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Extensions to IVX Methods of Inference For Return Predictability*

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Abstract

The contribution of this paper is threefold. First, we demonstrate that, provided either a suitable bootstrap implementation is employed or heteroskedasticity-consistent standard errors are used, the IVX-based predictability tests of Kostakis *et al.* (2015) retain asymptotically valid inference under the null hypothesis under considerably weaker assumptions on the innovations than are required by Kostakis *et al.* (2015). Second, under the same assumptions, we develop asymptotically valid bootstrap implementations of the IVX tests. Monte Carlo simulations show that the bootstrap tests deliver considerably more accurate finite sample inference than the asymptotic implementations of the tests under certain problematic parameter constellations, most notably for one-sided testing, and where multiple predictors are included. Third, we show how sub-sample implementations of the IVX approach can be used to develop asymptotically valid one-sided and two-sided tests for the presence of temporary windows of predictability.

Keywords: predictive regression; IVX estimation; (un)conditional heteroskedasticity; subsample tests; unknown regressor persistence; endogeneity; residual wild bootstrap.

JEL classification: C12, C22, G17

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1 Motivation

Predictive regression methods are a very important part of the statistical toolbox used in empirical finance, providing a framework to investigate whether a given series can be predicted by other lagged financial and macroeconomic variables. Two important applications are attempts to predict returns on financial assets, most notably equity returns (see, for example, the literature review in Campbell and Yogo, 2006), and developing regression-based tests for efficiency in foreign exchange markets (see Fama, 1984). In both of these applications the returns variable we wish to predict resembles a (near) martingale difference sequence (MDS), while the predictors used are often characterised by high persistence with a significant correlation existing between the predictive regression error and the innovations driving the predictors; see, among others, Campbell and Yogo (2006), Welch and Goyal (2008), Nelson and Kim (1993), Stambaugh (1999) and Pavlidis *et al.* (2017). In these circumstances, standard regression estimation and inference methods, including conventional regression t-tests, are rendered invalid; see, among others, Cavanagh *et al.* (1995), Campbell and Yogo (2006), Jansson and Moreira (2006) and Phillips and Magdalinos (2008).

As a result, a number of likelihood-based procedures have been developed for the case where the predictor is endogenous and displays strong persistence within the local-to-unity class of processes; see, in particular, Cavanagh *et al.* (1995), Campbell and Yogo (2006), Jansson and Moreira (2006) and Elliott *et al.* (2015). Excepting Elliott *et al.* (2015), a major practical drawback with these approaches is that they are invalid if the predictor is weakly persistent. An alternative approach is to base predictability tests on methods of estimating the predictive regression which are robust to the properties of the regressor. Various approaches have been considered, arguably the most successful is Kostakis *et al.* (2015) who estimate the predictive regression using the extended instrumental variable [IVX] procedure of Phillips and Magdalinos (2009); see also, Phillips and Lee (2013), Breitung and Demetrescu (2015), Lee (2016), Demetrescu and Hillmann (2022) and Demetrescu *et al.* (2022), among others. In the IVX approach each predictor in the predictive regression has an associated stochastic instrument formed by constructing a mildly integrated variable from the first differences of the predictor. The IVX instrument, by construction, has lower persistence than a near-integrated variable and, as a consequence, delivers predictability statistics with asymptotically pivotal limiting null distributions.

Kostakis *et al.* (2015) establish, under certain regularity conditions on the model innovations, an asymptotic mixed normality result for their IVX estimator and show that the associated IVX-based predictability statistics possess standard (pivotal) limiting null distributions, regardless of whether the predictor is local-to-unity or weakly dependent (stationary). The asymptotic theory for IVX predictability statistics can, however, provide a very poor approximation to their finite sample behaviour, particularly for highly persistent and endogenous predictors which, as noted above, is arguably the case of most practical relevance. To ameliorate these finite sample distortions from the asymptotic theory, Kostakis *et al.* (2015) suggest a finite sample modification to the standard errors used in computing the IVX statistics. While this correction appears to work well for tests against two-sided alternatives reported in the simulation study for the case of a single regressor in Kostakis *et al.* (2015), as we will show in this paper, tests against one-sided alternatives remain very badly size-distorted for highly persistent and endogenous regressors. Moreover, Xu and Guo (2020) present simulation evidence which suggests that the quality of the prediction from the asymptotic theory, even with the finite sample correction employed, also markedly deteriorates as the number of regressors in the predictive regression increases.

The regularity conditions adopted in Kostakis *et al.* (2015) include an assumption of unconditional homoskedasticity in the vector of innovations driving the predictive model. Kostakis *et al.* (2015) allow for some forms of conditional heteroskedasticity in the innovation vector (provided heteroskedasticity-consistent standard errors are used) although these conditions are rather restrictive in practice. In particular, while a relatively weak martingale difference assumption is placed on the innovations driving the regressors, the errors in the predictive regression equations are assumed to follow a finite-order parametric GARCH model. The latter precludes the conditional variance of the regression errors, as a function of the past, from involving any direct contributions of the lagged values of the innovations driving the predictors, a likely unrealistic restriction for many commonly posited predictors of stock returns. Moreover, while GARCH models are very widely used in empirical finance, their usefulness for returns data is not uncontentious; e.g., Carnero *et al.* (2004) argue that the class of autoregressive stochastic volatility [ARSV] models is better suited to capturing the main empirical properties of the volatility of financial returns series.

A major contribution of this paper is to address the foregoing issues with the practical implementation of the IVX tests. First, regarding the regularity conditions needed, we show that the IVX predictability statistics of Kostakis *et al.* (2015) (implemented with heteroskedasticity-consistent standard errors) continue to admit standard pivotal limiting null distributions (again regardless of the degree of persistence or endogeneity of the regressors) under essentially the same set of regularity conditions on the innovations as adopted by Demetrescu *et al.* (2022) for establishing that the (over-identified) two-stage least squares (2SLS) based predictability test statistics of Breitung and Demetrescu (2015) have standard pivotal limiting null distributions.[1] These conditions allow for quite general patterns of unconditional time heteroskedasticity in the innovations, allowing for time-varying innovation variances and the possibility of time-varying correlations between the innovations. The conditions also allow for a much larger martingale difference class of innovations than considered in Kostakis *et al.* (2015) with no need to exclude interdependence between the conditional variances of the innovations in the model. Moreover, no parametric model needs to be assumed for either the conditional or unconditional time-variation in the innovations.

The 2SLS tests of Breitung and Demetrescu (2015) are based on an (over-identified) regression where two instruments are used for each predictor: a Type-I instrument which by design has a lower degree of persistence than the predictor (an example is the IVX instrument), and a strictly exogenous Type-II instrument (such as a sine function of time). The Type-I (Type-II) instrument is asymptotically dominated under strong (weak) persistence by the type-II (Type-I) instrument; see Demetrescu *et al.* (2022). Consequently, to establish the limiting distribution of the 2SLS statistic under strong persistence one does not need to determine the large sample properties of the IVX instrument under strong persistence, only the large sample behaviour of the Type-II instrument is required and this is a relatively easy task under the regularity conditions on the innovations considered in this paper; see Demetrescu *et al.* (2022). In contrast, to develop limiting distribution theory for the (just-identified) IVX statistics of Kostakis *et al.* (2015) one must establish invariance principles and asymptotic independence results (see Equation (11) below) for the IVX-filtered predictor in the strongly persistent case, which is a much more involved task. Showing that the allowable regularity conditions to establish these results can be weakened from those adopted by Kostakis *et al.* (2015) to essentially the same (albeit higher-level) conditions as Demetrescu *et al.* (2022) use for the 2SLS statistics constitutes a major technical contribution to the literature.

[1] The conditions we adopt are *higher-level* assumptions than those in Demetrescu *et al.* (2022) and, as such, avoid the *ad hoc* rate condition imposed on the fourth (mixed) moments by the latter; see Remark 5.

Establishing that the 2SLS tests of Breitung and Demetrescu (2015) and the IVX tests of Kostakis *et al.* (2015) are asymptotically valid under the same set of regularity conditions implies the practitioner can choose which of these tests to use, agnostic of the properties of the data. This has important practical ramifications. First, as shown in, among others, Harvey *et al.* (2021, pp. 208–212) for the univariate case, two-sided IVX tests are more powerful than the 2SLS tests with certain type-I instruments, and increasingly so as the persistence of the predictor decreases. Second, the randomness of the sign of the correlation between the Type-II instrument and the predictor entails that the 2SLS tests can only be validly implemented as two-sided tests (one reason for this is given in Remark 4 of Breitung and Demetrescu, 2015, p. 364), while the IVX tests can be validly implemented as either one-sided or two-sided tests. This forms an important practical distinction between the tests as several studies have found that imposing an economically motivated structure, such as a known slope sign, on the predictive regression model can lead to better and more accurate findings of return predictability. For example, Campbell and Thompson (2008) find that, among other things, imposing positive predictability (so that the sign of the predictor is imposed to be positive under the alternative) almost always improves the out-of-sample predictability obtained for the predictors considered for equity returns in Welch and Goyal (2008). Similarly, in a Bayesian setting, Pettenuzzo *et al.* (2014) find that imposing a non-negative prior on the Sharpe ratio aids identification of return predictability. Moreover, Pavlidis *et al.* (2017) consider an application of predictive regression to testing for bubbles in foreign exchange markets where a natural positive sign restriction applies under the alternative hypothesis.

Second, and associatedly, to improve on their finite sample performance we also discuss bootstrap implementations of the IVX tests which are shown to be asymptotically valid under the same set of regularity conditions on the innovations. Although there are papers already in the literature that consider the problem of bootstrapping mildly integrated variables, see Fan and Lee (2019) and Smeekes and Westerlund (2019), neither of these are capable of allowing for the generality of time-variation in the variance matrix of the vector of innovations we consider here. Moreover, neither of these approaches is concerned with partial-sums based statistics. More relevant to the IVX tests of Kostakis *et al.* (2015) considered in this paper, Demetrescu *et al.* (2022), develop subsample implementations of the 2SLS-based predictability tests of Breitung and Demetrescu (2015) and base inference on a fixed regressor wild bootstrap [FRWB] resampling scheme. In this approach the regressor (and instrument in the case of Breitung and Demetrescu, 2015) is treated as fixed in the resampling exercise, while the series being predicted is resampled using a wild bootstrap. Demetrescu *et al.* (2022) demonstrate that the FRWB approach correctly replicates the first-order limiting null distributions of the temporary predictability statistics they propose under both conditional and unconditional heteroskedasticity. The FRWB is also used by Georgiev *et al.* (2018, 2019) who develop bootstrap tests for structural change in the predictive regression model.

The FRWB can also be used to successfully replicate the first-order limiting null distribution of the full sample IVX statistics under the conditions on the innovations considered in this paper. However, in Monte Carlo simulations we find that it does not address the finite sample distortions with the asymptotic IVX tests discussed above, most notably the distortions that occur when the regressor is highly persistent and endogenous. This is perhaps unsurprising given that the FRWB does not replicate in the bootstrap data the contemporaneous correlation present between the model's innovations. We therefore also discuss an alternative residual wild bootstrap [RWB] resampling scheme which is designed to replicate this correlation. Here we jointly wild resample the residuals from the fitted predictive regression model and a parametric autoregressive model fitted to the predictor. We also investigate the conditions under which the RWB-based IVX predictability tests are first-order asymptotically valid, and show that these deliver substantial improvements in finite sample behaviour relative to the asymptotic IVX tests.

Although the main application of the IVX methodology has been to predictive regressions for forecasting stock returns, it has also recently been applied to Fama regressions in the context of detecting episodic bubble-type behaviour in foreign exchange markets by Pavlidis *et al.* (2017) who consider a rolling subsample-based implementation of one-sided (right-tailed) IVX tests of Kostakis *et al.* (2015) proposing a test which rejects the null hypothesis of no bubble if any of the subsample statistics in the rolling sequence exceeds a given critical value. To avoid multiple testing bias, Pavlidis *et al.* (2017) base their approach on a conservative critical value obtained using a Bonferroni correction which adjusts the nominal significance level for the number of statistics in the rolling sequence. Pavlidis *et al.* (2017) note that this approach is likely to deliver a conservative test and suggest that a bootstrap implementation might deliver more powerful size controlled tests.

Tests based on the suprema of rolling and recursive subsample sequences of the 2SLS statistics of Breitung and Demetrescu (2015) have also been implemented recently in the context of detecting temporary periods of stock return predictability (so-called *pockets of predictability*) by Demetrescu *et al.* (2022). As noted above, Demetrescu *et al.* (2022) use a FRWB to implement these tests. The final contribution of this paper is to show that both the RWB and FRWB approaches can also be implemented in the context of the corresponding tests from sequences of subsample IVX statistics and that these are asymptotically valid under the same regularity conditions on the innovations as are required for the corresponding bootstrap implementations of the full sample tests. As with the full sample tests, this allows practitioners to implement one-sided tests for temporary windows of predictability allowing for cases where the direction of predictability under the alternative is known, as with the foreign exchange rate bubble testing problem considered in Pavlidis *et al.* (2017), something not possible with the 2SLS-based tests of Demetrescu *et al.* (2022). This also ensures that a rejection of the null is only associated with a window of predictability with the anticipated sign. In the Fama-regression setting, as discussed by Pavlidis *et al.* (2017), the slope parameter is likely to be estimated to be negative over many subsamples of the data and so the tests of Demetrescu *et al.* (2022) are likely to reject in practice even when no bubble is present.

The remainder of the paper is organised as follows. Section 2, introduces the predictive regression model we consider together with the assumptions needed for our analysis. Section 3 reviews the full sample IV-based predictability tests of Kostakis *et al.* (2015) and details subsample implementations of these statistics. Representations for the limiting distributions of these statistics under both the null and local alternatives are provided. These are shown to depend in general on any heteroskedasticity present. Moreover, the form of these limiting distributions depends on whether the predictor is weakly or strongly persistent, even under homoskedasticity. In the context of the full sample IVX statistic, however, the use of Eicker-White standard errors is shown to deliver a standard pivotal limiting null distribution regardless of the predictor's persistence. Section 4 discusses bootstrap implementations of the IVX tests and demonstrates the first-order asymptotic validity of these. Section 5 presents the results from a Monte Carlo analysis into the finite sample behaviour of both the full sample and subsample IVX tests, while empirical applications to stock returns and exchange rate data are reported in Section 6. Concluding comments including some suggestions for further research are provided in Section 7. Detailed proofs of the technical results given in the paper along with other supporting material appear in a supplementary appendix.

In what follows we use $\mathbb{I}(\cdot)$ to denote the indicator function, equal to one when its argument is true, zero otherwise, and $\|\cdot\|$ to denote the matrix norm $[Trace((\cdot)'(\cdot))]^{1/2}$ for square matrices. We denote by $\mathcal{D}^k$ the space of càdlàg real functions on $[0,1]^k$ equipped with the Skorokhod topology, and abbreviate $\mathcal{D}^1$ to $\mathcal{D}$. The weak convergence of probability measures on $\mathcal{D}^k$ and on $\mathbb{R}^k$ is denoted by $\Rightarrow$. We use the notation P, E, etc. for probability, expectation etc. with respect to the distribution of the original data and use P^*, E^*, etc. for probability, expectation, etc. induced by the data and the wild bootstrap multipliers (which we shall denote $\{R_t\}$) conditionally on the data. If w_T, w ($T \in \mathbb{N}$) are random elements of metric spaces, the weak-in-probability convergence $w_T \overset{w}{\Rightarrow}_p w$ means that $\mathrm{E}^* f(w_T) \overset{p}{\to} \mathrm{E} f(w)$ for all continuous bounded real functions with matching domain. Finally, the O_p and o_p symbols have their usual meaning.

2 The Predictive Regression Model and Assumptions

Consider the predictive regression model for returns, y_t, allowing for time-variation in the slope coefficient on a lagged predictor, x_{t-1}, of the form

$$y_t = \alpha + \beta_t x_{t-1} + u_t, \qquad t = 1, \dots, T, \tag{1}$$

where x_t satisfies the additive component model

$$x_t = \mu_x + \xi_t, \qquad t = 0, \dots, T, \tag{2}$$

$$\xi_t = \rho \xi_{t-1} + w_t, \quad A(L) w_t = v_t, \qquad t = 1, \dots, T, \tag{3}$$

in which u_t and v_t are serially uncorrelated (martingale difference [MD]) innovations, precise conditions on which are given in Assumption 3 below, and $A(L) := (1 - a_1 L - a_2 L^2 - \dots - a_p L^p)$ is a stable autoregressive polynomial in the conventional lag operator, L. We define $\omega := 1/A(1)$ and, for the case where x_t also follows a stable autoregression, we let κ^2 denote the sum of the squared coefficients of the filter $((1 - \rho L) A(L))^{-1}$. In our exposition and technical analysis we follow the bulk of the literature and focus attention on the case of a single predictor, so that x_{t-1} in (1) is a scalar. Extensions to allow for multiple predictors will be discussed at various points in the text, although we leave a detailed treatment of this case for future research.

The DGP in (1) generalises the constant parameter predictive regression model considered in Kostakis *et al.* (2015) by allowing for the possibility that the slope coefficient on x_{t-1} varies over time, allowing for changes over time in the predictive content of the regressor x_{t-1}. The constant parameter predictive regression model obtains by setting a constant slope parameter such that $\beta_t = \beta$, for all $t = 1, \dots, T$. The tests we consider in this paper are all for the null hypothesis, H_0, that $(y_t - \alpha)$ is a MD and, hence, that y_t is not predictable by x_{t-1}, which entails that $\beta_t = 0$, for all $t = 1, \dots, T$, in (1).[2] The full-sample IVX tests of Kostakis *et al.* (2015) test the same null hypothesis, H_0, against the alternative that y_t is predictable by x_{t-1} with a constant slope parameter holding across the whole sample; that is, $\beta_t = \beta \neq 0$ for all $t = 1, \dots, T$. The subsample implementations of IVX we discuss will be used to test against alternatives such that $\beta_t \neq 0$ for some t but without imposing constancy on β_t. In any case, some structure needs to be placed on the class of alternative hypotheses we may consider and this will be formalised below.

[2]The methods which we outline in this paper could equally well be used to test the null hypothesis that $\beta_t = \beta_0$ for all $t = 1, \dots T$, but as the focus in both equity forecasting and Fama regressions is on testing the null hypothesis of a zero coefficient on the lagged predictor we will restrict our discussion to $\beta_0 = 0$.

The degree of persistence of the regressor, x_t, is controlled via the parameter ρ. We allow x_t to be either weakly or strongly persistent through the following assumption.

Assumption 1 *$A(L)$ is a finite-order ($p < \infty$) polynomial with all of the roots of $A(z) = 0$ lying outside the unit circle, $|z| = 1$. The initial condition, ξ_0, is a mean zero $O_p(1)$ variate. Moreover, exactly one of the two following conditions holds on the autoregressive parameter ρ in (3):*

1. ***Weakly persistent regressor:*** *ρ is fixed and bounded away from unity, $|\rho| < 1$.*

2. ***Strongly persistent regressor:*** *ρ is parameterised to be local-to-unity; that is, $\rho := 1 - cT^{-1}$, where c is a finite constant. This allows for pure $I(1)$ predictors ($c = 0$), locally stable predictors ($c > 0$), and locally explosive predictors ($c < 0$).*

Remark 1. Assumption 1 imposes that the errors w_t in (3) follow a finite-order autoregression. This is imposed for the purposes of facilitating the RWB implementations of the full sample and subsample IVX tests in section 4. Asymptotic versions of these tests (i.e. tests based on critical values from the limiting null distributions of the statistics) could equally well be based on a linear process assumption for w_t of the form considered in Assumption INNOV of Kostakis *et al.* (2015, p. 1512) or the slightly weaker Assumption M of Magdalinos (2020); in particular, Proposition 1 of this paper would remain valid. The FRWB implementations of the IVX tests discussed in section 4 would also be asymptotically valid under a linear process assumption of this form. $\diamond$

Remark 2. We follow the bulk of the predictive regression literature in considering regressors that follow either stable (weakly dependent) processes, see Amihud and Hurvich (2004), or are near-integrated, see Campbell and Yogo (2006), without assuming knowing which of these holds. As we shall see, the limiting behavior of the IVX statistics can differ under the two types of persistence, but this can be consistently replicated (to asymptotic first order) by the bootstrap procedures we propose. An alternative framework, which we do not consider here, is to characterise the persistence of the predictors as lying in the class of fractionally integrated processes. Important contributions in predictability testing with fractionally integrated predictors include, Maynard and Phillips (2001), Maynard and Shimotsu (2009), Bauer and Maynard (2012), and Andersen and Varneskov (2021a; 2021b). The approach taken in Andersen and Varneskov (2021a; 2021b) is based on the concept of the local spectrum (LCM) methodology and allows for multivariate regressors with any mix of (fractional) integration degrees along with strong forms of endogeneity. The LCM methodology can therefore be viewed as complementary to the IVX approach. Andersen and Varneskov (2021a, pp. 227-228) demonstrate that the LCM approach remains asymptotically valid in the case where the predictors are observed with measurement error which is of a lower degree of persistence than the true predictor. A similar result holds in the set-up considered here; in particular, with an $I(0)$ measurement error, the large sample results given in this paper will continue to hold for strongly persistent predictors satisfying Assumption 1.2. Within the framework of Andersen and Varneskov (2021a), Andersen and Varneskov (2021c) consider the case of "imperfect" predictors, in the sense of Pastor and Stambaugh (2009), where a component of the conditional mean of the returns series exists that is not linearly spanned by the chosen predictor(s). Georgiev *et al.* (2019) consider essentially the same setting but in the context of standard linear predictive regression tests with predictors satisfying Assumption 1. In such cases the standard predictability tests should be interpreted not as tests for a perfect linear relation, but rather as tests of linear predictive power. Indeed, both Andersen and Varneskov (2021c) and Georgiev *et al.*

(2019) develop tests that allow practitioners to distinguish between the "imperfect" and "perfect" regressor scenarios. $\diamond$

The basic idea underlying the IVX procedure of Phillips and Magdalinos (2009) is to instrument the regressor x_{t-1} by a variable of controlled persistence, constructed as

$$z_0 = 0 \quad \text{and} \qquad z_t = (1 - \varrho L)_+^{-1} \Delta x_t := \sum\nolimits_{j=0}^{t-1} \varrho^j \Delta x_{t-j}, t = 1, \ldots, T, \tag{4}$$

and where $\varrho := 1 - aT^{-\eta}$ with $a > 0$ and $0 < \eta < 1$. The IVX scale and exponent parameters, a and η respectively, are tuning parameters set by the practitioner; Kostakis *et al.* (2015) recommend $a = 1$ and $\eta = 0.95$. Where x_t is near-integrated satisfying Assumption 1.2, z_t is approximately a mildly integrated process and therefore of lower persistence than x_t. Moreover, where x_t is weakly dependent satisfying Assumption 1.1, we have that $z_t \approx x_t$. As a result, Kostakis *et al.* (2015) demonstrate that the IVX full-sample estimator of the slope parameter in (1) is asymptotically (mixed) Gaussian under H_0 and under their Assumption INNOV regardless of whether Assumption 1.1 or 1.2 holds and that, consequently, the full-sample instrumental variable tests for H_0 have standard limiting null distributions regardless of the degree of persistence or endogeneity of x_t.

For our purposes we follow Demetrescu *et al.* (2022) and conduct our theoretical analysis of the large sample properties of both the full-sample and sub-sample IVX predictability statistics under local alternatives such that the slope parameter β_t is local-to-zero for an asymptotically non-vanishing set of the sample observations. This is an important generalisation of the large sample results presented for the full sample IVX-based tests in Kostakis *et al.* (2015) and Magdalinos (2020) which only apply under H_0. The localisation rate (or Pitman drift) is such that β_t is specified to lie in a neighbourhood of zero which shrinks with the sample size, T, at a rate which depends on which of Assumption 1.1 and Assumption 1.2 holds in (3). Specifically,[3]

Assumption 2 *In (1)–(3), let $\beta_t := n_T^{-1} b(t/T)$, where $b(\cdot)$ is a piecewise Lipschitz-continuous real function on $[0, 1]$, with $n_T = \sqrt{T}$ under Assumption 1.1, and $n_T = T^{1/2+\eta/2}$ under Assumption 1.2, recalling that η, $0 < \eta < 1$, is the IVX exponent used in the construction of z_t in (4).*

Under the structure of Assumption 2, the null hypothesis H_0 that $\beta_t = 0$, for all $t = 1, \ldots, T$, can be expressed as

$$H_0 : \textit{The function } b(\cdot) \textit{ is identically zero on } [0, 1], \tag{5}$$

while the alternative hypothesis can be written as

$$H_{1,b(\cdot)} : \textit{The function } b(\cdot) \textit{ is non-zero over at least one non-empty open subinterval of } [0, 1]. \tag{6}$$

The latter entails that at least one subset of the sample observations (this need not be a strict subset, so it could contain all of the sample observations) comprising contiguous observations exists

[3] Notice that while the Pitman drift rate considered in Assumption 2 is required to obtain non-degenerate asymptotic (local) limiting distributions for the IVX predictability statistics (it ensures that the left and right hand sides of the predictive regression in (1) are always asymptotically "balanced", possessing the same order of magnitude in probability), such local-to-zero magnitude alternative models should not necessarily be taken to have any inherent economic meaning. Indeed, in the case of excess returns, asset valuation theory stipulates that valuation ratios should have a fixed rather than shrinking magnitude coefficient in the predictive regression; see, for example, Campbell and Thompson (2008), and the references therein, who derive specific results for the case of the dividend price ratio based on the Gordon (1962) growth model. Moreover, dynamic asset pricing models often allow the equity premium to depend on persistent state variables, as for example in the long run risk model of Bansal and Yaron (2004).

for which $\beta_t \neq 0$, and where the size of this subset is proportional to the sample size T. One-sided alternatives that $\beta_t > 0$ ($\beta_t < 0$) in some subset(s) of the data can be considered simply by defining $b(\cdot)$ to be a non-negative (non-positive) function.

We conclude this section by detailing in Assumption 3 the conditions we will place on the innovations u_t and v_t in (1) and (3), respectively. Subsequently we will discuss these conditions relative to other sets of regularity conditions that have been adopted in the literature, before providing the key multivariate invariance principles [MIPs] that hold under these conditions.

Assumption 3 *Let*

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} := \mathbf{H}\left(\frac{t}{T}\right)\begin{pmatrix} a_t \\ e_t \end{pmatrix},$$

where:

1. $\mathbf{H}(\cdot) := \begin{pmatrix} h_{11}(\cdot) & h_{12}(\cdot) \\ h_{21}(\cdot) & h_{22}(\cdot) \end{pmatrix}$ *is a matrix of piecewise Lipschitz-continuous bounded functions on* $(-\infty, 1]$*, which is of full rank at all but a finite number of points;*

2. $\boldsymbol{\psi}_t := (a_t, e_t)'$ *is a* L_4*-bounded stationary and ergodic MD sequence satisfying* $\mathrm{E}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t') = \mathbf{I}_2$ *and*

$$\mathrm{E}\left\|\mathrm{E}_0\sum_{t=1}^{T}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2)\right\|^2 = O(T^{2\epsilon}) \tag{7}$$

for some $\epsilon < \frac{1}{2}$*, with* $\mathrm{E}_\tau(\cdot)$*, denoting expectation conditional on the* σ*-algebra generated by* $\{\boldsymbol{\psi}_{\tau-i}\}_{i=0}^{\infty}$*, and where* $\mathbf{I}_k$ *denotes the* $k \times k$ *identity matrix.*

Remark 3. As discussed in Demetrescu *et al.* (2022), Assumption 3.1 allows for unconditional time heteroskedasticity of quite general form in the innovations through the function $\mathbf{H}$, the unconditional covariance matrix of $(u_t, v_t)'$ being $\mathbf{H}(t/T)\mathbf{H}'(t/T)$. This allows u_t and v_t to display time-varying unconditional variances and both contemporaneous and time-varying (unconditional) correlation between u_t and v_t, including single or multiple (co-) variance shifts, (co-)variances following a broken trend, and smooth transition (co-) variance shifts. In contrast, Assumption INNOV of Kostakis *et al.* (2015, p.1512) and Assumption M of Magdalinos (2020) impose a constant unconditional variance matrix on $(u_t, v_t)'$, but do allow for conditional (stochastic) heteroskedasticity. Assumption 3.2 imposes a MD structure on $\boldsymbol{\psi}_t$ thereby also allowing for conditional heteroskedasticity. In common with Assumption INNOV of Kostakis *et al.* (2015) and Assumption M of Magdalinos (2020), Assumption 3.2 imposes finite fourth-order moments on $\boldsymbol{\psi}_t$. $\diamondsuit$

Remark 4. A crucial difference between the IVX tests of Kostakis *et al.* (2015) and the 2SLS tests of Breitung and Demetrescu (2015) is that in order to establish the large sample properties of the former in the strong persistence case we need to establish a weak convergence result for the partial sum process, $\frac{1}{\sqrt{T^{1+\eta}}}\sum_{t=1}^{[\tau T]} z_{t-1}u_t$ (see (11) below). This is not required for the over-identified 2SLS statistics because, as discussed in section 1, the Type-II instrument used in the case of these statistics asymptotically dominates the Type-I instrument (e.g., the IVX instrument) under strong persistence. For the case of full-sample sums, Kostakis *et al.* (2015) and Magdalinos (2020) need to make the parametric assumption that u_t is generated by a stationary finite-order GARCH(p, q) model with finite fourth moments. This assumption therefore has the consequence to preclude the conditional variance of u_t, as a function of the past, to involve contributions of lagged v_t, an arguably unrealistic restriction for many potential predictors of stock returns; see Example 1 in the supplementary appendix for further discussion. Moreover, a number of authors, including Carnero *et al.* (2004) and Johannes *et al.* (2014) argue that ARSV models capture the main empirical properties of the volatility of financial returns series better than GARCH models. To eliminate the need to choose a specific parametric volatility model, Assumption 3.2 instead adopts an explicit assumption of martingale approximability whereby $\mathrm{E}\|\mathrm{E}_0 \sum_{t=1}^{T}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2)\|^2 = O(T^{2\epsilon})$ for some $\epsilon < \frac{1}{2}$, see Merlevède *et al.* (2006). The exponent ϵ controls the degree of persistence permitted in the conditional variances of the innovations. Stationary vector GARCH processes with finite fourth-order moments satisfy Assumption 3.2 with $\epsilon = 0$, but the assumption is considerably more general as it also allows for asymmetric effects in the conditional variance. Stationary ARSV processes as, for example, are assumed in Johannes *et al.* (2014) also satisfy Assumption 3.2. $\diamondsuit$

Remark 5. It is instructive to compare the regularity conditions in Assumption 3 with those used in Demetrescu *et al.* (2022) in the context of establishing the large sample behaviour of the 2SLS-based predictability statistics of Breitung and Demetrescu (2015) for the case where the Type-I instrument used is set to be the IVX filter in (4). In doing so, it is important to note that the relevant regularity conditions in Demetrescu *et al.* (2022) are spread across Assumptions 3-6 of that paper; see, in particular, Lemma 1 of Demetrescu *et al.* (2022). Both sets of conditions impose the conditions in Assumption 3.1 and also require that the MD sequence $\boldsymbol{\psi}_t$ (in the notation of this paper) defined in Assumption 3.2 is strictly stationarity and ergodic. There are, however, some differences concerning the restrictions they place on the amount of serial dependence allowed in the conditional second order moments of the sequence $\boldsymbol{\psi}_t$. In particular, while the current submission assumes the single "high-level" condition in (7), Demetrescu *et al.* (2022) require that

$$\sup_{t\in\mathbb{N}} \mathrm{E}\|\mathrm{E}_{t-m}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2)\| \to 0 \quad \text{as} \quad m \to \infty, \tag{8}$$

and that

$$\sup_{t=1,\ldots,T;T\in\mathbb{N}} \left|\mathrm{E}\left(\left(v_t^2 - \mathrm{E}(v_t^2)\right) v_{t-k}v_{t-j}\right)\right| \leq \frac{C}{(jk)^{1/2+\vartheta/2}} \tag{9}$$

for some $\vartheta > 0$ and any $k, j > 0$, where v_t are the shocks to the predictor (the second element of $\mathbf{H}(t/T)\boldsymbol{\psi}_t$, denoted by $\tilde{v}_t$ in Demetrescu *et al.*, 2022). These conditions appear similar, and indeed coincide for many popular parametric models, such as for example finite-order stationary vector GARCH models or infinite-order ARCH models, for which if $\boldsymbol{\psi}_t$ satisfies (7) then it will also satisfy (8) and (9), and *vice versa*. Crucially, condition (7) replaces both condition (8) and the *ad hoc* rate of decay condition on the fourth order mixed moments stipulated under (9), and also allows us to make a strong connection with the probabilistic literature; see the discussion in Remark 4. Compared to (9), the condition in (7) is more easily interpretable, allowing us to state that the cumulated sum of $\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2$ is 'almost' a martingale. This type of assumption is arguably more appealing in the context of modelling financial data where finance theory often predicts that the innovations in the model should have the MD property, rather than the conventional approach, typified by the assumptions in Demetrescu *et al.* (2022) and taken from the classical time series literature, which is to impose generic rate of decay conditions on higher-order moments. $\diamondsuit$

Under Assumption 1.1 (weak persistence), $\xi_t = (1-\rho L)_+^{-1} A(L)^{-1}v_t + \rho^t\xi_0$ (recall that $(1-\rho L)_+^{-1} := \sum_{j=0}^{t-1}\rho^j L^j$), which, given the exponential decay of the coefficients, is asymptotically equivalent to the process $(1-\rho L)^{-1} A(L)^{-1}v_t$, and with a slight abuse of notation, we will write $\xi_t = (1-\rho L)^{-1} A(L)^{-1}v_t$ in what follows, ignoring the asymptotically negligible term. Given

Assumption 3, the normalised partial sums of $(u_t, v_t, \xi_{t-1}u_t)$ then satisfy the MIP,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor \tau T\rfloor}\begin{pmatrix} u_t \\ v_t \\ \xi_{t-1}u_t \end{pmatrix} \Rightarrow \int_0^\tau \mathbf{G}(s)\mathrm{d}\boldsymbol{B}(s) := \begin{pmatrix} M_u(\tau) \\ M_v(\tau) \\ M_{\xi u}(\tau) \end{pmatrix} \tag{10}$$

on $\mathcal{D}^3$, where $\mathbf{G}(\tau)$ is a 3×6 matrix of piecewise Lipschitz functions whose elements are formed from the elements of $\mathbf{H}(\tau)$, and where $\boldsymbol{B}(\tau)$ is a 6-dimensional Brownian motion. Explicit expressions for the covariance matrix of $\boldsymbol{B}(\tau)$ and for $\mathbf{G}(\tau)$ are provided in Lemma 4 in the supplement, where the result in (10) is formally established. Using the Phillips-Solo device, it is straightforwardly obtained from (10) that the normalised partial sums of ξ_t weakly converge to $\omega/(1-\rho)M_v$.

Remark 6. The MIP in (10) coincides with that given in Equation (2.6) and Lemma 1.1 of Demetrescu *et al.* (2022) for a weakly persistent predictor. The limiting processes M_u, M_v and $M_{\xi u}$ are individually variance-transformed Brownian motions; cf. Davidson (1994, section 29.4). They are, in general, correlated under Assumption 3, and indeed this correlation can be time-varying; see the supplementary appendix for precise expressions. Under conditional homoskedasticity, $M_{\xi u}$ can be seen to be uncorrelated with either M_u or M_v. Under conditional heteroskedasticity, however, M_v and $M_{\xi u}$ are in general dependent (as are M_u and $M_{\xi u}$), even where $\mathbf{H}(\tau)$ is constant, because $\mathrm{Cov}\,(\xi_{t-1}u_t\,, v_t)$ is not necessarily zero if the conditional correlation between u_t and v_t is nonzero. Where $\mathbf{H}(\tau)$ is constant, such that $(u_t, v_t)'$ is unconditionally homoskedastic, $\int_0^\tau \mathbf{G}(s)\mathrm{d}\boldsymbol{B}(s)$ reduces to a usual Brownian motion process. Where $\mathbf{H}(\tau)$ is non-constant the variance profiles of M_u, M_v and $M_{\xi u}$ will, in general, differ (we define the variance profile of a generic stochastic process $W(s)$ as $[W](s)/[W](1)$ where $[W](s)$ denotes the quadratic variation process of $W(s)$). Even in the special case where $\mathbf{H}(\tau)$ is a scalar multiple of the identity matrix, although M_u and M_v will share the same variance profile, this will not in general coincide with variance profile of $M_{\xi u}$ because the variance of its increments is a polynomial of degree four in the elements of $\mathbf{H}(\tau)$, while those of M_u and M_v are both polynomials of degree two (see the proof of Lemma 4 in the supplement). $\diamondsuit$

Under Assumption 1.2 (strong persistence), the normalized partial sums of (u_t, v_t) converge as previously to (M_u, M_v), where M_u and M_v are the same limiting processes as in (10). Moreover,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{[\tau T]}\begin{pmatrix} v_t \\ \frac{1}{\sqrt{T^\eta}}z_{t-1}u_t \end{pmatrix} \Rightarrow \begin{pmatrix} M_v(\tau) \\ M_{zu}(\tau) \end{pmatrix} \tag{11}$$

on $\mathcal{D}^2$, with $M_{zu}(\tau) := \frac{\omega}{\sqrt{2a}}\int_0^\tau \sqrt{[M_v]'(s)[M_u]'(s)}\mathrm{d}B(s)$, where B is a standard Brownian motion independent of M_v, and where $[M_v]'(s)$ and $[M_u]'(s)$ denote the derivatives (with respect to s) of $[M_v](s)$ and $[M_u](s)$, respectively. These derivatives are well-defined at all but finitely many $s \in [0,1]$, see Lemma 3 in the supplementary appendix. Convergence (11) is established in Lemma 5 in the supplementary appendix. Under strong persistence, the levels of ξ_t satisfy the weak convergence result $T^{-1/2}\xi_{\lfloor \tau T\rfloor} \Rightarrow \omega J_{c,\mathrm{H}}(\tau)$ on $\mathcal{D}$, where $J_{c,\mathrm{H}}(\tau)$ is an Ornstein-Uhlenbeck-type process driven by $M_v(\tau)$; that is, $J_{c,\mathrm{H}}(\tau) := \int_0^\tau e^{-c(\tau-s)}\mathrm{d}M_v(s)$.

Remark 7. The limiting process M_{zu} in (11) is also a variance-transformed Brownian motion. An important difference between the MIPs in (10) and (11) is that M_{zu} is independent of M_v irrespective of any conditional heteroskedasticity while, as discussed in Remark 6, $M_{\xi u}$ and M_v are in general dependent. Another important difference is that the processes $M_{\xi u}$ and M_{zu}, despite being determined by the same innovations, can have quite different behaviour depending on the pattern of conditional and unconditional heteroskedasticity present in $\boldsymbol{\psi}_t$. To illustrate, under unconditional heteroskedasticity the variance profiles of $M_{\xi u}$ and M_{zu} will in general differ where conditional heteroskedasticity is also present; see Example 2 in the Supplementary Appendix. $\diamondsuit$

Remark 8. The MIP in (11) generalises the corresponding convergence results in Kostakis *et al.* (2015) and Magdalinos (2020) in two ways. First, as discussed in Remarks 3 and 4, it establishes the existence of a MIP under much weaker conditions on the innovations than are allowed in Kostakis *et al.* (2015) and Magdalinos (2020). Second, Kostakis *et al.* (2015) and Magdalinos (2020) only provide a convergence result for the full sample quantity $\frac{1}{\sqrt{T^{1+\eta}}}\sum_{t=1}^{T} z_{t-1}u_t$, whereas the MIP in (11) establishes the joint limiting distribution of the corresponding sequence of quantities across all possible subsamples. The result in (11) also provides the necessary keystone for deriving the large sample properties of statistics arising in other settings involving IVX instrumentation of strongly persistent variables, under much weaker conditions than the extant literature allows. $\diamondsuit$

3 IVX-based Predictability Tests

3.1 Full-sample IVX tests

The full-sample IVX-based t-ratio, proposed in Kostakis *et al.* (2015), for testing the null hypothesis $H_0 : \beta_t = 0$ for all $t = 1, ..., T$ in (1) is given by[4]

$$t_{zx} := \frac{\hat{\beta}_{zx}}{s.e.(\hat{\beta}_{zx})}, \quad \hat{\beta}_{zx} := \frac{\sum_{t=1}^{T} z_{t-1}(y_t - \bar{y})}{\sum_{t=1}^{T} z_{t-1}(x_{t-1} - \bar{x}_{-1})} \tag{12}$$

$$s.e.(\hat{\beta}_{zx}) = \frac{\sqrt{\hat{\sigma}_u^2 \sum_{t=1}^{T} z_{t-1}^2}}{\sum_{t=1}^{T} z_{t-1}(x_{t-1} - \bar{x}_{-1})} \tag{13}$$

with $\bar{y} := T^{-1}\sum_{t=1}^{T} y_t$, $\bar{x}_{-1} := T^{-1}\sum_{t=1}^{T} x_{t-1}$, and $\hat{\sigma}_u^2 := T^{-1}\sum_{t=1}^{T} \hat{u}_t^2$.[5] A variety of choices for the residuals $\hat{u}_t$ is possible. Breitung and Demetrescu (2015) and Kostakis *et al.* (2015) recommend the OLS residuals from estimating (1) on the grounds that they come from the best linear projection of y_t on x_{t-1} regardless of the persistence of the putative predictor, and that their finite-sample behaviour appears to be more stable than that of the corresponding IV residuals. One could also use residuals computed under the null; that is, $\hat{u}_t := y_t - \frac{1}{T}\sum_{s=1}^{T} y_s$. Under the local alternatives considered in Assumption 2, these two possible choices are asymptotically equivalent in so far as the behaviour of the resulting IVX statistic is concerned. The IV residuals also have reduced convergence rates compared to the two possible choices above, so we will not consider them further.

