+2}(\mathbb{R}) \,,\ \lambda \in \mathbb{C} \,, \mathfrak{q}\left(A(\lambda)\right) = A(\lambda) \right\},$$

see [15,32]. At $x \to \pm\infty$, assuming boundary conditions $\mathbf{u} \to \mathbf{0}$ (which is typical for soliton-type solutions), $\mathcal{U}(\lambda) \to \lambda J$. In this limit, $P(\lambda)$ with initial condition $P(x = 0, \lambda) = \mathbb{1}$ behaves as $P(\lambda) \sim \mathrm{e}^{\lambda x J}$. This asymptotic behaviour suggests the transformation

$$P(\lambda) = \mathcal{M}(\lambda)\mathrm{e}^{\lambda x J} \,, \tag{21}$$

where $\mathcal{M}(\lambda) = P(\lambda)\mathrm{e}^{-\lambda x J} \in LG^{\langle \mathfrak{q} \rangle}$. Therefore, we have that the Lax matrices $\mathcal{V}_k(\lambda)$ are of the form $\mathcal{V}_k(\lambda) = \left(\mathcal{M}(\lambda)e^{\lambda x J} C(\lambda) e^{-\lambda x J} \mathcal{M}(\lambda)^{-1}\right)_+$. Substituting (21) in the differential equation (20) we obtain the following equation for $\mathcal{M}(\lambda)$

$$D_x \mathcal{M} + \lambda\, [\mathcal{M}, J] = U\mathcal{M} \,. \tag{22}$$

Proposition 1. *Equation (22) admits a formal solution $\mathcal{M}(\lambda) \in LG^{\langle \mathfrak{q} \rangle}$ of the form*

$$\mathcal{M}(\lambda) = \mathbb{1} + \lambda^{-1}\mathcal{M}_1 + \cdots \,, \tag{23}$$

where the coefficients $\mathcal{M}_n$ can be found recursively.

Proof. We substitute expression (23) in (22) and comparing powers of λ we obtain the following equations for $\mathcal{M}_n$

$$\mathrm{ad}_J(\mathcal{M}_1) = -U \,, \quad \mathrm{ad}_J(\mathcal{M}_{n+1}) = D_x \mathcal{M}_n - U\mathcal{M}_n \,, \quad n = 1, 2, \ldots \,. \tag{24}$$

Additionally, since $\mathcal{M}(\lambda) \in LG^{\langle \mathfrak{q} \rangle}$, we have that $\mathcal{M}(\lambda)\mathcal{M}(\lambda)^T = \mathbb{1}$ from which we obtain

$$\mathcal{M}_1 + \mathcal{M}_1^T = 0 \,, \quad \mathcal{M}_n + \mathcal{M}_n^T = -\sum_{j=1}^{n-1} \mathcal{M}_{n-j}\mathcal{M}_j^T \,, \quad n = 2, \ldots \,. \tag{25}$$

Given that $U \in \mathrm{Im}\,\mathrm{ad}_J$, from the first equation in (24) we find that $P_{\mathrm{Im}}(\mathcal{M}_1) = -\mathrm{ad}_J^{-1}(U)$. Then, applying projectors P_{Im} and P_{Ker} given in (11) to the second equation in (24) we obtain

$$P_{\mathrm{Im}}(\mathcal{M}_{n+1}) = \mathrm{ad}_J^{-1}(D_x\mathcal{M}_n - U\mathcal{M}_n)\,, \quad P_{\mathrm{Ker}}(D_x\mathcal{M}_n - U\mathcal{M}_n) = 0\,,$$

which can then be solved recursively for $P_{\mathrm{Im}}(\mathcal{M}_{n+1})$ and $P_{\mathrm{Ker}}(\mathcal{M}_n)$ together with equations (25). In particular, the first equation in (25) is automatically satisfied. In each step, from the second equation above we obtain non-local terms in $\mathbf{u}$ and its derivatives in x by formally inverting D_x. □

Equation (22) can be re-written as

$$\mathcal{L}(\lambda)\mathcal{M}(\lambda) = \mathcal{M}(\lambda)\mathcal{L}_0(\lambda) \tag{26}$$

where $\mathcal{L}_0(\lambda) = D_x - \lambda J$. This means that the matrix $\mathcal{M}(\lambda)$ is a formal Darboux-dressing matrix [22,42] of the trivial $\mathcal{L}_0(\lambda)$ operator which corresponds to the potential $\mathbf{u}_0 = \mathbf{0}$. In Section 4, we construct closed form Darboux matrices $\mathcal{M}(\lambda)$ which map $\mathbf{u}_0$ to the one-soliton and one-breather solutions of the vmKdV hierarchy. Next we prove that the formal Darboux matrix $\mathcal{M}(\lambda)$ admits a factorisation which simplifies the construction of the vmKdV hierarchy and also provides its conservation laws. Such factorisations also appear in e.g. [9,26].

Proposition 2. *The Darboux matrix $\mathcal{M}(\lambda)$ can be factorised as*

$$\mathcal{M}(\lambda) = W(\lambda)H(\lambda)\,, \tag{27}$$

where $W(\lambda)$, $H(\lambda) \in LG^{\langle \mathfrak{q}\rangle}$ and are of the form

$$W(\lambda) = \mathbb{1} + \frac{1}{\lambda}W_1 + \cdots\,, \quad H(\lambda) = \mathbb{1} + \frac{1}{\lambda}H_1 + \cdots\,, \tag{28}$$

satisfying $H(\lambda)JH(\lambda)^{-1} = J$ and

$$D_xW + Wh + \lambda[W, J] = UW\,, \quad h = H_xH^{-1} \in \mathfrak{K}((\lambda^{-1}))^{\langle \mathfrak{q}\rangle}\,. \tag{29}$$

Proof. If the Darboux matrix (23) is written as the product of $W(\lambda)$ and $H(\lambda)$, which are elements of $LG^{\langle \mathfrak{q}\rangle}$ and of the form (28), then equation (22) takes the form (29), assuming that $H(\lambda)JH(\lambda)^{-1} = J$. It follows from (28) that

$$h(\lambda) = H_xH^{-1} = \frac{1}{\lambda}h_1 + \frac{1}{\lambda^2}h_2 + \cdots\,, \tag{30}$$

with $h_i \in \mathfrak{K}^{(i \bmod 2)}$. Comparing powers of λ in equation (29), we obtain the following relation for the coefficients of $W(\lambda)$ and $h(\lambda)$ at λ^{-n}

$$\sum_{i=0}^{n} W_ih_{n-i} + \left[W_{n+1}, J\right] - UW_n + D_xW_n = 0\,, \quad n = 0, 1, \ldots\,, \tag{31}$$

with $W_0 = \mathbb{1}$ and $h_0 = 0$. Hence, the coefficients W_i and h_i can be recursively determined from (31) using the projectors P_{Ker}, P_{Im} and the orthogonality condition for $W(\lambda)$, i.e. $W(\lambda)W(\lambda)^T = \mathbb{1}$, as in the proof of Proposition 1. The factorisation (27) is not unique since we have that the transformation

$$(W(\lambda), H(\lambda)) \mapsto (\widetilde{W}(\lambda), \widetilde{H}(\lambda)) = (W(\lambda)S(\lambda), S(\lambda)^{-1}H(\lambda))\,,$$

with $S(\lambda) = S_0 + \frac{1}{\lambda} S_1 + \cdots$ and $[S(\lambda), J] = 0$ leaves $\mathcal{M}(\lambda)$ invariant. Additionally, $h \mapsto \widetilde{h} = -S^{-1}S_x + S^{-1}hS$, which is of the form (30) and commutes with J. Finally, it follows that equation (31) is invariant under the transformation. We can use this gauge freedom to fix the asymptotic behaviour of $W(\lambda)$ and $H(\lambda)$ at $\lambda = \infty$, and any undetermined components in $W(\lambda)$. □

The following proposition determines the Lax matrices $\mathcal{V}_k(\lambda)$ and effectively defines the vmKdV hierarchy.

