Lock-in through passive connections*

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Abstract

We consider a model of social coordination and network formation where agents decide on an action in a coordination game and on whom to establish costly links to. We study the role of passive connections; these are links that other agents form to a given agent. Such passive connections may create an endogenously arising form of lock-in where agents don't switch actions and links, as this may result in a loss of payoff received through them. When agents are constrained in the number of links they form, the set of Nash equilibria includes action-heterogenous strategy profiles, where different agents choose different actions. Depending on the precise parameters of the model, risk-dominant, payoff-dominant, or action-heterogenous strategy profiles are stochastically stable.

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1 Introduction

We propose a novel explanation for why we sometimes observe multiple technology standards being adopted at the same time. This explanation does not require any sort of heterogeneity among agents but instead is based on the idea that agents may become locked into their action choices when the interaction structure among them is determined by a process of network formation. To solidify ideas, consider an agent deciding on which kind of technology standard to adopt. Typically, this agent is better off if she interacts with somebody using the same technology standard, thus giving rise to a coordination game. In addition to her action chosen, her payoff depends on the choices of her interaction partners. These interaction partners can be distinguished in two groups, those she actively chooses to interact with and those who actively choose to interact with her, i.e. those who she passively interacts with. While agents have a say over the composition of the former group, they typically have much less control over who belongs to the latter.

The relative importance of benefits received through passive interaction depends on the context of the interaction among agents.[1] For instance, consider a set of agents who can decide whether to adopt a VHS recorder or use the Betamax standard. Forming a link in this context represents borrowing a video cassette from another agent. While this act carries positive payoff to the borrower (the active side of the interaction) there is little or no benefit to the lender (the passive side of the interaction). In other circumstances there are, however, clear benefits for the passive side of an interaction. For instance, consider software developers who can choose among multiple programming languages (e.g. Python and Java). In this context, forming a link indicates initiating a collaboration. Clearly, both parties benefit, regardless of who initiated the link, and the benefits are higher if both use the same programming language.[2]

We study the role of payoff received through such passive connections in a model of social coordination and network formation similar to the ones presented in Goyal & Vega-Redondo (2005) and Staudigl & Weidenholzer (2014). In contrast to previous work, action-heterogenous network configurations, where different actions are adopted by different agents, are Nash equilibria and sometimes may even be stochastically stable. More specifically, in our model there is a population of agents who decide on an action in a 2×2 coordination game and who choose their active interaction partners via establishing costly links to them.[3] The coordination game captures a conflict between efficiency- and risk- considerations, encapsulated by one equilibrium being payoff dominant and the other being risk dominant. In line with Staudigl & Weidenholzer (2014) and in

[1] Bala & Goyal (2000) distinguish between one-way and two-way flow of benefits.

[2] Trade and communication constitute further important examples where both sides of an interaction benefit and are also ripe with coordination problems, involving trading conventions (such as system of measurements or unit of exchange used) or communication technology standards (such as IOS vs. Android).

[3] Our main analysis focuses on low linking costs so that every potential interaction carries a positive net payoff. In section 6.3 we discuss higher linking costs where certain interactions among agents become unprofitable.

contrast to Goyal & Vega-Redondo (2005), we focus on a scenario where agents are constrained in the number of links they may form, thereby reflecting technological constraints in link formation or decreasing marginal benefits from socializing. For instance, in software development this is the case if the number of feasible collaborations is small compared to the pool of potential collaborators. Unlike Staudigl & Weidenholzer (2014) and in line with Goyal & Vega-Redondo (2005) agents also receive payoff from passive connections.

We argue that under constrained interactions the payoff received from passive connections may create an endogenous form of lock-in. To fix ideas, consider an agent's optimal choice in a population where both actions are used by other agents. If this agent has no passive connections (or there is no payoff earned from these as in Staudigl & Weidenholzer (2014)), she will choose the payoff dominant action and exclusively link up to agents choosing it (as the payoff dominant action gives the highest possible per interaction payoff). In contract, if this agent has passive connections from agents choosing the risk dominant action, then it may be optimal to choose the risk dominant action too and link up to other agents choosing it. The reason for this is that the payoff dominant action could result in a lower payoff received through passive connections. In this sense passive connections may endogenously create a situation where agents become locked into choosing the risk dominant action. When interactions are constrained, the resulting network may feature separated network components. Hence, agents may face different distributions of actions among their neighbors and consequently find it optimal to choose different actions. Thus, under passive payoffs and constrained interactions action-heterogenous profiles may be Nash equilibria (see Proposition 1). In contrast, this cannot happen if interactions are either unconstrained and payoff is earned from passive connections, as in Goyal & Vega-Redondo (2005), or if there is no payoff from passive connections (regardless whether interactions are constrained or not) as in Staudigl & Weidenholzer (2014). In the former case, the network will be fully connected. In the latter case, the actions of passive connections do not enter into payoffs. In both cases, all agents face the same distribution of actions among their (potential) interaction partners and necessarily have to choose the same action. It is thus the combination of constrained interactions *and* passive payoffs that leads to the emergence of action-heterogenous profiles as Nash equilibria.

We proceed by considering a myopic best response process in discrete time where a revising agent chooses links and actions so to maximize the payoff from the previous period. We characterize the absorbing sets of this dynamic process and, in doing so, provide a refinement of the set of Nash equilibria. In addition to profiles where all agents choose the same action, certain action-heterogenous profiles turn out to be absorbing (see Proposition 2). Interestingly, in action-heterogenous profiles the subnetwork among agents choosing the risk dominant action has to be fully connected, putting an upper bound on the number of agents choosing the risk dominant action. This is because all risk dominant agents need to receive sufficiently many passive links to be locked in.

Finally, we provide a discussion on the impact of passive connections on the long run outcome of our model. To this end, we consider a perturbed version of the process where agents with small probability make mistakes and choose actions and links different to the ones specified by the myopic best response. Following Kandori, Mailath & Rob (1993) and Young (1993) we identify stochastically stable states by assessing the relative robustness of the absorbing states to mistakes. The combination of passive payoffs and constrained interactions has important consequences for the dynamics of the model and for the transition among the various absorbing sets. To appreciate our results it is again useful to consider the models of Goyal & Vega-Redondo (2005) and Staudigl & Weidenholzer (2014) as a benchmark. When interactions are unconstrained the network will be fully connected regardless of whether there is payoff from passive interactions or not. Thus, as in Kandori et al. (1993), the question which convention will be selected comes down to a comparison of the size of the basin of attraction of the two actions; a comparison won by the risk dominant action. When interactions are constrained and there is no payoff from passive connections, payoff dominant profiles are stochastically stable: whenever there is a small cluster of agents choosing the payoff dominant action, agents want to choose the payoff dominant action and link up to agents using it. As some agents may not have any incoming links, this mechanism is also present in our paper. However, there is a further force at play. Starting from profiles where everybody chooses the payoff dominant action, risk dominant actions are able to spread contagiously through parts of the network. This is similar to the spread of risk dominant actions in fixed interactions structures (see Ellison (1993, 2000) and Morris (2000)).[4] The crucial difference is that in the present context the interaction structure among agents arises endogenously. Moreover, the network evolves at the same time as agents adjust their actions. This constitutes another difference to the fixed interaction case and has important consequences for the number of agents who change their action as a result of the initial mistake; effectively putting an upper bound on the size of a subnetwork through which a risk dominant action may spread. In this sense, in the present setting the network plays a crucial role for the propagation of actions and evolves as agents adjust their links. In contrast the transition among various absorbing states in previous work are rather mechanic with no functional role for the network. In both Goyal & Vega-Redondo (2005) and Staudigl & Weidenholzer (2014) first a certain fraction of the population switches and then everybody follows suit.

Our main results are obtained by determining the relative importance of these dynamic forces for the transition among the various absorbing sets. We identify parameter ranges for which payoff dominant-, risk dominant-, and action heterogenous- are (uniquely) stochastically stable. The exact prediction depends on the size of the population, the number of links agents may support, the basin of attraction of the risk dominant action, and on the degree of payoff dominance. In a nutshell, whenever the payoff dominant action earns a sufficiently high payoff when played against itself, then network configurations where everybody chooses it are stochastically stable (Proposition 3).

[4]See also Weidenholzer (2010) for a survey.

If the basin of attraction of the risk dominant action is sufficiently large, either risk dominant- or action-heterogenous- network configurations are stochastically stable (Proposition 4). In this case, action-heterogenous network configurations are uniquely stochastically stable if the population is large enough. This is because the transition from action-heterogenous network- to risk dominant network- configurations becomes more difficult in large populations. We further demonstrate (by means of Example 2) that sometimes even action-heterogenous networks where there are links between agents choosing different actions are stochastically stable.

Our results are in stark contrast to previous work where universal coordination on one convention always occurs and may offer an explanation why sometimes multiple technology standards are observed at the same time. Interestingly, this happens in a world where agents are ex-ante homogenous (so that in principle there shouldn't be a reason to expect action-heterogenous outcomes) and is driven by the endogenously formed interaction structure among agents. Our results draw on several crucial factors. Firstly, interactions are constrained so that agents only interact with a small subset of the overall population. Such constrained interactions may arise endogenously when linking costs are convex or benefits of interaction are concave in the number of links (as e.g. in the extensions discussed in Jackson & Watts (2002) or Staudigl & Weidenholzer (2014)) or simply be the result of technical restrictions in the linking technology (as is sometimes the case on online platforms such as facebook or Twitter). There is also ample of empirical evidence (see e.g. chapter 3 in Jackson (2008)) suggesting that usually the number of links agents form is small relative to the overall population. In this light, constrained interactions often seem to be a better description of agents' linking choices. Secondly, the emergence of action-heterogenous network configurations rests upon a conflict between payoff- and risk- dominance. Payoff dominance means that one technology is inherently better than the other (when used against itself). Risk dominance means that a technology offers a payoff advantage when used against a mixed group of interaction partners. As such, risk dominance is (in addition to its inherent properties) induced by the benefits received by users interacting with users using different standards as captured by the notions of inward- and outward compatibility.[5] Lastly, there have to be benefits from passive connection. As argued above, this will depend on the nature of the interacting under consideration.

1.1 Related literature

In addition to Goyal & Vega-Redondo (2005) and Staudigl & Weidenholzer (2014), discussed above, Hojman & Szeidl (2006) also employs a non-cooperative network formation model in the spirit of Bala & Goyal (2000) where agents unilaterally decide on whom to link to. In Hojman & Szeidl (2006) agents do not benefit from passive connections, but obtain payoff from indirect neighbors (i.e. agents connected through a directed path). As the resulting network is connected,

[5] See the discussion in Kandori & Rob (1998).

each agent faces the same distribution of actions among her (potential) opponents. Thus, as in Goyal & Vega-Redondo (2005) and Staudigl & Weidenholzer (2014), all agents will have to choose the same action in any Nash equilibrium and consequently in any stochastically stable network.[6] Bilancini & Boncinelli (2018) study a model of constrained interactions where different types of agents cannot observe each other's type and incur a cost of interacting with agents of the other type. If this cost is high, action-heterogenous profiles may be Nash equilibria and stochastically stable. Intuitively, agents can avoid interacting with agents of the other type by utilizing chosen actions as signals for the underlying types. In contrast, in our model there are no types and the co-existence of conventions arises with homogenous agents. Jackson & Watts (2002) differ from the contributions above by considering a process of network formation based on pairwise stability, due to Jackson & Wolinsky (1996). Further, links and actions cannot be adjusted simultaneously. Since link formation requires the consent of both parties and both parties pay for it, there is no distinction between active and passive interactions. While under constrained interactions action-heterogenous profiles may sometimes be pairwise stable, they are never stochastically stable.

An alternative branch of the literature models endogenous neighborhood formation through the choice of a location where agents interact. The premise that underlies these models is more radical, in the sense that agents can only change their interaction partners by uprooting from their current location and moving to a new one. In contrast, the choice of interaction partners in the current model of network formation is much more granular. An additional difference lies in the way in which payoffs of individual interactions are aggregated. While in network formation models agents receive the sum of payoffs from individual interactions, in the location literature agents typically receive average payoffs. Thus, whereas in network formation models agents prefer large over small neighborhoods, this force is absent in location models. In location models when there are either capacity constraints at locations or immobile agents, profiles where agents on different locations choose different actions may constitute Nash equilibria and sometimes may be stochastically stable (see e.g. Anwar (2002), Shi (2013) and Pin, Weidenholzer & Weidenholzer (2017)).[7] Intuitively, under such exogenous frictions agents may get stuck at locations where risk dominant actions are played as they are not able to move to their preferred location. In contrast, in the present paper there are no exogenous frictions but agents do not switch actions as this would entail a loss in the payoff earned through passive interactions. Moreover, the transitions among the various states in location models involve first a fraction of the population switching and then others following suit. This property is shared with most of the network literature. In the present paper, however, transitions are less mechanic and the interaction structure plays an important role for the spread of actions. Finally, while in location models all agents at a given location choose the same action and

[6]Which action will be stochastically stable depends on the exact payoff parameters and the level of linking costs.

[7]If there are no restrictions on mobility, payoff dominant outcomes are stochastically stable (see Oechssler 1997, 1999 and Ely 2002).

consequently there is no interaction between agents using different actions, our model may also feature interaction between agents using different actions.

Lock-in is also prominently discussed in industrial organisation. There it captures the phenomenon that consumers may be locked into certain choices through switching costs or by the choices of the agents they interact with.[8] The latter mechanism is referred to as network effects which conceptually correspond to coordination games. Under such network effects consumers benefit from complementarity of their chosen product with the product chosen by others, implying that the benefit of adopting a product is increasing in the size of its user base. While the classic literature on network effects has been predominantly concerned with strategic decision making of firms,[9] the present paper and the literature on coordination and network formation complements this literature by studying the implications of network effects on the interaction among consumers.

2 The model

We consider N agents who play a symmetric 2×2 coordination game against each other. Let $I = \{1, 2, \cdots, N\}$ denote the set of all agents. In addition to choosing an action in the coordination game, agents can choose interaction partners by forming links.

The coordination game is defined as follows. Each player can select an action from the set $\{A, B\}$ to be used in all of her interactions. Let $u(a, a^{'}) > 0$ denote the payoff of a player with action a against another player with action $a^{'}$. The payoffs are given in the following table:

	A	B
A	a, a	c, d
B	d, c	b, b

We assume that $a > d$, $b > c$ so that (A, A) and (B, B) are strict Nash equilibria. Further, we assume $b > a$ so that the equilibrium (B, B) is payoff-dominant and $a + c > b + d$ so that the equilibrium (A, A) is risk dominant in the sense of Harsanyi & Selten (1988); that is, A is the unique best response against an opponent playing both actions with equal probability. Note that this assumption and the payoff dominance of the equilibrium (B, B) together imply that $c > d$. Finally, we assume that $a > c$ implying that an A-player prefers playing against another A-player over playing against a B-player. These assumptions imply the following ordering of payoffs,

$$b > a > c > d > 0.$$

We denote by

$$p^* = \frac{b - c}{a + b - c - d}$$

[8] See Farrell & Klemperer (2007) for a comprehensive review.

[9] See e.g. Farrell & Saloner (1985) and Katz & Shapiro (1985) for two seminal contributions in this tradition.

the smallest fraction of an agent's opponents playing A such that playing A gives at least as high a payoff as playing B.[10] Correspondingly, $1-p^*$ represents the size of the basin of attraction of A. Risk dominance of (A,A) implies $p^* < \frac{1}{2}$, meaning that A has a larger basin of attraction than B.

In addition to their action choice in the coordination game, agents can decide on whom to link to. If agent i forms a link with agent j, we write $g_{ij}=1$, and if agent i does not form a link with agent j, we write $g_{ij}=0$. Agents do not link to themselves, $g_{ii}=0$ for all $i \in I$. We focus on a scenario where each agent may at most support k links, $\sum_{j\in I} g_{ij} \leq k$ for all $i \in I$. Agents are, however, not constrained in the number of links they can passively receive.

In the following we introduce some additional notation and provide a number of definitions, most of which are standard in the literature. The linking decision of agent i can be summarized as an N-tuple $g_i = (g_{i1}, g_{i2}, \cdots, g_{iN}) \in \mathcal{G}_i$ where $\mathcal{G}_i$ denotes the set of all permissible linking decisions of agent i, i.e. $\mathcal{G}_i = \big\{g_i \in \prod_{j\in I}\{0,1\} : g_{ii} = 0 \text{ and } \sum_{j\in I} g_{ij} \leq k\big\}$. The network induced by the linking decisions of all agents is denoted by $g = \big(I, \{g_{ij}\}_{i,j\in I}\big) \in \mathcal{G}$ where $\mathcal{G} = \prod_{i\in I}\mathcal{G}_i$ is the set of all permissible networks. We denote by $g_{I'} = \big(I', \{g_{ij}\}_{i,j\in I'}\big)$ the network defined on a subset of the population $I' \subseteq I$ and refer to it as *sub-network*.

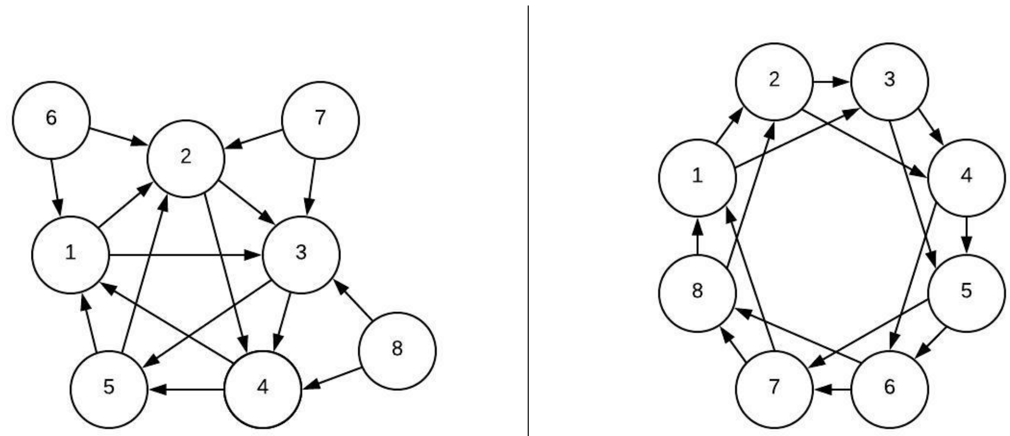
We denote by $N_i^{out}(g) = \{j \in I : g_{ij} = 1\}$ the set of agents to whom agent i forms a link and by $N_i^{in}(g) = \{j \in I : g_{ji} = 1\}$ the set of agents who form a link with agent i. We refer to the agents in the set $N_i^{out}(g)$ as active neighbors and to agents in the set $N_i^{in}(g)$ as passive neighbors. Further, $d_i^{out} = \sum_{j\in I} g_{ij}$ denotes the out-degree of agent i and $d_i^{in} = \sum_{j\in I} g_{ji}$ denotes the in-degree of agent i.

We denote by $I_A = \{i \in I | a_i = A\}$ the set of A-players and by $I_B = \{i \in I | a_i = B\}$ the set of B-players in the population. The number of A-players is given by $m = \big|I_A\big|$ and the number of B-player is $N-m$. We denote by $I_{AB} = \{i \in I_A : \sum_{j\in I_B} g_{ij} > 0\}$ the set of A-players who form links to B-players and by $I_{AA} = I_A \setminus I_{AB}$ the set of A-players who only form links to A-players. We denote by $m_i^{out} = \big|\{j \in N_i^{out}(g) | a_j = A\}\big|$ the number of A-players agent i actively connects to and by $m_i^{in} = \big|\{j \in N_i^{in}(g) | a_j = A\}\big|$ the number of A-agents that connect to i. Consequently the number of B-players among i's active neighbors is given by $d_i^{out} - m_i^{out}$ and the number of B-players among i's passive neighbors is $d_i^{in} - m_i^{in}$.

A network g is *essential* if $g_{ij}g_{ji} = 0$ for any two distinct agents $i,j \in I$, i.e. there is no duplication of links. The sub-network $g_{I'}$ is *fully connected* if $g_{ij} + g_{ji} = 1$ for any two distinct agents $i,j \in I'$. Note that to fully connect all agents from I', $\frac{|I'|(|I'|-1)}{2}$ links are needed while $k|I'|$ links are available. Thus, the sub-network g' can only be fully connected if $|I'| \leq 2k+1$. Hence, the size of the largest possible fully connected subnetwork is proportional to the number of maximally allowed links.

Two particular (sub-)network configurations play an important role in our analysis: We say that the sub-network $g_{I'}$ defines a *core-periphery network* if there exists a subset of cardinality $2k+1$,

[10] This corresponds to the critical mass on action A in the mixed strategy equilibrium of the coordination game.



a) Core-periphery network. Agents 1-5 form the core and agents 6-8 are in the periphery.

b) Circle network with $\kappa = k = 2$. Since $N > 2k+1$, it is not fully connected.

Figure 1: Network configurations with $N = 8$ and $k = 2$.

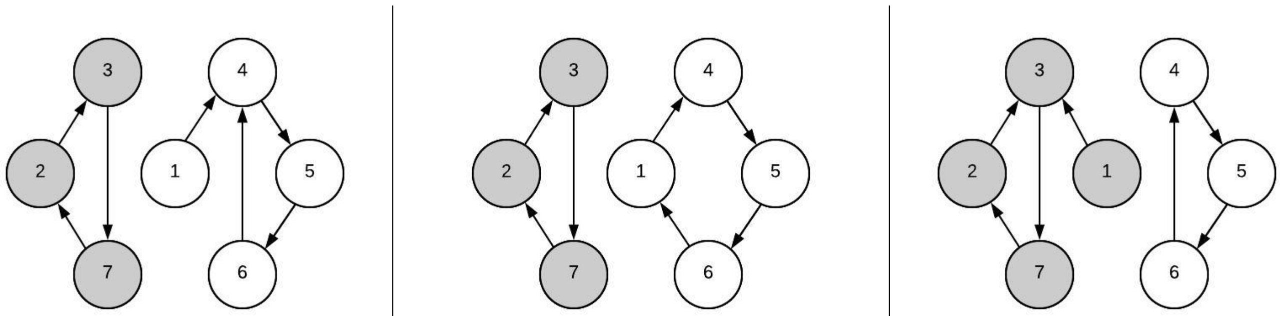
$I^{''} \subsetneqq I^{'}$, such that the sub-network $g_{I^{''}}$ is fully connected and all agents $i \in I^{'} \setminus I^{''}$ form k links to agents in $I^{''}$ (see Figure 1a) for an illustration of a core-periphery network with eight players and $k = 2$).[11] Another important sub-network features agents from a subset of the population $I^{'} \subseteq I$ arranged to form a circle where all agent connects to their κ immediate neighbors on one side. More formally, a subset of agents $I^{'}$ is said to form a *circle* of width κ if the agents in $I^{'}$ are arranged as $\{i_1, \cdots, i_\ell\}$, the sub-network $g_{I^{'}}$ is essential and each agent $i_j \in I^{'}$ forms κ links to agents $i_{j+1}, \cdots, i_{j+\kappa}$, where $j + \kappa'$, $1 \leq \kappa' \leq \kappa$, is understood as modulo ℓ.[12] Figure 1 b) illustrates a circle network of width 2. Note that to define a circle of width κ, the set $I^{'}$ has to contain at least $2\kappa + 1$ agents. In fact, since each agent $i \in \{i_1, \cdots, i_\kappa\}$ forms a link with agent $i_{\kappa+1}$, agent $i_{\kappa+1}$ has to form links with κ agents different to $i_1, \cdots, i_\kappa$ for the sub-network g' to be essential.

A pure strategy of an agent i consists of her action choice $a_i \in \{A, B\}$ and her linking decision $g_i \in \mathcal{G}_i$ and is denoted by $s_i = (a_i, g_i) \in \mathcal{S}_i = \{A, B\} \times \mathcal{G}_i$. A strategy profile or state is an N-tuple specifying a pure strategy for every agent, $s = (s_i)_{i \in I} \in \mathcal{S} = \prod_{i \in I} \mathcal{S}_i$. The strategy profile of all agents except i is an $(N-1)$-tuple and is denoted by $s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_N) \in \mathcal{S}_{-i} = \prod_{j \in I \setminus \{i\}} \mathcal{S}_j$.

Each agent has to pay a cost of γ, with $0 < \gamma < d$, for supporting each of her active

[11] This is similar to the definition in Galeotti & Goyal (2010). The difference to our definition is that in Galeotti & Goyal (2010) all players in the periphery connect to all players in the core and there are no restrictions on the number of players in the core. See also Borgatti & Everett (2000) for a related definition covering undirected networks.

[12] The wheel network in Bala & Goyal (2000) is a special case of a circle subnetwork with $I^{'} = I$ and $\kappa = k = 1$.



a) Non-NE network | b) Action-heterogenous NE network | c) Absorbing NE network

Figure 2: Various network configurations. White nodes indicate agents using the risk dominant action and grey nodes the payoff dominant action, respectively.

links.[13,14] There is no cost for receiving links. Given the strategy profile of other agents, $s_{-i} = (s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_N)$, the payoff of agent i from playing strategy s_i is given by

$$\pi_i(s_i, s_{-i}) = \sum_{j \in N_i^{out}(g)} u(a_i, a_j) + \sum_{j \in N_i^{in}(g)} (1 - g_{ij}) u(a_i, a_j) - \gamma d_i^{out}.$$

The first term on the right-hand side specifies the payoff from interacting with agents whom agent i is actively linked to and the second term gives the payoff of interacting with agents who are passively linked to agent i. The term $1 - g_{ij}$ implies that agent i does not receive payoff from passively interacting with j when she already receives payoff through actively interacting with j, $g_{ij} = 1$.[15] The description of payoffs concludes the specification of the game $\big(I, \{S_i\}_{i \in I}, \{\pi_i\}_{i \in I}\big)$.

3 The static game

3.1 Optimal action and link Choice

In a first step we will analyze the agents' best response. This will consist of an optimal linking choice and action choice. In order to provide insights into the main mechanism at play in our model we will introduce a simple example which we will also revisit later on to discuss key results.

Example 1. *Consider a population of seven agents each of whom can at most support one link. Suppose that the network among agents and their action choices are given by Figure 2a). Consider*

[13]We exclude the case $\gamma = 0$ from our main analysis. This ensures that agents will not form any superfluous links and implies that Nash equilibrium networks are essential. All of our remaining results do however also hold for $\gamma = 0$. See the discussion in online appendix D.3.

[14]We discuss higher linking costs in section 6.3.

[15]Thus, active and passive links are substitutes just as in Goyal & Vega-Redondo (2005). Dropping the term $1 - g_{ij}$ would yield a specification where active and passive links are complements.

agent 1 who plays A and forms a link to agent 4 who also plays A. She currently earns a payoff of $a - \gamma$. By deleting the link and forming a link to agent 3 while keeping her current action, her payoff would decrease to $c - \gamma$. Moreover, by switching to action B and keeping her link her payoff would decrease to $d - \gamma$. Switching to B and linking to some B-player would instead give her a payoff of $b - \gamma$. It is thus optimal for her to switch to action B and form link optimally. This is similar to the behavior in Staudigl & Weidenholzer (2014) where there is no payoff from passive links. Now consider agent 5 who plays action A, links to A-agent 6, and has an incoming link from A-agent 4. Keeping her current action and links gives her a payoff of $2a - \gamma$. However, switching to B and linking to another B-agent would decrease her payoff to $b + d - \gamma$. She will thus be better off keeping her current action and links. Note that the crucial difference between agents 1 and 5 was that agent 5 has an incoming link from another A-player which means that switching to B becomes less attractive as it entails a loss in the payoff received through this passive link.

In order to formally study the best reply of agents in a more general context, we first consider the optimal linking strategy keeping the current action of the agent fixed. Following this, we identify conditions under which either of the two actions gives a higher payoff. The first part corresponds to the derivation of link optimized payoffs (LOP) in Staudigl & Weidenholzer (2014), taking into account the role of passive connections.