Kostakis *et al.* (2015) also consider a variant of the t_{zx} statistic based on the use of heteroskedasticity-robust (Eicker-White) standard errors. This is given by

$$t_{zx}^{EW} := \frac{\hat{\beta}_{zx}}{s.e.^{EW}(\hat{\beta}_{zx})}, \quad s.e.^{EW}(\hat{\beta}_{zx}) := \frac{\sqrt{\sum_{t=1}^{T} z_{t-1}^2 \hat{u}_t^2}}{\sum_{t=1}^{T} z_{t-1}(x_{t-1} - \bar{x}_{-1})}. \tag{14}$$

[4] As discussed in Kostakis *et al.* (2015, p.1514), $\hat{\beta}_{zx}$ is invariant to whether z_{t-1} is demeaned or not.

[5] To ameliorate the finite sample effects of estimating the intercept term in (1), Kostakis *et al.* (2015, p. 1516) recommend the use of a finite-sample correction term. This entails replacing the numerator of (13) by $\sqrt{\hat{\sigma}_u^2 \sum_{t=1}^{T} z_{t-1}^2 - \Xi}$ where $\Xi := T\bar{z}_{-1}^2(\hat{\sigma}_u^2 - \hat{\sigma}_{uw}^2\hat{\sigma}_w^{-2})$, with $\bar{z}_{-1} := T^{-1}\sum_{t=1}^{T} z_{t-1}$, and where $\hat{\sigma}_w^2$ and $\hat{\sigma}_{uw}$ are estimates of the long-run variance of w_t, and of the long-run covariance between u_t and w_t, respectively; a discussion on the practical choice of these estimators is provided in Kostakis *et al.* (2015, pp. 1513 and 1524). The inclusion of the correction term, Ξ, does not alter any of the large sample results that follow.

As we show in section 3.3, t_{zx}^{EW} has a standard normal limiting null distribution under unconditional and/or conditional heteroskedasticity satisfying Assumption 3, regardless of whether x_t is strongly or weakly persistent. Kostakis *et al.* (2015) and Magdalinos (2020) have previously shown that this result holds under unconditional homoskedasticity and for the form of conditional heteroskedasticity they assume which, as discussed in section 2, is a special case of our Assumption 3.2. The same result holds for t_{zx} in the strongly persistent case when the innovations are unconditionally homoskedastic, but does not hold in general otherwise. The finite sample correction term Ξ discussed in footnote 5 can also be applied to the numerator of $s.e.^{EW}(\hat{\beta}_{zx})$ in (14).

One-sided tests based on either t_{zx} or t_{zx}^{EW} can be formed by rejecting against the right-sided alternative that $\beta_t = \beta > 0$, for all $t = 1, ..., T$, for large positive values of the statistics and against the left-sided alternative that $\beta_t = \beta < 0$, for all $t = 1, ..., T$, for large negative values of the statistics. The latter can be equivalently implemented as right-sided tests simply by replacing the predictor x_{t-1} by $-x_{t-1}$. Two-sided tests can be formed by rejecting against the alternative that $\beta_t = \beta \neq 0$, for all $t = 1, ..., T$, for large positive values of either $(t_{zx})^2$ or $(t_{zx}^{EW})^2$.

Remark 9. Kostakis *et al.* (2015) consider the more general set-up of multiple predictive regressions of the form $y_t = \alpha + \boldsymbol{\beta}'\boldsymbol{x}_{t-1} + u_t$, $t = 1, \ldots, T$, where $\boldsymbol{\beta} := (\beta_1, ..., \beta_K)'$ and where $\boldsymbol{x}_t := (x_{1,t}, ..., x_{K,t})'$ is such that $\boldsymbol{x}_t = \boldsymbol{\mu_x} + \boldsymbol{\xi}_t$ where $\boldsymbol{\xi}_t$ satisfies the K-dimensional generalisation of (3), $\boldsymbol{\xi}_t = \boldsymbol{\Gamma}\boldsymbol{\xi}_{t-1} + \boldsymbol{v}_t$, $t = 1, \ldots, T$, and where $\boldsymbol{\mu_x}$ is a K-vector of constants. In common with Kostakis *et al.* (2015), the dimension K of $\boldsymbol{x}_t$ is assumed to be fixed (does not increase with T). Kostakis *et al.* (2015) specify the matrix $\boldsymbol{\Gamma}$ to be diagonal with ith diagonal element ρ_i, $i = 1, ..., K$, and assume that the predictors all lie within the same persistence class; that is, the $x_{i,t}$, $i = 1, ..., K$, either all satisfy Assumption 1.1, or they all satisfy Assumption 1.2. Generating the set of K instruments, $\boldsymbol{z}_t := (z_{1,t}, ..., z_{K,t})'$, from the predictors $x_{i,t}$, $i = 1, ..., K$, each generated according to (4), a two-sided Wald-type IVX based test rejects the null $\mathbf{R}\boldsymbol{\beta} = \mathbf{0}$, where $\mathbf{R}$ is a known $q \times K$ matrix of full row rank, for large values of $W_{zx}^{\mathbf{R}} := \hat{\boldsymbol{\beta}}_{zx}'\mathbf{R}'(\mathbf{R}\widehat{\mathrm{Cov}(\hat{\boldsymbol{\beta}}_{zx})}\mathbf{R}')^{-1}\mathbf{R}\hat{\boldsymbol{\beta}}_{zx}$ where $\hat{\boldsymbol{\beta}}_{zx} := \boldsymbol{A}_T^{-1}\boldsymbol{C}_T$ with $\boldsymbol{A}_T := \sum_{t=1}^T \boldsymbol{z}_{t-1}(\boldsymbol{x}_{t-1} - \bar{\boldsymbol{x}}_{-1})'$, $\boldsymbol{C}_T := \sum_{t=1}^T \boldsymbol{z}_{t-1}(y_t - \bar{y})$, $\bar{\boldsymbol{x}}_{-1} := T^{-1}\sum_{t=1}^T \boldsymbol{x}_{t-1}$, and where $\widehat{\mathrm{Cov}(\hat{\boldsymbol{\beta}}_{zx})} := \hat{\sigma}_u^2 \boldsymbol{A}_T^{-1}\boldsymbol{B}_T(\boldsymbol{A}_T^{-1})'$ with $\boldsymbol{B}_T := \sum_{t=1}^T \boldsymbol{z}_{t-1}\boldsymbol{z}_{t-1}'$, $\hat{\sigma}_u^2 := T^{-1}\sum_{t=1}^2 \hat{u}_t^2$ and $\hat{u}_t$ being the OLS residuals of the estimated predictive regression. An Eicker-White version of $W_{zx}^{\mathbf{R}}$ can be formed by replacing $\hat{\sigma}_u^2\mathbf{B}_T$ in the expression of $\widehat{\mathrm{Cov}(\hat{\boldsymbol{\beta}}_{zx})}$ with $\mathbf{D}_T := \sum_{t=1}^T \boldsymbol{z}_{t-1}\boldsymbol{z}_{t-1}'\hat{u}_t^2$. A finite sample correction term can again be used; see Kostakis *et al.* (2015, p. 1515) for precise details. IVX (partial) t-type tests of the null hypothesis $\beta_i = 0$, $i \in \{1, ..., K\}$, can also be considered. $\diamondsuit$

3.2 Subsample IVX Tests

As we will subsequently show in Proposition 1, the full-sample test based on t_{zx} has non-trivial asymptotic local power against $H_{1,b(\cdot)}$ of (6) for both weakly and strongly persistent regressors. However, these tests are clearly designed for the case where the function $b(\cdot)$ of Assumption 2 is such that $b(t/T) = b$, $t = 1, ..., T$. If it were known that a pocket of predictability might occur only over the particular subsample $t = \lfloor \tau_1 T \rfloor + 1, \ldots, \lfloor \tau_2 T \rfloor$, such that $b(t/T) = b$ for $t = \lfloor \tau_1 T \rfloor + 1, \ldots, \lfloor \tau_2 T \rfloor$ but was zero elsewhere, then it would be more logical to base a test for this on the IVX statistic computed only on the subsample $t = \lfloor \tau_1 T \rfloor + 1, \ldots, \lfloor \tau_2 T \rfloor$, viz,

$$t_{zx}(\tau_1, \tau_2) := \frac{\hat{\beta}_{zx}(\tau_1, \tau_2)}{s.e.(\hat{\beta}_{zx}(\tau_1, \tau_2))} \tag{15}$$

where

$$\hat{\beta}_{zx}(\tau_1, \tau_2) \quad := \quad \frac{\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}\left(y_t - \bar{y}(\tau_1, \tau_2)\right)}{\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}\left(x_{t-1} - \bar{x}_{-1}(\tau_1, \tau_2)\right)} \tag{16}$$

$$s.e.(\hat{\beta}_{zx}(\tau_1, \tau_2)) \quad := \quad \frac{\hat{\sigma}_u(\tau_1, \tau_2)\sqrt{\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2}}{\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}\left(x_{t-1} - \bar{x}_{-1}(\tau_1, \tau_2)\right)} \tag{17}$$

with $\bar{y}(\tau_1, \tau_2) := (T^*)^{-1}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} y_t$ and $\bar{x}_{-1}(\tau_1, \tau_2) := (T^*)^{-1}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} x_{t-1}$, where $T^* := (\lfloor\tau_2 T\rfloor - \lfloor\tau_1 T\rfloor)$, and $\hat{\sigma}_u(\tau_1, \tau_2)^2$ is the analogue of $\hat{\sigma}_u^2$ in (13) computed for the subsample $t = \lfloor\tau_1 T\rfloor + 1, \ldots, \lfloor\tau_2 T\rfloor$. The corresponding subsample analogue of the full sample Eicker-White t_{zx}^{EW} statistic in (14) can be defined similarly and will be denoted $t_{zx}^{EW}(\tau_1, \tau_2)$.

In practice τ_1 and τ_2 are unknown and so, like in Demetrescu *et al.* (2022), we base tests on suitable functionals of sequences of subsample statistics. These need to be agnostic of the data to avoid any endogenous selection bias and any test formed from them must be such that multiple testing issues are also avoided. Given we are testing the null of no predictability against the alternative of predictability in at least one subsample of the data, an approach based on the maximum (in the case of two-sided and right-tailed tests) or minimum (in the case of left-sided tests) of the sequence of subsample predictability statistics would seem appropriate. Common choices of such agnostic sequences of statistics include forward and reverse recursive sequences and rolling sequences. Tests based on the forward recursive sequence of statistics are designed to detect pockets of predictability which begin at or near the start of the full sample period, while those based on the reverse recursive sequence are designed to detect end-of-sample pockets of predictability. For a given window width, tests based on a rolling sequence of statistics are designed to pick up a window of predictability, of (roughly) the same length, within the data.

The subsample IVX tests we propose are formally defined as follows. We will outline these for the case of IVX statistics computed with conventional standard errors, but these can also be implemented with Eicker-White standard errors by replacing $t_{zx}(\cdot, \cdot)$ with $t_{zx}^{EW}(\cdot, \cdot)$ throughout.

- The sequence of *forward recursive* statistics is given by $\{t_{zx}(0, \tau)\}_{\tau_L \leq \tau \leq 1}$, where the *warm-in* parameter $\tau_L \in (0, 1)$ is chosen by the user. The forward recursive regression approach uses $\lfloor T\tau_L\rfloor$ start-up observations and calculates the sequence of subsample predictive regression statistics $t_{zx}(0, \tau)$ for $t = 1, ..., \lfloor\tau T\rfloor$, with τ travelling across the interval $[\tau_L, 1]$. An upper-tailed test can then be based on the maximum taken across this sequence, *viz*,

$$\mathcal{T}_U^F := \max_{\tau_L \leq \tau \leq 1} \{t_{zx}(0, \tau)\}. \tag{18}$$

The corresponding left-tailed test can be based on the minimum across this sequence, denoted $\mathcal{T}_L^F$, and a two-tailed test can be based on the corresponding maximum taken over the sequence of $(t_{zx}(0, \tau))^2$ statistics, denoted $\mathcal{T}_2^F$.

- The sequence of *backward recursive* statistics is given by $\{t_{zx}(\tau, 1)\}_{0 \leq \tau \leq \tau_U}$ with $\tau_U \in (0, 1)$ chosen by the user. Here one calculates the sequence of subsample predictive regression statistics $t_{zx}(\tau, 1)$ for $t = \lfloor\tau T\rfloor + 1, ..., T$, with τ travelling across the interval $[0, \tau_U]$. Analogously to the forward recursive case, an upper-tailed test can again be based on the maximum from this sequence,

$$\mathcal{T}_U^B := \max_{0 \leq \tau \leq \tau_U} \{t_{zx}(\tau, 1)\} \tag{19}$$

while corresponding lower-tailed tests and two-sided tests can be formed from the statistics $\mathcal{T}_L^B$ and $\mathcal{T}_2^B$, defined analogously to the forward recursive case.

- The sequence of *rolling* statistics is given by $\{t_{zx}(\tau, \tau + \Delta\tau)\}_{0 \leq \tau \leq 1-\Delta\tau}$ where the user-defined *window fraction* $\Delta\tau \in (0, 1)$. Here one calculates the sequence of subsample statistics $t_{zx}(\tau, \tau+\Delta\tau)$ for $t = \lfloor \tau T \rfloor + 1, ..., \lfloor \tau T \rfloor + \lfloor T\Delta\tau \rfloor$, where $\lfloor T\Delta\tau \rfloor$ is the window width, with τ travelling across the interval $[0, 1 - \Delta\tau]$. An upper-tailed test can again be based on the maximum from this sequence,

$$\mathcal{T}_U^R := \max_{0 \leq \tau \leq 1-\Delta\tau} \{t_{zx}(\tau, \tau + \Delta\tau)\} \tag{20}$$

while corresponding lower-tailed tests and two-sided tests can again be formed from the statistics $\mathcal{T}_L^R$ and $\mathcal{T}_2^R$, defined analogously to the recursive cases.

Remark 10. The full sample IVX statistic t_{zx} of (12) is contained within the forward recursive, backward recursive, and rolling sequences of statistics, by setting $\tau = 1$, $\tau = 0$, and $\Delta\tau = 1$. $\diamondsuit$

Remark 11. Subsample implementations of the multiple predictor IVX Wald tests discussed in Remark 9 can also be defined in an analogous fashion to $\mathcal{T}_U^F$, $\mathcal{T}_U^B$ and $\mathcal{T}_U^R$ of (18), (19) and (20), respectively. Here, defining the subsample analogue of the IVX Wald statistic $W_{zx}^{\mathbf{R}}$ computed over the data subsample $t = \lfloor \tau_1 T \rfloor + 1, \ldots, \lfloor \tau_2 T \rfloor$, as $W_{zx}^{\mathbf{R}}(\tau_1, \tau_2)$, we can consider tests which reject for large values of the maxima from analogous forward recursive, backward recursive and rolling sequences of such subsample statistics, which we will denote $\mathcal{W}_F^{\mathbf{R}}$, $\mathcal{W}_B^{\mathbf{R}}$ and $\mathcal{W}_R^{\mathbf{R}}$, respectively. $\diamondsuit$

Tests based on sequences of subsample statistics have also been proposed in the literature on testing for episodic bubbles; see, for example, Phillips *et al.* (2015). Pavlidis *et al.* (2017) propose tests for episodic bubbles in foreign exchange markets based on the right-sided IVX t-ratios of Kostakis *et al.* (2015) applied to Fama regressions estimated over a rolling sequence of subsamples of the data. Their proposed test rejects the no bubble null hypothesis if any of the subsample statistics in the rolling sequence exceeds a given critical value. For size-controlled inference, they base their test on a conservative critical value obtained using a Bonferroni correction, adjusting the nominal significance level for the number of statistics in the sequence. Given that this number will generally be quite large (for given T, the number of statistics in the sequence will be larger the smaller the rolling window width, $\lfloor T\Delta\tau \rfloor$), Pavlidis *et al.* (2017) acknowledge that this approach will deliver a conservative test. The subsample maximum tests outlined above avoid the need for conservative testing methods and so would be expected to deliver more powerful bubble detection tests than those proposed in Pavlidis *et al.* (2017).

Demetrescu *et al.* (2022) also consider tests for episodic predictability based on the maxima from corresponding sequences of rolling and recursive subsample implementations of a 2SLS predictability statistic proposed in Breitung and Demetrescu (2015). It can be seen from Lemma S.6 of Demetrescu *et al.* (2022) and its proof that, under strong persistence and for any subsample of the type analysed there, the 2SLS t statistic with heteroskedasticity-consistent standard errors is distributed under the null in large samples as the product of a $\chi(1)$ variate[6] and a random sign that are statistically dependent in general, where the distribution of the random sign depends on the localisation parameter c. Moreover, neither the bootstrap studied by Demetrescu *et al.* (2022) nor the residual wild bootstrap studied in this paper can be validly applied to the 2SLS t statistic as the former fails to replicate the dependence structure of the $\chi(1)$ variate and the random sign,

[6]If a random variable is $\chi^2(1)$ distributed, then its positive square root obeys a $\chi(1)$ distribution.

whereas the latter cannot mimic the distribution of the random sign as a function of c. Valid (asymptotic and bootstrap) tests can be based on the *squared* 2SLS t statistics (as this eliminates the random sign), but doing so obviously precludes testing against one-sided alternatives. In contrast, the subsample IVX-based tests proposed here can be validly used to test against either one-sided or two-sided alternatives as we show below, and so can be used to test against directed alternatives in cases where the predictor has a natural sign predicted by theory, as with the testing problem considered in Pavlidis *et al.* (2017) where a bubble implies a positive slope coefficient.

3.3 Asymptotic Theory

In this section we now provide limiting distribution theory for the IVX statistics from sections 3.1 and 3.2.

Proposition 1 *Consider the model in (1)–(3) and let Assumptions 2 and 3 hold. Then under the local alternative $H_{1,b(\cdot)}$ of (6):*

(i) *Under Assumption 1.1, as $T \to \infty$*

$$\begin{aligned}
t_{zx}(\tau_1,\tau_2) &\Rightarrow \frac{M_{\xi u}(\tau_2) - M_{\xi u}(\tau_1) + \kappa^2 \int_{\tau_1}^{\tau_2} b(s)\mathrm{d}[M_v](s)}{\sqrt{\frac{\kappa^2}{\tau_2-\tau_1}\left([M_u](\tau_2)-[M_u](\tau_1)\right)\left([M_v](\tau_2)-[M_v](\tau_1)\right)}} := G_1(b,\tau_1,\tau_2);\\
\mathcal{T}_U^F &\Rightarrow \sup_{\tau\in[\tau_L,1]} \{G_1(b,0,\tau)\} := G_{1,U}^F(b);\\
\mathcal{T}_U^B &\Rightarrow \sup_{\tau\in[0,\tau_U]} \{G_1(b,\tau,1)\} := G_{1,U}^B(b);\\
\mathcal{T}_U^R &\Rightarrow \sup_{\tau\in[0,1-\Delta\tau]} \{G_1(b,\tau,\tau+\Delta\tau)\} := G_{1,U}^R(b).
\end{aligned}$$

(ii) *Under Assumption 1.2, and with $\epsilon < \min\{1-\eta, \frac{1}{2}\eta\}$ in Assumption 3,*

$$\begin{aligned}
t_{zx}(\tau_1,\tau_2) &\Rightarrow \frac{M_{zu}(\tau_2) - M_{zu}(\tau_1)}{\sqrt{\frac{1}{\tau_2-\tau_1}\left([M_u](\tau_2)-[M_u](\tau_1)\right)\left([M_v](\tau_2)-[M_v](\tau_1)\right)}}\\
&\quad + \sqrt{\frac{2\omega^2}{a}}\frac{\int_{\tau_1}^{\tau_2} b(s)\mathrm{d}[M_v](s) + \int_{\tau_1}^{\tau_2}(J_{c,\mathrm{H}}(s) - \overline{J}_{c,\mathrm{H}}(\tau_1,\tau_2))b(s)\mathrm{d}J_{c,\mathrm{H}}(s)}{\sqrt{\frac{1}{\tau_2-\tau_1}\left([M_u](\tau_2)-[M_u](\tau_1)\right)\left([M_v](\tau_2)-[M_v](\tau_1)\right)}} := G_2(b,\tau_1,\tau_2);\\
\mathcal{T}_U^F &\Rightarrow \sup_{\tau\in[\tau_L,1]} \{G_2(b,0,\tau)\} := G_{2,U}^F(b);\\
\mathcal{T}_U^B &\Rightarrow \sup_{\tau\in[0,\tau_U]} \{G_2(b,\tau,1)\} := G_{2,U}^B(b);\\
\mathcal{T}_U^R &\Rightarrow \sup_{\tau\in[0,1-\Delta\tau]} \{G_2(b,\tau,\tau+\Delta\tau)\} := G_{2,U}^R(b),
\end{aligned}$$

where a and η are the parameters defining the IVX filter in (4), ω and κ^2 are as defined in section 2, and $\overline{J}_{c,\mathrm{H}}(\tau_1,\tau_2) := \frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2} J_{c,\mathrm{H}}(s)\mathrm{d}s$. The results for $t_{zx}(\tau_1,\tau_2)$ hold for any given fixed values of τ_1 and τ_2, $0 \le \tau_1 < \tau_2 \le 1$.

Remark 12. Corresponding representations for the limiting distributions of the left-sided $\mathcal{T}_L^F$, $\mathcal{T}_L^B$ and $\mathcal{T}_L^R$ statistics under the conditions of Proposition 1 can be obtained by replacing the sup operator by the inf operator in the representations given in Proposition 1, and with an obvious notation we denote these limiting distributions as $G^F_{j,L}(b)$, $G^B_{j,L}(b)$ and $G^R_{j,L}(b)$, $j = 1,2$, respectively. Similarly, representations for the limiting distributions of the two-sided statistics $\mathcal{T}_2^F$, $\mathcal{T}_2^B$ and $\mathcal{T}_2^R$, denoted $G^F_{j,2}(b)$, $G^B_{j,2}(b)$ and $G^R_{j,2}(b)$, $j = 1,2$, respectively, can be obtained by squaring the limiting quantities over which the supremum is taken in the expressions in Proposition 1. $\diamondsuit$

Remark 13. Part (ii) of Proposition 1, which relates to the case where x_t is strongly dependent, imposes a further restriction on the degree of persistence permitted in the conditional variances via the additional requirement that $\epsilon < \min\{1-\eta, \frac{1}{2}\eta\}$. This restriction therefore entails that $\epsilon < 1/3$ (with this maximum upper bound for ϵ corresponding to the use of an IVX filter with $\eta = 2/3$). Recalling, for example, that parametric GARCH models are such that $\epsilon = 0$, it seems likely that this additional restriction would not be restrictive in practice. $\diamondsuit$

Remark 14. The results in Proposition 1 establish the asymptotic local power functions of the tests based on the subsample and full sample IVX-based statistics (the latter obtained by setting $\tau_2 = 1$ and $\tau_1 = 0$ in the limiting representations for $t_{zx}(\tau_1, \tau_2)$) from sections 3.1 and 3.2, respectively, under the local alternative $H_{1,b(\cdot)}$. These local power functions depend, in general, on any heteroskedasticity and/or weak autocorrelation (short-run dynamics) present in the errors and differ according to whether x_t is weakly or strongly persistent. In the strongly persistent case they also depend on the parameter a used in the IVX filter and on the local-to-unity parameter, c. For the full sample t_{zx} test these results therefore complement those provided in Kostakis *et al.* (2015) and Magdalinos (2020) which apply only under the null hypothesis. From Proposition 1 it can be seen that the full sample t_{zx} test exhibits non-trivial power against the class of time-varying local alternatives we consider in this paper; that is, it has power to detect predictive episodes. In the case where $b(s) = b$, for some constant b, the results in Proposition 1 provide the asymptotic local power functions of the tests in the case where (local) predictability holds across the full sample; in this case the limiting process $Z_b(\tau)$ in part (ii) of Proposition 1 simplifies to $bJ_{c,\mathrm{H}}(\tau)$. $\diamondsuit$

Remark 15. The limiting null distributions of the statistics obtain from the results in Proposition 1 on setting $b(s) = 0$ for all s (whereby $Z_b(\tau)$ collapses to zero). Doing so, the limiting null distributions of the individual statistics $t_{zx}(\tau_1, \tau_2)$ can be seen to be (pointwise) normal. For example, under strong persistence, we have for the full-sample statistic that

$$t_{zx} \Rightarrow \frac{M_{zu}(1)}{\sqrt{[M_u](1)[M_v](1)}} = \frac{\int_0^1 \sqrt{[M_u]'(s)[M_v]'(s)}\mathrm{d}B(s)}{\sqrt{[M_u](1)[M_v](1)}} \overset{d}{=} N\left(0, \frac{\int_0^1 [M_u]'(s)[M_v]'(s)\mathrm{d}s}{\int_0^1 [M_u]'(s)\mathrm{d}s \int_0^1 [M_v]'(s)\mathrm{d}s}\right).$$

It can then be seen that in the unconditionally homoskedastic case where $\mathbf{H}$ is constant, the limiting null distribution of t_{zx} is standard normal under strong persistence, and hence that of $(t_{zx})^2$ is χ^2_1. This holds regardless of any conditional heteroskedasticity present in the innovations. In the weakly persistent case, however, we have that

$$t_{zx} \Rightarrow \frac{M_{\xi u}(1)}{\sqrt{\kappa^2[M_u](1)[M_v](1)}} \overset{d}{=} N\left(0, \frac{[M_{\xi u}](1)}{\kappa^2[M_u](1)[M_v](1)}\right)$$

whereby it follows that the variance of the limiting distribution of t_{zx} will in general depend on any conditional heteroskedasticity and/or short-run dynamics (the latter through the parameter κ^2) present, even where $\mathbf{H}$ is constant. On the other hand, κ^2 drops out of this expression under conditional homoskedasticity of $\boldsymbol{\psi}_t$, even if $\mathbf{H}$ is time-varying. For further details see the proof of Lemma 4 in the supplementary appendix. $\diamondsuit$

Remark 16. The limiting null distributions of the subsample-based statistics, $\mathcal{T}_j^F$, $\mathcal{T}_j^B$ and $\mathcal{T}_j^R$, $j \in \{U, L, 2\}$, all depend, in general, in a highly complicated way on nuisance parameters arising from any heteroskedasticity and (in the weakly dependent case) serial correlation present in $(u_t, v_t)'$ and on whether x_t is strongly or weakly persistent. While, as we show below in Proposition 2, these dependencies can be removed from the limiting null distribution of the full sample statistic by using Eicker-White standard errors, this is not true of the subsample-based statistics. $\diamondsuit$

As discussed in Remark 15, the standard t_{zx} statistic, while having a limiting null distribution that is free of nuisance parameters when x_t is strongly persistent and the innovations are unconditionally homoskedastic, does not in general have a pivotal limiting null distribution when x_t is weakly persistent. The non-pivotal nature of the limiting null distribution of t_{zx} under conditional heteroskedasticity in the case of a weakly persistent predictor motivated Kostakis *et al.* (2015) to also consider the Eicker-White statistic t_{zx}^{EW} in (14). In Proposition 2 we demonstrate that the limiting (marginal) null distribution of the subsample Eicker-White statistic $t_{zx}^{EW}(\tau_1, \tau_2)$ has a standard normal limiting null distribution under the conditions of Proposition 1 and regardless of whether x_t is weakly dependent or near-integrated.

Proposition 2 *Under the conditions of Proposition 1, and for any given fixed values of τ_1 and τ_2, $t_{zx}^{EW}(\tau_1, \tau_2) \Rightarrow N(0, 1)$, and hence $(t_{zx}^{EW}(\tau_1, \tau_2))^2 \Rightarrow \chi_1^2$, under the null hypothesis, H_0, regardless of whether Assumption 1.1 or Assumption 1.2 holds.*

Remark 17. As a consequence of Proposition 2 the full-sample t_{zx}^{EW} statistic of (14) is seen to have a standard normal limiting null distribution under H_0 regardless of whether x_t is weakly or strongly persistent. The standard normality of the limiting null distribution of t_{zx}^{EW} has previously been shown to hold by Kostakis *et al.* (2015) under their Assumption INNOV and by Magdalinos (2020) under his Assumption M, both of which assume unconditional homoskedasticity. The result in Proposition 2 therefore establishes that this result holds under the much more general conditions of Assumption 3, which includes: (i) the case where $\mathbf{H}$ is non-constant such that the innovations are unconditionally heteroskedastic, and (ii) the case where the sequence $\boldsymbol{\psi}_t$ exhibits conditional heteroskedasticity of very general form; see again the discussion in Remarks 4 and 5. $\diamondsuit$

Remark 18. Provided the vector $(u_t, \boldsymbol{v}_t')'$ satisfies an obvious $(k+1)$-dimensional generalisation of Assumption 3, then the multiple predictor full sample Wald statistic, $W_{zx}^{\mathbf{R}}$ of Remark 9, when implemented with Eicker-White standard errors, can be shown to have a χ_q^2 limiting null distribution regardless of whether $\boldsymbol{x}_t$ is strongly or weakly persistent. The limiting null distributions of the corresponding subsample-based statistics, $\mathcal{W}_F^{\mathbf{R}}$, $\mathcal{W}_B^{\mathbf{R}}$ and $\mathcal{W}_R^{\mathbf{R}}$, of Remark 11 will, like the corresponding subsample-based tests for a scalar predictor, x_t, discussed in this section, have limiting null distributions which will, in general, depend in a highly complicated way on nuisance parameters arising from any heteroskedasticity and (in the weakly dependent case) serial correlation present in $(u_t, \boldsymbol{v}_t')'$ and on whether $\boldsymbol{x}_t$ is strongly or weakly persistent. $\diamondsuit$

4 Bootstrap IVX Tests

As the results in section 3.3 show, implementing tests based on either the full sample t_{zx} statistic from section 3.1 or the subsample-based $\mathcal{T}_j^F$, $\mathcal{T}_j^B$ and $\mathcal{T}_j^R$, $j = U, L, 2$, statistics from section 3.2 will require us to address the fact that their limiting null distributions will, in general, depend on nuisance parameters arising from heteroskedasticity and/or serial correlation present in the data, and on whether the predictor x_{t-1} is weakly dependent or near-integrated.

To that end, we will consider two bootstrap resampling schemes for t_{zx}, $\mathcal{T}^F$, $\mathcal{T}^B$ and $\mathcal{T}^R$. The first, a residual wild bootstrap [RWB], is outlined in Algorithm 1. In Algorithm 2 we then outline how the fixed regressor wild bootstrap [FRWB] employed by Demetrescu *et al.* (2022) can also be used with the full sample and subsample IVX statistics discussed in this paper.[7] Although we will not formally establish large sample validity here, the RWB could also be validly employed in connection with the corresponding 2SLS tests of Demetrescu *et al.* (2022).

Algorithm 1 (Residual Wild Bootstrap)

Step 1: *Fit the predictive regression to the sample data $(y_t, x_{t-1})'$ to obtain the residuals $\hat{u}_t$, $t = 1, ..., T$, using any of the two choices outlined below (13).*

Step 2: *Fit by OLS an autoregression of order $p+1$ to x_t; viz,*

$$x_t = \hat{m} + \sum_{j=1}^{p+1} \hat{a}_j x_{t-j} + \hat{v}_t$$

and compute the OLS residuals $\hat{v}_t$, $t = p+1, \ldots, T$. Set $\hat{v}_t = 0$ for $t = 1, \ldots, p$.

Step 3: *Generate bootstrap innovations $(u_t^*, v_t^*)' := (R_t \hat{u}_t,\ R_t \hat{v}_t)'$, $t = 1 \ldots, T$, where R_t, $t = 1, ..., T$, is a scalar i.i.d.$(0,1)$ sequence with $\mathrm{E}(R_t^4) < \infty$, which is independent of the sample data.*

Step 4: *Define the bootstrap data $(y_t^*, x_{t-1}^*)'$, where $y_t^* = u_t^*$ (so that the null hypothesis is imposed on the bootstrap y_t^*) and where x_t^* is generated according to the recursion*

$$x_t^* = \sum_{j=1}^{p+1} \hat{a}_j x_{t-j}^* + v_t^*, \quad t = 1, ..., T$$

with initial conditions $x_0^* = \ldots = x_{-p}^* = 0$. Create the associated bootstrap IVX instrument, z_t^*, via $z_0^* = 0$ and $z_t^* = \sum_{j=0}^{t-1} \varrho^j \Delta x_{t-j}^*$, $t = 1, \ldots, T$, where ϱ is the same value used in constructing the original IVX instrument, z_t.

Step 5: *Using the bootstrap sample data, $\left(y_t^*, x_{t-1}^*, z_{t-1}^*\right)'$, in place of the original sample data, $(y_t, x_{t-1}, z_{t-1})'$, construct the bootstrap analogues of the $t_{zx}(\tau_1, \tau_2)$, $\mathcal{T}_j^F$, $\mathcal{T}_j^B$ and $\mathcal{T}_j^R$, $j = U, L, 2$, statistics from section 3.2. Denote these bootstrap statistics as $t_{zx}^*(\tau_1, \tau_2)$, $\mathcal{T}_j^{*,F}$, $\mathcal{T}_j^{*,B}$ and $\mathcal{T}_j^{*,R}$, $j = U, L, 2$.*

Step 6: *Taking $\mathcal{T}_U^F$ to illustrate, a bootstrap p-value is then computed as $p_{1,T}^* := 1 - G_{1,T}^*(\mathcal{T}_U^F)$, where $G_{1,T}^*(\cdot)$ denotes the conditional (on the original sample data) cumulative distribution function (cdf) of $\mathcal{T}_U^{*,F}$. The bootstrap test, run at the λ significance level, based on $\mathcal{T}_U^F$ is therefore defined such that it rejects H_0 if $p_{1,T}^* < \lambda$. Bootstrap p-values for the other tests are similarly obtained.*

[7] To save space we outline our proposed bootstrap procedures for the case where conventional standard errors are used and where the finite sample correction term of Kostakis *et al.* (2015) is not employed; cf. footnote 5. Bootstrap implementations of the tests with the finite sample correction term can instead be used without altering any of the large sample properties given in this section. Moreover, bootstrap implementations of the IVX tests based around Eicker-White standard errors may also be considered and share the same asymptotic validity properties as the bootstrap tests based on conventional standard errors.

Algorithm 2 (Fixed Regressor Wild Bootstrap)

Step 1: *As Step 1 in Algorithm 1.*

Step 2: *Generate bootstrap innovations* $u_t^* := R_t \hat{u}_t$, $t = 1 \ldots, T$, *where* R_t *satisfies the same conditions as given in Step 3 of Algorithm 1*

Step 3: *For* $t = 1, \ldots, T$, *define the bootstrap data* $y_t^* = u_t^*$ *(so that the null hypothesis is imposed on the bootstrap* y_t^**).*

Step 4: *As detailed in Step 5 of Algorithm 1, but where the original sample data,* $(y_t, x_{t-1}, z_{t-1})'$ *are instead replaced by the fixed regressor bootstrap sample data,* $(y_t^*, x_{t-1}, z_{t-1})'$.

Step 5: *As Step 6 of Algorithm 1*

Remark 19. A key difference between the RWB and FRWB outlined in Algorithms 1 and 2, respectively, surrounds the generation of the bootstrap analogue data for x_t and, hence, z_t. In the FRWB scheme one calculates the bootstrap statistics in Step 4 using the data $(y_t^*, x_{t-1}, z_{t-1})'$; that is, y_t^* is generated exactly as in Algorithm 1, but the observed outcomes on $\mathbf{x} := [x_0, x_1, ..., x_T]'$ and $\mathbf{z} := [z_0, z_1, ..., z_T]'$ are treated as a fixed regressor and fixed instrument vector, respectively. As such, while the RWB rebuilds into the bootstrap data (an estimate of) the correlation between the innovations u_t and v_t through Step 3 of Algorithm 1 (it is crucial in doing so that the same R_t is used to multiply both $\hat{u}_t$ and $\hat{v}_t$), the FRWB does not. This is an important distinction because, as the simulation results we report in section 5 will show, the finite sample behaviour of the IVX statistics is heavily dependent on the correlation between u_t and v_t in the case where x_t is strongly persistent. As a result we find that the RWB delivers considerably better finite sample performance than the FRWB in the case where x_t is strongly persistent. $\diamondsuit$

Remark 20. A further difference between the RWB and the FRWB is that because one creates bootstrap analogues of x_t and z_t, x_t^* and z_t^* respectively, one implicitly has to use an estimate of ρ in doing so. Under Assumption 1.2 (strong persistence) it is well known that the associated local-to-unity parameter, c, cannot be consistently estimated. Consequently, when x_t is strongly persistent the bootstrap data on x_t^* will not be generated with the same local-to-unity parameter as the original data x_t. For the FRWB this issue does not arise as the original data on x_t are used in calculating the bootstrap statistics. However, the IVX statistics instrument x_{t-1} by z_{t-1}, and their bootstrap analogue statistics instrument x_{t-1}^* by z_{t-1}^*, where z_t and z_t^* are, by construction, both mildly integrated processes regardless of the value of c under Assumption 1.2. There is therefore no necessity for the estimate of c from Step 2 to be consistent in order to validly implement the RWB in Algorithm 1. Notice that this would not be true under Assumption 1.2 if we were bootstrapping the standard OLS t-statistic from (1) because this statistic does not instrument x_t by a variable of lower persistence and, as result, has a limiting null distribution which depends on c. $\diamondsuit$

Remark 21. It could also be possible to implement a moving block bootstrap [MBB] based scheme, similar to that used in Fan and Lee (2019), for the IVX-based tests considered here. An outline of this algorithm can be found in the Supplementary Appendix. We conjecture that this MBB procedure is asymptotically valid provided $\mathbf{H}$ is constant such that the innovations were unconditionally homoskedastic. To account for unconditional heteroskedasticity a block wild adaptation of this bootstrap could be employed and this is also outlined in the Supplementary

Appendix. We will not pursue either of these methods further here as in unreported simulations we found them to perform poorly in finite samples relative to the RWB-based tests. $\diamondsuit$

Remark 22. With simple modifications, the RWB of Algorithm 1 can be implemented for the multiple regressor full sample Wald statistic, $W_{zx}^{\mathbf{R}}$ of Remark 9, and the corresponding subsample-based statistics, $\mathcal{W}_F^{\mathbf{R}}$, $\mathcal{W}_B^{\mathbf{R}}$ and $\mathcal{W}_R^{\mathbf{R}}$, of Remark 11. In Step 2 of Algorithm 1 a vector autoregression of order $p+1$ is fitted to $\boldsymbol{x}_t$ to obtain the residuals $\hat{\boldsymbol{v}}_t$ with the residuals from these collected into $\hat{\boldsymbol{v}}_t$. In Step 3 one then calculates the bootstrap innovations $(u_t^*, \boldsymbol{v}_t^{*\prime})' = (R_t\hat{u}_t, R_t\hat{\boldsymbol{v}}_t')'$, $t = 1, \ldots, T$. In Step 4 one generates the bootstrap data $y_t^* = u_t^*$ imposing the null, together with the bootstrap predictor vector, $\boldsymbol{x}_t^*$, by the recursion based on the coefficient estimates obtained in Step 2. The bootstrap instruments, $\boldsymbol{z}_t^*$, are derived from $\boldsymbol{x}_t^*$ according to the same IVX filter used to obtain $\boldsymbol{z}_t$ from $\boldsymbol{x}_t$. The RWB statistics are then computed from the bootstrap sample data, $\left(y_t^*, \boldsymbol{x}_{t-1}^*, \boldsymbol{z}_{t-1}^*\right)'$. The FRWB of Algorithm 2 can also be modified to allow for multiple regressors by using the bootstrap sample data, $(y_t^*, \boldsymbol{x}_{t-1}, \boldsymbol{z}_{t-1})'$ in Step 4. Provided the conditions outlined in Remark 18 hold, the FRWB and RWB-based tests for multiple regressors will share analogous asymptotic validity properties to the bootstrap tests in the case of a single regressor established below. $\diamondsuit$

In Proposition 3 we demonstrate the large sample validity of the RWB and FRWB implementations of the IVX tests from Algorithms 1 and 2, respectively. In particular, these are shown to correctly replicate the first order asymptotic null distributions of the IVX statistics under both the null and local alternatives. For the RWB-based tests this result requires a further restriction to hold on the fourth moments of the innovations in the case where x_t is weakly persistent. This additional restriction is not required for the asymptotic validity of the FRWB-based tests.

