Government Bonds, Bank Liquidity and Non-Neutrality of Monetary Policy in the Steady State

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Abstract

This paper studies non-neutrality of monetary policy in a model where fiat money is used by banks to meet liquidity demand and a government bond to collateralize reserve borrowing. It finds that if some banks are liquidity constrained, any monetary policy that alters the bond-to -fiat money ratio moves the interbank rate and is non-neutral in the steady state. Moreover, the effect for liquidity unconstrained banks is the opposite of that for the maximally constrained. Lastly, if the expansion of digital ways of payment eliminates depositor withdrawals, fiat money will stop circulation and a bullion standard will probably return.

Key words: Government bond, bank liquidity, non-neutrality of monetary policy, digital ways of payment.

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1 Introduction

How can a difference in economic performance be made by someone moving a mouse or printing a special type of paper, that is, the central bank's nominal operations? This fundamental question concerning monetary policy has been long fascinating the economics profession and inspiring many important works. This paper discovers a new mechanism: Monetary policy produces real effects *in the steady state* by relaxing or tightening the liquidity constraint of commercial banks. Moreover, the effects for different types of banks are different, even opposite. While most of the existing works on the question focus on fiat money's role as media of exchange, this paper shares with recent studies by Bianchi and Bigio (2017), De Fiore et al (2018) and Piazzesi and Schneider (2018) in ceding the central stage to fiat money's role in banks' liquidity management, while having bank liability serve as media of exchange. Indeed, in a modern economy, the major form of media of exchange for goods and services is not fiat money, but bank liability, and fiat money is extensively used by banks to meet their liquidity needs, that is, to meet the withdrawal demand of depositors and to settle interbank liabilities. The importance of this role of fiat money is evidenced by the fact that a substantial fraction of fiat money is typically held by banks.[1] Moreover, this paper also models the important role that government bonds play for banks' liquidity management. To meet liquidity needs, often banks have to borrow reserves – that is, fiat money – and for this borrowing government bonds are often used as the collateral. Indeed, banks typically hold a large position of government bonds,[2] and the importance of public debt as collateral for obtaining liquidity in general is well established in the economics literature.[3]

The model economy lasts for an infinite number of periods, populated by a contin-

[1] Take the Bank of England as an example. Between June 2016 and June 2017 (i.e. before the crisis), according to Rule (2015), on its liability side, there are about as much of reserves and cash ratio deposits as the Bank's notes. The former is held by commercial banks only, which also hold a substantial fraction of the latter. It is hence safe to say that more than half of the fiat money that the Bank has created is held by commercial banks.

[2] Gennaioli et. al. (2018) document that on average 9% of their assets is government bonds.

[3] See Woodford (1990), Aiyagari and McGrattan (1998), Holmstrom and Tirole (1998), and recently Angeletos, Collard, and Dellas(2016). Some empirical evidence is provided by Grobéty (2018).

uum of banks, entrepreneurs, and workers. Banks issue nominal demandable liability (i.e. a promise to pay fiat money at demand). Entrepreneurs borrow it as a means of payment for hiring workers. Workers then deposit it back to the banking system. Due to this circulation of money, a bank sees a fraction of its liability deposited into another bank, becoming an interbank liability owed to the latter.[4] Banks face a liquidity risk: At an interim stage of each period, they might encounter a substantial withdrawal demand. To gather fiat money to meet the demand, a bank can demand settlement of interbank liabilities owed to it. Liquidity demand is thus passed on along the interbank liability links. Consequently, *banks with a bigger fraction of liability outflow need to meet a larger liquidity demand*. If a bank finds its fiat-money position inadequate to meet the liquidity demand, it has to borrow reserves from other banks. We assume that this borrowing is collateralized with a permanent government bond.[5] As a result, banks face a liquidity constraint: Their reserve-borrowing capacity is equal to the value of their bond positions. Monetary policy is modelled as a change to the aggregate nominal-asset portfolio composed of both fiat money and the bond.

These two nominal assets are a perfect substitute to one another for individual banks: The bond can be swiftly converted into fiat money via collateralized borrowing; and fiat money earns interest on the interbank reserves market as the bond earns dividend, at the same rate in equilibrium. For individual banks, therefore, the composition of their asset portfolios is not determined while the value is. However, the bond-to-fiat money ratio of the aggregate portfolio determines the property of the steady state because it determines the portfolio's real value in the steady state. This point can be easily understood in the scenario where the stead-state value of fiat money is invariant with its quantity. Suppose the ratio rises because fiat money's quantity falls. In this

[4]In reality, a substantial part of interbank liabilities formed during daily transactions are cancelled, but certainly not all of them. The unsettled part will add to the liquidity burden of the originating banks. This way of forming interbank liabilities is considered by Freixas, Parigi and Rochet (2000), Parlour, Rajan and Walden (2017), and Piazzesi and Schneider (2018), among others.

[5]Certainly, unsecured borrowing also plays an important part for banks' liquidity management, but that does not deny the importance of secured interbank markets, which is evidenced by their sheer sizes – e.g. $10 trillion in the United States in August 2007 according to Heider and Hoerova (2009). Moreover, De Fiore et al (2018) document that the former is declining, the latter growing, in the U.S. over the past 15 years.

scenario, such a fall leaves the real value of fiat money unchanged and causes the unit real value of money to rise. As the bond pays a fixed stream of nominal dividend, its real value rises. Altogether, the aggregate real value of liquid assets rises with the bond-to-fiat money ratio. Indeed, if this ratio is above a threshold, then liquid assets abound and no banks face a binding liquidity constraint. In this Never-Binding Regime, fiat money is neutral: A change in its quantity affects nothing but the nominal price (so long as the steady state remains in the regime), that is, the steady state is in the aforementioned scenario.

If the bond-to-fiat money ratio is below the threshold, the liquidity constraint is binding for banks with an outflow fraction above a critical level because their liquidity burdens are the heaviest. In this Binding Regime, any monetary policy that alters the bond-to-fiat money ratio, such as open market operations, moves the interbank interest rate in the *steady state*. This movement results from the synthesis of several effects but, ultimately, is driven by a chain of three links. First, if the bond-to-fiat money ratio changes, as discussed above, so does the abundance, that is, the aggregate real value, of liquid assets. Second, such a change alters the maximal tightness of the binding liquidity constraint. Third, this alteration moves the reserve borrowing rate. To see why, note that liquidity constrained banks derive two benefits from holding fiat money: It earns interest on the interbank reserve market; and it helps relax these banks' liquidity constraint, the benefit of which is measured by the tightness of the constraint. Hence, only banks facing the tightest constraint hold fiat money and the sum of the two benefits is equal to the marginal cost of holding it (due to the time preference) in equilibrium. If the maximal tightness of the liquidity constraint moves in one direction, the interbank interest rate must move in the opposite direction. Observe that in this mechanism nominal rigidity plays no role, nor does incomplete information regarding monetary shocks.

Furthermore, we find that an expansionary policy, by reducing the reserve borrowing rate, decreases the lending rates of liquidity unconstrained banks, but *increases* those of the maximally constrained; a contractionary policy does the opposite. Intuition is as follows. An expansionary policy reduces the reserve borrowing rate and thereby diminishes the funding costs of all banks. This is the only effect for liquidity unconstrained banks, which accordingly *decrease* their lending rates. Maximally liquidity-constrained banks, however, receive another, offsetting, effect: The reserve borrowing rate is reduced exactly because the tightness of these banks' liquidity constraint (i.e. the maximal tightness) is increased. This increase induces them to raise the lending rate in order to contract the lending size. This effect dominates the funding-cost effect. Hence in net, their lending rates *increase,* and their lending scales decrease, with an expansionary policy.

Lastly, we find that the steady-state real value of money approaches nil if the probability of depositor withdrawals goes to zero. That is because in this economy the ultimate use of fiat money is to meet the withdrawal demand; after all a bank demands settlement of its interbank credit positions only when it needs the fiat money to meet its own withdrawal demand. Otherwise, why should it want fiat money, which pays no return, rather than the credit positions, which pay the interbank rate? Hence, if the expansion of digital ways of payment puts an end to depositor withdrawals, then fiat money is useless and hence worthless. It will then stop circulation. Banks will issue real liabilities, such as a promise to pay gold. That is, a bullion standard will probably return.

Literature

This paper joins the long theoretical discussion on the non-neutrality of monetary policy. The role of fiat money as media of exchange is abstracted from in the New Keynesian literature, but is modelled in the Cash-In-Advance literature; see Walsh (2010) for a survey of both strands of literature. These strands of literature, in order to have the non-neutrality of fiat money, usually resort to an assumption of either nominal rigidity (e.g. menu costs or delay in changing nominal portfolios); or incomplete information on monetary shocks;[6] or exogenous rules on banks' holding of excess reserves.[7] To none of these we resort. The media-of-exchange role of fiat money is endogenized by the search-matching literature following the seminal work of Kiyotaki and Wright (1989), using physical frictions of trading. This literature develops into the New Mon-

[6]See the seminal work of Lucas (1972). Angeletos and Lian (2016) provide a survey and Mao (2019) a recent development.

[7]See among others Chen (2018) and Mishkin (2016).

etarism, a survey of which is provided by Lagos et al (2017) and Rocheteau and Nosal (2017).[8] Within the New Monetarism, fiat money is typically neutral (though not super neutral),[9] but Williamson (2012) shows that a change in the composition of fiat money and the bond can produce permanent real effects.[10] Altogether, all the three strands of literature consider fiat money's role as media of exchange, whereas we focus on its role in banks' liquidity management.[11]

Closer to this paper are the recent studies on the effects or effectiveness of monetary policy that put banks' liquidity management at the central stage; see Bianchi and Bigio (2017), De Fiore et al (2018) and Piazzesi and Schneider (2018). In all the four papers (including the present one), banks face a withdrawal shock and use interbank reserve markets for liquidity management. Bianchi and Bigio (2017) and the present paper share the interest in the non-neutrality of nominal operations and find complementary mechanisms for it. They underline the search friction on the interbank market and the policy works by altering the trade-off that banks face in allocating funding between assets of different liquidity, while in the present paper the friction is the collateral constraint on reserve borrowing and the policy works by altering the tightness of this constraint. This liquidity constraint plays a key part in De Fiore et al (2018) as well. They also consider a leverage constraint as well as unsecured borrowing, which we do not. Indeed, they underline that in different quarters of the parameter space different constraints bind and discerning the binding ones is important for the policy decision of the central bank. Different to the present paper, theirs features no nominal side or interbank liabilities, nor are they interested in policy non-neutrality. Piazzesi and Schneider (2018) underline the importance of the institutional features of the payment

[8] See Lahcen (2019) for a recent development that combines the labour search model of Mortensen and Pissarides (1994) with the classic New Monetarist model of Lagos and Wright (2005).

[9] Observe that in certain money-search models, such as Kiyotaki and Wright (1989) and Cavalcanti et. al. (1999), a change in the fraction of population that holds a unit of fiat money affects the real allocation. However, this effect is, to a large extent, driven by the assumption that each agent holds at most one unit of fiat money.

[10] In the seminal work of Wallace (1981) it is irrelevant.

[11] The problem of banks' reserve holding is also considered by studies unrelated to the non-neutrality of monetary policy, e.g. Cavalcanti et. al. (1999) and Ennis (2018). This problem is abstracted from in studies that resort to a binding liquidity constraint for pinning down banks' lending size; see e.g. Goodfriend and McCallum (2007).

system for the interplay between security prices, inflation, and policy transmission. In their work, the key friction is the leverage cost, which varies with the collateral ratio. Besides, different to the other three studies, we model the ex ante bank heterogeneity in the outflow fraction and derive that monetary policy produces heterogeneous, even opposite, effects for banks with different outflow fractions in the steady state.

This paper is built on a general equilibrium analysis of money creation by banks. Analysis of this kind is also to be found in a recent strand of literature; see, among others, Jakab and Kumhof (2015), Kumhof and Wang (2018), Mendizábal (forthcoming) and Morrison and Wang (2018).[12]. Jakab and Kumhof (2015) examine the quantitative implications of the fact that bank liability circulates as money. Kumhof and Wang (2018) consider the implications of this fact in a New Keynesian model where inflation determinacy is obtained by using menu costs, Morrison and Wang (2018) its implications for banks' liquidity management. And Mendizábal (forthcoming) examines the implication of 100% reserve bank for bank lending. These studies have enriched our understanding of banks' money creation activity. Unlike the present paper, however, none of these studies is concerned with the non-neutrality of monetary policy.

2 Model

The time $t \in \mathbf{T} = \{0, 1, 2...\}$. There is a continuum $[0,1]$ of banks. Bank $i \in [0,1]$ is specialized to lend to sector i which consists of a continuum $[0,1]$ of entrepreneurs. There are more workers than can be hired by entrepreneurs. All economic agents are risk-neutral. Bankers live forever, with a discount factor of $\beta < 1$. Entrepreneurs and workers live for one period. Hence, new members of them enter and exit each period. Entrepreneurs enter with an endowment of ξ units of human capital, workers with one unit of labour. If an entrepreneur hires l workers, then he produces $Y = A\xi^{1-\alpha}l^{\alpha}$ units of the consumption good, corn, within the period, where $0 < \alpha < 1$. Corn is perishable.[13] We normalize $\xi = 1$. Workers not hired by entrepreneurs each produce w

[12]See also Donaldson et al (2018) and Wang (2019) in which banks issue real liability – i.e. promises to pay real goods – rather than nominal liability.