Proposition 3. *If $\mathcal{V}_k(\lambda)$ is given by*

$$\mathcal{V}_k(\lambda) = \left(\lambda^k \mathcal{M}(\lambda) J \mathcal{M}(\lambda)^{-1}\right)_+ = \left(\lambda^k W(\lambda) J W(\lambda)^{-1}\right)_+, \quad k = 1, 3, \dots, \tag{32}$$

where $W(\lambda)$ is as in Proposition 2, then the Lax equations (15) are local evolutionary PDEs which admit the Lie symmetry

$$(x, t_k, \mathbf{u}) \mapsto (\widetilde{x}, \widetilde{t}_k, \widetilde{\mathbf{u}}) = (e^{\epsilon} x, e^{k\epsilon} t_k, e^{-\epsilon}\mathbf{u}), \quad k = 1, 3, ..., \tag{33}$$

respectively.

Proof. The members of the vmKdV hierarchy are given by the Lax equations (15), which can also be written as $-U_{t_k} = [\mathcal{V}_k, \mathcal{L}]$. For those $\mathcal{V}_k(\lambda)$ of the form $\mathcal{V}_k(\lambda) = a(\lambda)_+ = (P(\lambda)C(\lambda)P(\lambda)^{-1})_+$, with $a(\lambda)$ as described in Lemma 3.1, we obtain consistent Lax equations (15). Moreover, we have that $a(\lambda) = \lambda^k J + \cdots$, and from (20) we obtain $C(\lambda) = \lambda^k J + \cdots$. If we choose $C(\lambda) = \lambda^k J$, then, following Lemma 3.1, we can write the Lax equation in the form $U_{t_k} = \left[J, a_{-1}\right]$, with

$$a_{-1} = \operatorname*{Res}_{\lambda=0} a(\lambda) = \sum_{i+j=k+1} W_i J W_j^T ,$$

where W_i are calculated according to Proposition 2. Moreover, equation (31) admits the Lie symmetry

$$(x, U, W_n, h_n) \mapsto (\widetilde{x}, \widetilde{U}, \widetilde{W}_n, \widetilde{h}_n) = (e^{\epsilon} x, e^{-\epsilon} U, e^{-n\epsilon} W_n, e^{-(n+1)\epsilon} h_n) .$$

Since W_n is homogeneous of degree n, it follows that a_{-1} is homogeneous of degree $k+1$. Therefore the term U_{t_k} is of the same degree. This implies that the expression $U_{t_k} - \left[J, a_{-1}\right]$ is homogeneous of degree $k+1$, hence each Lax equation (15) admits the symmetry (33). □

Example. For $k = 1$ we find

$$\mathcal{V}_1(\lambda) = \left(\lambda W(\lambda) J W(\lambda)^T\right)_+ = \lambda J + \left(J W_1^T + W_1 J\right) ,$$

and since $W_1^T = -W_1$ it follows that $\mathcal{V}_1(\lambda) = \lambda J - \mathrm{ad}_J(W_1)$. Solving equations (31) recursively we find $W_1 = -\mathrm{ad}_J^{-1}(U)$. Hence, we have that $\mathcal{V}_1(\lambda) = \lambda J + U = \mathcal{U}(\lambda)$. Therefore, the Lax equation (15) provides the first equation of the hierarchy, $\mathbf{u}_{t_1} = \mathbf{u}_x$. In a similar manner, the case $k = 3$ provides the next member of the hierarchy. In particular, we obtain the following matrix

$$\mathcal{V}_3 = \begin{pmatrix} 0 & \lambda^3 - \lambda\frac{\|\mathbf{u}\|^2}{2} & -\lambda\mathbf{u}_x^T \\ -\lambda^3 + \lambda\frac{\|\mathbf{u}\|^2}{2} & 0 & \lambda^2\mathbf{u}^T - \frac{\|\mathbf{u}\|^2}{2}\mathbf{u}^T - \mathbf{u}_{xx}^T \\ \lambda\mathbf{u}_x & -\lambda^2\mathbf{u} + \frac{\|\mathbf{u}\|^2}{2}\mathbf{u} + \mathbf{u}_{xx} & \mathbf{u}\mathbf{u}_x^T - \mathbf{u}_x\mathbf{u}^T \end{pmatrix}, \tag{34}$$

such that the compatibility of $\mathcal{A}_3(\lambda) = D_{t_3} - \mathcal{V}_3(\lambda)$ and $\mathcal{L}(\lambda)$ is equivalent to the vmKdV equation (1).

The flows defined by the Lax equations (15) commute, namely $[D_{t_n}, D_{t_m}]\mathbf{u} = \mathbf{0}$, for $n, m = 1, 3, \ldots$. We prove this below following the ideas in [9], also taking into account the reduction group generated by (4)-(6).

Lemma 3.2. *The Lax operators $\mathcal{A}_k$, $k = 1, 3, \ldots$, commute.*

Proof. The Lax equation $[\mathcal{A}_k, \mathcal{L}] = 0$, in view of relation (26), takes the form

$$[\mathcal{M}^{-1}\mathcal{A}_k\mathcal{M}, \mathcal{L}_0] = 0\,, \quad \text{with} \quad \mathcal{M}^{-1}\mathcal{A}_k\mathcal{M} = D_{t_k} - \mathcal{M}^{-1}\mathcal{V}_k\mathcal{M} + \mathcal{M}^{-1}\mathcal{M}_{t_k}\,.$$

Since $\mathcal{V}_k(\lambda) = (\lambda^k \mathcal{M}(\lambda) J \mathcal{M}(\lambda)^{-1})_+$, it follows that $\mathcal{M}^{-1}\mathcal{A}_k\mathcal{M} = D_{t_k} - \lambda^k J - \hat{\mathcal{V}}_k(\lambda)$, with $\hat{\mathcal{V}}_k(\lambda) = -\mathcal{M}^{-1}\mathcal{M}_{t_k} - \mathcal{M}^{-1}(\lambda^k \mathcal{M} J \mathcal{M}^{-1})_- \mathcal{M} \in \mathfrak{g}((\lambda^{-1}))^{\langle \mathfrak{q} \rangle}$ and of the form

$$\hat{\mathcal{V}}_k(\lambda) = \frac{1}{\lambda}\hat{V}_1 + \frac{1}{\lambda^2}\hat{V}_2 + \cdots\,.$$

The commutativity relation $\left[\mathcal{M}^{-1}\mathcal{A}_k\mathcal{M}, \mathcal{L}_0\right] = 0$ is equivalent to $D_x \hat{\mathcal{V}}_k = \lambda[J, \hat{\mathcal{V}}_k]$. Comparing powers of λ we obtain

$$\mathrm{ad}_J \hat{V}_1 = 0\,, \quad D_x \hat{V}_i = \mathrm{ad}_J \hat{V}_{i+1}\,, \quad i = 1, 2, \ldots\,.$$

Given that $\hat{\mathcal{V}}_k(\lambda) \in \mathfrak{g}((\lambda^{-1}))^{\langle \mathfrak{q} \rangle}$, from the first equation above we have that $\hat{V}_1 \in \mathfrak{K}^{(1)}$. Also $\mathrm{ad}_J \hat{V}_2 \in \mathfrak{I}^{(1)}$, therefore from the second equation it follows that $D_x \hat{V}_1 = 0$ and $\hat{V}_2 \in \mathfrak{K}^{(0)}$. Recursively, in a similar manner we find that $D_x \hat{V}_i \in \mathfrak{K}^{(i \bmod 2)}$ while $\mathrm{ad}_J \hat{V}_{i+1} \in \mathfrak{I}^{(i \bmod 2)}$, therefore $D_x \hat{\mathcal{V}}_k(\lambda) = 0$ and $\hat{\mathcal{V}}_k(\lambda) \in \mathfrak{K}$. We have that $[\mathcal{A}_n, \mathcal{A}_m] = [\mathcal{A}_n, \mathcal{A}_m]_+$, as the Lax operators $\mathcal{A}_n$, $\mathcal{A}_m$ are polynomial in λ, hence

$$[\mathcal{A}_n, \mathcal{A}_m]_+ = \left(\mathcal{M}(D_{t_m}\hat{\mathcal{V}}_n - D_{t_n}\hat{\mathcal{V}}_m + [\hat{\mathcal{V}}_n, \hat{\mathcal{V}}_m])\mathcal{M}^{-1}\right)_+ = 0\,. \quad \square$$