The LOP of agent i with action a_i is the payoff when she links up optimally given a strategy profile of the other agents s_{-i}. It is given by

$$v(a_i, m, d_i^{in}, m_i^{in}) = \max_{g_i \in \mathcal{G}_i} \pi_i\left((a_i, g_i), s_{-i}\right).$$

To determine the optimal linking strategy note that payoffs from active and passive connections are substitutes. Thus, agents will not actively form links to agents they are already passively linked to. Since we are considering linking costs $0 < \gamma < d$, each connection carries a positive payoff. Consequently, provided there are at least k other agents who are not linked to i, $N - d_i^{in} - 1 \geq k$, agent i will form all of her k links. Further, since we are considering a coordination game (where $a > c$ and $b > d$) agents prefer to link up to agents using the same action as they do. Agents will only link to agents with a different action if they are already linked to all agents with the same action they are using. Formally, the set of optimal linking decisions of an A-player is given by

$$\Big\{ g_i \in \mathcal{G}_i \quad : \quad d_i^{out} = \min\{N - d_i^{in} - 1, k\}, m_i^{out} = \min\{m - m_i^{in} - 1, k\} \Big\}.$$

Likewise, the set of optimal linking decisions of a B-player is characterized by

$$\Big\{ g_i \in \mathcal{G}_i \quad : \quad d_i^{out} = \min\{N - d_i^{in} - 1, k\}, d_i^{out} - m_i^{out} = \min\{N - m - (d_i^{in} - m_i^{in}) - 1, k\} \Big\}.$$

Given the optimal linking strategies we now compute the payoff received from playing the two actions. The LOP of an A-player when there are m A-players and $N - m$ B-players in the population and when m_i^{in} of her passive neighbors choose A and $d_i^{in} - m_i^{in}$ choose B is given by

$$\begin{aligned} v(A, m, d_i^{in}, m_i^{in}) &= a \min\{m - m_i^{in} - 1, k\} \\ &+ c\big(\min\{N - d_i^{in} - 1, k\} - \min\{m - m_i^{in} - 1, k\}\big) \\ &+ [m_i^{in} a + (d_i^{in} - m_i^{in})c] - \gamma \min\{N - d_i^{in} - 1, k\}. \end{aligned}$$

The first term on the RHS captures the payoff received by actively linking up to other A-agents, the second term captures payoff received by actively linking up to B-agents, the third term captures all payoff received through passive connections, and the last term captures the linking costs. Similarly, the LOP of a B-player is given by

$$\begin{aligned} v(B, m, d_i^{in}, m_i) &= b \min\{N - m - (d_i^{in} - m_i^{in}) - 1, k\} \\ &+ d\big(\min\{N - d_i^{in} - 1, k\} - \min\{N - m - (d_i^{in} - m_i^{in}) - 1, k\}\big) \\ &+ [(d_i^{in} - m_i^{in})b + m_i^{in} d] - \gamma \min\{N - d_i^{in} - 1, k\}. \end{aligned}$$

We finally characterize conditions under which an agent will choose either of the two actions. Together with the previously derived optimal linking choice this defines the best response of agents. An A-player i, strictly prefers A over B if $v(A, m, d_i^{in}, m_i^{in}) > v(B, m-1, d_i^{in}, m_i^{in})$, prefers B over A if $v(A, m, d_i^{in}, m_i^{in}) < v(B, m-1, d_i^{in}, m_i^{in})$, and is indifferent if $v(A, m, d_i^{in}, m_i^{in}) = v(B, m-1, d_i^{in}, m_i^{in})$. Likewise, a B-player prefers B if $v(A, m+1, d_i^{in}, m_i^{in}) < v(B, m, d_i^{in}, m_i^{in})$, prefers A if $v(A, m+1, d_i^{in}, m_i^{in}) > v(B, m, d_i^{in}, m_i^{in})$ and is indifferent otherwise.[16] Tables A.1 and A.2 in the online appendix report conditions under which A- and B- players will keep their action for the various different cases that can occur.

3.2 Nash equilibrium networks

We can now proceed to provide a characterization of Nash equilibrium. A strategy profile s^* is a Nash equilibrium of the social game $\big(I, \{S_i\}_{i \in I}, \{\pi_i\}_{i \in I}\big)$ if and only if

- $\pi_i\big((a_i^*, g_i^*), s_{-i}^*\big) = v(a_i^*, m, d_i^{in}, m_i^{in})$ for any agent $i \in I$, and
- $v(A, m, d_i^{in}, m_i^{in}) \geq v(B, m-1, d_i^{in}, m_i^{in})$ for every player $i \in I_A$ and $v(B, m, d_j^{in}, m_j^{in}) \geq v(A, m+1, d_j^{in}, m_j^{in})$ for every player $j \in I_B$.

The first condition says that for each agent i, g_i^* is an optimal linking decision, and the second condition states that agent i with action a_i^* cannot improve her payoff by switching to the other action and choosing links optimally. We denote the set of Nash equilibria by S^*.

[16] Agents have to exclude themselves when calculating payoffs, i.e. an A-player faces $m-1$ A-players and $N-m$ B-players and a B-player faces m A-players and $N-m-1$ B-players. Further, agents switching actions have to consider the impact on the distribution of actions. An A-player switching to B faces $m-1$ A-players and $N-m$ B-players after switching. A switching B-player faces m A-players and $N-m-1$ B-players after the switch.

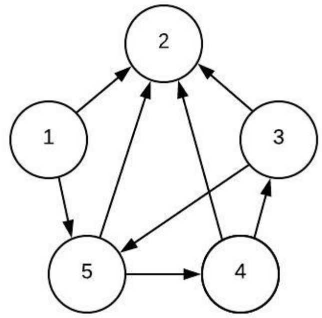


Figure 3: Nash equilibrium with $N = 5$ and $k = 2$. All agents play action A.

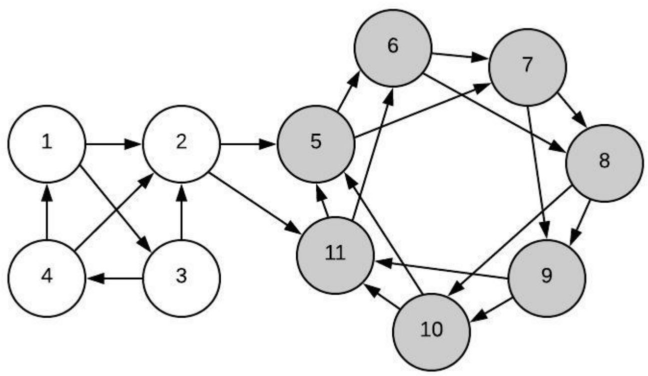


Figure 4: Nash equilibrium for $N = 11$ and $k = 2$ in a game when $3a \geq 2b+d$ and $4b+d \geq 3a+2c$. Agents $6 - 10$ cannot improve by switching to A. Agents $1, 3, 4$ will not switch to B provided $3a \geq 2b + d$, agent 2 will not switch if $3a + 2c \geq 2b + 3d$ (which is implied by $3a \geq 2b + d$ and $c > d$), and agents 5 and 11 will not switch to A since $4b + d \geq 3a + 2c$.

There is a variety of different Nash equilibria. First, strategy profiles where all players choose the same action and form all links are Nash equilibria. There may also be equilibria where agents do not form all of their links, as Figure 3 shows. This example also illustrates that Nash equilibrium networks do not have to be fully connected (even if there are sufficiently many links available). We refer to equilibria where agents use the same action as *action-homogenous* equilibria. Interestingly, there may exist Nash equilibria where agents choose different actions. We refer to such equilibria as *action-heterogenous*. In Figure 2b) and c) we exhibit two such equilibria for Example 1. In these equilibria each A-player receives one link from another A-player. Switching to B and linking up optimally would increase their payoff received through the active link but even more decrease the payoff from the passive link. Moreover, action-heterogenous equilibria may even feature agents using different actions interacting with each other, as the example in Figure 4 demonstrates.

This is in contrast to Goyal & Vega-Redondo (2005) and Staudigl & Weidenholzer (2014) where all agents have to choose the same action in equilibrium. The reason for this is that in Goyal & Vega-Redondo (2005) agents are unconstrained in their linking choice, $k = N - 1$. Thus, agents will be fully connected and therefore in every Nash equilibrium all have to choose the same action.[17] Similarly, in the model of Staudigl & Weidenholzer (2014) where there is no payoff from passive links, all agents want to link up to the same set of agents and consequently have to choose the same action in equilibrium. The reason why there exist action-heterogenous Nash equilibria in the present setting is that under constrained interactions the resulting interaction structures can be segregated. Thus, agents potentially may face different distributions of actions in their neighborhoods. Further, agents choosing the risk-dominant action may have sufficiently many passive links from agents choosing the risk-dominant action so that they are locked-in and switching to the payoff-dominant action and connecting to agents using it does not pay off.

The next proposition provides a condition when action-heterogenous Nash equilibria always exist and a condition when they never exist.

Proposition 1. *All Nash equilibrium networks are essential. Action-homogeneous Nash equilibrium networks always exist. If $N \leq k+2$, all Nash equilibrium networks are action-homogeneous. If $N \geq 4k+2$, there is always an action-heterogenous Nash equilibrium.*

Proof. Note that any Nash equilibrium network has to be essential. For, if otherwise an agent forming superfluous links would benefit from their deletion. Further, strategy profiles where each agent chooses the same action and either forms all of her k links or is connected to all other agents are clearly Nash equilibria.

First consider $N = k+1$ where the network will be fully connectedi. The claim follows without modification from Proposition 3.2 in Goyal & Vega-Redondo (2005). Now consider $N = k + 2$ and assume there exists an action-heterogenous Nash equilibrium. There have to be at least two A-players and two B-players. Note that there have to exist at least two agents, call them i and j, who are not connected to each other. Otherwise the network would be fully connected and as in Goyal & Vega-Redondo (2005) all agents have the same action as their best response. Each of these agents will form all of her k links and thus will have to be connected to all agents using the same action. Thus, i and j must choose different actions; if not they would be connected. However, since they are not fully connected it must be true that both have no incoming links, $d_i^{in} = d_j^{in} = 0$. As argued by Staudigl & Weidenholzer (2014) in lemma 1, agents with no incoming links have to choose the same action, yielding a contradiction.

Now consider the case $N \geq 4k + 2$. In this case the population can be arranged in two circles of width k each, one playing A and the other playing B (see Figure 2b) for an illustration). In such

[17] If linking costs are high enough so that interactions with agents using the other action carry negative payoff, $\gamma > c > d$, strategy profiles where there are two fully connected and separate components where each action is played may form a Nash equilibrium in Goyal & Vega-Redondo (2005).

profiles each A-player receives a payoff of $2ka - \gamma k$. By switching to B and forming active links to B-players a payoff of $k(b+d) - \gamma k$ can be achieved. Since $k(b+d) < k(a+c) < 2ka$, the deviation does not pay off. Similarly, a B-player who earns $2kb - \gamma k$ can at most earn $k(a+c) - \gamma k$ by switching to A and changing her links optimally. Thus, if $N \geq 4k+2$ there exists a Nash equilibrium where agents use different actions. □

In the range $N \in [k+3, 4k+1]$ the question whether there may be action-heterogenous Nash equilibria or not is more complicated and depends on the parameters of the coordination game. The main complication in this range arises as any (potential) action heterogenous Nash equilibrium will necessarily feature interaction of agents using different actions.

For the remainder we will focus on the case where $N > 4k+2$, ensuring that there exist action-heterogenous Nash equilibria. The rest of this section discusses a number of properties of action-heterogenous equilibria. These properties do not only provide insights into the structure of Nash equilibria but are also crucial for our later analysis.[18] The first lemma reveals a straightforward insight on the conditions when agents connect to agents using different actions.

Lemma 1. *In any Nash equilibrium s^* every A-player i who actively links to B-players, $d_i^{out} - m_i^{out} > 0$, is connected to all other A-players, $N_i^{out}(g) \cup N_i^{in}(g) \supseteq I_A \setminus \{i\}$. Likewise, every B-player i who actively links to A-players, $m_i^{out} > 0$, is connected to all other B-players, $N_i^{out}(g) \cup N_i^{in}(g) \supseteq I_B \setminus \{i\}$.*

This lemma follows from the following observation: In a coordination game an agent is always better off when interacting with somebody using the same action. Thus, an agent, who is not connected to all other agents using her action, can improve her payoff by deleting links to agents choosing a different action and by linking up to unconnected agents with the same action.

In the absence of payoff from passive connections A-players would switch actions and link up to B-players provided there is a sufficiently large number of them. When there is payoff from passive connections, A-players have to take into account that when they switch actions they will receive lower payoffs from their existing passive contacts. The next lemma identifies how many passive links A-players at least have to have for a switch not to occur, i.e. it provides conditions when A-players are locked-in their current action choice.

Lemma 2. *Consider an action-heterogenous Nash equilibrium s^* where B-players only link to other B-players. Then,*

i) for every A-player i who does not link to B-players, $d_i^{out} - m_i^{out} = 0$, $m_i^{in} \geq \frac{b-a}{a-d}k$ holds, and

[18] Lemma 21 in the online appendix In a coordination game an agent is always better off a more detailed characterization of action-heterogenous Nash equilibria for the special case where $k = 1$.

ii) for every A-player i who links to B-players, $d_i^{out} - m_i^{out} > 0$, $m_i^{in} \geq \max\left\{\frac{(b-c)k-(a-c)(m-1)}{c-d}, \frac{b-a}{a-d}k\right\}$ *holds.*

The proof of this lemma can be found in the appendix. The first part considers A-players who only link to other A-players and the second part considers the case of A-players who also form links to B-players. Intuitively, the more incoming links from other A-players they have, the less prone they are to switch. Further, as A-players who link to other B-players obtain less payoff from active links to A-players, they will require more passive links from A-players to be locked-in.

The following lemma provides lower bounds for the number of A-players and B-players in action-heterogenous Nash equilibria.

Lemma 3. *For every action-heterogenous Nash equilibrium* s^**,*

i) $m \geq \left\lceil \frac{2(b-c)}{2a-c-d}k \right\rceil + 1 := \underline{m}$,

ii) $N - m \geq \left\lceil \frac{2(a-d)}{2b-c-d}k \right\rceil + 1 := N - \overline{m}$.

The proof of this lemma is relegated to the appendix. The first part provides a lower bound for the number of A-players and the second part provides a lower bound for the number of B-players. Intuitively, there have to be sufficiently many agents using each of the two actions such that supporting an action-heterogenous Nash equilibrium is possible. The particular bounds are obtained by examining conditions on the network (i.e. incoming and outgoing links) such that agents do not find it worthwhile to switch to the other action. Note that this lemma also provides upper bounds for the number of A- and B-players in an action-heterogenous Nash equilibrium, $m \leq \overline{m}$ and $N - m \leq N - \underline{m}$.[19]

4 Myopic best response learning

We consider a myopic best-response learning process à la Kandori et al. (1993) and Young (1993). Each period one agent receives the opportunity to update her strategy. When an agent is presented with such a revision opportunity she chooses action and links as a best response to the distribution of play in the previous period. The implication of this simultaneous adjustment is that agents who consider switching actions can optimally adjust their links.[20]

[19] One can verify that sometimes these bounds are attainable and sometimes they are not (see examples 3 and 4 in the online appendix).

[20] This formulation is in-line with previous work by Goyal & Vega-Redondo (2005), Hojman & Szeidl (2006) and Staudigl & Weidenholzer (2014) but differs from the model of Jackson & Watts (2002) where agents cannot adjust their links optimally when switching actions. We discuss the potential implications of an alternative adjustment process where revising agents can either adjust links *or* actions in section 6.2.

Formally, the adjustment process is defined in the following way. Each period $t = 0, 1, 2, \cdots$, one agent i is randomly chosen to update her strategy with probability $\nu(i)$ where $\sum_{j \in I} \nu(j) = 1$ and $\nu(j) > 0$ for all $j \in I$.[21] Agent i chooses an action and linking decision to maximize her payoff in the previous period. More formally,

$$s_i(t+1) \in \arg\max_{s_i \in \mathcal{S}_i} \pi_i\big(s_i, s_{-i}(t)\big).$$

Whenever there is more than one element in the set on the right hand side, we assume that agent i chooses one element at random.[22] The strategy revision process defines a Markov chain $\{S(t)\}_{t \in \mathbb{N}}$. The state space of this Markov chain is given by the set of strategy profiles $\mathcal{S}$. We call this process the *unperturbed dynamics*.[23] An *absorbing set* is a minimal subset $\mathcal{S}' \subset \mathcal{S}$ such that once the dynamics is there, the probability of leaving it is zero. Absorbing sets may contain more than one element and the unperturbed dynamics may reach any two states, s and s', contained in a given absorbing set $\mathcal{S}'$ from one another with positive probability. We denote the set of absorbing sets of the unperturbed process by S^{**}.

We now proceed to formally analyze the dynamics and characterize the absorbing sets. This exercise does not only provide a refinement of the set of Nash equilibria, but is also a necessary step for our stochastic stability analysis in Section 5. Our first lemma shows that the dynamics converges to a Nash equilibrium, starting from any initial configuration.

Lemma 4. *From every state $s \in \mathcal{S}$ the unperturbed dynamics with positive probability reaches a Nash equilibrium s^*.*

The proof is relegated to the online appendix. While it is hardly surprising that our best response process may reach a Nash equilibrium with positive probability, the proof is not straightforward as we have to rule out cyclical behavior. We do so by constructing sequences of strategy revisions of individual agents, at the end of which nobody has a strict incentive to change strategy.

Before characterizing the absorbing sets of our process we revisit Example 1 to illustrate an important property of our model, namely that from action-heterogenous Nash equilibria the process will reach profiles where all A-players are fully connected, implying that their number is bounded.

Example 1 revisited. *To see that in action-heterogenous absorbing sets the subnetwork among A-players is fully connected, consider a Nash equilibrium where A-players are not fully connected,*

[21] If multiple agents were to update at the same time the resulting process is not guaranteed to settle at a Nash equilibrium. To see this point assume each agent can only support one link, $k = 1$, and consider an action-homogenous Nash equilibrium where $g_{ik} = 1$ and $g_{j\ell} = 1$. Since agents i and j are indifferent they may link up to any other agents using the same action. Thus, with positive probability we reach a strategy profile where both i and j support a link to each other, $g_{ij} = g_{ji} = 1$, which is clearly not a Nash equilibrium. While this complication would not change the long run prediction, assuming that only one agent revises at a time avoids it altogether.

[22] Section 6.1 discusses the potential implications of considering a process where indifferent agents stick to their current links and actions.

[23] In Section 5 we will introduce the possibility of mistakes, thus giving rise to a *perturbed dynamics*.

as in Figure 2b). Since agent 6 is indifferent between forming a link to agents 1 and 4, the dynamic with positive probability moves to a state where agent 6 replaces the link to 1 to a link to 4 (see Figure 2a)). Agent 1 now has no passive links and (when given revision opportunity) will switch to action B and link up to some agent using it (see Figure 2c)). Also note that each A-player now chooses a unique best response in terms of action and links and that B-players will not change action but may form alternative links to different B-players. This implies that the process may reach alternative configurations where the subnetwork among B-players varies but the one among A-players remains constant. All such profiles constitute an action-heterogenous absorbing set. Interestingly and in contrast to Nash equilibria, there can at most be three A-players in such a set.

We now move on to analyze the process more formally, demonstrating the previous observation in a more general framework and recovering additional important properties. The following lemma identifies a particular subclass of Nash equilibria the process may reach.

Lemma 5. *From every Nash equilibrium s^* the unperturbed dynamics with positive probability either reaches an action-homogenous Nash equilibrium where all agents support all of their k links or an action-heterogenous Nash equilibrium where all agents support all of their k links that fulfills the following properties,*

i) there are no more than $\overline{\overline{m}} = 2k+1$ and no less than $\underline{\underline{m}} = k+2+\max\{\lfloor \frac{b-a}{a-d}k \rfloor, k - \lceil \frac{b-a}{a-d}k \rceil\}$ fully connected A-players,

ii) each B-player forms all k links to other B-players,

iii) the number of A-players who form links to B-players, $|I_{AB}|$ is strictly less than $\frac{b-a}{a-d}k$.

The proof of this lemma is fairly technical and is reported in the online appendix. We proceed by providing some intuition for the various properties here. The logic why there can be at most $2k+1$ players choosing A generalizes the reasoning in Example 1: For profiles where more than $2k+1$ agents choose A it is possible to exhibit a path of revisions of individual agents, at the end of which at least one A-agent has no incoming links. In such a situation, similar to Staudigl & Weidenholzer (2014), it is optimal for such an agent to delete existing links, switch to B, and link up to B-players. The finding that the A-players have to be fully connected derives from constructing a series of strategy revisions in which any possible links from A- to B- players are exchanged by A to A links. The insight that the process may reach equilibria where each B-player forms all k links to other B-players rests on the fact that, since there are no more than $2k+1$ agents playing A, there have to be at least $2k+1$ agents who choose B. Consequently, it is in theory possible that each B-player forms all of her k links to other B-players. In the proof we show that the process indeed may reach such profiles. Property iii) captures another interesting aspect of our dynamic model, namely that the process may reach profiles where the number of A-agents supporting links to B-agents is small. Intuitively, if there are many A-players forming links to B-players, the process can move to a state where all of these agents form links to a B-player without incoming links from other B-agents. This agent will then be compelled to switch to action A. This observation also has consequences for the minimal number of A-players as expressed in the characterization of the minimal number of A-players in an absorbing set, $\underline{\underline{m}}$.

We denote by $ab[n]$ a profile that satisfies the properties in lemma 5 and where there are n A-players, with $\underline{\underline{m}} \leq n \leq \overline{\overline{m}}$. Note that while $ab[n]$ is a Nash equilibrium, it is not strict as the A-players are indifferent about which of the B-players (if any) to connect to and B-players also may have multiple potential sets of link recipients. Thus, starting from a Nash equilibrium $ab[n]$ the process with positive probability reaches any other Nash equilibrium $ab'[n]$ with the same set of A-players and the same subnetwork among these agents.[24] We collect all such profiles in the set $\overrightarrow{ab}[n]$ and remark that the process cannot leave this set, i.e. $\overrightarrow{ab}[n]$ is an absorbing set.

Note that even when keeping the identities of A-players constant there may be multiple absorbing sets differing in the subnetwork among A-players. Likewise, keeping the number of A-players n constant, there are several absorbing sets differing in the identities of- and subnetworks among- A-players. In order to subsume these absorbing sets into one category, we denote by $\overrightarrow{AB}[n]$ the set of all action-heterogenous absorbing sets with n A-players. The set of all action-heterogenous absorbing sets is then given by $\overrightarrow{AB} = \bigcup_{\underline{\underline{m}} \leq n \leq \overline{\overline{m}}} \overrightarrow{AB}[n]$. We finally denote by $\overrightarrow{A}$ and $\overrightarrow{B}$ the set of action-homogenous Nash equilibria where everybody supports k links and chooses A and B, respectively. We are now able to provide the following result.

Proposition 2. *The absorbing sets are given by* $S^{**} = \overrightarrow{A} \bigcup \overrightarrow{B} \bigcup \overrightarrow{AB}$.

Proof. Lemma 5 shows that the process will eventually reach a set in S^{**}.

Let us now assume that the process is in a Nash equilibrium in such a set. Start with a profile in $\overrightarrow{A}$. Whenever an agent receives revision opportunity she will not change her action. However, she can potentially link up to other A-players she is currently not linked to. Accordingly, the process may reach all states in the absorbing set $\overrightarrow{A}$. The same argument applies for states in $\overrightarrow{B}$.

Let us now consider the case where the process is in an action-heterogenous Nash equilibrium $ab[n]$. Since the A-players are fully connected, every A-player will always face k or less other A-agents who are not actively connected to him. Thus, a revising agent will always link up to those agents and the network among A-players will not change. Further, since there are strictly less than $\frac{b-a}{a-d}k$ agents connecting to B-players, no B-player will change from B to A. Since B-players support all of their links to other B-players every revising B-player will always have at least k other B-players to link to. Thus, while the network among B-players may change no B-player will ever link to an A-player. Hence, A-players will not change their action either.

It thus follows that while the process may move among the various states $ab[n]$ in an absorbing set $\overrightarrow{ab}[n]$, it may never leave the absorbing set $\overrightarrow{ab}[n]$. □

[24] More formally, let the set of A agents in $ab[n]$ and $ab'[n]$ be given by I_A and I'_A respectively. Then i) $I_A = I'_A$ and ii) $g_{I_A} = g_{I'_A}$.

We conclude this section by noting that lemmata 3 and 5 imply that the minimal number of A-agents in an absorbing set is no smaller than in any Nash equilibrium, $\underline{\underline{m}} \geq \underline{m}$. Likewise, the maximal number of A-agents is lower in an absorbing set than in any Nash equilibrium, $\overline{\overline{m}} < \overline{m}$. It follows that the absorbing sets constitute a proper subset of the set of Nash equilibria.

5 Stochastically stable networks

In the previous section we have argued that the unperturbed process will converge to a certain subset of Nash equilibria, the absorbing sets. This refinement step has excluded several network configurations and has also trimmed down the number of agents we may expect to use each action. There is however still a multiplicity of profiles that are absorbing, encompassing action-homogenous and action-heterogenous profiles. In order to assess which kind of behaviour is more likely to arise in the long run we will now study which kind of profiles are *stochastically stable*. In order to do so we complement the myopic best response process with the possibility of mistakes.

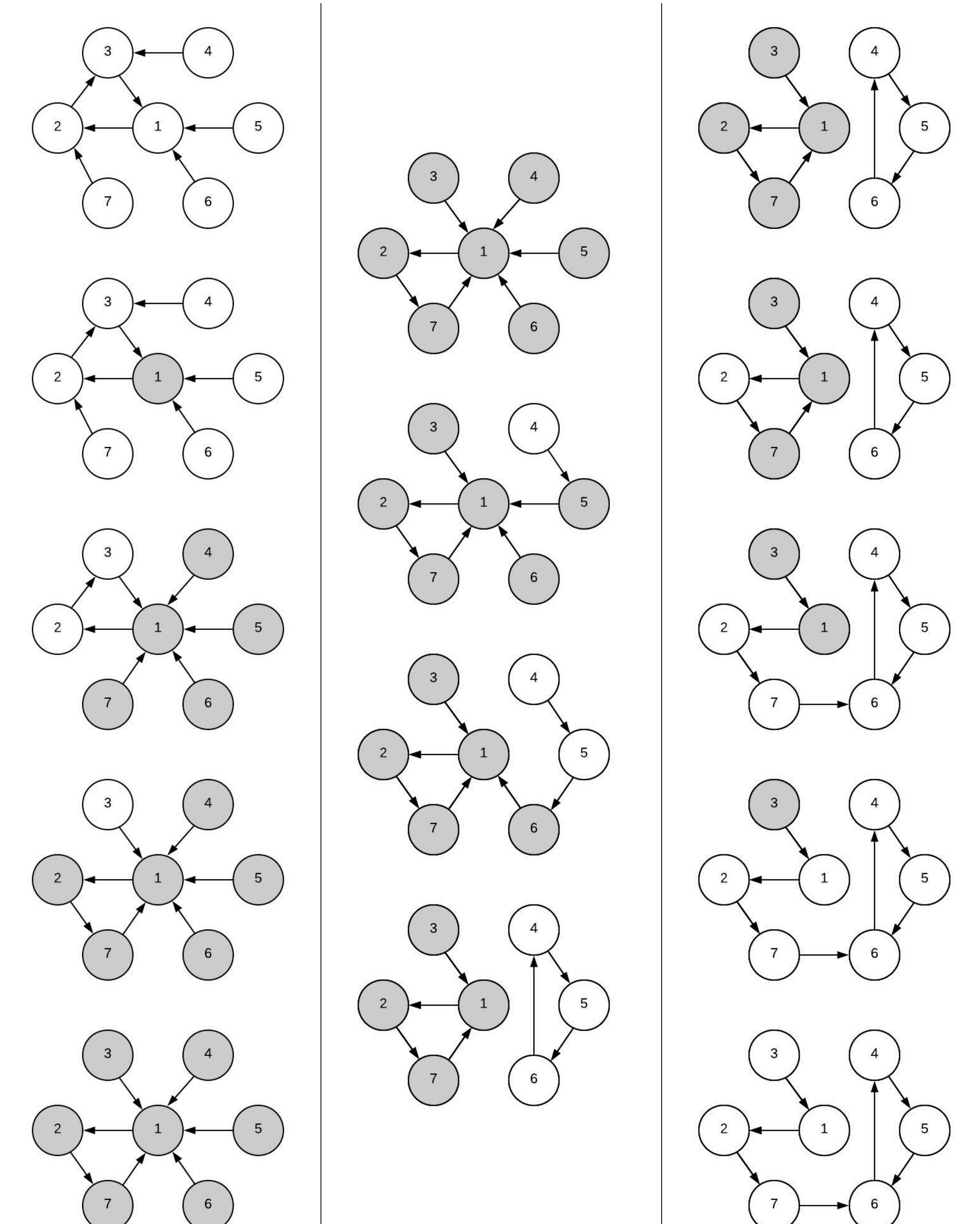
With fixed probability $\epsilon \in (0, 1)$, independent across time and agents, the selected agent ignores the prescription of the adjustment process and chooses a strategy (action and links) at random from the set $\mathcal{S}_i$, assigning positive probability to each of its elements. The process with mistakes, $\{S^\epsilon(t)\}_{t\in\mathbb{N}}$, is referred to as *perturbed dynamics*. For each $\epsilon > 0$, $\{S^\epsilon(t)\}_{t\in\mathbb{N}}$ is an irreducible, aperiodic Markov chain and has a unique invariant distribution $\mu(\varepsilon)$. We are interested in the *limit invariant distribution* as the error rate goes to zero, $\mu^* = \lim_{\epsilon\to 0} \mu(\varepsilon)$. This invariant distribution exists and provides a prediction for the long-run outcome of the perturbed process in the sense that when ϵ is small enough, the play in the long run corresponds to the distribution of play described by μ^*.[25] States in the support of μ^*, are referred to as stochastically stable states. We denote the set of stochastically stable states by $S^{***} = \{s \in \mathcal{S} \mid \mu^*(s) > 0\}$. Following Kandori et al. (1993) and Young (1993) we identify stochastically stable states by assessing the relative robustness of the absorbing states to mistakes.