[13]Even if it is storable, it will not be saved over periods in the steady state, due to the discount factor $\beta < 1$.

units of corn in autarky within the period.

In terms of assets, the economy has H units of fiat money and B units of permanent government bond. Fiat money is to be used by banks to meet both depositor withdrawal and to settle interbank liabilities, hence representing both cash and bank reserves in reality. The government bond is to be used by banks as the collateral for borrowing fiat money (i.e. reserves), hence representing all the liquid assets of this use. The specific payment structure of the bond does not matter; we choose to model it as a Lucas tree. Each unit of the bond pays out d units of money as the dividend at the end of each period, financed with a lump sum tax on entrepreneurs. Let

$$\delta := \frac{B}{H}$$

denote the ratio of the bond to fiat money held by the private sectors. This bond-to-fiat money ratio characterizes the aggregate nominal asset portfolio:

$$\mathbf{A} := (H, B) = H \times (1, \delta) .$$

We will model a monetary policy as a change to $\mathbf{A}$. Entrepreneurs and workers are endowed with neither of the assets when they enter and will carry neither when they exit. Therefore, all assets are held by banks.

Workers and entrepreneurs own all the factors of production and form the real side of the economy. What makes banks matter for the resource allocation is the following friction of payment.

Assumption 1: Workers do not accept entrepreneurs' promise to pay as a means of wage payment, but they accept money, which is either fiat money or banks' promise to pay fiat money.

The assumption captures the real-life observation that banks' liabilities are widely accepted as a means of payment, whereas rarely so are non-bank firms or natural persons.[14] To hire workers, therefore, entrepreneurs need to borrow either fiat money or some banks' liabilities. Because no banks default on their liabilities in the model, these two forms of money are equivalent for entrepreneurs and workers: One unit of fiat

[14] Kiyotaki and Moore (2001) provides a foundation for this assumption, as is explained in detail by Wang (2019), who has made a similar assumption.

money is worth the same as one unit of bank liability, defined as a promise to pay one unit of fiat money. For the time being, we assume that banks keep fiat money and lend only their liabilities to entrepreneurs; later, we will show this is an optimal decision.[15]

We assume that entrepreneurs of sector $i \in [0, 1]$ can borrow only from bank i which is specialized to lend to the sector.[16] As a result, banks have monopolistic power over their borrower entrepreneurs. Later, we will introduce bank heterogeneity. Due to the monopolistic power, all banks will stay in businesses and their lending rates are a mark up of their lending costs, which, we will show, are affected by monetary policy.

Entrepreneurs use borrowed bank liabilities to hire workers and workers might deposit their wage incomes with other banks than their employers'. As a result, a fraction of money lent by one bank flows out into others. This outflow fraction, denoted by x, defines the bank's permanent type. In reality, myriads of transactions can cause one bank's deposits to circulate into another, as Bianchi and Bigio (2017) underline, and the outflow fraction is an attribute of banks. For example, it is bigger for banks sparsely branched than for those extensively branched because the more extensive are a bank's branches, the more likely is the money lent out by it circulating back to itself. In the model economy, the outflow fraction is the only source of bank heterogeneity. This heterogeneity is important: The effect of monetary policy can be opposite for banks of different outflow fractions, as we will show. In aggregation, the cumulative distribution function of x across banks over $[0, 1]$ is $F(\cdot)$. Given that the identities of banks are less important than their types, we describe a representative type x bank in what follows.

Suppose at period t, a type x bank lends out M_{xt} units of liability. Then xM_{xt} units of it are deposited with other banks. When one unit of one bank's liability is deposited with another bank, what the receiving bank does is as follows. It adds one unit of credit to the depositor's account on the liability side. On the asset side, given that it now holds the originating bank's promise to pay one unit of fiat money, the former holds a credit position of this value to the latter. To simplify the exposition, we assume that

[15] In reality, indeed, what banks lend to the real economy is typically their liabilities.

[16] A justification for this assumption is that banks need to screen borrowers before making lending, and the cost of screening is much lower if the borrower is from within the specialized sector than from without. Bank specialization has been well documented in empirical research: see among others Jonghe et. al. (2016), Liu and Pogach (2016), Ongena and Yu (2017), and Paravisini et. al. (2014).

the liability of any bank flows out to all the other banks evenly. As a result, to each bank, the quantity of other banks' liabilities deposited in is

$$\Upsilon_{It} = \int_0^1 x M_{xt} dF(x). \tag{1}$$

Let ρ_t denote the interest rate of interbank liabilities, q_t the nominal price of the bond after the dividend payment and R_{xt} the gross lending rate charged by the type x bank. Then after lending and depositing, the bank's balance sheet in nominal terms is as follows.

Assets	**Liabilities**
Loans to entrepreneurs $(M_{xt}R_{xt})$	To depositors of its own liability $((1-x)M_{xt})$
Credit to other banks whose liabilities are deposited $(\Upsilon_{It}\times(1+\rho_t))$	To depositors of other banks' liabilities (Υ_{It})
Fiat money (h_{xt})	To other banks that hold its liability $(xM_{xt}\times(1+\rho_t))$
The bond $((q_t+d)\,b_{xt})$	Equity

Table 1: The balance sheet of the type x bank after lending and depositing

It follows that the quantity of the type x bank's deposits at period t is

$$D_{xt} = \Upsilon_{It} + (1-x) M_{xt}. \tag{2}$$

and its net interbank credit position is

$$\Upsilon_{xt} = \Upsilon_{It} - x M_{xt}. \tag{3}$$

Observe that for any t,

$$\int_0^1 D_{xt} dF(x) = \int_0^1 M_{xt} dF(x) \tag{4}$$

$$\int_0^1 \Upsilon_{xt} dF(x) = 0. \tag{5}$$

Namely, the aggregate of deposits equals the aggregate of loans, because all the money deposited into the banking system comes from banks' lending in the first place; and the aggregate net interbank position is zero because one bank's credit position is the counterparty's liability.

In reality, the major form of banks' liabilities that are widely accepted as a means of payment is demand deposits. We assume that in the model economy, the bank liability that workers accept as a means of wage payment takes this form, bearing the right to withdraw on demand. This exposes banks to liquidity risk. To model this liquidity risk, we assume at the middle stage of each period t, $\widetilde{\omega}$ fraction of a bank's depositors demand to withdraw their claims. Ex ante at the beginning of the period, $\widetilde{\omega} = \omega$ with probability $\mu > 0$ and $\widetilde{\omega} = 0$ with probability $1 - \mu > 0$, and its realisation is independent across banks and over time. To obtain fiat money to meet the withdrawal demand, a bank can demand settlement of interbank liabilities owed to it. Therefore, the liquidity demand can be passed on along the interbank liability links. If a bank fails to meet all the withdrawal and settlement demand, it is in a liquidity crisis. We assume this crisis is very costly so that it is avoided in equilibrium. At period t, the withdrawal demand of the type x bank is $\widetilde{\omega} D_{xt}$, the bank has h_{xt} units of fiat money, and its net interbank credit position is Υ_{xt}. Contingent on the realization of $\widetilde{\omega}$, the bank's *net reserve position* Λ_t is hence

$$\Lambda_t \left(\widetilde{\omega}, x\right) = h_{xt} + \Upsilon_{xt} - \widetilde{\omega} D_{xt}. \tag{6}$$

With equations (2) and (3),

$$\Lambda_t \left(\widetilde{\omega}, x\right) = \left(h_{xt} + \left(1 - \widetilde{\omega}\right) \Upsilon_{It}\right) - \left[\widetilde{\omega} \left(1 - x\right) + x\right] M_{xt}. \tag{7}$$

Here term $\widetilde{\omega} \left(1 - x\right) + x$ represents the quantity of liquidity that the type x bank needs to service the lending of one unit of liability. First, x fraction of the unit of liability flows to other banks and becomes an interbank liability, to settle which the bank needs x unit of fiat money. Second, of the rest $1 - x$ unit, $\widetilde{\omega}$ fraction is in demand to be withdrawn. To meet this demand the bank needs $\widetilde{\omega} \left(1 - x\right)$ unit of fiat money. Altogether, the marginal liquidity burden to service the bank's lending is $\widetilde{\omega} \left(1 - x\right) + x$. Let $\omega_e := \mu\omega$ denote the average probability of withdrawal. Then

$$\begin{aligned} \overline{\tau}_x &: = \omega \left(1 - x\right) + x \\ \tau_x^e &: = \omega_e \left(1 - x\right) + x \end{aligned}$$

are respectively the maximum and average marginal liquidity burden of the type x bank.

We previously assumed that banks lend out their liabilities rather than fiat money. Now it is time to explain that is the optimal decision of banks. If a bank lends out fiat money (instead of its liability), then one unit of fiat money services only one unit of loan. By contrast, if the bank lends out its liability instead of fiat money, then one unit of fiat money services $1/\overline{\tau}_x > 1$ units of loans. Hence, it is in banks' interest to lend out their liabilities rather than fiat money, as we observe in practice.

If the net reserve position $\Lambda_t(\widetilde{\omega}, x) < 0$, the bank needs the capacity to borrow $-\Lambda_t$ units of fiat money to cover the liquidity shortfall. We assume that borrowing of reserves must be collateralized with the government bond. The interest rate of borrowing fiat money is equal to that of interbank liabilities, ρ_t, because the former can be used to settle the latter. Given the bond position b_{xt} of the bank, its borrowing capacity – the maximum quantity of fiat money that it can borrow – is $(q_t + d)\, b_{xt}/(1 + \rho_t)$. Therefore, the following liquidity constraint is to be honoured:

$$-\Lambda_t(\widetilde{\omega}, x) \leq \frac{q_t + d}{1 + \rho_t} b_{xt}.$$

This constraint is tighter if the bank encounters the liquidity shock and $\widetilde{\omega} = \omega$. With (7), the type x bank is subject to the following liquidity constraint:

$$\overline{\tau}_x M_{xt} \leq \left[h_{xt} + \frac{q_t + d}{1 + \rho_t} b_{xt}\right] + (1 - \omega)\, \Upsilon_{It}. \tag{8}$$

On its left hand side is the quantity of liquidity that the bank needs in order to service its lending M_{xt} when it experiences the liquidity shock. On the right hand side, first,

$$h_{xt} + \frac{q_t + d}{1 + \rho_t} b_{xt} := V_t(x) \tag{9}$$

is the quantity of liquidity that the bank can obtain with its asset holding (h_{xt}, b_{xt}), or the bank's *liquidity position* $V_t(x)$. Second, $(1 - \omega)\, \Upsilon_{It}$ is the part of the interbank credit that the bank can use to meet its liquidity need because out of Υ_{It} units of other banks' liabilities flowing into this bank, ω fraction is withdrawn. Altogether, the liquidity constraint (8) says that banks meet their liquidity demand either with their liquid assets or their interbank credit positions.

Towards the end of each period t, entrepreneurs produce corn and sell it for money – bank liability or fiat money. Among the buyers of corn first are the workers employed by entrepreneurs, who hold either bank liability as their wage payment, or fiat money if they have withdrawn their deposits. They spend all their money buying corn because they will exit in this period. Second, banks also issue new liability and use it to buy corn from entrepreneurs. Indeed, that must occur in equilibrium. Otherwise, there will not be enough money for entrepreneurs to repay bank loans: The aggregate money holding by depositors $\int_0^1 D_{xt} dF(x) = \int_0^1 M_{xt} dF(x) < \int_0^1 R_{xt} M_{xt} dF(x)$, where the inequality is because loans all command a gross return rate $R_{xt} > 1$; that is, the supply of money is smaller than the demand and the market is not clear, untrue in equilibrium. Let p_t be the market-clearing price of corn, that is, $1/p_t$ is the real unit value of money.

After selling corn, entrepreneurs use the money from the sales revenue to repay loans to the banks and pay the lump sum tax to the government, which then distributes the tax revenue as the dividend to bondholders. Observe that entrepreneurs might have exchanged corn with money that is the liabilities of other banks than their lenders. Hence, after the repayment of loans and the distribution of the bond dividend, some banks' liabilities might flow to others another round. These newly formed interbank liabilities, and those formed at the beginning of the period (due to depositing), altogether are then netted and settled with fiat money. Afterwards, the market opens where banks trade the bond with fiat money at price q_t, accordingly choosing their asset positions (h_{t+1}, b_{t+1}) for the next period. We assume that if a bank fails to settle its interbank liabilities, it will be out of business forever. As a result, no banks issue new liability for the purchase of corn in such a scale that they will default.[17]

Lastly, the economic agents consume the corn that they have obtained; for banks, this consumption means dividend to their shareholders. Then, workers and entrepreneurs of this period exit and the next period dawns on banks.

The timing of events at period t can be illustrated as follows.

[17] Monnet and Sanches (2015) provide an analysis of how the market discipline stops banks from over-issuance if the cost is endogenous.