Proposition 3 *Consider the model in* (1)–(3) *and let Assumptions 2 and 3 hold. Then under the local alternative $H_{1,b(\cdot)}$ of* (6)*:*

(i) *Under Assumption 1.1,*

- **(a)** *For the bootstrap statistics generated according to the RWB scheme in Algorithm 1, provided $\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1') \otimes (\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')] = 0$ for all natural $i \neq j$, it holds that $t_{zx}^*(\tau_1, \tau_2) \overset{w}{\Rightarrow}_p G_1(0, \tau_1, \tau_2)$ for fixed $0 \leq \tau_1 < \tau_2 \leq 1$, and $\mathcal{T}_j^{*,F} \overset{w}{\Rightarrow}_p G_{1,j}^F(0)$, $\mathcal{T}_j^{*,B} \overset{w}{\Rightarrow}_p G_{1,j}^B(0)$ and $\mathcal{T}_j^{*,R} \overset{w}{\Rightarrow}_p G_{1,j}^R(0)$, in each case for $j = U, L, 2$.*
- **(b)** *For the bootstrap statistics generated according to the FRWB scheme in Algorithm 2, $t_{zx}^*(\tau_1, \tau_2) \overset{w}{\Rightarrow}_p G_1(0, \tau_1, \tau_2)$ for fixed $0 \leq \tau_1 < \tau_2 \leq 1$, and $\mathcal{T}_j^{*,F} \overset{w}{\Rightarrow}_p G_{1,j}^F(0)$, $\mathcal{T}_j^{*,B} \overset{w}{\Rightarrow}_p G_{1,j}^B(0)$ and $\mathcal{T}_j^{*,R} \overset{w}{\Rightarrow}_p G_{1,j}^R(0)$, in each case for $j = U, L, 2$.*

(ii) *Under Assumption 1.2, and with $\epsilon < \min\{\eta, \frac{1}{2}\}$ in Assumption 3, and regardless of whether the bootstrap statistics are generated according to the RWB scheme in Algorithm 1 or the FRWB scheme in Algorithm 2, $t_{zx}^*(\tau_1, \tau_2) \overset{w}{\Rightarrow}_p G_2(0, \tau_1, \tau_2)$ for fixed $0 \leq \tau_1 < \tau_2 \leq 1$, and $\mathcal{T}_j^{*,F} \overset{w}{\Rightarrow}_p G_{2,j}^F(0)$, $\mathcal{T}_j^{*,B} \overset{w}{\Rightarrow}_p G_{2,j}^B(0)$ and $\mathcal{T}_j^{*,R} \overset{w}{\Rightarrow}_p G_{2,j}^R(0)$, in each case for $j = U, L, 2$.*

Remark 23. A comparison of the limiting results for the bootstrap statistics in Proposition 3 with those given for the corresponding statistics in Proposition 1 demonstrates the usefulness of the RWB and FRWB procedures from Algorithms 1 and 2, respectively; as the number of observations increases, the bootstrapped statistics have the same first-order limiting null distributions as the corresponding original test statistic.[8] For this result to hold for the RWB statistics, however, it is seen that fourth moments of the form $\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\otimes(\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')]$ for $i\neq j$ should not contribute to the quadratic variation of the process $M_{\xi u}$. The reason is that in the RWB world the mixed fourth moments $\mathrm{E}^*[(R_t^2\boldsymbol{\psi}_t\boldsymbol{\psi}_t')\otimes(R_{t-i}R_{t-j}\boldsymbol{\psi}_{t-i}\boldsymbol{\psi}_{t-j}')] = 0$ by construction for all natural $i\neq j$, and hence, these do not contribute to the quadratic variation of the RWB analogue of $M_{\xi u}$. As with the conditions placed on $\{\boldsymbol{\psi}_t\}$ by Assumption 3.2, this assumption is not tied to any specific parametric model. Even where this condition is violated, the impact on the (asymptotic) size of the resulting RWB test might still be relatively small, given that the quantities $\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\otimes(\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')]$, for all natural $i\neq j$, only constitute part of the quadratic variation of $M_{\xi u}$ and it is this latter quantity which the bootstrap limit needs to reproduce. A well known class of models which violate this condition are GARCH models with non-zero leverage effects. We will explore the impact of such a model on the finite sample size behaviour of the RWB tests in section 5. $\diamondsuit$

Remark 24. In the case of the RWB, the asymptotic validity result in Proposition 3 requires knowledge of the true autoregressive lag length, p, used in Step 2 of Algorithm 1. In practice p, will be unknown. This can be selected in the usual way using a consistent information criterion such as the Bayes Information Criterion (BIC) or Hannan-Quinn [HQ] information criterion without affecting the stated asymptotic validity results for the RWB. A less parsimonious information criterion, such as the Akaike Information Criterion [AIC] could also be used. Furthermore, we conjecture that the RWB tests would still be asymptotically valid for more general linear process innovations of the form discussed in Remark 1, provided a sieve-type device is used in Step 2 whereby the truncation lag for the fitted autoregression is allowed to increase at a suitable rate with the sample size, e.g. $\lfloor\kappa(T/100)^{1/4}\rfloor$, for a positive constant κ. A formal proof of this conjecture is beyond the scope of the present paper but constitutes an interesting topic for future research. Along these lines, in unreported simulations we found that the lag length fitted in Step 2 has rather little bearing on the power of the resulting bootstrap tests. It should also be stressed that no choice of truncation lag is required in connection with the FRWB outlined in Algorithm 2. $\diamondsuit$

Remark 25. Although, as Proposition 3 shows, the RWB and FRWB are asymptotically equivalent, to first-order, to each other and to the limiting null distributions of the corresponding asymptotic statistics, they can be shown to differ in higher-order terms. In particular, Demetrescu and Hosseinkouchack (2021) demonstrate that in the strongly persistent case the second-order term in a Taylor expansion of the full-sample IVX statistic is a function of c. In preliminary work we have found that the FRWB fails to replicate this second-order term entirely, while the RWB, conditional on the data, replicates a similar functional to the second-order term but with $\hat{c}$ (the implied estimate of c obtained from Step 2 of Algorithm 1) replacing the true c. So although the RWB does not correctly replicate the second-order term from the limiting null distribution of the IVX statistic it replicates part of it and this would be anticipated to effect a reduced sensitivity to c in the finite sample size properties of the RWB test relative to the FRWB test, a prediction borne out by the simulations results in section 5. A full treatment of this issue is beyond the scope of the present paper, but constitutes a useful topic for further research. $\diamondsuit$

Remark 26. A consequence of the results in Proposition 3, using the same arguments as in the proof of Theorem 5 in Hansen (2000), is that for each of the tests the bootstrap p-values are

[8]Observe that the condition placed on ϵ in part (ii) of Proposition 3 is less restrictive than that imposed for part (ii) of Proposition 1 regardless of the value of η used in the IVX filter and therefore this result holds for all DGPs such that Proposition 1 holds.

(asymptotically) uniformly distributed under the unit root null hypothesis, H_0, leading to tests with (asymptotically) correct size, thereby establishing the asymptotic validity of the bootstrap tests. In the case of the FRWB, this validity result is achieved without the practitioner needing to have knowledge of whether x_t is weakly or strongly persistent and holds regardless of any autocorrelation or heteroskedasticity present in u_t and v_t satisfying Assumption 3. For the RWB this is also true, provided the condition $\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1') \otimes (\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')] = 0$ for all natural $i \neq j$ holds. A further consequence of the result in Proposition 3 for $t_{zx}^*(\tau_1, \tau_2)$, setting $\tau_1 = 0$ and $\tau_2 = 1$, is therefore that under the null the RWB and FRWB bootstrap implementations of the full sample t_{zx} test deliver asymptotically valid (by which we mean that the bootstrap p-values are asymptotically invariant to any nuisance parameters under the null) inference under Assumption 3 (or the restricted version thereof in the case of the RWB scheme) without the need for Eicker-White standard errors. $\diamondsuit$

Remark 27. A further implication of Proposition 3 is that the bootstrap IVX tests from Algorithms 1 and 2 will admit the same asymptotic local power functions under $H_{1,b(\cdot)}$ as the corresponding (infeasible) size-adjusted tests based on the corresponding original IVX statistic. $\diamondsuit$

Remark 28. As discussed in Remark 23, a key difference between the large sample properties of the RWB and FRWB is that the former can only be validly applied in the case where x_t is weakly persistent if the mixed fourth moments $\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1') \otimes (\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')]$ with $i \neq j$ do not contribute to the quadratic variation of the process $M_{\xi u}$. However, as we will see in the simulations in section 5, the RWB delivers considerably better finite sample performance than the FRWB when x_t is strongly persistent, while the two display similar performance when the degree of persistence in x_t is weaker. In principle one might use the sample data on x_t to decide which of the RWB and FRWB to use. In particular, one could adopt the RWB of Algorithm 1 unless the sample data suggested the persistence in x_t was relatively weak. This idea has previously been advocated in the predictability testing literature by Elliott *et al.* (2015) who propose a procedure which switches between a weighted average power test where x_t is strongly persistent and the standard OLS t-test from (1) when x_t is weakly persistent. The switching mechanism they adopt is to use the OLS t-test when $\hat{c} \geq 130$ and the weighted average power test otherwise, where $\hat{c}$ is an estimate of the local-to-unity parameter, c. A similar rule could be used here, whereby we use the RWB unless $\hat{c}$ exceeds some specified value in which case we use the FRWB. An obvious estimate of c, based on the autoregressive estimates from Step 2 of Algorithm 1, is $\hat{c} := T(1 - \sum_{j=1}^{p} \hat{a}_j)$. This rule ensures that, with probability approaching one, the RWB would not be chosen in large samples when x_t was weakly dependent, and so this hybrid bootstrap will share the asymptotic validity result enjoyed by the FRWB in the weak persistence case. $\diamondsuit$

Remark 29. In practice the cdf $G_{1,T}^*(\cdot)$ of the bootstrap $\mathcal{T}_U^{*,F}$ statistic, and the corresponding cdfs for the other statistics, required in Step 6 of Algorithm 1 and Step 5 of Algorithm 2 will be unknown but can be approximated in the usual way through numerical simulation. To illustrate, again for the case of the $\mathcal{T}_U^F$ statistic, this is achieved by generating B bootstrap (conditionally) independent statistics, say $\mathcal{T}_{U,b}^{*,F}$, $b = 1, ..., B$, each computed as in Algorithm 1 above. The simulated bootstrap p-value for the test is then computed as $\tilde{p}_{1,T}^* = B^{-1} \sum_{b=1}^{B} \mathbb{I}\left(\mathcal{T}_{U,b}^{*,F} > \mathcal{T}_U^F\right)$ and is such that $\tilde{p}_{1,T}^* \stackrel{a.s.}{\to} p_{1,T}^*$ as $B \to \infty$, where $\stackrel{a.s.}{\to}$ denotes almost sure convergence. An approximate standard error for $\tilde{p}_{1,T}^*$ is given by $(\tilde{p}_{1,T}^*(1 - \tilde{p}_{1,T}^*)/B)^{1/2}$. Simulated bootstrap critical values can also be obtained; e.g. for the $\mathcal{T}_U^F$ statistic, a λ level bootstrap critical value, $cv_{\lambda,B}$ say, can be calculated as the upper tail λ percentile from the order statistic formed from the B bootstrap statistics, $\mathcal{T}_{U,b}^{*,F}$, $b = 1, ..., B$. The resulting bootstrap test, which rejects H_0 if $\mathcal{T}_U^F > cv_{\lambda,B}$, will have asymptotic size that for sufficiently large B will be as close as desired to λ. $\diamondsuit$

5 Monte Carlo Simulations

We now discuss the results from a detailed Monte Carlo study into the finite sample properties of IVX-based predictability tests. In section 5.1 for the case of a single predictor and section 5.2 for multiple predictors, we compare the properties of the full sample tests of Kostakis *et al.* (2015) based on asymptotic critical values with their RWB and FRWB bootstrap implementations developed in this paper. A comparison of the subsample bootstrap IVX tests proposed in this paper with their 2SLS counterparts from Demetrescu *et al.* (2022) is made in section 5.3. For all statistics, OLS residuals are used in computing the standard errors. All simulations are preformed in MATLAB, versions R2018b and R2020a, using the Mersenne Twister random number generator. Results are reported for tests run at the 5% nominal significance level. Unless otherwise stated, the results are based on $B = 999$ bootstrap replications, and 10,000 Monte Carlo replications.

5.1 Single Predictor Regressions - Full Sample Tests

We first consider the case where a single predictor, x_{t-1}, is included in the predictive regression. Results are reported for the IVX test of Kostakis *et al.* (2015) both with and without Eicker-White corrected standard errors, t_{zx}^{EW} and t_{zx}, respectively; these statistics were computed exactly as detailed in section 3.1 with the finite sample correction term, Ξ, (see footnote 5) included using a Bartlett kernel with bandwidth $T^{1/3}$ as recommended by Kostakis *et al.* (2015) - this choice was made in all of the numerical experiments and empirical applications reported in this paper. We will compare these with their RWB and FRWB bootstrap analogues, denoted $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$, described in Algorithms 1 and 2, respectively. In the context of the RWB the autoregressive lag length used in Step 2 of Algorithm 1 was chosen applying the BIC over $p \in \{0, ..., \lfloor 4(T/100)^{0.25} \rfloor\}$. The bootstrap statistics are all based on conventional standard errors and all include the finite sample correction term. Our analysis consists of testing the null hypothesis of no predictability, $H_0 : \beta = 0$, in (1) in the context of a constant parameter prediction model, so that $\beta_t = \beta$, for all $t = 1, ..., T$. We will consider tests directed against both one-sided alternatives, left-tailed tests for $H_1 : \beta < 0$, and right-tailed tests for $H_1 : \beta > 0$, together with two-sided tests for $H_1 : \beta \neq 0$.

5.1.1 Empirical Size

To investigate the finite sample size properties of t_{zx}, t_{zx}^{EW}, $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ under the null hypothesis of no predictability, we generate data according to (1)-(3) with $\beta_t = \beta = 0$ for all $t = 1, ..., T$. We initialised the autoregressive process characterising the dynamics of the putative predictor, x_t, in (3) at $\xi_0 = 0$, and considered a wide range of values for the autoregressive parameter ρ in (3) covering stationary, near-integrated and mildly explosive predictors; in particular, we set $\rho = 1 - c/T$ with $c \in \{-5, -2.5, 0, 2.5, 5, 10, 25, 50, 75, 100, 125, 150, 200, 250\}$. All results reported, both in the main text and in the supplementary appendix, are for sample sizes $T = 250$ and $T = 1000$. In total, for the single predictor case, we consider 11 distinct classes of DGP. For the sake of space we will present Tables of results for two of these DGPs in this section. A summary of the results for the other 9 DGPs will also be given, with the full details of these DGPs and the associated tables of results for these cases given in the supplementary appendix.

Main Results

The first DGP (DGP1) we will consider corresponds to (1)-(3) with the innovation vector $(u_t, v_t)'$ drawn from an i.i.d. bivariate Gaussian distribution with mean vector zero and covariance matrix $\mathbf{\Sigma}= \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}$, where ϕ corresponds to the correlation between u_t and v_t. Results from DGP1 for $\phi = -0.95$, -0.90, and -0.50 are reported in Table 1.[9] Results for tests run at the 1% and 10% significance levels are qualitatively similar and can be found in Tables D.1–D.3 of the supplementary appendix. Additional results for $\phi = 0$ can also be found in Table D.4.

The second DGP (DGP2) we will consider is one designed to be such that the regularity conditions needed for the validity of the RWB when x_t is weakly persistent are violated. The DGP we consider is a well known model where the conditional variance of the innovations $(u_t, v_t)'$ follows a stationary ARCH model with leverage effects and is of the form

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \boldsymbol{\psi}_t \tag{21}$$

with

$$\boldsymbol{\psi}_t = \begin{pmatrix} a_t \\ e_t \end{pmatrix} = \begin{pmatrix} \varepsilon_{1t}\sqrt{1 + \frac{1}{2}a_{t-1}^2 \mathbb{I}_{\{a_{t-1}<0\}}} \\ \varepsilon_{2t} \end{pmatrix}$$

and $(\varepsilon_{1t}, \varepsilon_{2t})' \sim NIID(\mathbf{0}, \mathbf{I_2})$. The AR parameter ρ is again set equal to $1 - c/T$.

DGP2 satisfies our assumptions of finite fourth moments of $\boldsymbol{\psi}_t$ and martingale approximability of $\boldsymbol{\psi}_t \boldsymbol{\psi}_t'$ (with $\epsilon = 0$). However, and crucially, the quadratic variation of $M_{\xi u}$ depends on,

$$\begin{aligned} h_{11}^2 h_{21}^2 b_1 b_2 \,\mathrm{E}(a_t^2 a_{t-1} a_{t-2}) &= \rho^3 \,\mathrm{E}(a_t^2 a_{t-1} a_{t-2}) \\ &= \frac{\rho^3}{8} \,\mathrm{E}\,|\varepsilon_1|^3 \,\mathrm{E}\Big\{ |a_1| \Big[\sqrt{\Big(1 + \frac{1}{2}a_1^2\Big)^3} - 1 \Big] \Big\} > 0 \end{aligned} \tag{22}$$

for $\rho > 0$; see the proof of Lemma 4. This model therefore violates the limiting condition that $M_{\xi u}^* \stackrel{d}{=} M_{\xi u}$ which is necessary and sufficient for the validity of the RWB in the case where x_t is weakly persistent. Specifically, the non-zero term in (22) is absent from the quadratic variation of $M_{\xi u}^*$ in the limiting distribution of the RWB bootstrap statistic when x_t is weakly persistent; cf. Remark 23. Therefore, results will be reported only for $c \in \{5, 10, 25, 50, 75, 100, 125, 150, 200, 250\}$. Recall, however, that this limiting condition is not required for the asymptotic validity of the FRWB statistic. Results from DGP2 are reported in Table 2; additional results for tests run at the 1% and 10% significance levels can be found in Table D.5 of the supplementary appendix.

Consider first the results for the homoskedastic DGP1. A comparison of the results in Table 1 for $\phi = -0.95$, -0.90, and -0.50, respectively, shows that when the innovations are homoskedastic the endogeneity correlation parameter, ϕ, has relatively little impact on the size properties of the two-sided tests, at least for cases where the autoregressive parameter c is positive and not close to zero. Here there is relatively little difference between the tests based on asymptotic critical values and the corresponding RWB and FRWB bootstrap tests. For all of these cases the two-sided tests display finite sample size close to the nominal level. However, where x_t is mildly explosive with $c = -5$ there is a tendency to undersize in t_{zx}, t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ for $\phi = -0.95$ and $\phi = -0.90$

[9] Notice that, because we report results for both left-tailed, right-tailed and two-tailed tests, it is not necessary to report results for positive values of ϕ; cf. Campbell and Yogo (2006, p. 30).

which is largely redressed by $t_{zx}^{*,RWB}$. For $0 \leq c \leq 10$ slight oversizing is also seen for both $\phi = -0.95$ and $\phi = -0.90$ with t_{zx}, t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ which is again largely eliminated by $t_{zx}^{*,RWB}$.

A different picture emerges for the one-sided implementations of the tests. The one-sided t_{zx}, t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ tests display severe size distortions for $c < 50$ when $\phi = -0.95$. Specifically, for $\phi = -0.95$ the left-tailed t_{zx}, t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ tests display very significant undersizing, while their right-tailed counterparts are severely oversized (for instance when $c < 10$ empirical size is in most cases more than double the nominal size). The size distortions observed for these one-sided tests decrease, other things equal, as $|\phi|$ decreases, but significant size distortions are still observed even for $\phi = -0.5$. We also observe that the empirical rejection frequencies of the one-sided t_{zx}, t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ tests under DGP1 are all very similar to each other for given values of ϕ and c. Consequently, the FRWB based implementations of the one-sided IVX tests do not appear to offer any tangible improvement on the finite sample size properties of the asymptotic tests, as might be expected in the light of Remark 19. In contrast, both the left-sided and right-sided tests implemented with the RWB offer empirical size properties close to the nominal level throughout.

Consider next the results in Table 2 for DGP2 where the conditional variance of $(u_t, v_t)'$ follows an ARCH model with leverage effects. The results show that in general the two-sided versions of the t_{zx}^{EW}, $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ tests all display reasonable size control throughout. In contrast, significant size distortions are seen for the two-sided t_{zx} test. The latter finding is consistent with our discussion in Remark 15 on the non-pivotal nature of the limiting null distribution of t_{zx} under conditional heteroskedasticity when x_t is weakly dependent. Large size distortions are also seen for the one-sided t_{zx} tests. Moreover, and as observed with DGP1, although the two-sided t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ tests show decent finite sample size control the same is not true of the one-sided versions of these tests. In contrast the one-sided $t_{zx}^{*,RWB}$ tests deliver decent finite sample size control for all values of c and regardless of the sample size. Consequently, although the limiting condition $M_{\xi u}^* \stackrel{d}{=} M_{\xi u}$ formally required for the asymptotic validity of the RWB tests is not met by DGP2, the results in Table 2 suggest that $t_{zx}^{*,RWB}$ nonetheless displays arguably the most reliable finite sample size control among the tests considered for data generated according to DGP2.

Summary of Additional Results

We also investigated the impact on the finite sample performance of the IVX statistics and their bootstrap implementations from a variety of additional empirically relevant models. Full details of the simulation DGPs considered and the tabulated results (which appear in Tables D.7 - D.42) are given in the supplementary appendix. In what follows we provide a summary of these results:

(1). We repeated the experiments in Table 1 for the case where v_t follows either a positively autocorrelated (DGP3) or a negatively autocorrelated (DGP4) stationary AR(1) process. These results, which can be found in Tables D.7 - D.14, were qualitatively very similar to those reported for serially uncorrelated v_t in Table 1.

(2). We consider two DGPs which include a contemporaneous one-time break of equal magnitude in the unconditional variances of u_t and v_t, as in Georgiev *et al.* (2018) and Demetrescu *et al.* (2022). The first, labelled DGP5, contains an upward change in the unconditional variances of u_t and v_t at the sample midpoint (Tables D.15 - D.18), while the second, labelled DGP6, contains a corresponding downward change in the unconditional variances of u_t and v_t (Tables D.19–D.22).

The results reported in Tables D.15 to D.22 reveal that, as expected, the two-sided IVX test with conventional standard errors, t_{zx}, displays significant size distortions. For example, for a 5%

significance level and $\phi = -0.95$ the rejection frequencies observed across all values of c considered, when an upward change in variance occurs (Table D.15) are in the range $[0.064, 0.095]$ for $T = 250$ and $[0.066, 0.097]$ for $T = 1000$. For a downward change in variance (Table D.19) results are similar ($[0.017, 0.098]$ for $T = 250$ and $[0.018, 0.091]$ for $T = 1000$), except for cases where $c < 0$ (mildly explosive predictors) in which case some undersizing is observed. The magnitude of these size distortions are relatively stable across ϕ.

In contrast, for the one-sided versions of t_{zx} the empirical size distortions for the former worsen, other things equal, as $|\phi|$ increases. For example, for DGP5 with $T = 250$ and $\phi = -0.95$ the range of empirical rejection frequencies for the left-sided tests is $[0.003, 0.075]$ and for the right-sided tests $[0.085, 0.151]$; see Table D.15. On the other hand, for $\phi = 0$ the left and right-sided tests rejection frequencies' range is $[0.064, 0.081]$; see Table D.18.

The size distortions seen in the two-sided t_{zx} test for DGP5 and DGP6 are significantly ameliorated by using Eicker-White standard errors (t_{zx}^{EW}) when $c \geq -2.5$. However, the one-sided (left and right-sided) t_{zx}^{EW} tests do not improve much relative to t_{zx} when $c \leq 25$; see Tables D.15 to D.22.

The RWB and FRWB bootstrap implementations of the two-sided t_{zx} test both do a very good job of controlling finite sample size in the presence of unconditional heteroskedasticity. For the one-sided tests, $t_{zx}^{*,RWB}$ displays empirical rejection frequencies which are again in general close to the nominal significance level considered, regardless of the values of c and ϕ. In contrast, the one-sided $t_{zx}^{*,FRWB}$ test displays significant size distortions for values of $c \leq 25$; these improve as $|\phi|$ decreases, as anticipated by the discussion in Remark 19.

(3). To further evaluate the impact of conditional heteroskedasticity we considered three further volatility specifications: i) a GARCH(1,1) model with either $N(0, 1)$ (DGP7) or Student-t distributed innovations with 5 degrees of freedom $[t_5]$ (DGP8); ii) a GoGARCH(1,1) model, see for example Van der Weide (2002), allowing for either $N(0, 1)$ (DGP9) or t_5 innovations (DGP10); and iii) an autoregressive stochastic volatility process (DGP11).

As observed earlier, the non-pivotal nature of the t_{zx} statistic's limiting null distribution under GARCH type conditional heteroskedasticity is also apparent in the results in Tables D.23 to D.26 and D.27 to D.30 corresponding to DGP7 and DGP8, respectively. These results highlight that the size distortion of the two-sided t_{zx} statistic increases as $|\phi|$ increases regardless of whether $N(0, 1)$ (Tables D.23 to D.26) or Student-t innovations (Tables D.27 to D.30) are used in generating the data. The magnitude of the size distortions is, however, considerably exacerbated when the innovations are heavy tailed (DGP8). For instance, for $N(0, 1)$ innovations, $T = 250$, $\phi = -0.95$ and for a 5% significance level the range of the empirical rejection frequencies for t_{zx} is $[0.042, 0.082]$, whereas for t_5 innovations the range is $[0.081, 0.167]$. The Eicker-White correction does a good job in correcting the size distortion of the two-sided t_{zx} test regardless of whether the innovations are $N(0, 1)$ or Student-t distributed. In the previous example, the ranges of the rejection frequencies of t_{zx}^{EW} when the innovations are $N(0, 1)$ and t_5 distributed is $[0.047, 0.066]$ and $[0.062, 0.068]$, respectively. The results also show that the RWB and FRWB both display good empirical size properties in a two-sided hypothesis testing context. However, for one-sided testing $t_{zx}^{*,RWB}$ delivers significantly better finite sample size control than $t_{zx}^{*,FRWB}$ when x_t is strongly persistent, while they display similar performance for weaker levels of persistence in x_t. Overall $t_{zx}^{*,RWB}$ is the best performing test regardless of the nominal significance levels used and regardless of the underlying distribution of the innovations. All of the other one-sided tests display serious size distortions when the predictor is strongly persistent ($c < 25$), for both $N(0, 1)$ or t_5 distributed innovations.

For the GoGARCH models (DGP9 and DGP10 in Tables D.31 to D.34 and Tables D.35 to D.38, respectively), qualitatively similar conclusions can be drawn to those discussed above for the GARCH(1,1) case albeit the magnitude of the size distortions observed for the $t_{zx}^{*,FRWB}$, t_{zx}^{EW} and t_{zx} tests are generally smaller.

Finally, for stochastic volatility (DGP11), the results in Tables D.39 to D.42 suggest that all of the two-sided tests display adequate finite sample size control, with the exception of t_{zx}^{EW} which is oversized for $T = 250$, although its size properties are improved for $T = 1000$. For the one-sided tests, similar conclusions are drawn as for the GARCH and GoGARCH specifications. Specifically, $t_{zx}^{*,FRWB}$, t_{zx}^{EW} and t_{zx} are considerably oversized when the predictor is strongly persistent and $\phi = -0.95$, but $t_{zx}^{*,RWB}$ displays reliable empirical rejection frequencies, across c.

5.1.2 Finite Sample Local Power

We next provide a brief analysis of the finite sample local power properties of one-sided and two-sided implementations of the IVX tests from section 3.1 together with their bootstrap analogues from section 4 and compare these with the 2SLS predictability tests of Breitung and Demetrescu (2015). To that end, we simulate data from DGP1 under a variety of local alternatives. For the sake of space, we only report results for $\phi = \{-0.95, -0.50\}$, for a sample of size $T = 250$ and for four values of the persistence parameter, c, associated with x_t; specifically, $c = \{0, 10, 25, 50\}$. The slope parameter β is parameterised in (1) as $\beta = b/T$, with results reported for $b \in \{-20, -19, ..., 19, 20\}$.

Due to the large finite sample size distortions seen with the one-sided t_{zx}, t_{zx}^{EW} and $t_{zx}^{*,FRWB}$ tests discussed in section 5.1.1 for these combinations of c and ϕ, we only report local power results results for the two-sided t_{zx}, $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ tests and for the one-sided $t_{zx}^{*,RWB}$ test all of which have well controlled empirical size under DGP1. Results are also reported for the two-sided test of Breitung and Demetrescu (2015), denoted $t_{zx}^{*,2SLS}$, implemented with a fixed regressor wild bootsrap (the asymptotic validity of which is established in Demetrescu *et al.*, 2022) using the choice of instruments recommended by Breitung and Demetrescu (2015), namely the type-I fractionally integrated instrument $z_{1t} := (1-L)_+^{0.51} x_t$ and type-II sine instrument $z_{2t} := \mathsf{sin}(\frac{\pi}{2} t/T)$. The finite sample local power curves of these tests are graphed in Figure 1. Recalling from Remark 27 that the RWB and FRWB implementations of the IVX tests share the same asymptotic local power functions as the corresponding (size-adjusted) asymptotic IVX test, Figure 1 shows that this prediction from the limiting theory is borne out well even for a sample of size $T = 250$ with the power curves of the bootstrap and asymptotic two-sided tests being almost indistinguishable from each other for all of the values of c considered. The power dominance of the IVX tests, both one-sided and two-sided, over the two-sided 2SLS $t_{zx}^{*,2SLS}$ test is clearly seen in Figure 1, confirming the findings of Harvey *et al.* (2021). For alternatives where $\beta < 0$ the gains from using the left-tailed IVX tests over the two-sided IVX tests are also clearly seen in Figure 1, with the magnitude of the power gains from using the one-sided tests generally larger for $\phi = -0.95$ *vis-à-vis* $\phi = -0.5$, and greater the larger is c. For alternatives where $\beta > 0$, the gains from using a right-tailed IVX test over the two-tailed IVX test are much less obvious than for testing against $\beta < 0$, but are nonetheless still apparent for $c \geq 10$ when $\phi = -0.50$ and for $c \geq 25$ when $\phi = -0.95$.

5.2 Multiple Predictors - Full Sample Tests

We now investigate the finite sample behaviour of the asymptotic IVX test and its RWB and FRWB bootstrap counterparts in cases where multiple predictors are included in the predictive regression. For our analysis we use the same DGP as is considered in Xu and Guo (2020); that is,

$$y_t = \alpha + \boldsymbol{x}'_{t-1}\boldsymbol{\beta} + u_t, \qquad t = 1, \ldots, T, \tag{23}$$

$$\boldsymbol{x}_t = \boldsymbol{\Gamma}\boldsymbol{x}_{t-1} + \boldsymbol{v}_t, \qquad t = 0, \ldots, T, \tag{24}$$

where $\boldsymbol{x}_t := (x_{1,t}, ..., x_{K,t})'$ is a $K \times 1$ vector of predictor variables, $\boldsymbol{\beta}$ is a $K \times 1$ vector of parameters, $\alpha = 0.25$, $\boldsymbol{\Gamma}$ is the $K \times K$ diagonal matrix $\boldsymbol{\Gamma} := \mathrm{diag}(\rho, ..., \rho)$, and $(u_t, \boldsymbol{v}'_t)' \sim NIID(\boldsymbol{0}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_u^2 & \sigma_{u,v_1} & 0 & \cdots & 0 \\ \sigma_{u,v_1} & \sigma_{v_1}^2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_{v_2}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{v_K}^2 \end{pmatrix} \tag{25}$$

with $\sigma_u^2 = 0.037$, $\sigma_{u,v_1} = -0.035$, $\sigma_{v_1}^2 = ... = \sigma_{v_K}^2 = 0.045$. Notice, therefore, that the first predictor, $x_{1,t}$ is endogenous (with an endogeneity correlation parameter $\phi_1 = -0.83$), while the remaining predictors $x_{2,t}, ..., x_{K,t}$ are exogenous. For the autoregressive parameter we again consider $\rho = 1 - c/T$ with $c \in \{-5, 2.5, 0, 2.5, 5, 10, 25, 50, 75, 100, 125, 150, 200, 250\}$.

Table 3 reports the empirical rejection frequencies at a 5% significance level, for $T = 250$ and $T = 1000$ and for $K \in \{1, 3, 5, 10\}$, for the Wald-type IVX tests W_{zx} and W_{zx}^{EW} discussed in Remark 9, together with the RWB and FRWB bootstrap implementations of W_{zx}, denoted $W_{zx}^{*,RWB}$ and $W_{zx}^{*,FRWB}$, respectively, computed as described in Remark 22 (results for 1% and 10% significance levels are reported in Table D.6 of the supplementary appendix). In the context of $W_{zx}^{*,RWB}$, in Step 2 of the multivariate version of Algorithm 1 autoregressions of length $p + 1$ were fitted to each element of $\boldsymbol{x}_t$ with p selected in each case by BIC using the same range of values of p as were used in the simulations for a single predictor.

For $K = 1$ (the single predictor case), and in line with what was observed in section 5.1.1 for the two-sided tests based under DGP1, all of the Wald-based IVX statistics display empirical rejection frequencies close to the nominal level. Again, $W_{zx}^{*,RWB}$ displays the smallest size distortions among the tests considered. For instance, for a 5% significance level the rejection frequencies of $W_{zx}^{*,RWB}$ are in the range [0.042, 0.056] for $T = 250$ and [0.038, 0.056] for $T = 1000$, whereas for $W_{zx}^{*,FRWB}$, W_{zx}^{EW} and W_{zx} these are [0.037, 0.058], [0.045, 0.064], and [0.040, 0.060], respectively, when $T = 250$ and [0.034, 0.060], [0.036, 0.060] and [0.035, 0.059], respectively, when $T = 1000$.

However, it is as K increases that the significant advantage of the RWB becomes clear, particularly in the case where the predictors are strongly persistent. It is clear from the results that the $W_{zx}^{*,FRWB}$, W_{zx}^{EW} and W_{zx} tests are not reliable when the predictors are strongly persistent. The rejection frequencies we observe for W_{zx} are in line with those reported in Xu and Guo (2020) who also show that the quality of the prediction from the asymptotic theory deteriorates as the number of regressors, K, specified in the predictive regression increases. For instance, for $K = 3$ and $c < 0$; for $K = 5$ and $c < 2.5$; and for $K = 10$ and $c < 25$, even for $T = 1000$ all three of these tests display rejection frequencies larger than 15% at a 5% nominal level. For the smaller sample, $T = 250$, qualitatively similar size behaviour is observed (but with distortions of larger magnitude) for $W_{zx}^{*,FRWB}$ and W_{zx}. However, W_{zx}^{EW} becomes severely oversized as K increases, for all values of c. For instance, for $K = 10$, $T = 250$ and a 5% significance level, the smallest empirical rejection frequencies seen for this statistic is more than double the significance level considered. To illustrate the severity of the size distortions, observe from Table 3 that, for $K = 10$ unit root predictors ($c = 0$) and a 5% significance level, the empirical rejection frequencies of $W_{zx}^{*,FRWB}$, W_{zx}^{EW} and W_{zx} are 30.6%, 40.6% and 32.4%, respectively, for $T = 250$, and 29.5%, 30.0% and 28.0%, respectively for $T = 1000$. For mildly explosive predictors, the situation is even worse with empirical size in the region of 70% for each of $W_{zx}^{*,FRWB}$, W_{zx}^{EW} and W_{zx} when $K = 10$ and $c = -5$.

In contrast, the residual wild bootstrap based test, $W_{zx}^{*,RWB}$, controls empirical size much better than the other tests with empirical rejection frequencies acceptably close to the nominal level for all of the values of K considered. Some size distortions remain for values of $c \leq 5$, albeit unlike with the other tests these do not get appreciably worse as K increases. Moreover, in those cases where size distortions are seen with the $W_{zx}^{*,RWB}$ test, these are very much smaller than those seen for those cases with the other tests. Indeed, there are no entries in Table 3 where $W_{zx}^{*,RWB}$ displays an empirical size in excess of 10%, which compares very favourably with the other tests.

Finally, although not reported here, we also investigated the finite sample behaviour of the partial IVX t-type tests discussed in Remark 9. To summarise our findings, for both one-sided and two-sided implementations, the t-type tests associated with the exogenous predictors, $x_{2,t}, ..., x_{K,t}$, all displayed qualitatively similar finite sample size properties to those which were observed for the single predictive regression case for DGP1 with $\phi = 0$ (see Table D.6 of the supplementary appendix), i.e., the t-type tests display empirical rejection frequencies which are close to the nominal significance levels considered. For the t-type tests associated with the endogenous predictor, $x_{1,t}$, both one-sided and two-sided versions of the RWB implementation of the tests continued to display good finite sample size control, regardless of the number of predictors, K, and the value of c. In contrast, the empirical sizes of the other implementations of the tests, including the FRWB tests, deteriorated very badly as K increased, rendering these tests highly unreliable in practice.

5.3 Subsample Tests

We next summarise the findings from a Monte Carlo study of the finite sample performance of the subsample IVX-based predictability tests proposed in section 3.2. The full set of results can be found in the supplementary appendix: the empirical size results in Tables D.43–D.48, and the empirical local power results in Figures D.1–D.3. Results are reported for two-sided and one-sided tests, implemented with either the RWB or FRWB, together with the corresponding rolling, forward and backward recursive two-sided 2SLS-based tests of Demetrescu *et al.* (2022) which use a type-I instrument constructed as in (4) and the type-II sine instrument, $z_{2t} := \sin(\frac{\pi}{2}t/T)$, and are implemented using the FRWB. For the recursive sequences we set $\tau_L = 1/3$ and $\tau_U = 2/3$, respectively, and for the rolling sequences we set $\Delta\tau = 1/3$. The empirical size and local power simulations are based on 3000 and 1000 Monte Carlo replications, respectively, and $B = 399$ bootstrap replications. All other computational aspects are as outlined previously.

5.3.1 Empirical Size

The results in Tables D.43–D.48 pertain to data generated from DGP1, as described in Section 5.1.1, setting $\beta_t = 0$ for all t and $\rho = 1-c/T$, with $c \in \{-5, -2.5, 0, 2.5, 5, 10, 25, 50, 75, 100, 125, 150, 200, 250\}$, and $T \in \{250, 1000\}$.

Consider first the rolling tests. The results in Tables D.43–D.44 suggest that the correlation parameter ϕ has relatively little impact on the empirical size properties of the two-sided rolling tests. For $c \geq 0$ there is relatively little difference between the two-sided 2SLS test and the two-sided IVX-based tests, regardless of whether a RWB or FRWB implementation is used with the latter, with all of the tests displaying good finite sample size control. For $c < 0$ (locally explosive predictors) the two-sided FRWB IVX-based test is rather conservative while the 2SLS test is slightly over-sized. Turning to the one-sided rolling tests, both the lower- and upper-tail RWB based IVX tests display decent finite sample size control (the empirical sizes of the nominal 5% level lower-tailed and upper-tailed statistics across all the values of c considered lie in the range [0.026, 0.063] and [0.029, 0.064], respectively). In contrast, the rolling upper-tail FRWB IVX-based test displays a tendency to over-sizing, most notably when $c \geq 0$ and $\phi = -0.95$ (rejection frequencies at a nominal 5% level when $T = 250$ are between [0.065, 0.117] and between [0.090, 0.129] for $T = 1000$). This over-sizing moderates for smaller ϕ; however, Table D.43 shows that for c close to zero (regardless of the sample size) significant over-sizing is still observed in the FRWB IVX-based test. The lower-tailed FRWB IVX-based rolling test also displays notable under-sizing when the predictor is strongly persistent, which is severe for locally explosive processes. Notice also that these patterns of size-distortion in the one-sided FRWB IVX-based tests closely mirror the full sample FRWB (and asymptotic) IVX-based tests under DGP1 reported in section 5.1.1.

Turning to the recursive tests, the results in Tables D.45 - D.48 show similar patterns to those observed for the rolling tests. Specifically, for $c \geq 0$ there is little to choose between the two-sided 2SLS and IVX-based tests, with all of these tests displaying decent size control throughout. Differences again surface between the one-sided IVX-based tests: the RWB-based implementations of the tests displaying good size control across c, while the lower- and upper-tailed forward and backward FRWB IVX-based tests display size distortions. To illustrate the latter, the upper-tailed forward and backward recursive FRWB IVX-based tests tend to be over-sized, particularly when for a strongly persistent predictor with a high endogeneity correlation (e.g., for $\phi = -0.95$, the forward recursive test displays size between [0.071, 0.088] for $0 \leq c \leq 25$ and the backward recursive test between [0.074, 0.091] for $0 \leq c \leq 10$). The lower-tailed FRWB IVX-based recursive tests are again correspondingly undersized. As with the full sample IVX tests, our simulation results strongly support using RWB implementations of the subsample IVX tests rather than the FRWB.

5.3.2 Finite Sample Local Power

Figure D.1 (rolling tests) and Figures D.2 and D.3 (forward and backward recursive tests) graph finite sample local power functions of the subsample tests for data generated from DGP (1)–(3) under local alternatives satisfying $H_{1,b(\cdot)}$ of (6). We report results for $\phi = -0.95$ and $\phi = -0.5$, for the homoskedastic case, $\sigma^2_{ut} = \sigma^2_{vt} = 1$, for $T = 250$ and for $c = \{0, 10, 25, 50\}$. The slope parameter in (1) is parameterised as $\beta_t = b_t/T$, with $b_t = b$ with $b \in \{-40, -39, ..., -1, 1, ..., 39, 40\}$ for $t = 1, ..., \lfloor T/3 \rfloor$ and $b_t = 0$ for $t = \lfloor T/3 \rfloor + 1, ..., T$, such that a window of predictability occurs in the first third of the sample. We would therefore expect the rolling and forward recursive tests to display more power against this DGP than the backward recursive tests.

Consider the results for the rolling tests in Figure D.1. For testing against right-tailed alternatives (or equally, against left-tailed alternatives when ϕ is positive) there is little difference in general between those tests which are size-controlled. Exceptions occur for the case where $c = 0$ and $\phi = -0.95$ where the (FRWB) two-sided 2SLS-based test is slightly more powerful than the other tests, and where $c = 50$ and $\phi = -0.50$ where the one-sided RWB IVX-based test is slightly more powerful than the two-sided 2SLS and IVX tests. The pattern is very different when testing against left-tailed alternatives (right-tailed alternatives when ϕ is positive) where clear power gains over the two-sided tests are achieved by the one-sided IVX tests, except for $c = 0$ where the one-sided IVX test has almost identical power to the two-sided 2SLS-based test. The power gains for the left-sided IVX tests over the two-sided tests are larger for $\phi = -0.95$ *vis-à-vis* $\phi = -0.50$.

Turning to the forward and backward recursive tests in Figures D.2–D.3, as expected the forward recursive tests are the more powerful against this DGP and so we will focus our discussion on those tests. In contrast to the rolling tests discussed above, the gains to using the one-tailed IVX tests are most obvious when testing against right-sided alternatives, and again these power gains are larger for $\phi = -0.95$ *vis-à-vis* $\phi = -0.50$. The one-sided RWB IVX-based test clearly dominates the other tests against right-sided alternatives (noting that the FRWB IVX-based test is not size-controlled). For left-sided alternatives the one-sided IVX tests are significantly more powerful than the two-sided IVX tests but roughly as powerful as the two-sided 2SLS tests; albeit for $c = 0$ and $\phi = -0.95$ the 2SLS test is somewhat more powerful than the one-sided IVX test, although it should be recalled from Table D.47 that the 2SLS test is a little over-sized in this case.