Before presenting our main results, we revisit Example 1 to i) showcase the nature of transitions between the various classes of absorbing sets and to ii) develop intuition for our later findings.

Example 1 revisited. *We will now determine the relative robustness of the various network/action configurations to mistakes. Figure 5 illustrates these transitions. In a first step, we study the transition from $\overrightarrow{A}$ to $\overrightarrow{B}$. Assume the dynamics has reached a network configuration such as the one in the first row of Figure 5a). In this core-periphery network agents 1,2, and 3 form the core and are fully connected to each other.*[26] *The remaining agents in the periphery connect to these core agents. Assume now that agent 1 makes a mistake and switches to B, but keeps her links*

[25]See Foster & Young (1990), Kandori et al. (1993) or Young (1993).

[26]Lemma 18 in the online appendix shows that the dynamics with positive probability may reach such core-periphery networks.



a) Risk- to payoff- dominant network

b) Payoff- dominant to action-heterogenous network

c) Action-heterogenous to risk dominant network

Figure 5: Transitions among absorbing sets.

unchanged. Following this, the periphery agents 4,5,6, and 7 will switch to B. In a next step, agent 2 who has one incoming link from agent 1 choosing B will follow suit, switch actions, delete the link to agent 3, and form a link to another agent using B, say agent 7. Finally, agent 3 will switch actions. With one mistake we have thus reached a state where everybody chooses B.

Now consider $\overrightarrow{B}$ and assume that the process has reached a core-periphery network such as the one in the first row of Figure 5b). Assume that agent 4 makes a mistake, switches to A and replaces the link to agent 1 with a link to agent 5. Now agent 5 has one passive link from an A-player. Thus, she will find it optimal to switch to A. As there are no A-players she is not already linked to, she will link to a periphery agent choosing B, say agent 6. Now agent 6 will find it optimal to switch actions. However, she can now link to an agent choosing A, namely agent 4. We have thus reached an action-heterogenous absorbing network. Hence action A is able to spread contagiously through parts of the network. This is similar to the spread of actions in fixed networks (see Ellison (1993, 2000) and Morris (2000)). In the present setting, however, the network among agents arises endogenously and evolves as agents adjust their actions. This has important consequences for the number of agents who change their action as a result of the initial mistake; effectively putting an upper bound on the size of a subnetwork through which risk dominant actions may spread.

Finally, consider an action-heterogenous absorbing set such as the one in the first row of Figure 5c). Assume that agent 2 by mistake switches to A. This will prompt agent 7 to switch actions and link up to a player using A. Now agent 1 will switch actions which in turn makes the remaining agent 3 also switch. With one mistake we have thus reached a network configuration in $\overrightarrow{A}$.

We have thus shown that in the present example, where $N = 7$, $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{\mathcal{AB}}$ can be reached from each other via a chain of single mistakes and are consequently all stochastically stable. Note, however, that if there would be eight instead of seven agents one mistake is not enough to induce the transition from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{A}$ as agent 1 would connect to this eighth agent and keep playing action B. Thus, the number of mistakes required to reach $\overrightarrow{A}$ is increasing in the population size. This in turn implies that for $N > 7$ states in $\overrightarrow{A}$ are not stochastically stable.

The finding that $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{\mathcal{AB}}$ are stochastically stable is fundamentally different to Goyal & Vega-Redondo (2005) who predict risk dominant networks and Staudigl & Weidenholzer (2014) predicting payoff dominance. Even more interesting, in the present setting the network plays a crucial role for the propagation of actions and evolves as agents adjust their links. In contrast the transition among various states in Goyal & Vega-Redondo (2005) and Staudigl & Weidenholzer (2014) is rather mechanic with no functional role for the network. First a certain fraction of the population switches and then everybody follows suit.

We now proceed to generalize these insights. This serves three different purposes: i) We demonstrate that the results and mechanisms identified in the example are not just a curiosity that arises in the special case where everybody may only support one link. ii) We are able to identify parameter ranges such that payoff dominant- , risk dominant-, and action-heterogenous network configurations are (uniquely) stochastically stable. The predictions of the general model are, thus, not a "anything goes" result. iii) We are able to demonstrate that sometimes action-heterogenous profiles in which agents with different actions interact with each other are stochastically stable.

In the two main Propositions of this section we identify conditions under which each of our candidates $\overrightarrow{A}$, $\overrightarrow{B}$, and $\overrightarrow{\mathcal{AB}}$ is (uniquely) stochastically stable. We have relegated the proofs to the appendix. There we first characterize transitions among the various absorbing sets in a series of lemmata and then identify circumstances under which particular absorbing sets are most robust to mistakes. We have not been able to fully pin down the transition from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{A}$.[27] Thus, our characterization of stochastically stable sets is necessarily not complete either. However, we are nonetheless able to provide the following two propositions covering a significant range of parameters. We start with the payoff dominant network configuration $\overrightarrow{B}$:

Proposition 3. *There exists b^* such that for $b \geq b^*$ we have $\overrightarrow{B} \subseteq S^{***}$. Further, for $k \geq \frac{a-d}{a-c}$ we have $S^{***} = \overrightarrow{B}$.*

The logic that underlies this proposition is fairly straightforward. Whenever B offers a sufficiently large advantage over the risk dominant action (measured in terms of b, the payoff it earns when matched against itself), only a few agents choosing it will entice those with no or only a few passive links to switch actions. This makes it rather easy to leave risk dominant or action-heterogenous network configurations. At the same time, the more attractive B is, the more difficult it is to leave payoff dominant network configurations. Consequently, for b large enough payoff dominant network configurations are stochastically stable. When k is small, it may nonetheless happen that the transitions into and out of $\overrightarrow{B}$ are equally costly in terms of the number of mistakes required (as these are measured in integers). In this cases $\overrightarrow{B}$ is not uniquely stochastically stable, as Example 1 illustrates. When k is sufficiently large, reaching $\overrightarrow{B}$ is strictly easier than leaving it, implying that $\overrightarrow{B}$ will be uniquely stochastically stable.

Proposition 4. *There exists a $\tilde{p}$ such that for $p^* \leq \tilde{p}$ and $k \geq \frac{b-d}{a-c}$ we have $S^{***} \subseteq \overrightarrow{A} \bigcup \overrightarrow{\mathcal{AB}}$. Further, for N sufficiently large we have $S^{***} = \overrightarrow{\mathcal{AB}}$ and for N small we have $S^{***} = \overrightarrow{A}$.*

Let us provide some technical intuition for this result. Note that if p^* is small, the basin of attraction of the risk dominant action in the underlying coordination game is large. As we have seen in Example 1, when passive connections matter, the risk dominant action may spread through a subset of the population. The smaller p^*, the easier it is to trigger such a contagious spread. Consequently, for small p^* it is relatively easy to move from $\overrightarrow{B}$ to $\overrightarrow{\mathcal{AB}}$. At the same time a large basin of attraction of A makes $\overrightarrow{A}$ and $\overrightarrow{\mathcal{AB}}$ resilient to agents experimenting with payoff dominant actions. Moving from $\overrightarrow{\mathcal{AB}}$ on to $\overrightarrow{A}$ is easier under small levels of p^* too, but becomes more difficult as the population size N increases. For, a fixed set of agents switching to A, just as in

[27] Instead we provide upper and lower bounds on this transition cost (see lemma 14).

example 1, may at most induce a limited set of agents to switch. Consequently, the larger the population the more difficult it is to move from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{A}$. Our results are obtained by weighing each of these factors. For small p^* and a relatively small population risk dominant and action-heterogenous network configurations may be stochastically stable. However, for a large enough population action-heterogenous network configurations are uniquely stochastically stable.[28]

So, unlike previous results where risk- and payoff- dominance per se determined which profiles is stochastically stable, the predictions are not clear cut in present framework. Instead, which profile will emerge in the long run depends on the *degree* of risk dominance (as measured by the size of the basin of attraction of the risk dominant action), the *degree* of payoff dominance (as determined by the payoff advantage the payoff dominant action offers when played against itself) and on the size of the population. Most notably, and also in contrast to previous work, action-heterogenous profiles may be stochastically stable.

A further interesting implication of Proposition 4 is that action-heterogenous network configurations where A-players connect to B-players may be stochastically stable. This occurs since all absorbing sets $\overrightarrow{\mathcal{AB}}$ can be connected via a chain of single mistakes, implying that if any absorbing set in $\overrightarrow{\mathcal{AB}}$ is stochastically stable, so are all the others. Whether sets where agents with different action interact are included then translates into the question whether such sets are absorbing. This in turn boils down to whether there exists absorbing sets where the subnetwork among A-players is not fully connected, $\underline{\underline{m}} < 2k + 1$. While one can show that this never holds for $k \leq 3$, it may very well be the case for larger k, as the following example demonstrates.

Example 2. *Consider $N = 22$, $k = 4$ and a coordination game with parameters $[a, b, c, d] = [34, 45, 33, 1]$. We have that the minimal number of A-players in a absorbing set is $\underline{\underline{m}} = 8 < 9 = 2k + 1$. Thus, network configurations where a single A-agent links to B-agents are absorbing. In addition, one can check that the transitions from $\overrightarrow{A}$ to $\overrightarrow{B}$, from $\overrightarrow{A}$ to $\overrightarrow{\mathcal{AB}}$ and from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{A}$ take three mistakes each, while the transitions from $\overrightarrow{B}$ to $\overrightarrow{\mathcal{AB}}$ and from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{B}$ are possible with only two mistakes. Thus, $S^{***} = \overrightarrow{A} \bigcup \overrightarrow{B} \bigcup \overrightarrow{\mathcal{AB}}$. Note that the latter of these sets includes network configurations where some A-player links to B-players. In fact, for larger k and N we can exhibit examples where more than one A-player does so.*

6 Extensions and discussion

6.1 Tie breaking

In line with previous contributions we assume that when an agent's best response is not unique she will randomize among all action- and link- combinations giving the highest payoff. This assumption is particulary appropriate when agents don't care with which of any two agents to interact

[28]This result again requires k to be large enough, so to avoid special cases as those encountered in Example 1.

(provided they both use the same action) and there is no preference for sticking to previous interaction partners. It is generally understood that it is a fairly harmless assumption and does not gravely change the nature of results (other than in non-generic cases). In the present context, however, it may influence which profiles are absorbing and how the dynamics moves among them. This in turn may have consequences for stochastic stability.[29] For instance, with random tie breaking the unperturbed process may move among states where agents change links so that eventually some A-agents have no incoming links from other A-agents and will switch to B. This puts an upper bound on the number of A-players in any action-heterogenous absorbing set (see lemma 5). In contrast, without tie breaking all Nash equilibria are absorbing. This also has important consequences for stochastic stability as it may be easier to leave the (now larger) set of action-heterogenous absorbing states. In online appendix D.1 we highlight the potential consequences discussing the case where $k = 1$. It turns out that all Nash equilibria can be connected to each other with a chain of single mistakes and are thus stochastically stable, regardless of the population size. This contrasts our main discussion where i) only a subset of Nash equilibria is absorbing and stochastically stable and ii) $S^{***} = \overrightarrow{B} \bigcup \overrightarrow{\mathcal{AB}}$ for $N > 7$. We anticipate that in a more general setting, where $k > 1$, a strict subset of Nash equilibria will be stochastically stable and, depending on the parameter configurations, this may include action-heterogenous profiles.

6.2 Synchronous updating of actions and links

A further premise that underlies our adjustment process is that links and actions are chosen simultaneously. This is also the approach taken by most of the related non-cooperative network formation literature. It postulates that agents are flexible enough to adopt their links following a change in action. In contrast, Jackson & Watts (2002), in their model based on pairwise stability, posit that links and actions are adjusted independently. As a result, agents cannot adjust their links optimally when changing actions and the process that governs network formation becomes more rigid. Clearly, which approach is more appropriate depends on the situation one has in mind.[30] Intuitively, the mechanisms that govern transitions on fixed interaction structures play an important role in this setting. In online appendix D.2 we discuss the implications of an asynchronous (non-cooperative) updating process for the case where $k = 1$. We find that not only are the set of Nash equilibria and the absorbing sets different to the benchmark case but also the stochastically stable set differs, as payoff-dominant network configurations are no longer stochastically stable. Intuitively, as the process governing the formation of the interaction structure becomes less flexible, payoff dominant actions are at a disadvantage. We expect that in a general setting where $k > 1$

[29] A modified adjustment process where indifferent agents keep their links and actions with probability $\alpha \in (0, 1)$ and randomize with probability $1 - \alpha$ would give exactly the same results as in our main discussion.

[30] Staudigl & Weidenholzer (2014) propose an extension where only a fraction of links may be adjusted in each round. Depending on the degree of inertia in link adaption different outcomes may arise.

this may also have implications for the stochastic stability of action-heterogenous profiles.

6.3 Linking costs

We have focused on the case where linking costs are low enough so that the net benefit of any interaction is positive and agents find it worthwhile to link up to agents using different actions. For higher linking costs this is not the case and agents will not form links to agents with a different action. For linking costs in the range $d \leq \gamma \leq c$, B-players will not form links to A-players while A-players will still form links to B-players. Intuitively, while B-players forgo a payoff of d by not linking to an A-player, they also save the linking cost γ by not doing so. Since $\gamma > d$, this makes the payoff dominant action relatively more attractive than the risk dominant action. In online appendix D.3 we show that action-heterogenous network configurations are stochastically stable for the same parameter range as in the benchmark case. For higher linking costs in the range $c < \gamma \leq a$ no agent will form a link to another agent using a different action. Our analysis of this case in the online appendix reveals that with a chain of single mistakes it is possible to move from action-heterogenous absorbing sets to payoff dominant profiles while at the same time it is not possible to leave payoff dominant profiles with only one mistake. As a consequence action-heterogenous profiles are no longer stochastically stable for high linking costs.

7 Conclusion

Most notably, our results offer a novel explanation for why we may observe agents adopting different actions or technology standards at the same time. Since action-heterogenous profiles are sometimes stochastically stable, co-existence is not merely a temporal phenomena but may persist in the long run. Our results do not require heterogeneity of preferences such as in Neary (2012), exogenous given locations allowing agents to separate themselves from others as in Anwar (2002), or feature adopter technologies as in Goyal & Janssen (1997) or Alós-Ferrer & Weidenholzer (2007).[31] Instead, coexistence arises as agents become locked into their action choices through their passive connections. Passive connections may further lead to agents receiving lower payoffs as compared to the relevant benchmark case of Staudigl & Weidenholzer (2014) where agents only receive payoffs from active links. Thus, just as in the classic industrial organization literature lock-in (through passive connection) may lead to the persistence of inefficient technology standards and adverse effects for consumer welfare.

We believe that there are several dimensions that may potentially be fruitful to study. The

[31] In Goyal & Janssen (1997) agents located in a circle network may at an additional cost use two actions at the same time, thus allowing strings of agents using the two different actions to co-exist. In Alós-Ferrer & Weidenholzer (2007) there are more than two actions with some of them acting as buffers between agents using different actions.

first concerns the interplay between active and passive links. In the present contribution active and passive links are substitutes in the sense that duplication of a link between two agents only increases the cost incurred by the two agents involved but does not result in higher payoff. It is also plausible to think of scenarios where duplication leads to a stronger link between the two agents and carries a higher payoff. This avenue could be studied by considering the case where active and passive links are perfect complements or by introducing some weighting between the two. A further interesting question concerns the fraction of players who choose each action in the long run. Our results put an upper limit on the number of agents choosing the risk dominant action in the long run and this upper is independent of the population size. Further, only one component of the network may choose the risk dominant action. Clearly, this is at odds with casual empiricism suggesting a much richer distribution of actions. It would thus be interesting to study under which conditions multiple clusters of agents using different action may arise. Studying a model where each agent may possibly only interact with a certain subset of the population (i.e. those known to her) may potentially be able to achieve this goal.

A Appendix

A.1 Proofs of Section 3

Proof of Lemma 2: Let us start with i). Agent i interacts with other A-players via active links or passive links. The number of active links is k and the number of passive links is m_i^{in}. Note that since B-players only link to other B-players, there are at least $2k+1$ of them. Therefore, the LOP of action A is $a(m_i^{in}+k)-\gamma k$. On the other hand, the LOP of action B is $dm_i^{in}+bk-\gamma k$ which can be attained by forming k links with B-players. As s^* is a Nash equilibrium, it has to be true that $a(m_i^{in}+k)-\gamma k \geq dm_i^{in}+bk-\gamma k$. That is, $m_i^{in} \geq \frac{b-a}{a-d}k$.

Then consider ii). By lemma 1, every A-player i who actively links to B-players must be connected to all other A-players. Thus, i forms $m-1-m_i^{in}$ links with A-players and $k-(m-1-m_i^{in})$ links with B-players. The LOP of action A is thus given by

$$a(m-1-m_i^{in})+c[k-(m-1-m_i^{in})]+am_i^{in}-\gamma k=(a-c)(m-1)+ck+cm_i^{in}-\gamma k$$

and the LOP of action B is $bk+dm_i^{in}-\gamma k$ which can be attained by forming k links with B-players. Since s^* is a Nash equilibrium, it has to be true that $(a-c)(m-1)+ck+cm_i^{in} \geq dm_i^{in}+bk$. It follows that,

$$m_i^{in} \geq \frac{(b-c)k-(a-c)(m-1)}{c-d}. \tag{1}$$

Further, if $m-1 \leq \frac{b-d}{a-d}k$, then $m_i^{in} \geq \frac{(b-c)k-(a-c)(m-1)}{c-d} \geq \frac{(b-c)k-(a-c)\frac{b-d}{a-d}k}{c-d} = \frac{(b-c)k-(a-c)(1+\frac{b-a}{a-d})k}{c-d} = \frac{(b-a)k-(a-c)\frac{b-a}{a-d}k}{c-d} = \frac{b-a}{a-d}k$; if $m-1 > \frac{b-d}{a-d}k$, then $m_i^{in} > (m-1)-k \geq \frac{b-a}{a-d}k$. It follows that

$m_i^{in} \geq \max\left\{\frac{(b-c)k-(a-c)(m-1)}{c-d}, \frac{b-a}{a-d}k\right\}$. □

Proof of Lemma 3: We start with part i) of the lemma. Note that Proposition 1 shows that there always exists an action-heterogenous Nash equilibrium with $m = 2k+1$. It is sufficient for us to only consider the action-heterogenous Nash equilibrium s^* with $m \leq 2k$.

To show that $\underline{m}$ is a lower bound for the number of A-players in s^*, let us focus on A-player i_0 with the fewest passive links from other A-players. If i_0 does not form all of her links in s^*, $d_{i_0}^{out} \leq k-1$, then i_0 is already interacting with all other agents. Thus, for A to be optimal at least a fraction of p^* of all other agents has to choose A. We consequently have that $m-1 \geq (N-1)p^* = (N-1)\frac{b-c}{a+b-c-d} > 4k\frac{b-c}{a+b-c-d} > \frac{2(b-c)}{2a-c-d}k$ and the claim follows for this case.

Now consider the case $d_{i_0}^{out} = k$. There are two cases. In the first case agent i_0, when considering to switch to B, can potentially form all her k links to B-players, $N - m - (d_{i_0}^{in} - m_{i_0}^{in}) \geq k$. In the second case this is not possible.

Consider the first case. There are two subcases to consider, in the first i_0 does not form links to B-players in s^* and in the second she does. In the first subcase, where $d_{i_0}^{out} - m_{i_0}^{out} = 0$, the LOP of action A is $ak + am_{i_0}^{in} + c(d_{i_0}^{in} - m_{i_0}^{in}) - \gamma k$ while the LOP of action B is $bk + dm_{i_0}^{in} + b(d_{i_0}^{in} - m_{i_0}^{in}) - \gamma k$. It follows that $m_{i_0}^{in} \geq \frac{b-a}{a-d}k$. Consequently, $m \geq m_{i_0}^{in} + k + 1 \geq \frac{b-a}{a-d}k + k + 1 = \frac{b-d}{a-d}k + 1 > \frac{2(b-c)}{2a-c-d}k + 1$. In the second subcase where i_0 forms links to B-players in s^*, $d_{i_0}^{out} - m_{i_0}^{out} > 0$, we proceed as follows. The LOP of action A is $a(m-1) + c[k - (m-1-m_{i_0}^{in})] + c(d_{i_0}^{in} - m_{i_0}^{in}) - \gamma k = (a-c)(m-1) + ck + cm_{i_0}^{in} + c(d_{i_0}^{in} - m_{i_0}^{in}) - \gamma k$ while the LOP of action B is $bk + m_{i_0}^{in}d + (d_{i_0}^{in} - m_{i_0}^{in})b - \gamma k$. Rearranging terms yields

$$m_{i_0}^{in} \geq \frac{(b-c)k - (a-c)(m-1) + (d_{i_0}^{in} - m_{i_0}^{in})(b-c)}{c-d} \geq \frac{(b-c)k - (a-c)(m-1)}{c-d}.$$

Further, recall that i_0 is the agent with the fewest passive links from A-players. She consequently must have the most active links to A-agents. Thus, there is no A-player who does not support links to B-players. It follows that all A-agents have to be fully connected. Now note that to connect all A-players $\frac{m(m-1)}{2}$ links are needed. This implies that $m_{i_0}^{in} \leq \frac{m-1}{2}$. It follows that

$$\frac{m-1}{2} \geq m_{i_0}^{in} \geq \frac{(b-c)k - (a-c)(m-1)}{c-d}$$

Rearranging terms yields $m \geq \frac{2(b-c)}{2a-c-d}k + 1$.

Now consider the second case where i_0 can potentially form at most $k-1$ links to B-players when she consider switching to action B, $N - m - (d_{i_0}^{in} - m_{i_0}^{in}) < k$. Again we distinguish two subcases. In the first i_0 does not form links to B-players in s^* and in the second she does. In the first subcase, where $d_{i_0}^{out} - m_{i_0}^{out} = 0$, the LOP of action A is $a(k + m_{i_0}^{in}) + c(d_{i_0}^{in} - m_{i_0}^{in}) - \gamma k$ while the LOP of action B is $b(N-m) + dm_{i_0}^{in} + d\{k - [(N-m) - (d_{i_0}^{in} - m_{i_0}^{in})]\} - \gamma k =$

$(b-d)(N-m)+d(k+m_{i_0}^{in})+d(d_{i_0}^{in}-m_{i_0}^{in})-\gamma k$. We have that,

$$m-1 \geq k+m_{i_0}^{in} \geq \frac{(b-d)(N-m)-(c-d)(d_{i_0}^{in}-m_{i_0}^{in})}{a-d} \geq \frac{(b-c)(N-m)}{a-d}$$

where the last inequality follows from the fact that the number of passive links from B-players cannot exceed the number of B-players, $N-m \geq d_{i_0}^{in}-m_{i_0}^{in}$. Note that $N-m > 2k$, which implies that $m-1 > 2\frac{b-c}{a-d}k \geq \frac{2(b-c)}{2a-c-d}k$. Consider the second subcase where i_0 links to B-players in s^*, $d_{i_0}^{out}-m_{i_0}^{out} > 0$. The LOP of action A is $a(m-1)+c[k-(m-1-m_{i_0}^{in})]+c(d_{i_0}^{in}-m_{i_0}^{in})-\gamma k = (a-c)(m-1)+c(k+d_{i_0}^{in})-\gamma k$ while the LOP of action B is $b(N-m)+dm_{i_0}^{in}+d\{k-[(N-m)-(d_{i_0}^{in}-m_{i_0}^{in})]\}-\gamma k = (b-d)(N-m)+d(k+d_{i_0}^{in})-\gamma k$. Thus,

$$(a+b-c-d)(m-1) \geq (b-d)(N-1)-(c-d)(k+d_{i_0}^{in}) \geq (b-c)(N-1)$$

where the last inequality follows from the fact that $k+d_{i_0}^{in} = d_{i_0}^{out}+d_{i_0}^{in} \leq N-1$. Therefore, $m-1 \geq (N-1)\frac{b-c}{a+b-c-d} = (N-1)p^* > \frac{2(b-c)}{2a-c-d}k$.

Now part ii) of the lemma. To show that $\lceil \frac{2a-2d}{2b-c-d}k \rceil + 1$ is a lower bound for the number of B-players in s^*, let us focus on B-player j_0 with the most passive links from other B-players. If j_0 does not form all of her links in s^*, $d_{j_0}^{out} \leq k-1$, then j_0 is already interacting with all other agents. Thus, for B to be optimal at least a fraction of $1-p^*$ of all other agents has to choose B. We consequently have that $N-m-1 \geq (N-1)(1-p^*) = (N-1)\frac{1}{2} > 2k \geq \lceil k\frac{2a-2d}{2b-c-d} \rceil$ and the claim follows for this case.

Now consider the case $d_{j_0}^{out} = k$. There are two cases. In the first case agent j_0 forms all her k links to B-players in s^*, $N-m-1-(d_{j_0}^{in}-m_{j_0}^{in}) \geq k$. In the second case this is not possible.

Consider the first case. Since B-player j_0 has the most passive links from other B-players. if j_0 forms all of her k links to B-players in s^*, then any other B-player also forms all of her k-links to B-players. As a result, for all B-players, there are $(N-m)k$ links among them while at most $\frac{(N-m)(N-m-1)}{2}$ links can be accommodated. It implies that $\frac{(N-m)(N-m-1)}{2} \geq (N-m)k$. It follows that $N-m-1 \geq 2k \geq \lceil k\frac{2a-2d}{2b-c-d} \rceil$.