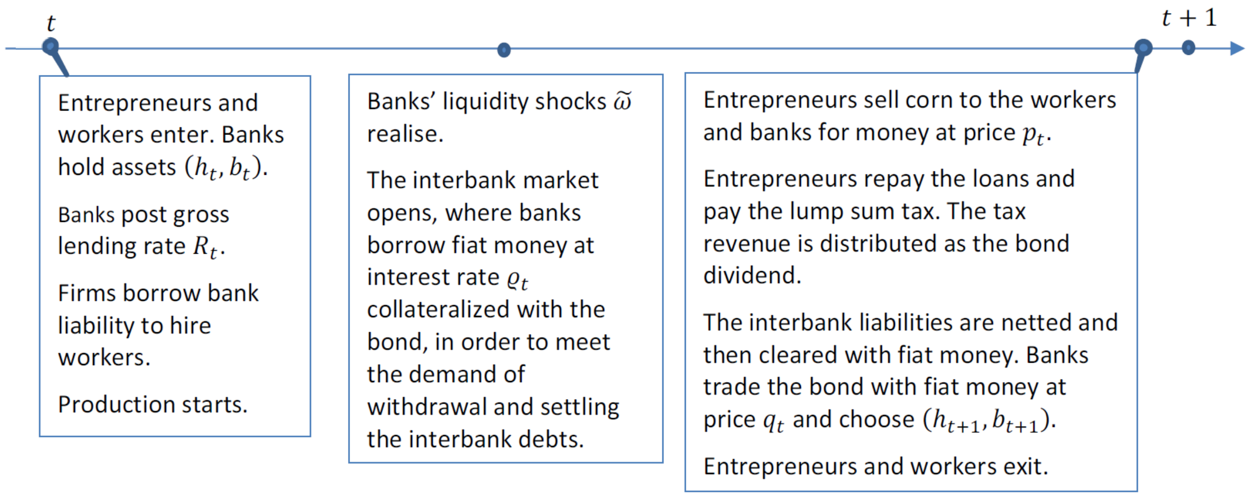

Figure 1: The timing of events in period t: At the beginning banks lend demandable nominal liability to entrepreneurs who use it to hire workers. At the middle stage, the liquidity shocks realise and the interbank reserve market opens. At the end of the period, the interbank liabilities are cleared and banks choose asset positions for the next period.

Two technical conditions are assumed. First,

$$\beta > \alpha > \frac{1}{2}.\text{[18]} \tag{10}$$

The assumption implies that

$$\frac{1-\beta}{\beta} < \frac{1-\alpha}{\alpha} < 1 < \frac{\alpha}{1-\alpha}. \tag{11}$$

Second, we assume

$$\frac{\omega - \omega_e}{1-\omega} < \frac{(\beta-\alpha)\,\beta}{1-\beta} \tag{12}$$

and

$$\frac{\omega}{1-\omega} < E(x)\,\beta^{\frac{1}{1-\alpha}}, \tag{13}$$

both of which command the size of the liquidity shock ω is not too big.

Passing on to the analysis of the market equilibrium, we examine the social planner's allocation as a benchmark, which concerns the number l of workers that each entrepreneur employs. Considering that the opportunity cost of a worker in this employment is

[18] This assumption is satisfied in the macroeconomics literature where the labour share α is around 2/3 and the discount factor β around 0.9.

w, the social planner's problem is

$$\max_{l} Al^{a} - wl.$$

Hence, in the first-best allocation, each entrepreneur hires l^{SB} workers, where

$$l^{SB} = \left(\frac{A\alpha}{w}\right)^{\frac{1}{1-\alpha}}. \tag{14}$$

3 The Equilibrium

We begin with entrepreneurs' problem, then banks' problem, and lastly the clearing of three markets: The interbank reserve market at the interim stage, and the corn market and the bond market towards the end of each period. Given that we are interested in the non-neutrality of monetary policy *in the steady state*, we consider the steady state only. The steady state of a variable z_t is denoted by z; e.g. the steady-state nominal price is p. Moreover, for a nominal variable z, its real counterparty is denoted by $\widetilde{z}$, that is, $\widetilde{z} := z/p$.

3.1 Entrepreneurs' decision

In each period, given that there are more workers than can be employed by entrepreneurs, workers earn an equilibrium real wage that is equal to their output in autarky, w. The nominal wage is hence wp. If an entrepreneur borrows M units of money, then it employs $l = M/(wp)$ workers. Given the gross lending rate R, the entrepreneur's decision problem is:

$$\max_{M} \frac{1}{p}\left[pA\left(\frac{M}{wp}\right)^{a} - MR - \text{ the lump-sum tax}\right].$$

The nominal demand function of money is hence

$$M(R) = p\left(\frac{A\alpha}{w^{\alpha}R}\right)^{\frac{1}{1-\alpha}}, \tag{15}$$

whereby the real demand function is

$$\widetilde{M}(R) = \left(\frac{A\alpha}{w^{\alpha}R}\right)^{\frac{1}{1-\alpha}}. \tag{16}$$

The number of workers employed by the entrepreneur is $\widetilde{M}/w$ and thus equal to:

$$l = l^{SB} \times R^{-\frac{1}{1-\alpha}}. \tag{17}$$

Therefore, the efficiency of the equilibrium allocation depends only on banks' lending rates R.

3.2 Banks' decision

In the steady state, a type x bank enters period t with portfolio (h_x, b_x). During the period, the bank earns three incomes. First, if it charges lending rate R, then it lends out $M(R)$ units of money and thereby earns a profit of $M(R)(R-1)$. Second, contingent on the realisation of $\widetilde{\omega}$, it has a reserve position $\Lambda(\widetilde{\omega}, x)$ given by (7), which earns it interest $\Lambda(\widetilde{\omega}, x)\rho$ on the interbank reserve market. Third, it earns dividend db_x from its bond position. The incomes are spent either paying dividend υ_t or adjusting its asset position. The bank's nominal budget constraint is thus,

$$\upsilon_t + [h_{t+1} + qb_{t+1} - (h_x + qb_x)] = M(R)(R-1) + \Lambda(\widetilde{\omega}, x)\rho + db_x. \tag{18}$$

Hence, the bank's problem is:

$$V(h_x, b_x) := \max_{R, \upsilon_t, h_{t+1} \geq 0, b_{t+1} \geq 0} E_{\widetilde{\omega}} \frac{\upsilon_t}{p} + \beta V(h_{t+1}, b_{t+1}), \tag{19}$$

subject to (18) and the liquidity constraint (8), or its real version:

$$\overline{\tau}_x \widetilde{M} \leq \left[\widetilde{h}_x + \frac{q+d}{1+\rho}\widetilde{b}_x\right] + (1-\omega)\widetilde{\Upsilon}_I. \tag{20}$$

In the steady state, type x chooses $(h_{t+1}, b_{t+1}) = (h_x, b_x)$. We first find the optimal lending rate R of the bank and then its optimal portfolio choice (h_x, b_x).

Let $\lambda(x)$ be the Lagrangian multiplier for the liquidity constraint (20). Then $\lambda(x)$ measures the marginal benefit of fiat money to the bank by relaxing its liquidity constraint and the marginal cost of an increase in its liquidity burden by tightening the constraint. The optimal lending rate is given by the following lemma.

Lemma 1 *The optimal lending rate of a type x bank is:*

$$R = \frac{1}{\alpha} \times \left(1 + \rho\tau_x^e + \lambda(x)\overline{\tau}_x\right) := R(x, \lambda(x), \rho). \tag{21}$$

Of this formula, the term $1/\alpha$ is the mark-up factor due to the monopolistic power that the bank has over its borrower entrepreneurs, and

$$C = 1 + \rho\tau_x^e + \lambda(x)\,\overline{\tau}_x \tag{22}$$

is the marginal cost of lending. To understand why so, first, what the bank lends out is its liability. Hence one unit of loans entails one unit of liability that the bank is obliged to repay, which costs 1. Second, each unit of loan issued by the type x bank needs τ_x^e unit of liquidity on average to service and the cost of liquidity is ρ per unit. Hence the liquidity cost per unit of loan is $\rho\tau_x^e$. Lastly, under the liquidity shock, one more unit of loan increases the liquidity burden by $\overline{\tau}_x$ unit and thereby incurs a cost of $\lambda(x)\,\overline{\tau}_x$. Altogether, equation (22) obtains.

Consider now the type x bank's portfolio choice (h_x, b_x) in the steady state. To begin with, for individual banks, the bond is as liquid as fiat money: The former can be *fully* converted into the latter via collateralized borrowing on the interbank reserves market. Therefore, these two assets should deliver an inter-temporal return at the same rate.

Proposition 1 *The net return rate of holding the bond over a period is equal to that of holding fiat money, that is,*

$$\frac{d}{q} = \rho. \tag{23}$$

As a result of this proposition, the bond is a perfect substitute for fiat money to individual banks. Hence, the composition of the type x bank's portfolio is indeterminate; what is determined is its market value, $h_x + qb_x$. This market value is equal to the quantity V_x of liquidity that can be obtained with this portfolio: By equation (23),

$$h_x + qb_x = h_x + \frac{q+d}{1+\rho}b_x = V(x).$$

This equation holds because the bond is as liquid as fiat money in the model economy.

Banks hold these assets for liquidity reasons. In the steady state, the marginal cost of holding a unit of liquid assets, such as fiat money, is $1/\beta - 1$ due to the time preference, while the marginal benefit is twofold. First, liquid assets earn a return at rate ρ in the next period, as said by Proposition 1. The other, it relaxes the liquidity constraint, of which the marginal benefit is $\lambda(x)$ to the type x. Together, therefore, the Kuhn-Tucker conditions regarding the asset holding are

$$\rho + \lambda(x) \leq \frac{1}{\beta} - 1; \tag{24}$$

and $V_x > 0$ – type x banks hold liquid assets – only if (24) holds with equality. Define

$$\overline{\lambda} := \max_{x \in [0,1]} \lambda(x).$$

It follows that

$$\overline{\lambda} = \frac{1-\beta}{\beta} - \rho; \tag{25}$$

and that $V(x) > 0$ only if $\lambda(x) = \overline{\lambda}$. That is, only banks that face the tightest liquidity constraint hold liquid assets and the total marginal benefit of doing so $\rho + \overline{\lambda}$ is equal to the marginal cost $(1-\beta)/\beta$. This simple observation actually bears a surprising implication.

Proposition 2 *The lending rates of liquidity-unconstrained banks increase with the interbank interest rate* ρ*, but those of the maximally constrained decrease with it.*

Proof. If a bank is liquidity unconstrained, then $\lambda = 0$. By (22) the bank's lending cost $C = 1 + \rho\tau_x^e$ increases with ρ. Hence, so does its lending rate $R = \frac{1}{\alpha}C$. However, if a bank faces the tightest liquidity constraint, then $\lambda = \overline{\lambda} = \frac{1-\beta}{\beta} - \rho$. Substitute $\lambda = \frac{1-\beta}{\beta} - \rho$ into (22), the bank's lending cost $C = 1 + \frac{1-\beta}{\beta}\overline{\tau}_x - \rho\left(\overline{\tau}_x - \tau_x^e\right)$. This cost strictly decreases with ρ – hence so does its lending rate – because $\overline{\tau}_x - \tau_x^e = (1-\mu)\,\omega\,(1-x) > 0$. ■

If the interbank rate ρ rises, banks' funding costs rise, and hence their lending rates should rise as well – so it is widely believed. This is indeed true for liquidity unconstrained banks. For banks maximally constrained, however, the opposite is true, by the proposition. Intuition is suggested by the proof. A rise in the cost ρ of liquidity indeed increases *any* type x bank's funding cost at a rate of τ_x^e because each unit of loan needs *average* τ_x^e unit of liquidity to service. Besides this effect, however, for a type x that is maximally liquidity constrained, a rise in ρ is associated with an additional, countervailing, effect. Because $\rho = (1-\beta)/\beta - \overline{\lambda}$, a rise in ρ must be companied with a fall in the tightness $\overline{\lambda}$ of this type's liquidity constraint. This fall reduces the lending cost at a rate of $\overline{\tau}_x$ by equation (22) because it is concerned with the liquidity burden in the contingency when the shock has happened. This rate is greater than the rising rate τ_x^e of the funding cost because the latter is due to the average liquidity burden. Hence, in net, the lending costs – and thus the lending rates as well – of the maximally liquidity-constrained banks actually decrease with the interbank rate ρ.

Proposition 2 highlights the importance of the tightness of a bank's liquidity constraint. This tightness λ depends on the bank's liquidity burden, the right-hand side of the liquidity constraint (20). Given the real demand function $\widetilde{M}(R)$ in (15) and the lending rate in (21), the type x bank's real liquidity burden is represented by the following function.

$$\widetilde{S}(x,\lambda,\rho) := \overline{\tau}_x \widetilde{M}(R(x,\lambda,\rho)) = \overline{\tau}_x \left(\frac{A\alpha^2}{w^\alpha(1+\rho\tau_x^e+\lambda\overline{\tau}_x)}\right)^{\frac{1}{1-\alpha}} \tag{26}$$

It has the following properties.

Lemma 2 *$\frac{\partial \widetilde{S}}{\partial x} > 0$, $\frac{\partial \widetilde{S}}{\partial \lambda} < 0$ and $\frac{\partial \widetilde{S}}{\partial \rho} < 0$.*

That $\partial\widetilde{S}/\partial x > 0$ is driven by the fact that a unit of flowing-out liability becomes an interbank liability and thus needs 1 unit of liquidity to service, whereas a unit of liability deposited back needs ω unit of liquidity to service under the liquidity shock; thus the outflow of liability increases the bank's liquidity burden. If the liquidity constraint is tighter – i.e. λ rises – or the interbank rate ρ is higher, then the marginal cost of lending is higher; consequently, the bank increases the lending rate R, which lowers its lending size M and thus reduces its liquidity burden S.