6 Empirical Applications

6.1 Testing for Predictability in the Equity Premium

We first re-evaluate the predictability of the equity premium based on the predictors considered in the empirical case studies of Welch and Goyal (2008) and Campbell and Yogo (2006), using both one-sided and two-sided bootstrap and asymptotic tests. Specifically, we will first test for predictability in the log excess stock return, which is computed as the log of the monthly return on the S&P 500 index (including dividends) minus the log of the risk-free rate as our dependent variable, based on the predictors in Welch and Goyal, 2008), namely: the log dividend-price ratio (dp); the log dividend yield (dy); the log earnings-price ratio (ep); the log dividend-payout ratio (de); the stock return variance (svar); the book-to-market ratio (bm); the net equity expansion (ntis); the treasury bill rate (tbl); the long-term yield (lty); the long-term return (ltr); the term spread (tms); the default yield spread (dfy); the default return spread (dfr); and inflation (infl); see Welch and Goyal (2008) for full details on how these predictors are generated. We then subsequently revisit the empirical analysis of Campbell and Yogo (2006) who also test for predictability in excess returns using as putative predictors: the dividend-price ratio (dp); the earnings-price ratio (ep); the three-month T-bill rate (r_3), and a measure of the long-short yield spread ($\mathsf{y} - \mathsf{r}_1$); see Campbell and Yogo (2006) for full data descriptions.[10]

Table 4 presents bootstrap and asymptotic p-values for both one-sided and two-sided IVX predictability tests and the two-sided 2SLS predictability tests of Breitung and Demetrescu (2015) (again using their recommended fractional type-I and sine type-II instruments), in each case computed from predictive regressions with a single predictor based on Welch and Goyal's monthly data (Panel A) and on Campbell and Yogo's data (Panel B). The results in Panel A are obtained from a sample of monthly data from January 1927 to December 2020 ($T = 1128$) and in Panel B for samples of annual data from 1926 to 2002 ($T = 77$), quarterly data from the 4th quarter of 1926 to the 4th quarter of 2002 ($T = 305$) and monthly data from December 1926 to December 2002 ($T = 913$). The asymptotic IVX and 2SLS tests are computed with Eicker-White standard errors to allow for heteroskedasticity in the innovations. The bootstrap 2SLS test is implemented with the fixed regressor wild bootstrap while the bootstrap IVX tests are implemented using the residual wild bootstrap, in each case using conventional standard errors. For each predictor, Table 4

[10]The Welch-Goyal data (updated data up to 2020) were obtained from `https://sites.google.com/view/agoyal145/` and the Campbell-Yogo data from `https://sites.google.com/site/motohiroyogo/research/asset-pricing`.

also reports both OLS and IVX estimates of the predictive regression slope parameter, β (denoted $\hat{\beta}_{OLS}$ and $\hat{\beta}_{IVX}$, respectively), together with OLS estimates of the dominant AR root ($\hat{\rho}$) for each predictor and estimates ($\hat{\phi}$) of the endogeneity correlation.[11]

Consider first Panel A. The estimated endogenity correlation is relatively small (between -0.297 and 0.185) for all of the predictors, except dp, ep and bm for which $\hat{\phi}$ is large and negative: -0.980, -0.767 and -0.821, respectively. The dominant estimated AR root is also close to unity for most of the predictors predictors: $\hat{\rho} \in [0.962, 1.006]$, with the exception of svar, ltr, dfr and infl for which $\hat{\rho}$ is 0.577, 0.043, -0.102 and 0.480, respectively. Turning to the outcomes of the predictability tests, for both dy and ep we see that the right-sided asymptotic IVX test, $t_{zx}^{EW(+)}$, yields significant evidence of positive predictability at the 10% level, while in both cases the two-sided t_{zx}^{EW} test fails to reject at the 10% level. For both of these series the right-sided RWB $t_{zx}^{*,RWB(+)}$ fails to reject at the 10% level suggesting that these are most likely spurious rejections attributable to the finite-sample oversize of the $t_{zx}^{EW(+)}$ test seen in the simulations in Table 1 for strongly persistent predictors. Among the other putative predictors, the left-sided RWB $t_{zx}^{*,RWB(-)}$ test find evidence of (negative) predictability at the 5% level for tbl and lty and at the 10% level for ntis, in each case statistically stronger evidence than is provided by the corresponding two-sided $t_{zx}^{*,RWB}$ test. Similar conclusions are drawn for the tbl and lty predictors using the asymptotic IVX tests, while no rejections are seen with either the two-sided or one-sided asymptotic IVX tests in the case of ntis. For ltr (infl) both the right-sided RWB $t_{zx}^{*,RWB(+)}$ (left-sided RWB $t_{zx}^{*,RWB(-)}$) test and the right-sided asymptotic $t_{zx}^{EW(+)}$ (left-sided asymptotic $t_{zx}^{EW(-)}$) test find evidence of positive (negative) predictability at the 10% level, not found in the corresponding two-sided tests. To summarise the results in Panel A, we find rather stronger evidence of predictability when using one-sided tests, with 5 (2) of the predictors being found to be statistically significant at the 10% (5%) level using one-sided RWB bootstrap tests, compared to 2 (0) when using the two-sided RWB bootstrap tests. Finally, consistent with the local power results in Figure 1, the 2SLS tests are insignificant at the 10% level for all of the predictors, regardless of whether asymptotic or bootstrap p-values are used.

Consider next Panel B. As with the predictors in Panel A, the Campbell and Yogo (2006) predictors again appear to be strongly persistent in general. Moreover, although $\hat{\phi}$ is relatively small for both r_3 and $\mathsf{y} - \mathsf{r}_1$ (in line with the corresponding Welch and Goyal predictors), for dp and ep very strong negative endogeneity correlations are estimated: for monthly data $\hat{\phi}$ for dp and ep is -0.954 and -0.987, respectively (reducing to -0.721 and -0.957, respectively for annual data). With annual data, the null hypothesis of no predictability is rejected for both dp and ep at the 5% level, regardless of whether one uses right-sided or two-sided tests and for both the asymptotic and RWB implementations of the IVX test. As the data frequency increases the strength of these rejections tends to decline. For both quarterly and monthly data in the case of ep, both the RWB and asymptotic tests now only reject at the 10% level for the two-sided tests and at around the 5% level for right-sided tests. For dp, both the right-sided and two-sided RWB IVX-based tests fail to reject at the 10% level; some rejections are still seen for both quarterly and monthly data with the right-sided asymptotic IVX test, albeit we should treat the results from the latter with a degree of caution given the high persistence and large negative endogeneity correlation observed for dp (cf Table 1). For the monthly data the left-sided RWB and asymptotic IVX tests both indicate rejection of the null hypothesis of no predictability for r_3 at the 10% level, while the right-sided

[11] Here, $\hat{\rho}$ is computed from an error correction parameterisation of an $AR(p)$ model fitted to the predictor, in which p is chosen applying the BIC over $p \in \{0, ..., \lfloor 4(T/100)^{0.25} \rfloor\}$, while $\hat{\phi}$ is the OLS estimate of the correlation of the predictive regression residuals and the residuals from the fitted $AR(p)$ model.

RWB test also indicates rejection of the null for $\mathsf{y} - \mathsf{r}_1$ at the 10% level; in none of these cases does the corresponding two-sided version of the test signal predictability at the 10% level. Finally, as with the results in Panel A, the 2SLS tests are insignificant at the 10% level for all of the predictors considered, regardless of whether asymptotic or bootstrap p-values are used.

6.2 Testing for Bubbles in Foreign Exchange Rates

We next re-visit the problem of testing for speculative bubbles in the U.K. £ to U.S. $ foreign exchange market considered in Pavlidis *et al.* (2017), using monthly data (downloaded from the Bank of England, `www.bankofengland.co.uk`) on spot and forward rates for the period from January 1999 to July 2021 ($T = 271$), the start date coinciding with the introduction of the Euro.

Fama (1984) proposes the following regression as a basis for testing for efficiency in foreign exchange markets,

$$s_{t+h} - f_{t,h} = \alpha_h + \beta_h(f_{t,h} - s_t) + u_{t+h}, \tag{26}$$

where $s_{t+h} - f_{t,h}$ is typically referred to as the excess return (or forecast error) (see, for example, Maynard, 2006) and $f_{t,h} - s_t$ is the forward premium, where s_t is (the log of) the spot exchange rate at time t and $f_{t,h}$ is (the log of) the forward rate at time t for maturity at time $t + h$, $h \geq 1$.

In the context of (26), as discussed in Pavlidis *et al.* (2017), the efficient market hypothesis corresponds to $\beta_h = 0$, while if an exchange rate bubble is present in any time period then $\beta_h > 0$. Pavlidis *et al.* (2017) therefore apply right-tailed rolling subsample implementations of the IVX tests of Kostakis *et al.* (2015) to test the null hypothesis $\beta_h = 0$ against the alternative hypothesis that β_h in (26) is positive in at least one subsample of the data. In the context of this testing problem it is important to use only right-tailed tests because, as is well known in this literature, the estimate of β_h can suffer from a severe negative finite sample bias when $\beta_h = 0$; the so-called *forward bias puzzle*. A number of explanations have been posited for this phenomenon including the negative correlation between the risk premium and the forward premium; see Maynard (2003). Consequently, a two-sided test might be inappropriate as a rejection could be due to either a downward bias effecting large negative statistics in some subsamples, or a genuine bubble episode.

Like Pavlidis *et al.* (2017) we report results for the three periods to maturity available in the dataset, namely, one, three, and six months: $h = \{1, 3, 6\}$. Where $h > 1$ we follow Pavlidis *et al.* (2017, Appendix A.2, pp. 1221-1223) and estimate the parameters of (26) using a reverse regression (Phillips and Lee, 2013) type approach. Fitting an autoregressive model (with a constant), we found the forward premium, $(f_{t,h} - s_t)$, to be a strongly persistent time series regardless of the maturity period; in particular the dominant autoregressive root was estimated to be $\hat{\rho} = 0.9635$ for $h = 1$, $\hat{\rho} = 0.9821$ for $h = 3$, and $\hat{\rho} = 0.9880$ for $h = 6$. The estimated correlation parameter, $\hat{\phi}$, was found to be relatively small and negative for $h = 1$, but increases in absolute value as h increases: $\hat{\phi} = -0.0861$ for $h = 1$, $\hat{\phi} = -0.3100$ for $h = 3$, and $\hat{\phi} = -0.3686$ for $h = 6$. The estimates of ρ and ϕ where calculated as outlined in footnote 11.

Table 5 reports bootstrap (both RWB and FRWB) p-values for the maximum rolling, and forward and backward recursive subsample IVX statistics from section 3.2, in each case implemented as upper-tailed (right-sided) tests, using $B = 9999$ bootstrap replications. Results are reported for four values of the tuning parameters $\Delta\tau$ (the *window fraction* used for the sequence of rolling statistics) and τ_L and $(1 - \tau_U)$ (the *warm-in* parameters for the forward and backward recursive sequences, respectively), namely 1/6, 1/4, 1/3, and 1/2.

The results in Table 5 show that for $h = 1$ none of the subsample IVX tests provide evidence, at any conventional significance level, of exuberant behaviour in the foreign exchange rate. However, for the longer maturities considered, $h = 3$ and $h = 6$, statistically significant results are found in the case of the rolling tests, suggesting the presence of a potential bubble. Specifically, for both $\Delta\tau = 1/6$ and $1/2$ the rolling tests signal the presence of exuberant behaviour in the foreign exchange rate, although they do not for $\Delta\tau = \{1/4, 1/3\}$. The strongest rejections are observed for $h = 6$. In this case, the RWB based rolling tests find evidence of exuberant behaviour for all four window widths. The FRWB based rolling tests also reject the null that $\beta_6 = 0$ for all window widths, except $\Delta\tau = 1/4$. The forward and backward recursive tests do not reject the null that $\beta_h = 0$ for any of the warm-in parameters and maturities considered, suggesting that the start and end points of the bubble episode are likely bounded away from the end points of the full sample.

Pavlidis *et al.* (2017) find no evidence of speculative bubbles for any of the maturity periods considered for data covering the period January 1979 to December 2013, and so, although we consider a different sample period, it would seem instructive to investigate where in the sample the rejections that we find above occur. To that end, Figure 2 plots the sequence of rolling IVX subsample statistics computed for a window fraction $\Delta\tau = 1/2$ and maturity period $h = 6$. Plotted on the graph are the 5% and 10% RWB critical values for the maximum of the rolling test statistics together with the upper-tail 5% and 10% pointwise RWB critical values. This plot indicates that the rejection of the null hypothesis $H_0 : \beta_6 = 0$ by the maximum of the rolling tests occurs between January 2016 and November 2016 (after the end of the sample period considered in Pavlidis *et al.* (2017)) when a 10% significance level is considered and between February 2016 and September 2016 for a 5% significance level. The sequence of rolling statistics displays a steady and sustained increase in magnitude from mid-2013 onwards, with the statistics exceeding the pointwise 10% (5%) significance level between June 2015 and May 2019 (September 2015 and August 2018). These findings are, on the face of it, consistent with a bubble episode in the U.K. pound - U.S. dollar exchange rate which collapsed at or around the time of the Brexit vote in summer 2016.

7 Conclusions

We have extended the IVX-based predictability tests of Kostakis *et al.* (2015) in three directions. First, we have shown that, provided either a suitable bootstrap implementation is employed or Eicker-White standard errors are used, these tests still deliver asymptotically valid inference, regardless of the degree of persistence or endogeneity of the predictor, under considerably weaker assumptions on the innovations, including quite general forms of conditional and unconditional heteroskedasticity, than required by Kostakis *et al.* (2015) in their analysis. Second, we have developed asymptotically valid residual and fixed regressor wild bootstrap implementations of the IVX tests. Simulation evidence has been provided which demonstrates that tests based around a residual wild bootstrap resampling scheme perform particularly well in finite samples. Third, we have shown how sub-sample implementations of the IVX approach can be used to develop asymptotically valid one-sided and two-sided tests for temporary windows of predictability.

We finish with three suggestions for further research. First, we have focused on the case of a single predictive regressor. As we have noted, the methods we have discussed readily extend to the case of multiple regressors, provided, as assumed in Kostakis *et al.* (2015), they all belong to the same persistence class. However, based on the results in this paper, we conjecture that our bootstrap IVX tests should also retain asymptotic validity in the scenario where some of the regressors are weakly persistent and others strongly persistent, and where the strongly persistent regressors could cointegrate. The practitioner would not need to know which of the regressors were weakly persistent and which were strongly persistent, or the form of any cointegrating relations present. A formal proof of this conjecture is likely very involved but constitutes an important next step in this research agenda, with technical material in this paper providing important groundwork.

Second, unlike Kostakis *et al.* (2015), we have not discussed the case of mildly integrated regressors. While we hold it to be plausible that our results may be extended to cover the mildly integrated case, the corresponding derivations may be quite lengthy and we leave them for further work. Among other things, one would have to consider several distinct cases depending on whether the IVX filter depends on a coefficient which is closer to or further away from unity than the largest autoregressive root of the predictor, together with the interplay of the true regressor's persistence with the mixed fourth moments of the innovations series, which, as shown by Proposition 3, is quite different under weak and strong persistence.

Third, the finite sample efficacy of the residual wild bootstrap IVX tests proposed in this paper will depend, in part, on the finite sample properties of the autoregressive parameter estimates obtained in Step 2 of Algorithm 1. The OLS estimates we have employed are known to suffer from non-negligible finite sample biases. It might be useful to explore a refinement of Algorithm 1 based on the bootstrap-after-bootstrap approach of Kilian (1998) to investigate if this can further improve on the finite sample properties of our proposed bootstrap tests.

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Table 1: Empirical rejection frequencies at 5% significance level of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad \phi; \quad \phi \quad 1]$.

		Left-sided tests								**Right-sided tests**								**Two-sided tests**							
ϕ	c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
		$T = 250$				$T = 1000$				$T = 250$				$T = 1000$				$T = 250$				$T = 1000$			
-0.95	-5	0.046	0.004	0.004	0.003	0.045	0.002	0.003	0.003	0.046	0.074	0.080	0.073	0.039	0.064	0.065	0.064	0.048	0.038	0.044	0.039	0.040	0.030	0.032	0.031
	-2.5	0.045	0.000	0.000	0.001	0.046	0.000	0.000	0.000	0.041	0.094	0.097	0.093	0.041	0.092	0.092	0.091	0.038	0.040	0.048	0.044	0.037	0.042	0.043	0.042
	0	0.041	0.001	0.001	0.001	0.042	0.001	0.001	0.001	0.053	0.105	0.114	0.110	0.050	0.103	0.104	0.102	0.047	0.051	0.057	0.053	0.041	0.050	0.050	0.049
	2.5	0.062	0.005	0.005	0.005	0.060	0.006	0.006	0.006	0.064	0.112	0.116	0.115	0.059	0.107	0.108	0.107	0.053	0.058	0.062	0.060	0.050	0.056	0.057	0.058
	5	0.068	0.010	0.011	0.010	0.068	0.013	0.014	0.013	0.062	0.107	0.116	0.112	0.059	0.105	0.106	0.106	0.054	0.058	0.063	0.060	0.052	0.056	0.058	0.058
	10	0.064	0.019	0.019	0.018	0.064	0.022	0.021	0.021	0.062	0.097	0.102	0.099	0.059	0.097	0.098	0.098	0.055	0.060	0.066	0.060	0.055	0.061	0.063	0.062
	25	0.057	0.029	0.030	0.028	0.061	0.030	0.030	0.031	0.057	0.078	0.084	0.080	0.055	0.081	0.082	0.081	0.056	0.056	0.060	0.058	0.052	0.056	0.057	0.056
	50	0.056	0.034	0.036	0.035	0.057	0.035	0.035	0.035	0.052	0.067	0.072	0.067	0.051	0.070	0.071	0.070	0.051	0.051	0.054	0.052	0.051	0.053	0.054	0.053
	75	0.056	0.037	0.038	0.037	0.055	0.039	0.039	0.038	0.053	0.064	0.068	0.065	0.051	0.067	0.068	0.067	0.049	0.047	0.052	0.049	0.051	0.052	0.054	0.052
	100	0.054	0.038	0.040	0.038	0.055	0.040	0.040	0.040	0.053	0.061	0.065	0.062	0.051	0.065	0.066	0.065	0.049	0.048	0.052	0.050	0.051	0.051	0.053	0.052
	125	0.054	0.039	0.042	0.041	0.054	0.041	0.042	0.041	0.052	0.060	0.063	0.060	0.051	0.063	0.063	0.063	0.050	0.049	0.053	0.051	0.050	0.051	0.052	0.052
	150	0.055	0.043	0.046	0.042	0.053	0.043	0.043	0.043	0.053	0.056	0.060	0.059	0.051	0.061	0.061	0.061	0.051	0.049	0.054	0.052	0.050	0.052	0.053	0.051
	200	0.054	0.046	0.048	0.045	0.053	0.044	0.044	0.042	0.050	0.054	0.056	0.053	0.050	0.059	0.060	0.059	0.050	0.048	0.054	0.050	0.052	0.051	0.052	0.051
	250	0.054	0.048	0.051	0.048	0.053	0.044	0.044	0.044	0.051	0.051	0.055	0.053	0.050	0.059	0.059	0.059	0.049	0.048	0.053	0.050	0.050	0.051	0.051	0.051
-0.90	-5	0.049	0.004	0.005	0.004	0.047	0.003	0.003	0.003	0.044	0.074	0.079	0.073	0.041	0.066	0.067	0.064	0.045	0.036	0.044	0.038	0.041	0.032	0.034	0.033
	-2.5	0.047	0.000	0.001	0.001	0.048	0.000	0.000	0.000	0.042	0.093	0.100	0.094	0.040	0.092	0.091	0.090	0.037	0.042	0.047	0.044	0.037	0.040	0.043	0.041
	0	0.039	0.001	0.001	0.001	0.040	0.002	0.002	0.002	0.054	0.104	0.112	0.108	0.051	0.098	0.101	0.100	0.048	0.052	0.057	0.053	0.043	0.048	0.049	0.048
	2.5	0.059	0.005	0.005	0.005	0.058	0.007	0.007	0.007	0.063	0.110	0.114	0.112	0.058	0.103	0.106	0.105	0.055	0.059	0.062	0.061	0.050	0.056	0.057	0.057
	5	0.066	0.010	0.011	0.010	0.065	0.014	0.014	0.014	0.062	0.106	0.114	0.109	0.059	0.100	0.102	0.102	0.054	0.059	0.064	0.060	0.051	0.058	0.059	0.058
	10	0.063	0.019	0.020	0.020	0.063	0.023	0.022	0.022	0.061	0.094	0.100	0.096	0.059	0.094	0.095	0.094	0.055	0.057	0.065	0.061	0.054	0.059	0.061	0.060
	25	0.055	0.030	0.032	0.030	0.059	0.031	0.031	0.031	0.058	0.078	0.082	0.079	0.055	0.080	0.080	0.078	0.054	0.055	0.060	0.057	0.052	0.056	0.056	0.055
	50	0.054	0.033	0.036	0.034	0.056	0.036	0.036	0.036	0.053	0.067	0.071	0.068	0.051	0.069	0.070	0.069	0.049	0.050	0.054	0.052	0.051	0.053	0.054	0.052
	75	0.054	0.038	0.038	0.037	0.054	0.039	0.040	0.039	0.052	0.064	0.067	0.064	0.052	0.066	0.066	0.066	0.048	0.049	0.053	0.049	0.051	0.053	0.053	0.053
	100	0.050	0.037	0.040	0.039	0.054	0.040	0.041	0.040	0.052	0.062	0.065	0.062	0.053	0.065	0.064	0.064	0.049	0.048	0.052	0.050	0.052	0.052	0.052	0.053
	125	0.053	0.041	0.043	0.041	0.055	0.044	0.043	0.043	0.053	0.059	0.062	0.059	0.051	0.060	0.062	0.062	0.052	0.050	0.054	0.052	0.051	0.052	0.053	0.053
	150	0.054	0.043	0.045	0.044	0.055	0.045	0.044	0.044	0.052	0.056	0.059	0.058	0.051	0.061	0.060	0.060	0.049	0.050	0.054	0.052	0.051	0.052	0.054	0.052
	200	0.054	0.046	0.049	0.047	0.054	0.045	0.046	0.046	0.051	0.055	0.058	0.055	0.053	0.059	0.061	0.061	0.049	0.048	0.052	0.050	0.049	0.050	0.053	0.051
	250	0.055	0.048	0.051	0.048	0.052	0.046	0.046	0.046	0.048	0.049	0.053	0.051	0.051	0.058	0.059	0.060	0.049	0.049	0.054	0.049	0.050	0.050	0.051	0.050
-0.50	-5	0.053	0.019	0.024	0.019	0.050	0.018	0.018	0.018	0.046	0.072	0.079	0.072	0.047	0.070	0.072	0.070	0.048	0.043	0.055	0.042	0.047	0.042	0.043	0.041
	-2.5	0.050	0.006	0.007	0.005	0.048	0.004	0.004	0.004	0.053	0.101	0.107	0.101	0.047	0.095	0.096	0.094	0.049	0.050	0.059	0.051	0.043	0.047	0.047	0.045
	0	0.030	0.005	0.006	0.006	0.032	0.009	0.008	0.008	0.061	0.097	0.102	0.096	0.056	0.091	0.090	0.091	0.048	0.051	0.057	0.053	0.045	0.049	0.050	0.048
	2.5	0.045	0.016	0.017	0.016	0.049	0.018	0.018	0.018	0.061	0.090	0.096	0.090	0.058	0.087	0.087	0.086	0.052	0.054	0.059	0.055	0.050	0.052	0.053	0.052
	5	0.050	0.023	0.024	0.023	0.054	0.024	0.024	0.025	0.061	0.081	0.087	0.085	0.057	0.082	0.081	0.080	0.052	0.055	0.059	0.057	0.050	0.053	0.054	0.053
	10	0.052	0.030	0.031	0.030	0.053	0.031	0.031	0.031	0.056	0.074	0.078	0.076	0.053	0.073	0.074	0.074	0.052	0.053	0.057	0.055	0.050	0.052	0.054	0.054
	25	0.050	0.036	0.038	0.036	0.053	0.039	0.039	0.039	0.053	0.065	0.069	0.067	0.052	0.064	0.065	0.063	0.049	0.050	0.054	0.052	0.049	0.050	0.052	0.051
	50	0.048	0.039	0.040	0.039	0.051	0.042	0.042	0.043	0.054	0.063	0.066	0.064	0.049	0.057	0.059	0.058	0.049	0.050	0.054	0.050	0.054	0.052	0.053	0.053
	75	0.050	0.042	0.045	0.043	0.053	0.045	0.045	0.045	0.055	0.060	0.065	0.062	0.049	0.056	0.057	0.057	0.049	0.049	0.054	0.052	0.053	0.053	0.054	0.053
	100	0.049	0.043	0.045	0.043	0.052	0.047	0.046	0.045	0.054	0.059	0.063	0.061	0.049	0.056	0.059	0.057	0.050	0.048	0.054	0.052	0.052	0.052	0.053	0.052
	125	0.051	0.044	0.046	0.045	0.051	0.045	0.046	0.045	0.055	0.059	0.061	0.059	0.049	0.054	0.056	0.055	0.051	0.050	0.055	0.050	0.052	0.052	0.052	0.053
	150	0.051	0.046	0.048	0.047	0.052	0.046	0.046	0.046	0.055	0.057	0.061	0.058	0.050	0.055	0.055	0.054	0.051	0.049	0.054	0.052	0.052	0.052	0.053	0.054
	200	0.052	0.047	0.050	0.048	0.053	0.047	0.046	0.046	0.053	0.054	0.057	0.054	0.050	0.054	0.054	0.054	0.049	0.049	0.055	0.052	0.052	0.052	0.053	0.053
	250	0.053	0.049	0.052	0.051	0.052	0.048	0.047	0.046	0.051	0.052	0.055	0.051	0.050	0.053	0.054	0.055	0.049	0.049	0.053	0.050	0.050	0.050	0.051	0.051

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4.

Table 2: Empirical rejection frequencies at 5% significance level of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP2 (ARCH with Leverage Effects):** $\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \boldsymbol{\psi}_t$ with $\boldsymbol{\psi}_t = (a_t; e_t)' = (\varepsilon_{1t}\sqrt{1 + \frac{1}{2}a_{t-1}^2 \mathbb{I}_{\{a_{t-1}<0\}}}; \varepsilon_{2t})'$ and $(\varepsilon_{1t}, \varepsilon_{2t})' \sim NIID(0, \mathbf{I}_2)$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	$T = 250$				$T = 1000$			
				Left-sided tests				
5	0.062	0.099	0.106	0.107	0.060	0.097	0.101	0.102
10	0.059	0.088	0.096	0.099	0.059	0.090	0.092	0.093
25	0.059	0.076	0.081	0.092	0.055	0.076	0.075	0.080
50	0.058	0.067	0.074	0.089	0.053	0.066	0.068	0.074
75	0.060	0.062	0.069	0.090	0.055	0.064	0.066	0.075
100	0.060	0.061	0.067	0.089	0.057	0.061	0.063	0.076
125	0.059	0.060	0.066	0.088	0.058	0.061	0.062	0.078
150	0.059	0.058	0.063	0.085	0.059	0.061	0.063	0.080
200	0.057	0.056	0.060	0.082	0.061	0.060	0.063	0.085
250	0.053	0.053	0.057	0.078	0.062	0.062	0.063	0.088
				Right-sided tests				
5	0.058	0.014	0.015	0.015	0.057	0.013	0.013	0.013
10	0.059	0.021	0.022	0.024	0.057	0.021	0.020	0.021
25	0.061	0.030	0.030	0.039	0.057	0.030	0.030	0.034
50	0.062	0.037	0.038	0.053	0.058	0.037	0.037	0.045
75	0.061	0.039	0.041	0.061	0.061	0.041	0.040	0.052
100	0.059	0.037	0.040	0.065	0.060	0.041	0.041	0.055
125	0.057	0.039	0.041	0.067	0.061	0.042	0.042	0.058
150	0.059	0.040	0.042	0.071	0.060	0.042	0.042	0.062
200	0.056	0.041	0.045	0.072	0.062	0.042	0.042	0.067
250	0.054	0.043	0.047	0.075	0.062	0.042	0.044	0.071
				Two-sided tests				
5	0.053	0.055	0.065	0.063	0.051	0.056	0.058	0.058
10	0.051	0.053	0.060	0.062	0.053	0.056	0.058	0.059
25	0.056	0.052	0.057	0.069	0.053	0.053	0.055	0.059
50	0.059	0.050	0.056	0.081	0.054	0.051	0.052	0.062
75	0.061	0.051	0.057	0.090	0.059	0.052	0.054	0.069
100	0.063	0.053	0.058	0.095	0.061	0.055	0.054	0.074
125	0.062	0.051	0.059	0.098	0.064	0.054	0.055	0.079
150	0.060	0.051	0.057	0.097	0.064	0.054	0.056	0.084
200	0.058	0.050	0.056	0.096	0.068	0.053	0.056	0.089
250	0.053	0.051	0.054	0.092	0.068	0.052	0.055	0.097

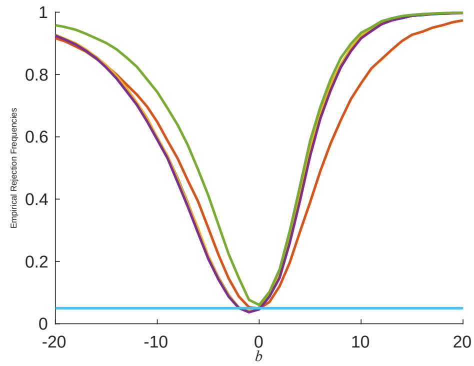
Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4.

Table 3: Empirical rejection frequencies at 5% significance level of Wald-type IVX-based tests for predictability in a multiple predictive regression context with $K \in \{1, 3, 5, 10\}$ predictors, for sample sizes $T = 250$ and $T = 1000$.

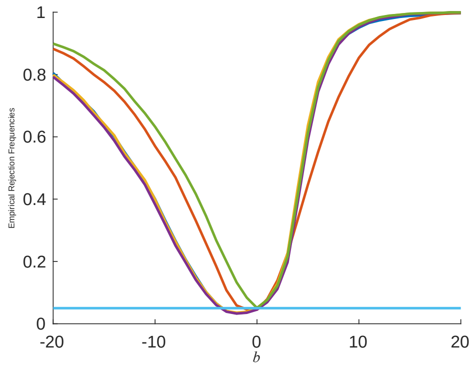
K	c	$W_{zx}^{*,\mathrm{RWB}}$	$W_{zx}^{*,\mathrm{FRWB}}$	W_{zx}^{EW}	W_{zx}	$W_{zx}^{*,\mathrm{RWB}}$	$W_{zx}^{*,\mathrm{FRWB}}$	W_{zx}^{EW}	W_{zx}
		$T = 250$				$T = 1000$			
1	-5	0.048	0.037	0.045	0.040	0.043	0.034	0.036	0.035
	-2.5	0.042	0.045	0.051	0.048	0.038	0.042	0.043	0.043
	0	0.050	0.053	0.059	0.056	0.045	0.051	0.051	0.050
	2.5	0.054	0.058	0.062	0.059	0.051	0.054	0.056	0.056
	5	0.056	0.058	0.064	0.060	0.054	0.059	0.060	0.058
	10	0.054	0.056	0.061	0.056	0.055	0.060	0.060	0.059
	25	0.055	0.057	0.060	0.057	0.056	0.059	0.059	0.059
	50	0.054	0.055	0.059	0.056	0.055	0.059	0.059	0.058
	75	0.055	0.055	0.060	0.056	0.056	0.057	0.058	0.058
	100	0.055	0.054	0.059	0.055	0.056	0.056	0.057	0.058
	125	0.055	0.054	0.058	0.054	0.056	0.056	0.057	0.056
	150	0.053	0.051	0.056	0.052	0.055	0.053	0.055	0.055
	200	0.052	0.051	0.056	0.052	0.054	0.054	0.053	0.052
	250	0.051	0.049	0.055	0.051	0.053	0.052	0.053	0.053
3	-5	0.085	0.352	0.385	0.366	0.083	0.346	0.354	0.346
	-2.5	0.097	0.176	0.193	0.177	0.092	0.162	0.159	0.155
	0	0.075	0.105	0.117	0.104	0.071	0.096	0.096	0.095
	2.5	0.067	0.086	0.103	0.090	0.063	0.081	0.084	0.083
	5	0.059	0.077	0.095	0.083	0.060	0.076	0.079	0.077
	10	0.054	0.066	0.083	0.071	0.057	0.072	0.078	0.075
	25	0.052	0.061	0.075	0.066	0.056	0.064	0.067	0.065
	50	0.053	0.057	0.070	0.061	0.053	0.058	0.061	0.059
	75	0.053	0.053	0.069	0.058	0.050	0.055	0.058	0.055
	100	0.051	0.053	0.069	0.057	0.048	0.052	0.054	0.051
	125	0.052	0.054	0.070	0.058	0.048	0.049	0.054	0.050
	150	0.052	0.054	0.069	0.058	0.047	0.049	0.052	0.049
	200	0.052	0.055	0.071	0.059	0.046	0.048	0.051	0.048
	250	0.053	0.055	0.071	0.060	0.046	0.048	0.050	0.048
5	-5	0.074	0.402	0.466	0.421	0.074	0.398	0.408	0.403
	-2.5	0.091	0.239	0.281	0.241	0.091	0.237	0.238	0.230
	0	0.082	0.157	0.186	0.156	0.085	0.152	0.154	0.148
	2.5	0.069	0.120	0.156	0.129	0.069	0.118	0.126	0.117
	5	0.063	0.105	0.138	0.116	0.063	0.104	0.110	0.105
	10	0.062	0.086	0.120	0.098	0.058	0.089	0.096	0.092
	25	0.053	0.067	0.100	0.080	0.052	0.069	0.077	0.071
	50	0.052	0.059	0.089	0.069	0.049	0.057	0.064	0.059
	75	0.051	0.055	0.085	0.063	0.050	0.055	0.062	0.057
	100	0.049	0.053	0.082	0.062	0.051	0.056	0.060	0.057
	125	0.049	0.053	0.080	0.062	0.052	0.055	0.060	0.057
	150	0.046	0.052	0.078	0.061	0.051	0.055	0.059	0.056
	200	0.047	0.051	0.079	0.060	0.051	0.052	0.057	0.054
	250	0.044	0.049	0.077	0.058	0.051	0.052	0.058	0.054
10	-5	0.058	0.513	0.635	0.559	0.060	0.502	0.526	0.501
	-2.5	0.072	0.398	0.505	0.425	0.076	0.384	0.394	0.371
	0	0.087	0.306	0.406	0.324	0.091	0.295	0.300	0.280
	2.5	0.075	0.238	0.342	0.262	0.078	0.229	0.244	0.224
	5	0.067	0.191	0.301	0.225	0.068	0.188	0.211	0.191
	10	0.060	0.141	0.244	0.175	0.061	0.147	0.166	0.151
	25	0.050	0.089	0.174	0.118	0.057	0.101	0.119	0.108
	50	0.048	0.067	0.142	0.091	0.057	0.081	0.096	0.085
	75	0.046	0.060	0.129	0.081	0.055	0.071	0.085	0.077
	100	0.046	0.056	0.120	0.077	0.055	0.066	0.080	0.071
	125	0.043	0.053	0.117	0.074	0.055	0.061	0.077	0.067
	150	0.042	0.052	0.116	0.071	0.052	0.058	0.075	0.064
	200	0.039	0.049	0.116	0.070	0.053	0.057	0.070	0.063
	250	0.036	0.050	0.116	0.072	0.051	0.055	0.070	0.061

Note: W_{zx} and W_{zx}^{EW} are the Wald-type IVX-based statistics discussed in Remark 9 of the main text, and $W_{zx}^{*,RWB}$ and $W_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) versions of W_{zx} computed as described in Algorithms 1 and 2 of Section 4 of the main text.

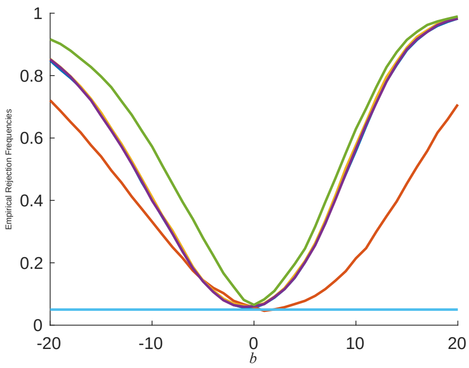
Figure 1: Power plots for two-sided $t_{zx}^{*,FRWB}$, $t_{zx}^{*,2SLS}$, t_{zx}, $t_{zx}^{*,RWB}$ tests and the one-sided $t_{zx}^{*,RWB}$ tests for predictability. Data generated from DGP1 with $\phi = \{-0.95, -0.50\}$ and for $T = 250$.



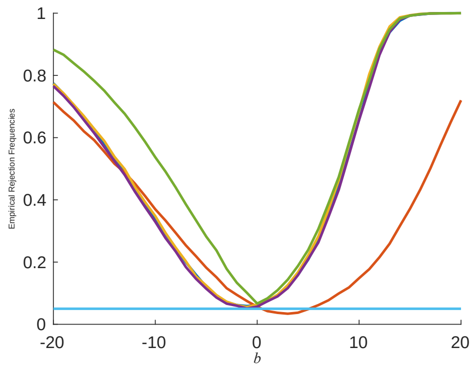
(a) $c = 0$, $\phi = -0.50$.



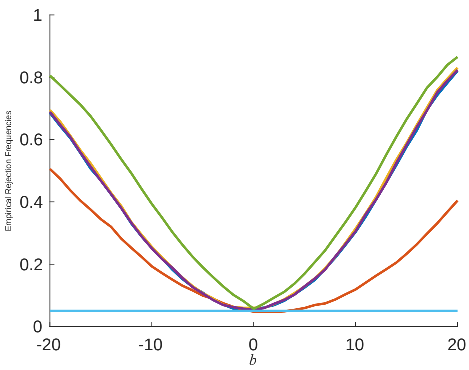
(b) $c = 0$, $\phi = -0.95$.



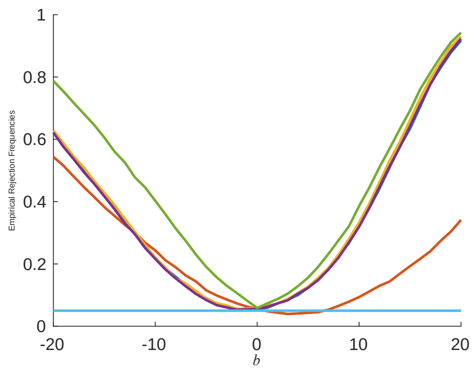
(c) $c = 10$, $\phi = -0.50$.



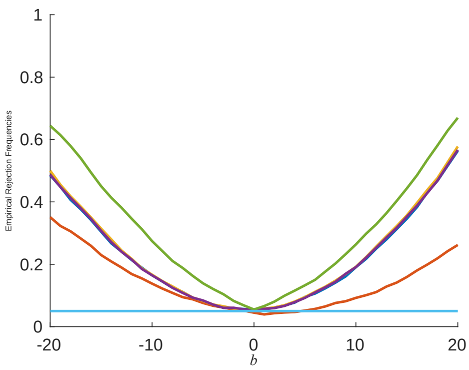
(d) $c = 10$, $\phi = -0.95$.



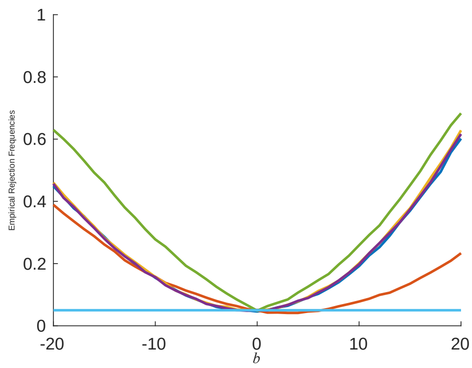
(e) $c = 25$, $\phi = -0.50$.



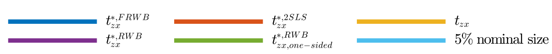
(f) $c = 25$, $\phi = -0.95$.



(g) $c = 50$, $\phi = -0.50$.



(h) $c = 50$, $\phi = -0.95$.



Note: The green line corresponds to the rejection frequencies of the left-sided and right-sided RWB-based t-tests in the relevant tail: i.e., for $b < 0$ the line corresponds to the left-sided test, while for $b > 0$ it corresponds to the right-sided test.

Table 4: Empirical results for the Welch and Goyal (2008) and Campbell and Yogo (2006) predictive regressions.