Consider the second case. There are two subcases to consider, in the first, agent j_0, when considering to switch to A, can potentially form all her k links to A-players, $m-m_{j_0}^{in} \geq k$, and in the second subcase this is not possible. In the first subcase, the LOP of action A is $ak+am_{j_0}^{in}+c(d_{j_0}^{in}-m_{j_0}^{in})-\gamma k$. The LOP of action B is $b(N-m-1)+d[k-(N-m-1)+(d_{j_0}^{in}-m_{j_0}^{in})]+dm_{j_0}^{in}-\gamma k$. Since s^* is a Nash equilibrium, it has to be true that

$$b(N-m-1)+d[k-(N-m-1)+(d_{j_0}^{in}-m_{j_0}^{in})]+dm_{j_0}^{in} \geq ak+am_{j_0}^{in}+c(d_{j_0}^{in}-m_{j_0}^{in}).$$

Rearranging terms yields

$$\begin{aligned}(b-d)(N-m-1) &\geq (a-d)k+(c-d)(d_{j_0}^{in}-m_{j_0}^{in})+(a-d)m_{j_0}^{in} \\ &\geq (a-d)k+(c-d)\frac{N-m-1}{2},\end{aligned}$$

where the last inequality follows from the fact that $d_{j_0}^{in} - m_{j_0}^{in} \geq \frac{N-m-1}{2}$. This can be written as $N - m \geq k\frac{2a-2d}{2b-c-d} + 1$. In the second subcase where j_0 can not form all her k links to A-players, when considering to switch to A, we proceed as follows. The LOP of action A is $am + c[k - (m - m_{j_0}^{in})] + c(d_{j_0}^{in} - m_{j_0}^{in}) - \gamma k = (a-c)m + ck + cm_{j_0}^{in} + c(d_{j_0}^{in} - m_{j_0}^{in}) - \gamma k$. The LOP of action B is $b(N-m-1) + d[k - (N-m-1) + (d_{j_0}^{in} - m_{j_0}^{in})] + dm_{j_0}^{in} - \gamma k = (b-d)(N-m-1) + dk + d(d_{j_0}^{in} - m_{j_0}^{in}) + dm_{j_0}^{in} - \gamma k$. Since s^* is a Nash equilibrium, it has to be true that

$$\begin{aligned} & (a-c)m + ck + cm_{j_0}^{in} + c(d_{j_0}^{in} - m_{j_0}^{in}) \\ \leq \; & (b-d)(N-m-1) + dk + d(d_{j_0}^{in} - m_{j_0}^{in}) + dm_{j_0}^{in} \end{aligned}$$

which can be rewritten as

$$\begin{aligned} & (a-c)m + (c-d)k + (c-d)m_{j_0}^{in} + (c-d)(d_{j_0}^{in} - m_{j_0}^{in}) \\ \leq \; & (b-d)(N-m-1). \end{aligned}$$

Consequently, $(a-d)m \leq (a-c)m + (c-d)m \leq (a-c)m + (c-d)k + (c-d)m_{j_0}^{in} \leq (b-d)(N-m-1)$ where the second inequality follows from the fact that $m \leq k + m_{j_0}^{in}$. It follows that $(N-m-1) \geq (N-1)\frac{a-d}{a+b-2d} > 4k\frac{a-d}{a+b-2d} > k\frac{2a-2d}{2b-c-d}$. □

A.2 Transition costs and proofs of propositions 3 and 4

The proof of Propositions 3 and 4 uses the Freidlin & Wentzell (1988) algorithm to identify stochastically stable sets.[32] Consider two absorbing sets $\mathcal{S}'$ and $\mathcal{S}''$. Let the *transition cost* $c(\mathcal{S}', \mathcal{S}'') > 0$ be the minimal number of mistakes or mutations required for the transition from $\mathcal{S}'$ to $\mathcal{S}''$. An $\mathcal{S}$-tree is a directed rooted tree with root $\mathcal{S}$ where the nodes of the tree are given by all absorbing sets in S^{**}. The cost of a tree is given by the sum of the costs of transition on each edge. As shown by Freidlin & Wentzell (1988) an absorbing set $\mathcal{S}$ is stochastically stable if and only if there exists an $\mathcal{S}$-tree the cost of which is minimal among all trees.

We start by calculating the transition costs among the absorbing sets. The next lemma shows that in fact all action-heterogenous absorbing sets can be connected via a chain of single mutations.

Lemma 6. *Any two absorbing sets $\overrightarrow{ab}[n]$ and $\overrightarrow{ab}'[n']$, with $\overrightarrow{ab}[n], \overrightarrow{ab}'[n'] \in \overrightarrow{\mathcal{AB}}$, can be accessed from each other via a chain of single mistakes.*

Thus, all absorbing sets in $\overrightarrow{\mathcal{AB}}$ form a mutation connected component in the sense of Nöldeke & Samuelson (1993). Lemma 6 has two important consequences. First, it allows us to subsume all action-heterogenous absorbing sets into the class $\overrightarrow{\mathcal{AB}}$. This greatly simplifies the analysis. Consequently, we denote by $c(\overrightarrow{\mathcal{AB}}, \mathcal{S})$ the minimal transition cost from some absorbing set $\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}$ to

[32]See also Samuelson (1997) for a textbook exposition.

the absorbing set $\mathcal{S}$ and by $c(\mathcal{S}, \overrightarrow{\mathcal{AB}})$ the minimal transition cost from the absorbing set $\mathcal{S}$ to some absorbing set $\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}$.[33] Second, if any absorbing set in $\overrightarrow{\mathcal{AB}}$ turns out to be stochastically stable, so are all other action-heterogenous absorbing sets contained in $\overrightarrow{\mathcal{AB}}$.

Proof. The proof of this lemma follows from the combination of a series of lemmata discussed below. Lemma 7 shows that all absorbing sets in $\overrightarrow{AB}[2k+1]$ can be connected with one another via a chain of single mutations. Lemma 8 shows this for absorbing sets in $\overrightarrow{AB}[n]$, keeping $n < 2k+1$ fixed. Lemma 9 shows that (within the class of action-heterogenous absorbing sets) one can move to some absorbing sets with one more A-player at the cost of one mistake. Lemma 10 makes clear that (within the class of action-heterogenous absorbing sets) it is also possible to move to an absorbing set with one less A-player. □

Lemma 7. *For any two distinct absorbing sets $\overrightarrow{ab}[2k+1], \overrightarrow{ab}'[2k+1] \in \overrightarrow{AB}[2k+1]$, there is a sequence of absorbing sets $(\overrightarrow{ab}_0[2k+1], \cdots, \overrightarrow{ab}_\ell[2k+1])$ such that (1) $\overrightarrow{ab}_{\ell'}[2k+1] \in \overrightarrow{AB}[2k+1]$ for all $0 \leq \ell' \leq \ell$; (2) $\overrightarrow{ab}_0[2k+1] = \overrightarrow{ab}[2k+1]$ and $\overrightarrow{ab}_\ell[2k+1]) = \overrightarrow{ab}'[2k+1]$ and (3) to move from $\overrightarrow{ab}_{\ell'}[2k+1]$ to $\overrightarrow{ab}_{\ell'+1}[2k+1]$ one single mutation is enough for all $0 \leq \ell' \leq \ell - 1$.*

Proof. Consider the case where $I_A \neq I'_A$. Note that $|I_A| = |I'_A| = 2k+1$. This implies that $|I'_A \setminus I_A| = |I_A \setminus I'_A|$. We focus on the case where the identities of A-players differ only by one agent $I'_A \backslash I_A = \{i\}$ and $I_A \backslash I'_A = \{i'\}$. The more general case applies the same argument iteratively and is omitted. Assume the process has reached a state in $\overrightarrow{ab}[2k+1]$ where the B-players' linking strategies define a core-periphery network and i is a periphery agent. Assume that agent i makes a mistake, switches to A and forms k links to all agents in $N^{out}_{i'}(g_{I_A})$. Then, each agent in $N^{in}_{i'}(g_{I_A})$, upon receiving revision opportunity, deletes the link to i' and forms a link to i. In a next step, agent i', who has no passive links, receives revision opportunity. Thus, agent i' switches to action B and forms k links to B-players. We have thus reached a new absorbing set which has the same set of A-players as $\overrightarrow{ab}'[2k+1]$. The network among those A-players may be still different, though.

Consider now the case where $I_A = I'_A$ but the subnetwork among A-players is different, $g_{I_A} \neq g'_{I'_A}$. Note that in both cases the subnetwork among A-players has to be fully connected. This in turn implies that there are at least two agents i and i' for whom the link connecting them points in a different direction in the two cases, that is for g_{I_A} we have $g_{ii'} = 1$ and for $g'_{I'_A}$ we have $g'_{i'i} = 1$. Thus, in g_{I_A} agent i' is connected to all other A-players except i via $k-1$ passive links and k active links while in $g'_{I'_A}$ agent i' is connected to all other A-players except i via k passive links and $k-1$ active links. For agent i', there must thus be another A-player i'' whom she actively connects to in g_{I_A} and is passively connected to in $g'_{I'_A}$. Put differently, $i'' \in I_A$ such that $g_{i'i''} = 1$ in the subnetwork g_{I_A} and $g'_{i''i'} = 1$ in the subnetwork $g'_{I'_A}$. We can apply this reasoning iteratively to

[33] More formally, $c(\overrightarrow{\mathcal{AB}}, \mathcal{S}) = \min_{\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}} c(\overrightarrow{ab}, \mathcal{S})$ and $c(\mathcal{S}, \overrightarrow{\mathcal{AB}}) = \min_{\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}} c(\mathcal{S}, \overrightarrow{ab})$.

show that there in fact has to exist a sequence of links among A-players in the set $\{i_1, \cdots, i_{m'}\}$, the direction of which is different in the two subnetworks g_{I_A} and $g'_{I'_A}$. More formally, for the sequence $(i_1, \cdots, i_{m'})$ it has to be true that under g_{I_A} we have $g_{i_1 i_2} = \ldots = g_{i_{m'-1} i_{m'}} = g_{i_{m'} i_1} = 1$; and under $g'_{I'_A}$ we have $g'_{i_{m'} i_{m'-1}} = \cdots = g'_{i_2 i_1} = g'_{i_1 i_{m'}} = 1$. Because the A-players are fully connected and because of the finiteness of I_A it has to be true that there exists such a sequence that starts at $i_1 = i$ and finishes at $i_{m+1} = i$. Further, note that the length of the sequence has to be strictly larger than 2 and does not exceed $2k + 1$.

We start with the case where the length of the path $m' \leq k + 1$. In a first step the periphery B agent j makes a mistake, switches to A and forms links to agents in the set $\{i_2, \ldots, i_{m'}\}$. Following this, agent i_1 deletes the link to i_2 and forms a link to j. In a next step, agent i_2 deletes the link to i_3 and forms a link to i_1. Since the number of passive links of i_2 remains unchanged, she will not switch to B. We can reiterate this argument, thus reversing the direction of the cycle and making j switch back to B.

We now proceed to discuss the case where $m' > k + 1$. We do this by discussing the case $m' = k + 2$ and remarking that the argument carries over to the more general case. Again, the periphery B-agent j makes a mistake, switches to action A and forms k links to $i_2, \cdots, i_{k+1}$. In a next step, agent i_1 continues to play action A, deletes one link to i_2 and forms a link to j. In the same manner, each agent i_ℓ in the set $\{i_2, \cdots, i_{k+1}\}$ sequentially receives revision opportunity, continues to play A and replaces the link to agent $i_{\ell+1}$ with a link to $i_{\ell-1}$.

Note that now agent i_{k+2} has one less passive link. We thus still need to ensure that i) she will not switch to B and ii) she will eventually form the link to i_{k+1}. This is achieved in the following way and can easily be extended to include the case $m' > k+2$. We assume that all agents in the set $N^{in}_{i_1}(g_{I_A}) \setminus \{i_{k+2}\}$ replace the link to i_1 with a link to j. Now j has k passive links from A-players and will not switch back to B. She may thus delete the link to i_2 and form the link to i_{k+2}. Then i_{k+2} has k passive links from A-players and may delete the link to i_1 and form the link to i_{k+1}. Now all agents in $N^{in}_{i_1}(g_{I_A}) \setminus \{i_{k+2}\}$ delete the link to j and restore the link to i_1. In a final step, i_1 deletes the link to j and forms the link to i_{k+2}, inducing j to switch back to B. We have thus reversed the direction of the cycle. □

Lemma 8. *For any two distinct absorbing sets $\overrightarrow{ab}[n], \overrightarrow{ab}'[n] \in \overrightarrow{AB}[n]$, $n \leq 2k$, there is a sequence of absorbing sets $(\overrightarrow{ab}_0[n], \cdots, \overrightarrow{ab}_\ell[n])$ such that (1) $\overrightarrow{ab}_{\ell'}[n] \in \overrightarrow{AB}[n]$ for all $0 \leq \ell' \leq \ell$; (2) $\overrightarrow{ab}_0[n] = \overrightarrow{ab}[n]$ and $\overrightarrow{ab}_\ell[n] = \overrightarrow{ab}'[n]$ and (3) to move from $\overrightarrow{ab}_{\ell'}[n]$ to $\overrightarrow{ab}_{\ell'+1}[n]$ one single mutation is enough for all $0 \leq \ell' \leq \ell - 1$.*

Proof. In a first step, note that if $k \leq 2$ there are no absorbing sets with $n \leq 2k$. To see this note that for $k = 1$ in any action-heterogenous absorbing set there are exactly three A-agents. Consider $k = 2$. If $\frac{b-a}{a-d}k \leq 1$ then $\frac{b-a}{a-d}k > |I_{AB}|$ implies that $|I_{AB}| = 0$; if $\frac{b-a}{a-d}k > 1$, then each A-player has to have at least two passive links from A-players.

The following class of absorbing sets will play an important role. We define $\overrightarrow{\mathcal{AB}}[n]$ to be the set of absorbing sets such that for each element in $\overrightarrow{\mathcal{AB}}[n]$, i) each agent in I_{AA} forms $|I_{AB}|$ links to agents in I_{AB} and $k-|I_{AB}|$ links to agents in I_{AA} and ii) the agents in the set I_{AB} can be organized as $i_1,\ldots,i_{|I_{AB}|}$ where $i_\ell \in I_{AB}$ forms links to A-agents in $\{i_{\ell+1},\ldots,i_{|I_{AB}|}\}$.

First, consider the case of moving from $\overrightarrow{ab}[n] \in \overrightarrow{\mathcal{AB}}[n]$ to $\overrightarrow{ab}'[n] \in \overrightarrow{\mathcal{AB}}[n]$. Starting from $\overrightarrow{ab}[n]$ we can apply the same logic as in the proof of lemma 7 to move to a state in $\overrightarrow{ab}''[n] \in \overrightarrow{\mathcal{AB}}[n]$ where $I_{AA} = I'_{AA}$ and $I_{AB} = I'_{AB}$. Let g_{I_A} denote the subnetwork over I_A in $\overrightarrow{ab}''[n]$ and let g'_{I_A} denote the subnetwork over I_A in $\overrightarrow{ab}'[n]$.

Now consider the subnetwork among agents in I_{AA}. Since $g_{I_{AA}}$ is fully connected and each agent in I_{AA} forms $k-|I_{AB}|$ links to agents in I_{AA} and receives $k-|I_{AB}|$ passive links from agents in I_{AA}, we can apply the same argument as in the proof of lemma 7 to show that we can move among sets with different subnetworks of players in I_{AA} via a chain of single mutations.

We now turn towards the subnetwork defined over players in I_{AB}. Note that $|I_{AB}| < \frac{b-a}{a-d}k < k$. Since $\overrightarrow{ab}[n] \in \overrightarrow{\mathcal{AB}}[n]$ it follows that each agent in I_{AB} forms at most $|I_{AB}|-1 \leq k-2$ links to A-players and at least two links to B-players. If the subnetworks among agents in I_{AB} are different from another, there have to be two agents $i, i' \in I_{AB}$ such that the link between them is in a different direction in the two subnetworks, i.e. $g_{ii'} = 1$ under subnetwork $g_{I_{AB}}$ and $g'_{i'i} = 1$ under subnetwork $g'_{I_{AB}}$.

Assume now that i' receives revision opportunity, makes a mistake, and replaces a link to some B-player with a link to i. In a next step, agent i gets revision opportunity. Now agent i has one more passive link from A-agents, which implies that the LOP of action A has increased by c, and the LOP of action B has increased by d. Thus, she continues to play A and replaces the (redundant) link to i' with a link to some B-player. Iterating this argument, we end up at a new action-heterogenous absorbing set with the same sub-network over I_{AB} as $\overrightarrow{ab}'[n]$. We can thus move to the absorbing set $\overrightarrow{ab}'[n]$ via a chain of single mutations.

Consider the case that $\overrightarrow{ab}[n] \notin \overrightarrow{\mathcal{AB}}[n]$ and $\overrightarrow{ab}'[n] \in \overrightarrow{\mathcal{AB}}[n]$. Consider some agent $j_0 \in I_{AB}$. If there exists an A-agent j who forms links to B-players and receives one passive link from j_0, we can apply the argument in the above paragraph to show that with one mutation we can move to a new absorbing set where the link $g_{j_0 j} = 1$ is replaced by the link $g_{jj_0} = 1$, agent j has one less links to B-players and j_0 has one more link to B-players. Iterating the argument, we can end up at a new absorbing set where agent j_0 receives links from all other agents in I_{AB}. If j_0 still supports links to A-players in I_{AA} we proceed as follows:

Consider one such agent $j \in I_{AA}$ with $g_{j_0 j} = 1$. Note that the number of links from A-players to B-players is given by $nk - \frac{n(n-1)}{2} = \frac{1}{2}[n(2k+1) - n^2] \geq k$ and that j_0 forms at most $k-1$ links to B-players. Thus, there exists another A-agent $j_1 \neq j_0$ in I_{AB}.

Note that i) there are $k+1$ A-players in $\{j\} \cup N_j^{out}(g_{I_A})$ and ii) agent j_1 forms at most $k-1$ links to agents in I_A. Since A-players are fully connected there have to exist at least two agents in

$\{j\} \cup N_j^{out}(g_{I_A})$ who support a link to agents j_1. Thus, either $g_{jj_1} = 1$ or there is another agent $j^{'} \in I_A$ such that $g_{jj'} = g_{j'j_1} = 1$. Consider the case $g_{jj'} = g_{j'j_1} = 1$ and $j^{'} \in I_{AA}$. (The other cases where either $g_{jj'} = g_{j'j_1} = 1$ and $j^{'} \in I_{AB}$ or $g_{jj_1} = 1$ hold derive from modified (simpler) arguments).

Let ℓ be a periphery B-agent. First, agent ℓ makes a mistake, switches from action B to action A and forms three links to agents j, $j^{'}$ and j_1. In a next step, agent j_0 receives revision opportunity and replaces the link to j with a link to ℓ. Next, agent j continues to play action A, deletes the link to $j^{'}$ and forms a link to j_0. Following this, $j^{'}$ deletes the link to j_1 and forms a link to j. Then, agent j_1 deletes a link to some B-player and forms a link to $j^{'}$. Note that in this construction the number of passive links for agents in $\{j, j^{'}, j_0, j_1\}$ does not change, implying that none of them will switch to action B at some point. Now consider agent ℓ, who only has one passive link from j_0. The following argument establishes that she will switch back to B. Consider the case where ℓ requires one passive link to remain an A-player. By lemma 2 any A-player requires at least $\frac{b-a}{a-d}k$ passive links. We thus would have $\frac{b-a}{a-d}k \leq 1$. However, then we also have $2k \geq n \geq \underline{\underline{m}} > k + 1 + \max\{\frac{b-a}{a-d}k, k - \frac{b-a}{a-d}k\} = k + 1 + k - \frac{b-a}{a-d}k \geq 2k$ where the equality follows from the fact that $k \geq 3$. Thus, it has to be the case that $\frac{b-a}{a-d}k > 1$. As a result, agent ℓ switches back to action B and forms k links with B-players. Iterating the argument, we end up at a new absorbing set where A-agent j_0 forms k links to B-players. Now j_0 corresponds to the last agent in the set $I_{AB} = \{i_1, \ldots, i_{|I_{AB}|}\}$ in the definition of $\overrightarrow{\mathcal{AB}}[n]$. In the same manner, we can exhibit a chain of single mutations at the end of which some agent x will support one link to agent $i_{|I_{AB}|}$ and $k-1$ links to B-players, and who thus will serve in the role of agent $i_{|I_{AB}|-1}$. In this manner we can move from any absorbing set in $\overrightarrow{AB}[n]$ to an absorbing set in $\overrightarrow{\mathcal{AB}}[n]$.

One can in fact reverse the above argument (by appropriately changing the set of agents the mutant ℓ connects to and by flipping the order in which they receive revision opportunity) to exhibit transition paths from any absorbing set in $\overrightarrow{\mathcal{AB}}[n]$ to any absorbing set in $\overrightarrow{AB}[n]$. Thus all absorbing sets in $\overrightarrow{AB}[n]$ can be connected via a chain of single mutations. □

Lemma 9. *$C(\overrightarrow{AB}[n], \overrightarrow{AB}[n+1]) = 1$ for any $\underline{\underline{m}} \leq n \leq 2k$.*

Proof. Note that $|I_{AB}| < \frac{b-a}{a-d}k$, which implies that $|I_{AB}| \leq \left\lfloor \frac{b-a}{a-d}k \right\rfloor$. The inequality $n \geq \underline{\underline{m}} = k + \max\left\{ \left\lfloor \frac{b-a}{a-d}k \right\rfloor, k - \left\lceil \frac{b-a}{a-d}k \right\rceil \right\} + 2$ implies that $|I_{AA}| \geq k + 2$. Without loss of generality, assume that $I_{AA} = \{1, \ldots, n^{'}\}$ and $I_{AB} = \{n^{'} + 1, \cdots, n\}$ where $n^{'} \geq k + 2$. Further assume that the process has reached a state s where the B-players' linking strategies form a core-periphery network. Denote by $n + 1$ a periphery B-agent.

First, assume that agent $n+1$ receives revision opportunity and, by mistake, switches to A and forms k links to agents in $\{1, \cdots, k\} \subset I_{AA}$. Proceed by giving revision opportunity to agents in $I_{AB} = \{n^{'} + 1, \cdots, n\}$. In comparison to the initial state s, there is one more A-player and the number of passive link has not changed. Thus, none of them will switch to B. Instead, each agent in I_{AB} will continue to play A and replace one of her links to B-players with a link to $n + 1$.

Now consider agents $k+1, \cdots, n^{'}$ who are not connected to $n+1$. If an agent j in this set forms a link to an A-player ℓ who still forms links to B-players, we can proceed in the following manner. Agent j, upon receiving revision opportunity, deletes the link to ℓ and forms a link to $n+1$, leaving her payoff unchanged. Then, ℓ receives revision opportunity. Since ℓ is passively connected to one less A-player than before we need to verify that she does not switch to B. To this end, note that ℓ is connected to all A-players except j and forms at most $k-1$ links to A-players. Further, ℓ has no less than $(n-1)-(k-1) = n-k > \frac{b-a}{a-d}k$ passive links from A-players and the number of A-players is $n+1 > \frac{b-d}{a-d}k+1$. Using lemma 2 we can check that ℓ continues to play A and replaces one B-link with a link to j. Iterating the argument, the dynamics reaches a state $s^{'}$ where every agent who is not connected to $n+1$ does not form a link to any agent in I_{AB}.

Now consider the case where there is still an agent x who is not connected to $n+1$. Denote one A-player who forms links to B-players by z. Agent x must form all of her k links to agents in I_{AA} and does not form a link to z. Since z has to be fully connected to A-players, she can form at most $k-1$ links to agents in I_{AA}. Since $|I_{AA}| > k$ there exists an agent $w \in I_{AA}$ who receives a link from x and supports a link to z. Then x can delete the link to w and form a link to $n+1$. Following this, we can apply the same logic as above. □

Lemma 10. *There exists an absorbing set* $\tilde{ab}[n] \in \overrightarrow{AB}[n]$ *such that* $C(\tilde{ab}[n], \overrightarrow{AB}[n-1]) = 1$ *for* $\underline{\underline{m}}+1 \leq n \leq 2k+1$.

Proof. We start by considering an absorbing set $\tilde{ab}[n] \in \overrightarrow{AB}[n]$ with agents $I_{AA} = \{1, \ldots, 2(n-k-1)+1\}$ and $I_{AB} = \{2(n-k-1)+2, \ldots, n\}$. The linking decisions of agents in I_{AA} form a circle of width $n-k-1$ and each agent in I_{AA} forms $k-n+k+1 = 2k+1-n = |I_{AB}|$ links to all agents in I_{AB}. Each agent in $i \in I_{AB}$ forms links to the A agents $i+1, \ldots, n$ and some B-players.

We now show that one mistake is enough to move to an absorbing set $\overrightarrow{ab}[n-1]$. To this end, assume that agent $2(n-k-1)+1$ makes a mistake, keeps her action, deletes the links to agents in I_{AA} and forms these $n-k-1$ links to B-players. Next, each agent $i \in \{1, \ldots, n-k-1\}$ deletes the link to agent $i+(n-k-1)$ and forms the link to $2(n-k-1)+1$. Note that since all players in the set $\{1, \ldots, n-k-1\}$ were initially fully connected to all other A-players, each agent i in the set $\{1, \ldots, n-k-1\}$ has now $n-2-k$ passive links. Since $n-2-k > \frac{b-a}{a-d}k$ it follows that none of them will switch to B. In a next step, each agent i in the set $\{(n-k-1)+1, \ldots, 2(n-k-1)-1\}$ deletes the link to agent $2(n-k-1)$ and forms a link to $i+n-k-2$ (which is understood modulo $2(n-k-2)+1$). As above, none of the agents in the set $\{(n-k-1)+1, \ldots, 2(n-k-1)-1\}$ will switch. We have, thus, reached a profile where $2(n-k-2)+1$ agents in the set $I_{AA} \setminus \{2(n-k-1), 2(n-k-1)+1\}$ are fully connected and form a circle of width $n-k-2$, agent $2(n-k-1)+1$ forms links to B players and still finds it optimal to choose A, and agent $2(n-k-1)$ has no incoming links. Thus, agent $2(n-k-1)$ will switch to action B, implying that with one mutation we have reached a new absorbing set with one less A-player. □

Our next lemma analyzes transitions out of the risk dominant convention.

Lemma 11. $c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) = c(\overrightarrow{A}, \overrightarrow{B}) = \lceil \frac{a-d}{b-d} k \rceil$

Proof. With positive probability the process reaches a state where the linking decisions of agents form a core-periphery network. Without loss of generality assume that the agents in the core are given by $\{1, \dots, 2k+1\}$. Let x denote the minimal number of agents switching from A to B such that any other agent finds it optimal to switch from A to B. Since agents in the periphery do not have any incoming links they would be easiest to switch. In particular, an agent in the periphery will switch with positive probability whenever $xb + (k-x)d \geq ak$. We thus have $x = \lceil \frac{a-d}{b-d} k \rceil \leq k$.

In a next step let us consider the transition to some state in $\overrightarrow{\mathcal{AB}}$. Assume that x mutations happen among periphery agents. Now all other periphery agents will find it optimal to switch to B and link up to the mutants. Agents in the core still each have k passive links from other A-agents and will thus not switch. We have thus reached a state in $\overrightarrow{AB}[2k+1]$.

Finally, consider the transition to $\overrightarrow{B}$. Now assume that the x mutations happen among the core players $\{1, \dots, x\}$ and that those players do not change their links. As before, the periphery agents will switch to B. Let us thus consider the remaining core agents, starting with agent $x+1$. This agent has now x passive links from B-players and $k-x$ passive links from A-players. Her LOP of playing A is $ka + (k-x)a + xc - \gamma k$ and her LOP from action B is $kb + xb + (k-x)d - \gamma k$. She will thus switch if $x \geq \frac{2a-b-d}{b-d+a-c} k$. Pointing out that $\frac{2a-b-d}{b-d+a-c} k < \lceil \frac{a-d}{b-d} k \rceil = x$, show that she will indeed switch. Iterating this argument shows that in fact also all A-agents in the core will switch.

Note that this transition cost is the same as in Staudigl & Weidenholzer (2014) where there is no payoff from passive connections. The reason for this is that agents in the periphery have no incoming and that mistakes within the core turn out to be sufficient for core agents to switch. $\square$

We proceed by discussing the transition from $\overrightarrow{B}$ to $\overrightarrow{\mathcal{AB}}$.

Lemma 12. $c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}) = \lceil \frac{b-c}{a-d} k \rceil$ *and* $c(\overrightarrow{B}, \overrightarrow{A}) \geq \lceil \frac{b-c}{a-d} k \rceil$

Proof. Let y denote the minimal number of agents switching from B to A such that any other agent finds it optimal to switch from B to A. Now consider an B-agent i. The impact to this agent of others switching will be the larger, the higher the fraction of A agents among her neighbors. Let us thus assume that i only has y incoming links, all of whom switch to A. Her LOP from playing A is $ay + kc - \gamma k$ and her LOP from action B is $bk + dy - \gamma k$. We thus have $y = \lceil \frac{b-c}{a-d} k \rceil \leq k$. Thus, $c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}) \geq \lceil \frac{b-c}{a-d} k \rceil$ and $c(\overrightarrow{B}, \overrightarrow{A}) \geq \lceil \frac{b-c}{a-d} k \rceil$.