The commutativity of the flows $[D_{t_n}, D_{t_m}]\mathbf{u} = \mathbf{0}$ follows from the Lax equations (15) and the Jacobi identity. Indeed, from (15) we have that

$$[D_{t_n}, D_{t_m}]\mathcal{L} = D_{t_n}[\mathcal{V}_m, \mathcal{L}] - D_{t_m}[\mathcal{V}_n, \mathcal{L}]\,.$$

Expanding the right hand side, using the Lax equations once again, and the Jacobi identity we obtain

$$[D_{t_n}, D_{t_m}]\mathcal{L} = [D_{t_n}\mathcal{V}_m - D_{t_m}\mathcal{V}_n + [\mathcal{V}_m, \mathcal{V}_n], \mathcal{L}]\,,$$

therefore, $[D_{t_n}, D_{t_m}]\mathcal{L} = [[\mathcal{A}_n, \mathcal{A}_m], \mathcal{L}] = 0$ from the above lemma.

3.1. The recursion operator

In this section we construct the recursion operator $\mathcal{R}$ for the vmKdV hierarchy (15). The concept of the recursion operator was first introduced in [30] as a recursive way to construct an infinite number of symmetries for evolutionary equations. Such an operator can be used to directly generate the vmKdV hierarchy $\mathbf{u}_{t_{2n+1}} = K_{2n+1}[\mathbf{u}]$, namely $\mathbf{u}_{t_{2n+1}} = \mathcal{R}\mathbf{u}_{t_{2n-1}}$, starting from the seed $\mathbf{u}_{t_1} = \mathbf{u}_x$. The recursion operator for the vmKdV hierarchy was first derived in [33] and later in [2] using the bi-Hamiltonian formalism. Here we present an alternative construction, following [16], using the Lax matrices of the hierarchy (32) and their reduction group (4)-(6).

Proposition 4. *The recursion operator* $\mathcal{R}$ *for the vmKdV hierarchy is given by*

$$\mathcal{R}\mathbf{f} = -D_x^2\mathbf{f} - \|\mathbf{u}\|^2\mathbf{f} - \mathbf{u}_x D_x^{-1}\left(\mathbf{u}^T\mathbf{f}\right) - D_x^{-1}(\mathbf{u}_x \wedge \mathbf{f})\mathbf{u}\,, \tag{35}$$

where $\mathbf{a} \wedge \mathbf{b} = \mathbf{a}\mathbf{b}^T - \mathbf{b}\mathbf{a}^T$.

Proof. We split (32) in polynomial and purely negative powers in λ as follows

$$\mathcal{V}_{2n+1}(\lambda) = \left(\lambda^2\lambda^{2n-1}WJW^T\right)_+ = \left(\lambda^2\mathcal{V}_{2n-1}(\lambda)\right)_+ + \left(\lambda^2\left(\lambda^{2n-1}WJW^T\right)_-\right)_+ .$$

Hence we can write the following recursive expression for the Lax matrices $\mathcal{V}_k(\lambda)$, for $k = 1, 3, \ldots,$

$$\mathcal{V}_{2n+1}(\lambda) = \lambda^2\mathcal{V}_{2n-1}(\lambda) + \lambda A_{2n-1} + B_{2n-1}\,. \tag{36}$$

Since $\mathcal{V}_{2n+1}(\lambda)\,, \mathcal{V}_{2n-1}(\lambda) \in \mathfrak{g}[\lambda]^{\langle\mathfrak{q}\rangle}$ and λ^2 is invariant under $\lambda \mapsto -\lambda$, it follows that $A_{2n-1} \in \mathfrak{g}^{(1)}$ and $B_{2n-1} \in \mathfrak{g}^{(0)}$. Then from the Lax equations (15) we have

$$\mathcal{L}_{t_{2n+1}} = \lambda^2\mathcal{L}_{t_{2n-1}} + \lambda[A_{2n-1}, \mathcal{L}] + [B_{2n-1}, \mathcal{L}]\,,$$

from which, comparing powers of λ, we obtain

$$\begin{aligned} U_{t_{2n+1}} &= D_x B_{2n-1} + [B_{2n-1}, U]\,, \\ D_x A_{2n-1} &= [U, A_{2n-1}] + [J, B_{2n-1}]\,, \\ U_{t_{2n-1}} &= [J, A_{2n-1}]\,. \end{aligned} \tag{37}$$

The above equations provide the relation between two members of the vmKdV hierarchy, while fixing A_{2n-1}, B_{2n-1} in terms of $\mathbf{u}$ and $\mathbf{u}_{t_{2n-1}}$. In particular, we have that

$$\mathbf{u}_{t_{2n+1}} = -D_x^2\,\mathbf{u}_{t_{2n-1}} - \|\mathbf{u}\|^2\mathbf{u}_{t_{2n-1}} - \mathbf{u}_x D_x^{-1}\left(\mathbf{u}^T\mathbf{u}_{t_{2n-1}}\right) - D_x^{-1}(\mathbf{u}_x \wedge \mathbf{u}_{t_{2n-1}})\mathbf{u}\,.$$

Hence, we can write $\mathbf{u}_{t_{2n+1}} = \mathcal{R}\mathbf{u}_{t_{2n-1}}$, with $\mathcal{R}$ given in (35). □

Example. Acting with the recursion operator (35) on the first equation of the hierarchy, $\mathbf{u}_{t_1} = \mathbf{u}_x$, we obtain the vmKdV equation

$$\mathbf{u}_{t_3} = \mathcal{R}\mathbf{u}_{t_1} = \mathcal{R}\mathbf{u}_x = -\mathbf{u}_{xxx} - \tfrac{3}{2}\|\mathbf{u}\|^2\mathbf{u}_x\,.$$

Accordingly, we find the next equation of the hierarchy to be

$$\begin{aligned} \mathbf{u}_{t_5} = \mathbf{u}_{xxxxx} &+ \tfrac{5}{2}\left(\mathbf{u}^T\mathbf{u}\right)\mathbf{u}_{xxx} + \tfrac{5}{2}\left(\mathbf{u}_x^T\mathbf{u}_x\right)\mathbf{u}_x + 5\left(\mathbf{u}^T\mathbf{u}_x\right)\mathbf{u}_{xx} \\ &+ 5\left(\mathbf{u}^T\mathbf{u}_{xx}\right)\mathbf{u}_x + \tfrac{15}{8}\left(\mathbf{u}^T\mathbf{u}\right)^2\mathbf{u}_x\,. \end{aligned}$$