	$t^{*,2SLS}_{zx}$	$t^{2SLS,EW}_{zx}$	$t^{*,RWB(-)}_{zx}$	$t^{*,RWB(+)}_{zx}$	$t^{*,RWB}_{zx}$	$t^{EW(-)}_{zx}$	$t^{EW(+)}_{zx}$	t^{EW}_{zx}	$\hat{\beta}_{OLS}$	$\hat{\beta}_{IVX}$	$\hat{\rho}$	$\hat{\phi}$
PANEL A: Welch and Goyal (2008) monthly data: January 1927 - December 2020												
dp	0.5125	0.4916	0.5366	0.4634	0.6124	0.7450	0.2550	0.5099	0.002	0.003	0.994	-0.980
dy	0.3395	0.2578	0.8355	0.1645	0.2868	0.9254	**0.0746**	0.1492	0.000	0.000	1.006	-0.060
ep	0.9116	0.9107	0.8271	0.1729	0.2719	0.9142	**0.0858**	0.1716	0.005	0.006	0.989	-0.767
de	0.2721	0.2684	0.2236	0.7764	0.4647	0.3013	0.6987	0.6025	-0.004	-0.004	0.991	-0.073
svar	0.5812	0.5821	0.3511	0.6489	0.7107	0.3856	0.6144	0.7712	-0.159	-0.189	0.577	-0.297
bm	0.5819	0.5805	0.6720	0.3280	0.4530	0.7181	0.2819	0.5939	0.008	0.007	0.987	-0.821
ntis	0.5083	0.5061	**0.0784**	0.9216	0.1609	0.1170	0.8830	0.2339	-0.142	-0.141	0.981	-0.047
tbl	0.3212	0.3135	***0.0340***	0.9660	**0.0874**	***0.0365***	0.9635	**0.0730**	-0.001	-0.001	0.994	-0.053
lty	0.1145	0.1073	***0.0278***	0.9722	**0.0703**	***0.0272***	0.9728	**0.0543**	-0.001	-0.001	0.997	-0.088
ltr	0.4045	0.3992	0.9176	**0.0824**	0.1707	0.9180	**0.0820**	0.1639	0.001	0.001	0.043	0.055
tms	0.8831	0.8855	0.7883	0.2117	0.4144	0.7434	0.2566	0.5132	0.002	0.001	0.962	-0.002
dfy	0.7160	0.7121	0.4588	0.5412	0.9939	0.5019	0.4981	0.9962	0.000	0.000	0.975	-0.265
dfr	0.2367	0.2316	0.8166	0.1834	0.3740	0.8029	0.1971	0.3943	0.002	0.002	-0.102	0.185
infl	0.5870	0.5887	**0.0959**	0.9041	0.1874	**0.0567**	0.9433	0.1113	-0.004	-0.005	0.480	0.033
PANEL B: Campbell and Yogo (2006) data: 1926 - 2002												
Annual data												
dp	0.2610	0.1853	0.9807	***0.0193***	***0.0205***	0.9983	***0.0017***	***0.0035***	0.158	0.166	0.932	-0.721
ep	0.3683	0.3175	0.9780	***0.0220***	***0.0255***	0.9967	***0.0033***	***0.0065***	0.162	0.161	0.855	-0.957
r_3	0.6242	0.6035	0.1178	0.8822	0.2247	**0.0977**	0.9023	0.1954	-0.934	-0.914	0.908	0.091
y-r_1	0.5576	0.5490	0.7721	0.2279	0.4361	0.7824	0.2176	0.4351	1.743	1.570	0.626	-0.248
Quarterly data												
dp	0.6747	0.6470	0.8967	0.1033	0.1296	0.9506	***0.0494***	**0.0987**	0.034	0.035	0.963	-0.942
ep	0.5878	0.5579	0.9558	***0.0442***	**0.0544**	0.9544	***0.0456***	**0.0913**	0.047	0.047	0.958	-0.986
r_3	0.6767	0.6566	0.1267	0.8733	0.2742	0.1206	0.8794	0.2412	-0.233	-0.228	0.965	-0.050
y-r_1	0.6590	0.6517	0.8179	0.1821	0.3378	0.7550	0.2450	0.4900	0.556	0.502	0.800	-0.119
Monthly data												
dp	0.6534	0.6398	0.7974	0.2026	0.2675	0.9092	0.0908	0.1817	0.008	0.008	0.990	-0.954
ep	0.8686	0.8548	0.9467	**0.0533**	**0.0656**	0.9607	***0.0393***	**0.0787**	0.013	0.013	0.989	-0.987
r_3	0.4100	0.4074	**0.0834**	0.9166	0.1775	**0.0833**	0.9167	0.1666	-0.086	-0.086	0.991	-0.058
y-r_1	0.1723	0.1744	0.9348	**0.0652**	0.1304	0.8641	0.1359	0.2719	0.239	0.222	0.938	-0.065

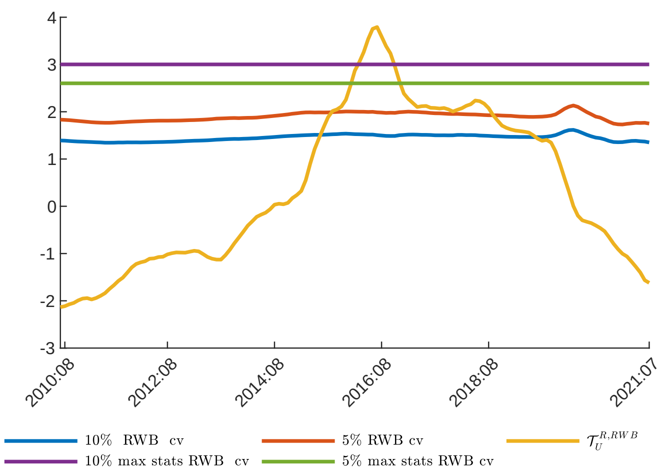
Note: The columns headed $t^{*;2SLS}_{zx}$ and t^{2SLS}_{zx} provide p-values for the bootstrap based 2SLS tests and for the corresponding 2SLS test based on asymptotic critical values, respectively. The columns $t^{*,RWB(-)}_{zx}$, $t^{*,RWB(+)}_{zx}$ and $t^{*,RWB}_{zx}$ correspond to the p-values of the left-sided, right-sided and two-sided residual wild bootstrap based IVX t-tests, respectively. The columns labeled $t^{(-)}_{zx}$, $t^{(+)}_{zx}$ and t_{zx} provide the p-values of the left-sided, right-sided and two-sided IVX tests computed using asymptotic critical values. $\hat{\beta}_{OLS}$ and $\hat{\beta}_{IVX}$ are the OLS and IVX estimates of the predictive regression slope parameter β, $\hat{\rho}$ is the estimate of the largest root from an AR model fitted to the predictor, and $\hat{\phi}$ is the OLS estimate of the correlation of the predictive regression residuals and the residuals from an AR model fitted to the predictor. All bootstrap p-values were computed using 9999 bootstrap replications. Test outcomes significant the 10% (5%) level are highlighted in bold (bold italic).

Table 5: Testing for bubbles in exchange rates between 1999 and 2021. P-values for the rolling, and forward and backward recursive statistics.

$\Delta\tau$, τ_L, $(1-\tau_U)$	$\mathcal{T}_U^{R,RWB}$	$\mathcal{T}_U^{R,FRWB}$	$\mathcal{T}_U^{F,RWB}$	$\mathcal{T}_U^{F,FRWB}$	$\mathcal{T}_U^{B,RWB}$	$\mathcal{T}_U^{B,FRWB}$
			$\boldsymbol{h=1}$			
1/6	0.1770	0.4929	0.8851	0.5917	0.8817	0.6292
1/4	0.2321	0.6094	0.8085	0.4214	0.8124	0.6035
1/3	0.2913	0.6451	0.7530	0.4059	0.7837	0.5769
1/2	0.1547	0.3251	0.4454	0.3517	0.6795	0.5592
			$\boldsymbol{h=3}$			
1/6	***0.0498***	**0.0847**	0.8152	0.8719	0.3858	0.3733
1/4	0.2443	0.3973	0.7681	0.7932	0.3512	0.3551
1/3	0.1126	0.2198	0.7364	0.7328	0.3275	0.3348
1/2	**0.0588**	***0.0347***	0.5129	0.6676	0.2866	0.3230
			$\boldsymbol{h=6}$			
1/6	***0.0169***	***0.0078***	0.8915	0.8628	0.2125	0.1745
1/4	**0.0950**	0.1390	0.8484	0.7848	0.1799	0.1497
1/3	***0.0347***	**0.0587**	0.8129	0.7575	0.1752	0.1427
1/2	***0.0121***	***0.0010***	0.6205	0.6768	0.1631	0.1419

Notes: The columns headed $\mathcal{T}_U^{R,k}$, $k = RWB, FRWB$, provide p-values for the residual (RWB) and fixed regressor ($FRWB$) wild bootstrap based rolling (R) upper tail tests. The columns $\mathcal{T}_U^{F,k}$, and $\mathcal{T}_U^{B,k}$, $k = RWB, FRWB$, correspond to forward (F) and backward (B) recursive residual (RWB) and fixed regressor ($FRWB$) wild bootstrap based upper tail tests, respectively. All bootstrap p-values were computed using 9999 bootstrap replications. Test outcomes significant at the 10% (5%) level are highlighted in bold (bold italic).

Figure 2: Plot of the sequence of upper-tailed rolling statistics for testing the null hypothesis of no bubble in the U.K. pound - U.S. dollar foreign exchange market for a six month maturity ($h = 6$).



On-Line Supplementary Appendix

to

"Extensions to IVX Methods of Inference for Return Predictability"

by

Matei Demetrescu, Iliyan Georgiev, Paulo Rodrigues and Robert Taylor

Summary of Contents

This supplement contains four sections. Section A contains Examples 1 and 2 referred in Remarks 4 and 6 of the main paper. Section B outlines how moving blocks bootstrap methods can be applied to the setting considered in this paper. Section C contains detailed proofs of Propositions 1-3. Section D reports additional supporting Monte Carlo results to those reported in section 5 of the paper.

A Additional material

Example 1 *Let*

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_t \\ e_t \end{pmatrix} \tag{A.1}$$

where, with $a_0, b_0 > 0$,

$$\begin{aligned} a_t &= \sqrt{a_0}\nu_{t,1} \\ e_t &= \sqrt{b_0 + b_1 e_{t-1}^2}\nu_{t,2} \end{aligned}$$

with $\{\nu_{t,1}\}$ and $\{\nu_{t,2}\}$ two mutually independent zero-mean unit-variance IID sequences. Assume $\nu_{t,1}$, $\nu_{t,2}$ to be uniformly L_4 bounded, and $b_1^2 < 1/\mathrm{E}(\nu_{t,2}^4)$ to ensure that e_t does itself have finite 4th moment. The process $v_t = e_t$ is therefore a stationary ARCH(1) process whenever $0 \leq b_1 < 1$, whereas a_t is conditionally homoskedastic (a_t is an IID sequence).

The natural filtration is $\mathcal{F}_t = \{(\nu_{t1};\nu_{t2}), (\nu_{t-1,1};\nu_{t-1,2}), \ldots\}$, and the conditional variance of u_t is easily seen to be

$$\mathrm{E}\left(u_t^2|\mathcal{F}_{t-1}\right) = a_0 + \gamma^2\left(b_0 + b_1 v_{t-1}^2\right).$$

In this model, the conditional variance of u_t obviously does not depend on the past innovations v_t when $\gamma = 0$; however, this restriction also implies the absence of any contemporaneous correlation between u_t and v_t, inconsistent with the conditions ordinarily expected to hold in a predictive regression model for financial variables.

The model outlined above satisfies our Assumption 3.2, (see remark 4), but violates assumption INNOV of Kostakis et al. (2015) because u_t from (A.1) cannot have a so-called strict finite-order GARCH representation (i.e. with IID shocks) in general:

1. *If u_t did have such a strict GARCH representation, it would hold that $u_t = \sqrt{h_t}\eta_t$ where η_t is an IID sequence, and $h_t = \alpha_0 + \alpha_1 u_{t-1}^2 + \beta_1 h_{t-1}$ (we show below that u_t^2 has an ARMA(1,1) representation, such that u_t itself can only have a GARCH(1,1) representation). The ARMA(1,1) representation of this squared GARCH model equation is then*

$$u_t^2 = \alpha_0 + (\alpha_1 + \beta_1) u_{t-1}^2 + \vartheta_t - \beta_1 \vartheta_{t-1}$$

where $\vartheta_t = h_t\left(\eta_t^2 - 1\right)$. The errors ϑ_t in the squared GARCH model equation must be conditionally heteroskedastic martingale differences with the particular conditional variance, $\mathrm{E}\left(\vartheta_t^2|\mathcal{F}_t\right) = h_t^2\,\mathrm{E}\left(\left(\eta_t^2-1\right)^2\right)$.

2. *The squared u_t implied by the model in (A.1) is given as*

$$\begin{aligned} u_t^2 &= a_0 + \gamma^2 b_0 + \gamma^2 b_1 v_{t-1}^2 + \left(a_t^2 - a_0\right) + \gamma^2\left(v_t^2 - \left(b_0 + b_1 v_{t-1}^2\right)\right) + 2\gamma a_t v_t \\ &= a_0 + \gamma^2 b_0 + b_1 \gamma^2 v_{t-1}^2 + \xi_t \end{aligned}$$

where

$$\xi_t = \left(a_t^2 - a_0\right) + \gamma^2\left(b_0 + b_1 v_{t-1}^2\right)\left(\nu_{t,2}^2 - 1\right) + 2\gamma a_t v_t$$

is a MD sequence w.r.t. $\mathcal{F}_t$. Furthermore,

$$\gamma^2 v_{t-1}^2 = u_{t-1}^2 - a_{t-1}^2 - 2\gamma a_{t-1} v_{t-1}$$

such that, plugging this in, we obtain

$$\begin{aligned} u_t^2 &= \left(a_0 + \gamma^2 b_0\right) + b_1\left(u_{t-1}^2 - a_{t-1}^2 - 2\gamma a_{t-1} v_{t-1}\right) + \xi_t \\ &= \left(a_0 + \gamma^2 b_0 - b_1 a_0\right) + b_1 u_{t-1}^2 + \pi_t \end{aligned}$$

where

$$\begin{aligned} \pi_t &= \xi_t - b_1\left(a_{t-1}^2 - a_0\right) - 2b_1\gamma a_{t-1} v_{t-1} \\ &= \left(a_t^2 - a_0 + 2\gamma a_t v_t\right) - b_1\left(a_{t-1}^2 - a_0 + 2\gamma a_{t-1} v_{t-1}\right) \\ &\quad + \gamma^2\left(b_0 + b_1 v_{t-1}^2\right)\left(\nu_{t,2}^2 - 1\right) \end{aligned}$$

is a weakly stationarity process and therefore possesses a linear representation. The autocovariance function of π_t is obtained as follows,

$$\begin{pmatrix} \pi_t \\ \tilde{\pi}_t \end{pmatrix} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} L\right)\begin{pmatrix} a_t^2 - a_0 + 2\gamma a_t v_t \\ \gamma^2\left(b_0 + b_1 v_{t-1}\right)\left(\nu_{t,2}^2 - 1\right) \end{pmatrix}$$

where $\begin{pmatrix} a_t^2 - a_0 + 2\gamma a_t v_t \\ \gamma^2 (b_0 + b_1 v_{t-1}) (\nu_{t,2}^2 - 1) \end{pmatrix}$ *is easily seen to be a zero-mean white noise sequence under our assumptions, such that* $\begin{pmatrix} \pi_t \\ \tilde{\pi}_t \end{pmatrix}$ *is a vector MA(1) process. Therefore,* π_t *does has a* marginal *MA(1) representation – but one where the innovations are uncorrelated, and not MD sequences in general. In turn, this does make* u_t^2 *an ARMA(1,1) process, but not necessarily one with MD innovations, so, in general, the model (A.1) does not have a GARCH representation where the driving shocks are IID.*

Example 2 *Consider the following particular case where* $A(L) = 1$ *but* $\rho \neq 0$ *is fixed and bounded away from unity and* $\boldsymbol{\psi}_t$ *is conditionally heteroskedastic. Assume also that* $h_{12}(\tau) = 0\ \forall \tau$. *Then,* $\xi_t = \sum_{j=0}^{\infty} \rho^j v_{t-j}$ *such that*

$$\operatorname{Var}(\xi_{t-1} u_t) = \mathrm{E}\left(h_{11}^2(t/T) a_t^2 \left(\sum_{j=0}^{\infty} \rho^j [h_{21}((t-1-j)/T) a_{t-1-j} + h_{22}((t-1-j)/T) e_{t-1-j}] \right)^2 \right),$$

where some algebra shows that

$$\operatorname{Var}(\xi_{t-1} u_t) = \mathrm{E}\left(h_{11}^2(t/T) a_t^2 \left(h_{21}(t/T) \sum_{j=0}^{\infty} \rho^j a_{t-1-j} + h_{22}(t/T) \sum_{j=0}^{\infty} \rho^j e_{t-1-j} \right)^2 \right) + o(1).$$

One therefore obtains

$$\operatorname{Var}(\xi_{t-1} u_t) = h_{11}^2(t/T) \left(C_1 h_{21}^2(t/T) + C_2 h_{22}^2(t/T) + C_3 h_{21}(t/T) h_{22}(t/T) \right) + o(1)$$

where $C_1 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \rho^j \rho^k \mathrm{E}\left(a_t^2 a_{t-1-j} a_{t-1-k}\right)$, $C_2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \rho^j \rho^k \mathrm{E}\left(a_t^2 e_{t-1-j} e_{t-1-k}\right)$ *and* $C_3 = 2\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \rho^j \rho^k \mathrm{E}\left(a_t^2 a_{t-1-j} e_{t-1-k}\right)$. *Therefore, we have at all differentiability points*

$$[M_{\xi u}]'(s) = h_{11}^2(s) \left(C_1 h_{21}^2(s) + C_2 h_{22}^2(s) + C_3 h_{21}(s) h_{22}(s) \right).$$

At the same time, it follows analogously that

$$[M_u]'(s) = h_{11}^2(s) \qquad [M_v]'(s) = h_{21}^2(s) + h_{22}^2(s)$$

such that

$$[M_{zu}]'(s) = h_{11}^2(s) \left(h_{21}^2(s) + h_{22}^2(s) \right).$$

Summing up, the quadratic variation (and thus the variance profile) of M_{zu} *is in general different from that of* $M_{\xi u}$.

B Moving blocks bootstrap

Following Fan and Lee (2019), one could employ a block-bootstrap scheme. This amounts, in their notation, to the following algorithm:

1. Let b be an integer block length and let $B(t) = (\boldsymbol{w}_t, \boldsymbol{w}_{t+1}, \ldots, \boldsymbol{w}_{t+b-1})$ denote a data block with starting point $t \in \{1, \ldots, T-b+1\}$, where the data to be resampled stacks $\boldsymbol{w}_t = (y_t, z_t)'$.

2. The total number of possible blocks and the number of blocks in one bootstrapped sample are denoted by q and m. The letter ℓ indicates the bootstrapped sample size, $T = q + b - 1$ and $\ell = mb$. (Intuitively, one should choose m such that $\ell \approx T$; Fan and Lee (2019) only require $m = O(T)$ and $T = O(m)$.)

3. Sample m blocks randomly with replacement from $\{B(t) : t = 1, \ldots, n-b+1\}$: the resulting bootstrap sample $\boldsymbol{w}_1^*, \ldots, \boldsymbol{w}_\ell^*$ is $(B(I_1), \ldots, B(I_m))$ with I_i are IID discrete uniform variables on $\{1, ..., n - b + 1\}$.

4. Compute e.g. the full-sample bootstrap IVX t-statistic,

$$t_{zx}^* = \frac{\sum_{t=1}^{\ell} \tilde{z}_{t-1}^* \tilde{y}_t^*}{\sqrt{\hat{\sigma}_{u^*}^2} \sqrt{\sum_{t=1}^{\ell} \left(\tilde{z}_{t-1}^*\right)^2}};$$

   this step is different from the corresponding step of Fan and Lee (2019), since they work in a quantile regression framework.

5. Use quantiles of distribution of t_{zx}^* for inference rather than quantiles of the standard normal.

The above procedure does not replicate the null hypothesis in the bootstrap data, so one would need to either construct confidence intervals and invert them to obtain a test, or replace y_t with the OLS residuals $\hat{u}_t := y_t - \hat{\alpha} - \hat{\beta} x_{t-1}$ in the definition of $\boldsymbol{w}_t$ in Step 1 to ensure that the null is imposed on the bootstrap data.

Block wild bootstrap

To account for unconditional heteroskedasticity, Step 3 of the above MBB scheme could be replaced with a block wild bootstrap. In this case, one needs to impose the null when resampling, i.e. replace y_t with the OLS residuals $\hat{u}_t := y_t - \hat{\alpha} - \hat{\beta} x_{t-1}$ in the definition of $\boldsymbol{w}_t$ in step 1.

We do not provide theoretical results for either moving block bootstrap.

C Technical appendix

We denote by P^*, E^* and Var^* respectively probability, expectation and variance conditional on the original data. Further, we use E_{t-1}^* for expectation conditional on the data and $\{R_s\}_{s=1}^{t-1}$. Weak in-probability convergence is denoted by $\overset{w}{\Rightarrow}_p$. If w is a degenerate (deterministic) element, an alternative notation to $w_T \overset{w}{\Rightarrow}_p w$ is $w_T \overset{p}{\Rightarrow}_p w$. If the metric space of interest is a linear space with zero element 0, we use $w_T \overset{w}{\Rightarrow}_p 0$ interchangeably with $w_T = o_p^*(1)$. For instance, $w_T \overset{p}{\Rightarrow}_p w$ is equivalent to $d(w_T, w) = o_p^*(1)$ for the metric d of the underlying space. We introduce $w_T = O_p^*(1)$ by the standard property that for every $\epsilon > 0$ there exists a $K_\epsilon \in \mathbb{R}$ such that $\mathrm{P}(\mathrm{P}^*(d(w_T, 0) > K_\epsilon) < \epsilon) > 1 - \epsilon$ for all $T \in \mathbb{N}$. As usual, $o_p^*(T^\alpha) := T^\alpha o_p^*(1)$ and $O_p^*(T^\alpha) := T^\alpha O_p^*(1)$. The o_p and O_p symbols retain their usual meaning. For r.v.'s w we write $\|w\|_r$ for $(E|w|^r)^{1/r}$, $r > 0$. Finally, C is an unspecified positive constant whose value may change across the expressions where it appears.

C.1 Toolbox

We start with some results that structure our approach to the derivation of the main theory.

Martingale approximation

Assumption 3.2 implies that the components of $\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2$ are well approximated by martingale differences. Specifically, let

$$\begin{pmatrix} \Psi_T^a & \Psi_T^{ae} \\ \Psi_T^{ae} & \Psi_T^e \end{pmatrix} := \sum_{t=1}^{T}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2).$$

Then the condition $\mathrm{E}\|\mathrm{E}_0\sum_{t=1}^T(\boldsymbol{\psi}_t\boldsymbol{\psi}_t' - \mathbf{I}_2)\|^2 = O(T^{2\epsilon})$ with $\epsilon \in (0, \frac{1}{2})$ ensures, by Jensen's inequality, that component-wise $\|\mathrm{E}(\Psi_T^a|\mathcal{F}_0^a)\|_2 = O(T^\epsilon)$, $\|\mathrm{E}(\Psi_T^e|\mathcal{F}_0^e)\|_2 = O(T^\epsilon)$ and $\|\mathrm{E}(\Psi_T^{ae}|\mathcal{F}_0^{ae})\|_2 = O(T^\epsilon)$ for $\mathcal{F}_0^c := \sigma(c_{-i} : i \in \mathbb{N}\cup\{0\})$, $c \in \{a, e, ae\}$ and for the same ϵ. Together with the stationarity of $\boldsymbol{\psi}_t$ and the finite fourth moment of its components, this implies that the martingale approximation results of Merlevède *et al.* (2006) are applicable to Ψ_T^a, Ψ_T^e and Ψ_T^{ae}. The Lipschitz-by-parts property of the function $\mathbf{H}$ transfers this behavior to the sequences $u_t^2 - \sigma_{ut}^2$ and $v_t^2 - \sigma_{vt}^2$, where $\sigma_{ut}^2 := \mathrm{E}u_t^2 = h_{11}^2(t/T) + h_{12}^2(t/T)$ and similarly for σ_{vt}^2. Some implications are collected in the next lemma.

Lemma 1 *Let $S^u_{T(t+1,r)} := \sum_{s=t+1}^r(u_s^2 - \sigma_{us}^2)$ and $S^v_{T(t+1,r)} := \sum_{s=t+1}^r(v_s^2 - \sigma_{vs}^2)$ for $1 \le t < r \le T$. Under Assumption 3 it holds that:*

(a) $\max_{1\le t\le T}|T^{-1/2}S^u_{T(1,t)}| = O_p(1)$ *and* $\max_{1\le t\le T}|T^{-1/2}S^v_{T(1,t)}| = O_p(1)$

(b) $\mathrm{E}\left[\max_{1\le t<r\le T}(S^u_{T(t+1,r)})^2\right] = O(T)$

(c) $\max_{1\le t<r\le T}|\mathrm{E}[(u_t^2 - \sigma_{ut}^2)S^u_{T(t+1,r)}]| = O(T^\epsilon)$, $\max_{1\le t<r\le T}|\mathrm{E}[(v_t^2 - \sigma_{vt}^2)S^v_{T(t+1,r)}]| = O(T^\epsilon)$ *and* $\max_{1\le s<t<r\le T}|\mathrm{E}(v_tv_sS^v_{T(t+1,r)})| = O(T^\epsilon)$.

Exponential averaging

For an arbitrary real sequence w_t, partial summation produces

$$\left|\sum_{t=1}^r \varrho^{t-1}w_t\right| = \left|\varrho^{r-1}\sum_{t=1}^r w_t + (1-\varrho)\sum_{s=1}^{r-1}\varrho^{s-1}\sum_{t=1}^s w_t\right| \le \max_{1\le s\le r}\left|\sum_{t=1}^s w_t\right|. \tag{C.1}$$

Some implications of this estimate (and not only) are collected next. Here and in what follows, E_t denotes expectation conditional on $\sigma(\boldsymbol{\psi}_{-i} : i \in \mathbb{N}\cup\{0, -1, ..., -t\})$.

Lemma 2 *Let w_{Tt} be an array of r.v.'s.*

(a) If $T^{-\alpha}\sum_{t=1}^{\lfloor T\tau\rfloor} w_{Tt} \Rightarrow W(\tau)$ in the sense of weak convergence of probability measures on $\mathcal{D}$, then $\max_{1\le s\le T}\left|\sum_{t=1}^s \varrho^{t-1}w_{Tt}\right| = O_p(T^\alpha)$;

(b) $\max_{1\le s\le T}\mathrm{E}\left|\sum_{t=1}^s \varrho^{t-1}w_{Tt}\right| \le \max_{1\le s\le T}\mathrm{E}\left|\sum_{t=1}^s w_{Tt}\right|$;

(c) If w_{Tt} is an MD array with $\mathrm{E}|w_{Tt}|^p < \infty$ for some $p > 2$, then

$$\max_{1\le t\le T}\left\|\sum_{j=0}^{t-1}\varrho^j w_{T,t-j}\right\|_p = O(T^{\eta/2})\left(\max_{t\le T}\mathrm{E}|w_{Tt}|^p\right)^{1/p}$$

$$\max_{1\le t\le T}\left|\sum_{j=0}^{t-1}\varrho^j w_{T,t-j}\right| = o_p(T^{1/2})\left(\max_{t\le T}\mathrm{E}|w_{Tt}|^p\right)^{1/p}.$$

In the following parts, let Assumption 3 hold. Then:

(d) $\max_{1\le t\le T}|\sum_{r=t+1}^T \varrho^{2(r-t-1)}(u_r^2-\sigma_{ur}^2)| = O_p(T^{1/2})$*;*

(e) $\max_{1\le t\le T}\|\mathrm{E}_t\sum_{r=t+1}^T \varrho^{2(r-t-1)}(u_r^2-\sigma_{ur}^2)\|_2 = O(T^{\epsilon})$ *and* $\max_{1\le t\le T}\left\|\mathrm{E}_t\sum_{r=t+1}^T \varrho^{r-t}u_r v_r\right\|_2 = O(T^{\epsilon})$*;*

(f) $T^{-1-\eta}\sum_{s=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2\right)u_t^2 = T^{-1-\eta}\sum_{s=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2\right)\sigma_{ut}^2 + O_p(T^{(\epsilon-\eta)/2})$ *pointwise;*

(g) If $\epsilon < \eta$, then $T^{-1-\eta}\sum_{s=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2\right)\sigma_{ut}^2 \xrightarrow{p} \int_0^\tau [M_u(s)]'[M_v(s)]'\mathrm{d}s$ *pointwise and uniformly (the derivatives exist everywhere except at finitely many points and are continuous on the intervals where they exist).*

The space $\mathcal{D}(T_\Delta)$

Let $T_\Delta = [0,1]^2 \cap \{(\tau_1,\tau_2)\in\mathbb{R}^2 : \tau_2-\tau_1 \ge \Delta_\tau\}$ for some $\Delta_\tau \in (0,1)$. Let $\mathcal{D}(T_\Delta)$ be the set of real functions on T_Δ which are continuous from the 'right' (i.e., $f(\tau_1^{(n)},\tau_2^{(n)}) \to f(\tau_1,\tau_2)$ when $\tau_i^{(n)} \downarrow \tau_i$, $i=1,2$, for $(\tau_1^{(n)},\tau_2^{(n)})$, $(\tau_1,\tau_2)\in T_\Delta$ and $f\in\mathcal{D}(T_\Delta)$) and have limits from within each of the four right angles $[A_1\times A_2]\cap T_\Delta$, $A_i \in \{[0,\tau_i),[\tau_i,1]\}$, $i=1,2$, when the angles are non-empty. For clarity, note that all bivariate cdf's with domain restricted to T_Δ belong to $\mathcal{D}(T_\Delta)$. It is well-known (e.g. Bickel and Wichura, 1971, p. 1662) that $\mathcal{D}(T_\Delta)$ can be equipped with a Skorokhod-like metric which makes it a separable and complete metric space such that stochastic process with values in $\mathcal{D}(T_\Delta)$ are measurable w.r.t. the resulting Borel σ-algebra. Moreover, the resulting topology relativised to $\mathcal{C}(T_\Delta)\subset\mathcal{D}(T_\Delta)$, the subspace of continuous real functions on T_Δ, coincides with the uniform topology. As we will only be interested in convergence to limits in $\mathcal{C}(T_\Delta)$, in what follows convergence and continuity issues involving elements of $\mathcal{D}(T_\Delta)$ are always discussed w.r.t. the uniform metric on $\mathcal{D}(T_\Delta)$. It is then straightforward to see that the function from $\mathcal{D}^2$ to $\mathcal{D}(T_\Delta)$ which associates to every $(f_1,f_2)\in\mathcal{D}^2$ the element $(\tau_1,\tau_2)\mapsto f_2(\tau_2)-f_1(\tau_1)$ of $\mathcal{D}(T_\Delta)$ is continuous on the subspace of continuous functions $\mathcal{C}^2$ of $\mathcal{D}^2$. Moreover, linearly combining functions in $\mathcal{D}(T_\Delta)$, multiplication of functions in $\mathcal{D}(T_\Delta)$ and division of functions in $\mathcal{D}(T_\Delta)$ (for denominators bounded away from zero) are continuous transformations of the product subspace $\mathcal{C}(T_\Delta)\times\mathcal{C}(T_\Delta)$ of $\mathcal{D}(T_\Delta)\times\mathcal{D}(T_\Delta)$. The evaluation functional is certainly continuous on $\mathcal{C}(T_\Delta)\times\mathcal{C}(T_\Delta)$. Finally, also the functionals $\sup_{A^s}|f|$, $s\in\{F,B,R\}$, are continuous on $\mathcal{C}(T_\Delta)\times\mathcal{C}(T_\Delta)$, where $A^F=\{0\}\times[\tau_L,1]$, $A^B=[0,\tau_U]\times\{1\}$ and $A^R=\{(\tau,\tau+\Delta_\tau):\tau\in[0,1-\Delta_\tau]\}$ with $\tau_L\ge\Delta_\tau$ and $1-\tau_U\ge\Delta_\tau$.

C.2 Asymptotics on the space of the original data

The first result is independent of the persistence properties of x_t.

Lemma 3 *Under Assumption 3, it holds as $T \to \infty$ that*

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau \rfloor}\begin{pmatrix} u_t^2 \\ v_t^2 \end{pmatrix} = \frac{1}{T}\sum_{t=1}^{\lfloor T\tau \rfloor}\begin{pmatrix} \sigma_{ut}^2 \\ \sigma_{vt}^2 \end{pmatrix} + o_p(T^{-1/2}) \xrightarrow{p} \begin{pmatrix} [M_u] \\ [M_v] \end{pmatrix}(\tau) = \int_0^\tau \begin{pmatrix} h_{11}^2(s) + h_{12}^2(s) \\ h_{21}^2(s) + h_{22}^2(s) \end{pmatrix} \mathrm{d}s$$

uniformly over $\tau \in [0,1]$.

We now turn to the weakly persistent case.

Lemma 4 *Under Assumptions 1.1 and 3, we have as $T \to \infty$:*

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor \tau T \rfloor}\begin{pmatrix} u_t \\ v_t \\ u_t\xi_{t-1} \end{pmatrix} \Rightarrow \int_0^\tau \mathbf{G}(s)\mathrm{d}\boldsymbol{B}(s)$$

on $\mathcal{D}^3$, where

$$\mathbf{G}(\tau) = \begin{pmatrix} h_{11} & h_{12} & 0 & 0 & 0 & 0 \\ h_{21} & h_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{11}h_{21} & h_{11}h_{22} & h_{12}h_{21} & h_{12}h_{22} \end{pmatrix}(\tau)$$

and $\boldsymbol{B}(\tau)$ is a 6-variate Brownian motion of covariance matrix defined in the proof.

The next lemma collects some product-moment limits in the strongly persistent case.

Lemma 5 *Under Assumptions 1.2 and 3 with $\epsilon < \min\{1-\eta, \frac{1}{2}\eta\}$, the following convergence facts hold jointly on $\mathcal{D}$ as $T \to \infty$:*

(a) $\frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1} \Rightarrow \frac{\omega}{a}J_{c,\mathrm{H}}(\tau) = \frac{\omega}{a}\int_0^\tau e^{-c(\tau-s)}\mathrm{d}M_v(s)$

(b) $\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1}x_{t-1} \Rightarrow \frac{\omega^2}{a}\left(J_{c,\mathrm{H}}^2(\tau) - \int_0^\tau J_{c,\mathrm{H}}(s)\mathrm{d}J_{c,\mathrm{H}}(s)\right)$

(c) $\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1}^2 \xrightarrow{p} \frac{\omega^2}{2a}[M_v](\tau)$

(d) $\frac{1}{T^{1/2+\eta/2}}\sum_{t=1}^{[\tau T]} z_{t-1}u_t \Rightarrow \frac{\omega}{\sqrt{2a}}\int_0^\tau \sqrt{[M_u]'(s)[M_v]'(s)}\mathrm{d}B(s)$ *where B is a standard Brownian motion independent of M_v (and thus, of $J_{c,\mathrm{H}}$).*

(e) $\frac{1}{T^{1/2+\eta}}\sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1}b(t/T) \Rightarrow \frac{\omega}{a}\int_0^\tau b(s)\mathrm{d}J_{c,\mathrm{H}}(s)$.

(f) $\frac{1}{T^{1+\eta}}\sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1}b(t/T)x_{t-1} \Rightarrow \frac{\omega^2}{a}\left(\int_0^\tau J_{c,\mathrm{H}}(s)b(s)\mathrm{d}J_{c,\mathrm{H}}(s) + \int_0^\tau b(s)\mathrm{d}[M_v](s)\right)$.

Moreover, by sample-path continuity of the limiting processes in Lemma 5, the previous convergence facts hold jointly also on $\mathcal{D}$ equipped with the uniform metric.

Proof of Proposition 1. For the space $\mathcal{D}(T_\Delta)$ and our approach to the weak convergence of probability measures on it, see Section C.1. In particular, to prove Proposition 1, it suffices to establish the weak convegence of $t_{zx}(\tau_1, \tau_2)$ on $\mathcal{D}(T_\Delta)$ to the limits stated in the two parts of the proposition.

We have

$$t_{zx}(\tau_1,\tau_2)=\frac{\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor} z_{t-1}\left(u_t-\bar{u}(\tau_1,\tau_2)\right)}{\hat{\sigma}_u(\tau_1,\tau_2)\sqrt{\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor} z_{t-1}^2}}+\frac{\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor} z_{t-1}\beta_t\left(\xi_{t-1}-\bar{\xi}_{-1}(\tau_1,\tau_2)\right)}{\hat{\sigma}_u(\tau_1,\tau_2)\sqrt{\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor} z_{t-1}^2}}.$$

Under Assumption 1.1, we notice that, given our moment restrictions and the absolute summability of the Wold coefficients of ξ_t, $\sup_t|\xi_{t-1}|=O_p\left(T^{1/4}\right)$; also, $\hat{\alpha}$ and $\hat{\beta}$ are easily shown to be $\sqrt{T}$-consistent, so

$$\hat{u}_t=u_t-(\hat{\alpha}-\alpha)-\left(\hat{\beta}-\beta\right)x_{t-1}=u_t+o_p(1)$$

uniformly in t. (The same is easily shown to hold for the residuals computed under the null and we omit the details.) Then,

$$\begin{aligned}\hat{\sigma}_u^2(\tau_1,\tau_2) &= \frac{1}{T}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}\hat{u}_t^2=\frac{1}{T}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}u_t^2+\frac{1}{T}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}\left(\hat{u}_t^2-u_t^2\right)\\ &= \frac{1}{T}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}\sigma_{ut}^2+\frac{1}{T}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}\left(u_t^2-\sigma_{ut}^2\right)+o_p(1)\end{aligned}$$

uniformly in τ_1 and τ_2 with $0\le\tau_1<\tau_2\le 1$, such that, thanks e.g. to Lemma 3,

$$\hat{\sigma}_u^2(\tau_1,\tau_2)\Rightarrow\frac{1}{\tau_2-\tau_1}\left([M_u](\tau_2)-[M_u](\tau_1)\right). \tag{C.2}$$

Moving on, we have like in the proof of Lemma 6 that $z_t=\xi_t+R_{t,T}$ where the rest term $R_{t,T}$ vanishes as $T\to\infty$ and can be controlled for in the relevant sums, such that we may conclude that, uniformly in τ_1 and τ_2 with $0\le\tau_1<\tau_2\le 1$,

$$\begin{aligned}\frac{1}{T}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}z_{t-1}^2 &= \frac{1}{T}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}\xi_{t-1}^2+o_p(1)\\ &\Rightarrow \kappa^2\left([M_v](\tau_2)-[M_v](\tau_1)\right).\end{aligned} \tag{C.3}$$

Similarly, we have uniformly in τ_1 and τ_2 with $0\le\tau_1<\tau_2\le 1$ that

$$\begin{aligned}\frac{1}{\sqrt{T}}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}z_{t-1}\left(u_t-\bar{u}(\tau_1,\tau_2)\right) &= \frac{1}{\sqrt{T}}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}\xi_{t-1}\left(u_t-\bar{u}(\tau_1,\tau_2)\right)+o_p(1)\\ &= \frac{1}{\sqrt{T}}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}\xi_{t-1}u_t-\left(\frac{1}{T(\tau_2-\tau_1)}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}u_t\right)\left(\frac{1}{\sqrt{T}}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}\xi_{t-1}\right)+o_p(1),\end{aligned}$$

where the weak convergence of the partial sums of ξ_t and u_t implies

$$\frac{1}{\sqrt{T}}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}\xi_{t-1}=O_p(1)\qquad\frac{1}{\sqrt{T}}\sum_{t=\lfloor\tau_1T\rfloor+1}^{\lfloor\tau_2T\rfloor}u_t=O_p(1)$$

uniformly in τ_1 and τ_2 with $0\le\tau_1<\tau_2\le 1$. The weak convergence of the partial sums of $\xi_{t-1}u_t$

therefore implies

$$\frac{1}{\sqrt{T}}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}\left(u_t-\bar{u}(\tau_1,\tau_2)\right)\Rightarrow M_{\xi u}(\tau_2)-M_{\xi u}(\tau_1).$$

To assess the drift term under the local alternative $\beta_t = T^{-1/2}b(t/T)$, write like above

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}\beta_t\left(\xi_{t-1}-\bar{\xi}_{-1}(\tau_1,\tau_2)\right)\\
&\quad=\frac{1}{T}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} b(t/T)\xi_{t-1}^2-\bar{\xi}_{-1}(\tau_1,\tau_2)\frac{1}{\sqrt{T}}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor}\xi_{t-1}b(t/T)+o_p(1)
\end{aligned}$$

uniformly in τ_1 and τ_2 with $0\le\tau_1<\tau_2\le 1$. It is then not difficult to establish analogously to Lemma 3 that

$$\frac{1}{T}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} b(t/T)\xi_{t-1}^2\Rightarrow\kappa^2\int_{\tau_1}^{\tau_2} b(s)\mathrm{d}[M_v](s)$$

and we omit the details. Finally,

$$\frac{1}{\sqrt{T}}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor}\xi_{t-1}b(t/T)=O_p(1)\qquad\bar{\xi}_{-1}(\tau_1,\tau_2)=O_p\left(1/\sqrt{T}\right)$$

as required for the convergence of $t_{zx}(\tau_1,\tau_2)$ on $\mathcal{D}(T_\Delta)$ in Proposition 1(i). The complete set of conclusions there follows by continuity considerations (see Section C.1).

Moving on to the part concerning Assumption 1.2, $\hat{\sigma}_u^2(\tau_1,\tau_2)$ is easily shown to have the same behavior as under the stable regressor case considering that the OLS residuals satisfy

$$\begin{aligned}
\hat{u}_t &= u_t-(\hat{\alpha}-\alpha)-\left(\hat{\beta}-\beta\right)x_{t-1}\\
&= u_t+O_p\left(T^{-1/2}\right)
\end{aligned}$$

uniformly in t since $\hat{\alpha}-\alpha=O_p\left(T^{-1/2}\right)$, $\hat{\beta}-\beta=O_p\left(T^{-1}\right)$ and $\sup_{1\le t\le T}|x_{t-1}|=O_p\left(\sqrt{T}\right)$ given the weak convergence of $T^{-1/2}x_{[\tau T]}$ to an a.s. continuous process. (An analogous argument applies for the residuals computed under the null). Then, under Assumption 1.2, Lemma 5 part (c) then leads to

$$\frac{1}{T^{1+\eta}}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2\Rightarrow\frac{\omega^2}{2a}\left([M_v](\tau_2)-[M_v](\tau_1)\right). \tag{C.4}$$

Lemma 5 parts (a) and (d) furthermore imply

$$\frac{1}{T^{1/2+\eta/2}}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}\left(u_t-\bar{u}(\tau_1,\tau_2)\right)\Rightarrow\frac{\omega}{\sqrt{2a}}\left(M_{zu}(\tau_2)-M_{zu}(\tau_1)\right),$$

and, given the weak convergence of $\xi_{[\tau T]} = x_{[\tau T]} - \mu_x$ and also Lemma 5 part (e),

$$\frac{\bar{\xi}_{-1}(\tau_1,\tau_2)}{\sqrt{T}} \frac{1}{T^{1/2+\eta}} \sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1} b(t/T) \Rightarrow \tfrac{1}{\tau_2-\tau_1} \tfrac{\omega^2}{a} \int_{\tau_1}^{\tau_2} J_{c,\mathrm{H}}(s)\mathrm{d}s \int_{\tau_1}^{\tau_2} b(s)\mathrm{d}J_{c,\mathrm{H}}(s)$$
$$= \tfrac{\omega^2}{a} \int_{\tau_1}^{\tau_2} \overline{J}_{c,\mathrm{H}}(\tau_1,\tau_2) b(s)\mathrm{d}J_{c,\mathrm{H}}(s).$$

Finally, Lemma 5 part (f) leads to

$$\frac{1}{T^{1+\eta}} \sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1} b(t/T) \xi_{t-1} \Rightarrow \frac{\omega^2}{a}\left(\int_0^{\tau_2} J_{c,\mathrm{H}}(s) b(s)\mathrm{d}J_{c,\mathrm{H}}(s) + \int_0^{\tau_2} b(s)\mathrm{d}[M_v](s)\right)$$
$$-\frac{\omega^2}{a}\left(\int_0^{\tau_1} J_{c,\mathrm{H}}(s) b(s)\mathrm{d}J_{c,\mathrm{H}}(s) + \int_0^{\tau_1} b(s)\mathrm{d}[M_v](s)\right)$$

such that the convergence of $t_{zx}(\tau_1,\tau_2)$ on $\mathcal{D}(T_\Delta)$ to the limit asserted in Proposition 1(ii) follows by the continuous mapping theorem. □

Proof of Proposition 2.