Now we turn to show that y mutations are indeed also sufficient for the transition from $\overrightarrow{B}$ to $\overrightarrow{AB}[n]$. To this end, assume that the process has reached a core-periphery network and without loss of generality assume that agents $\{N-2k, \dots, N\}$ form the core. Thus all other agents do not have any incoming links. Now assume that agents $\{1, \dots, y\}$ switch to A and each agent i in this set links up to agents $\{i+1, \dots, i+k\}$. Now consider agent $y+1$. Since, $y \leq k$, she now has y incoming A-links and will switch to A. With positive probability she will link up to agents $\{y+2,\ldots,y+k+1\}$. Now agent $y+2$ has at least y incoming links and we can reiterate the argument. Note that in this construction all agents up to agent $k+2$ have incoming links from all A-agents and will only link to B-agents. Agent $k+2$, however, is not linked to agent 1 and, thus, forms links to the B-agents $\{k+3,\ldots,2k+1\}$ and to the A-agent 1. More generally, when given revision opportunity in this construction, an agent $j \in \{k+2,\ldots,2k+1\}$ will link to the B-agents $\{j+1,\ldots,2k+1\}$ and to the A-agents $\{1,\ldots,j-k-1\}$. With y mutations we have thus reached a state in $\overrightarrow{AB}[2k+1]$. □

Note that lemma 12 also provides a lower bound for the transition cost from $\overrightarrow{B}$ to $\overrightarrow{A}$. Since the indirect transition via $\overrightarrow{\mathcal{AB}}$ will in total require no more mutations than the direct transition, knowing the exact value of $c(\overrightarrow{B},\overrightarrow{A})$ is not required for our purposes.

In a next step we consider transitions out of i $\overrightarrow{\mathcal{AB}}$, starting with the transition to $\overrightarrow{B}$.

Lemma 13. $c(\overrightarrow{\mathcal{AB}},\overrightarrow{B}) = \left\lceil \left(\underline{\underline{m}} - k - 1 - \frac{b-a}{a-d}k\right)(1-p^*)\right\rceil$

Proof. In a first step we show $x(n)$ mutations are necessary. Consider an absorbing set $\overrightarrow{ab}[n]$. Note that in $\overrightarrow{ab}[n]$, all A-players are fully connected. Consider a mutant ℓ who switches from A to B. The LOP of an A-player i can be affected in three possible ways,

i) if $g_{\ell i} = 1$ then i's LOP from action A decreases by $a-c$ and the LOP from action B increases by $b-d$;

ii) if $g_{i\ell} = 1$ then i's LOP from action A decreases by $a-c$ and the LOP from action B does not change; and

iii) if $g_{i\ell} = g_{\ell i} = 0$ then i's LOP from action A decreases by a and the LOP from action B decreases by d.

where in the last case ℓ deletes the link to i. Thus, the effect of one single mutations is largest in the first case, where ℓ supports a link to i. Thus, to minimize the overall number of mutations required for a transition, we focus on the case where the mutants are actively connected to a given agent i.

Consider an A-player $i \in I_{AA}$. Since, i forms k links to A-players and since all A-players are fully connected, this player has to have $n-k-1$ incoming links from A-players. Now assume that x of her passive neighbors mutate to B. The LOP of action A is $a(n-k-1-x)+cx+ak-\gamma k$ and the LOP of action B is $d(n-k-1-x)+bx+bk-\gamma k$. Thus, for i to switch it has to be true that $d(n-k-1-x)+bx+bk \geq a(n-k-1-x)+cx+ak$. It follows that $x \geq \left\lceil \left(n-k-1-\frac{b-a}{a-d}k\right)(1-p^*)\right\rceil := x(n)$.

Now consider an A-player $j \in I_{AB}$ and assume that y A-players who form links to j make mistakes, and switch to action B while keeping their linking strategies. For agent j, the LOP of action A is $a(n-1-y)+cy+c[k-(n-1-m_j^{in})]-\gamma k$ and the LOP of action B is

$d(m_j^{in} - y) + by + bk - \gamma k$. The LOP of action B exceeds the LOP of action A if $d(m_j^{in} - y) + by + bk \geq a(n-1-y) + cy + c[k - (n-1-m_j^{in})]$. This can be rewritten as

$$\begin{aligned}
(b-d+a-c)y &> (c-d)m_j^{in} - (b-c)k + (a-c)(n-1) \\
&= (a-d)m_j^{in} - (b-a)k - (a-c)[k - (n-1-m_j^{in})] \\
&= [(a-d)(n-k-1) - (b-a)k] \\
&- (a-d)(n-k-1) + (a-d)m_j^{in} - (a-c)[k - (n-1-m_j^{in})] \\
&= [(a-d)(n-k-1) - (b-a)k] + (c-d)[k - (n-1-m_j^{in})].
\end{aligned}$$

Note that $x(n)$ is the smallest integer such that $(b-d+a-c)x > (a-d)(n-k-1) - (b-a)k$. Since $(c-d)[k - (n-1-m_j^{in})] > 0$ it follows that $y \geq x$.

Thus, for players in I_{AA} and in I_{AB} at least $x(n)$ mutations are required to prompt a player to switch to B.

In the following we show that $x(n)$ mutations are indeed sufficient starting from an appropriate absorbing set (which can be connected to all other action-heterogenous absorbing sets via a chain of single mutations, see lemma 6). In particular, we consider an absorbing set $\tilde{ab}[n] \in \overrightarrow{AB}[n]$ where i) $I_{AA} = \{1, \ldots, 2(n-k-1)+1\}$ and $I_{AB} = \{2(n-k-1)+2, \ldots, n\}$, ii) the linking decisions of agents in I_{AA} form a circle of width $n-k-1$ and each agent in I_{AA} forms $k-n+k+1 = 2k+1-n = |I_{AB}|$ links to all agents in I_{AB}, and iii) each agent in $i \in I_{AB}$ forms links with the A agents $i+1, \ldots, n$ and some B-players.

Assume now that all agents $1, \ldots, x(n)$ mutate to B and keep their linking strategy. Agent $x(n)+1$ has now $x(n)$ incoming links from B-players and will -given the argument provided above- switch to B. By the same reasoning, the remainder of the A-players in I_{AA} will, one-by-one, switch to B.

Now consider agent $2(n-k-1)+2$ (who belongs to I_{AB}). As she has no passive links, she will switch to B and connect to B-players. In the same manner the remainder of the agents in I_{AB} will iteratively switch to B. $\square$

The next lemma discusses the transition from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{A}$. While a full characterization of this transition cost has eluded us, we were able to provide the following important properties.

Lemma 14. *The transition cost $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A})$ fulfills the following properties:*

i) $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \geq \lceil \frac{b-a}{a-d} k \rceil$.

ii) There exists N^* such that for $N \leq N^*$ we have $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \leq \lceil \frac{2b-a-c}{a+b-c-d} k \rceil$.

iii) For any integer $x > 0$ there exists a population size $N^{**}(x)$ such that for $N \geq N^{**}(x)$ we have $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \geq x$.

Proof. Consider i). B-agents receive only links from other B-agents. Let us consider the conditions under which any of them switches to A. (In all other scenarios the dynamics will move back to a state in $\overrightarrow{AB}[2k+1]$ with certainty.) Denote the B-agent under consideration by i. In the most favourable case i is only passively linked to A-players after the mistakes have occurred. Thus, assume that ℓ agents make a mistake, change their action to A and link to agent i. Agent i's LOP of playing A is now given by $ak+a\ell-\gamma k$. Her LOP of action B is now $bk+d\ell-\gamma k$. She will thus switch if and only if $\ell \geq \lceil \frac{b-a}{a-d}k \rceil$. It thus follows that we need at least $\lceil \frac{b-a}{a-d}k \rceil$ mistakes to move from $\overrightarrow{AB}[2k+1]$ to $\overrightarrow{A}$. Further note that one can find examples such that $\lceil \frac{b-a}{a-d}k \rceil$ mistakes are also sufficient. See example 5 in the online appendix. We were however not able to show sufficiency for the general case.

We now proceed to property ii). Here we show that for N small $\lceil \frac{2b-a-c}{a+b-c-d}k \rceil$ mistakes are sufficient for a transition. Note that Example 5 in the online appendix demonstrates that this bound is (at least under certain conditions) not tight. Assume that the dynamics has reached a state where all B-players $1, \dots, N-2k-1$ form a circle of width k where each player i forms links to agents $i+1, \dots, i+k$ (understood modulu $N-2k-1$). First, assume that agents $1, \dots, y$ with $y \leq k$ make a mistake, switch to action A and form links to agents $y+1, \dots, y+k$. Now agent $y+1$ has y passive links from A-players and $k-y$ passive links from B-players. She will switch if the LOP of A exceeds the LOP from B, i.e. $ak+ay+c(k-y)-\gamma k \geq bk+dy+b(k-y)-\gamma k$. It thus follows that this construction requires that the number of mistakes y has to be larger than or equal to $\lceil \frac{2b-a-c}{a+b-c-d}k \rceil$. Further note that $y \leq k$, so that forming a circle was indeed possible.

In this construction now $y+1$ will switch to A and delete the links to agents $y+2, \dots, y+k+1$. Iterating this, all remaining agents in the set $\{y+1, \dots, y+k\}$ will switch to A and delete their links to B-players. As a result, we have now $y+k$ new A-agents none of which links to B-agents. Consider the remaining agents in the set $R = \{y+k+1, \dots, N-2k-1\}$. In particular, agent $N-2k-1$ has at most k-passive links from B-agents. Denote by w the number of her passive links. We distinguish two cases $w \leq k-1$ and $w = k$. In the first case agent $N-2k-1$ is passively connected to all other B-player and thus will form all of her k-links to A-players. We, thus, have that the LOP of actions A and B are given by $wc+ka-\gamma k$ and $wb+dk-\gamma k$, respectively. Thus, agent $N-2k-1$ will switch to action A, followed by player $N-2k-2$, and so forth, until no B-player is left. Now consider the second case where $w = k$. If this agent continues to choose B she will form $\min\{k, z\}$ links to B-players, where $z = N-2k-1-y-k-(k+1)$ is the number of B-players she is not linked to. Clearly, if $z \geq k$ this agent will not switch. Thus, consider the case where $z < k$. The LOP of action A is given by $ak+ck-\gamma k$ and the LOP of action B is $bz+d(k-z)+bk-\gamma k$. Solving for z reveals that agent $N-2k-1$ will switch to A with positive probability whenever $z \leq \frac{a+c-b-d}{b-d}k$. Note that if $N-2k-1$ switches then also the remaining agents in the set R will switch. This shows that with $\lceil \frac{2b-a-c}{a+b-c-d}k \rceil$ mistakes we can at least make $\lceil \frac{2b-a-c}{a+b-c-d}k \rceil + 2k + 1 + \lfloor \frac{a+c-b-d}{b-d}k \rfloor$ agents switch from B to A. Thus, for

$N \leq N^* = \lceil \frac{2b-a-c}{a+b-c-d} k \rceil + 4k + 2 + \lfloor \frac{a+c-b-d}{b-d} k \rfloor$ we have $c(\overrightarrow{AB}[2k+1], \overrightarrow{A}) \leq \lceil \frac{2b-a-c}{a+b-c-d} k \rceil$.

We finally proceed to discuss iii). Assume that $x \geq \lceil \frac{b-a}{a-d} k \rceil = \ell$ agents make a mistake and switch from B- to A. We know from the argument above that for any other agent i to switch it has to be case that $m_i^{in} \geq \ell$. Further, note that any agent who switches to A (not by mistake) will form all of her k links to A-players. It, thus, follows that when the total number of links going from A- to B-agents is xk, at most $\lceil \frac{xk}{\ell} \rceil$ agents will switch as a direct result of x mutations. Now consider the set of remaining B-players R. These players will only switch to action A if they cannot form sufficiently many of their links to other B-agents. Consider a B-agent who can form no links to B-agents. She, thus, has to be passively connected to all other B-agents in R. The LOP of action A is $ak + c(|R|-1) - \gamma k$ and the LOP of action B is $dk + b(|R|-1) - \gamma k$. Thus, for $|R| > \lceil \frac{a-d}{b-c} k \rceil + 1$ an agent without any active links to B-players will not switch. Similarly, we can show that agents with some active links to B-players and less passive links from B-players will not switch to A. It thus follows that if $|R| > \lceil \frac{a-d}{b-c} k \rceil + 1$ then x mistakes are not sufficient and the dynamics will move back to a state in $\overrightarrow{AB}[2k+1]$. It follows that if $N \geq N^{**} = x + \lceil \frac{xk}{\ell} \rceil + \lceil \frac{a-d}{b-c} k \rceil + 2k + 2$ then at least $x+1$ mistakes are required. □

Proof of Proposition 3: We first show that we can restrict our analysis to reduced trees defined over the vertices $\overrightarrow{A}$, $\overrightarrow{B}$ and $\overrightarrow{\mathcal{AB}}$. Note that by lemma 6 all absorbing sets in $\overrightarrow{\mathcal{AB}}$ can be connected to each other via a chain of single mutations. The transition costs involving the set of action-heterogenous absorbing sets $\overrightarrow{\mathcal{AB}}$ now refer to minimum costs out/into this class, i.e. $c(\overrightarrow{\mathcal{AB}}, \mathcal{S}) = \min_{\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}} c(\overrightarrow{ab}, \mathcal{S})$ and $c(\mathcal{S}, \overrightarrow{\mathcal{AB}}) = \min_{\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}} c(\mathcal{S}, \overrightarrow{ab})$. It is straightforward to see that i) if there exists a reduced minimum cost $\overrightarrow{A}$- or $\overrightarrow{B}$-tree, then there also exists a (non-reduced) minimum cost $\overrightarrow{A}$- or $\overrightarrow{B}$-tree and ii) if there exists a reduced $\overrightarrow{\mathcal{AB}}$-tree, then for each absorbing set $\overrightarrow{ab} \subset \overrightarrow{\mathcal{AB}}$ there exists a (non-reduced) $\overrightarrow{ab}$-tree.

First, we show that if $\frac{b-a}{a-d} + \frac{b-a}{b-d} \geq 1$, then $\overrightarrow{B} \subseteq S^{***}$. Note that $\frac{b-a}{a-d} + \frac{b-a}{b-d} \geq 1$ implies $\frac{b-a}{a-d} > \frac{1}{2}$, which in turn implies that $\left\lfloor \frac{b-a}{a-d} k \right\rfloor + \left\lceil \frac{b-a}{a-d} k \right\rceil \geq k$.[34] Now note that the minimum number of A-players in any action-heterogenous absorbing is given by $\underline{\underline{m}} = k+2+\max\left\{ \left\lfloor \frac{b-a}{a-d} k \right\rfloor, k - \left\lceil \frac{b-a}{a-d} k \right\rceil \right\}$. By the previous observation, $\max\left\{ \left\lfloor \frac{b-a}{a-d} k \right\rfloor, k - \left\lceil \frac{b-a}{a-d} k \right\rceil \right\} = \left\lfloor \frac{b-a}{a-d} k \right\rfloor$. It follows that

$$c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) = \left\lceil \left(k + 2 + \left\lfloor \frac{b-a}{a-d} k \right\rfloor - k - 1 - \frac{b-a}{a-d} k\right)(1-p^*) \right\rceil = 1 \leq \left\lceil \frac{b-c}{a-d} k \right\rceil = c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$$

Thus, for any reduced $\overrightarrow{\mathcal{AB}}$-tree, a reduced $\overrightarrow{B}$-tree with cost no larger than the original tree can be obtained by deleting the branch leaving $\overrightarrow{B}$ and adding the branch from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{B}$. Further, note that $\frac{a-d}{b-d} = 1 - \frac{b-a}{b-d} \leq \frac{b-a}{a-d} < \frac{b-c}{a-d}$. It follows that

$$c(\overrightarrow{A}, \overrightarrow{B}) = c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) = \left\lceil \frac{a-d}{b-d} k \right\rceil \leq \left\lceil \frac{b-c}{a-d} k \right\rceil$$

[34]To see this note that for k odd, $\left\lfloor \frac{b-a}{a-d} k \right\rfloor + \left\lceil \frac{b-a}{a-d} k \right\rceil \geq \lfloor \frac{k}{2} \rfloor + \lceil \frac{k}{2} \rceil = \frac{k-1}{2} + \frac{k+1}{2} = k$ and that for k even, $\left\lfloor \frac{b-a}{a-d} k \right\rfloor + \left\lceil \frac{b-a}{a-d} k \right\rceil \geq \lfloor \frac{k}{2} \rfloor + \lceil \frac{k}{2} \rceil = \frac{k}{2} + \frac{k}{2} = k$.

Note that to leave the basin of attraction of $\overrightarrow{B}$, at least $\lceil \frac{b-c}{a-d} k \rceil$ mutations are needed. Thus, for any reduced $\overrightarrow{A}$-tree, a reduced $\overrightarrow{B}$-tree with cost less or equal to the original one can be obtained by deleting the branch out of $\overrightarrow{B}$ and adding the branch from $\overrightarrow{A}$ to $\overrightarrow{B}$. Thus, if $\frac{b-a}{a-d} + \frac{b-a}{b-d} \geq 1$ there exists a $\overrightarrow{B}$-tree of minimum cost.

Now note $\overrightarrow{B}$ is uniquely stochastically stable whenever all $\overrightarrow{A}$- and $\overrightarrow{\mathcal{AB}}$- trees have strictly larger cost. This is the case if $1 \leq \lceil \frac{a-d}{b-d} k \rceil < \lceil \frac{b-c}{a-d} k \rceil$.

$$\begin{aligned}
\left\lceil \frac{a-d}{b-d} k \right\rceil &< \frac{a-d}{b-d} k + 1 = \left(k - \frac{b-a}{b-d} k \right) + 1 \\
&\leq \left(k - \frac{b-a}{b-d} k \right) + \frac{a-c}{a-d} k \leq \frac{b-a}{a-d} k + \frac{a-c}{a-d} k = \frac{b-c}{a-d} k \\
&\leq \left\lceil \frac{b-c}{a-d} k \right\rceil
\end{aligned}$$

where the second inequality follows from $\frac{a-c}{a-d} k \geq 1$ and the third inequality follows from $\frac{b-a}{a-d} + \frac{b-a}{b-d} \geq 1$. Thus, whenever $k \geq \frac{a-d}{a-c}$ we have $S^{***} = \overrightarrow{B}$.

Finally, note that solving $\frac{b-a}{a-d} + \frac{b-a}{b-d} \geq 1$ for b yields $b \geq \frac{\sqrt{5}+1}{2} a - \frac{\sqrt{5}-1}{2} d := b^*$. □

Proof of Proposition 4: In the first part of the proof we show that if $p^* \leq \tilde{p}$, no reduced $\overrightarrow{B}$-tree can have a cost smaller or equal than the reduced minimum cost $\overrightarrow{A}$- or $\overrightarrow{ab}$-trees. To this end note that every reduced $\overrightarrow{B}$-tree either has a branch from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{B}$ or a branch from $\overrightarrow{A}$ to $\overrightarrow{B}$. In the former case, when $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) > c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$, we can delete the branch going from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{B}$ and add a branch from $\overrightarrow{B}$ to $\overrightarrow{\mathcal{AB}}$, thus obtaining a reduced $\overrightarrow{ab}$-tree of strictly smaller cost. Similarly, in the latter case where there exists a branch from $\overrightarrow{A}$ to $\overrightarrow{B}$, there has to exist a branch from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{A}$. If $c(\overrightarrow{A}, \overrightarrow{B}) > c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$, we can delete the branch from $\overrightarrow{A}$ to $\overrightarrow{B}$ and add a branch from $\overrightarrow{B}$ to $\overrightarrow{\mathcal{AB}}$, thus exhibiting a lower cost reduced $\overrightarrow{A}$-tree. It follows that if $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) > c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$ and $c(\overrightarrow{A}, \overrightarrow{B}) > c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$, then there exists no reduced $\overrightarrow{B}$-tree of minimal cost.

To this end assume that $1 - 3p^* \geq \frac{p^*}{1-p^*}$. This is equivalent to $p^* \leq \frac{1}{6}\left(5 - \sqrt{13}\right) := \tilde{p}$. Note that $\frac{1-3p^*}{1-p^*} \geq \frac{p^*}{(1-p^*)^2} > 0$. It then follows that $1 - 2\frac{b-a}{a-d} > 1 - 2\frac{p^*}{1-p^*} = \frac{1-3p^*}{1-p^*} > 0$. This in turn implies that $\frac{b-a}{a-d} < \frac{1}{2}$. We thus have that

$$\underline{\underline{m}} = k + \max\left\{ \left\lfloor \frac{b-a}{a-d} k \right\rfloor, k - \left\lceil \frac{b-a}{a-d} k \right\rceil \right\} + 2 = 2k - \left\lceil \frac{b-a}{a-d} k \right\rceil + 2.$$

It follows that

$$c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) = \left\lceil \left(k - \left\lceil \frac{b-a}{a-d} k \right\rceil + 1 - \frac{b-a}{a-d} k \right) (1 - p^*) \right\rceil \leq \left\lceil \frac{a-d}{b-d} k \right\rceil = c(\overrightarrow{A}, \overrightarrow{B})$$

since $\frac{a-d}{b-d}k = k - \frac{b-a}{b-d}k \geq k - \lceil \frac{b-a}{a-d}k \rceil + 1 - \frac{b-a}{a-d}k$. Further, we have

$$\begin{aligned}
c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) - c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}) &= \left\lceil \left(k - \left\lceil \frac{b-a}{a-d}k \right\rceil + 1 - \frac{b-a}{a-d}k \right)(1-p^*) \right\rceil - \left\lceil \frac{p^*}{1-p^*}k \right\rceil \\
&> \left(k - 2\frac{b-a}{a-d}k \right)(1-p^*) - \frac{p^*}{1-p^*}k - 1 \\
&= \left[(1 - 2\frac{b-a}{a-d})(1-p^*) - \frac{p^*}{1-p^*} \right] k - 1 \\
&= \left[(1 - 2\frac{b-c}{a-d})(1-p^*) - \frac{p^*}{1-p^*} \right] k + 2\frac{a-c}{a-d}(1-p^*)k - 1 \\
&= \left[(1-3p^*) - \frac{p^*}{1-p^*} \right] k + 2\frac{a-c}{a+b-c-d}k - 1 \\
&> \left[(1-3p^*) - \frac{p^*}{1-p^*} \right] k + \frac{a-c}{b-d}k - 1 \\
&\geq 0.
\end{aligned}$$

Consequently, for $p^* \leq \tilde{p}$ and $\frac{a-c}{b-d}k \geq 1$ we have $c(\overrightarrow{A}, \overrightarrow{B}) \geq c(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) > c(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$ and no reduced $\overrightarrow{B}$-tree can be of minimal cost.

We now turn to the second part of the proof. Consider any reduced $\overrightarrow{A}$-tree. Note that the cheapest way to enter the basin of attraction of $\overrightarrow{A}$ is starting at an absorbing set in $\overrightarrow{\mathcal{AB}}$. The direct transition from $\overrightarrow{B}$ features a strictly higher cost. By part iii) of lemma 14 we have that, for any integer $x > 0$ there exists a population size $N^{**}(x)$ such that for $N \geq N^{**}(x)$ we have $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \geq x$. Thus, provided N is sufficiently large $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) > c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) = \lceil \frac{a-d}{b-d}k \rceil$. By reversing the branch from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{A}$, we can thus construct a reduced $\overrightarrow{\mathcal{AB}}$-tree of minimum cost.

Consider now the last part of the proposition. Part ii) of lemma 14 shows that for $N \leq N^* = \lceil \frac{2b-a-c}{a+b-c-d}k \rceil + 4k + 2 + \lfloor k\frac{a+c-b-d}{b-d}k \rfloor$ we have $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \leq \lceil \frac{2b-a-c}{a+b-c-d}k \rceil$. Thus, if $\lceil \frac{2b-a-c}{a+b-c-d}k \rceil < \lceil \frac{a-d}{b-d}k \rceil = c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}})$ we can always construct a minimum cost $\overrightarrow{A}$-tree by reversing the relevant branch. Now note that

$$\begin{aligned}
& c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) - c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \\
> \; & \frac{a-d}{b-d}k - \frac{2b-a-c}{a+b-c-d}k - 1 \\
= \; & \left(k - \frac{b-a}{b-d}k \right) - \left(\frac{b-c}{a+b-c-d}k + \frac{b-a}{a+b-c-d}k \right) - 1 \\
> \; & \left(k - \frac{b-c}{a-d}k + \frac{a-c}{b-d}k \right) - \left(p^*k + \frac{b-c}{a-d}k \right) - 1 \\
= \; & 2\left(\frac{1}{2} - \frac{p^*}{2} - \frac{p^*}{1-p^*} \right) k + \frac{a-c}{b-d}k - 1.
\end{aligned}$$

We can solve $\frac{1}{2} - \frac{p^*}{2} \geq \frac{p^*}{1-p^*}$ for p^* to obtain $p^* \leq 2 - \sqrt{3}$. Since $2 - \sqrt{3} > \frac{1}{6}\left(5 - \sqrt{13}\right) = \tilde{p}$ this inequality holds in the relevant range. It follows that $c(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) - c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) > \frac{a-c}{b-d}k - 1$. Thus, for $\frac{a-c}{b-d}k \geq 1$ the reduced $\overrightarrow{A}$-tree has the unique lowest cost. □

References

Alós-Ferrer, C. & Weidenholzer, S. (2007), 'Partial Bandwagon Effects and Local Interactions', Games and Economic Behavior **61**, 179–197.

Anwar, A. W. (2002), 'On the Co-existence of Conventions', Journal of Economic Theory **107**, 145–155.

Bala, V. & Goyal, S. (2000), 'A noncooperative model of network formation', Econometrica **68**, 1181 – 1229.

Bilancini, E. & Boncinelli, L. (2018), 'Social coordination with locally observable types', Economic Theory **65**(4), 975–1009.

Borgatti, S. P. & Everett, M. G. (2000), 'Models of core/periphery structures', Social networks **21**(4), 375–395.

Ellison, G. (1993), 'Learning, Local Interaction, and Coordination', Econometrica **61**, 1047–1071.

Ellison, G. (2000), 'Basins of Attraction, Long-Run Stochastic Stability, and the Speed of Step-by-Step Evolution', Review of Economic Studies **67**, 17–45.

Ely, J. C. (2002), 'Local Conventions', Advances in Theoretical Economics **2**, 1–30.

Farrell, J. & Klemperer, P. (2007), 'Coordination and lock-in: Competition with switching costs and network effects', Handbook of industrial organization **3**, 1967–2072.

Farrell, J. & Saloner, G. (1985), 'Standardization, compatibility, and innovation', the RAND Journal of Economics pp. 70–83.

Foster, D. P. & Young, P. (1990), 'Stochastic evolutionary game dynamics', Theoretical Population Biology **38**, 219–232.

Freidlin, M. & Wentzell, A. (1988), Random Perturbations of Dynamical Systems, 2nd Ed., Springer Verlag, New York.

Galeotti, A. & Goyal, S. (2010), 'The law of the few', American Economic Review **100**(4), 1468–92.

Goyal, S. & Janssen, M. C. (1997), 'Non-exclusive conventions and social coordination', Journal of Economic Theory **77**(1), 34–57.

Goyal, S. & Vega-Redondo, F. (2005), 'Network formation and social coordination', Games and Economic Behavior **50**, 178–207.

Harsanyi, J. & Selten, R. (1988), A General Theory of Equilibrium Selection in Games, The MIT Press, Cambridge, MA.

Hojman, D. & Szeidl, A. (2006), 'Endogenous networks, social games, and evolution', Games and Economic Behavior **55**, 112–130.

Jackson, M. O. (2008), Social and Economic Networks, Princeton University Press.

Jackson, M. O. & Watts, A. (2002), 'On the formation of interaction networks in social coordination games', Games and Economic Behavior **41**, 265–291.