That $\partial\widetilde{S}/\partial x > 0$ suggests that the tightness $\lambda(x)$ of the liquidity constraint increases with the outflow fraction x. That is indeed true. The liquidity constraint (20) is equivalent to

$$\widetilde{S}(x,\lambda,\rho) \leq \widetilde{V}(x) + (1-\omega)\widetilde{\Upsilon}_I. \tag{27}$$

If we define

$$x^L\left(\rho,\widetilde{\Upsilon}_I\right) := \min\left\{x|\widetilde{S}(x,0,\rho) \geq (1-\omega)\widetilde{\Upsilon}_I\right\}, \tag{28}$$

then, for $x < x^L$, we have $\widetilde{S}(x,0,\rho) < (1-\omega)\widetilde{\Upsilon}_I$. That is, the liquidity constraint (27) is non-binding even if $\widetilde{V}(x) = 0$ and the bank holds no liquid assets. Hence $\lambda(x) = 0$ for

$x < x^L$. Second, a bank holds liquid assets, we have seen, only if $\lambda = \overline{\lambda} = (1-\beta)/\beta - \rho$. Hence, a type x bank needs liquid assets to meet the liquidity constraint (27) if and only if $\widetilde{S}\left(x, \overline{\lambda}, \rho\right) > (1-\omega)\widetilde{\Upsilon}_I$, which is equivalent to $x > x^H$, where

$$x^H\left(\rho, \widetilde{\Upsilon}_I\right) := \min\left\{x|\widetilde{S}\left(x, (1-\beta)/\beta - \rho, \rho\right) \geq (1-\omega)\widetilde{\Upsilon}_I\right\}. \tag{29}$$

That is, if and only if $x > x^H$, the bank holds liquid assets and $\lambda(x) = \overline{\lambda}$. If $\overline{\lambda} > 0$, the liquidity constraint (27) is binding, which determines the asset holding:

$$\widetilde{V}(x) = \widetilde{S}\left(x, (1-\beta)/\beta - \rho, \rho\right) - (1-\omega)\widetilde{\Upsilon}_I. \tag{30}$$

These two thresholds are ordered as follows.

Lemma 3 *$x^H \geq x^L > 0$ and the first inequality holds in the strict form if $\overline{\lambda} > 0$.*

In the case of $\overline{\lambda} > 0$, for a bank of types $x \in \left(x^L, x^H\right)$, on the one hand, $x < x^H$ and hence the bank holds no liquid assets, i.e. $\widetilde{V}(x) = 0$. On the other hand, $x > x^L$ and hence the liquidity constraint is not non-binding with $\widetilde{V}(x) = 0$. Together, $\widetilde{V}(x) = 0$ and the liquidity constraint (27) is binding, which leads to

$$\widetilde{S}(x, \lambda, \rho) = (1-\omega)\widetilde{\Upsilon}_I. \tag{31}$$

This equation determines $\lambda(x)$ and implies $\lambda'(x) > 0$ over $\left(x^L, x^H\right)$. Observe that $\overline{\lambda} = 0$ if and only if $\rho = (1-\beta)/\beta$. The discussion leads to the following proposition.

Proposition 3 *(a) If $\rho = \frac{1-\beta}{\beta}$ and hence the maximal tightness $\overline{\lambda} = 0$, then $\lambda(x) = 0$ for all x and $\widetilde{V}(x)$ is indeterminate so long as it satisfies the liquidity constraint (27):*

$$\widetilde{V}(x) \geq \max\left(0, \widetilde{S}\left(x, 0, \frac{1-\beta}{\beta}\right) - (1-\omega)\widetilde{\Upsilon}_I\right).$$

(b) If $\rho < \frac{1-\beta}{\beta}$ and hence $\overline{\lambda} = \frac{1-\beta}{\beta} - \rho > 0$, then $\lambda(x) = 0$ for $x \leq x^L$; $\lambda(x)$ is determined by (31) and $\lambda'(x) > 0$ for $x \in \left(x^L, x^H\right)$; and lastly $\lambda = \overline{\lambda}$ for $x \geq x^H$. Moreover,

$$\widetilde{V}(x) = \begin{Bmatrix} 0 \text{ if } x \leq x^H \\ \widetilde{S}\left(x, (1-\beta)/\beta - \rho, \rho\right) - (1-\omega)\widetilde{\Upsilon}_I \text{ if } x > x^H \end{Bmatrix}. \tag{32}$$

According to this proposition, while $V(x)$ is indeterminate if $\rho = (1-\beta)/\beta$, the tightness $\lambda(x)$ of the liquidity constraint is a function of $(\rho, \widetilde{\Upsilon}_I)$, and can therefore be written as $\lambda\left(x, \rho, \widetilde{\Upsilon}_I\right)$. Then the real lending scale of a type x bank is $\widetilde{M}\left(R\left(x, \lambda\left(x, \rho, \widetilde{\Upsilon}_I\right), \rho\right)\right)$. By equation (1), the aggregate interbank liability inflow $\widetilde{\Upsilon}_I$ to each bank satisfies:

$$\widetilde{\Upsilon}_I = \int_0^1 x\widetilde{M}\left(R\left(x, \lambda\left(x, \rho, \widetilde{\Upsilon}_I\right), \rho\right)\right) dF(x) := T\left(\widetilde{\Upsilon}_I, \rho\right) \tag{33}$$

That is, $\widetilde{\Upsilon}_I$ is a fixed point of $T(\cdot, \rho)$. The function T denotes the aggregate of interbank liability outflow if banks have $\widetilde{\Upsilon}_I$ units of the interbank credit to lean on for their liquidity needs, given the interbank rate ρ. In equilibrium, the aggregate interbank outflow equals the aggregate inflow and $T = \widetilde{\Upsilon}_I$. If $\widetilde{\Upsilon}_I$ is very small, that is, if $\widetilde{\Upsilon}_I \leq S(0, \overline{\lambda}, \rho)$, even type $x = 0$ need hold liquid assets to meet the liquidity demand, and thus $\lambda = \overline{\lambda} = (1-\beta)/\beta - \rho$ for all x and $T\left(\widetilde{\Upsilon}_I, \rho\right) = \mathbf{E}_x\{x\widetilde{M}(R(x, \overline{\lambda}, \rho))\}$. If $\widetilde{\Upsilon}_I$ is very large, that is if $\widetilde{\Upsilon}_I \geq S(1, 0, \rho)$, even type $x = 1$ can meet its liquidity demand wholly with the interbank credit and need hold no liquid assets, and thus $\lambda = 0$ for all x and $T(\widetilde{\Upsilon}_I, \rho) = \mathbf{E}_x\{x\widetilde{M}(R(x, 0, \rho))\}$. In between, $\lambda'_{\widetilde{\Upsilon}_I} \leq 0$ because the more abundant the interbank credit, the looser the liquidity constraint. As a result, $T'_{\widetilde{\Upsilon}_I} \geq 0$. Therefore, for any $\rho \in [0, (1-\beta)/\beta]$, the fixed point problem of (33) has a solution, denoted by $\widetilde{\Upsilon}_I(\rho)$, as illustrated in Figure 2 below.

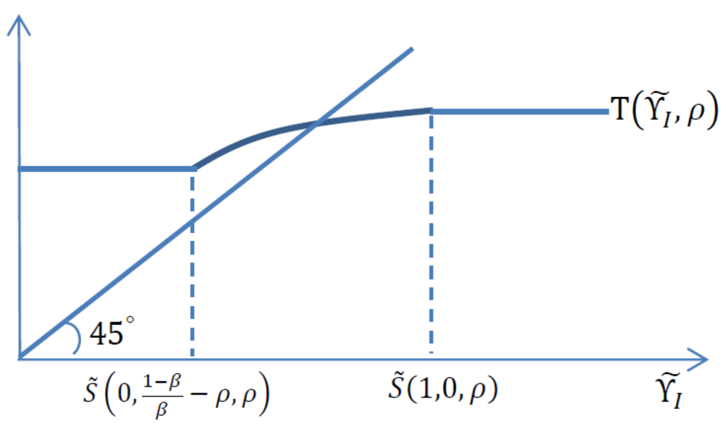

Figure 2: The equilibrium interbank credit $\widetilde{\Upsilon}_I$ is the fixed point of function $T(\cdot, \rho)$.

In particular, at $\rho = (1-\beta)/\beta$, $\lambda = 0$ for all $\left(x, \widetilde{\Upsilon}_I\right)$ and $T(\cdot, \rho)$ is a constant. Hence,

$$\widetilde{\Upsilon}_I\left(\frac{1-\beta}{\beta}\right) = \mathbf{E}_x\left\{x\widetilde{M}\left(R\left(x, 0, \frac{1-\beta}{\beta}\right)\right)\right\}.$$

Having examined banks' decision and found $\widetilde{\Upsilon}_I = \widetilde{\Upsilon}_I(\rho)$, we move on to consider the clearing of markets and define the steady-state equilibrium.

3.3 Market clearing and equilibrium definition

In each period three markets open in the model economy: The interbank reserve market at the interim stage, and the corn and the liquid-asset markets at the end of the period. To pin down the equilibrium, we choose to consider the clearing of the first and third of these three markets, beginning with the latter. As was said, the composition of the liquid assets held by individual banks is indeterminate and what is determined is their value. The real value of the liquid assets demanded by a type x bank is $\widetilde{V}(x)$, given by Proposition 3. The aggregate demand is hence $\int_0^1 \widetilde{V}(x)dF(x)$. The aggregate real supply – the real value of the aggregate liquid-asset portfolio is $\widetilde{H}+q\widetilde{B} = \widetilde{H}(1+d/\rho\times\delta)$ because $q = d/\rho$ (by equation 23) and $B = \delta H$. The clearing of the liquid-asset market commands

$$\int_0^1 \widetilde{V}(x)dF(x) = \widetilde{H}\left(1+\frac{d}{\rho}\delta\right). \tag{34}$$

Regarding the interbank reserve market, one bank's interbank interest income must be another's expense. That is, $\mathbf{E}_{\widetilde{\omega},x}\{\Lambda(\omega,x)\rho\} = 0$. By (23), $\rho = d/q > 0$. Hence, $\mathbf{E}_{\widetilde{\omega},x}\{\Lambda(\widetilde{\omega},x)\} = 0$.

Lemma 4 $\mathbf{E}_{\widetilde{\omega},x}\{\Lambda(\omega,x)\rho\} = 0$ *is equivalent to*

$$\omega_e \int_0^1 \widetilde{M}\left(R\left(x,\lambda\left(x,\rho,\widetilde{\Upsilon}_I(\rho)\right),\rho\right)\right)dF(x) = \widetilde{H}. \tag{35}$$

As $\widetilde{H} = H/p$, Equation (35) pins down the price level p in the steady state. In the model economy, the means of payment is served by banks' liabilities. The economic function of fiat money is to meet banks' liquidity demand, which is equal to the aggregate of depositor withdrawals because the interbank claims are netted out in aggregation. Hence Equation (35).

Use equation (35) to cancel $\widetilde{H}$ in equation (34) and we arrive at a single equilibrium condition:

$$\int_0^1 \widetilde{V}(x)dF(x) = \left(1+\frac{d}{\rho}\delta\right)\omega_e \int_0^1 \widetilde{M}\left(R\left(x,\lambda\left(x,\rho,\widetilde{\Upsilon}_I(\rho)\right),\rho\right)\right)dF(x). \tag{36}$$

We can define the steady-state equilibrium in real terms as follows.

Definition 1 *A steady-state equilibrium is a profile of* $\left\{\left\{\widetilde{V}(x)\right\}_{x\in[0,1]}, \rho\right\}$ *that satisfies the following conditions.*

(I): Given the interbank interest rate ρ *and thus the interbank credit* $\widetilde{\Upsilon}_I = \widetilde{\Upsilon}_I(\rho)$, $\widetilde{V}(x)$ *is the optimal asset holding by type* x *banks, as given by Proposition 3.*

(II) Given banks' asset holdings $\left\{\widetilde{V}(x)\right\}_{x\in[0,1]}$, *the interbank interest rate* ρ *satisfies condition (36).*

It follows from Equation (35) that

$$p^{-1} = \omega_e \times \frac{1}{H}\int_0^1 \widetilde{M}\left(R\left(x, \lambda\left(x, \rho, \widetilde{\Upsilon}_I(\rho)\right), \rho\right)\right) dF(x). \tag{37}$$

Obviously, the aggregate real lending $\int_0^1 \widetilde{M}\left(R\left(x, \lambda\left(x, \rho, \widetilde{\Upsilon}_I(\rho)\right), \rho\right)\right) dF(x)$ can never go to infinity.[19] The following proposition regarding the real unit value of money p^{-1} is hence self-evident.

Proposition 4 $\lim_{\omega_e\to 0} p^{-1} = 0$ *and* $\partial p^{-1}/\partial\omega_e > 0$ *at* $\omega_e \gtrsim 0$. *That is, fiat money is worthless in the steady state if there is no random withdrawal; and the steady-state price level* p *rises if the average withdrawal probability* ω_e *is very small and falling.*

The proposition is intuitive. In this economy, the ultimate use of fiat money is to meet depositors' demand for withdrawals. As a result, the higher the withdrawal demand, the greater the economic value of fiat money. In the extreme case where depositors never randomly withdraw (i.e. $\omega_e = 0$), fiat money is useless and thus worthless. In this case, fiat money will not circulate, nor will the nominal liabilities of banks, and they will have to issue real liabilities, that is, a promise to pay real goods, probably bullion. Due to the decades of advancement of digital ways of payment, nowadays the probability of random withdrawals – namely withdrawals for reasons other than concerns about the banks' default risks – is indeed very small and getting even smaller. This paper hence predicts that the continuing advancement of digital ways of payment, by reducing the withdrawal probability, will raise the nominal price level;

[19] Indeed by (21), the lending rates $R > 1/\alpha$, and hence the real lending scales $\widetilde{M} < \left(\frac{A\alpha^2}{w^\alpha}\right)^{\frac{1}{1-\alpha}}$ by (16).

and if it completely eliminates random withdrawals, fiat money will stop circulation and a gold or silver standard will probably return.