From equation (36) we obtain a recursion relation for the Lax matrices in terms of A_{2n-1} and B_{2n-1}, which we determine using equations (37). In particular, starting from the last equation in (37), and since $U \in \mathfrak{I}$ and $A_{2n-1} \in \mathfrak{g}^{(1)}$, we obtain

$$A_{2n-1} = \mathrm{ad}_J^{-1}(U_{t_{2n-1}}) + \alpha J\,,$$

where α is a scalar to be determined. Projecting the second equation in (37) in $\mathfrak{K}$ we find that $\alpha = -D_x^{-1}(\mathbf{u}^T \mathbf{u}_{t_{2n-1}})$, while from the projection in the $\mathfrak{I}$ part in the same equation we obtain $P_{\mathfrak{I}}(B_{2n-1})$. Finally, from the first equation we find $P_{\mathfrak{K}}(B_{2n-1})$. Overall, we have that

$$\begin{aligned} A_{2n-1} &= -\mathrm{ad}_J(U_{t_{2n-1}}) - D_x^{-1}(\mathbf{u}^T \mathbf{u}_{t_{2n-1}})J\,, \\ B_{2n-1} &= -D_x U_{t_{2n-1}} - D_x^{-1}(\mathbf{u}^T \mathbf{u}_{t_{2n-1}})U + D_x^{-1}\left[D_x U_{t_{2n-1}}, U\right], \end{aligned}$$

where we have used the relation $\mathrm{ad}_J^{-1}(U) = -\mathrm{ad}_J(U)$. Hence, the recursion (36) can be written in matrix form as follows for $n = 1, 2, \ldots$

$$\begin{aligned} \mathcal{V}_{2n+1}(\lambda) &= \lambda^2 \mathcal{V}_{2n-1}(\lambda) + \lambda \begin{pmatrix} 0 & -D_x^{-1}(\mathbf{u}^T \mathbf{u}_{t_{2n-1}}) & -\mathbf{u}^T_{t_{2n-1}} \\ D_x^{-1}(\mathbf{u}^T \mathbf{u}_{t_{2n-1}}) & 0 & \mathbf{0}^T \\ \mathbf{u}_{t_{2n-1}} & \mathbf{0} & \mathbb{O} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & -D_x(\mathbf{u}^T_{t_{2n-1}}) - D_x^{-1}(\mathbf{u}^T_{t_{2n-1}}\mathbf{u})\mathbf{u}^T \\ \mathbf{0} & D_x(\mathbf{u}_{t_{2n-1}}) + D_x^{-1}(\mathbf{u}^T \mathbf{u}_{t_{2n-1}})\mathbf{u} & D_x^{-1}(\mathbf{u} \wedge D_x(\mathbf{u}_{t_{2n-1}})) \end{pmatrix}, \end{aligned}$$

with $\mathcal{V}_1(\lambda) = \lambda J + U$ and $\mathbf{u}_{t_1} = \mathbf{u}_x$.

3.2. Conservation laws

The vmKdV hierarchy admits an infinite number of conservation laws. In the present section we construct a generating function for conservation laws using the Lax representation of the hierarchy as presented previously. The conservation laws for the scalar mKdV equation first appeared in [29], while the first few conserved densities and corresponding fluxes related to the vmKdV equation first appeared in [20]. We also point the reader to [3], where the complex mKdV and Sasa-Satsuma equations are treated.

Proposition 5. *The vmKdV hierarchy admits an infinite number of scalar conservation laws, with the* $(1, 2)$ *element of matrix* $h(\lambda)$ *in* (30) *being a generating function of the corresponding densities.*

Proof. Given the factorisation (27) of the Darboux matrix $\mathcal{M}(\lambda)$, the Lax equations $[\mathcal{A}_k, \mathcal{L}] = 0$ can be written as $[W^{-1}\mathcal{A}_k W, H\mathcal{L}_0 H^{-1}] = 0$, where $H\mathcal{L}_0 H^{-1} = D_x - \lambda J - h(\lambda)$ and $W^{-1}\mathcal{A}_k W = D_{t_k} - \lambda^k J - \widetilde{\mathcal{V}}_k(\lambda)$, with $\widetilde{\mathcal{V}}_k(\lambda) \in \mathfrak{g}((\lambda^{-1}))^{\langle \mathfrak{q} \rangle}$ of the form

$$\widetilde{\mathcal{V}}_k(\lambda) = \frac{\widetilde{V}_{k,1}}{\lambda} + \frac{\widetilde{V}_{k,2}}{\lambda^2} + \cdots .$$

We have that $\widetilde{\mathcal{V}}_k = H\hat{\mathcal{V}}_k H^{-1} + H_{t_k} H^{-1}$, with $\hat{\mathcal{V}}_k(\lambda) \in \mathfrak{K}$ as in Lemma 3.2 and H as in Proposition 2. Therefore, $\widetilde{\mathcal{V}}_k(\lambda) \in \mathfrak{K}$, and thus $[W^{-1}\mathcal{A}_k W, H\mathcal{L}_0 H^{-1}] = 0$ takes the form

$$D_{t_k} h(\lambda) - D_x \widetilde{\mathcal{V}}_k(\lambda) + [h(\lambda), \widetilde{\mathcal{V}}_k(\lambda)] = 0\,. \tag{38}$$

Given that $h(\lambda)\,, \widetilde{\mathcal{V}}_k(\lambda) \in \mathfrak{K}$ and in view of relations (13), it follows from (38) that

$$D_{t_k}\big(h(\lambda)_{12}\big) = D_x\left(\widetilde{\mathcal{V}}_k(\lambda)_{12}\right) \quad \text{for} \quad k = 1, 3, \ldots ,$$

hence the element $h(\lambda)_{12}$ is a generating function of conserved scalar densities for the vmKdV hierarchy, with the corresponding fluxes given by $\widetilde{\mathcal{V}}_k(\lambda)_{12}$. □

The first three conserved densities, up to an element of $\mathrm{Im}(D_x)$, are given by

$$f_1 = \tfrac{1}{2}\|\mathbf{u}\|^2\,, \quad f_2 = -\tfrac{1}{8}\|\mathbf{u}\|^4 + \tfrac{1}{2}\|\mathbf{u}_x\|^2\,,$$

$$f_3 = \tfrac{1}{2}\|\mathbf{u}\|^6 + 4\|\mathbf{u}_{xx}\|^2 + 6\|\mathbf{u}\|^2\mathbf{u}^T\mathbf{u}_{xx} + 8(\mathbf{u}^T\mathbf{u}_x)^2\,.$$

Remark. While the conserved densities obtained from the generating function $h(\lambda)_{12}$ are O_N-invariant, there exist additional conserved densities which are O_N-invariant modulo terms in $\mathrm{Im}(D_x)$. Indeed, there exists a matrix conservation law for the vmKdV hierarchy given by

$$D_{t_k}(h_2) = D_x(\widetilde{V}_{k,2})\,,$$

with $h_2\,,\ \widetilde{V}_{k,2} \in \mathfrak{K}^{(0)}$, where h_2 is calculated using expression (31) in the proof of Proposition 2. The corresponding densities are $u_i D_x(u_j)$. Additionally, we observe that contrary to the case of the scalar mKdV equation, the vmKdV hierarchy does not admit $\mathbf{u}$ as a conserved density. This implies that the conservation of mass no longer holds in the vectorial case. See also [3] for a discussion.

3.3. *Special case:* $\mathfrak{g} = \mathfrak{so}_4(\mathbb{R})$

The current section is dedicated to the case where $N = 2$, so $\mathbf{u} = (u_1, u_2)^T$. In this case the vmKdV equation (1) is equivalent to the system of two equations

$$u_{1t} + u_{1xxx} + \tfrac{3}{2}(u_1^2 + u_2^2)u_{1x} = 0\,, \quad u_{2t} + u_{2xxx} + \tfrac{3}{2}(u_1^2 + u_2^2)u_{2x} = 0\,.$$

After setting $v = u_1 + iu_2$, we find that the field $v = v(x,t)$ satisfies the complex mKdV equation, also known as Hirota equation [18],

$$v_t + v_{xxx} + \tfrac{3}{2}|v|^2 v_x = 0\,, \quad \text{where} \quad |v|^2 = v^* v = u_1^2 + u_2^2\,. \tag{39}$$

In Proposition 3 we showed that the Lax equations $D_{t_k}\mathcal{L} = [\mathcal{V}_k, \mathcal{L}]$ define the vmKdV hierarchy for $\mathcal{V}_k(\lambda) = \left(\lambda^k W J W^T\right)_+$, with k odd. For generic N, if k is even then the coefficients of $\mathcal{V}_k(\lambda)$ contain non-local elements, hence leading in general to non-local Lax equations. More specifically, the first non-local elements appear in the coefficient of λ^{k-2} and are of the form $D_x^{-1}(\mathbf{u}\mathbf{u}_x^T\mathbb{A} + \mathbb{A}\mathbf{u}_x\mathbf{u}^T)$, with $\mathbb{A} \in \mathfrak{so}_N$. However, when $N = 2$ we obtain local equations for even k, since in this case the term $\mathbf{u}\mathbf{u}_x^T\mathbb{A} + \mathbb{A}\mathbf{u}_x\mathbf{u}^T$ is in the image of D_x and proportional to $\mathbb{A}$, as $\mathfrak{so}_2$ is one dimensional.