Jackson, M. O. & Wolinsky, A. (1996), 'A Strategic Model of Social and Economic Networks', Journal of Economic Theory **71**, 44–74.

Kandori, M., Mailath, G. J. & Rob, R. (1993), 'Learning, Mutation, and Long Run Equilibria in Games', Econometrica **61**, 29–56.

Kandori, M. & Rob, R. (1998), 'Bandwagon Effects and Long Run Technology Choice', Games and Economic Behavior **22**, 30–60.

Katz, M. L. & Shapiro, C. (1985), 'Network externalities, competition, and compatibility', The American economic review **75**(3), 424–440.

Morris, S. (2000), 'Contagion', Review of Economic Studies **67**, 57–78.

Neary, P. R. (2012), 'Competing conventions', Games and Economic Behavior **1**(76), 301–328.

Nöldeke, G. & Samuelson, L. (1993), 'An Evolutionary Analysis of Backward and Forward Induction', Games and Economic Behavior **5**, 425–454.

Oechssler, J. (1997), 'Decentralization and the Coordination Problem', Journal of Economic Behaviour and Organization **32**, 119–135.

Oechssler, J. (1999), 'Competition among Conventions', Mathematical and Computational Organization Theory **5**, 31–44.

Pin, P., Weidenholzer, E. & Weidenholzer, S. (2017), 'Constrained mobility and the evolution of efficient outcomes', Journal of Economic Dynamics and Control **82**, 165–175.

Samuelson, L. (1997), Evolutionary Games and Equilibrium Selection, The MIT Press, Cambridge, Massachusetts.

Shi, F. (2013), 'Comment on "on the co-existence of conventions"[j. econ. theory 107 (2002) 145–155]', Journal of Economic Theory **1**(148), 418–421.

Staudigl, M. & Weidenholzer, S. (2014), 'Constrained interactions and social coordination', Journal of Economic Theory **152**, 41–63.

Weidenholzer, S. (2010), 'Coordination games and local interactions: A survey of the game theoretic literature', Games **1**(4), 551–585.

Young, P. (1993), 'The Evolution of Conventions', Econometrica **61**, 57–84.

Online Appendix for "Lock-In Through Passive Connections"

Z. Cui and S. Weidenholzer

Table of Contents

A Switching thresholds

The first section of the tables gives the various cases. The second section provides the payoffs of A- and B-players for each of these cases and the last section provides the conditions under which the current action will be kept.

Table A.1: Non-switching thresholds for A-players

	$N-d_i^{in}-1<k$	$N-d_i^{in}-1\geq k$			
		$m-m_i^{in}-1\leq k$ $N-m-(d_i^{in}-m_i^{in})\leq k$	$m-m_i^{in}-1\leq k$ $N-m-(d_i^{in}-m_i^{in})>k$	$m-m_i^{in}-1>k$ $N-m-(d_i^{in}-m_i^{in})\leq k$	$m-m_i^{in}-1>k$ $N-m-(d_i^{in}-m_i^{in})>k$
$v(A,m,d_i^{in},m_i^{in})$	$a(m-m_i^{in}-1)+$ $c[N-m-(d_i^{in}-m_i^{in})]-$ $\gamma(N-d_i^{in}-1)+$ $[m_i a+(d_i^{in}-m_i^{in})c]$	$a(m-m_i^{in}-1)+$ $c(k-m+m_i^{in}+1)-\gamma k+$ $[m_i^{in}a+(d_i^{in}-m_i^{in})c]$	$a(m-m_i^{in}-1)+$ $c(k-m+m_i^{in}+1)-\gamma k+$ $[m_i^{in}a+(d_i^{in}-m_i^{in})c]$	$ak-\gamma k+$ $[m_i^{in}a+(d_i^{in}-m_i^{in})c]$	$ak-\gamma k+$ $[m_i^{in}a+(d_i^{in}-m_i^{in})c]$
$v(B,m-1,d_i^{in},m_i^{in})$	$b[N-m-(d_i^{in}-m_i^{in})]+$ $d(m-m_i^{in}-1)-$ $\gamma(N-d_i^{in}-1)+$ $[(d_i^{in}-m_i^{in})b+m_i^{in}d]$	$b[N-m-(d_i^{in}-m_i^{in})]+$ $d[k-N+m+(d_i^{in}-m_i^{in})]-$ $\gamma k+$ $[(d_i^{in}-m_i^{in})b+m_i^{in}d]$	$bk-\gamma k+$ $[(d_i^{in}-m_i^{in})b+m_i^{in}d]$	$b[N-m-(d_i^{in}-m_i^{in})]+$ $d[k-N+m+(d_i^{in}-m_i^{in})]-$ $\gamma k+$ $[(d_i^{in}-m_i^{in})b+m_i^{in}d]$	$bk-\gamma k+$ $[(d_i^{in}-m_i^{in})b+m_i^{in}d]$
$v(A,m,d_i^{in},m_i^{in})\geq$ $v(B,m-1,d_i^{in},m_i^{in})$	$m-1\geq(N-1)p^*$	$m\geq\frac{(N-d_i^{in}-1)(b-d)-k(c-d)}{a+b-c-d}+$ $d_i^{in}p^*+1$	$m-m_i^{in}\geq\frac{b-c}{a-c}k+$ $\frac{a+b-c-d}{a-c}(d_i^{in}p^*-m_i^{in})+1$	$m-m_i^{in}\geq(N-d_i^{in})-\frac{a-d}{b-d}k+$ $+\frac{a+b-c-d}{b-d}(d_i^{in}p^*-m_i^{in})$	$m_i^{in}\geq\frac{(b-a)k}{a+b-c-d}+d_i^{in}p^*$

Table A.2: Non-switching thresholds for B-players

	$N-d_i^{in}-1<k$	$N-d_i^{in}-1\geq k$			
		$m-m_i^{in}\leq k$ $N-m-(d_i^{in}-m_i^{in})-1\leq k$	$m-m_i^{in}\leq k$ $N-m-(d_i^{in}-m_i^{in})-1>k$	$m-m_i^{in}>k$ $N-m-(d_i^{in}-m_i^{in})-1\leq k$	$m-m_i^{in}>k$ $N-m-(d_i^{in}-m_i^{in})-1>k$
$v(A,m+1,d_i^{in},m_i^{in})$	$a(m-m_i^{in})+$ $c[N-m-(d_i^{in}-m_i^{in})-1]-$ $\gamma(N-d_i^{in}-1)+$ $[m_i^{in}a+(d_i^{in}-m_i^{in})c]$	$a(m-m_i^{in})+$ $c(k-m+m_i^{in})-\gamma k+$ $[m_i^{in}a+(d_i^{in}-m_i^{in})c]$	$a(m-m_i^{in})+$ $c(k-m+m_i^{in})-\gamma k+$ $[m_i^{in}a+(d_i^{in}-m_i^{in})c]$	$ak-\gamma k+$ $[m_i^{in}a+(d_i^{in}-m_i^{in})c]$	$ak-\gamma k+$ $[m_i^{in}a+(d_i^{in}-m_i^{in})c]$
$v(B,m,d_i^{in},m_i^{in})$	$b[N-m-(d_i^{in}-m_i^{in})-1]+$ $d(m-m_i^{in})-$ $\gamma(N-d_i^{in}-1)+$ $[(d_i^{in}-m_i^{in})b+m_i^{in}d]$	$b[N-m-(d_i^{in}-m_i^{in})-1]+$ $d[k-N+m+(d_i^{in}-m_i^{in})+1]-$ $\gamma k+$ $[(d_i^{in}-m_i^{in})b+m_i^{in}d]$	$bk-\gamma k+$ $[(d_i^{in}-m_i^{in})b+m_i^{in}d]$	$b[N-m-(d_i^{in}-m_i^{in})-1]+$ $d[k-N+m+(d_i^{in}-m_i^{in})+1]-$ $\gamma k+$ $[(d_i^{in}-m_i^{in})b+m_i^{in}d]$	$bk-\gamma k+$ $[(d_i^{in}-m_i^{in})b+m_i^{in}d]$
$v(A,m+1,d_i^{in},m_i^{in})\leq$ $v(B,m,d_i^{in},m_i^{in})$	$m\leq(N-1)p^*$	$m\leq\frac{(N-d_i^{in}-1)(b-d)-k(c-d)}{a+b-c-d}+$ $d_i^{in}p^*$	$m-m_i^{in}\leq\frac{b-c}{a-c}k+$ $\frac{a+b-c-d}{a-c}(d_i^{in}p^*-m_i^{in})$	$m-m_i^{in}\leq(N-d_i^{in})-\frac{a-d}{b-d}k+$ $\frac{a+b-c-d}{b-d}(d_i^{in}p^*-m_i^{in})-1$	$m_i^{in}\leq\frac{(b-a)k}{a+b-c-d}+d_i^{in}p^*$

B Additional Proofs

Proof of Lemma 4: The proof proceeds by constructing a positive probability path leading to a Nash equilibrium from each possible initial state s. Throughout this construction we assume that agents will not replace a current link to an A-player j with a link to another A-player j' or a link to a B-player ℓ with a link to another B-player ℓ'. We construct a sequence of revisions for individual players leading to a Nash equilibrium. This sequence consists of multiple rounds where in each of these rounds a certain subset of agents receives revision opportunity. Note that since each player receives revision opportunity with positive probability, this sequence also occurs with positive probability.

Let $I_A(0,0)$ denote the set of A-players in the initial state s. In the first round, each agent in $I_A(0,0)$, one by one, receives revision opportunity. Let $I_A(1,0) \subseteq I_A(0,0)$ denote the set of agents who still find it optimal to choose A after this first round of revisions. In case $I_A(1,0) \subset I_A(0,0)$ we proceed to the second round, where each agent in $I_A(1,0)$, one by one, is selected to update her strategy. Let $I_A(2,0) \subseteq I_A(1,0)$ be the set of remaining A-players. According to the finiteness of $I_A(0,0)$, after a finite number of t_1 rounds, the unperturbed dynamics reaches a state where no A-player has an incentive to switch her strategy, $I_A(t_1,0) = I_A(t_1+1,0)$.

Now, we turn to B-agents who are contained in the set $I_B(t_1,0) = I \setminus I_A(t_1,0)$. In round t_1+1 of the revision sequence, each agent in $I_B(t_1,0)$, one by one, receives revision opportunity. Let $I_B(t_1,1) \subseteq I_B(t_1,0)$ denote the set of remaining B-players. Note that for all A-agents $i \in I \setminus I_B(t_1,1)$, the only possible change, in comparison to the most recent strategy revision, is to have more passive links from A-players who previously played action B. Hence, all A-agents in the set $I \setminus I_B(t_1,1)$ will still find it optimal to play action A. In the second round of revisions, agents in $I_B(t_1,1)$, one by one, are selected to update their strategy. Let $I_B(t_1,2) \subseteq I_B(t_1,1)$ be the set of remaining B-players. Since $I_B(t_1,0)$ is finite, after a finite number of t_2 rounds of revisions, the unperturbed dynamics reaches a state where no B-player has an incentive to switch her strategy. Let $I_B(t_1,t_2)$ be the set of all these B-players.

Now, each A-player i in $I \setminus I_B(t_1,t_2)$, upon receiving revision opportunity, may improve her payoff by replacing links to B-players with links to A-players who played B the last time i was selected. Finally, note that each agent in $I_B(t_1,t_2)$ still chooses a best-response. In fact, for each agent i in $I_B(t_1,t_2)$, the only possible change is a loss of passive links from A-players or the addition of new links to A-players. For each lost passive link, the LOP of action A decreases by a while the LOP of action B decreases by d. Since $a > d$, agent i does not have an incentive to switch her strategy. In the later case, agent i was connected to all other B-players. When an active link to A-players is added, which replaces one passive link to A-players, the LOPs of action A and action B both decrease by γ. Thus, we have reached an equilibrium profile where neither A- nor B- players have strict incentives to change their actions and/or links. $\square$

Proof of Lemma 5: The proof of this lemma follows from the combination of a series of lemmas discussed below. Lemma 15 shows that from every action-homogenous Nash equilibrium the process with positive probability reaches an action-homogenous Nash equilibrium where all agents form k links. Lemma 16 shows that from any action-heterogenous Nash equilibrium the process either reaches an action-homogenous essential Nash equilibrium where all agents choose action B or an action-heterogenous Nash equilibrium with no more than $\overline{\overline{m}} = 2k + 1$ A-players who are fully connected. A consequence of this is that there will be at least $2k + 1$ B-players. Lemma 17 then shows that from such action-heterogenous Nash equilibria the process may reach a Nash equilibrium where each B-player forms all k links to other B-players or an equilibrium where all agents choose action B. Lemma 18 then shows that the process may reach profiles where the sub-network among B-players is core-periphery. This is an auxiliary result that allows us to prove lemma 19 which shows that the process with positive probability reaches Nash equilibria where the number of A-players supporting links to B-players is strictly smaller than $\frac{b-a}{a-d}k$. Lemma 20 then makes clear that in such an equilibrium we have $m \geq \underline{\underline{m}} = k + 2 + \max\{\lfloor\frac{b-a}{a-d}k\rfloor, k - \lceil\frac{b-a}{a-d}k\rceil\}$. The combination of these lemmas implies that the process with positive probability reaches a Nash equilibrium fulfilling properties i)-iii).

Lemma 15. *From every action-homogenous Nash equilibrium s^* the unperturbed dynamics with positive probability reaches an action-homogenous Nash equilibrium where all agents form k links.*

Proof. Consider a profile where agent i does not form all of her links. This means that she must be connected to all other agents. Thus, at least $N - 1 - (k - 1)$ agents form links to i. Since at most $2k+1$ agents can be fully connected but there are at least $N-1-(k-1)$ in the set $N_i^{in}(g)$, there are at least two agents j and ℓ who are not linked. When receiving revision opportunity with positive probability agent j deletes to link to i and forms a link to ℓ. When i receives revision opportunity she will form the link to j. We can iterate this argument to reach another Nash equilibrium where all agents form k links. □

Lemma 16. *From every action-heterogenous Nash equilibrium s^* the unperturbed dynamics with positive probability either reaches an action-homogenous Nash equilibrium where all agents choose action B or an action-heterogenous Nash equilibrium with no more than $\overline{\overline{m}} = 2k + 1$ A-players who are fully connected.*

Proof. In the first step, we consider an action-heterogenous Nash equilibrium s^* with $m \leq 2k + 1$ where the A-players are not fully connected, so that there exist at least two agents, i and j, who are not connected, $g_{ij} = 0$. We denote by I_A^* the set of A-players in s^*. The following argument establishes that the dynamics will with positive probability reach a Nash equilibrium where agents i and j are connected or an equilibrium with strictly fewer A-players.

Note, that both i and j have to form all of their k links to other A-players. For, otherwise they could improve their payoff by linking up to each other. Furthermore, note that to fully connect m A-players, $\frac{m(m-1)}{2}$ links are required. Since all A-players in total have $km \geq \frac{m(m-1)}{2}$ links available, fully connecting all A-players is possible. The absence of the link between i and j, thus, implies that there has to exist at least one A-player, ℓ, who forms active links to B-players. It is straightforward to see that agent ℓ has to be (either actively or passively) connected to all other A-players. Without loss of generality, assume that agent ℓ forms k links in s^*.

First, consider the case where the A-player, ℓ, is passively connected to either i or j. Denote one of the B-players ℓ links up to by x. We have $\ell \in N_i^{out}(g) \cup N_j^{out}(g)$ and $g_{\ell x} = 1$ for some B-player x. Without loss of generality, assume that $g_{i\ell} = 1$. Assume i receives revision opportunity. Since she is indifferent between linking to either ℓ or j, she may substitute the link $g_{i\ell} = 1$ with the link $g_{ij} = 1$. Note that agent j now has one more A-link and thus will not switch to B either. Now ℓ, who has one link less from the other A-players, may receive revision opportunity. Note that since s^* was a Nash equilibrium and ℓ was choosing A, we must have had $v(A, m, d_\ell^{in}, m_\ell^{in}) \geq v(B, m, d_\ell^{in}, m_\ell^{in})$. After the change of i the LOP of ℓ for action A is given by $v(A, m, d_\ell^{in}, m_\ell^{in}) - c + a - a$ which can be attained by deleting a link to some B-player and forming the link to i, and the LOP of action B is $v(B, m, d_\ell^{in}, m_\ell^{in}) - d$. Since, $c > d$ it is not clear whether agent ℓ would switch to action B. In case agent ℓ does not switch to action B, the unperturbed dynamics has reached a Nash equilibrium with one more link among the A-players. If, however, agent ℓ switches to action B, we can apply the same construction as in the proof of Lemma 4. In each round of revisions, all A-players are selected to update their strategy. When during these revisions an A-player switches to action B, this influences B-players in the following way: they either have more passive links from other B-players or passive links from A-players are replaced by passive links from B-players. In both cases the LOP of action B increases while the LOP of action A does not increase. Thus, each B-player will continue to play action B. It, thus, follows that the unperturbed dynamics will reach another Nash equilibrium s^{**} where the set of A-players I_A^{**} satisfies that $i)$ $I_A^{**} \subset I_A^*$ and $ii)$ for any two agents i' and j' from I_A^{**}, if i' forms a link to j' in s^*, then i' also forms a link to j' in s^{**}.

In the next step, we consider the case where no agent in $N_i^{out}(g) \cup N_j^{out}(g)$ supports a link to a B-player. In this case the A-player ℓ, forming active links to B-players, does not belong to the set $N_i^{out}(g) \cup N_j^{out}(g)$. Note that ℓ has to actively connect to i, j, and a B-player. Since i is using all of her k links to other A-players it follows that ℓ is at most actively connected to $k-3$ of the k players in $N_i^{out}(g)$. Since ℓ is fully connected to all other A-players, it follows that there are at least three players in $N_i^{out}(g)$ who actively connect to ℓ. Denote by y one such player. Note that we have $y \in N_\ell^{in}(g)$ and $y \in N_i^{out}(g)$. Assume now i receives revision opportunity and changes the active link g_{iy} to the link g_{ij}. Now consider agent y, who has one passive link less from A-players. If y does not switch to action B, she will with positive probability delete the link to ℓ and form a link to

i. Now we consider ℓ. As above ℓ will either remain an A-player and form the link to y or she will switch to B and link up accordingly. In the former case we have reached a Nash equilibrium with one more link among A-players. In the latter case and in case agent y from above switches to B, we proceed as before to show that the dynamics reaches a Nash equilibrium with less A-players.

In the second step, we consider an action-heterogenous Nash equilibrium s^* with $m > 2k+1$. First, consider the case where there exists an subset $I' \subset I_A^*$ where $|I'| = 2k+1$ and all agents in I' are fully connected. Without loss of generality, denote these agents by $1, \dots, 2k+1$ and the remaining A-players by $2k+2, \cdots, m$. Now we proceed in the following manner. First we give revision opportunity to the B-agents and change all of their links to A-players outside $|I'|$ to agents in $|I'|$.

With positive probability agent $2k+2$ receives revision opportunity. Note that the removal of a passive link from B-players, decreases the LOP of action A by c and the LOP of action B by b. In case she prefers to choose action A, she may also change her links from $I_A^* \setminus I'$ to links in I'. In case she prefers to choose action B she switches and links up accordingly (where if it is necessary for her to form links with A-players, she only form these links to agents in I'). Applying this procedure to agents $2k+3, \dots, m$ we arrive at a state where all agents in I' are fully connected and all links of any other A-agent left are to agents I'. Denote the set of all A-players outside of I' by I''. Agents in I'' don't have any incoming links. Provided $N - m \geq k$, the LOP of an agent $i \in I''$ when choosing action B is given by $bk - \gamma k$. Since the LOP of A is only $ak - \gamma k$ all agents in I'' will switch to B. If $N - m < k$, the LOP of an agent $i \in I''$ when choosing action B is given by $b(N-m) + d(k-N+m) - \gamma k$. Thus, an agent would switch if $(b-d)(N-m) \geq (a-d)k$, requiring $N - m \geq \frac{a-d}{b-d}k$. Since s^* was an action-heterogenous Nash equilibrium, Lemma 3 provides a lower bound for the number of B-players, $N - m \geq \left\lceil k \frac{2a-2d}{2b-c-d} \right\rceil + 1$. Since $\frac{2a-2d}{2b-c-d}k > \frac{a-d}{b-d}k$, the A-players in I'' will also find it optimal to switch to action B and link up accordingly.

We have thus reached a state where there are $2k+1$ fully connected A-players and the remainder of the population chooses B. In order to ensure that this is also a Nash equilibrium we need to switch all current links from B- agents to A- agents in I' to other B-agents. Lemma 17 characterizes a series of revision opportunities that does so.[1] At the end of these transitions, B-players will support all of their k links and there will be no links from B- to A-players.

Finally, consider the case where for any subset $I' \subset I_A^*$ with $|I'| = 2k+1$, all agents in I' are not fully connected. Without loss of generality, denote all A-players by $1, \dots, m$. Agents $1, \dots, 2k+1$ are not fully connected and there are links from these agents to agents in $2k+2, \cdots, m$ or B-players. As in the above case where $m \leq 2k+1$, we proceed by adding links among agents $1, \dots, 2k+1$ and deleting the links from $1, \dots, 2k+1$ to other agents. Then, the unperturbed dynamics reaches a Nash equilibrium with strictly less A-players or a Nash equilibrium with $2k+1$

[1] Note that since there are no links from A-players to B-players it is not necessary to consider the possibility that B-agents may be influenced by passive A-links, as Lemma 17 does.

fully connected A-players. □

Lemma 17. *From action-heterogenous Nash equilibrium s^* with $2k+1$ or more B-players the unperturbed dynamics with positive probability reaches a Nash equilibrium where each B-player forms all k links to other B-players or all agents choose action B.*

Proof. We start by considering a Nash equilibrium s^* where there exists at least one B-player, ℓ, who supports active links to A-players. In this exposition we consider the case where ℓ supports all of her k links. The case where ℓ supports less than k links follows the same logic and is omitted.[2] Note that, as above, fully connecting all B-players is possible if and only if $N-m \leq 2k+1$. Since we consider $N-m \geq 2k+1$ and since at least one B-player forms active links to A-players, there have to exist two B-players, i and j, who are not linked, $g_{ij} = g_{ji} = 0$. Note that i and j have to form all k links to other B-players, otherwise they could improve their payoff by linking up to each other.

First, consider the case where either i or j are actively linked to ℓ and denote an A-player ℓ links to by x. Formally, $\ell \in N_i^{out}(g) \cup N_j^{out}(g)$ and $g_{\ell x} = 1$ with $x \in I_A$. Without loss of generality, assume that $g_{i\ell} = 1$. When i receives revision opportunity she may delete the link to ℓ and form a link to j, leaving her payoff unaffected. Agent j now has one more B-link and thus will not switch to A either. In a next step, ℓ may receive revision opportunity. If she deletes the link to x and establishes a link to i, her payoff will be given by $v(B, m, d_\ell^{in}, m_i) - d$. If she instead switches to A and links up optimally her payoff is $v(A, m, d_\ell^{in}, m_i) - c$. Since we originally had $v(B, m, d_\ell^{in}, m_i) \geq v(A, m, d_\ell^{in}, m_i)$ and since $c > d$, agent ℓ will keep her action.

Second, consider the case where all agents in $N_i^{out}(g) \cup N_j^{out}(g)$ are actively connected only to other B-players. Thus, $\ell \notin N_i^{out}(g) \cup N_j^{out}(g)$. By Lemma 1, ℓ has to be connected to all B-players, $N_\ell(g) \supset I_B \setminus \{\ell\}$. Since ℓ is already connected to i, j, and an A-player she can at most have $k-3$ links to the k players in the set $N_i^{out}(g)$. It follows that there has to be a player y in $N_i^{out}(g)$ who actively links to ℓ, i.e. $y \in N_\ell^{in}(g)$ and $y \in N_i^{out}(g)$. Assume now i receives revision opportunity and changes the active link to y to an active link to j. Now we consider agent y. Note that y only forms active links to B-players. If y forms $k-1$ or less links, she will form the link to i in the next step and we reach a state with one more link among the B-players.

If however, y currently forms all k links to B-players we proceed in the following manner: Assume that each A-player $z \in N_y^{in}(g)$ receives revision opportunity and changes the link to y to a link to another B-player $z' \in N_i^{out}(g) \cup N_j^{out}(g) \cup \{i, j\}$.[3] In a next step, player y who now has no incoming links from A-players, upon receiving revision opportunity, deletes the link to ℓ and

[2]In this case, when one passive link from a B-player j to another B-player i is deleted, agent i, upon receiving revision opportunity, forms a new link to j rather than replacing one link to an A-player or B-player by the link to j. The relative comparison of LOPs and hence action choice is the same as in the case analyzed here.

[3]Note that in comparison with the initial state s^*, the only change is that agent z may have less passive links from B-players, and as a result z has no incentive to switch to action B.

forms a link to i. Then, each A-player $z \in N_y^{in}(g)$ receives revision opportunity, and changes the link to $z^{'}$ back to a link to y. As before, agent ℓ will delete the link to x and form a link to y.

Now consider the A-player x who has one less incoming link from B-players. If x forms $k-1$ or less links in s^*, both LOPs of action A and action B decrease by γ; if x forms k links in s^*, the LOP of action A decreases by c and the LOP of action B decreases by d if she is unable to form k links to B-players in s^* and decreases by b if she is able to form k links to B-players in s^*. If x is motivated to switch to action B, we can apply the same construction as in the proof of Lemma 4. Iterating the argument, we end up at an equilibrium where there are more links among B-players or more agents choosing action B.

Finally, note that once there are no links from B-players to A-players, they will not reappear under the dynamics. The reason for this is that in such equilibria all B-players are using all of their k links to B-players. Whenever, a B-player receives revision opportunity, there are thus at least k B-players she may actively link to. □

Lemma 18. *From every action-heterogenous Nash equilibrium s^* with more than $2k+1$ B-players the unperturbed dynamics with positive probability reaches a state where the linking choices of the B-players form a core-periphery network.*

Proof. In this exposition we only present the case where $N - m = 2k + 2$. The case where $N - m > 2k + 2$ iteratively applies the same arguments and is omitted.

Note that by Lemma 17 the unperturbed dynamics reaches a state where all B-players form k links to other B-players. The following construction allows us to ignore the role of incoming connections from A-players for the remainder of the argument. First, note that $N-m = 2k+2 > k$ and since B-players do not link to A-players, every A-player has multiple potential B-players to link to. In order to avoid the issue of B-players switching to A at some point in the process, we assume that just before some B-player i receiving revision opportunity, all of her incoming A-links are switched to another player $k \in I_B \setminus \{i\}$. After i has adjusted her strategy, the A-players are assumed to reestablish their links to i.

Assume that the set of B-players is given by $\{1, \cdots, 2k + 2\}$. If the linking decisions of B-players do not form a core-periphery network, agent $2k + 2$ receives at least one passive link from agents in the set $I' = \{1, \cdots, 2k + 1\}$.

Note that to fully connect agents in I' we need $k(2k + 1)$ links. However, since at least one of these agents links to agent $2k + 2$, this set is not fully connected in s^*. Thus, there are two distinct agents $i, j \in I'$ who are not linked. If either of these agents is actively connected to $2k + 2$, we can proceed in the following manner. Without loss of generality, assume that $g_{i(2k+2)} = 1$. When i receives revision opportunity she deletes the link to $2k + 2$ and forms a link to j. Iterating this argument the dynamics reaches a state where every B-player who was initially linked to $2k + 2$ but not to all agents in the set I' now supports all of her links to agents in this set. If there is no missing link among agents in the set I', the proof is completed.

If there is still a missing link, the B-player $x \in I'$ who forms a link to $2k+2$ has to be linked to all other B-players. Denote the two agents who are not linked by $i^{'}, j^{'} \in I'$. If either of these agents forms a link to x, we can proceed in the following manner. Without loss of generality, assume that $g_{i'x} = 1$. When $i^{'}$ receives revision opportunity she deletes the link to x and forms the link to j'. Then x deletes the link to $2k+2$ and links to i'. We have reached a state with one more link among the players in I'.