We move on to examine the other properties of the steady state in the next section, focusing on the real effects of monetary policy in the steady state. We define the central bank of the economy a unique entity that can produce or retire fiat money costlessly and manages a large stock of the government bond. Then monetary policy is modelled as a change $\Delta := (\Delta_H, \Delta_B)$ to the aggregate liquid-asset portfolio $\mathbf{A} = (H, B)$ held by the private sectors at a certain period τ. Below are three policies that this modelling covers.

Policy 1 (Change in Fiat Money Holding): The central bank changes the fiat-money holding of a type x bank by N_x units, where $N_x > 0$ represents transfer and $N_x < 0$ tax, at the beginning of period τ. Then the aggregate change in the quantity of fiat money is $N := \sum_{x=0,\kappa} f_x N_x$ and the policy is $\Delta = (N, 0)$.

Policy 2 (Open Market Operations): At the beginning of period τ, the central bank announces that it will trade O units of the government bond at the market price q_τ on the liquid-asset market of the period, where $O > 0$ represents the central bank buying and $O < 0$ selling. Hence, $\Delta = (q_\tau O, -O)$. Observe that the bond price q_τ will be affected by the operation.

Policy 3 (Repo or Reverse Repo): At the beginning of period τ, the central bank announces that it will lend (or borrow) fiat money at a policy rate ρ^p to any banks who want to be the counterparty, using the government bond as the collateral, on the interbank reserve market of the period. Suppose that as a result of the policy, the central bank lends (or borrows) N units of fiat money. At the end of the period the counterparty banks pay to (or receive from) the central bank interest of $N\rho^p$ units of fiat money in total. Hence, $\Delta = (-N\rho^p, 0)$ if we let $N > 0$ represents the central bank lending (i.e. repo) and $N < 0$ borrowing (i.e. reverse repo).

We will show that monetary policy can produce real effects in the steady state, in which case, as suggested by Proposition <ref>Prop_Opp</ref>, the effects can be opposite to different types of banks.

4 Non-Neutrality of Monetary Policy in the Steady State

As shown in Proposition 3, the steady state can be in two regimes. One is the Never-Binding Regime, in which $\rho = (1-\beta)/\beta$ so that $\overline{\lambda} = 0$ and the liquidity constraint is non-binding for any banks. The other is the Binding Regime, in which $\rho < (1-\beta)/\beta$ so that $\overline{\lambda} > 0$ and the liquidity constraint is binding for banks of type $x > x^L$. We begin with the Never-Binding Regime, which is easier to characterize.

In the Never-Binding Regime $\overline{\lambda} = 0$ for all x and $\rho = (1-\beta)/\beta$. Then the lending rate of any type x is $R(x, 0, (1-\beta)/\beta)$, independent of both the quantity of fiat money H and that of the bond B. A marginal change to B affects nothing, while that to H moves only the price level p proportionally because by (37)

$$p \times \omega_e \int_0^1 \widetilde{M}(R(x, 0, (1-\beta)/\beta))dF(x) = H.$$

Hence the following proposition.

Proposition 5 *If the steady state is in the Never-Binding Regime, then*

(i) The Quantity Theory of Money holds: A marginal change in fiat money's quantity H *affects nothing but the the nominal price level* p *and* $d\ln p/d\ln H = 1$*.*

(ii) The government bond is neutral too: A marginal change in its quantity B *affects nothing.*

As a result of the proposition, in the Never-Binding Regime, monetary policy produces no real results, *unless it pushes the steady state out of the Never-Binding Regime into the Binding Regime.* A substantial change to (H, B)can cause this regime change. To see this, we investigate under which conditions is the steady state in the Never-Binding Regime. By Proposition 3, in the Never-Binding Regime, the real liquidity position $\widetilde{V}(x)$ of any type x satisfies:

$$\widetilde{V}(x) \geq \max\left(0, \widetilde{S}\left(x, 0, \frac{1-\beta}{\beta}\right) - (1-\omega)\,\widetilde{\Upsilon}_I\left(\frac{1-\beta}{\beta}\right)\right). \tag{38}$$

Namely, the steady state is in the Never-Binding Regime if each type x can maintain such a real liquidity position $\widetilde{V}(x)$ that its liquidity constraint is non-binding at interbank rate $\rho = (1-\beta)/\beta$. The aggregation of all the banks' positions, by equilibrium condition (34), is $\widetilde{H}(1+\delta d/\rho) = \widetilde{H}(1+\delta d\beta/(1-\beta))$. We can specify a profile $\left\{\widetilde{V}(x)\right\}_{x\in[0,1]}$ that satisfies inequality (38) for any x, and hence the steady state is in the Never-Binding Regime, if and only if the following condition holds:

$$\underbrace{\widetilde{H}\left(1+\frac{d\beta}{1-\beta}\delta\right)}_{\text{Supply of liquid assets}} \geq \underbrace{\int_0^1 \max\left(0, \widetilde{S}\left(x, 0, \frac{1-\beta}{\beta}\right) - (1-\omega)\,\widetilde{\Upsilon}_I\left(\frac{1-\beta}{\beta}\right)\right) dF(x)}_{\text{quantity needed for no banks to face a binding liquidity constraint}}. \tag{39}$$

Essentially, the inequality says the aggregate supply of liquid assets $\widetilde{H}(1+\delta d\beta/(1-\beta))$ is sufficient to meet the liquidity needs of all banks for them to face a nonbinding liquidity constraint at $\rho = (1-\beta)/\beta$. As the real value of fiat money $\widetilde{H}$ is a constant due to the Quantity Theory of Money, the aggregate supply of liquid asset is proportional to the bond-to-fiat money ratio δ. If δ is large enough, the liquid assets abound. The following proposition is hence intuitive

Proposition 6 *There exists* $\delta_c \in \left(0, \frac{(1-\beta)(1-\omega_e)}{d\beta\omega_e}\right)$ *such that the steady state is in the Never-Binding Regime if* $\delta \geq \delta_c$ *and it is in the Binding Regime if* $\delta < \delta_c$.

The bond-to-fiat money ratio δ characterizes the composition of the aggregate liquid asset portfolio $(H, B) = H(1, \delta)$. While we saw from Proposition 1 that for individual banks the composition of their portfolios is indeterminate, Proposition 6 shows that the composition δ of the aggregate portfolio is a determinant of the property of the steady state. Intuitively, for single banks, having more fiat money and less bond makes no difference as the two assets are a perfect substitute for one another. At the aggregate level, in contrast, having a higher quantity H of fiat money and a lower quantity B of the bond makes a substantial difference for the value of the aggregate liquid-asset portfolio in the steady state. The mechanism for that can be most cleanly explained in the Never-Binding Regime, where the return rate of liquid assets $\rho = (1-\beta)/\beta$ is independent of H and B. In the steady state of this regime, a rise in H has no effect on the real value of fiat money, but lowers the unit real value p^{-1} of money proportionally. As a result, the real value of dividend per unit of the bond, which is of a fixed nominal value d, decreases proportionally. So does the unit real value of the bond, given that the discount factor applied to value it, determined by its return rate ρ, is fixed. Therefore, a rise in H and fall in B decreases the real value of the aggregate liquid-asset portfolio in the steady state. Of course, in the Never-Binding Regime this decrease produces no real effect because there no banks face a binding liquidity constraint.

Things are completely different in the Binding Regime, which arises if$\delta < \delta_c$. By Proposition 3, in this regime, the real liquidity position of type x banks is given by Equation (32). This equation substituted into equilibrium condition (36), we arrive at the condition that pins down the steady state interbank rate ρ in the Binding Regime:

$$\int_{x^H\left(\rho,\widetilde{\Upsilon}_I(\rho)\right)}^{1} \widetilde{S}\left(x,(1-\beta)/\beta-\rho,\rho\right)-(1-\omega)\widetilde{\Upsilon}_I\left(\rho\right)dF(x)$$
$$= \left(1+\frac{d}{\rho}\delta\right)\omega_e\int_0^1 \widetilde{M}\left(R\left(x,\lambda\left(x,\rho,\widetilde{\Upsilon}_I\left(\rho\right)\right),\rho\right)\right)dF(x). \tag{40}$$

It is obvious from this condition that the aggregate liquid-asset portfolio $\mathbf{A}=(H,B)$ affects the real allocation – i.e. banks' lending rates – only via the impact of the bond-to-fiat money ratio $\delta = B/H$ on the interbank rate ρ. Hence the following proposition is self evident.

Proposition 7 *A monetary policy – i.e. a change* (Δ_H,Δ_B) *to the aggregate liquid-asset portfolio* $\mathbf{A}=(H,B)$ *– produces real effects in the steady state if and only if it alters the portfolio's composition, that is,*

$$\frac{B+\Delta_B}{H+\Delta_H}\neq\frac{B}{H}. \tag{41}$$

and if and only if it moves the steady-state interbank interest rate ρ.

Three implications immediately follow. First, any change in the quantity of fiat money H alone produces real effects in the steady state. In contrast, no real effect is produced if the aggregate liquid-asset portfolio changes from $\mathbf{A}$ to $z\mathbf{A}$ for any $z>0$. That is, *fiat money is not neutral in the Binding Regime, but the aggregate liquid-asset portfolio always is.* Second, given that all the three policies listed preceding Section 4 – change to banks' fiat money positions, open market operations, repo (or inverse repo) – satisfy Condition (41), they all produce real effects in the steady state in the Binding Regime. Third, together with Propositions 2 and 3, the follow corollary follows:

Corollary 1 *If a monetary policy* (Δ_H,Δ_B) *satisfies condition (41) and thus moves the interbank rate* ρ*, then it moves the lending rates of type* $x\leq x^L$ *banks in the opposite direction to its moving the rates of type* $x\geq x^H$ *banks.*

Proof. By Proposition 2, the lending rates of liquidity unconstrained banks are increasing with ρ, those of the maximally constrained decreasing. By Proposition 3, the former is banks of types $x \leq x^L$, the latter of types $x \geq x^H$. ■

To explain the mechanism in which a change in δ moves ρ, we rewrite Condition (40) as follows

$$\int_{x^H\left(\rho,\widetilde{\Upsilon}_I(\rho)\right)}^{1} \widetilde{S}\left(x,(1-\beta)/\beta-\rho,\rho\right)-(1-\omega)\widetilde{\Upsilon}_I\left(\rho\right)dF(x)=\left(1+\frac{d}{\rho_b}\delta\right)\omega_e\widetilde{H}\left(\rho\right), \quad (42)$$

where the real value of fiat money $\widetilde{H}\left(\rho\right)=\omega_e\mathbf{E}_x\left\{\widetilde{M}\left(R\left(x,\lambda\left(x,\rho,\widetilde{\Upsilon}_I\left(\rho\right)\right),\rho\right)\right)\right\}$ (by 35) and ρ_b is the discount rate applied to evaluate the bond, which equals its return rate ρ. The left-hand side of (42) represents the aggregate demand of liquid assets (which only banks of type $x \geq x^H$ hold), the right-hand side the aggregate supply, which now we denote by $\widetilde{\mathbf{V}}$. From this condition, we can see that via both direct and indirect effects does a change –say a rise – in δ moves the interbank interest rate ρ. The direct effect concerns the movement of ρ if $\left(\widetilde{H},\rho_b,\widetilde{\Upsilon}_I\right)$ is fixed. Given that

$$\partial\widetilde{S}\left(x,(1-\beta)/\beta-\rho,\rho\right)/\partial\rho=\overline{\tau}_x\cdot\underbrace{\widetilde{M}'(R)}_{<0}\cdot\underbrace{\partial R\left(x,(1-\beta)/\beta-\rho,\rho\right)/\partial\rho}_{<0}>0,$$

the direct effect of a rise in δ is that ρ moves up. Intuitively, this effect consists of three links. First, given $\left(\widetilde{H},\rho_b,\widetilde{\Upsilon}_I\right)$, a rise in δ increases the aggregate steady-state value $\widetilde{\mathbf{V}}$ of liquid assets. This link has been explained in the discussion for Proposition 6. Second, the increase in $\widetilde{\mathbf{V}}$, that is, the increased abundance of liquid assets, reduces the maximal tightness $\overline{\lambda}$ of the liquidity constraint. Third, as $\rho=(1-\beta)/\beta-\lambda$ in equilibrium, the interbank rate ρ rises. This is only a part of the full picture, however, because this rise in ρ produces three indirect effects by moving $\widetilde{H}\left(\rho\right)$, ρ_b and $\widetilde{\Upsilon}_I\left(\rho\right)$. The first two affects the supply $\widetilde{\mathbf{V}}$ of liquid assets, the last the demand. Given the discount rate $\rho_b=\rho$, the rise in ρ reduces the bond price $q=d/\rho_b$ for sure. However, the direction of the movement in $\widetilde{H}$ or $\widetilde{\Upsilon}_I$ is unclear. The reason lies in Propositions 2 and 3, by which the real lending scales $\widetilde{M}_x$ of types $x \leq x^L$ are decreasing with ρ, those of types $x \geq x^H$ increasing. The relationship of ρ with $\widetilde{H}\left(\rho\right)=\omega_e\mathbf{E}_x\left\{\widetilde{M}_x\right\}$ and that with the interbank credit $\widetilde{\Upsilon}_I=\mathbf{E}_x\left\{x\widetilde{M}_x\right\}$ are thus unclear. As a result, although we know $\partial\rho/\partial\delta\neq0$, the sign of it is unclear. Note, however, that the indirect effects are all driven by a change in ρ, which occurs only because of the direct effect in the first place. Hence, ultimately, it is by changing the aggregate value of liquid assets and thereby relaxing or tightening banks' liquidity constraint that a change in the bond-to-fiat money ratio δ moves the interbank rate ρ, although the direction of the movement is unclear so far.