Under the null hypothesis,

$$t_{zx}^{EW}(\tau_1,\tau_2) = \frac{\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}\left(u_t - \bar{u}(\tau_1,\tau_2)\right)}{\sqrt{\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2 \hat{u}_t^2}}$$

and we only need to tackle the limiting behavior of the denominator, for which we have that

$$\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2 \hat{u}_t^2 = \sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2 u_t^2 + \sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2\left(\hat{u}_t^2 - u_t^2\right).$$

We recall from the proof of Proposition 1 that $\sup_{1\le t\le T}\left|\hat{u}_t^2 - u_t^2\right| = o_p(1)$ under both Assumptions 1.1 and 1.2.

Under Assumption 1.1, we have

$$\left|\frac{1}{T}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2\left(\hat{u}_t^2 - u_t^2\right)\right| \le \sup_{1\le t\le T}\left|\hat{u}_t^2 - u_t^2\right| \frac{1}{T}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2 = o_p(1)$$

see Equation (C.4), and, using the same argument leading to Equation (C.3), we obtain

$$\frac{1}{T}\sum_{t=+1}^{\lfloor\tau T\rfloor} z_{t-1}^2 u_t^2 = \frac{1}{T}\sum_{t=+1}^{\lfloor\tau T\rfloor} \xi_{t-1}^2 u_t^2 + o_p(1)$$

where $\frac{1}{T}\sum_{t=+1}^{\lfloor\tau T\rfloor} \xi_{t-1}^2 u_t^2 \Rightarrow [M_{\xi u}](\tau)$ is a byproduct of establishing the weak convergence of the partial sums of $\xi_{t-1}u_t$.

Under Assumption 1.2, we then immediately have thanks to Lemma 5 part (c) that

$$\left|\frac{1}{T^{1+\eta}}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2\left(\hat{u}_t^2 - u_t^2\right)\right| \le \sup_t\left|\hat{u}_t^2 - u_t^2\right| \frac{1}{T^{1+\eta}}\sum_{t=\lfloor\tau_1 T\rfloor+1}^{\lfloor\tau_2 T\rfloor} z_{t-1}^2 = o_p(1),$$

while the quadratic variation

$$\frac{1}{T^{1+\eta}} \sum_{t=+1}^{\lfloor \tau T \rfloor} z_{t-1}^2 u_t^2 \Rightarrow [M_{zu}](\tau)$$

is dealt with in the proof of Lemma 5 part (d). □

C.3 Bootstrap asymptotics

The next lemma establishes the asymptotics of the processes in the numerator and the denominator of the bootstrap statistic t_{zx}^* in the weakly persistent case.

Lemma 6 *Let Assumptions 1.1 and 3 hold. Let B be a standard Brownian motion on $[0,1]$ and $\mathbf{H}_{1\cdot}$, $\mathbf{H}_{2\cdot}$ denote the rows of $\mathbf{H}$. Then, as $T \to \infty$:*

(a) $T^{-1/2} \sum_{t=1}^{\lfloor T\tau \rfloor} z_{t-1} u_t^* \overset{w}{\Rightarrow}_p M_{\xi u}(\tau) = \int_0^\tau \chi(s)^{1/2} \mathrm{d}B(s)$ *on $\mathcal{D}$, with*

$$\chi(s) = \sum_{i,j \geq 0} b_i b_j \mathrm{E}[\mathbf{H}_{1\cdot}(s)(\boldsymbol{\psi}_1 \boldsymbol{\psi}_1')\mathbf{H}_{1\cdot}(s)'\mathbf{H}_{2\cdot}(s)(\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')\mathbf{H}_{2\cdot}(s)'];$$

(b) $T^{-1/2} \sum_{t=1}^{\lfloor T\tau \rfloor} z_{t-1}^* u_t^* \overset{w}{\Rightarrow}_p M_{\xi u}^*(\tau) := \int_0^\tau \chi^*(s)^{1/2} \mathrm{d}B(s)$ *on $\mathcal{D}$, with*

$$\chi^*(s) = \sum_{j \geq 0} b_j^2 \mathrm{E}[\mathbf{H}_{1\cdot}(s)(\boldsymbol{\psi}_1 \boldsymbol{\psi}_1')\mathbf{H}_{1\cdot}(s)'\mathbf{H}_{2\cdot}(s)(\boldsymbol{\psi}_{-j}\boldsymbol{\psi}_{-j}')\mathbf{H}_{2\cdot}(s)'];$$

(c) $T^{-1-\eta} \sum_{t=1}^{\lfloor T\tau \rfloor} (z_{t-1}^*)^2 \overset{p}{\Rightarrow}_p \kappa^2 [M_v](\tau)$ *on $\mathcal{D}$;*

(d) $\hat{\sigma}_u^{2*}(0,\tau) = T^{-1} \sum_{t=1}^{\lfloor T\tau \rfloor} (\hat{u}_t^*)^2 \overset{p}{\Rightarrow}_p [M_u](\tau)$ *on $\mathcal{D}$.*

We now turn to the case of a strongly persistent posited predictor variable and discuss the process $N_T^*(\tau) := T^{-(1+\eta)/2} \sum_{t=1}^{\lfloor T\tau \rfloor} z_{t-1}^* u_t^*$ in steps similar to those of Magdalinos (2020). First, we approximate $N_T^*(\tau)$ by $\tilde{N}_T^*(\tau) := T^{-(1+\eta)/2} \sum_{t=1}^{\lfloor T\tau \rfloor} \zeta_{t-1}^* \tilde{u}_t$ for $\zeta_t^* := \omega \sum_{j=0}^{t-1} \varrho^j v_{t-j}^*$ and $\tilde{u}_t := u_t R_t$. Second, we discuss the predictable quadratic variation of $\tilde{N}_T^*$ conditional on the data,

$$\tilde{V}_T^*(\tau) := T^{-1-\eta} \sum_{t=1}^{\lfloor T\tau \rfloor} \mathrm{E}_{t-1}^* (\zeta_{t-1}^* \tilde{u}_t)^2 = T^{-1-\eta} \sum_{t=1}^{\lfloor T\tau \rfloor} (\zeta_{t-1}^*)^2 u_t^2,$$

whose asymptotics determine those of N_T^*.

Lemma 7 *Under Assumptions 1.2 and 3, it holds that*

(a) $\sup_{[0,1]} |N_T^* - \tilde{N}_T^*| = o_p^*(1)(1 + \sup_{[0,1]} |\tilde{N}_T^*|)$*;*

(b) $\tilde{V}_T^*(\tau) = \tilde{V}(\tau) + o_p^*(1)(1 + \tilde{V}(1))$ *pointwise for* $\tilde{V}(\tau) := T^{-1-\eta}\omega^2 \sum_{s=1}^{\lfloor T\tau \rfloor} \left(\sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2 \right) u_t^2$*;*

(c) If $\epsilon < \eta$ in Assumption 3.2, then $\tilde{V}_T^*(\tau) \overset{p}{\Rightarrow}_p \frac{\omega^2}{2a} \int_0^\tau [M_u(s)]'[M_v(s)]' \mathrm{d}s$ *on $\mathcal{D}$.*

We are now in a position to establish the asymptotic behaviour of the processes in the numerator and the denominator of the bootstrap statistic t_{zx}^* in the strongly persistent case.

Lemma 8 *Under Assumptions 1.2 and 3 with $\epsilon < \eta$ it holds that*

(a) $N_T^*(\tau) \overset{w}{\Rightarrow}_p N(\tau) = \frac{|\omega|}{\sqrt{2a}} \int_0^\tau \sqrt{[M_v(s)]'[M_u(s)]'}\mathrm{d}B(s)$ on $\mathcal{D}$;

(b) $T^{-1-\eta} \sum_{t=1}^{\lfloor T\tau \rfloor} (z_{t-1}^*)^2 \overset{p}{\Rightarrow}_p \frac{\omega^2}{2a}[M_v](\tau)$ on $\mathcal{D}$;

(c) $T^{-1}\sum_{t=1}^{\lfloor T\tau \rfloor} (\hat{u}_t^*)^2 = T^{-1} \sum_{t=1}^{\lfloor T\tau \rfloor} \hat{u}_t^2 + o_p^*(1) \overset{p}{\Rightarrow}_p [M_u](\tau)$ on $\mathcal{D}$.

Proof of Proposition 3. For the space $\mathcal{D}(T_\Delta)$, see Section C.1.

Using the limits in Lemma 6 and a CMT for weak convergence in probability (e.g., Theorem 10 of Sweeting, 1989), it follows that under Assumption 1.1,

$$t_{zx}^{*,FR}(\tau_1, \tau_2) \overset{w}{\Rightarrow}_p \frac{\sqrt{\tau_2 - \tau_1}(M_{\xi u}(\tau_2) - M_{\xi u}(\tau_1))}{|\kappa|\sqrt{\{[M_u](\tau_2) - [M_u](\tau_1)\}\{[M_v](\tau_2) - [M_v](\tau_1)\}}},$$
$$t_{zx}^{*,RB}(\tau_1, \tau_2) \overset{w}{\Rightarrow}_p \frac{\sqrt{\tau_2 - \tau_1}(M_{\xi u}^*(\tau_2) - M_{\xi u}^*(\tau_1))}{|\kappa|\sqrt{\{[M_u](\tau_2) - [M_u](\tau_1)\}\{[M_v](\tau_2) - [M_v](\tau_1)\}}}$$

on $\mathcal{D}(T_\Delta)$, respectively for the FRWB and the RWB t-processes. Similarly, using the limits in Lemma 8 and a CMT for weak convergence in probability, it follows that under Assumption 1.2,

$$t_{zx}^{*,RB}(\tau_1, \tau_2) \overset{w}{\Rightarrow}_p \frac{\sqrt{\tau_2 - \tau_1}\int_0^\tau \sqrt{[M_u]'[M_v]'}\mathrm{d}B}{\sqrt{\{[M_u](\tau_2) - [M_u](\tau_1)\}\{[M_v](\tau_2) - [M_v](\tau_1)\}}}$$

on $\mathcal{D}(T_\Delta)$. From here the 'weak in probability' limits of $\mathcal{T}^x$, $x \in \{R, F, B\}$, follow by a further application of the same CMT, except for the fixed-regressor bootstrap statistics under Assumption 1.2. These latter limits follow along the lines of Demetrescu *et al.* (2022).

We notice that the condition $M_{\xi u}^* \overset{d}{=} M_{\xi u}$, which is necessary and sufficient for the validity of the residual-based fixed-regressor bootstrap under Assumption 1.1, is satisfied iff $\sum_{i,j\geq 0} \mathbb{I}_{\{i\neq j\}} b_i b_j \mathrm{E}[(\boldsymbol{\psi}_1 \boldsymbol{\psi}_1') \otimes (\boldsymbol{\psi}_{-i} \boldsymbol{\psi}_{-j}')] = 0$. For the latter to hold, it suffices that $\mathrm{E}[(\boldsymbol{\psi}_1 \boldsymbol{\psi}_1') \otimes (\boldsymbol{\psi}_{-i} \boldsymbol{\psi}_{-j}')] = 0$ for all natural $i \neq j$. $\square$

C.4 Proofs of the auxiliary results

We observe for use throughout the proofs that $(u_t, v_t)'$ inherit the uniform L_4-boundedness of $(a_t, e_t)'$ inasmuch as $\sup_{t\leq T} \mathrm{E}u_t^4 \leq C \|\mathbf{H}\|_\infty (\sup_t \mathrm{E}a_t^4 + \sup_t \mathrm{E}e_t^4)$ with $\|\mathbf{H}\|_\infty := \sup_{r\in(-\infty,1]} \|\mathbf{H}(r)\| < \infty$, and similarly for $\sup_{t\leq T} \mathrm{E}v_t^4$.

Proof of Lemma 1. In parts (a)-(c) we provide a proof for the sequences constructed from u_t, as for those constructed from v_t the argument is analogous. It holds that

$$S_{T(1,t)}^u = \sum_{s=1}^t h_{11}^2(\frac{s}{T})\Delta\Psi_s^a + 2\sum_{s=1}^t h_{11}(\frac{s}{T})h_{12}(\frac{s}{T})\Delta\Psi_s^{ae} + \sum_{t=1}^T h_{12}^2(\frac{s}{T})\Delta\Psi_s^e.$$

In part (a) we find by partial summation that

$$\left|\sum_{s=1}^t h_{11}^2(\frac{s}{T})\Delta\Psi_s^a\right| = \left|\Psi_t^a h_{11}^2(\frac{t}{T}) - \sum_{s=2}^t \Psi_{s-1}^a \Delta h_{11}^2(\frac{s}{T})\right| \leq C \max_{1\leq s\leq t} |\Psi_s^a|,$$

and similarly for the other two summations in the decomposition of $S^u_{T(1,t)}$, with the constant C depending on the global Lipschitz constant of $\mathbf{H}$. Therefore,

$$\max_{1\leq t\leq T}|S^u_{T(1,t)}| \leq C\left(\max_{1\leq t\leq T}|\Psi^a_t| + 2\max_{1\leq t\leq T}|\Psi^{ae}_t| + \max_{1\leq t\leq T}|\Psi^e_t|\right). \tag{C.5}$$

The three maxima on the right-hand side are all $O_p(T^{1/2})$ by Theorem 11 of Merlevède *et al.* (2006). Hence, also $\max_{1\leq t\leq T}|S^u_{T(1,t)}| = O_p(T^{1/2})$.

In part (b), by writing $(S^u_{T(t+1,r)})^2 = (S^u_{T(1,r)} - S^u_{T(1,t)})^2 \leq 4\max_{1\leq t\leq T}(S^u_{T(1,t)})^2$ and then using (C.5) we can conclude that

$$\mathrm{E}\left[\max_{1\leq t<r\leq T}(S^u_{T(t+1,r)})^2\right] \leq C\left(\mathrm{E}\left[\max_{1\leq t\leq T}(\Psi^a_t)^2\right] + \mathrm{E}\left[\max_{1\leq t\leq T}(\Psi^{ae}_t)^2\right] + \mathrm{E}\left[\max_{1\leq t\leq T}(\Psi^e_t)^2\right]\right).$$

Under Assumption 3 with $\epsilon < \frac{1}{2}$, the three expectations on the r.h.s. are $O(T)$ by Proposition 9 of Merlevède *et al.* (2006), and thus, so is the expectation on the l.h.s.

In part (c), $|\mathrm{E}[(u^2_t - \sigma^2_{ut})S^u_{T(t+1,r)}]| = |\mathrm{E}[(u^2_t - \sigma^2_{ut})\mathrm{E}_t S^u_{T(t+1,r)}]| \leq \|u^2_t - \sigma^2_{ut}\|_2\|\mathrm{E}_t S^u_{T(t+1,r)}\|_2$, where

$$\begin{aligned}\left\|\mathrm{E}_t S^u_{T(t+1,r)}\right\|_2 \leq &\left\|\mathrm{E}_t\left(\sum_{s=t+1}^{r} h^2_{11}(\tfrac{s}{T})\Delta\Psi^a_s\right)\right\|_2 + 2\left\|\mathrm{E}_t\left(\sum_{s=t+1}^{r} h_{11}(\tfrac{s}{T})h_{12}(\tfrac{s}{T})\Delta\Psi^{ae}_s\right)\right\|_2\\ &+\left\|\mathrm{E}_t\left(\sum_{s=t+1}^{r} h^2_{12}(\tfrac{s}{T})\Delta\Psi^e_s\right)\right\|_2,\end{aligned}$$

and, using partial summation and the stationarity of a_t,

$$\begin{aligned}\left\|\mathrm{E}_t\left(\sum_{s=t+1}^{r} h^2_{11}(\tfrac{s}{T})\Delta\Psi^a_s\right)\right\|_2 &= \left\|\mathrm{E}_t\left[(\Psi^a_r - \Psi^a_t)h^2_{11}(\tfrac{r}{T}) - \sum_{s=t+2}^{r}(\Psi^a_{s-1} - \Psi^a_t)\Delta h^2_{11}(\tfrac{s}{T})\right]\right\|_2\\ &= \left\|\mathrm{E}_0\left(\Psi^a_{r-t}h^2_{11}(\tfrac{r}{T}) - \sum_{s=t+2}^{r}\Psi^a_{s-t-1}\Delta h^2_{11}(\tfrac{s}{T})\right)\right\|_2\\ &\leq h^2_{11}(\tfrac{r}{T})\left\|\mathrm{E}_0\Psi^a_{r-t}\right\|_2 + \sum_{s=t+2}^{r}\left\|\mathrm{E}_0\Psi^a_{s-t-1}\right\|_2\left|\Delta h^2_{11}(\tfrac{s}{T})\right|\\ &\leq C\max_{1\leq t\leq T}\|\mathrm{E}_0\Psi^a_T\|_2 = O(T^\epsilon)\end{aligned}$$

uniformly in r, t, and similarly for the other two conditional expectations in the upper bound for $|\mathrm{E}[(u^2_t - \sigma^2_{ut})S^u_{T(t+1,r)}]|$, with the constant C depending on the global Lipschitz constant of $\mathbf{H}$. We conclude that $\|\mathrm{E}_t S^u_{T(t+1,r)}\|_2 = O(T^\epsilon)$ uniformly in r, t. As $\|u^2_t - \sigma^2_{ut}\|_2$ is a bounded sequence, part (c) follows. □

Proof of Lemma 2. In part (a), by using (C.1), we find that

$$\max_{s\leq T}\left|T^{-\alpha}\sum_{t=1}^{s}\varrho^{t-1}w_t\right| \leq \max_{s\leq T}\left|T^{-\alpha}\sum_{t=1}^{s}w_t\right| \Rightarrow \sup_{\tau\in[0,1]}|W(\tau)|$$

by the CMT, from where the magnitude order of $\max_{s\leq T}\left|T^{-\alpha}\sum_{t=1}^{s}\varrho^{t-1}w_t\right|$ follows.

In part (b), for every $r \in \{1, ..., T\}$ (C.1) yields

$$\mathrm{E}\left|\sum_{t=1}^{r} \varrho^{t-1} w_t\right| \leq \varrho^{r-1}\mathrm{E}\left|\sum_{t=1}^{r} w_t\right| + (1-\varrho)\sum_{s=1}^{r-1} \varrho^{s-1}\mathrm{E}\left|\sum_{t=1}^{s} w_t\right|$$
$$\leq (\varrho^{r-1} + (1-\varrho)\sum_{s=1}^{r-1} \varrho^{s-1}) \max_{1\leq s\leq T} \mathrm{E}\left|\sum_{t=1}^{s} w_t\right| = \max_{1\leq s\leq T} \mathrm{E}\left|\sum_{t=1}^{s} w_t\right|$$

and the conclusion follows by taking maxima over r.

We turn to part (c) and discuss the nontrivial case $m_T := \max_{t\leq T} \mathrm{E}|w_{Tt}|^p > 0$. If w_{Tt} is an MD array with $\mathrm{E}|w_{Tt}|^p < \infty$ for some $p > 2$, then

$$\mathrm{E}\left|\sum_{j=0}^{t-1} \varrho^j w_{T,t-j}\right|^p \leq C\left(\sum_{j=0}^{t-1} \varrho^{2j} \left(\mathrm{E}|w_{T,t-j}|^p\right)^{2/p}\right)^{p/2}$$

by Lemma 2.5.2 of Giraitis et al. (2012). Further,

$$\mathrm{E}\left|\sum_{j=0}^{t-1} \varrho^j w_{T,t-j}\right|^p \leq Cm_T\left(\sum_{j=0}^{T} \varrho^{2j}\right)^{p/2} \leq Cm_T T^{\frac{\eta p}{2}},$$

such that $\sum_{j=0}^{t-1} \varrho^j w_{T,t-j} = O_p(m_T^{1/p}T^{\eta/2}) = o_p(m_T^{1/p}T^{1/2})$ for every fixed $t \leq T$. To obtain the same infinitesimality order uniformly, we apply Billingsley's (1968, Theorem 15.6) tightness criterion to $m_T^{-1/p}T^{-1/2}W_T(\tau)$ with $W_T(\tau) := \sum_{j=0}^{\lfloor T\tau\rfloor-1} \varrho^j w_{T,\lfloor T\tau\rfloor-j}$. For $0 \leq \tau_1 < \tau < \tau_2 \leq 1$, it holds that

$$\mathrm{E}[|W_T(\tau_2) - W_T(\tau)|^{p/2}|W_T(\tau) - W_T(\tau_1)|^{p/2}] \leq \sqrt{\mathrm{E}|W_T(\tau_2) - W_T(\tau)|^p\mathrm{E}|W_T(\tau) - W_T(\tau_1)|^p}$$

where

$$\mathrm{E}|W_T(\tau_2) - W_T(\tau)|^p = \mathrm{E}\left|\sum_{j=0}^{\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor-1} \rho^j w_{T,\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor-j} + (\varrho^{\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor} - 1)\sum_{j=0}^{\lfloor\tau T\rfloor-1} \rho^j w_{T,\lfloor\tau T\rfloor-j}\right|^p$$
$$\leq \left[\sum_{j=0}^{\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor-1} \rho^{2j}(\mathrm{E}|w_{T,\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor-j}|^p)^{2/p}\right.$$
$$\left. + (\varrho^{\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor} - 1)^2\sum_{j=0}^{\lfloor\tau T\rfloor-1} \rho^{2j}(\mathrm{E}|w_{T,\lfloor\tau T\rfloor-j}|^p)^{2/p}\right]^{p/2}.$$

by Lemma 2.5.2 of Giraitis et al. (2012), then

$$\mathrm{E}|W_T(\tau_2) - W_T(\tau)|^p \leq m_T(1-\rho^2)^{-p/2}\left[1 - \rho^{2(\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor)} + (\varrho^{\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor} - 1)^2(1 - \varrho^{2\lfloor\tau T\rfloor})\right]^{p/2}$$
$$= m_T(1-\rho^2)^{-p/2}(1-\rho^{\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor})^{p/2}\left[1 + \rho^{\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor}\right.$$
$$\left. + (1-\rho^{\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor})(1-\varrho^{2\lfloor\tau T\rfloor})\right]^{p/2}$$
$$\leq m_T(1-\rho^2)^{-p/2}(1-\rho^{\lfloor\tau_2 T\rfloor-\lfloor\tau T\rfloor})^{p/2}3^{p/2}.$$

and by Bernoulli's inequality,

$$\mathrm{E}|W_T(\tau_2) - W_T(\tau)|^p \leq m_T \frac{(3a)^{p/2}(\lfloor \tau_2 T\rfloor - \lfloor \tau T\rfloor)^{p/2}}{T^{\eta p/2}(1-\rho^2)^{p/2}} \leq C m_T (\lfloor \tau_2 T\rfloor - \lfloor \tau T\rfloor)^{p/2},$$

such that, with a similar estimate for $\mathrm{E}|W_T(\tau) - W_T(\tau_1)|^p$, eventually

$$\begin{aligned} m_T^{-1} T^{-p/2} \mathrm{E}[|W_T(\tau_2) - W_T(\tau)|^{p/2}|W_T(\tau) - W_T(\tau_1)|^{p/2}] &\leq C T^{-p/2}(\lfloor \tau_2 T\rfloor - \lfloor \tau T\rfloor)^{p/4}(\lfloor \tau_1 T\rfloor - \lfloor \tau_1 T\rfloor)^{p/4} \\ &\leq C(\frac{\lfloor \tau_2 T\rfloor - \lfloor \tau_1 T\rfloor}{T})^{p/2} \leq C(\tau_2 - \tau_1)^{p/2}. \end{aligned}$$

Since $p/2 > 1$, as required by Billingsley's criterion, it follows that $m_T^{-1/p} T^{-1/2} W_T(\tau)$ is tight.

In part (d), (C.1) yields $\max_{1\leq s\leq T} |\sum_{t=1}^{T-s} \varrho^{2(t-1)}(u_{s+t}^2 - \sigma_{u,s+t}^2)| \leq 2 \max_{1\leq s\leq T} |S^u_{T(1,s)}| = O_p(T^{1/2})$ by Lemma 1(a). Similarly, in part (e),

$$\max_{1\leq t\leq T} \left\| \mathrm{E}_t \sum_{r=t+1}^{T} \varrho^{2(r-t-1)}(u_r^2 - \sigma_{ur}^2) \right\|_2 \leq \max_{1\leq t<r\leq T} \left\| \mathrm{E}_t S_{T(t+1,r)} \right\|_2 = O(T^\epsilon)$$

by the proof of Lemma 1(c).

We turn to the proof of part (f). It holds that

$$\begin{aligned} \left[\sum_{s=1}^{\lfloor T\tau\rfloor} \left(\sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2\right)(u_t^2 - \sigma_{ut}^2)\right]^2 &= \left[\sum_{s=1}^{\lfloor T\tau\rfloor - 1} v_s^2 \sum_{t=s+1}^{T} \varrho^{2(t-s-1)}(u_t^2 - \sigma_{ut}^2)\right]^2 \\ &\leq \sum_{s=1}^{\lfloor T\tau\rfloor - 1} v_s^4 \sum_{s=1}^{\lfloor T\tau\rfloor - 1} \left[\sum_{t=s+1}^{T} \varrho^{2(t-s-1)}(u_t^2 - \sigma_{ut}^2)\right]^2. \end{aligned}$$

As $\sum_{s=1}^{\lfloor T\tau\rfloor - 1} v_s^4 = O_p(T)$ by Markov's inequality, part (c) will follow if

$$\sum_{s=1}^{\lfloor T\tau\rfloor - 1} \left[\sum_{t=s+1}^{T} \varrho^{2(t-s-1)}(u_t^2 - \sigma_{ut}^2)\right]^2 = O_p(T^{1+\eta+\epsilon}). \tag{C.6}$$

In the decomposition

$$\begin{aligned} \mathrm{E}\left[\sum_{t=s+1}^{T} \varrho^{2(t-s-1)}(u_t^2 - \sigma_{ut}^2)\right]^2 &= \sum_{t=s+1}^{T} \varrho^{4(t-s-1)} \mathrm{E}(u_t^2 - \sigma_{ut}^2)^2 \\ &\quad + 2 \sum_{t=s+1}^{T} \varrho^{2(t-s-1)} \sum_{r=t+1}^{T} \varrho^{2(r-t-1)} \mathrm{E}[(u_t^2 - \sigma_{ut}^2)(u_r^2 - \sigma_{ur}^2)] \end{aligned}$$

eq. (C.1) can be used to bound the mixed products as follows:

$$\begin{aligned} \left|\sum_{r=t+1}^{T} \varrho^{2(r-t-1)} \mathrm{E}[(u_t^2 - \sigma_{ut}^2)(u_r^2 - \sigma_{ur}^2)]\right| &\leq \max_{t+1\leq q\leq T} \left|\mathrm{E}\left[(u_t^2 - \sigma_{ut}^2) \sum_{r=t+1}^{q} (u_r^2 - \sigma_{ur}^2)\right]\right| \\ &\leq \max_{t+1\leq q\leq T} \left|\mathrm{E}[(u_t^2 - \sigma_{ut}^2) S^u_{T(t+1,q)}]\right|. \end{aligned}$$

As $\max_{1\le t\le T}\left\|u_t^2-\sigma_{ut}^2\right\|_2=O(1)$, it can be concluded that

$$\mathrm{E}\left[\sum_{t=s+1}^{T}\varrho^{2(t-s-1)}(u_t^2-\sigma_{ut}^2)\right]^2=O(T^\eta)+2\max_{1\le t\le q\le T}\left|\mathrm{E}[(u_t^2-\sigma_{ut}^2)S_{T(t+1,q)}^u]\right|\sum_{t=s+1}^{T}\varrho^{2(t-s-1)}$$

uniformly in $s\le T$, such that (C.6) follows by Markov's inequality and Lemma 1(c). This completes the proof of part (f).

Finally, to prove part (g), we first show that $\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\sigma_{ut}^2=o_p(T^{1+\eta})$ pointwise. In fact,

$$\left[\sum_{t=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right)\sigma_{ut}^2\right]^2\le\sum_{t=1}^{T}\left[\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right]^2\sum_{t=1}^{T}\sigma_{ut}^4,$$

where $\sum_{t=1}^{T}\sigma_{ut}^4=O(T)$, whereas the other factor on the right-hand side is $O_p(T^{1+\eta+\epsilon})$ similarly to an analogous expression in the proof of part (f). Specifically,

$$\begin{aligned}\mathrm{E}\left[\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right]^2&=\sum_{j=0}^{t-2}\varrho^{4j}\mathrm{E}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)^2\\&\quad+2\sum_{j=0}^{t-2}\sum_{i=j+1}^{t-2}\varrho^{2(j+i)}\mathrm{E}[v_{t-i-1}^2(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)],\end{aligned}$$

where $\sum_{j=0}^{t-2}\varrho^{4j}\mathrm{E}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)^2\le\max_{1\le t\le T}\left\|v_t^2-\sigma_{vt}^2\right\|_2^2\sum_{j=0}^{T}\varrho^{4j}=O(T^\eta)$ and

$$\sum_{j=0}^{t-2}\sum_{i=j+1}^{t-2}\varrho^{2(j+i)}\mathrm{E}[v_{t-i-1}^2(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)]=\sum_{s=1}^{t-1}\varrho^{4(t-s-1)}\sum_{r=s+1}^{t-1}\varrho^{2(s-r)}\mathrm{E}[v_s^2(v_r^2-\sigma_{vr}^2)]$$

with

$$\begin{aligned}\left|\sum_{r=s+1}^{t-1}\varrho^{2(s-r)}\mathrm{E}[v_s^2(v_r^2-\sigma_{vr}^2)]\right|&\le\max_{s+1\le q\le t-1}\left|\sum_{r=s+1}^{q}\mathrm{E}[v_s^2(v_r^2-\sigma_{vr}^2)]\right|\\&=\max_{s+1\le q\le t-1}\left|\mathrm{E}(v_s^2S_{T(s+1,q)}^v)\right|\\&\le\max_{1\le s<q\le T}\left|\mathrm{E}(v_s^2S_{T(s+1,q)}^v)\right|=O(T^\epsilon)\end{aligned}$$

using (C.1) and Lemma 1(c). As the upper bounds are uniform in $t=1,...,T$, it follows that

$$\mathrm{E}\sum_{t=1}^{T}\left[\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right]^2=O(T^{1+\eta})+O(T^\epsilon)\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varrho^{4(t-s-1)}=O(T^{1+\eta+\epsilon}).$$

This and Markov's inequality let us conclude that $\sum_{t=1}^{T}\left[\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right]^2=O_p(T^{1+\eta+\epsilon})$ and hence, $\sum_{t=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}(v_{t-j-1}^2-\sigma_{v,t-j-1}^2)\right)\sigma_{ut}^2=O_p(T^{1+(\eta+\epsilon)/2})=o_p(T^{1+\eta})$ for $\epsilon<\eta$. Equivalently, $\tilde V(\tau)=T^{-1-\eta}\omega^2\sum_{t=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}\sigma_{v,t-j-1}^2\right)\sigma_{ut}^2+o_p(1)$ pointwise.

Second, we discuss the convergence of the deterministic $\sum_{t=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}\sigma_{v,t-j-1}^2\right)\sigma_{ut}^2$ to an integral. Say for concreteness that the function $\mathbf{H}$ (determining the unconditional variance profile) of (u_t, v_t) is Lipschitz continuous on $[0,\lambda)$ and $(\lambda,1]$, the case of more than two (but finitely many) maximal intervals of Lipschitz continuity being analogous. Without loss of generality, let $\mathbf{H}$ be right-continuous at λ. Then, for $t < \lfloor T\lambda \rfloor$ it holds that

$$\sum_{j=0}^{t-2} \varrho^{2j} \left| \sigma_{v,t-j-1}^2 - \sigma_{v,t}^2 \right| \le C \sum_{j=0}^{t-2} \varrho^{2j} \left(\frac{j-1}{T} \right) = O(T^{2\eta-1})$$

uniformly in t, where C depends on the Lipschitz constant of the function $\mathbf{H}$, whereas for $t = \lfloor T\lambda \rfloor, ..., T$ the analogous estimate is

$$\begin{aligned} \left| \sum_{j=0}^{t-2} \varrho^{2j} (\sigma_{v,t-j-1}^2 - \sigma_{v,t}^2) - \sum_{j=t-\lfloor T\lambda \rfloor}^{t-2} \varrho^{2j} (\sigma_{v,\lfloor T\lambda \rfloor -1}^2 - \sigma_{v,t}^2) \right| &\le \sum_{j=0}^{t-\lfloor T\lambda \rfloor -1} \varrho^{2j} \left| \sigma_{v,t-j-1}^2 - \sigma_{v,t}^2 \right| \\ + \sum_{j=t-\lfloor T\lambda \rfloor}^{t-2} \varrho^{2j} \left| \sigma_{v,t-j-1}^2 - \sigma_{v,\lfloor T\lambda \rfloor -1}^2 \right| &\le C \sum_{j=0}^{t-\lfloor T\lambda \rfloor -1} \varrho^{2j} \left(\frac{j-1}{T} \right) \\ + C \sum_{j=t-\lfloor T\lambda \rfloor}^{t-2} \varrho^{2j} \left(\frac{j-\lfloor T\lambda \rfloor}{T} \right) &= O(T^{2\eta-1}). \end{aligned}$$

As a result,

$$\begin{aligned} \sum_{t=1}^{\lfloor T\tau \rfloor} \left(\sum_{j=0}^{t-2} \varrho^{2j} \sigma_{v,t-j-1}^2 \right) \sigma_{ut}^2 &= \sum_{t=1}^{\lfloor T\tau \rfloor} \left(\sum_{j=0}^{t-2} \varrho^{2j} \right) \sigma_{vt}^2 \sigma_{ut}^2 + \sum_{t=\lfloor T\lambda \rfloor}^{\lfloor T\tau \rfloor} \left(\sum_{j=t-\lfloor T\lambda \rfloor}^{t-2} \varrho^{2j} \right) (\sigma_{v,\lfloor T\lambda \rfloor -1}^2 - \sigma_{v,t}^2) \sigma_{ut}^2 \\ &+ O(T^{2\eta-1}) \sum_{t=1}^{\lfloor T\tau \rfloor} \sigma_{ut}^2 \\ &= \frac{1}{2a} T^{\eta} \sum_{t=1}^{\lfloor T\tau \rfloor} \sigma_{vt}^2 \sigma_{ut}^2 + O(T^{\eta}) \sum_{t=\lfloor T\lambda \rfloor}^{\lfloor T\tau \rfloor} \varrho^{2(t-\lfloor T\lambda \rfloor)} (\sigma_{v,\lfloor T\lambda \rfloor -1}^2 - \sigma_{v,t}^2) \sigma_{ut}^2 \\ &+ O(T^{2\eta}) = \frac{T^{1+\eta}}{2a} \int_0^{\tau} [M_v(s)]'[M_u(s)]' \mathrm{d}s + o(T^{1+\eta}) + O(T^{2\eta}) \end{aligned}$$

using the boundedness of σ_{ut}^2 and σ_{vt}^2. This establishes the pointwise limit asserted in part (g). As the involved processes are increasing and the limiting function is also continuous, the limit is a uniform one as well. $\square$

Proof of Lemma 3. It holds that

$$\frac{1}{T} \sum_{t=1}^{\lfloor T\tau \rfloor} \begin{pmatrix} u_t^2 - \sigma_{ut}^2 \\ v_t^2 - \sigma_{vt}^2 \end{pmatrix} = T^{-1} \begin{pmatrix} S_{T(1,\lfloor T\tau \rfloor)}^u \\ S_{T(1,\lfloor T\tau \rfloor)}^v \end{pmatrix} = O_p(T^{-1/2})$$

uniformly in τ, by Lemma 1(a). Further, $T^{-1} \sum_{t=1}^{\lfloor T\tau \rfloor} (\sigma_{ut}^2, \sigma_{vt}^2)'$ are Riemann sums of the limiting integral, which exists by the Lipschitz-by-parts property of $\mathbf{H}$, and convergence follows from the definition of the integral. The convergence is uniform because the involved coordinate functions are increasing and the limiting coordinate functions are continuous. $\square$

Proof of Lemma 4. With b_j the coefficients of $[A(L)(1-\rho L)]^{-1}$, where $|\rho|<1$ is bounded away from unity, let

$$\begin{aligned}\tilde{\xi}_{t-1} &= \sum_{j\geq 0} b_j \left(h_{21}(t/T)a_{t-1-j} + h_{22}(t/T)e_{t-1-j}\right) \\ &= h_{21}(t/T)\sum_{j\geq 0} b_j a_{t-1-j} + h_{22}(t/T)\sum_{j\geq 0} b_j e_{t-1-j}\end{aligned}$$

and note that

$$\xi_{t-1}-\tilde{\xi}_{t-1} = \sum_{j\geq 0} b_j \left(\left(h_{21}\left(\frac{t-1-j}{T}\right) - h_{21}\left(\frac{t}{T}\right)\right) a_{t-1-j} + \left(h_{22}\left(\frac{t-1-j}{T}\right) - h_{22}\left(\frac{t}{T}\right)\right) e_{t-1-j}\right).$$

Therefore,

$$\sum_{t=1}^{T} \mathrm{E}\left(\left|\xi_{t-1}-\tilde{\xi}_{t-1}\right|\right) \leq CT\sum_{j\geq 0}\frac{j+1}{T} b_j = O(1)$$

since the absolute moments are uniformly bounded, b_j are 1-summable (in fact they have exponential decay), and $h_{ij}(\cdot)$ are piecewise Lipschitz, where the discontinuities are accounted for along the lines of the proof of Lemma 2 (g). We may therefore write

$$\sup_{\tau\in[0,1]}\left|\sum_{t=1}^{\lfloor \tau T\rfloor} u_t\xi_{t-1} - \sum_{t=1}^{\lfloor \tau T\rfloor} u_t\tilde{\xi}_{t-1}\right| \leq \sum_{t=1}^{\lfloor \tau T\rfloor} |u_t|\left|\xi_{t-1}-\tilde{\xi}_{t-1}\right| \leq \sup_{1\leq t\leq T}|u_t|\sum_{t=1}^{T}\left|\xi_{t-1}-\tilde{\xi}_{t-1}\right| = o_p\left(\sqrt{T}\right)$$

thanks to Markov's inequality and the fact that uniformly bounded 4th order moments imply $\sup_{1\leq t\leq T}|u_t| = o_p\left(\sqrt{T}\right)$.

Then, uniformly in τ,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor \tau T\rfloor}\begin{pmatrix} u_t \\ v_t \\ u_t\xi_{t-1}\end{pmatrix} = \frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor \tau T\rfloor}\begin{pmatrix} u_t \\ v_t \\ u_t\tilde{\xi}_{t-1}\end{pmatrix} + o_p(1).$$

Now,

$$\begin{aligned} u_t\tilde{\xi}_{t-1} &= h_{11}(t/T)h_{21}(t/T)a_t\sum_{j\geq 0} b_j a_{t-1-j} + h_{11}(t/T)h_{22}(t/T)a_t\sum_{j\geq 0} b_j e_{t-1-j} \\ &\quad + h_{12}(t/T)h_{21}(t/T)e_t\sum_{j\geq 0} b_j a_{t-1-j} + h_{12}(t/T)h_{22}(t/T)e_t\sum_{j\geq 0} b_j e_{t-1-j}\end{aligned}$$

and we note (with all functions h_{ij} evaluated at t/T) that

$$\begin{pmatrix} u_t \\ v_t \\ u_t\tilde{\xi}_{t-1}\end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & 0 & 0 & 0 & 0 \\ h_{21} & h_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & h_{11}h_{21} & h_{11}h_{22} & h_{12}h_{21} & h_{12}h_{22}\end{pmatrix}\begin{pmatrix} a_t \\ e_t \\ a_t\sum_{j\geq 0} b_j a_{t-1-j} \\ a_t\sum_{j\geq 0} b_j e_{t-1-j} \\ e_t\sum_{j\geq 0} b_j a_{t-1-j} \\ e_t\sum_{j\geq 0} b_j e_{t-1-j}\end{pmatrix} = \mathbf{G}(t/T)\tilde{\boldsymbol{\psi}}_t.$$

Furthermore, the covariance matrix of $\tilde{\boldsymbol{\psi}}_t$ is constant and can be determined in a straightforward manner, e.g.

$$Cov\left(\tilde{\psi}_{t,3},\tilde{\psi}_{t,4}\right)=\sum_{j\geq 0}\sum_{k\geq 0}b_jb_k\mathrm{E}\left(a_t^2a_{t-1-j}e_{t-1-k}\right).$$

Finally, $\tilde{\psi}_t$ is easily seen to obey an invariance principle for stationary and ergodic square-integrable MDs, such that, summing up,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor\tau T\rfloor}\tilde{\psi}_t\Rightarrow\int_0^\tau\mathbf{G}(s)\mathrm{d}\boldsymbol{B}(s)$$

where $\boldsymbol{B}(\tau)$ is a 6-variate Brownian motion of covariance matrix $Cov\left(\tilde{\psi}_t\right)$.