Finally, consider the case where neither i' nor j' forms a link to x. We have that $i^{'}$ forms all k links to agents in I' and that x only forms $k-1$ links to I' and is linked to every other B-player. Thus, there are $k+1$ agents who actively link to x. It follows that $i^{'}$ must form a link to some B-player z who is actively linked to x. Now $i^{'}$ may delete the link to z and form a link to $j^{'}$. In a next step z deletes the link to x and forms the link to $i^{'}$. Following this, x deletes the link to $2k+2$ and forms the link to z. Again, we have reached a state with one more link among the players in I'. Iterating this argument, we end up at a state where agents in I' are fully connected and agent $2k+2$ forms all k links to agents in this set. □

Lemma 19. *From every action-heterogenous Nash equilibrium s^* with $m < 2k+1$ and $|I_{AB}| \geq \frac{b-a}{a-d}k$ the unperturbed dynamics with positive probability reaches a Nash equilibrium with $|I_{AB}| < \frac{b-a}{a-d}k$.*

Proof. First note that by lemma 18 the dynamics reaches a state s^* where the A-players are fully connected and find it optimal to choose A and the B-players are arranged in a core-periphery network. Denote an agent in the periphery by i_0. Note that i_0 has no incoming links from B-players.

Consider the A-players who support links to B-players, I_{AB}. With positive probability the dynamics reaches a state where they all support links to i_0. As a result, $m_{i_0}^{in} = |I_{AB}|$.

Now consider the case where $I_{AB} = I_A$. Following from Lemma 3, if every A-player is actively linked to B-players, $m = |I_A| = |I_{AB}| \geq \frac{2(b-c)}{2a-c-d}k + 1 > \frac{b-c}{a-d}k$. For agent i_0, the LOP of action A is $ck + am - \gamma k$ and the LOP of B is $bk + dm - \gamma k$. The inequality $m > \frac{b-c}{a-d}k$ implies that i_0 prefers to switch to action A. All A-players are still fully connected, and each of them has an additional active link pointed to the new A-player i_0 which implies that none has an incentive to switch to action B.

Then, consider the case where $I_{AB} \subsetneq I_A$ and $|I_{AA}| \leq k$. Recall that $m_{i_0}^{in} = |I_{AB}|$. In this case, for agent i_0, the LOP of action A is

$$a(m - |I_{AB}|) + c[k - (m - |I_{AB}|)] + a|I_{AB}| - \gamma k = (a-c)m + ck + c|I_{AB}| - \gamma k$$

which can be attained by forming $|I_{AA}| = m - |I_{AB}|$ links to A-players and $k - |I_{AA}|$ links to B-players. The LOP of B is $bk + d|I_{AB}| - \gamma k$. The non-emptiness of I_{AA} implies that $m \geq$

$\frac{b-a}{a-d}k + k + 1$. Then, it follows that the payoff advantage of A over B is

$$\begin{aligned} & \left[(a-c)m + ck + c|I_{AB}| - \gamma k\right] - \left[bk + d|I_{AB}| - \gamma k\right] \\ = \ & (a-c)m + (c-d)|I_{AB}| - (b-c)k \\ > \ & (a-c)\left(\frac{b-a}{a-d}k + k\right) + (c-d)\frac{b-a}{a-d}k - (b-c)k \\ = \ & (a-d)\frac{b-a}{a-d}k - (b-a)k = 0 \end{aligned}$$

As in the above case, i_0 switches to action A. All A-players are still fully connected, and each of them has no incentive to switch to action B.

Finally, consider the case where $I_{AB} \subsetneq I_A$ and $|I_{AA}| > k$. In this case, for agent i_0, the LOP of action A is $ak + am_{i_0}^{in} - \gamma k$ which can be attained by forming k links to agents in I_{AA} and the LOP of B is $bk + dm_{i_0}^{in} - \gamma k$. The inequality $m_{i_0}^{in} = |I_{AB}| > \frac{b-a}{a-d}k$ implies that $ak + am_{i_0}^{in} - \gamma k > bk + dm_{i_0}^{in} - \gamma k$. As in the above two cases, i_0 switches to action A. Let s^{**} denote the resulting state. Note that now there are k links from i_0 to agents in I_{AA}. Since $|I_{AA}| > k$, now A-players are not fully connected among themselves.

To fully connect A-players among each other, we now switch links A-agents support to B-agents to links to A-agents. Lemma 16 characterizes a series of revision opportunities that does so. In the present context we need to ensure here that no A-player will change her action.

Consider the case where in s^* we had $|I_{AA}| = k + 1$. Note that each agent in I_{AB} forms at most k links to other A-players. Since A-player were fully connected in s^* and $|I_{AA}| = k + 1$ it follows that each A-player in I_{AB} has at least one incoming link from agents in I_{AA}. Recall that now i_0 is missing a link to another A-player j_0 in I_{AA}. Denote by x_0 the agent in I_{AB} supporting a link to a B-player, denoted by z_0. If j_0 supports a link to x_0, we can proceed in the following manner: delete the link from j_0 to x_0, form the link from j_0 to i_0, delete the link from x_0 to z_0 and finally form the link from x_0 to j_0. If j_0 does not support a link to x_0, we have to slightly modify this argument. Note that in this case there has to be some agent w_0 supporting a link to x_0. We can now delete the link from i_0 to w_0 and form the link from i_0 to j_0. The argument above establishes a way to add the link from w_0 to i_0 (while deleting a link from an A-agent to a B-agent).

We have thus arrived at a state with one more link among A-players. Finally consider the case where $|I_{AA}| \geq k+2$ holds for s^*. Note since $m = |I_{AA}| + |I_{AB}| \geq k + 2 + \frac{b-a}{a-d}k$ that for all agents $i \in I_{AA}$ we have $m_i^{in} \geq \frac{b-a}{a-d}k + 1$. Thus, any agent i will continue to choose action A after the deletion of one passive link.

Iterating this argument shows the required result. □

Lemma 20. *If for a Nash equilibrium s^* it holds that $|I_{AB}| < \frac{b-a}{a-d}k$, then $m \geq \underline{\underline{m}} = k + 2 + \max\{\lfloor \frac{b-a}{a-d}k \rfloor, k - \lceil \frac{b-a}{a-d}k \rceil\}$.*

Proof. By lemma 19, we know that in any state contained in an absorbing set, it has to be the case that $|I_{AB}| < \frac{b-a}{a-d}k$. Further, by Lemma 3 we know that if every A-player is actively linked to

B-players, $m = |I_A| = |I_{AB}| \geq \frac{2(b-c)}{2a-c-d}k + 1 > \frac{b-c}{a-d}k + 1 > \frac{b-a}{a-d}k + 1$, yielding a contradiction. Thus, there has to exist at least one A-agent i who links only to A-agents, $|I_{AA}| > 0$. Lemma 2 implies that $m_i^{in} > \frac{b-a}{a-d}k$. Note that since an A-players in I_{AA} forms k links to other A-players and receives more than $\frac{b-a}{a-d}k$ links it has to be true that

$$m > k + \frac{b-a}{a-d}k + 1 \tag{2}$$

Further, each A-player $i \in I_{AA}$ forms at most $|I_{AB}|$ links to agents in I_{AB}. Thus, each A-player $i \in I_{AA}$ forms at least $k - |I_{AB}|$ links to agents in I_{AA}. As a result, there are at least $2(k - |I_{AB}|) + 1$ agents in I_{AA}. Thus, we have that

$$m = |I_{AA}| + |I_{AB}| \geq 2k + 1 - |I_{AB}| > 2k - \frac{b-a}{a-d}k + 1 \tag{3}$$

where the last inequality follows from the fact that $|I_{AB}| < \frac{b-a}{a-d}k$. Combining (2) and (3) yields the desired result. □

C Additional Examples

Example 3. *The following example identifies circumstances under which the upper and lower bounds identified in lemma 3 can not be attained in a Nash equilibrium. Consider $k = 1$ and a coordination game with parameters $[a, b, c, d] = [8, 10, 5, 1]$. We have that*

$$\underline{m} = \left\lceil \frac{2(b-c)}{2a-c-d}k \right\rceil + 1 = \left\lceil \frac{2(10-5)}{2 \times 8 - 5 - 1} \right\rceil + 1 = 2$$

and

$$N - \overline{m} = \left\lceil \frac{2(a-d)}{2b-c-d}k \right\rceil + 1 = \left\lceil \frac{2(8-1)}{2 \times 10 - 5 - 1} \right\rceil + 1 = 2.$$

It is straightforward to see that in every action-heterogenous Nash equilibrium, the number of agents choosing each action has to be at least three. Thus, the lower bounds of A- and B-players identified in lemma 3 can not be attained.

Example 4. *The following example identifies Nash equilibria where the upper and lower bounds in lemma 3 can be attained. Consider any strategy profile s^* where*

i) the A-players form a circle of width $\lceil \frac{b-c}{2a-c-d}k \rceil$;

ii) the B-players form a core-periphery network where all periphery B-players form their k links to the same k core B-players; and *iii) all A-players form their $k - \left\lceil \frac{b-c}{2a-c-d} k \right\rceil$ links to the same $k - \left\lceil \frac{b-c}{2a-c-d} k \right\rceil$ core B-players who receive passive links from all periphery B-players.*

We show that s^* is an action-heterogenous Nash equilibrium. For each A-player, the LOP of action A is $2a \left\lceil \frac{b-c}{2a-c-d} k \right\rceil + c \left(k - \left\lceil \frac{b-c}{2a-c-d} k \right\rceil \right) - \gamma k$, and the LOP of action B is $bk + d \left\lceil \frac{b-c}{2a-c-d} k \right\rceil - \gamma k$. The LOP of action A is larger than the LOP of action B, and each A-player chooses one of her best-response strategies. Periphery- and core- B-player who do not have incoming links from A-players do not interact with A-players and chooses one of their best-response strategies. By the proof of Lemma 3, the number of A-players is less than $(N-1)p^*$ when k is large. It follows that core B-player who receive passive links from A-players do not have an incentive to switch action. When $\left\lceil \frac{2(b-c)}{2a-c-d} k \right\rceil$ is even, it holds that $\left\lceil \frac{2(b-c)}{2a-c-d} k \right\rceil = 2 \left\lceil \frac{b-c}{2a-c-d} k \right\rceil$. Thus, in the action-heterogenous Nash equilibrium s^*, the number of A-players is exactly $\underline{m}$.

For the lower bound on the number of B-players, $N - \bar{m}$, the analysis can be performed in a similar way by identifying another action-heterogenous Nash equilibrium s^{**}, where

i) the B-players form a circle of width $\left\lceil \frac{a-d}{2b-c-d} k \right\rceil$;

ii) the A-players form a circle of width k; and

iii) all B-players form their $k - \left\lceil \frac{a-d}{2b-c-d} k \right\rceil$ links to the same A-players.

We show that s^{**} is an action-heterogenous Nash equilibrium. When $\left\lceil \frac{a-d}{2b-c-d} k \right\rceil = k$, it is trivial to see that s^{**} is a Nash equilibrium. Now consider the case that $\left\lceil \frac{a-d}{2b-c-d} k \right\rceil < k$. For each B-player, the LOP of action A is $ak + c \left\lceil \frac{a-d}{2b-c-d} k \right\rceil - \gamma k$, and the LOP of action B is $2b \left\lceil \frac{a-d}{2b-c-d} k \right\rceil + d \left(k - \left\lceil \frac{a-d}{2b-c-d} k \right\rceil \right) - \gamma k$. The LOP of action A is smaller than the LOP of action B, and each B-player chooses one of her best-response strategies. It is straightforward to see that any A-player who do not have incoming links from B-players chooses one of her best-response strategies. Consider every A-player i who receive incoming links from B-players. The LOP of action A is $2ak + c \left(2 \left\lceil \frac{a-d}{2b-c-d} k \right\rceil + 1 \right) - \gamma k$, and the LOP of action B is $2dk + b \left(2 \left\lceil \frac{a-d}{2b-c-d} k \right\rceil + 1 \right) - \gamma k$. Given that $\left\lceil \frac{a-d}{2b-c-d} k \right\rceil < k$, for agent i, the LOP of action A is strictly larger than the LOP of action B. Therefore, A-player i also chooses one of her best-response strategies. Further, when $\left\lceil \frac{2(a-d)}{2b-c-d} k \right\rceil$ is even, it holds that $\left\lceil \frac{2(a-d)}{2b-c-d} k \right\rceil = 2 \left\lceil \frac{a-d}{2b-c-d} k \right\rceil$. We have that in s^{**}, the number of B-players is exactly $N - \overline{m}$.

Example 5. *The following example illustrates that the transition cost $c(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) = \left\lceil \frac{b-a}{a-d} k \right\rceil$ identified in lemma 14 i) may sometimes be tight. Consider $N = 11$ and $k = 2$. Consider the absorbing set $\overrightarrow{AB}[5]$ where agents in the set $\{1, \ldots, 6\}$ choose action B. With positive probability the dynamics reaches a configuration where B-agents support the following links $N_1^{out}(g) = \{2, 3\}$, $N_2^{out}(g) = \{3, 4\}$, $N_3^{out}(g) = \{4, 6\}$, $N_4^{out}(g) = \{5, 6\}$, $N_5^{out}(g) = \{3, 6\}$, $N_6^{out}(g) = \{1, 2\}$. Figure C.1 depicts the subnetwork among B-players and illustrates the resulting dynamic. Now*

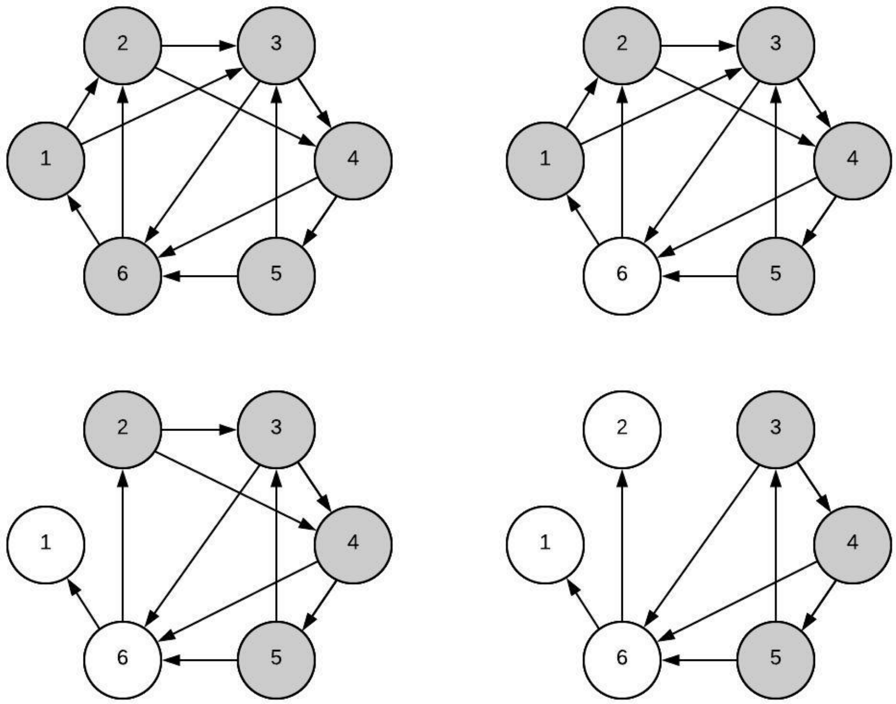


Figure C.1: Subnetwork and resulting dynamic among B-players in Example 5. Newly formed links of agents 1 and 2 to other A-players not depicted.

consider the case where $\ell = \lceil \frac{b-a}{a-d} k \rceil = 1 < 2 = \lceil \frac{2b-a-c}{a+b-c-d} k \rceil = y$. Assume that agent 6 makes a mistake, switches to A and keeps her linking strategy constant. Now agent 1 has $\ell = 1$ incoming links and will thus switch to A, delete her links to 2 and 3 and form links to some A-players. Consequently, agent 2 will switch actions, delete her links to 3 and 4 and form links to some other A-players. The dynamics has, thus, reached a state where the remaining agents $3, 4$ and 5 are arranged on a circle of width 1 and each forms one link to an A-player and one link to a B-player. They will, thus, also switch to action A if $2a + c \geq 2b + d$. Note that this condition is neither excluded nor implied by any of our previous conditions on the payoffs in the coordination game. Provided it holds, the dynamics moves to a state in $\overrightarrow{A}$.

D Extensions and Discussion

D.1 Tie breaking

In the main analysis we consider a myopic best response process where in case of payoff ties a revising agent randomizes among all strategies (i.e. actions and links) giving the highest payoff.

Importantly, this implies that provided there are more than $2k + 1$ agents using one action, the subnetwork among them will vary over time. These variations drive some of our convergence results and also are important for the transitions among profiles. In contrast, when agents stick to their current action and links in case of payoff ties some of our convergence results may no longer hold and the nature of transition may change. In order to highlight these points and to explore the robustness of our model in this dimension we explore a model without tie breaking for the case where $k = 1$.

First, note that when each agent may only support one link we are actually able to provide a more detailed description of the set of Nash equilibria than in the general case.

Lemma 21. *Consider an action-heterogenous Nash equilibrium s^* and the case where $k = 1$. Then,*

i) there are three or more A-players and three or more B-players,

ii) each agent will form her link, $d_i^{out} = 1$ for all $i \in I$,

iii) there are no links from A-players to B-players,

iv) all A-players are arranged in circles,

v) there is at most one link from B-players to A-player.

Proof. Consider property i) and note that clearly if there is either only one A- or one B-player then this player has a strict incentive to switch actions. Now consider the case where there are only two B-players. One of them has no incoming B-links and will have to link to A-players or has incoming links from all other players. In either case choosing A is optimal. Likewise, consider the case where there are two A-players. Consequently, there is one A-player without incoming A-links and this player has an incentive to switch to action B; or there is one B-player receiving one incoming link from A-player, and this agent has an incentive to switch to action A. To see ii) note that if this is not the case all agents will have to connect to some agent i_0. Note that since there are at least two agents with each action, there have to be agents among those linking to i_0 who choose a different action than i_0 and thus would have a strict incentive to link up. Concerning iii) note that for every A-player i it has to be true that $m_i^{in} > 0$ for otherwise she would switch to B. Thus, if one A-player forms her link to B-players it must be true that some other A-player has no incoming links from A-players and then would have incentives to switch. Property iv) follows from i) and the observation that each agent A-agent has to have (at least) one incoming A-link. Property v) follows from the observation that a B-player will only form links to A-players if she has incoming links from all other B-players. Clearly, given that $k = 1$ there can only be one such agent. □

Note that, in contrast to our main model, all Nash equilibria are absorbing as no player has a strict incentive to change her actions or links, $S^{**} = S^{*}$. Secondly, note that now all absorbing sets are singleton, as the dynamics no longer moves among sets of states where the network among agents varies. Now we establish that, in fact, every Nash equilibrium can be connected to every other Nash equilibrium via a chain of single mutations.

In the following we denote the set of absorbing states where everybody chooses A by $\overrightarrow{\texttt{A}}$, the set of absorbing states where everybody chooses B by $\overrightarrow{\texttt{B}}$, and the set of action-heterogenous absorbing states by $\overrightarrow{\texttt{AB}}$.

First, note that any two states contained in $\overrightarrow{\texttt{A}}$ and all states contained in $\overrightarrow{\texttt{B}}$ can be connected to each other via a chain of single mutations where in each step one agent changes her link.

Then, consider the transition from $\overrightarrow{\texttt{A}}$ to $\overrightarrow{\texttt{B}}$. To this end, note that with a chain of single mutations we can move to a state where all agents form a star and agent 1 is in the center of the star. Agent 1 has no outgoing link. If now 1, by mistake switches to B, all other agents will follow. We have, thus, reached a state in $\overrightarrow{\texttt{B}}$.

In a next step, consider an absorbing state s contained in $\overrightarrow{\texttt{B}}$ where agents $1, 2$ and 3 form a circle, and each agent $i \in \{4, \ldots, N\}$ forms her link to agents $4, \ldots, N$. Assume that agent 1 makes mistakes and switches to A. Following this, agents 2 and 3 will also switch to A. We have reached an action-heterogenous absorbing state, where there are three A-players.

Now we show that in fact any action-heterogenous Nash equilibrium s' can be reached from a state in $\overrightarrow{\texttt{A}}$ via a chain of single mutations. Denote by $g_{I'_A}$ the subnetwork among A-players in s' and by $g_{I'_B}$ the subnetwork among B-players. Recall that from lemma 21 we know that $|I'_A| \geq 3$, $|I'_B| \geq 3$ and that A-players are organized into one or multiple circles. Consider the following initial Nash equilibrium $s \in \overrightarrow{\texttt{A}}$ where the subnetwork among players in I'_A (who are supposed to play A in s') is given by $g_{I'_A}$ and the remaining players (in I'_B) connect to these players. Assume that one agent in I'_B makes a mistake, links up to another agent I'_B and switches to B. It follows that all remaining agents in I'_B will one-by-one switch to B and we have reached an absorbing state where no agent has an incentive to switch and the subnetwork among A-players is the same as the one as in s'. If the desired Nash equilibrium s' does not feature any links from B- to A-players, we can simply by single mutations change one link at a time and note that no B-player will switch actions in this construction. If there are B-players linking to A-players, lemma 21 tells us that there is at most one such B-agent. Label this agent i. When changing links we assume that i changes her links last, ensuring that she will not switch action in this construction. Thus, we can reach any Nash equilibrium s' starting from $\overrightarrow{\texttt{A}}$ with a chain of single mutations.

In a next step, note that we can always move from an action-heterogenous absorbing state with m A-players to another action-heterogenous absorbing state with $(m+1)$ A-players, provided that there were initially more than three B-agents. To see this, note that the A-agents are arranged in circles and label the agents in one of these circles by $1, \ldots, g$. Now, because $N - m \geq 4$ there either exists a B-agent (say agent $m+1$) without incoming links or we can reach with a single mutation chain an absorbing state where this is the case. Now assume that agent g makes a mistake and links to agent $m+1$. Agent $m+1$ will switch to A and form her link to 1. We have reached an absorbing state with one more A-player.

Now consider a state with three B-players. If any of them, by mistake, switches to A, there are only two B-players left. One of them will link to an A-player, have half of her neighbors choosing A, and consequently will switch to A too. The remaining B-player will have to switch to A as well and we have reached a state in $\overrightarrow{\mathtt{A}}$.

The above arguments show that we can find a single mutation chain connecting any two Nash equilibria. This implies that all Nash equilibria are stochastically stable, $S^{***} = \overrightarrow{\mathtt{A}} \bigcup \overrightarrow{\mathtt{B}} \bigcup \overrightarrow{\mathtt{AB}}$.

D.2 Synchronous updating of actions and links

We now discuss the implications of a process where actions and links are updated independently in the case where each agent may only support one link, $k=1$. The main difference to Jackson & Watts (2002) is that we focus on a non-cooperative process of network formation while theirs is based on pairwise stability. In particular, we assume in each period one agent is presented with a revision opportunity. When such an opportunity arrives she may either update her links with exogenously given probability $0 < \lambda < 1$ or her action with the remaining probability $1-\lambda$.

Optimal behaviour in this case is fairly different to our main model as agents revising their action take the network as given instead of optimally adjusting their links. It is straightforward to see that any Nash equilibrium has to fulfill the following two properties. Firstly, for each agent the action choice has to be optimal given the distribution of actions among her neighbors. This implies that for each A-player i it has to be true that $\frac{m_i^{out}+m_i^{in}}{d_i^{out}+d_i^{in}} \geq p$ and that for each B-player j $\frac{m_j^{out}+m_j^{in}}{d_j^{out}+d_j^{in}} \leq p$ holds. Secondly, the linking choice has to be optimal given the own action choice and the action choices of others. This implies that i) an agent will always form her link unless she receives incoming links from all other agents and ii) an agent will form her link to an agent using her own action and will link to agents with the other action if she receives links from all agents with her own action. Note that the set of Nash equilibria will be different to the one in our benchmark case, as e.g. now in action-heterogenous profiles A-agents without incoming links will not always switch to B.

In a next step, we consider which network configurations are absorbing. First, note that provided that $3 \leq m \leq N-3$, the process will move to a configuration where there are no links between agents choosing different actions. To see this consider a Nash equilibrium where agent i forms her link to j who chooses another action. It follows that it must be the case i has incoming links from all other agents choosing her action. Since there are two or more of them, it follows that these neighbors are not linked among themselves. Now the process with positive probability may move to a profile where a (previously missing) link among these agents is present, implying that i is now not linked to one of the agents with her action, say agent k. When i receives revision opportunity, she will delete the link to j and form the link to k. In this manner, we can move to configurations where there are only links among agents using the same action. While the network among agents using the same action may change (provided that there are more than three), no links to agents with other actions will reappear and nobody will change actions. Secondly, note that in any action heterogenous absorbing set it has to be true that there are at least three A-players and three B-players. It is straightforward to see the latter point as with two B-players one of them has to link to A-players or receive incoming links from all $N-2$ A-players and thus faces at least half of her opponents choosing A, and consequently will update her action to A when given revision opportunity. Now consider the case where there are only two A-agents with one of them linking to B agents and refer to this agent as i. Note that it is impossible that all other agents form links to an A-player when $m = 2$. If the B-agent has no incoming links from other B-agents she will have to switch to A and the original state was not absorbing. If there is another B-agent without incoming links the A-player i may form a link to her and she will switch to A. Otherwise, since there are more than four B-players, the network among them may change so that one of them has no incoming links. When the A-agent i forms a link to this B-player, she will switch actions. We have thus shown that agents will form only links to agents using the same action and that $3 \leq m \leq N-3$ has to hold in any action-heterogenous absorbing set.

Finally, consider transitions among the various absorbing sets. First, consider transitions among action-heterogenous networks. Provided that there are more than three A-players (B-players) we can always move to an absorbing set where there is one less A-player (B-player) with one mutation. We will outline the transition reducing A-players and remark that the transition for B-players works analogously. Since there are four or more A-players, the process with positive probability moves to a configuration where one of them does not have any incoming links. Assume that this player now makes a mistake and chooses action B. Assume that in a next step this agent receives the opportunity to change links. In this case she will link up to a B-player and we have reached an absorbing set with one less A-player. In a similar manner we can construct transitions where the number of A-players stays constant but their identity changes.

Now consider the transitions out of payoff dominant profiles. The process will with positive probability reach a profile where a group of agents forms a circle. If now one agent chooses action A by mistake, as in Ellison (1993), it will spread contagiously through the circle. Thus, depending on the size of the original circle, we can move from payoff dominant absorbing sets to action-heterogenous and risk dominant network configurations at the cost of one mistake.

In a next step, consider transitions out of action-heterogenous profiles. Consider first the transition to risk dominant action profiles. With positive probability, the dynamics reaches a profile where the B-players are arranged in a circle. Again, if one agent in this circle makes a mistake the entire circle will follow. Now consider the transition to payoff dominant absorbing sets. Note that all A-agents will only have links to other A-agents. As argued above, if $m > 3$ we can always reduce the number of A players by one at the cost of one mistake. Let us thus consider absorbing sets where $m = 3$, and label the A-players by $1, 2$ and 3. These three agents have to be connected. Without loss of generality assume that 1 links to 2 who links to 3, who links back to 1. One B-player making a mistake and linking up to any of these players can never change their behaviour. So assume any of the A-players, say 3, makes a mistake and switches to action B. In the most favourable case, after a series of revisions of B-players we may reach a profile where all B-agents link up to 3 and 3 finds it optimal to stay with B. We may thus reach configurations where 1 links to 2 who will have to form a link to a B-player. These configurations may be even be Nash equilibria. However, inevitably the dynamics will reach a state where the B-player 3 has no incoming B-links. This player will then switch to A and we are back in an absorbing set where $m = 3$. Note, however that if agents 2 and 3 make a mistake and switch to B, then player 1 will follow when given revision opportunity.