Regarding this direction, however, the conventional argument has a clear prediction, based on a consideration of demand-supply of bank reserves. That is, the argument goes, an expansionary monetary policy, by increasing the supply H of bank reserves, will decrease the cost ρ of borrowing it, while a contractionary policy, which decreases the supply H, will increase ρ. Given B, variable H moves in the opposite direction to $\delta = B/H$. Therefore, the conventional argument predicts $\partial\rho/\partial\delta > 0$. In the model economy, while the sign of $\partial\rho/\partial\delta$ is unclear in the general case, as we explained above, it is clearly determined in a special case where there are only two types of banks. Specifically we assume that the distribution of the outflow fraction x is as follows:

$$x = \left\{ \begin{array}{c} 0 \text{ with probability } f_0 \\ \kappa > 0 \text{ with probability } f_\kappa \end{array} \right\},$$

and $f_0 + f_\kappa = 1$. Because $x^L > 0$ by Lemma 3, type $x = 0$ is liquidity unconstrained. Then type κ must be maximally constrained in the Binding Regime. The steady state is in the Binding Regime, by Proposition 6, if and only if $\delta < \delta_c$.

Proposition 8 *In the two-type case, for any bond-to-fiat money ratio* $\delta \in (0, \delta_c]$, *there is a unique steady-state reserve borrowing rate* $\rho \in (0, (1-\beta)/\beta]$. *The function* $\rho(\delta)$ *satisfies:* $\partial\rho/\partial\delta > 0$; $\rho(\delta_c) = (1-\beta)/\beta$; *and* $\lim_{\delta\to 0}\rho(\delta) = 0$.

In this two-type case, therefore, the model makes the same prediction on the sign of $\partial\rho/\partial\delta$ as the conventional argument. That is, in the Binding Regime, an expansionary policy reduces the interbank interest rate ρ, a contractionary policy raising it. However, there are differences in two dimensions.

First, the mechanism is different. The conventional argument focuses on fiat money only, in particular its supply, whereas in this paper what matters is the composition δ of the aggregate liquid-asset portfolio; if and only if a monetary policy changes this composition, it produces real effects. Moreover, in the conventional argument banks' liquidity constraint plays no role, whereas in this paper the real effect is produced by loosening or tightening this constraint.

Second, the prediction on the effects for the real allocation is different. According to Corollary 1, the effects of non-neutral policy are heterogeneous, even opposite, for banks with different outflow fractions. In contrast, no such heterogeneity is to be found in the conventional argument. The argument is concerned with the funding costs only. Hence it predicts that banks' lending rates should all decrease with an expansionary policy and all increase with a contractionary one. Moreover, in this paper, the real effects that a non-neutral monetary policy produces occur in the steady state, whereas by the conventional argument, they are only transitional.

5 Conclusion

This paper considers the non-neutrality of monetary policy incorporating three fundamental facts about money in modern economies. One, the major form of media of exchange for real goods and services is bank liability rather than fiat money. Two, a major use of fiat money is to meet banks' liquidity needs. Three, banks extensively use government bonds as the collateral for liquidity borrowing. We underline the importance for bank liquidity of the outflow fraction, the fraction in which the money lent out by a bank circulates into other banks: The greater the outflow fraction, the heavier the liquidity burden and the tighter the liquidity constraint.

In this paper, monetary policy is modelled as a change to the aggregate liquid-asset portfolio, which consists of fiat money and a permanent government bond. The bond-to-fiat money ratio determines the property of the steady state. If this ratio is above a threshold, then in the steady state no banks are liquidity constrained and monetary policy is neutral so long as it keeps the steady state within this Never-Binding Regime. If the ratio is below the threshold, the liquidity constraint is binding for banks with an outflow fraction big enough. When the steady state is in this Binding Regime, any monetary policy that changes the bond-to-fiat money ratio moves the interbank interest rate, thereby moving banks' lending rates. Ultimately, the mechanism is that such a change alters the abundance – that is, the real value – of aggregate liquid assets, and hence tightens or relaxes banks' liquidity constraint. Moreover, a non-neutral monetary policy moves the lending rates of liquidity-unconstrained banks and those of the maximal constrained in opposite directions. Lastly, we find that if the advancement of digital ways of payment eliminates random withdrawals completely, then fiat money will stop circulation and a bullion standard will probably return.

References

Aiyagari, S. Rao, and Ellen McGrattan (1998), "The Optimum Quantity of Debt," Journal of Monetary Economics, 42(3): 447–469.

Andolfatto, D., A. Berentsen, and F. M. Martin, "Money, Banking and Financial Markets," forthcoming, Review of Economic Studies.

Angeletos, G, F. Collard and H. Dellas (2016), "Public Debt as Private Liquidity: Optimal Policy", NBER Working Paper No. 22794.

Angeletos, G. and C. Lian (2016), "Incomplete Information in Macroeconomics: Accommodating Frictions in Coordination," mimeo.

Bianchi, J., and S. Bigio (2017): "Banks, liquidity management and monetary policy," Federal Reserve Bank of Minneapolis Staff Report 503.

Cavalcanti, R., A. Erosa, and T. Temzelides. 1999. "Private Money and Reserve Management in a Random-Matching Model." Journal of Political Economy, 107, 929–45.

Chen, S. (2018), "The Credit Channel Transmission Mechanism and the Non-linear Growth and Welfare Effects of Inflation and Taxes," Economic Inquiry, 56, 724-744.

De Fiore, F., M. Hoerova, and H. Uhlig (2018). "Money Markets, Collateral and Monetary Policy". University of Chicago, Becker Friedman Institute for Economics Working Paper No. 2018-79.

Diamond, D. and P. Dybvig (1983). "Bank Runs, Deposit Insurance, and Liquidity," Journal of Political Economy, 91(3), 401-419.

Donaldson, J. D., G. Piacentino, and A. Thakor (2018), "Warehouse Banking", Journal of Financial Economics, 129, 250–67.

Ennis, H. (2018), "A simple general equilibrium model of large excess reserves", Journal of Monetary Economics, 98, 50–65

Faure, S. and H. Gersbach (2016), Money Creation and Destruction, CFS Working Paper, No. 555.

Freixas, X., Parigi, B. M., and Rochet, J.-C. (2000), "Systemic risk, interbank relations, and liquidity provision by the Central Bank", Journal of Money, Credit and Banking, 32, 611–638.

Gennaiolia, N., A. Martin and S. Rossi (2018), "Banks, Government Bonds, and Default: What do the Data Say?", Journal of Monetary Economics, 98, 98-113.

Goodfriend, M., and B. McCallum. (2007), "Banking and Interest Rates in Monetary Policy Analysis: A Quantitative Exploration." Journal of Monetary Economics, 54 (5): 1480–1507.

Grobéty, M. (2018), "Government debt and growth: The role of liquidity," Journal of International Money and Finance, 83, 1–22.

Heider, F. and M. Hoerova (2009). "Interbank lending, credit risk premia and collateral". ECB working paper NO 1107.

Holmström, Bengt, and Jean Tirole (1998), "Private and Public Supply of Liquidity," Journal of Political Economy, 106(1): 1–40.

Jakab, Z. and M. Kumhof (2015), "Banks are not intermediaries of loanable funds - and why this matters", Bank of England Working Paper No. 529.

Jonghe, O. D., H. Dewachterz, K. Mulier, S. Ongena and G. Schepens (2016), "Some borrowers are more equal than others: Bank funding shocks and credit reallocation," working paper.

Keynes, J. M. (1914), A Treatise on Money, Macmillan and Co. Limited, London, UK.

Kiyotaki, N., and J. Moore. 2001. "Evil is the Root of All Money." Clarendon Lecture No. 1, Oxford University.

Kiyotaki, N. and R. Wright (1989), "On Money as a Medium of Exchange", Journal of Political Economy, 97, 927-954.

Kumhof, Michaeal and Xuan Wang (2018),"Banks, Money and the Zero Lower Bound", Saïd Business School WP 2018-16.

Lahcen, M. (2019), "Informality and the Long Run Phillips Curve".

Lagos, R. , Rocheteau, G. , Wright, R. , 2017. Liquidity: a new monetarist perspective. Journal of Economic Literature, 55, 371–440 .

Lagos, R. and R. Wright (2005). "A unified framework for monetary theory and policy analysis", Journal of political economy, 113, 463-484.

Liu, E. X. and J. Pogach (2016), "Global Banks and Syndicated Loan Spreads: Evidence from U.S. Banks," working paper.

Lucas, R. E., Jr. (1972). "Expectations and the Neutrality of Money". Journal of Economic Theory. 4 (2): 103–124.

Mao, R. (2019), "Optimal Monetary Policy with Rationally Inattentive entrepreneurs." Mimeo.

Mendizábal, H. R., "Narrow Banking with Modern Depository Institutions: Is there a Reason to Panic?" forthcoming, International Journal of Central Banking.

Mortensen, D. and C. Pissarides (1994), "Job creation and job destruction in the theory of unemployment", Review of Economic Studies, 61(3), 397-415.

Mishkin, F. S. (2016). Economics of Money, Banking, and Financial Markets. 11th edition. Pearson Education.

Monnet, C., and D. Sanches (2015). "Private Money and Banking Regulation." Journal of Money, Credit and Banking, 47 (6): 1031–62.

Morrison, A. and T. Wang (2019), Bank liquidity crisis and the demandability of negotiable depositsj.

Ongena, S. and Y. Yu (2017), "entrepreneur Industry Affiliation and Multiple Bank Relationships", Journal of Financial Services Research, 51, 1, pp 1–17.

Paravisini, D., V. Rappoport, and P. Schnabl (2014), "Comparative Advantage and Specialization in Bank Lending," working paper.

Parlour, C., U. Rajan and J. Walden (2017), "Making Money: Commercial Banks, Liquidity Transformation and the Payment System," https://papers.ssrn.com/sol3/papers.cfm?abstract_

Piazzesi, M. and Schneider, M. (2018). "Payments, Credit and Asset Prices". mimeo.

Rocheteau, G. and E. Nosal (2017). Money, Payments and Liquidity, Cambridge, MA: MIT Press.

Rocheteau, G., R. Wright and S. X. Xiao (2018), "Open market operations", Journal of Monetary Economics, 98, 114–128.

Rule, Garreth (2015), "Understanding the central bank balance sheet", https://www.bankofengland.c /media/boe/files/ccbs/resources/understanding-the-central-bank-balance-sheet.

Walsh, Carl E. (2010), Monetary Theory and Policy, 3rd ed. MIT press, Cambridge, MA.

Wallace, N. (1981). "A Modigliani-Miller Theorem for Open-Market Operations". American Economic Review, 71(3), 267-274.

Wang, T. (2019), "Banks'Wealth, Banks Creation of Money, and Central Banking", International Journal of Central Banking, 15(3), 89-137.

Williamson, S. (2012). "Liquidity, Monetary Policy, and the Financial Crisis: A New Monetarist Approach." American Economic Review, 102(6), 2570–2605.

Woodford, Michael (1990), "Public Debt as Private Liquidity." American Economic Review, 80(2): 382–388.

Appendix

Proof of Lemma 1:

The first order condition concerning the lending size M is as follows.

$$\left(M\left(R-1\right)\right)'_M - \left(\rho\tau_{e,x} + \lambda\overline{\tau}_x\right) = 0. \tag{43}$$

As $M = M\left(R\right)$, we have $\frac{dR}{dM} = \frac{1}{M'(R)}$. Then

$$\begin{aligned}\left(M\left(R-1\right)\right)'_M &= \left(M\left(R\right)\left(R-1\right)\right)'_R \times \frac{dR}{dM} \\ &= \alpha R - 1.\end{aligned}$$

It follows from (43) that

$$\alpha R - 1 - \rho\tau_{e,x} - \lambda\overline{\tau}_x = 0,$$

Which lead to the lemma. Q.E.D.