Of particular importance is the quadratic variation (and implicitly the variance profile) of $M_{\xi u}(\tau)$ (the third component of $\int_0^\tau\mathbf{G}(s)\mathrm{d}\boldsymbol{B}(s)$), we have at all differentiability points

$$\frac{\mathrm{d}\left[M_{\xi u}\right](\tau)}{\mathrm{d}\tau}=\mathrm{Var}\left(u_{\lfloor\tau T\rfloor}\tilde{\xi}_{\lfloor\tau T\rfloor-1}\right)+O\left(\frac{1}{T}\right)$$

where (again with all functions h_{ij} evaluated at t/T),

$$\mathrm{Var}\left(u_t\tilde{\xi}_{t-1}\right)=\mathrm{E}\left((h_{11}a_t+h_{12}e_t)^2\left(h_{21}\sum_{j\geq 0}b_ja_{t-1-j}+h_{22}\sum_{j\geq 0}b_je_{t-1-j}\right)^2\right)$$

or

$$\begin{aligned}\mathrm{Var}\left(u_t\tilde{\xi}_{t-1}\right) &= h_{11}^2h_{21}^2\sum_{j\geq 0}\sum_{k\geq 0}b_jb_k\mathrm{E}\left(a_t^2a_{t-1-j}a_{t-1-k}\right)\\
&+2h_{11}^2h_{21}h_{22}\sum_{j\geq 0}\sum_{k\geq 0}b_jb_k\mathrm{E}\left(a_t^2a_{t-1-j}e_{t-1-k}\right)\\
&+h_{11}^2h_{22}^2\sum_{j\geq 0}\sum_{k\geq 0}b_jb_k\mathrm{E}\left(a_t^2e_{t-1-j}e_{t-1-k}\right)\\
&+2h_{11}h_{12}h_{21}^2\sum_{j\geq 0}\sum_{k\geq 0}b_jb_k\mathrm{E}\left(a_te_ta_{t-1-j}a_{t-1-k}\right)\\
&+4h_{11}h_{12}h_{21}h_{22}\sum_{j\geq 0}\sum_{k\geq 0}b_jb_k\mathrm{E}\left(a_te_ta_{t-1-j}e_{t-1-k}\right)\\
&+2h_{11}h_{12}h_{22}^2\sum_{j\geq 0}\sum_{k\geq 0}b_jb_k\mathrm{E}\left(a_te_te_{t-1-j}e_{t-1-k}\right)\\
&+h_{12}^2h_{21}^2\sum_{j\geq 0}\sum_{k\geq 0}b_jb_k\mathrm{E}\left(e_t^2a_{t-1-j}a_{t-1-k}\right)\\
&+2h_{12}^2h_{21}h_{22}\sum_{j\geq 0}\sum_{k\geq 0}b_jb_k\mathrm{E}\left(e_t^2a_{t-1-j}e_{t-1-k}\right)\\
&+h_{12}^2h_{22}^2\sum_{j\geq 0}\sum_{k\geq 0}b_jb_k\mathrm{E}\left(e_t^2e_{t-1-j}e_{t-1-k}\right).\end{aligned}$$

The previous variance is precisely $\chi(t/T)$ as defined in Lemma 6(a). □

In the proof of Lemma 5, z_t is frequently approximated by $\omega\zeta_t$, where $\zeta_{t-1}=(1-\varrho L)_+^{-1}v_{t-1}$ and the approximation error can be controlled for in most sums, but not all (see the partial sums of z_t). □

Proof of Lemma 5(a). It holds that

$$z_t = \sum_{j=0}^{t-1} \varrho^j w_{t-j} - (c/T) \sum_{j=0}^{t-1} \varrho^j \xi_{t-j-1} \tag{C.7}$$

where, by using the Beveridge-Nelson decomposition $w_t = A^{-1}(L)v_t = \omega v_t - \Delta \tilde{v}_t$ (which defines $\tilde{v}_t$) and (C.1),

$$\begin{aligned}\sum_{j=0}^{t-1} \varrho^j w_{t-j} &= \omega \sum_{j=0}^{t-1} \varrho^j v_{t-j} - \sum_{j=0}^{t-1} \varrho^j \Delta \tilde{v}_{t-j} \\ &= \omega \zeta_t - \tilde{v}_t + \varrho^{t-1} \tilde{v}_0 + (1-\varrho) \sum_{s=1}^{t-1} \varrho^{s-1} \tilde{v}_{t-s}\end{aligned}$$

with $\zeta_t = \sum_{j=0}^{t-1} \varrho^j v_{t-j}$. Write $\sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1} = \omega Z_1(\tau) + Z_2(\tau) - (c/a) T^{\eta-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \xi_{t-2}$ with, first,

$$\begin{aligned}Z_1(\tau) := \sum_{t=1}^{\lfloor \tau T \rfloor} \zeta_{t-1} &= \left(\sum_{j=0}^{\infty} \varrho^j \right) \sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} - \sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} \left(\sum_{j=\lfloor \tau T \rfloor - t+1}^{\infty} \varrho^j \right) \\ &= a^{-1} T^{\eta} \left(\sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} - \sum_{t=2}^{\lfloor \tau T \rfloor} \varrho^{\lfloor \tau T \rfloor - t+1} v_{t-1} \right) \\ &= a^{-1} T^{\eta} \left(\sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} - \zeta_{\lfloor \tau T \rfloor - 1} \right) = a^{-1} T^{\eta} \sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} + o_p(T^{\eta+1/2})\end{aligned}$$

uniformly in $\tau \in [0,1]$ because $\max_{t \leq T} |\zeta_t| = o_p(T^{1/2})$ by Lemma 2(c) with $w_{Tt} = v_t, p = 4$ and $\max_{1 \leq t \leq T} \mathrm{E} v_t^4 = O(1)$. Second,

$$Z_2(\tau) := \sum_{t=1}^{\lfloor \tau T \rfloor} \sum_{j=0}^{t-2} \varrho^j \Delta \tilde{v}_{t-1-j} = \sum_{j=0}^{\lfloor \tau T \rfloor - 2} \varrho^j \tilde{v}_{\lfloor \tau T \rfloor - 1 - j} - \tilde{v}_0 \sum_{j=0}^{\lfloor \tau T \rfloor - 2} \varrho^j = o_p(T^{\eta+1/2})$$

uniformly in $\tau \in [0,1]$ because because $\max_{t \leq T} |\sum_{j=0}^{t-1} \varrho^j \tilde{v}_{t-j}| = o_p(T^{1/2})$ by Lemma 2(c) with $w_{Tt} = \tilde{v}_t, p = 4$ and $\max_{1 \leq t \leq T} \mathrm{E} \tilde{v}_t^4 = O(1)$. By collecting the previous results, it follows that, uniformly in $\tau \in [0,1]$,

$$\frac{1}{T^{1/2+\eta}} \sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1} = \frac{\omega}{a} \frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor \tau T \rfloor} v_{t-1} - \frac{c}{a} \frac{1}{T^{3/2}} \sum_{t=2}^{\lfloor \tau T \rfloor} \xi_{t-2} + o_p(1) \Rightarrow \frac{\omega}{a} \left(M_v(\tau) - c \int_0^{\tau} J_{c,H}(s) \mathrm{d}s \right),$$

using in particular the continuity of the two summand processes. The latter limit is $\frac{\omega}{a} J_{c,H}(\tau)$ by the Ornstein-Uhlenbeck differential equation. □

Proof of Lemma 5(b). We first show that

$$\max_{t \leq T} \left| \sum_{j=0}^{t-1} \varrho^j \xi_{t-j} v_{t-j} \right| = o_p(T).$$

For any fixed $K > 0$, the following decomposition holds:

$$\begin{aligned}
\sum_{j=0}^{t-1} \varrho^j \xi_{t-j} v_{t-j} &= \sum_{j=0}^{t-1} \varrho^j \xi_{t-j-1} v_{t-j} + \sum_{j=0}^{t-1} \varrho^j \Delta\xi_{t-j} v_{t-j} \\
&= (1 - \frac{c}{T}) \sum_{j=0}^{t-1} \varrho^j \xi_{t-j-1} v_{t-j} + \sum_{j=0}^{t-1} \varrho^j w_{t-j} v_{t-j} \\
&= (1 - \frac{c}{T}) \left(\sum_{j=0}^{t-1} \varrho^j \mathbb{I}_{\{|\xi_{t-j-1}| \le T^{1/2}K\}} \xi_{t-j-1} v_{t-j} + \sum_{j=0}^{t-1} \varrho^j \mathbb{I}_{\{|\xi_{t-j-1}| > T^{1/2}K\}} \xi_{t-j-1} v_{t-j} \right) \\
&+ \sum_{j=0}^{t-1} \varrho^j (v_{t-j}^2 - \sigma_{v,t-j}^2) + \sum_{j=0}^{t-1} \varrho^j \sigma_{v,t-j}^2 + \sum_{j=0}^{t-1} \varrho^j v_{t-j} (w_{t-j} - v_{t-j}).
\end{aligned}$$

Here $\sum_{j=0}^{t-1} \varrho^j \mathbb{I}_{\{|\xi_{t-j-1}| \le T^{1/2}K\}} \xi_{t-j-1} v_{t-j} = o_p(T)$ by Lemma 2(c) with $w_{Tt} = \mathbb{I}_{\{|\xi_{t-j-1}| \le T^{1/2}K\}} \xi_{t-j-1} v_{t-j}$, $p = 4$ and $\max_{1 \le t \le T} \mathrm{E} w_{Tt}^4 = O(T^2)$. Since $\max_{t \le T} |\xi_t| = O_p(T^{1/2})$, it follows that, by choosing K sufficiently large, $\sum_{j=0}^{t-1} \varrho^j \mathbb{I}_{\{|\xi_{t-j-1}| > T^{1/2}K\}} \xi_{t-j-1} v_{t-j}$ can be made equal to zero with probability as close to one as desired. Next, by (C.1) and Lemma 1(a),

$$\max_{1 \le t \le T} \left| \sum_{j=0}^{t-1} \varrho^j (v_{t-j}^2 - \sigma_{v,t-j}^2) \right| \le \max_{1 \le t \le T} |S_{T(1,t)}^v| = O_p(T^{1/2}),$$

whereas $\sum_{j=0}^{t-1} \varrho^j \sigma_{v,t-j}^2 = O(T^\eta) = o(T)$ by the boundedness of σ_{vt}^2. Finally, for any fixed $L > 0$,

$$\sum_{j=0}^{t-1} \varrho^j v_{t-j} (w_{t-j} - v_{t-j}) = \sum_{j=0}^{t-1} \varrho^j \left[\mathbb{I}_{\{|w_{t-j} - v_{t-j}| \le T^{1/2}L\}} + \mathbb{I}_{\{|w_{t-j} - v_{t-j}| > T^{1/2}L\}} \right] v_{t-j} (w_{t-j} - v_{t-j}),$$

where $w_{t-j} - v_{t-j} = \sum_{i=1}^{\infty} b_i v_{t-j-i}$ is in the past of v_{t-j}. Thus, $\sum_{j=0}^{t-1} \mathbb{I}_{\{|w_{t-j} - v_{t-j}| \le T^{1/2}L\}} v_{t-j} (w_{t-j} - v_{t-j}) = o_p(T)$ by Lemma 2(c) with $w_{Tt} = \mathbb{I}_{\{|w_{t-j} - v_{t-j}| \le T^{1/2}L\}} v_{t-j} (w_{t-j} - v_{t-j})$, $p = 4$ and $\max_{1 \le t \le T} \mathrm{E} w_{Tt}^4 = O(T^2)$. As $\max_{t \le T} |w_{t-j} - v_{t-j}| = o_p(T^{1/2})$ because $\mathrm{E}|w_t - v_t|^4$ is a bounded sequence, by choosing L sufficiently large $\sum_{j=0}^{t-1} \mathbb{I}_{\{|w_{t-j} - v_{t-j}| > T^{1/2}L\}} v_{t-j} (w_{t-j} - v_{t-j})$ can be made equal to zero with probability as close to one as desired. By combining the previous conclusions, it follows that $\max_{t \le T} \left| \sum_{j=0}^{t-1} \varrho^j \xi_{t-j} v_{t-j} \right| = o_p(T)$.

We turn to the process of main interest in part (b). Similarly to part (a), it holds that $\sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1} x_{t-1} = \omega Z X_1(\tau) + Z X_2(\tau) - (c/a) T^{\eta - 1} \sum_{t=2}^{\lfloor \tau T \rfloor} \xi_{t-2} \xi_{t-1} + o_p(T^{1/2 + \eta})$ uniformly in $\tau \in [0, 1]$, with the remainder $\mu_x \sum_{t=1}^{\lfloor \tau T \rfloor} z_{t-1}$ discussed in part (a). The summands $Z X_i(\tau) := \sum_{t=2}^{\lfloor \tau T \rfloor} \Delta Z_i(\frac{t}{T}) \xi_{t-1}$ $(i = 1, 2)$ behave as follows. First,

$$\begin{aligned}
Z X_1(\tau) &= \sum_{t=2}^{\lfloor \tau T \rfloor} \zeta_{t-1} \xi_{t-1} = a^{-1} T^\eta \left(\sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} \xi_{t-1} - \sum_{t=2}^{\lfloor \tau T \rfloor} \varrho^{\lfloor \tau T \rfloor - t + 1} v_{t-1} \xi_{t-1} \right) \\
&= a^{-1} T^\eta \sum_{t=2}^{\lfloor \tau T \rfloor} v_{t-1} \xi_{t-1} + o_p(T^{1+\eta})
\end{aligned}$$

uniformly in $\tau \in [0,1]$ because $\max_{t\le T}\left|\sum_{j=0}^{t-1}\varrho^j\xi_{t-j}v_{t-j}\right| = o_p(T)$ as shown previously. Second,

$$
\begin{aligned}
ZX_2(\tau) :=& \sum_{t=2}^{\lfloor \tau T\rfloor}(\tilde{v}_{t-1}-(1-\varrho)\sum_{j=1}^{t-2}\varrho^{j-1}\tilde{v}_{t-1-j}-\varrho^{t-2}\tilde{v}_0)\xi_{t-1} \\
=& O_p(T)+(1-\varrho)O_p(T^{1+\eta})+O_p(T^{1/2+\eta}) = o_p(T^{1+\eta})
\end{aligned}
$$

uniformly in $\tau \in [0,1]$ because $T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}\tilde{v}_{t-1}\xi_{t-1}$ converges weakly in $\mathcal{D}$, $\sum_{t=2}^{\lfloor \tau T\rfloor}\sum_{j=1}^{t-2}\varrho^{j-1}\tilde{v}_{t-1-j}\xi_{t-1}$ is of the same form (and thus, uniform magnitude order) as $ZX_1(\tau)$, and $\max_{t\le T}|\xi_t| = O_p(T^{1/2})$. Recollecting the results about $ZX_i(\tau)$ $(i=1,2)$, we find that

$$
\frac{1}{T^{1+\eta}}\sum_{t=1}^{\lfloor \tau T\rfloor}z_{t-1}x_{t-1} = \frac{\omega}{a}\frac{1}{T}\sum_{t=2}^{\lfloor \tau T\rfloor}v_{t-1}\xi_{t-1} - \frac{c}{a}\frac{1}{T^2}\sum_{t=2}^{\lfloor \tau T\rfloor}\xi_{t-2}\xi_{t-1}+o_p(1),
$$

where the summations on the right-hand side are not affected by mild integration. It then follows by standard near-integration asymptotics that

$$
\begin{aligned}
\frac{1}{T^{1+\eta}}\sum_{t=1}^{\lfloor \tau T\rfloor}z_{t-1}x_{t-1} \Rightarrow& \frac{\omega^2}{a}\left(\int_0^\tau J_{c,H}dM_v+[M_v]_\tau - c\int_0^\tau J_{c,H}^2\right) \\
=& \frac{\omega^2}{a}\left(\int_0^\tau J_{c,H}dJ_{c,H}+[M_v]_\tau\right),
\end{aligned}
$$

the equality by the Ornstein-Uhlenbeck differential equation. It remains to note that $[M_v]_\tau = [J_{c,H}]_\tau$ and $J_{c,H}^2(\tau) - \int_0^\tau J_{c,H}dJ_{c,H} = \int_0^\tau J_{c,H}dJ_{c,H} + [J_{c,H}]_\tau$, the latter by the semimartingale property of $J_{c,H}$. (Alternative functional representations of the limit are, thus, are given by $\frac{\omega^2}{2a}(J_{c,H}^2(\tau)+[J_{c,H}]_\tau) = \frac{\omega^2}{2a}(J_{c,H}^2(\tau)+[M_v]_\tau)$. □

Proof of Lemma 5(c). Since the involved processes are increasing and the function $[M_v](\tau)$ is continuous, with the interval $[0,1]$ compact, it is sufficient to show that the asserted convergence holds pointwise in probability for $\tau \in [0,1]$.

First, we argue that terms involving the local parameter c and v_{-i}, $i \in \mathbb{N}\cup\{0\}$, are asymptotically negligible. Recall (C.7). Since

$$
\sum_{t=1}^{\lfloor \tau T\rfloor}\left(\sum_{j=0}^{t-1}\varrho^j\xi_{t-j-1}\right)^2 \le \max_{t=0,\dots,T}\xi_t^2\sum_{t=1}^{T}\left(\sum_{j=0}^{t-1}\varrho^j\right)^2 = O_p(T^{2+2\eta}) = o_p(T^{3+\eta})
$$

and $\theta_{\lfloor \tau T\rfloor} = \sum_{t=1}^{\lfloor \tau T\rfloor}\left\{\sum_{i=t-1}^{\infty}v_{t-1-i}\left(\sum_{j=0}^{t-1}\varrho^j b_{i-j}\right)\right\}^2 \ge 0$ with

$$
\begin{aligned}
\mathrm{E}\theta_{\lfloor \tau T\rfloor} =& \sum_{t=1}^{\lfloor \tau T\rfloor}\sum_{i=t-1}^{\infty}\mathrm{E}v_{t-1-i}^2\left(\sum_{j=0}^{t-1}\varrho^j b_{i-j}\right)^2 \le C\sum_{t=1}^{T}\sum_{j,k=0}^{t-1}\varrho^j\varrho^k\sum_{i=t-1}^{\infty}b_{i-j}b_{i-k} \\
\le& C\sum_{k=0}^{T-1}\sum_{j=0}^{k}\varrho^j\varrho^k\sum_{t=1}^{T-k}\sum_{i=t-1}^{\infty}|b_{i+k-j}||b_i| \le C\sum_{k=0}^{T-1}\sum_{j=0}^{k}\varrho^j\varrho^k\sum_{i=0}^{\infty}(i+1)|b_i| = O(T^{2\eta}),
\end{aligned}
$$

it follows using Markov's inequality that

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{\lfloor\tau T\rfloor} z_{t-1}^2 = \frac{1}{T^{1+\eta}}\sum_{t=1}^{\lfloor\tau T\rfloor-1} \tilde{z}_{t-1}^2 + o_p\left(\frac{1}{T^{1+\eta}}\sum_{t=1}^{T} z_{t-1}^2\right).$$

for $\tilde{z}_{t-1} := \sum_{j=0}^{t-2}\varrho^j\sum_{i=0}^{t-j-2} b_i v_{t-j-i-1}$.

Second, we establish the pointwise expansion

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]} \tilde{z}_{t-1}^2 = \frac{\omega^2}{T^{1+\eta}}(1+o_p(1))\sum_{t=1}^{[\tau T]}\zeta_{t-1}^2. \tag{C.8}$$

The following Beveridge-Nelson decomposition holds:

$$\tilde{z}_{t-1} - \omega\zeta_{t-1} = -\sum_{i=0}^{t-2}\tilde{b}_i v_{t-i-1} + (1-\varrho)\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\tilde{b}_i v_{t-i-j-2} \tag{C.9}$$

with

$$\mathrm{E}\sum_{t=1}^{T}(\sum_{i=0}^{t-2}\tilde{b}_i v_{t-i-1})^2 = \sum_{t=1}^{T}\sum_{i=0}^{t-2}\tilde{b}_i^2\mathrm{E}(v_{t-i-1}^2) \le C\sum_{t=1}^{T}\sum_{i=0}^{\infty}\tilde{b}_i^2 = O(T)$$

and $\mathrm{E}\sum_{t=1}^{T}(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\tilde{b}_i v_{t-i-j-2})^2 = O(T^{1+\eta})$ as shown next:

$$\begin{aligned}
\mathrm{E}\sum_{t=1}^{T}(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\tilde{b}_i v_{t-i-j-2})^2 &= \mathrm{E}\sum_{t=1}^{T}(\sum_{s=1}^{t-2}v_s\sum_{j=0}^{t-s-2}\varrho^j\tilde{b}_{t-s-j-2})^2\\
&= \sum_{t=1}^{T}\sum_{s=1}^{t-2}\mathrm{E}v_s^2(\sum_{j=0}^{t-s-2}\varrho^j\tilde{b}_{t-s-j-2})^2\\
&\le C\sum_{t=1}^{T}\sum_{s=1}^{t-2}\varrho^{2(t-s)} = O(T^{1+\eta})
\end{aligned}$$

using the exponential decay of $\tilde{b}_i$. As $1-\varrho = aT^{-\eta}$, we can conclude that $\mathrm{E}\sum_{t=1}^{T}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2 = o(T^{1+\eta})$ and $\sum_{t=1}^{T}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2 = o_p(T^{1+\eta})$, by Markov's inequality. This estimate and the bound

$$\left|\sum_{t=1}^{[\tau T]}(\tilde{z}_{t-1}^2 - \omega^2\zeta_{t-1}^2)\right| \le \sum_{t=1}^{T}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2 + 2|\omega|\sqrt{\sum_{t=1}^{T}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2\sum_{t=1}^{[\tau T]}\zeta_{t-1}^2}$$

establish (C.8).

Third, we consider

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]}\zeta_{t-1}^2 = \frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-1-j}^2 + \frac{2}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1}\sum_{s=t+1}^{[\tau T]-1}\varrho^{s-t}v_t v_s\sum_{j=0}^{[\tau T]-s-1}\varrho^{2j}$$

with the second addend on the r.h.s. being $o_p(1)$, as shown next. It holds that

$$\sum_{t=1}^{[\tau T]-1}\sum_{s=t+1}^{[\tau T]-1}\varrho^{s-t}v_t v_s\sum_{j=0}^{[\tau T]-s-1}\varrho^{2j}=(1-\varrho^2)\sum_{t=1}^{[\tau T]}\sum_{s=t+1}^{[\tau T]}\varrho^{s-t}v_t v_s(1-\varrho^{2([\tau T]-s)}) \tag{C.10}$$

with

$$\begin{aligned}\mathrm{E}\left(\sum_{t=1}^{[\tau T]-1}\sum_{s=t+1}^{[\tau T]-1}\varrho^{s-t}v_t v_s\right)^2 &= \sum_{t=1}^{[\tau T]-1}\sum_{s=t+1}^{[\tau T]-1}\varrho^{2(s-t)}\mathrm{E}(v_t^2 v_s^2)+2\sum_{t=1}^{[\tau T]-1}\sum_{r=t+1}^{[\tau T]-1}\sum_{s=r+1}^{[\tau T]-1}\varrho^{2s-t-r}\mathrm{E}(v_t v_r v_s^2)\\ &\le CT^{1+\eta}+2\sum_{t=1}^{[\tau T]-1}\sum_{r=t+1}^{[\tau T]-1}\varrho^{r-t}\sum_{s=r+1}^{[\tau T]-1}\varrho^{2(s-r)}\mathrm{E}(v_t v_r(v_s^2-\sigma_{vs}^2))\\ &=O(T^{1+\eta})+O(T^{1+\eta+\epsilon})\end{aligned}$$

because, by (C.1),

$$\begin{aligned}\left|\sum_{t=1}^{[\tau T]-1}\sum_{r=t+1}^{[\tau T]-1}\varrho^{r-t}\sum_{s=r+1}^{[\tau T]-1}\varrho^{2(s-r)}\mathrm{E}(v_t v_r(v_s^2-\sigma_{vs}^2))\right| &\le \max_{1\le t<r\le[\tau T]-2}|\mathrm{E}(v_t v_r S^v_{T(r+1,[\tau T]-1)})|\sum_{t=1}^{[\tau T]-1}\sum_{r=t+1}^{[\tau T]-1}\varrho^{r-t}\\ &=O(T^{1+\eta+\epsilon})\end{aligned}$$

using Lemma 1(c), such that $(1-\varrho^2)\sum_{t=1}^{[\tau T]}\sum_{s=t+1}^{[\tau T]}\varrho^{s-t}v_t v_s=O_p(T^{(1+3\eta+\epsilon)/2})$ by Chebyshev's inequality, and a similar estimate holds for the aggregare contribution of the terms in (C.10) involving $\varrho^{2([\tau T]-s)}$. Hence,

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]}\zeta_{t-1}^2=\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-1-j}^2+o_p(1)$$

under the assumption that $\eta+\epsilon<1$.

Fourth,

$$\begin{aligned}\sum_{t=1}^{\lfloor\tau T\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 &= \sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}\sigma_{v,t-j-1}^2+o_p(T^{1+\eta})=\sum_{t=1}^{\lfloor\tau T\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}\right)\sigma_{vt}^2+o_p(T^{1+\eta})\\ &=\frac{T^\eta}{2a}\sum_{t=1}^{\lfloor\tau T\rfloor}\sigma_{vt}^2+o_p(T^{1+\eta})\end{aligned}$$

by formally substituting σ_{ut}^2 with 1 in the proof of Lemma 2(f). The pointwise convergence $T^{-1-\eta}\sum_{t=1}^{\lfloor\tau T\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2\xrightarrow{p}\frac{1}{2a}[M_v](\tau)$ is now immediate. In conjunction with (C.8) this yields $T^{-1-\eta}\sum_{t=1}^{\lfloor\tau T\rfloor}z_{t-1}^2\xrightarrow{p}\frac{\omega^2}{2a}[M_v](\tau)$. As the involved processes are increasing and the limit function is continuous, the convergence is in fact uniform. □

Proof of Lemma 5(d). First, we expand $\sum_{t=1}^{[\tau T]}z_{t-1}u_t=\sum_{t=1}^{[\tau T]}\tilde z_{t-1}u_t+o_p(T^{1/2+\eta/2})$ uniformly in τ, and second, we continue the expansion as $\sum_{t=1}^{[\tau T]}\tilde z_{t-1}u_t=\omega\sum_{t=1}^{[\tau T]}\zeta_{t-1}u_t+o_p(T^{1/2+\eta/2})$, with

$\tilde{z}_t$ and ζ_{t-1} defined previously. Third, we show that

$$\frac{1}{T^{1/2}}\sum_{t=1}^{[\tau T]}\begin{pmatrix} v_t \\ \frac{1}{T^{\eta/2}}\zeta_{t-1}u_t \end{pmatrix} \Rightarrow \begin{pmatrix} M_v(\tau) \\ \frac{1}{\sqrt{2a}}\int_0^\tau [M_v]'(s)[M_u]'(s)\mathrm{d}B(s) \end{pmatrix} \tag{C.11}$$

by discussing its predictable variation and applying a Lindeberg-style martingale CLT. This is the most involved step of the proof and its structure will be detailed later.

First,

$$\begin{aligned}\sum_{t=1}^{[\tau T]}(z_{t-1}-\tilde{z}_{t-1})u_t &= -(c/T)\sum_{t=1}^{[\tau T]}\left(\sum_{j=0}^{t-2}\varrho^j\xi_{t-j-2}\right)u_t + \sum_{t=1}^{[\tau T]}\left[\sum_{i=t-1}^{\infty}v_{t-1-i}\left(\sum_{j=0}^{t-1}\varrho^j b_{i-j}\right)\right]u_t \\ &= -(c/T)ZU_1(\tau) + ZU_2(\tau).\end{aligned}$$

Choose $\delta = \frac{1}{4}(1-\eta) > 0$. As $\max_{t\le T}|\xi_t| = O_p(T^{1/2})$, the process $ZU_1(\tau)$ equals with probability approaching one the martingale $Z\tilde{U}_1(\tau) = \sum_{t=1}^{[\tau T]}\left(\sum_{j=0}^{t-2}\varrho^j\mathbb{I}_{\{|\xi_{t-j-2}|\le T^{1/2+\delta}\}}\xi_{t-j-2}\right)u_t$ with

$$\begin{aligned}\mathrm{Var}(Z\tilde{U}_1(1)) &= \sum_{t=1}^{T}\mathrm{E}\left[\left(\sum_{j=0}^{t-2}\varrho^j\mathbb{I}_{|\xi_{t-j-2}|\le T^{1/2+\delta}}\xi_{t-j-2}\right)^2 u_t^2\right] \le CT^{1+2\delta}\sum_{t=1}^{T}\left(\sum_{j=0}^{t-2}\varrho^j\right)^2\mathrm{E}\left[u_t^2\right] \\ &= O(T^{2+2\eta+2\delta}) = o(T^{3+\eta}),\end{aligned}$$

such that, by Doob's martingale inequality, $Z\tilde{U}_1(\tau) = o_p(T^{3/2+\eta/2})$ uniformly in $\tau\in[0,1]$, and the same magnitude order is inherited by $ZU_1(\tau)$. The process $ZU_2(\tau)$ is a martingale with

$$\begin{aligned}\mathrm{E}|ZU_1(1)| &\le \sum_{t=1}^{T}\sum_{i=t-1}^{\infty}\mathrm{E}|v_{t-1-i}u_t|\sum_{j=0}^{t-1}\varrho^j|b_{i-j}| \le C\sum_{t=1}^{T}\sum_{i=t-1}^{\infty}\sum_{j=0}^{t-1}\varrho^j|b_{i-j}| \\ &= C\sum_{j=0}^{T-1}\varrho^j\sum_{t=1}^{T-j}\sum_{i=t-1}^{\infty}|b_i| \le C\sum_{j=0}^{T-1}\varrho^j\sum_{i=0}^{\infty}(i+1)|b_i| = O(T^\eta) = o(T^{1/2+\eta/2})\end{aligned}$$

and, again by Doob's martingale inequality, $ZU_1(\tau) = o_p(T^{1/2+\eta/2})$ uniformly in $\tau\in[0,1]$. As a result, $\sum_{t=1}^{[\tau T]}(z_{t-1}-\tilde{z}_{t-1})u_t = o_p(T^{1+\eta})$ uniformly in $\tau\in[0,1]$.

Second, $\sum_{t=1}^{[\tau T]}(\tilde{z}_{t-1}-\omega\zeta_{t-1})u_t$ is a martingale with variance at 1 given by

$$\sum_{t=1}^{T}\mathrm{E}[(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2u_t^2] \le \sum_{t=1}^{T}\sqrt{\mathrm{E}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^4\mathrm{E}u_t^4} \le C\sum_{t=1}^{T}\sqrt{\mathrm{E}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^4},$$

where, by using (C.9) and Lemma 2.5.2 of Giraitis et al. (2012),

$$
\begin{aligned}
\mathrm{E}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^4 &\le C\left[\mathrm{E}\left(\sum_{i=0}^{t-2}\tilde{b}_i v_{t-i-1}\right)^4+(1-\varrho)^4\mathrm{E}\left(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\tilde{b}_i v_{t-i-j-2}\right)^4\right]\\
&\le C\left[\mathrm{E}\left(\sum_{i=0}^{t-2}\tilde{b}_i v_{t-i-1}\right)^4+T^{-4\eta}\mathrm{E}\left(\sum_{s=1}^{t-2}v_s\sum_{j=0}^{t-s-2}\varrho^j\tilde{b}_{t-s-j-2}\right)^4\right]\\
&\le C\left[\left(\sum_{i=0}^{t-2}\tilde{b}_i^2\sqrt{\mathrm{E}v_{t-i-1}^4}\right)^2+T^{-4\eta}\left(\sum_{s=1}^{t-2}\left(\sum_{j=0}^{t-s-2}\varrho^j\tilde{b}_{t-s-j-2}\right)^2\sqrt{\mathrm{E}v_s^4}\right)^2\right]\\
&\le C\left[\left(\sum_{i=0}^{\infty}\tilde{b}_i^2\right)^2+T^{-4\eta}\left(\sum_{s=1}^{t-2}\left(\sum_{j=0}^{t-s-2}\varrho^j\tilde{b}_{t-s-j-2}\right)^2\right)^2\right].
\end{aligned}
$$

Further, in view of the exponential decay of $\tilde{b}_i$,

$$
\mathrm{E}(\tilde{z}_{t-1}-\omega\zeta_{t-1})^4 \le C+O(T^{-4\eta})\left(\sum_{s=1}^{t-2}\varrho^{2(t-s)}\right)^2=O(1)
$$

uniformly in $t=1,...,T$, such that $\sum_{t=1}^{T}\mathrm{E}[(\tilde{z}_{t-1}-\omega\zeta_{t-1})^2u_t^2]=O(T)=o(T^{1+\eta})$ and, by Doob's martingale inequality, $\sum_{t=1}^{[\tau T]}(\tilde{z}_{t-1}-\omega\zeta_{t-1})u_t=o_p(T^{1+\eta})$ uniformly in $\tau\in[0,1]$.

Third, we establish that the predictable quadratic variation of the l.h.s. martingale in (C.11) satisfies

$$
\sum_{t=1}^{\lfloor T\tau\rfloor}\begin{pmatrix}\frac{1}{T}\mathrm{E}_{t-1}v_t^2 & \frac{1}{T^{1+\eta/2}}\zeta_{t-1}\mathrm{E}_{t-1}(u_tv_t)\\ \frac{1}{T^{1+\eta/2}}\zeta_{t-1}\mathrm{E}_{t-1}(u_tv_t) & \frac{1}{T^{1+\eta}}\zeta_{t-1}^2\mathrm{E}_{t-1}u_t^2\end{pmatrix}\xrightarrow{p}\begin{pmatrix}[M_v](\tau) & 0\\ 0 & \frac{1}{2a}\int_0^\tau[M_v]'(s)[M_u]'(s)\mathrm{d}s\end{pmatrix}. \tag{C.12}
$$

Indeed, only the entries in the second row require detailed discussion. The analysis of the off-diagonal entry relies on the martingale approximability of $\sum_{t=1}^{\lfloor T\tau\rfloor}u_tv_t$. We write

$$
\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}\mathrm{E}_{t-1}(u_tv_t)=\sum_{t=1}^{\lfloor T\tau\rfloor-1}v_t\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}u_sv_s-\sum_{t=1}^{\lfloor T\tau\rfloor-1}v_t\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}[u_sv_s-\mathrm{E}_{s-1}(u_sv_s)] \tag{C.13}
$$

where

$$
\mathrm{E}\left|\sum_{t=1}^{\lfloor T\tau\rfloor-1}v_t\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}u_sv_s\right|\le\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sqrt{\mathrm{E}v_t^2}\sqrt{\mathrm{E}\left(\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}u_sv_s\right)^2}
$$

with

$$\mathrm{E}\left(\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}u_sv_s\right)^2=\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{2(s-t-1)}\mathrm{E}(u_s^2v_s^2)+2\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{2(s-t-1)}\mathrm{E}\left(u_sv_s\mathrm{E}_s\sum_{r=s+1}^{\lfloor T\tau\rfloor-1}\varrho^{r-s}u_rv_r\right)$$
$$\le CT^\eta+2\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{2(s-t-1)}\sqrt{\mathrm{E}(u_s^2v_s^2)}\sqrt{\mathrm{E}\left(\mathrm{E}_s\sum_{r=s+1}^{\lfloor T\tau\rfloor-1}\varrho^{r-s}u_rv_r\right)^2}$$
$$\le CT^\eta+C\max_{1\le s\le T}\left\|\mathrm{E}_s\sum_{r=s+1}^{T}\varrho^{r-s}u_rv_r\right\|_2\sum_{s=0}^{T}\varrho^{2s}=O(T^{\eta+\epsilon})$$

by using Lemma 1(e) and the condition $\epsilon<\eta$. To deal with the possible nonexistence of a finite second moment of the second summation on the r.h.s. of (C.13), we notice first that it equals, with probability approaching one,

$$\sum_{t=1}^{\lfloor T\tau\rfloor-1}\mathbb{I}_{\{|v_t|\le T^{1/3}\}}v_t\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{s-t-1}[u_sv_s-\mathrm{E}_{s-1}(u_sv_s)]$$

because $\max_{1\le t\le T}|v_t|=o(T^{1/3})$ under uniform L_4-boundedness of v_t. By MD considerations, the variance of the quantity in the previous display is

$$\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{s=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{2(s-t-1)}\mathrm{E}\{\mathbb{I}_{\{|v_t|\le T^{1/3}\}}v_t^2[u_sv_s-\mathrm{E}_{s-1}(u_sv_s)]^2\}=O(T^{5/3+\eta})=o(T^{2+\eta}).$$

By combining the previous estimates and Markov's inequality, we can conclude that $\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}\mathrm{E}_{t-1}(u_tv_t)=o_p(T^{1+\eta/2})$ provided that $\epsilon<\eta$.

Based on the martingale approximability of $\sum_{t=1}^{\lfloor T\tau\rfloor}(u_t^2-\sigma_{ut}^2)$, we discuss next

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}^2\mathrm{E}_{t-1}u_t^2=\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}^2u_t^2+\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}^2(\mathrm{E}_{t-1}u_t^2-u_t^2),\tag{C.14}$$

where

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}^2u_t^2-\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2u_t^2=2\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\sum_{i=0}^{j-1}\varrho^{i+j}v_{t-j-1}v_{t-i-1}u_t^2$$
$$=2\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{s=1}^{t-1}\varrho^{t-s}v_sv_t\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}u_r^2$$
$$=2\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{s=1}^{t-1}\varrho^{t-s}v_sv_t\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}\sigma_{ur}^2\tag{C.15}$$
$$+2\varrho\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}v_t\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}(u_r^2-\sigma_{ur}^2).$$

The first term on the r.h.s. of (C.15) is $o_p(T^{1+\eta})$ by Chebyshev's inequality:

$$\begin{aligned}
\mathrm{E}\left(\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{s=1}^{t-1}\varrho^{t-s}v_s v_t\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}\sigma_{ur}^2\right)^2 &= \sum_{t=1}^{\lfloor T\tau\rfloor}\left(\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}\sigma_{ur}^2\right)^2\mathrm{E}(\sum_{s=1}^{t-1}\varrho^{t-s}v_s v_t)^2\\
&\le CT^{2\eta}\sum_{t=1}^{\lfloor T\tau\rfloor}\Bigg[\sum_{s=1}^{t-1}\varrho^{2(t-s)}\mathrm{E}(v_s^2v_t^2)\\
&\qquad+\sum_{s=1}^{t-1}\varrho^{t-s}\sum_{q=t+1}^{\lfloor T\tau\rfloor-1}\varrho^{2(q-t)}\mathrm{E}(v_t v_s v_q^2)\Bigg]\\
&\le CT^{1+3\eta}+CT^{2\eta}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varrho^{t-s}\max_{1\le s<t<r\le T}\left|\sum_{q=t+1}^{r}\mathrm{E}(v_t v_s v_q^2)\right|\\
&= CT^{1+3\eta}+CT^{2\eta}\sum_{t=1}^{T}\sum_{s=1}^{t-1}\varrho^{t-s}\max_{1\le s<t<r\le T}\left|\mathrm{E}(v_t v_s S^v_{T(t+1,r)})\right|\\
&= O(T^{1+3\eta+\epsilon}) = o(T^{2+2\eta})
\end{aligned}$$

for $\epsilon < 1-\eta$, using (C.1) and Lemma 1(c) for the estimate involving a maximum. For the discussion of the second term on the r.h.s. of (C.15) we define $\check{\zeta}_t = \zeta_t\mathbb{I}_{\{|\zeta_t|\le T^{1/2}\}}$, $\check{v}_t = v_t\mathbb{I}_{\{|v_t|\le T^{1/3}\}}$ and $A^{u\tau}_{t+1} := \sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}(u_r^2-\sigma_{ur}^2)$ and notice that, with probability approaching one,

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}v_t\sum_{r=t+1}^{\lfloor T\tau\rfloor}\varrho^{2(r-t-1)}(u_r^2-\sigma_{ur}^2) = \sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t A^{u\tau}_{t+1}$$

because $\max_{1\le t\le T}|\zeta_t| = o(T^{1/2})$ and $\max_{1\le t\le T}|v_t| = o(T^{1/3})$; the purpose of truncation is to ensure square integrability. Furthermore, we use the decomposition

$$\begin{aligned}
\sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t A^{u\tau}_{t+1} &= \sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t\iota_t(A^{u\tau}_{t+1}-\mathrm{E}_t A^{u\tau}_{t+1})\\
&\quad+\sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t\iota_t\mathrm{E}_t A^{u\tau}_{t+1}+\sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t(1-\iota_t)A^{u\tau}_{t+1},
\end{aligned}\tag{C.16}$$

where $\iota_t := \mathbb{I}\{\mathrm{E}_t[\max_{r\le T}(S^u_{T(t+1,r)})^2]\le T^{1+\delta}\}$ for some $\delta\in(0,\eta)$ (notice that $\mathrm{E}_t[(A^{u\tau}_{t+1}-\mathrm{E}_t A^{u\tau}_{t+1})^2]\le \mathrm{E}_t[(A^{u\tau}_{t+1})^2]\le \mathrm{E}_t[\max_{r\le T}(S^u_{T(t+1,r)})^2]$ a.s., by using eq. (C.1)). For the terms in the decomposition in (C.16), first, by MD considerations,

$$\begin{aligned}
\mathrm{E}\left[\sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t\iota_t(A_{t+1}^{u\tau}-\mathrm{E}_tA_{t+1}^{u\tau})\right]^2 &= \sum_{t=1}^{\lfloor T\tau\rfloor}\mathrm{E}\left[\check{\zeta}_{t-1}\check{v}_t\iota_t(A_{t+1}^{u\tau}-\mathrm{E}_tA_{t+1}^{u\tau})\right]^2\\
&\le \sum_{t=1}^{\lfloor T\tau\rfloor}\mathrm{E}\left[\check{\zeta}_{t-1}^2\check{v}_t^2\iota_t\mathrm{E}_t[(A_{t+1}^{u\tau}-\mathrm{E}_tA_{t+1}^{u\tau})^2]\right]\\
&\le \sum_{t=1}^{\lfloor T\tau\rfloor}\mathrm{E}\left[\check{\zeta}_{t-1}^2\check{v}_t^2\iota_t\mathrm{E}_t[\max_{r\le T}(S_{T(t+1,r)}^u)^2]\right]\\
&\le T^{1+\delta}\sum_{t=1}^{T}\|\zeta_{t-1}\|_4^2\,\|v_t\|_4^2 = O(T^{2+\eta+\delta})\\
&= o(T^{2+2\eta})
\end{aligned}$$

by Lemma 2(c) and given the choice of δ; second,

$$\begin{aligned}
\mathrm{E}\left|\sum_{t=1}^{\lfloor T\tau\rfloor}\check{\zeta}_{t-1}\check{v}_t\iota_t\mathrm{E}_tA_{t+1}^{u\tau}\right| &\le \max_{1\le t\le T}\left\|\mathrm{E}_tA_{t+1}^{u\tau}\right\|_2\sum_{t=1}^{T}\|\zeta_{t-1}\|_4\,\|v_t\|_4\\
&= O(T^{1+\eta/2+\epsilon}) = o(T^{1+\eta})
\end{aligned}$$

by Lemma 2(c) and given that $2\epsilon<\eta$, and third,

$$\mathrm{P}\left(\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}v_t(1-\iota_t)A_{t+1}^{u\tau}=0\right)\ge \mathrm{P}\left(\min_{t\le T}\iota_t=1\right)\to 1$$

because

$$\iota_t=\mathbb{I}\{\mathrm{E}_t[\max_{r\le T}(S_{T(t+1,r)}^u)^2]\le T^{1+\delta}\}\ge \mathbb{I}\{\max_{t\le T}\mathrm{E}_t[\max_{1\le s<r\le T}(S_{T(s+1,r)}^u)^2]\le T^{1+\delta}\}$$

for all $t=1,...,T$,such that

$$\begin{aligned}
\mathrm{P}\left(\min_{t\le T}\iota_t=1\right) &\ge 1-\mathrm{P}\left(\max_{t\le T}\mathrm{E}_t[\max_{1\le s<r\le T}(S_{T(s+1,r)}^u)^2]>T^{1+\delta}\right)\\
&\ge 1-T^{-1-\delta}\mathrm{E}\mathrm{E}_T[\max_{1\le s<r\le T}(S_{T(s+1,r)}^u)^2]\\
&= 1-T^{-1-\delta}\mathrm{E}[\max_{1\le s<r\le T}(S_{T(s+1,r)}^u)^2] = 1-O(T^{-\delta})
\end{aligned}$$

by Doob's martingale inequality applied to the martingale $\mathrm{E}_t[\max_{1\le s<r\le T}(S_{T(s+1,r)}^u)^2]$ $(t=1,...,T)$ and by Lemma 1(b). By collecting the previous three results, we conclude that also the second term on the r.h.s. of (C.15) is $o_p(T^{1+\eta})$, such that

$$T^{-1-\eta}\sum_{t=1}^{[\tau T]}\zeta_{t-1}^2u_t^2 = T^{-1-\eta}\sum_{t=1}^{[\tau T]}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2u_t^2+o_p(1)\xrightarrow{p}\frac{1}{2a}\int_0^\tau[M_v]'(s)[M_u]'(s)\mathrm{d}s$$

by Lemma 2(e,f).