Proposition 6. *When $\mathfrak{g} = \mathfrak{so}_4(\mathbb{R})$ the hierarchy generated by the Lax operator $\mathcal{L}(\lambda)$ in (2) contains the nonlinear Schrödinger equation.*

Proof. When $k = 2$ the Lax matrix $\mathcal{V}_k(\lambda)$ is of the form $\mathcal{V}_2(\lambda) = \lambda^2 a_2 + \lambda a_1 + a_0$, with $a_1 \in \mathfrak{g}^{(1)}$ and $a_0,\ a_2 \in \mathfrak{g}^{(0)}$. From the first equation in (18) we have that $a_2 \in \mathfrak{K}^{(0)}$. Further, splitting the second equation $D_x a_2 = [U, a_2] + [J, a_1]$ in $\mathfrak{K}$ and $\mathfrak{I}$ we find that $D_x a_2 = 0$ and also specify the $\mathfrak{I}$ component of a_1. Hence a_2 is of the form

$$a_2 = \begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \mathbb{A} \end{pmatrix}, \quad \text{with} \quad \mathbb{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{so}_2\,.$$

Solving equations (18) recursively we further find the following expressions for a_1 and a_0

$$a_1 = \begin{pmatrix} 0 & 0 & \mathbf{u}^T\mathbb{A} \\ 0 & 0 & \mathbf{0}^T \\ \mathbb{A}\mathbf{u} & \mathbf{0} & \mathbb{0} \end{pmatrix}, \quad a_0 = \begin{pmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{u}_x^T\mathbb{A} \\ \mathbf{0} & \mathbb{A}\mathbf{u}_x & -\frac{1}{2}\|\mathbf{u}\|^2\mathbb{A} \end{pmatrix}.$$

Then, the Lax equation (15) is equivalent to the system of equations

$$u_{1t} + u_{2xx} + \tfrac{1}{2}(u_1^2 + u_2^2)u_2 = 0\,, \quad u_{2t} - u_{1xx} - \tfrac{1}{2}(u_1^2 + u_2^2)u_1 = 0\,.$$

Setting $v = u_1 - iu_2$ we find that the complex field $v = v(x,t)$ satisfies the NLS equation

$$i v_t - v_{xx} - \tfrac{1}{2}|v|^2 v = 0\,, \quad \text{where} \quad |v|^2 = v^* v\,. \quad \square \tag{40}$$

Further, when $N = 2$ there also exist additional conservation laws for the vmKdV hierarchy. Indeed, since $\widetilde{\mathcal{V}}_k(\lambda), h(\lambda) \in \mathfrak{K}$ and $\mathfrak{K}$ is abelian when $N = 2$, (38) takes the form of a conservation law

$$D_{t_k} h(\lambda) = D_x \widetilde{\mathcal{V}}_k(\lambda) \quad \text{for} \quad k = 1, 2, 3, \ldots\,.$$

For example, we obtain the conserved density

$$f_4 = \tfrac{1}{2}\|\mathbf{u}\|^2 u_1 u_{2x} + u_1\left(u_{2xxx} + u_2^2 u_{2x}\right).$$

In addition, in the $N = 2$ case there exists a relation between the recursion operator $\mathcal{R}_{NLS}$ of the NLS equation and that of the vmKdV equation $\mathcal{R}$ given in (35). This relation was also remarked in [33].

Proposition 7. *The recursion operators $\mathcal{R}$ of the two-component vmKdV and $\mathcal{R}_{NLS}$ of the NLS equation (40) are related according to $\mathcal{R}_{NLS}^2 = \mathcal{R}$.*

Proof. The recursion operator of the NLS equation (40) is

$$\mathcal{R}_{NLS} f = -i D_x f - \tfrac{i}{2} u\, D_x^{-1}\left(f u^* + u f^*\right).$$

Setting $u = u_1 + iu_2$, and considering the action of $\mathcal{R}_{NLS}$ in real plane vectors, we can represent $\mathcal{R}_{NLS}$ in matrix form as

$$\mathcal{R}_{NLS} = \begin{pmatrix} u_2 D_x^{-1} u_1 & D_x + u_2 D_x^{-1} u_2 \\ -D_x - u_1 D_x^{-1} u_1 & -u_1 D_x^{-1} u_2 \end{pmatrix}.$$

The recursion operator of the vmKdV hierarchy (35) in the case where $N = 2$ and $\mathbf{u} = (u_1, u_2)^T$ can be expressed in matrix form as

$$\mathcal{R} = -\begin{pmatrix} D_x^2 + \|\mathbf{u}\|^2 + u_{1x} D_x^{-1} u_1 - u_2 D_x^{-1} u_{2x} & u_{1x} D_x^{-1} u_2 + u_2 D_x^{-1} u_{1x} \\ u_{2x} D_x^{-1} u_1 + u_1 D_x^{-1} u_{2x} & D_x^2 + \|\mathbf{u}\|^2 + u_{2x} D_x^{-1} u_2 - u_1 D_x^{-1} u_{1x} \end{pmatrix}.$$

Finally, it can be verified that $\mathcal{R}_{NLS}^2 = \mathcal{R}$. $\square$

4. Soliton solutions for the vector mKdV hierarchy

The vmKdV hierarchy admits both soliton and breather solutions. In this section we present the Darboux transformations and the corresponding dressing formulas that produce such solutions. A Darboux-dressing matrix $M(\lambda)$ defines the transformations

$$\widetilde{\mathcal{L}}(\lambda) = M(\lambda)\mathcal{L}(\lambda)M(\lambda)^{-1}, \quad \widetilde{\mathcal{A}}_k(\lambda) = M(\lambda)\mathcal{A}_k(\lambda)M(\lambda)^{-1}, \tag{41}$$

where $\mathcal{L}(\lambda)$, $\mathcal{A}_k(\lambda)$ are as in (2), (14), while $\widetilde{\mathcal{L}}(\lambda) = \mathcal{L}(\widetilde{\mathbf{u}}, \lambda)$, $\widetilde{\mathcal{A}}_k(\lambda) = \mathcal{A}_k(\widetilde{\mathbf{u}}, \lambda)$, and $\widetilde{\mathbf{u}}$ is the 'dressed' solution of the vmKdV hierarchy. In general, the Darboux matrix $M(\lambda)$ can be constant or dependent on $x, t_3, \ldots$, however for simplicity we write $M(\lambda)$. Additionally, the Darboux matrices $M(\lambda)$ of the vmKdV hierarchy satisfy

$$M(\lambda)M(\lambda)^T = \mathbb{1}, \quad M(\lambda) = QM(-\lambda)Q^{-1}, \quad M(\lambda) = M(\lambda^*)^*, \tag{42}$$

which ensure that the dressed operators $\widetilde{\mathcal{L}}(\lambda)$ and $\widetilde{\mathcal{A}}_k(\lambda)$ are invariant under the reduction group (4)-(6). Such types of Darboux transformations were also used in [27,28] for the study of soliton and breather solutions, their interactions, and related integrable structures associated with the vector sine-Gordon equation.

The Darboux matrices $M(\lambda)$ have rational dependence on the spectral parameter λ and have simple poles. The soliton solutions correspond to Darboux matrices with purely imaginary poles, while the Darboux matrices for the breather solutions have poles at generic points in the complex plane. Moreover, each of these Darboux matrices is parametrised by an element of the Grassmannian $Gr(s, \mathbb{C}^{N+2})$, with $s = 1, 2, \ldots, N+1$ in the case of a breather solution, and $s = 1$ for a soliton solution.