Proof of Proposition 1:

Regarding the holding (h_{t+1}, b_{t+1}) of liquid assets, the Kuhn-Tucker conditions and the Envelop Theorem together command:

$$\beta\left(1+\rho+\lambda\right) - 1 \leq 0;$$

$$h_{t+1}\left[\beta\left(1+\rho+\lambda\right) - 1\right] = 0;$$

$$h_{t+1} \geq 0.$$

$$\beta\left(q+d\right)\left(1+\frac{\lambda}{1+\rho}\right) - q \leq 0;$$

$$b_{t+1}\left(\beta\left(q+d\right)\left(1+\frac{\lambda}{1+\rho}\right) - q\right) = 0;$$

$$b_{t+1} \geq 0.$$

To prove the lemma, we first show that there exists one bank that finds it optimal to hold both fiat money and the bond. Otherwise, there exists a bank with $\lambda = \lambda_1$ finds it optimal to hold cash but not the bond, and another bank with $\lambda = \lambda_2$ finds it optimal to hold the bond, but not cash. The first order conditions for the λ_1-bank are

$$\begin{aligned}\beta\left(1+\rho+\lambda_1\right) - 1 &= 0; \\ \beta\left(q+d\right)\left(1+\frac{\lambda_1}{1+\rho}\right) &< q.\end{aligned}$$

It follows from the equality that $\beta\left(1+\frac{\lambda_1}{1+\rho}\right)=\frac{1}{1+\rho}$, substitute which into the inequality and we find

$$\frac{d}{q}<\rho. \tag{44}$$

The first order conditions for the λ_2 bank are

$$\begin{aligned}\beta\left(1+\rho+\lambda_2\right) &< p;\\ \beta\left(q+d\right)\left(1+\frac{\lambda_2}{1+\rho}\right)-q &= 0.\end{aligned}$$

It follows from the equality that $\beta\left(1+\rho+\lambda_2\right)=\left(1+\rho\right)\frac{q}{q+d}$, substitute which into the inequality and we find

$$\frac{d}{q}>\rho,$$

which contradicts with (44). Therefore, there exists a bank that finds it optimal to hold both fiat money and the bond. Then the first order conditions regarding (h_{t+1},b_{t+1}) for this bank are:

$$\begin{aligned}\beta\left(1+\rho+\lambda\right) &= 1\\ \beta\left(q+d\right)\left(1+\frac{\lambda}{1+\rho}\right) &= q,\end{aligned}$$

which together imply

$$\frac{d}{q}=\rho.$$

Q.E.D.

Proof of Lemma 2:

As $\widetilde{S}=\overline{\tau}_x\widetilde{M}\left(R\right)=\left[\omega\left(1-x\right)+x\right]\widetilde{M}\left(R\right)$, we have $\frac{\partial\widetilde{S}}{\partial x}=\left(1-\omega\right)\widetilde{M}\left(R\right)+\overline{\tau}\widetilde{M}'\left(R\right)\frac{\partial R}{\partial x}$. Because $\widetilde{M}'<0$, $\frac{\partial\widetilde{S}}{\partial x}>0$ is equivalent to $\left(1-\omega\right)\frac{\widetilde{M}}{\widetilde{M}'}+\overline{\tau}\frac{\partial R}{\partial x}<0\Leftrightarrow\left(1-\omega\right)\frac{1}{\left[\log\widetilde{M}(R)\right]'_R}+\overline{\tau}\frac{(\rho+\lambda)(1-\omega_e)-\lambda(\omega-\omega_e)}{\alpha}<0\Leftrightarrow-\left(1-\omega\right)\left(1-\alpha\right)R+\overline{\tau}\frac{(\rho+\lambda)(1-\omega_e)-\lambda(\omega-\omega_e)}{\alpha}<0$, which with (21), is equivalent to

$$\begin{aligned}\overline{\tau}\left[\rho\left(1-\omega_e\right)+\lambda\left(1-\omega\right)\right] &< \left(1-\omega\right)\left(1-\alpha\right)\left[1+\rho\tau_{e,x}+\lambda\overline{\tau}\right]\Leftrightarrow\\ \alpha\overline{\tau}\lambda\left(1-\omega\right)+\overline{\tau}\rho\left(1-\omega_e\right) &< \left(1-\omega\right)\left(1-\alpha\right)\left[1+\rho\tau_{e,x}\right]\Leftrightarrow\\ \alpha\overline{\tau}\lambda+\overline{\tau}\rho\frac{1-\omega_e}{1-\omega} &< \left(1-\alpha\right)\left[1+\rho\tau_{e,x}\right]\Leftrightarrow\\ \alpha\overline{\tau}\left(\lambda+\rho\right)+\left[\overline{\tau}\frac{1-\omega_e}{1-\omega}-\alpha\overline{\tau}-\left(1-\alpha\right)\tau_{e,x}\right]\rho &< 1-\alpha,\end{aligned}$$

which, as $\overline{\tau} > \tau_{e,x}$, follows from $\alpha\overline{\tau}(\lambda+\rho) + \left[\overline{\tau}\frac{1-\omega_e}{1-\omega} - \tau_{e,x}\right]\rho < 1-\alpha \Leftrightarrow \alpha\overline{\tau}(\lambda+\rho) + \frac{\omega-\omega_e}{1-\omega}\rho < 1-\alpha \Leftarrow (\lambda+\rho)\left(\alpha\overline{\tau} + \frac{\omega-\omega_e}{1-\omega}\right) < 1-\alpha$, which, as $\overline{\tau} < 1$ and $\rho+\lambda \leq \frac{1-\beta}{\beta}$, follows from $\frac{1-\beta}{\beta}\left(\alpha + \frac{\omega-\omega_e}{1-\omega}\right) < 1-\alpha \Leftrightarrow \frac{\omega-\omega_e}{1-\omega} < \frac{(\beta-\alpha)\beta}{1-\beta}$, which is assumed in (12). Hence, $\frac{\partial\widetilde{S}}{\partial x} > 0$.

$\frac{\partial\widetilde{S}}{\partial\lambda} = \overline{\tau}\widetilde{M}'(R)\frac{\partial R}{\partial\lambda} < 0$ because $\widetilde{M}'(R) < 0$ and $\frac{\partial R}{\partial\lambda} = \frac{\overline{\tau}_x}{\alpha} > 0$.

$\frac{\partial\widetilde{S}}{\partial\rho} = \overline{\tau}\widetilde{M}'(R)\frac{\partial R}{\partial\rho} < 0$ because $\widetilde{M}' < 0$ and $\frac{\partial R}{\partial\rho} > 0$. Q.E.D.

Proof of Lemma 3

Because $\frac{\partial\widetilde{S}}{\partial\lambda} < 0$, we have $\widetilde{S}(x,0,\rho) \geq \widetilde{S}\left(x,\overline{\lambda},\rho\right)$. Hence, $\left\{x|\widetilde{S}\left(x,\overline{\lambda},\rho\right) \geq (1-\omega)\widetilde{Y}_I\right\} \subset \left\{x|\widetilde{S}(x,0,\rho) \geq (1-\omega)\widetilde{Y}_I\right\}$. It follows that $x^H \geq x^L$. Moreover, if $\overline{\lambda} > 0$, the former set is a true subset of the latter and hence $x^H > x^L$.

We are left to prove that Assumption (13) implies $x^L > 0$. For this purpose, it suffice to show that $S(0,0,\rho) < (1-\omega)\Upsilon_I$. On the left hand side of this inequality $S(0,0,\rho) = \omega M(R)|_{R>\frac{1}{\alpha};M'<0} < \omega M\left(\frac{1}{\alpha}\right)$, while on the right hand side $\Upsilon_I = \int_0^1 xM(R(x))\,dF(x)$. Because $R = \frac{1+\rho\tau_{e,x}+\lambda\overline{\tau}}{\alpha}|_{\rho+\lambda\leq\frac{1-\beta}{\beta},\tau<1} < \frac{1}{\alpha\beta}$ and $M'(R) < 0$, we have $\Upsilon_I > \int_0^1 xM\left(\frac{1}{\alpha\beta}\right)dF(x) = M\left(\frac{1}{\alpha\beta}\right)E(x)$. Therefore, $S(0,0,\rho) < (1-\omega)\Upsilon_I \Leftarrow \omega M\left(\frac{1}{\alpha}\right) < (1-\omega)M\left(\frac{1}{\alpha\beta}\right)E(x) \Leftrightarrow \omega/(1-\omega) < E(x)\left(\frac{1}{\beta}\right)^{-\frac{1}{1-\alpha}} = E(x)\beta^{\frac{1}{1-\alpha}}$, namely, condition assumed in (13). Q.E.D.

Proof of Proposition 3

The Kuhn-Tucker conditions regarding the liquidity position $V(x)$ of a type x bank and the tightness $\lambda(x)$ of its liquidity constraint are:

$$V(x)\left(\overline{\lambda}-\lambda(x)\right) = 0 \tag{45}$$

$$\lambda(x)\left(V(x)+(1-\omega)\Upsilon_I - S(x,\lambda(x),\rho)\right) = 0, \tag{46}$$

(i) If $x < x^L$, because $\partial S/\partial\lambda < 0$, we have $S(x,\lambda,\rho) \leq S(x,0,\rho) \leq (1-\omega)\Upsilon_I$ for any $\lambda \geq 0$. Hence, $\lambda(x) = 0$. We next prove that if $x \geq x^H$, $\lambda(x) = \overline{\lambda}$. By definition, if $x \geq x^H$, then $S\left(x,\overline{\lambda},\rho\right) \geq (1-\omega)\Upsilon_I$. Suppose, if on the contrary, $\lambda(x) < \overline{\lambda}$. Then the bank holds no liquid assets: $V(x) = 0$ by (45). Moreover, by Lemma 2, $S(x,\lambda,\rho) > S\left(x,\overline{\lambda},\rho\right)$. Then, we have $S(x,\lambda,\rho) > (1-\omega)\Upsilon_I = (1-\omega)\Upsilon_I + V(x)$, that is, the liquidity constraint of the bank is violated, a contradiction. Therefore, if $x \geq x^H$, $\lambda = \overline{\lambda}$. If $\overline{\lambda} = 0$, we have found $\lambda(x)$ for any $x \in [0,1]$.

If $\overline{\lambda} > 0$, then by Lemma 3, $x^H > x^L$. For $x \in (x^L, x^H)$, by definition,

$$S\left(x, \overline{\lambda}, \rho\right) < (1-\omega)\Upsilon_I \quad (47)$$

$$S(x, 0, \rho) > (1-\omega)\Upsilon_I. \quad (48)$$

Then, two implications follows. First, inequality (47) commands that $\lambda(x) < \overline{\lambda}$, otherwise, $\lambda(x) = \overline{\lambda} > 0$ and hence by equation (46) $S\left(x, \overline{\lambda}, \rho\right) = (1-\omega)\Upsilon_I + V(x) \geq (1-\omega)\Upsilon_I$, which, however, contradicts inequality (47). Second, $\lambda(x) > 0$. Otherwise, $\lambda(x) = 0$ and by equation (45) the bank chooses $V(x) = 0$. But then by inequality (48) $S(x, \lambda(x), \rho) = S(x, 0, \rho) > (1-\omega)\Upsilon_I = (1-\omega)\Upsilon_I + V(x)$, that is, the liquidity constraint of the type x bank is violated. The two implications together, $\lambda(x) \in \left(0, \overline{\lambda}\right)$. Hence for the bank, the liquidity constraint is binding (i.e. $\lambda > 0$) and it holds no liquid assets (i.e. $\lambda < \overline{\lambda}$): $V(x) = 0$. These two claims combined mean that

$$S(x, \lambda(x), \rho) = \Upsilon_I,$$

as the proposition claims. By Lemma 2 $\frac{\partial S}{\partial x} > 0$ and $\frac{\partial S}{\partial \lambda} < 0$. The implicit function theorem implies that $\partial\lambda/\partial x > 0$.

(ii) $V(x) > 0$ only if $\lambda(x) = \overline{\lambda}$ by equation (45), that is, only banks with $\lambda = \overline{\lambda}$ holds liquid assets. By result (i), $\lambda(x) = \overline{\lambda}$ if and only if $x \geq x^H$. With $\overline{\lambda} > 0$, the liquidity constraint for banks of type $x \geq x^H$ is binding, namely $S\left(x, \overline{\lambda}, \rho\right) = (1-\omega)\Upsilon_I + V(x)$. Hence, the value of liquid assets hold by the type x satisfies

$$V(x) = S\left(x, \overline{\lambda}, \rho\right) - (1-\omega)\Upsilon_I.$$

Q.E.D.

Proof of Lemma 4:

For any t, by (7) $\Lambda(\widetilde{\omega}, x) = \left(h_{xt} + \widehat{\Upsilon}_{xt}\right) - \widetilde{\omega}D_{xt}$. The aggregate supply of fiat money $\mathbf{E}_x\{h_{xt}\} = H$, and by (5) $\mathbf{E}_x\left\{\widehat{\Upsilon}_{xt}\right\} = 0$ because the net interbank credit positions are cancelled out in aggregation. Moreover, the liquidity risk $\widetilde{\omega}$ is independent across banks and hence $\mathbf{E}_{\widetilde{\omega},x}\{\widetilde{\omega}D_{xt}\} = \omega_e\mathbf{E}_x\{D_{xt}\} = \omega_e\mathbf{E}_x\{M_{xt}\}$. Hence, altogether, $\mathbf{E}_{\widetilde{\omega},x}\{\Lambda(\widetilde{\omega}, x)\} = H - \omega_e\mathbf{E}_x\{M_{xt}\}$, from which the lemma follows. Q.E.D.