Now consider transitions out of risk dominant network configurations. Note that an A-player will only change her action only if strictly more than half of her neighbors choose B. So an agent i who changes actions as a result of another agent j switching to action B satisfies that $g_{ij} = 1$ and $m_i^{in} = 0$. Let us start with the transition to action-heterogenous profiles. Assume everybody plays A and that the dynamics has reached a state where agents $1 - 3$ form the core, agent 4 connects to 3 and everybody else links to 4. If now 4 switches by mistake to action B, agents 5 to N will follow switch to action B and form links among them. When given revision opportunity agent 4 will link up to some agent of the other B-players and we have reached an action-heterogenous absorbing set where $m = 3$. Finally, consider the direct transition to payoff dominant profiles. Note that in any absorbing set everybody supports a link. This implies that an agent may at most have $N - 2$ incoming links. Without loss of generality assume that this is agent 2 and she supports her link to agent 1, who has to link to somebody different than agent 2, say agent 3. Now the maximal effect of one mistake can be achieved if agent 2 switches to B, prompting agents 4 to N to follow. Agents 1 and 3, however, will keep their actions as at least half of their neighbors choose the risk dominant action. Eventually, agent 3 will link up to a B-player without incoming links and this agent will switch to A and will form a link to 1 at the next revision opportunity. Following one mistake, we will thus have to arrive at an action-heterogenous absorbing set with $m \geq 3$. If however, agent 1 in the above construction also switches, we will move to the payoff dominant absorbing set, establishing that two mistakes are sufficient for the transition.

Finally, assume that the number of action-heterogenous absorbing sets is L, so that together with the payoff dominant and the risk dominant absorbing sets there are $L + 2$ absorbing sets. The above arguments show that all action-heterogenous and the risk dominant absorbing sets have stochastic potential of $L + 1$. Payoff dominant network profiles, however, have stochastic potential of $L+2$. It follows that action-heterogenous and risk dominant network profiles are stochastically stable.

D.3 Linking costs

Our main analysis focused on the case where $0 < \gamma < d$ so that i) there is no duplication of links and ii) agents always find it worthwhile to establish a link to other agents, regardless of the recipient's action and so. In the following we discuss the implications of different linking costs.

Zero linking costs, $\gamma = 0$:

If $\gamma = 0$ an agent i who has more than $N-1-k$ incoming links from other agents, is indifferent between forming a link to another agent j who already supports a link to i and not forming this link, i.e. $g_{ij} = g_{ji} = 1$ can occur. Thus, Nash equilibrium networks do not have to be essential. Note, however, that for $N > 4k+2$ our best response process will converge to configurations where no agent has more than $N-1-k$ incoming links and that from such profiles we will never move back to profiles with superfluous links. It follows that the absorbing sets, the transition costs, and consequently the set of stochastically stable states are the same as in our original model.

Intermediate linking costs, $d \leq \gamma \leq c$:

In a first step, consider the case where $\gamma = d$. Now a B-player j who has more than $N-m-1-k$ incoming links from other B-players, is indifferent between forming a link to an A-player and not forming this link. Thus, the optimal linking decisions of agent j is characterized by $\min\{N-m-1-(d_j^{in}-m_j^{in}), k\}$ links to B-players and $0 \leq m_i^{out} \leq \min\{N-1-d_j^{in}, k\} - \min\{N-m-1-(d_j^{in}-m_j^{in}), k\}$ links to A-players. Note, however, that the LOP of action B is the same regardless of how many links j chooses to form (or not to form) to A-players. Note further that for $N > 4k+2$ our best response process will converge to configurations where each of $N-m > 2k+1$ B-players forms all of her k links to other B-players. It follows that the absorbing sets, the transition costs, and consequently the set of stochastically stable states are the same as in the benchmark case where $0 < \gamma < d$.

Now consider the case where $d < \gamma \leq c$. In this case, B-players will only form links to other B-players and never link up to A-players. A-players, however, will still link to B-players (provided that they are already linked to all other A-players). In the boundary case, where $\gamma = c$, A-players are in fact indifferent between forming links to B-players and not forming links.

Consider an action-heterogenous Nash equilibrium s^*. Note that the lower bound on the number of A-players identified in lemma 3 remains unchanged at $\left\lceil \frac{2(b-c)}{2a-c-d}k \right\rceil + 1$. When calculating the minimal number of B-players we will now have to take into account that B-players will never link to A-players. Replicating the steps in the proof of lemma 3 reveals that the lower bound for the number of B-players now is $\left\lceil \frac{2(a-\gamma)}{2b-\gamma-c}k \right\rceil + 1$ and consequently depends on the linking cost γ. The only difference to the benchmark case is that the payoff parameter d is replaced by the linking cost γ. Intuitively, while B-players forgo a payoff of d by not linking to an A-player, they also save the linking cost γ by not doing so.

Note that one can replicate the proofs of lemmata 4, 5 and of proposition 2 without modification for linking costs in the range $d < \gamma \leq c$. This implies that the absorbing sets remain unchanged. Note that in our original characterization of absorbing sets we showed that the dynamics converges to profiles where there are no links from B-players to A-players. Thus, intuitively one shouldn't expect there to be any differences to the scenario where B-players would never choose to form any such links.

Now consider transitions among the various absorbing sets and stochastic stability of the various absorbing sets. In order to distinguish the case $0 \leq \gamma \leq d$ in the benchmark case from the intermediate cost case where $d < \gamma \leq c$, we add the superscripts ℓ and m to the relevant transition costs.

First, the following lemma establishes that if for $\gamma \in [0, d]$ an A-player finds it optimal to switch to action B, she will also find it optimal to do so for $\gamma \in (d, c]$. Conversely, if a B-player finds it optimal to stay with her action for $\gamma \in [0, d]$, this is also optimal for $\gamma \in (d, c]$. Thus, when γ increase from $[0, d]$ to $(d, c]$ it becomes (weakly) easier to induce A-agents to switch and (weakly) more difficult to induce B-agents to switch.

Lemma 22. *The payoff advantage (disadvantage) of action B over action A weakly increases (decreases) over the range* $[0, c]$.

Proof. First note that when linking costs change, the payoff from passive links keeps unchanged. Therefore, it is sufficient for us to analyze the payoff from active links. Here we present the considerations for a player j who currently uses action B. The calculations for A-players follow exactly the same steps, with the only difference that m is replaced with $m-1$ and $N-m-1$ with $N-m$.

First, consider the case where there are sufficiently other non-linked players so that this agents will form all of her links, $N-1-d_j^{in} \geq k$. For any $\gamma \in [0, c] = [0, d] \cup (d, c]$, the LOP (disregarding payoff from passive links) from action A of player j is

$$a(\min\{k, m - m_j^{in}\}) + c(k - \min\{k, m - m_j^{in}\}) - \gamma k.$$

The LOP from action B is

$$b(\min\{k, N-m-1-(d_j^{in} - m_j^{in})\}) + d(k - \min\{k, N-m-1-(d_j^{in} - m_j^{in})\}) - \gamma k$$

for $\gamma \in [0, d]$ and is

$$b(\min\{k, N-m-1-(d_j^{in} - m_j^{in})\}) - \gamma \min\{k, N-m-1-(d_j^{in} - m_j^{in})\}$$

for $\gamma \in (d, c]$. If the B-player can form all of her links to other B-agents, $N-m-1-(d_j^{in}-m_j^{in}) \geq k$, the difference in LOPs between actions B and A remains constant, as γ changes. Now consider the case where the B-agent j can not fill all of her links with other B-agents, $N-m-1-(d_j^{in} - m_j^{in}) < k$. For $\gamma \in [0, d]$, the difference between the LOP from action B and the LOP from action A is

$$\begin{aligned} & -[a(\min\{k, m-m_j^{in}\}) + c(k - \min\{k, m-m_j^{in}\})] + b(\min\{k, N-m-1-(d_j^{in}-m_j^{in})\}) \\ + \quad & d(k - \min\{k, N-m-1-(d_j^{in}-m_j^{in})\})]. \end{aligned}$$

This is strictly smaller than the difference between the LOP from action B and the LOP from action A for any $\gamma \in (d, c]$, given by

$$\begin{aligned} & -[a(\min\{k, m-m_j^{in}\}) + c(k - \min\{k, m-m_j^{in}\})] + b(\min\{k, N-m-1-(d_j^{in}-m_j^{in})\}) \\ + \quad & \gamma(k - \min\{k, N-m-1-(d_j^{in}-m_j^{in})\})]. \end{aligned}$$

Note that we can analyse the case where agent j cannot fill her k links, $N-1-d_j^{in} < k$, in a similar manner as the argument above, replacing k with $N-1-d_j^{in}$. We have, thus, shown that the payoff advantage (disadvantage) of action B over action A weakly increases (decreases) over the range $[0, c]$. $\square$

Now consider the transition from any absorbing set $S' \in \{\overrightarrow{B}, \overrightarrow{\mathcal{AB}}\}$ to another absorbing set $S'' \in \{\overrightarrow{A}, \overrightarrow{\mathcal{AB}}\}$ with strictly more A-players. Lemma 22 implies that for $\gamma \in (d, c]$, it is weakly more difficult to induce B-agents to switch to action A in comparison with the benchmark case. Therefore,

$$c^m(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \geq c^\ell(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}), c^m(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}) \geq c^\ell(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}), \text{ and } c^m(\overrightarrow{B}, \overrightarrow{A}) \geq c^\ell(\overrightarrow{B}, \overrightarrow{A}).$$

Now consider the transition from $\overrightarrow{B}$ to $\overrightarrow{\mathcal{AB}}$ as analysed in lemma 12 for the low linking cost case. In this transition, B-players that are induced to switch are able to form k connections to other B-agents. Thus, the comparison of payoffs is the same as in lemma 12. As lemma 12 provides a lower bound for the number of mistakes that any agent will switch from A to B we also have $c^m(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}) \leq c^m(\overrightarrow{B}, \overrightarrow{A})$.

In a next step, consider the transition from any absorbing set $S' \in \{\overrightarrow{A}, \overrightarrow{\mathcal{AB}}\}$ to another absorbing set $S'' \in \{\overrightarrow{\mathcal{AB}}, \overrightarrow{B}\}$ with strictly more B-players. Lemma 22 implies that for $\gamma \in (d, c]$, it is weakly easier to induce A-agents to switch to action B in comparison with the benchmark case. Therefore,

$$c^m(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) \leq c^\ell(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}), c^m(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) \leq c^\ell(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}), \text{ and } c^m(\overrightarrow{A}, \overrightarrow{B}) \leq c^\ell(\overrightarrow{A}, \overrightarrow{B}).$$

Further, note that in the transition from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{B}$ characterized in lemma 13 for the low linking cost case, passive neighbors of A-players mutate to action B. The threshold identified in lemma 13 is obtained by considering circumstances under which the remaining A-players will switch to

B. When comparing payoffs from the two actions, A-players are potentially always able to fill up all of their k slots with B-players. Thus, the difference in LOPs between actions A and B remains constant, as γ changes. For this reason, the transition cost in lemma 13 does not change, $c^m(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) = c^\ell(\overrightarrow{\mathcal{AB}}, \overrightarrow{B})$.

Finally, consider the transitions out of $\overrightarrow{A}$. First, let us focus on the transition from $\overrightarrow{A}$ to $\overrightarrow{\mathcal{AB}}$. Lemma 11 can be modified in the following way. Let x denote the minimal number of agents switching from A to B such that any other agent finds it optimal to switch from A to B. Note that agents without incoming links are easiest to switch. In particular, an agent without incoming links will switch with positive probability whenever $xb - x\gamma \geq ak - \gamma k$. We thus have $x = \left\lceil \frac{a-\gamma}{b-\gamma} k \right\rceil = c^m(\overrightarrow{A}, \overrightarrow{\mathcal{AB}})$. Then, turn to the transition from $\overrightarrow{A}$ to $\overrightarrow{B}$. By the previous argument we know that $\left\lceil \frac{a-\gamma}{b-\gamma} k \right\rceil$ mistakes are necessary. Following the proof of the second part of lemma 11 we can establish that $y = \max\left\{ \left\lceil \frac{a-\gamma}{b-\gamma} k \right\rceil, \left\lceil \frac{2a-b-d}{a+b-c-d} k \right\rceil \right\}$ mistakes are sufficient for a transition. To this end, assume that the y mutations happen among the core players $\{1, \ldots, y\}$ and that those players do not change their links. Consider the remaining core agents, starting with agent $y+1$. This agent has now y passive links from B-players and $k-y$ passive links from A-players. Her LOP of playing A is $ka + (k-y)a + yc - \gamma k$ and her LOP from action B is $kb + yb + (k-y)d - \gamma k$. She will thus switch if $y \geq \frac{2a-b-d}{b-d+a-c} k$. Note that since $y \geq \left\lceil \frac{a-\gamma}{b-\gamma} k \right\rceil$ also agents in the periphery will switch and we have established that $y = \max\left\{ \left\lceil \frac{a-\gamma}{b-\gamma} k \right\rceil, \left\lceil \frac{2a-b-d}{a+b-c-d} k \right\rceil \right\}$ mistakes are sufficient for the transition to $\overrightarrow{B}$. We thus have that $\left\lceil \frac{a-\gamma}{b-\gamma} k \right\rceil \leq c^m(\overrightarrow{A}, \overrightarrow{B}) \leq \max\left\{ \left\lceil \frac{a-\gamma}{b-\gamma} k \right\rceil, \left\lceil \frac{2a-b-d}{a+b-c-d} k \right\rceil \right\}$. It further can be verified that,

$$\max\left\{ \left\lceil \frac{a-\gamma}{b-\gamma} k \right\rceil, \left\lceil \frac{2a-b-d}{a+b-c-d} k \right\rceil \right\} < \left\lceil \frac{a-d}{b-d} k \right\rceil = c^\ell(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) = c^\ell(\overrightarrow{A}, \overrightarrow{B}).$$

Now we turn to consider stochastic stability. For any $S' \in \{\overrightarrow{A}, \overrightarrow{\mathcal{AB}}, \overrightarrow{B}\}$, let the stochastic potential $\mathcal{C}(S')$ is the cost of the minimum cost S'-tree. Following Kandori et al. (1993) and Young (1993), an absorbing set S' is, thus, stochastically stable if $\mathcal{C}(S') = \min_{S'' \in \{\overrightarrow{A}, \overrightarrow{\mathcal{AB}}, \overrightarrow{B}\}} \mathcal{C}(S'')$. Again we use the superscripts ℓ and m to differentiate stochastic potentials in the low and intermediate linking costs cases.

In the following we show that if $\overrightarrow{\mathcal{AB}}$ is stochastically stable for low linking costs $0 \leq \gamma \leq d$, it is so too for intermediate linking costs $d < \gamma \leq c$. To this end, we argue that when $\overrightarrow{\mathcal{AB}}$ has the lowest stochastic potential under low linking costs, it still has the lowest stochastic potential under intermediate linking costs.

Following the discussion above, we have that transition costs are characterized by $c^\ell(\overrightarrow{\mathcal{AB}}, \overrightarrow{A}) \leq c^m(\overrightarrow{\mathcal{AB}}, \overrightarrow{A})$, $c^\ell(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) = c^m(\overrightarrow{\mathcal{AB}}, \overrightarrow{B})$, $c^\ell(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}) = c^m(\overrightarrow{B}, \overrightarrow{\mathcal{AB}})$ and $c^\ell(\overrightarrow{B}, \overrightarrow{A}) \leq c^m(\overrightarrow{B}, \overrightarrow{A})$. Further, the transition cost from $\overrightarrow{A}$ to $\overrightarrow{\mathcal{AB}}$ and $\overrightarrow{B}$ changes as follows

$$c^\ell(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) = \left\lceil \frac{a-d}{b-d} k \right\rceil > c^m(\overrightarrow{A}, \overrightarrow{\mathcal{AB}}) = \left\lceil \frac{a-\gamma}{b-\gamma} k \right\rceil$$

and

$$c^{\ell}(\overrightarrow{A},\overrightarrow{B})=\left\lceil\frac{a-d}{b-d}k\right\rceil>\max\left\{\left\lceil\frac{a-\gamma}{b-\gamma}k\right\rceil,\left\lceil\frac{2a-b-d}{a+b-c-d}k\right\rceil\right\}\geq c^{m}(\overrightarrow{A},\overrightarrow{B}).$$

First, note that since $c^m(\overrightarrow{B},\overrightarrow{\mathcal{AB}})\leq c^m(\overrightarrow{B},\overrightarrow{A})$ and $c^{\ell}(\overrightarrow{B},\overrightarrow{\mathcal{AB}})\leq c^{\ell}(\overrightarrow{B},\overrightarrow{A})$, A-trees with direct branches from $\overrightarrow{B}$ and from $\overrightarrow{\mathcal{AB}}$ into the root $\overrightarrow{A}$ will have cost no smaller than the tree from $\overrightarrow{B}$ to $\overrightarrow{\mathcal{AB}}$ and then into root $\overrightarrow{A}$. Thus, to determine the stochastic potential of $\overrightarrow{A}$, it is sufficient to compare the cost of two different $\overrightarrow{A}$-trees:

$$\begin{aligned}\mathcal{C}^m(\overrightarrow{A}) &= \min\{c^m(\overrightarrow{\mathcal{AB}},\overrightarrow{B})+c^m(\overrightarrow{B},\overrightarrow{A}),c^m(\overrightarrow{B},\overrightarrow{\mathcal{AB}})+c^m(\overrightarrow{\mathcal{AB}},\overrightarrow{A})\}\\ &\geq \min\{c^{\ell}(\overrightarrow{\mathcal{AB}},\overrightarrow{B})+c^{\ell}(\overrightarrow{B},\overrightarrow{A}),c^{\ell}(\overrightarrow{B},\overrightarrow{\mathcal{AB}})+c^{\ell}(\overrightarrow{\mathcal{AB}},\overrightarrow{A})\}\\ &= \mathcal{C}^{\ell}(\overrightarrow{A}).\end{aligned}$$

Further, for low and high linking costs, the stochastic potential of $\overrightarrow{\mathcal{AB}}$ is given by the sum of the cost of the branch from $\overrightarrow{A}$ to $\overrightarrow{\mathcal{AB}}$ and the cost of the branch from $\overrightarrow{B}$ to $\overrightarrow{\mathcal{AB}}$; thus,

$$\begin{aligned}\mathcal{C}^m(\overrightarrow{\mathcal{AB}}) &= c^m(\overrightarrow{A},\overrightarrow{\mathcal{AB}})+c^m(\overrightarrow{B},\overrightarrow{\mathcal{AB}})\\ &= c^{\ell}(\overrightarrow{A},\overrightarrow{\mathcal{AB}})+c^{\ell}(\overrightarrow{B},\overrightarrow{\mathcal{AB}})+\left(\left\lceil\frac{a-\gamma}{b-\gamma}k\right\rceil-\left\lceil\frac{a-d}{b-d}k\right\rceil\right)\\ &= \mathcal{C}^{\ell}(\overrightarrow{\mathcal{AB}})+\left(\left\lceil\frac{a-\gamma}{b-\gamma}k\right\rceil-\left\lceil\frac{a-d}{b-d}k\right\rceil\right).\end{aligned}$$

At last, note that $c^m(\overrightarrow{A},\overrightarrow{\mathcal{AB}})\leq c^m(\overrightarrow{A},\overrightarrow{B})$ and $c^{\ell}(\overrightarrow{A},\overrightarrow{\mathcal{AB}})=c^{\ell}(\overrightarrow{A},\overrightarrow{B})$. It follows that B-trees with direct branches from $\overrightarrow{A}$ and from $\overrightarrow{\mathcal{AB}}$ into the root $\overrightarrow{B}$ will have cost no smaller than the trees from $\overrightarrow{A}$ to $\overrightarrow{\mathcal{AB}}$ and then into root $\overrightarrow{B}$. To determine the stochastic potential of $\overrightarrow{B}$, we thus only have to compare the cost of two different $\overrightarrow{B}$-trees:

$$\begin{aligned}\mathcal{C}^m(\overrightarrow{B}) &= \min\{c^m(\overrightarrow{A},\overrightarrow{\mathcal{AB}})+c^m(\overrightarrow{\mathcal{AB}},\overrightarrow{B}),c^m(\overrightarrow{\mathcal{AB}},\overrightarrow{A})+c^m(\overrightarrow{A},\overrightarrow{B})\}\\ &\geq \min\{c^{\ell}(\overrightarrow{A},\overrightarrow{\mathcal{AB}})+c^{\ell}(\overrightarrow{\mathcal{AB}},\overrightarrow{B}),c^{\ell}(\overrightarrow{\mathcal{AB}},\overrightarrow{A})+c^{\ell}(\overrightarrow{A},\overrightarrow{B})\}\\ &\quad+\left(\left\lceil\frac{a-\gamma}{b-\gamma}k\right\rceil-\left\lceil\frac{a-d}{b-d}k\right\rceil\right)\\ &= \mathcal{C}^{\ell}(\overrightarrow{B})+\left(\left\lceil\frac{a-\gamma}{b-\gamma}k\right\rceil-\left\lceil\frac{a-d}{b-d}k\right\rceil\right)\end{aligned}$$

where the inequality follow from the observations that $c^m(\overrightarrow{A},\overrightarrow{\mathcal{AB}})-c^{\ell}(\overrightarrow{A},\overrightarrow{\mathcal{AB}})=\left\lceil\frac{a-\gamma}{b-\gamma}k\right\rceil-\left\lceil\frac{a-d}{b-d}k\right\rceil$ and $c^m(\overrightarrow{A},\overrightarrow{B})-c^{\ell}(\overrightarrow{A},\overrightarrow{B})\geq\left\lceil\frac{a-\gamma}{b-\gamma}k\right\rceil-\left\lceil\frac{a-d}{b-d}k\right\rceil$.

Note that

$$\left\lceil\frac{a-\gamma}{b-\gamma}k\right\rceil-\left\lceil\frac{a-d}{b-d}k\right\rceil<0.$$

It follows that for intermediate linking costs the stochastic potential of $\overrightarrow{\mathcal{AB}}$ decreases by $\lceil \frac{a-d}{b-d}k \rceil - \left\lceil \frac{a-\gamma}{b-\gamma}k \right\rceil$ and the stochastic potential of $\overrightarrow{\mathcal{B}}$ decreases by at most $\lceil \frac{a-d}{b-d}k \rceil - \left\lceil \frac{a-\gamma}{b-\gamma}k \right\rceil$, while the stochastic potential of $\overrightarrow{\mathcal{A}}$ weakly increases.

It follows that if a $\overrightarrow{\mathcal{AB}}$-tree is of minimal costs under low linking costs, it is also of minimum costs under intermediate linking costs. Thus, if $\overrightarrow{\mathcal{AB}}$ is (uniquely) stochastically stable for certain parameters as in the low linking cost case, it is also so for the intermediate linking cost case.

High linking costs, $c < \gamma \leq a$:

For linking costs in the range $c < \gamma \leq a$, agents will not form links to agents using another action. This implies that A-players will form $\min\{k, m - 1 - m_i^{in}\}$ links to other A-players and no links to B-players. B-players will form $\min\{k, N - m - 1 - (d_i^{in} - m_i^{in})\}$ links to other B-players and no links to A-players.

Consider an action-heterogenous Nash equilibrium s^*. Note that there do neither exist links from A- to B-players nor links from B- to A-players in any Nash equilibrium. We can apply the same logic as in lemma 3 to obtain that the lower bound on A-players is $\underline{m} = \left\lceil \frac{2(b-\gamma)}{2a-\gamma-d}k \right\rceil + 1$, and the lower bound on B-players is $N - \overline{m} = \left\lceil \frac{2(a-\gamma)}{2b-\gamma-c}k \right\rceil + 1$. Note that these lower bounds now depend on the specific value of γ.

Now consider the unperturbed process. It is still true that there can be at most $2k + 1$ players choosing A. For, if otherwise the process would reach a state where some A-players has have no incoming links from A-players and switch to B. Further, note that since there are at most $2k + 1$ agents choosing A, it follows that in an absorbing set A-players are fully connected and the best response of them is unique. B-players, however, are not fully connected and the dynamics will move among various states where the subnetwork among them differs. Now consider the lower bound $\underline{\underline{m}}$ identified in lemma 5 i). This lower bound is determined by A-players forming links to B-players without any incoming links and prompting them to switch. Since for linking costs in the range $c < \gamma \leq a$, there are no links from A-players to B-players this does not happen in the present context and the lower bound $\underline{\underline{m}}$ does not apply. In fact, any action-heterogenous Nash equilibrium with fully connected A-players is part of an absorbing set now.

In a next step, we consider the transition costs. We use the superscript h to differentiate the transition costs in the high linking cost case from the low and intermediate cost cases.

First, we consider transitions out of $\overrightarrow{\mathcal{AB}}$. Consider any action-heterogenous absorbing set. Assume that A-player i makes mistakes, switches to action B and forms k links to B-players. All A-players who form a link to i will delete these links. Now consider the LOP of the remaining A-players. For agents who had received a link from i the LOP of action A decreases by a and the LOP of action B decreases by d. For agents who were supporting a link to i the LOP of action A decreases by $a - \gamma$ and the LOP of action B stays the same. In either case action B becomes relatively more attractive. Depending on the exact circumstances remaining A-agents will either switch to B or keep their current strategy. Thus, with one mistake the process will either reach another action-heterogenous absorbing set with strictly less A-players or $\overrightarrow{B}$. Thus, $c^h(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) = 1$.

Now consider the transition out of $\overrightarrow{B}$. The following argument establishes that one mistake is not enough. Consider any state in $\overrightarrow{B}$ and assume that one agent, say agent 1, makes a mistake and switches to A. Without loss of generality assume that the agents she forms links with are $2, \ldots, k+1$. Note that no B-player $i \in \{k+2, \ldots, N\}$ will switch to A. To see this note that an agent who switches to A and links to 1 has to have incoming links from $N - 2 > 2k$ other agents. Otherwise in addition to agent 1 there are k agents who are not linked to i. If i switches to A, the payoff from active links is at most $(a - \gamma)$, while the loss from passive links is at least $2k(b - c)$. It is easy to see that $2k(b - c) > (a - \gamma)$. Therefore, i will not switch to action A. Thus, the only agents that may switch are those now receiving a link from 1. Thus, assume that 2 is the first to revise her strategy. Assume that she switches to A (which is e.g. the case when she is a periphery agent and $a \geq k(b-\gamma)+d$ holds) and note that she will not link to anybody. Note that again at this stage none of the B-agents in $\{k+2, \ldots, N\}$ has an incentive to switch to A and that agent 2 has a unique best response. Proceed to agent 3 and assume she switches to A. In this case she forms a link to 2. We can iterate this reasoning to show that the process may reach a state where i) 1 links to $2, \ldots, k+1$, 2 does not link to anybody, 3 links to 2, and so on until, $k+1$ links to $1, \ldots, k$, ii) at each stage the best response of A-players is unique and iii) B-players in $\{k+2, \ldots, N\}$ are never tempted to switch to A. To see this latter point note that for any B-player in $\{k+2, \ldots, N\}$, who considers switching to A and link to $1, \cdots, k+1$ the number of passive neighbors should be larger than $N - 1 - 2k > 2k$. For, otherwise, besides $1, \ldots, k+1$ there are at least k other agents who are not linked to i. The payoff from active links is at most $k(a - \gamma)$, while the loss from passive links is at least $2k(b - c)$. Since $2k(b - c) > k(a - \gamma)$ the agent will not switch to action A. Note that this implies that inevitably the process will have to reach some state where player 1 will have no incoming links and no A-player when given revision opportunity will link to her. At some point 1 will receive revision opportunity and switch back to B. This is followed by 2 switching back, and so forth until we are back in a state in $\overrightarrow{B}$. Thus, one mistake is not enough to leave the basin of attraction of $\overrightarrow{B}$, i.e. $c^h(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}) > 1$ and $c^h(\overrightarrow{B}, \overrightarrow{A}) > 1$

We have thus seen that $c^h(\overrightarrow{\mathcal{AB}}, \overrightarrow{B}) = 1 < \min\{c^h(\overrightarrow{B}, \overrightarrow{\mathcal{AB}}), c^h(\overrightarrow{B}, \overrightarrow{A})\}$. This implies that $\overrightarrow{\mathcal{AB}}$ can never be stochastically stable. To see this assume that there exists a minimum cost $\overrightarrow{\mathcal{AB}}$-tree. This tree has to have a branch going out of $\overrightarrow{B}$. We can delete this branch and add a branch from $\overrightarrow{\mathcal{AB}}$ to $\overrightarrow{B}$. We have thus arrived at a $\overrightarrow{B}$-tree of strictly smaller cost. Thus, $\overrightarrow{\mathcal{AB}}$ is not stochastically stable for high linking costs.