Given that $M(\lambda)$ satisfies (42), it follows that it will have poles at group orbits of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $\mathfrak{q}$ and $\mathfrak{r}$. We focus on Darboux matrices that have poles in only one orbit, and we call them elementary Darboux matrices. A Darboux matrix with poles at a generic orbit $\{\mu, -\mu, \mu^*, -\mu^*\}$ of the form

$$M(\lambda) = \mathbb{1} + \frac{M_0}{\lambda - \mu} - \frac{QM_0Q}{\lambda + \mu} + \frac{M_0^*}{\lambda - \mu^*} - \frac{QM_0^*Q}{\lambda + \mu^*}, \tag{43}$$

leads to a breather solution. A soliton solution corresponds to a degenerate orbit $\{i\mu, -i\mu\}$ formed by a purely imaginary pole with a Darboux matrix given by

$$M(\lambda) = \mathbb{1} + \frac{M_0}{\lambda - i\mu} - \frac{QM_0Q}{\lambda + i\mu}, \quad \text{with} \quad M_0 = -QM_0^*Q. \tag{44}$$

In expressions (43) and (44) M_0 does not depend on the spectral parameter λ. One can verify that matrices of the form (43) and (44) are invariant under $\mathfrak{q}$ and $\mathfrak{r}$. The orthogonality condition in (42) will be imposed separately for each of the Darboux matrices given above. We define the rank of a soliton or breather solution to be the rank of matrix M_0. In what follows we construct the Darboux-dressing matrix for the one-soliton and one-breather solutions for the vmKdV hierarchy, using the dressing formula

$$\widetilde{U} = U + \mathrm{ad}_J \left(\operatorname*{Res}_{\lambda=\infty} M(\lambda) \right). \tag{45}$$

Equation (45) follows from the fact that a Darboux matrix $M(\lambda)$ of the form (43) or (44) can be written as

$$M(\lambda) = \mathbb{1} + \frac{1}{\lambda}\sum_j \operatorname*{Res}_{\lambda=j} M(\lambda) + \ldots = \mathbb{1} - \frac{1}{\lambda}\operatorname*{Res}_{\lambda=\infty} M(\lambda) + \ldots ,$$

and then substituting in the first equation in (41) and comparing the constant in λ terms.

4.1. Soliton solution

In this section, we use the Darboux matrix (44) in order to derive the one-soliton solution for the vmKdV hierarchy. We find that the one-soliton solution is parametrised by a point in the imaginary line (the pole of the Darboux matrix (44)) and a point in the sphere $\mathbb{S}^N$.

Proposition 8. *A Darboux matrix with poles on a degenerate orbit, i.e. of the form* (44)*, satisfies the orthogonality condition* $M(\lambda)M(\lambda)^T = \mathbb{1}$ *if and only if*

$$M_0 = 2i\mu P \quad \textit{with} \quad P = \frac{Q\mathbf{q}\mathbf{q}^T}{\mathbf{q}^T Q\mathbf{q}} \quad \textit{and} \quad \mathbf{q}^T\mathbf{q} = 0\,, \tag{46}$$

where $\mathbf{q} \in \mathbb{CP}^{N+1}$.

Proof. The double pole at $\lambda = i\mu$ of the orthogonality condition implies that $M_0 M_0^T = 0$. It follows that M_0 is not of full rank, and together with the reality condition $M_0 = -QM_0^* Q$ we have that $\operatorname{rank}(M_0) = 1$. For a proof see [27], and [28] for a discussion. In this case, M_0 can be parametrised by two $(N+2)$-dimensional complex vectors $\mathbf{p}$, $\mathbf{q}$ as

$$M_0 = \mathbf{p}\mathbf{q}^T\,, \quad \text{with} \quad \mathbf{q}^T\mathbf{q} = 0\,.$$

The residue at $\lambda = i\mu$ of the relation $M(\lambda)M(\lambda)^T = \mathbb{1}$ implies that

$$\mathbf{p} = \frac{2i\mu}{\mathbf{q}^T Q\mathbf{q}} Q\mathbf{q}\,.$$

The matrix P is invariant under the scaling $\mathbf{q} \mapsto \alpha\mathbf{q}$ with α a complex valued function of the independent variables. Therefore, $\mathbf{q}$ is an element in the Grassmannian $Gr(1, \mathbb{C}^{N+2}) \simeq \mathbb{CP}^{N+1}$. □

Remark. From the proposition above it follows that the Darboux matrix in (44) takes the form

$$M(\lambda) = \mathbb{1} + \frac{2i\mu}{\lambda - i\mu} P - \frac{2i\mu}{\lambda + i\mu} QPQ\,, \quad \text{with} \quad P^* = QPQ\,. \tag{47}$$

We observe that the matrix P is a projector, i.e. satisfies $P^2 = P$.

From the double pole at $\lambda = i\mu$ of equations (41) we obtain that the vector $\mathbf{q}$ satisfies

$$\mathcal{L}(i\mu)\mathbf{q} = 0, \quad \mathcal{A}_k(i\mu)\mathbf{q} = 0\,.$$

Thus, we have that

$$\mathbf{q} = \Psi(i\mu)\mathbf{C}\,, \quad \text{with} \quad \mathbf{C}^T\mathbf{C} = 0\,, \tag{48}$$

where $\Psi(i\mu)$ is the common fundamental solution of the Lax operators of the vmKdV hierarchy at $\lambda = i\mu$.

Given the form (47) of the Darboux matrix $M(\lambda)$, the dressing formula (45) can be written as

$$\widetilde{U} = U - 2i\mu\, \mathrm{ad}_J \left(P - P^*\right). \tag{49}$$

Then, taking into account the structure of U in (3) and P in (46), the dressing formula (49) for the vmKdV hierarchy takes the form

$$\widetilde{u}_j = u_j - 4i\mu \frac{q_1\, q_{j+2}}{-q_1^2 + \sum_{k=2}^{N+2} q_k^2}, \quad j = 1, 2, \ldots, N, \tag{50}$$

with $q_j, \widetilde{u}_j$ the components of vector $\mathbf{q}$ and transformed vmKdV field $\widetilde{\mathbf{u}}$, respectively.

Example. We start with the trivial solution $\mathbf{u}_0 = \mathbf{0}$. In this case the Lax operators of the vmKdV hierarchy at $\lambda = i\mu$ take the form

$$\mathcal{L}(i\mu) = D_x - i\mu J, \quad \mathcal{A}_k(i\mu) = D_{t_k} - (i\mu)^k J, \quad \text{with} \quad k = 1, 3, \ldots.$$

Hence, the fundamental solution for the above systems of operators is

$$\Psi(i\mu) = \begin{pmatrix} \cosh\xi & i\sinh\xi & \mathbf{0}^T \\ -i\sinh\xi & \cosh\xi & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \mathbb{1} \end{pmatrix}, \quad \text{with} \quad \xi(\mu, t_k) = \sum_{n=0}^{\infty} (-1)^n \mu^{2n+1} t_{2n+1}.$$

Then, from expression (48) and the reality condition $P^* = QPQ$ in (47) we obtain that $\mathbf{C} = (i, c_0, \mathbf{c}^T)^T$, with $c_0 \in \mathbb{R}$, and $\mathbf{c} \in \mathbb{R}^N$ a constant vector such that $c_0^2 + c_1^2 + \cdots + c_N^2 = 1$. Thus, it follows that the vector $\mathbf{q}$ can be expressed as

$$\mathbf{q} = (i\cosh\xi + ic_0\sinh\xi, c_0\cosh\xi + \sinh\xi, \mathbf{c}^T)^T, \tag{51}$$

and then expression (50) leads to the one-soliton solution for the vmKdV hierarchy

$$\mathbf{u} = \frac{2\mu\,\mathbf{c}}{\cosh\xi + c_0\sinh\xi}.$$

In the case where $N = 1$ and $N = 2$ the above reduces to the one-soliton solution of the mKdV and complex mKdV equations, respectively, see also [3] and references therein. Higher soliton solutions can be recursively constructed using the dressing formula (49).