Proof of Proposition 6:

We have seen only bank of type $x > x^H$ needs liquid assets to meet the liquidity constraint; indeed by Definition 29, the "max" term on the right-hand side of 39 is positive if and only if $x > x^H$. Define

$$Q := \widetilde{H}\left(1+\frac{d\beta}{1-\beta}\delta\right) - \int_{x^H}^{1}\left[\widetilde{S}\left(x,0,\frac{1-\beta}{\beta}\right) - (1-\omega)\,\widetilde{\Upsilon}_I\left(\frac{1-\beta}{\beta}\right)\right] dF(x). \quad (49)$$

Then index Q measures the real surplus or deficit of the liquid assets relative to banks' liquidity needs in the Never-Binding Regime. Condition 39 holds, and hence the steady state is in the Never-Binding Regime, if and only if $Q \geq 0$. Let $\widetilde{M}^{NB}(x) := \widetilde{M}\left(R\left(x,0,(1-\beta)/\beta\right)\right)$ denote the real lending scale by a type x in this regime. Then,

$$\widetilde{M}^{NB}(x) = \left(\frac{A\alpha^2}{w^\alpha\left(1+\frac{1-\beta}{\beta}\left[\omega_e(1-x)+x\right]\right)}\right)^{\frac{1}{1-\alpha}}. \quad (50)$$

We first calculate Q. The real liquidity need of a type x bank in the never-binding regime is $\widetilde{S}^{NB}(x) = \overline{\tau}_x \widetilde{M}^{NB}(x) = \left[\omega(1-x)+x\right]\widetilde{M}^{NB}(x)$. Then:

$$\begin{aligned} Q = {} & \omega_e\left(1+\frac{\beta d\delta}{1-\beta}\right)\int_0^1 \widetilde{M}^{NB}(x)\,dF(x) \\ & + \left[1-F\left(x^H\right)\right](1-\omega)\int_0^1\left[x\widetilde{M}^{NB}(x)\right]dF(x) - \int_{x^H}^1 \widetilde{S}^{NB}(x)\,dF(x), \end{aligned} \quad (51)$$

where $x^H > 0$ is defined in Equation (29) and satisfies

$$\widetilde{S}^{NB}\left(x^H\right) = (1-\omega)\int_0^1 x\widetilde{M}^{NB}(x)\,dF(x), \quad (52)$$

which pins down x^H as a function of only exogenous variables. Obviously, Q linearly increases with δ. To prove the proposition, it suffices to prove that (a): $Q < 0$ at $\delta = 0$; and (b): $Q > 0$ if $\delta \geq \frac{(1-\beta)(1-\omega_e)}{\beta d\omega_e}$, or equivalently if

$$\omega_e\left(1+\frac{\beta d\delta}{1-\beta}\right) > 1. \quad (53)$$

For (a): Let $G(t) := \omega_e\int_0^1 \widetilde{M}^{NB}(x)\,dF(x) + \left[1-F(t)\right](1-\omega)\int_0^1\left[x\widetilde{M}^{NB}(x)\right]dF(x) - \int_t^1 \widetilde{S}^{NB}(x)\,dF(x)$. Then claim (a) is equivalent to $G\left(x^H\right) < 0$, to prove which we first show that $G\left(x^H\right) \leq G(0)$ and then that $G(0) < 0$. For the first claim, observe that

$$G'(t) = f(t)\left(\widetilde{S}^{NB}(t) - (1-\omega)\int_0^1\left[x\widetilde{M}^{NB}(x)\right]dF(x)\right).$$

By (52) $G'(x^H) = 0$. Moreover, because $\widetilde{S}^{NB}(t)$ increases with t by Lemma 2, $G'(t) \leq 0$ for $t < x^H$ and $G'(t) \geq 0$ for $t > x^H$. Altogether, it follows that $t = x^H$ is the minimum point of $G(t)$. Hence, $G\left(x^H\right) \leq G(0)$. For the second claim, that $G(0) < 0$, observe that

$$\begin{aligned}
G(0) &= \omega_e \int_0^1 \widetilde{M}^{NB}(x)\, dF(x) + (1-\omega)\int_0^1 \left[x\widetilde{M}^{NB}(x)\right] dF(x) - \int_0^1 \widetilde{S}^{NB}(x)\, dF(x) \\
&= \int_0^1 \left[\omega_e + (1-\omega)x - \left[\omega(1-x) + x\right]\right] \widetilde{M}^{NB}(x)\, dF(x) \\
&= \int_0^1 (\omega_e - \omega)\, \widetilde{M}^{NB}(x)\, dF(x) \\
&< 0.
\end{aligned}$$

For (b): By (51), $Q > \omega_e\left(1+\frac{\beta d\delta}{1-\beta}\right)\int_0^1 \widetilde{M}^{NB}(x)\, dF(x) - \int_0^1 \widetilde{S}^{NB}(x)\, dF(x) = \int_0^1 \left[\omega_e\left(1+\frac{\beta d\delta}{1-\beta}\right) - \left[\omega(1-x)+x\right]\right] \widetilde{M}^{NB}(x)\, dF(x)$. Observe that $g(x) := \omega_e\left(1+\frac{\beta d\delta}{1-\beta}\right) - \left[\omega(1-x)+x\right]$ decreases with x. Hence $g(x) \geq 0$ for $x \in [0,1]$ if $g(1) \geq 0$, that is, condition (53) holds. Q.E.D.

Proof of Proposition 8:

First, we determin δ_c. In the proof of Proposition 6, we show that the steady state is in the binding regime if and only if $Q < 0$ where Q is defined in (49). In this two-type case $Q < 0$, and hence the steady state is in the binding regime, if and only if

$$\omega_e\left(1+\frac{\beta\delta}{1-\beta}\right)\left[\left(\frac{1+\frac{1-\beta}{\beta}\tau_{e,\kappa}}{1+\frac{1-\beta}{\beta}\omega_e}\right)^{\frac{1}{1-\alpha}}\frac{f_0}{f_\kappa}+1\right] < (1-\omega)\kappa f_0 + \omega. \tag{54}$$

That is, $\delta < \delta_c$ if and only (54) holds. Therefore, δ_c is determined by equalization of the two sides of (54).

In the binding regime, considering that $\lambda(0) = 0$, we have $\lambda(\kappa) = \overline{\lambda} = 1/\beta - 1 - \rho$. Hence, by (33),

$$\widetilde{Y}_I = f_\kappa \kappa \widetilde{M}(\kappa, 1/\beta - 1 - \rho, \rho).$$

The equilibrium condition (36) becomes:

$$\begin{aligned}
f_\kappa\left[\widetilde{S}(\kappa, 1/\beta-1-\rho, \rho) - (1-\omega)\kappa f_\kappa \widetilde{M}(\kappa, 1/\beta-1-\rho,\rho)\right] &= \\
\left(1+\frac{\delta}{\rho}\right)\omega_e\left[\widetilde{M}(0,0,\rho) f_0 + \widetilde{M}(\kappa, 1/\beta-1-\rho,\rho) f_\kappa\right]&
\end{aligned} \tag{55}$$

Divide both sides by $f_\kappa \widetilde{M}(\kappa, 1/\beta - 1 - \rho, \rho)$, use $\widetilde{S}_\kappa / \widetilde{M}_\kappa = \overline{\tau}_\kappa$ by (**??**), and we arrive the single following equation that determines ρ.

$$\Phi(\rho,\delta) := \left(1+\frac{\delta}{\rho}\right)\omega_e\left[\frac{f_0}{f_\kappa}\frac{\widetilde{M}(0,0,\rho)}{\widetilde{M}(\kappa,1/\beta-1-\rho,\rho)}+1\right]+(1-\omega)\kappa f_\kappa-\overline{\tau}_\kappa=0. \quad (56)$$

Function $\widetilde{M}(x,\lambda,\rho)$ is given in equation (**??**). Use this equation and recall that $\lambda(0) = 0$ and $\lambda(\kappa) = \overline{\lambda}$ and $\overline{\lambda} = 1/\beta - 1 - \rho$, and we find:

$$\frac{\widetilde{M}(0,0,\rho)}{\widetilde{M}(\kappa,1/\beta-1-\rho,\rho)} = \left(\frac{1+\left(\frac{1}{\beta}-1\right)\overline{\tau}_\kappa-\rho\left(\overline{\tau}_\kappa-\tau_{e,\kappa}\right)}{1+\rho\omega_e}\right)^{\frac{1}{1-\alpha}}. \quad (57)$$

Substitute these into (56) and use the fact that $(1-\omega)\kappa f_\kappa - \overline{\tau}_\kappa = -\omega - (1-\omega)\kappa f_0$, and equation (56) is equivalent to:

$$\left(1+\frac{\delta}{\rho}\right)\omega_e\left[\frac{f_0}{f_\kappa}\left(\frac{1+\left(\frac{1}{\beta}-1\right)\overline{\tau}_\kappa-\rho\left(\overline{\tau}_\kappa-\tau_{e,\kappa}\right)}{1+\rho\omega_e}\right)^{\frac{1}{1-\alpha}}+1\right]-\omega-(1-\omega)\kappa f_0=0. \quad (58)$$

Let $\Phi(\rho,\delta)$ denote the left hand side of (58). We show that $\Phi(\rho,\delta) = 0$ has a unique solution over $\rho \in (0, (1-\beta)/\beta)$ if and only if condition (54) holds true. For this purpose, let

$$\phi(\rho) := \frac{f_0}{f_\kappa}\left(\frac{1+\left(\frac{1}{\beta}-1\right)\overline{\tau}_\kappa-\rho\left(\overline{\tau}_\kappa-\tau_{e,\kappa}\right)}{1+\rho\omega_e}\right)^{\frac{1}{1-\alpha}}+1. \quad (59)$$

Using this definition,

$$\Phi = \left(1+\frac{\delta}{\rho}\right)\omega_e\phi(\rho)-\omega-(1-\omega)\kappa f_0. \quad (60)$$

Obviously, $\phi(\rho) > 0$ and $\phi'(\rho) < 0$. It is straightforward to see that for any $\delta > 0$, $\Phi = \infty$ at $\rho = 0$. Moreover, $\Phi'_\rho = -\frac{\delta}{\rho^2}\omega_e\phi(\rho) + \left(1+\frac{\delta}{\rho}\right)\omega_e\phi'(\rho)$. Hence,

$$\Phi'_\rho < -\frac{\delta}{\rho^2}\omega_e\phi(\rho) < 0. \quad (61)$$

Together, therefore, $\Phi(\rho) = 0$ has a unique solution over $\rho \in (0, (1-\beta)/\beta)$ if and only if $\Phi((1-\beta)/\beta, \delta) < 0$, which is equivalent to condition (54), that is, $\delta < \delta_c$.

This calculation also shows that $\Phi((1-\beta)/\beta, \delta_c) = 0$. Hence at $\delta = \delta_c$, $\rho = (1-\beta)/\beta$. By the implicit function theorem, $\frac{\partial\rho}{\partial\delta} = -\frac{\Phi'_\delta}{\Phi'_\rho}$. By the definition of $\phi(\rho)$ in

(59), $\phi(\rho) > 0$ and $\phi'(\rho) < 0$. Hence $\Phi'_\rho = -\frac{\delta}{\rho^2}\omega_e\phi(\rho) + \left(1+\frac{\delta}{\rho}\right)\omega_e\phi'(\rho) < 0$, while $\Phi'_\delta = \frac{1}{\rho}\omega_e\phi(\rho) > 0$. Hence, $\frac{\partial\rho}{\partial\delta} = -\frac{\Phi'_\delta}{\Phi'_\rho} > 0$. We are left to prove that $\lim_{\delta\to 0}\rho(\delta) = 0$. By (60), $\Phi = 0 \Leftrightarrow$

$$\delta = \rho\left[\frac{\omega + (1-\omega)\kappa f_0}{\omega_e\phi(\rho)} - 1\right].$$

Hence, $\lim_{\delta\to 0}\rho = 0$ if

$$g(\rho) := \frac{\omega + (1-\omega)\kappa f_0}{\omega_e\phi(\rho)} - 1 > 0$$

for any $\rho \in [0, (1-\beta)/\beta]$. Given $\phi'(\rho) < 0$ and hence $g'(\rho) > 0$, the condition holds true if $g(0) > 0$. With ϕ defined in (59), $\phi(0) = \frac{f_0}{f_\kappa}\left(1 + \left(\frac{1}{\beta}-1\right)\overline{\tau}_\kappa\right)^{\frac{1}{1-\alpha}} + 1|_{\overline{\tau}_\kappa<1} < \frac{f_0}{f_\kappa}\left(\frac{1}{\beta}\right)^{\frac{1}{1-\alpha}} + 1$. Hence, $g(0) > 0 \Leftrightarrow \omega + (1-\omega)\kappa f_0 > \omega_e\phi(0) \Leftarrow \omega + (1-\omega)\kappa f_0 > \omega_e\left[\frac{f_0}{f_\kappa}\left(\frac{1}{\beta}\right)^{\frac{1}{1-\alpha}} + 1\right]$, which, as $\omega_e = \mu\omega$, is equivalent to

$$\kappa f_0 > \frac{\omega}{1-\omega} \times \left\{\mu\left[\frac{f_0}{f_\kappa}\left(\frac{1}{\beta}\right)^{\frac{1}{1-\alpha}} + 1\right] - 1\right\}. \tag{62}$$

To prove this inequality, observe that condition (13) in the special case becomes

$$\kappa f_\kappa \beta^{\frac{1}{1-\alpha}} > \frac{\omega}{1-\omega}. \tag{63}$$

As a result, inequality (62) follows from $\kappa f_0 \geq \kappa f_\kappa \beta^{\frac{1}{1-\alpha}} \times \left\{\mu\left[\frac{f_0}{f_\kappa}\left(\frac{1}{\beta}\right)^{\frac{1}{1-\alpha}} + 1\right] - 1\right\} \Leftrightarrow$ $\frac{f_0}{f_\kappa}\left(\frac{1}{\beta}\right)^{\frac{1}{1-\alpha}} \geq \mu\left[\frac{f_0}{f_\kappa}\left(\frac{1}{\beta}\right)^{\frac{1}{1-\alpha}} + 1\right] - 1 \Leftrightarrow \frac{f_0}{f_\kappa}\left(\frac{1}{\beta}\right)^{\frac{1}{1-\alpha}} + 1 \geq \mu\left[\frac{f_0}{f_\kappa}\left(\frac{1}{\beta}\right)^{\frac{1}{1-\alpha}} + 1\right] \Leftrightarrow 1 \geq \mu$, which certainly holds. Q.E.D.