4.2. Breather solution

Here, we consider a Darboux matrix $M(\lambda)$ of the form (43), and we present the corresponding dressing formula that leads to breather-type solutions for the vmKdV hierarchy. As an example, we derive the simplest one-breather solution, we express it as a ratio of determinants, and we find that it is parametrised by two constant unit vectors normal to each other.

Proposition 9. *A Darboux matrix with poles on a generic orbit, i.e. of the form* (43)*, satisfies the orthogonality condition* $M(\lambda)M(\lambda)^T = \mathbb{1}$ *if and only if*

$$M_0 = q^* B q^T + Q q C q^T + Q q^* D q^T, \quad \text{with} \quad q^T q = 0, \tag{52}$$

and $q \in Gr(s, \mathbb{C}^{N+2}) \simeq M_{N+2,s}(\mathbb{C})/GL_s(\mathbb{C})$*, with* $s = 1, 2, \ldots, N+1$*. Additionally,* $D, B, C \in M_{s,s}(\mathbb{C})$ *are of the form*

$$D = -\left(FH^{-1}F^* + G^*H^{*-1}G^* - H^*\right)^{-1}, \; B = DG^*H^{*-1}, \; C = -D^*F^*H^{*-1}, \tag{53}$$

where F, G, H *are given by*

$$F = \frac{q^T Qq}{2\mu}, \quad G = \frac{q^{*T} q}{\mu - \mu^*}, \quad H = \frac{q^{*T} Qq}{\mu + \mu^*}. \tag{54}$$

Proof. The double pole at $\lambda = \mu$ of the orthogonality condition implies that $M_0 M_0^T = 0$. Hence, M_0 is not of full rank, i.e. $\mathrm{rank}(M_0) = s$ with $s = 1, 2, \ldots, N + 1$. We parametrise M_0 in terms of two matrices $p, q \in M_{N+2,s}(\mathbb{C})$ as

$$M_0 = pq^T, \quad \text{with} \quad q^T q = 0.$$

From the residue at $\lambda = \mu$ of the relation $M(\lambda)M(\lambda)^T = \mathbb{1}$ it follows that

$$p = q^* B + QqC + Qq^* D, \tag{55}$$

with B, C, D as in (53). The matrix M_0 is invariant under transformation $q \mapsto qA$, where $A \in M_{s,s}(\mathbb{C})$ and $\det A \neq 0$, hence $q \in Gr(s, \mathbb{C}^{N+2})$. □

We write the matrix q in terms of $(N + 2)$-dimensional vectors $\mathbf{q}^l = (q_1^l, q_2^l, \ldots, q_{N+2}^l)^T$, for $l = 1, \ldots, s$, as $q = (\mathbf{q}^1, \mathbf{q}^2, \ldots, \mathbf{q}^s)$. Then, using relation (55) we express the components of p in terms of the components of matrix q according to

$$p_1^l = \sum_{k=1}^{s} \left(q_1^{*k} B_{kl} - q_1^k C_{kl} - q_1^{*k} D_{kl}\right) \tag{56}$$

$$p_i^l = \sum_{k=1}^{s} \left(q_i^{*k} B_{kl} + q_i^k C_{kl} + q_i^{*k} D_{kl}\right), \quad i = 2, \ldots, N + 2. \tag{57}$$

Similar to the soliton case considered in the previous section, from the double pole at $\lambda = \mu$ of equations (41) it follows that

$$\mathcal{L}(\mu)q = 0, \quad \mathcal{A}_k(\mu)q = 0,$$

thus, we can express the matrix q in terms of the fundamental solution of the linear problems $\mathcal{L}(\mu)\Psi = 0$, $\mathcal{A}_k(\mu)\Psi = 0$ as

$$q = \Psi(\mu)R, \quad \text{with} \quad R^T R = 0, \tag{58}$$

and $R \in M_{N+2,s}(\mathbb{C})$ a constant matrix.

Expression (45) leads to the following dressing formula

$$\widetilde{U} = U + [K - QKQ, J] \quad \text{with} \quad K = pq^T + p^* q^{*T} = 2\Re \sum_{l=1}^{s} \mathbf{p}^l \mathbf{q}^{lT}, \tag{59}$$

which can be written as

$$\widetilde{u}_j = u_j + 4\Re\left(\sum_{l=1}^{s} p_1^l q_{j+2}^l\right), \quad j = 1, 2, \ldots, N. \tag{60}$$

Using expression (56) we obtain the dressing transformation for the vmKdV hierarchy, leading to breather-type solutions,

$$\widetilde{u}_j = u_j - 4\Re \sum_{k,l=1}^{s} \begin{vmatrix} q_1^k & 0 & 0 \\ 0 & q_{j+2}^l & B_{kl}^* - D_{kl}^* \\ 0 & q_{j+2}^{*l} & C_{kl} \end{vmatrix}, \quad j = 1, 2, \ldots, N, \tag{61}$$

with B, C, D given in (53).

Example. In the case $s = 1$, expression (55) takes the form

$$\mathbf{p} = \frac{1}{\Delta}\left(G\mathbf{q}^* + F^* Q\mathbf{q} - HQ\mathbf{q}^*\right), \quad \Delta = G^2 - H^2 + |F|^2, \tag{62}$$

where G, F, H are given in (54) and are now scalar quantities. Then, the dressing transformation (61) becomes

$$\widetilde{u}_j = u_j - 4\Re\frac{\Delta_j}{\Delta}, \quad j = 1, 2, \ldots, N,$$

with Δ_j and Δ the following determinants

$$\Delta_j = \begin{vmatrix} q_1 & 0 & 0 \\ 0 & q_{j+2} & H - G \\ 0 & q_{j+2}^* & F^* \end{vmatrix}, \quad \Delta = \begin{vmatrix} F & H - G \\ G + H & F^* \end{vmatrix}.$$

Starting with the trivial solution $\mathbf{u}_0 = \mathbf{0}$ we have that

$$\Psi(\mu) = \begin{pmatrix} \cos\xi & \sin\xi & \mathbf{0}^T \\ -\sin\xi & \cos\xi & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \mathbb{1} \end{pmatrix}, \quad \text{with} \quad \xi = \sum_{n=0}^{\infty} \mu^{2n+1} t_{2n+1},$$

hence, using (58) and (60) we obtain the one-breather solution

$$\mathbf{u} = -\frac{4}{\Delta}\Re\big((R_1\cos\xi + R_2\sin\xi)(F^*\mathbf{r} + (G - H)\mathbf{r}^*)\big),$$

where $\mathbf{R} = (R_1, R_2, \mathbf{r}^T)^T$ such that $\mathbf{R}^T\mathbf{R} = 0$. The latter condition implies that the real and imaginary parts of vector $\mathbf{R}$ have the same length, and furthermore they are normal to each other. Using the fact that $\mathbf{R}$ is in $\mathbb{CP}^{N+1}$ we can normalise its real and imaginary parts and assume their length is equal to one. It follows that the one-breather solution for the vmKdV hierarchy is parametrised by a complex number (the pole of the Darboux matrix (43)) and an element of the unit tangent bundle $T_1\left(\mathbb{S}^{N+1}\right)$ of the sphere

$$T_1\left(\mathbb{S}^{N+1}\right) = \{(\mathbf{v}_1, \mathbf{v}_2) \in \mathbb{R}^{2(N+2)} | \ \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0, \ \|\mathbf{v}_1\| = \|\mathbf{v}_2\| = 1\}.$$

5. Conclusions

In this paper we studied the hierarchy and solutions of a vector mKdV equation which is not a member of the vector NLS hierarchy. An interesting characteristic of the vector mKdV equation that we studied here is that it is an example of an integrable equation admitting a Lax operator in which the constant, regular, semi-simple element of the Drinfel'd-Sokolov approach is replaced by a constant non-regular element. Furthermore, the Lax operator admits a reduction group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, as opposed to the case of the vector NLS equation. These properties of the Lax operator are important ingredients in the construction of the hierarchy and associated conservation laws, as well as the recursion operator. The reduction group plays also an important role in the construction of solutions to the hierarchy. Specifically, in this work we constructed Darboux transformations for soliton and breather solutions for the whole hierarchy, and presented general formulas for the one-soliton and higher-rank one-breather solution. An interesting direction of future study would be the construction of higher soliton and breather solutions and the study of their interactions, the expression of the general n-soliton solution in terms of determinants, and, consequently, connections with Hirota's τ functions and bilinear form for the hierarchy.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] P. Adamopoulou, A. Doikou, G. Papamikos, Darboux–Bäcklund transformations, dressing and impurities in multi-component NLS, Nucl. Phys. B 918 (2017) 91–114.
[2] S.C. Anco, Hamiltonian flows of curves in $G/SO(N)$ and vector soliton equations of mKdV and sine-Gordon type, SIGMA 2 (2006) 044.
[3] S.C. Anco, N.T. Ngatat, M. Willoughby, Interaction properties of complex modified Korteweg-de Vries (mKdV) solitons, Physica D 240 (2011) 1378–1394.
[4] C. Athorne, A.P. Fordy, Generalised KdV and mKdV equations associated with symmetric spaces, J. Phys. A, Math. Gen. 20 (1987) 1377–1386.
[5] A.A. Belavin, V.G. Drinfel'd, Solutions of the classical Yang-Baxter equation for simple Lie algebras, Funct. Anal. Appl. 16 (1982) 159–180.
[6] F. Calogero, A. Degasperis, New integrable equations of nonlinear Schrödinger type, Stud. Appl. Math. 113 (2004) 91–137.
[7] M.F. de Groot, T.J. Hollowood, J.L. Miramontes, Generalized Drinfel'd-Sokolov hierarchies, Commun. Math. Phys. 145 (1992) 57–84.
[8] V.G. Drinfel'd, V.V. Sokolov, Equations of Korteweg-de Vries type, and simple Lie algebras, Dokl. Akad. Nauk SSSR 258 (1981) 11–16.
[9] V.G. Drinfel'd, V.V. Sokolov, Lie algebras and equations of Korteweg-de Vries type, J. Sov. Math. 30 (1985) 1975–2036.
[10] L. Fehér, J. Harnad, I. Marshall, Generalized Drinfeld-Sokolov reductions and KdV type hierarchies, Commun. Math. Phys. 154 (1993) 181–214.
[11] V. Fenchenko, E. Khruslov, Nonlinear dynamics of solitons for the vector modified Korteweg-de Vries equation, J. Math. Phys. Anal. Geom. 14 (2018) 153–168.
[12] A.P. Fordy, P.P. Kulish, Nonlinear Schrödinger equations and simple Lie algebras, Commun. Math. Phys. 89 (1983) 427–443.
[13] V.S. Gerdjikov, The generalized Zakharov-Shabat system and the soliton perturbations, Theor. Math. Phys. 99 (1994) 593–598.
[14] V.S. Gerdjikov, G.G. Grahovski, A.V. Mikhailov, T.I. Valchev, Polynomial bundles and generalised Fourier transforms for integrable equations on A.III-type symmetric spaces, SIGMA 7 (2011) 096.
[15] M.A. Guest, Harmonic Maps, Loop Groups, and Integrable Systems, Cambridge University Press, 1997.

[16] M. Gürses, A. Karasu, V.V. Sokolov, On construction of recursion operators from Lax representation, J. Math. Phys. 40 (1999) 6473.
[17] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, American Mathematical Society, 2001.
[18] R. Hirota, Exact envelope-soliton solutions of a nonlinear wave equation, J. Math. Phys. 14 (1973) 805–809.
[19] M. Iwao, R. Hirota, Soliton solutions of a coupled modified KdV equations, J. Phys. Soc. Jpn. 66 (1997) 577–588.
[20] J. Jackaman, G. Papamikos, T. Pryer, The design of conservative finite element discretisations for the vectorial modified KdV equation, Appl. Numer. Math. 137 (2019) 230–251.
[21] G. Marí Beffa, J.A. Sanders, J.P. Wang, Integrable systems in three-dimensional Riemannian geometry, J. Nonlinear Sci. 12 (2002) 143–167.
[22] V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons, Springer-Verlag, Berlin, 1991.
[23] A.V. Mikhailov, Integrability of the two-dimensional generalization of Toda chain, JETP Lett. 30 (7) (1979) 414–418.
[24] A.V. Mikhailov, Reduction in integrable systems. The reduction group, JETP Lett. 32 (1980) 174–178.
[25] A.V. Mikhailov, The reduction problem and the inverse scattering method, Physica D 3 (1981) 73–117.
[26] A.V. Mikhailov, Formal diagonalisation of Lax-Darboux schemes, Model. Anal. Inf. Syst. 22 (6) (2015) 795–817.
[27] A.V. Mikhailov, G. Papamikos, J.P. Wang, Dressing method for the vector sine-Gordon equation and its soliton interactions, Phys. D, Nonlinear Phenom. 325 (2016) 53–62.
[28] A.V. Mikhailov, G. Papamikos, J.P. Wang, Darboux transformation for the vector sine-Gordon equation and integrable equations on a sphere, Lett. Math. Phys. 106 (2016) 973–996.
[29] R.M. Miura, C.S. Gardner, M.D. Kruskal, Korteweg-de Vries equation and generalizations. II. Existence of conservation laws and constants of motion, J. Math. Phys. 9 (1968) 1204–1209.
[30] P.J. Olver, Evolution equations possessing infinitely many symmetries, J. Math. Phys. 18 (1977).
[31] D.E. Pelinovsky, Y.A. Stepanyants, Helical solitons in vector modified Korteweg-de Vries equations, Phys. Lett. A 382 (2018) 3165–3171.
[32] A. Pressley, G. Segal, Loops Groups, Clarendon Press, Oxford, 1986.
[33] J.A. Sanders, J.P. Wang, Integrable systems in n-dimensional Riemannian geometry, Mosc. Math. J. 3 (2003) 1369–1393.
[34] I.P. Shestakov, V.V. Sokolov, Multi-component generalizations of mKdV equation and non-associative algebraic structures, arXiv:1905.01016 [nlin.SI], 2019.
[35] V.V. Sokolov, T. Wolf, Classification of integrable polynomial vector evolution equations, J. Phys. A, Math. Gen. 34 (2001) 11139–11148.
[36] S.I. Svinolupov, Jordan algebras and integrable systems, Funct. Anal. Appl. 27 (1993) 257–265.
[37] S.I. Svinolupov, V.V. Sokolov, Vector-matrix generalizations of classical integrable equations, Theor. Math. Phys. 100 (1994) 959–962.
[38] T. Tsuchida, M. Wadati, The coupled modified Korteweg-de Vries equations, J. Phys. Soc. Jpn. 67 (1998) 1175–1187.
[39] T. Tsuchida, Multisoliton solutions of the vector nonlinear Schrödinger equation (Kulish-Sklyanin model) and the vector mKdV equation, arXiv:1512.01840 [nlin.SI].
[40] G. Wilson, The modified Lax and two-dimensional Toda lattice equations associated with simple Lie algebras, Ergod. Theory Dyn. Syst. 1 (1981) 361–380.
[41] V.E. Zakharov, A.V. Mikhailov, Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method, Sov. Phys. JETP 47 (1978) 1017–1027.
[42] V.E. Zakharov, A.B. Shabat, Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II, Funct. Anal. Appl. 13 (1979) 166–174.