In view of (C.14), to establish the convergence of the predictable quadratic variation as stated in (C.12), it remains to show that

$$\sum_{t=1}^{[\tau T]} \zeta_{t-1}^2(u_t^2 - \mathrm{E}_{t-1}u_t^2) = \sum_{t=1}^{[\tau T]} v_t^2 B_{t+1}^{u\tau} + 2\varrho \sum_{t=1}^{\lfloor T\tau \rfloor} \zeta_{t-1} v_t B_{t+1}^{u\tau} = o_p(T^{1+\eta}),$$

where $B_{t+1}^{u\tau} := \sum_{r=t+1}^{\lfloor T\tau \rfloor} \varrho^{2(r-t-1)}(u_r^2 - \mathrm{E}_{r-1}u_r^2)$. This can be achieved similarly to the discussion of (C.16), though with some simplifications due to the martingale difference property $\mathrm{E}_t B_{t+1}^{u\tau} = 0$. We skip the details but mention that $S_{T(t+1,r)}^u$ could be replaced by $\tilde{S}_{T(t+1,r)}^u := \sum_{s=t+1}^{r}(u_s^2 - \mathrm{E}_{s-1}u_s^2)$, with $\mathrm{E}[\max_{1\le s<r\le T}(\tilde{S}_{T(s+1,r)}^u)^2] = O(T)$ as a consequence of the martingale property of $\tilde{S}_{T(1,r)}^u$ and the uniform L_4-boundedness of u_t (e.g. Proposition 9 of Merlevède *et al.* (2006) asserts this under much weaker conditions).

Finally, to complete the proof of convergence (C.11), a conditional Lindeberg condition now suffices, by the function-space version of Corollary 3.1 of Hall and Heyde (1980). The following conditional Lindeberg condition can be established along the lines of Lemma 3.5(ii) of Magdalinos (2020):

$$\begin{aligned}
\sum_{t=1}^{T} \mathrm{E}_{t-1}\left[\left(\frac{v_t^2}{T} + \frac{\zeta_{t-1}^2 u_t^2}{T^{1+\eta}}\right)\mathbb{I}\left\{\sqrt{\frac{v_t^2}{T} + \frac{\zeta_{t-1}^2 u_t^2}{T^{1+\eta}}} > 2\delta\right\}\right] &\le \sum_{t=1}^{T} \mathbb{I}\{|\zeta_{t-1}| > T^{\frac{1}{2}}\}\mathrm{E}_{t-1}\left(\frac{v_t^2}{T} + \frac{\zeta_{t-1}^2 u_t^2}{T^{1+\eta}}\right)\\
&+ \sum_{t=1}^{T} \mathrm{E}_{t-1}\left[\left(\frac{v_t^2}{T} + \frac{\delta^2 u_t^2}{T^{\eta}}\right)\mathbb{I}\{|v_t| > T^{1/2}\delta\}\right]\\
&+ \sum_{t=1}^{T} \mathrm{E}_{t-1}\left[\left(\frac{v_t^2}{T} + \frac{\zeta_{t-1}^2 u_t^2}{T^{1+\eta}}\right)\mathbb{I}\{|u_t| > T^{\frac{\eta}{2}}\}\right]\\
&= o_p(1)
\end{aligned}$$

for any $\delta > 0$. In fact, the first and the second term on the majorant side are zero with probability approaching one, respectively because $\max_{t\le T}|\zeta_{t-1}| = o_p(T^{1/2})$ (by Lemma 2(c) with $w_{Tt} = v_t, p = 4$ and $\max_{1\le t\le T}\mathrm{E}v_t^4 = O(1)$) and $\max_{1\le t\le T}|v_t| = o_p(T^{1/2})$ (because $\max_{1\le t\le T}\mathrm{E}v_t^4 = O(1)$). The third term on the majorant side is $o_p(1)$ because it is non-negative and its expectation is bounded by

$$\begin{aligned}
2\sum_{t=1}^{T}&\left(\frac{\sqrt{\mathrm{E}v_t^4}}{T}\sqrt{\mathrm{P}(|u_t| > T^{\eta/2})} + \frac{\sqrt{\mathrm{E}\zeta_{t-1}^4}}{T^{1+\eta}}\sqrt{\mathrm{E}(u_t^4\mathbb{I}\{|u_t| > T^{\eta/2}\})}\right)\\
&\le 2\sum_{t=1}^{T}\left(\frac{\sqrt{\mathrm{E}v_t^4\mathrm{E}u_t^4}}{T^{1+\eta}} + \frac{\sqrt{\mathrm{E}\zeta_{t-1}^4}}{T^{1+\eta}}\sqrt{\mathrm{E}(u_t^4\mathbb{I}\{|u_t| > T^{\eta/2}\})}\right) = o(1)
\end{aligned}$$

by Markov's inequality, by Lemma 2(c) for $\max_{t\le T}\mathrm{E}\zeta_{t-1}^4 = O(T^{2\eta})$ and because u_t^4 are uniformly integrable (a property inherited from the uniformly L_4-bounded and stationary sequence $\boldsymbol{\psi}_t$ because $\mathbf{H}$ is bounded).

Convergence (C.14) and the conditional Lindeberg condition imply convergence (C.11). In view of the first two steps of this proof, also the convergence

$$\frac{1}{T^{1/2}}\sum_{t=1}^{[\tau T]}\begin{pmatrix} v_t \\ \frac{1}{T^{\eta/2}} z_{t-1}u_t \end{pmatrix} \Rightarrow \begin{pmatrix} M_v(\tau) \\ \frac{1}{\sqrt{2a}}\int_0^\tau [M_v]'(s)[M_u]'(s)\mathrm{d}B(s) \end{pmatrix}$$

follows, making the convergence of $T^{-1/2-\eta/2}\sum_{t=1}^{[\tau T]} z_{t-1}u_t$ joint with the one established in part (c). $\square$

Proof of Lemma 5(e).

First, assume that b is Lipschitz on $[0,1]$. Apply the partial summation formula to obtain

$$\frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1}b(t/T) = b([\tau T]/T)\frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]-2} z_t - \frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]-1}\left(\sum_{j=1}^{t} z_j\right)\left(b\left(\frac{t+1}{T}\right)-b\left(\frac{t}{T}\right)\right).$$

We know from part (a) that $\frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1} \Rightarrow \frac{\omega}{a}J_{c,\mathrm{H}}(\tau)$, such that the first summand converges to $\frac{\omega}{a}b(\tau)J_{c,\mathrm{H}}(\tau)$. Moreover, the functional $f \mapsto \int_0^\tau f(s)\mathrm{d}b(s)$ is well-defined and continuous on $\mathcal{C}$ for a Lipschitz function b. Since the Ornstein-Uhlenbeck process $J_{c,\mathrm{H}}(\tau)$ is pathwise continuous, the second summand converges to the Stieltjes integral $\frac{\omega}{a}\int_0^\tau J_{c,\mathrm{H}}(s)\mathrm{d}b(s)$. By continuity of the limiting processes involved, it follows that

$$\frac{1}{T^{1/2+\eta}}\sum_{t=1}^{\lfloor\tau T\rfloor} z_{t-1}b(t/T) \Rightarrow \frac{\omega}{a}\left(b(\tau)J_{c,\mathrm{H}}(\tau) - \int_0^\tau J_{c,\mathrm{H}}(s)\mathrm{d}b(s)\right) = \frac{\omega}{a}\int_0^\tau b(s)\mathrm{d}J_{c,\mathrm{H}}(s),$$

the latter equality by applying pathwise; cf. Theorem 7.6 of Apostol (1981).

Second, let $0 < s_1 < \ldots < s_k < 1$ delimit the intervals of Lipschitz continuity of b. As $\max_{1\le t\le T}|z_{t-1}| = o_p(T^{1/2+\eta})$, e.g. by Lemma 5(c), it follows that modifying the value of the bounded function b at its finitely many discontinuity points has an asymptotically negligible effect on $T^{-1/2-\eta}\sum_{t=1}^{[\tau T]} z_{t-1}b(t/T)$, uniformly in τ. Therefore, with no loss in generality, we can take b to be of the form $b = b^c + \sum_{i=1}^k b_i\mathbb{I}(\cdot \ge s_i)$, where $b^c : [0,1]\to\mathbb{R}$ is Lipschitz and $b_i\in\mathbb{R}$, $i=1,\ldots,k$. Then

$$\frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1}b(t/T) = \frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1}b^c(t/T) + \sum_{i=1}^{k}\frac{b_i}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]}\mathbb{I}(\frac{t}{T}\ge s_i)z_{t-1}$$

and, from the previous discussion of the continuous case and Lemma 5(a) it can be concluded that

$$\frac{1}{T^{1/2+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1}b(t/T) \Rightarrow \frac{\omega}{a}\left(\int_0^\tau b^c(s)\mathrm{d}J_{c,\mathrm{H}}(s) + \sum_{i=1}^{k}\mathbb{I}(\tau\ge s_i)b_i(J_{c,\mathrm{H}}(\tau)-J_{c,\mathrm{H}}(s_i))\right) = \frac{\omega}{a}\int_0^\tau b(s)\mathrm{d}J_{c,\mathrm{H}}(s),$$

using also the continuity of the limiting processes involved. $\square$

Proof of Lemma 5(f).

We show below that

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{\lfloor\tau T\rfloor} z_{t-1}b(t/T)x_{t-1} \Rightarrow \frac{\omega^2}{a}\left(J_{c,\mathrm{H}}(\tau)Z_b(\tau) - \int_0^\tau Z_b(s)\mathrm{d}J_{c,\mathrm{H}}(s)\right)$$

for $Z_b(\tau) := \int_0^\tau b(s)\mathrm{d}J_{c,\mathrm{H}}(s)$. Stochastic integration by parts then yields

$$J_{c,\mathrm{H}}(\tau)Z_b(\tau) - \int_0^\tau Z_b(s)\mathrm{d}J_{c,\mathrm{H}}(s) = \int_0^\tau J_{c,\mathrm{H}}(s)\mathrm{d}Z_b(s) + [Z_b, J_{c,\mathrm{H}}](\tau),$$

where the quadratic covariation $[Z_b, J_{c,\mathrm{H}}](\tau)$ equals $\int_0^\tau b(s)\mathrm{d}[J_{c,\mathrm{H}}](s) = \int_0^\tau b(s)\mathrm{d}[M_v](s)$. This establishes the desired result.

Apply the partial summation formula to obtain

$$\begin{aligned}\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]} z_{t-1}b(t/T)x_{t-1} &= x_{[\tau T]-1}\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1} z_t b((t+1)/T) \\ &\quad -\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1}\left(\sum_{j=1}^{t} z_{j-1}b(j/T)\right)\left(w_t-\frac{c}{T}\xi_{t-1}\right)\end{aligned}$$

where the limit of the first summand on the r.h.s. follows with with part (e) and the weak convergence of $T^{-1/2}\xi_{[\tau T]}$.

Then, following the arguments in the proof of part (b), it straightforward to show that, uniformly in τ,

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1}\left(\sum_{j=1}^{t} z_{j-1}b(j/T)\right) w_t = \omega\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1}\left(\sum_{j=1}^{t} z_{j-1}b(j/T)\right) v_t + o_p(1)$$

such that, with v_t orthogonal to $\sum_{j=1}^{t} z_{j-1}b(j/T)$, we obtain as required

$$\frac{1}{T^{1+\eta}}\sum_{t=1}^{[\tau T]-1}\left(\sum_{j=1}^{t} z_j b((j+1)/T)\right)\left(w_t-\frac{c}{T}\xi_{t-1}\right) \Rightarrow \frac{\omega^2}{a}\left(\int_0^{\tau} Z_b(s)\mathrm{d}M_v(s) - c\int_0^{\tau} Z_b(s)J_{c,\mathrm{H}}(s)\mathrm{d}s\right).$$

□

We now turn to the derivation of the limit bootstrap distributions.

Under Assumption 1.1, let $\hat{A}(z) := 1 - \sum_{i=1}^{p+1} \hat{a}_i z^i$, whereas under Assumption 1.2, let $\hat{A}(z) := 1 - \sum_{i=1}^{p} \tilde{a}_i z^i$ for $\tilde{a}_i$ as in $\Delta x_t^* = \hat{\phi} x_{t-1}^* + \sum_{i=1}^{p} \tilde{a}_i \Delta x_{t-i}^* + v_t^*$. Let further $\sum_{i=0}^{\infty} \hat{b}_i z^i = (\hat{A}(z))^{-1}$ with $\hat{b}_0 = 1$. As the coefficients of $\hat{A}(z)$ estimate consistently those of $(1-\rho z)A(z)$ and $A(z)$ respectively under Assumption 1.1 and Assumption 1.2, and since $(1-\rho z)A(z)$ and $A(z)$, under the respective assumptions, have their roots outside a complex disk of radius $1+2\delta'$ for some $\delta' > 0$, it follows that with probability approaching one $\hat{A}(z)$ has its roots outside the complex disk of radius $1+\delta'$, such that the coefficients of the power series $\sum_{i=0}^{\infty} \hat{b}_i z^i$ decrease exponentially ($|\hat{b}_i| \leq C\delta^i$ for some $\delta \in (0,1)$,with probability approaching one). Since we are interested in results 'in probability', in the proof of such results we proceed, without loss of generality, as if the roots of $\hat{A}(z)$ were a.s. outside the complex disk of radius $1+\delta'$. Thus, as x_t^* is initialized with zero initial values, under Assumption 1.1 we write $x_t^* = \sum_{i=0}^{t-1} \hat{b}_i v_{t-i}^*$, where $\hat{b}_i$ a.s. decay at an exponential rate which is uniform over T. Similarly, under Assumption 1.1, we write

$$\Delta x_t^* = \sum_{i=0}^{t-1} \hat{b}_i (\hat{\phi} x_{t-i-1}^* + v_{t-i}^*),$$

$$x_t^* = \sum_{i=0}^{t-1} ((\hat{A}(1))^{-1} + \hat{b}_i^*)(\hat{\phi} x_{t-i-1}^* + v_{t-i}^*),$$

where $\hat{b}_i$ and the Beveridge-Nelson coefficients $\hat{b}_i^*$ a.s. decay at an exponential rate which is uniform over T.

We often use the estimates $\max_{1\leq t\leq T} |\hat{v}_t - v_t| = O_p(T^{-1/4})$, $\max_{1\leq t\leq T} |\hat{v}_t^2 - v_t^2| = O_p(1)$, $\max_{1\leq t\leq T} |\hat{u}_t - u_t| = O_p(T^{-1/2})$ and $\max_{1\leq t\leq T} |\hat{u}_t^2 - u_t^2| = O_p(T^{-1/4})$ which hold as a result of consistent parameter estimation and the assumptions on (u_t, v_t). Their standard implications where $(\hat{u}_t, \hat{v}_t)$ are approximated by (u_t, v_t) are usually used without explicit justification, e.g., $\sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} \hat{v}_{t-j-1}^2 = \sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2 + o_p(T^{1+\eta})$. In this specific case, a possible justification would be

$$\left| \sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} \hat{v}_{t-j-1}^2 - \sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2 \right| \leq 2 \max_{1\leq t\leq T} |\hat{v}_t - v_t| \sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} |v_{t-j-1}| \qquad \text{(C.17)}$$
$$+ \max_{1\leq t\leq T} (\hat{v}_t - v_t)^2 \sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} = o_p(T^{1+\eta})$$

because $\sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} |v_{t-j-1}| = O_p(T^{1+\eta})$ by Markov's inequality.

Proof of Lemma 6. As μ_x of the DGP of x_t cancels out in the definition of z_t,we assume that $\mu_x = 0$, without loss of generality. Also without loss of generality when distributional results are concerned, we regard the independent sequences $\boldsymbol{\psi}_t$ and R_t as defined on a product probability space with a generic outcome (ω, ω^*).

In part (a) it holds that

$$\sum_{t=1}^{\lfloor T\tau \rfloor} z_{t-1} u_t^* = \sum_{t=1}^{\lfloor T\tau \rfloor} x_{t-1} u_t R_t + \sum_{t=1}^{\lfloor T\tau \rfloor} (z_{t-1} - x_{t-1}) u_t R_t + \sum_{t=1}^{\lfloor T\tau \rfloor} z_{t-1} (\hat{u}_t - u_t) R_t,$$

where $\sum_{t=1}^{\lfloor T\tau \rfloor} (z_{t-1} - x_{t-1}) u_t R_t$ and $\sum_{t=1}^{\lfloor T\tau \rfloor} z_{t-1} (\hat{u}_t - u_t) R_t$ conditionally on the data are martingales in τ with, first,

$$\mathrm{E}^*\left(\sum_{t=1}^{T}(z_{t-1}-x_{t-1})u_tR_t\right)^2=\sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}-x_{t-1})^2u_t^2\le\sqrt{\sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}-x_{t-1})^4\sum_{t=1}^{T}u_t^4}=o_p(T)$$

because $\sum_{t=1}^{T}u_t^4=O_p(T)$ and $\|z_t-x_t\|_4=O_p(T^{-\eta/2}+\varrho^t)$ under Assumption 1.2 (see e.g. Lemma 4 (a) in Demetrescu and Hillmann, 2022), and second,

$$\mathrm{E}^*\left(\sum_{t=1}^{T}z_{t-1}(\hat{u}_t-u_t)R_t\right)^2=\sum_{t=1}^{T}z_{t-1}^2(\hat{u}_t-u_t)^2\le\max_{1\le t\le T}|\hat{u}_t-u_t|^2\sum_{t=1}^{T}z_{t-1}^2=o_p(T)$$

because $\sum_{t=1}^{T}z_{t-1}^2=O_p(T)$ by Markov's inequality and the uniform L_4-boundedness of z_t. By using Doob's martingale inequality, it follows that

$$\sum_{t=1}^{\lfloor T\tau\rfloor}z_{t-1}u_t^*=\sum_{t=1}^{\lfloor T\tau\rfloor}x_{t-1}u_tR_t+o_p^*(T^{1/2})=\sum_{t=1}^{\lfloor T\tau\rfloor}\xi_{t-1}u_tR_t+o_p^*(T^{1/2})$$

uniformly over $\tau\in[0,1]$. Then, by using the Lipschitz-by-parts property of $\mathbf{H}$,

$$\max_{\tau\in[0,1]}\left|\sum_{t=1}^{\lfloor T\tau\rfloor}\xi_{t-1}u_tR_t-\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}u_tR_t\right|=o_p(T^{1/2})$$

by the same argument as in the proof of Lemma 4, with $\tilde{\xi}_t$ defined there. As convergence in probability to zero becomes $\overset{p}{\Rightarrow}_p$ convergence upon conditioning, the previous estimates holds weakly in probability conditionally on the data, such that $\sum_{t=1}^{\lfloor T\tau\rfloor}z_{t-1}u_t^*=\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}u_tR_t+o_p^*(T^{1/2})$ uniformly over $\tau\in[0,1]$. The limit of $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}z_{t-1}u_t^*$ asserted in part (a) will then follow if we show that this limit holds for $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}u_tR_t$, as we do next.

Similarly to the proof of Lemma 4, consider the representation

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}u_tR_t=\sum_{t=1}^{\lfloor T\tau\rfloor}\begin{pmatrix}\mathbf{H}_{1\cdot}(\tfrac{t}{T}) & \mathbf{H}_{2\cdot}(\tfrac{t}{T})\end{pmatrix}\left[(\boldsymbol{\psi}_tR_t)\otimes\sum_{j\ge0}b_j\boldsymbol{\psi}_{t-1-j}\right]\tag{C.18}$$

where $\boldsymbol{\psi}_t$ is as in Assumption 3. Notice that, by the ergodic theorem and the dominated convergence theorem,

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t')\otimes\sum_{i,j\ge0}b_ib_j\boldsymbol{\psi}_{t-1-i}\boldsymbol{\psi}_{t-1-j}'\overset{a.s.}{\to}\tau\sum_{i,j\ge0}b_ib_j\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\otimes(\boldsymbol{\psi}_{-i}\boldsymbol{\psi}_{-j}')]:=\tau\Omega,\tag{C.19}$$

as it was already used in the proof of Lemma 4. Moreover, the convergence holds in the functional sense (on $\mathcal{D}$) given that the involved functions are increasing and the limit function is also continuous. For $\check{\psi}_t:=(\boldsymbol{\psi}_tR_t)\otimes\sum_{j\ge0}b_j\boldsymbol{\psi}_{t-1-j}$, let $\mathcal{B}$ be an almost certain event in the factor space of the data such that $\mathrm{E}_{\omega^*}\|\check{\psi}_t(\omega,\omega^*)\|^2<\infty$ for every fixed $\omega\in\mathcal{B}$, where the expectation is taken w.r.t. the probability measure on the factor space of the bootstrap multipliers; such a $\mathcal{B}$ exists by the L_4-boundedness of $\boldsymbol{\psi}_t$. Let g_{TN} be measurable functions from $\mathbb{R}^\infty$ to $\mathbb{R}$ such that

$g_{TN}(\boldsymbol{\psi}_t, \boldsymbol{\psi}_{t-1}, ...)$ are versions of $\mathrm{E}^*[\|\check{\psi}_t\|^2 \mathbb{I}_{\{\|\check{\psi}_t\|>N\}}]$ and the equalities

$$g_{TN}(\boldsymbol{\psi}_t(\omega), \boldsymbol{\psi}_{t-1}(\omega), ...) = \mathrm{E}_{\omega^*}[\|\check{\psi}_t(\omega,\omega^*)\|^2 \mathbb{I}_{\{\|\check{\psi}_t(\omega,\omega^*)\|>N\}}]$$

are satisfied for all $T, N \in \mathbb{N}$ and $\omega \in \mathcal{B}$; such g_{TN} exist by the ergodicity of $\boldsymbol{\psi}_t$ and the product structure of the underlying probability space. By the ergodic theorem, it holds that

$$\frac{1}{T}\sum_{t=1}^{T} g_{TN}(\boldsymbol{\psi}_t, \boldsymbol{\psi}_{t-1}, ...) \overset{a.s.}{\to} \mathrm{E}\left[\|\check{\psi}_1\|^2 \mathbb{I}_{\{\|\check{\psi}_1\|>N\}}\right] \tag{C.20}$$

for every $N \in \mathbb{N}$. Let $\mathcal{A} \subset \mathcal{B}$ be an almost certain event in the factor space of the data such that the countably many convergence facts (C.19) and (C.20) (with (C.19) counted as a single functional convergence) hold simultaneously for every $\omega \in \mathcal{A}$. Then, for every fixed $\omega \in \mathcal{A}$, the process

$$T^{-1/2}\sum_{t=1}^{\lfloor T\tau \rfloor}(\boldsymbol{\psi}_t(\omega)R_t(\omega^*)) \otimes \sum_{j\geq 0} b_j \boldsymbol{\psi}_{t-1-j}(\omega) \tag{C.21}$$

is a martingale with variance function

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau \rfloor}(\boldsymbol{\psi}_t(\omega)\boldsymbol{\psi}_t'(\omega)) \otimes \sum_{i,j\geq 0} b_i b_j \boldsymbol{\psi}_{t-1-i}(\omega)\boldsymbol{\psi}_{t-1-j}'(\omega) \overset{a.s.}{\to} \tau\Omega$$

and, moreover, for every $n, N \in \mathbb{N}$, it holds for large T that

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}_{\omega^*}\left[\|\check{\psi}_t(\omega,\omega^*)\|^2 \mathbb{I}_{\{\|\check{\psi}_t(\omega,\omega^*)\|>\sqrt{T}/n\}}\right] &\leq \frac{1}{T}\sum_{t=1}^{T}\mathrm{E}_{\omega^*}\left[\|\check{\psi}_t(\omega,\omega^*)\|^2 \mathbb{I}_{\{\|\check{\psi}_t(\omega,\omega^*)\|>N\}}\right] \\
&= \frac{1}{T}\sum_{t=1}^{T} g_{TN}(\boldsymbol{\psi}_t(\omega), \boldsymbol{\psi}_{t-1}(\omega), ...) \\
&\to \mathrm{E}\left[\|\check{\psi}_1\|^2 \mathbb{I}_{\{\|\check{\psi}_1\|>N\}}\right].
\end{aligned}$$

Since $\mathrm{E}[\|\check{\psi}_1\|^2 \mathbb{I}_{\{\|\check{\psi}_1\|>N\}}]$ can be made arbitrarily small by choosing N large, it follows that the Lindeberg condition

$$\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}\left[\|\check{\psi}_t(\omega,\omega^*)\|^2 \mathbb{I}_{\{\|\check{\psi}_t(\omega,\omega^*)\|>\sqrt{T}/n\}}\right] \to 0$$

is satisfied for every $n \in \mathbb{N}$, and therefore, for every $\omega \in \mathcal{A}$, the process (C.21) weakly converges to a quadrivariate Brownian motion B_Ω with variance matrix at unity Ω. Then, using the Lipshits-by-parts property of $\mathbf{H}$ and representation (C.18), we can conclude that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor T\tau \rfloor}\tilde{\xi}_{t-1}(\omega)u_t(\omega)R_t(\omega^*) \Rightarrow \int_0^\tau [\mathbf{H}_{1\cdot}(s) \otimes \mathbf{H}_{2\cdot}(s)]\mathrm{d}B_\Omega(s) \overset{d}{=} \int_0^\tau \sqrt{\chi(s)}\mathrm{d}B(s)$$

for every fixed $\omega \in \mathcal{A}$, where B is a standard univariate Brownian motion. As the probability of $\mathcal{A}$ is one and the underlying probability space has a product structure, the previous convergence yields

$$T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}u_tR_t \overset{w}{\Rightarrow}_{a.s.} \int_0^{\tau}\sqrt{\chi(s)}\mathrm{d}B(s).$$

By the earlier discussion, the same limit is inherited by $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}u_tR_t$ in the weak-in-probability mode.

We turn to part (b). Notice for further reference that, for $s\geq t$, it holds (a.s., without loss of generality) that

$$|\mathrm{E}^*(x_s^*x_t^*)| \leq \sum_{i=0}^{t-1}|\hat{b}_i||\hat{b}_{i+s-t}|\hat{v}_{t-i}^2 \leq C\delta^{s-t}\sum_{i=0}^{t-1}\delta^{2i}\hat{v}_{t-i}^2. \tag{C.22}$$

Let $\xi_t^* := \sum_{i=0}^{t-1}b_iv_{t-i}R_{t-i}$. Then

$$\begin{aligned}\sum_{t=1}^{\lfloor T\tau\rfloor}z_{t-1}^*u_t^* &= \sum_{t=1}^{\lfloor T\tau\rfloor}\xi_{t-1}^*u_tR_t\\ &\quad+\sum_{t=1}^{\lfloor T\tau\rfloor}(x_{t-1}^*-\xi_{t-1}^*)u_tR_t+\sum_{t=1}^{\lfloor T\tau\rfloor}x_{t-1}^*(\hat{u}_t-u_t)R_t-(1-\varrho)\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-1}\varrho^jx_{t-j-2}^*u_t^*\end{aligned}$$

where, conditionally on the data, the processes $\sum_{t=1}^{\lfloor T\tau\rfloor}(x_{t-1}^*-\xi_{t-1}^*)u_tR_t$, $\sum_{t=1}^{\lfloor T\tau\rfloor}x_{t-1}^*(\hat{u}_t-u_t)R_t$ and $\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-1}\varrho^jx_{t-j-2}^*u_t^*$ are martingales in τ with, first,

$$\begin{aligned}\mathrm{E}^*\left(\sum_{t=1}^{T}(x_{t-1}^*-\xi_{t-1}^*)u_tR_t\right)^2 &= \sum_{t=1}^{T}\mathrm{E}^*[(x_{t-1}^*-\xi_{t-1}^*)^2]u_t^2 = \sum_{t=1}^{T}\sum_{i=0}^{t-2}(\hat{b}_i\hat{v}_{t-i-1}-b_iv_{t-i-1})^2u_t^2\\ &\leq 2\sum_{t=1}^{T}\sum_{i=0}^{t-2}\hat{b}_i^2(\hat{v}_{t-i-1}-v_{t-i-1})^2u_t^2+2\sum_{t=1}^{T}\sum_{i=0}^{t-2}(\hat{b}_i-b_i)^2v_{t-i-1}^2u_t^2\\ &\leq 2\max_{1\leq t\leq T}(\hat{v}_t-v_t)^2\sum_{i=0}^{\infty}\hat{b}_i^2\sum_{t=1}^{T}u_t^2+C\max_{1\leq t\leq T}|\hat{b}_i-b_i|\sum_{t=1}^{T}\sum_{i=0}^{t-2}\delta^iv_{t-i-1}^2u_t^2\\ &= o_p(T)\end{aligned}$$

by Markov's inequality for $\sum_{t=1}^{T}\sum_{i=0}^{t-2}\delta^iv_{t-i-1}^2u_t^2$; second,

$$\begin{aligned}\mathrm{E}^*\left(\sum_{t=1}^{T}x_{t-1}^*(\hat{u}_t-u_t)R_t\right)^2 &= \sum_{t=1}^{T}\mathrm{E}^*[(x_{t-1}^*)^2](\hat{u}_t-u_t)^2 = \sum_{t=1}^{T}\sum_{i=0}^{t-2}\hat{b}_i^2\hat{v}_{t-i-1}^2(\hat{u}_t-u_t)^2\\ &\leq \max_{1\leq t\leq T}(\hat{u}_t-u_t)^2\sum_{i=0}^{\infty}\hat{b}_i^2\sum_{t=1}^{T}\hat{v}_t^2 = o_p(1)\sum_{t=1}^{T}v_t^2 = o_p(T)\end{aligned}$$

and third, using (C.22),

$$\mathrm{E}^*\left(\sum_{t=1}^{T}\sum_{j=0}^{t-1}\varrho^j x^*_{t-j-2}u^*_t\right)^2 = \sum_{t=1}^{T}\hat{u}_t^2\sum_{i,j=0}^{t-1}\varrho^{j+i}\mathrm{E}^*(x^*_{t-i-2}x^*_{t-j-2})$$
$$\leq C\sum_{t=1}^{T}\hat{u}_t^2\sum_{i=0}^{t-1}\sum_{j=0}^{i-1}\varrho^{j+i}\delta^{i-j}\sum_{k=i+2}^{t-1}\delta^{2(k-i-2)}\hat{v}_{t-k}^2$$
$$\leq C\sum_{t=1}^{T}\hat{u}_t^2\sum_{i=0}^{t-1}\varrho^i\sum_{k=i+2}^{t-1}\delta^{2(k-i-2)}\hat{v}_{t-k}^2$$
$$\leq C(1+o_p(1))\sum_{t=1}^{T}u_t^2\sum_{i=0}^{t-1}\varrho^i\sum_{k=i+2}^{t-1}\delta^{2(k-i-2)}v_{t-k}^2 = O_p(T^{1+\eta})$$

by Markov's inequality, as $\mathrm{E}(u_t^2 v_{t-k}^2)$ are bounded uniformly in t,k and

$$\sum_{t=1}^{T}\sum_{i=0}^{t-1}\varrho^i\sum_{k=i+2}^{t-1}\delta^{2(k-i-2)} = O(T^{1+\eta}).$$

Hence, by Doob's martingale inequality,

$$\sum_{t=1}^{\lfloor T\tau\rfloor} z^*_{t-1}u^*_t = \sum_{t=1}^{\lfloor T\tau\rfloor}\xi^*_{t-1}u_tR_t + o_{p^*}(T^{1/2})$$

uniformly in τ. Further, similarly to the proof of Lemma 4, let

$$\tilde{\xi}^*_{t-1} \quad = \quad h_{21}(\frac{t}{T})\sum_{j\geq 0}b_j a_{t-1-j}R_{t-1-j} + h_{22}(\frac{t}{T})\sum_{j\geq 0}b_j e_{t-1-j}R_{t-1-j}.$$

Then, as in the proof of Lemma 4, the Lipschitz-by-parts property of h_{21}, h_{22} can be used to check that

$$\max_{\tau\in[0,1]}\left|\sum_{t=1}^{\lfloor T\tau\rfloor}\xi^*_{t-1}u_tR_t - \sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}^*_{t-1}u_tR_t\right| = o_p(T^{1/2}).$$

As convergence to zero in probability becomes $\overset{p}{\Rightarrow}_p$ convergence upon conditioning, it follows that $\sum_{t=1}^{\lfloor T\tau\rfloor} z^*_{t-1}u^*_t = \sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}^*_{t-1}u_tR_t + o_{p^*}(T^{1/2})$ and part (b) will be proved if we show that $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}^*_{t-1}u_tR_t$ conditionally on the data converges weakly in probability to the asserted limit of $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor} z^*_{t-1}u^*_t$. We show this convergence next.

Consider the representation

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}^*_{t-1}u_tR_t = \sum_{t=1}^{\lfloor T\tau\rfloor}\begin{pmatrix}\mathbf{H}_{1\cdot}(\frac{t}{T}) & \mathbf{H}_{2\cdot}(\frac{t}{T})\end{pmatrix}\left[(\boldsymbol{\psi}_tR_t)\otimes\sum_{j\geq 0}b_j\boldsymbol{\psi}_{t-1-j}R_{t-1-j}\right]. \tag{C.23}$$

Notice that, by the ergodic theorem and the dominated convergence theorem,

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}(\boldsymbol{\psi}_t\boldsymbol{\psi}_t')\otimes\sum_{j\geq 0}b_j^2\boldsymbol{\psi}_{t-1-j}\boldsymbol{\psi}_{t-1-j}' \overset{a.s.}{\to} \tau\sum_{j\geq 0}b_j^2\mathrm{E}[(\boldsymbol{\psi}_1\boldsymbol{\psi}_1')\otimes(\boldsymbol{\psi}_{-j}\boldsymbol{\psi}_{-j}')] := \tau\Omega^*$$

and

$$\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}^*\left[\|\boldsymbol{\psi}_t^*\|^2\mathbb{I}_{\{\|\boldsymbol{\psi}_t^*\|>N\}}\right]\overset{a.s.}{\to}\mathrm{E}\left[\|\boldsymbol{\psi}_1^*\|^2\mathbb{I}_{\{\|\boldsymbol{\psi}_1^*\|>N\}}\right]$$

where $\boldsymbol{\psi}_t$ is as in Assumption 3 and $\boldsymbol{\psi}_t^* := (\boldsymbol{\psi}_t R_t)\otimes\sum_{j\geq 0} b_j\boldsymbol{\psi}_{t-1-j}R_{t-1-j}$. As a result, similarly to the proof of part (a), in the factor space of $\boldsymbol{\psi}_t$ there exists an event $\mathcal{A}^*$ of probability one such that, for every fixed $\omega\in\mathcal{A}^*$, the process

$$T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}(\boldsymbol{\psi}_t(\omega)R_t(\omega^*))\otimes\sum_{j\geq 0} b_j\boldsymbol{\psi}_{t-1-j}(\omega)R_{t-1-j}(\omega^*) = T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}\boldsymbol{\psi}_t^*(\omega,\omega^*), \qquad \text{(C.24)}$$

with randomness originating from ω^* alone, is a martingale with variance function

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}(\boldsymbol{\psi}_t(\omega)\boldsymbol{\psi}_t'(\omega))\otimes\sum_{j\geq 0} b_j^2\boldsymbol{\psi}_{t-1-j}(\omega)\boldsymbol{\psi}_{t-1-j}'(\omega)\to\tau\Omega^*$$

and satisfies the Lindeberg condition

$$\frac{1}{T}\sum_{t=1}^{T}\mathrm{E}_{\omega^*}\left[\|\boldsymbol{\psi}_t^*(\omega,\omega^*)\|^2\mathbb{I}_{\{\|\boldsymbol{\psi}_t^*(\omega,\omega^*)\|>\sqrt{T}/n\}}\right]\to 0$$

for all $n\in\mathbb{N}$. By a martingale FCLT it follows that the process (C.24) converges weakly to a quadrivariate Brownian motion B_Ω^* defined on $[0,1]$ and having variance matrix Ω^*. This fact and representation (C.23), together with the Lipschitz-by-parts property of $\mathbf{H}$, imply that

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}^*(\omega,\omega^*)u_t(\omega)R_t(\omega^*)\Rightarrow\int_0^\tau[\mathbf{H}_{1\cdot}(s)\otimes\mathbf{H}_{2\cdot}(s)]\mathrm{d}\boldsymbol{B}_{\boldsymbol{\Omega}}^*(s)\overset{d}{=}\int_0^\tau\sqrt{\chi^*(s)}\mathrm{d}B(s)$$

for every fixed $\omega\in\mathcal{A}^*$, where B is a standard Brownian motion. As the probability of $\mathcal{A}^*$ is one, and given the product structure of the probability space, the previous convergence implies that

$$T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}\tilde{\xi}_{t-1}^* u_t R_t\overset{w}{\Rightarrow}_{a.s.}\int_0^\tau\sqrt{\chi^*(s)}\mathrm{d}B(s)$$

conditionally on the data, and hence, the same convergence holds also weakly in probability. By the discussion earlier in this proof, the convergence is inherited by $T^{-1/2}\sum_{t=1}^{\lfloor T\tau\rfloor}z_{t-1}^*u_t^*$ as asserted in part (b).

In part (c), we first find that

$$\sum_{t=1}^{\lfloor T\tau\rfloor-1}|(z_t^*)^2-(x_t^*)^2|\leq\sum_{t=1}^{\lfloor T\tau\rfloor-1}(z_t^*-x_t^*)^2+2\left[\sum_{t=1}^{\lfloor T\tau\rfloor-1}(z_t^*-x_t^*)^2\right]^{1/2}\left[\sum_{t=1}^{\lfloor T\tau\rfloor-1}(x_t^*)^2\right]^{1/2},$$

where, using (C.22),

$$\mathrm{E}^* \sum_{t=1}^{\lfloor T\tau\rfloor-1} (z_t^* - x_t^*)^2 = (1-\varrho)^2 \sum_{t=1}^{\lfloor T\tau\rfloor-1} \mathrm{E}^* \left(\sum_{j=0}^{t-2} \varrho^j x_{t-j-1}^* \right)^2 = a^2 T^{-2\eta} \sum_{t=1}^{\lfloor T\tau\rfloor-1} \sum_{i,j=0}^{t-2} \varrho^{i+j} \mathrm{E}^*(x_{t-i-1}^* x_{t-j-1}^*)$$

$$= O_p(T^{-2\eta}) \sum_{t=1}^{T} \sum_{i=0}^{t-2} \sum_{j=0}^{i} \varrho^{i+j} \delta^{i-j} \sum_{k=0}^{t-i-2} \delta^{2i} \hat{v}_{t-i-k-1}^2$$

$$= O_p(T^{-2\eta}) \sum_{t=1}^{T} \sum_{i=0}^{t-2} \varrho^i \sum_{k=0}^{t-i-2} \delta^{2i} v_{t-i-k-1}^2 + O_p(T^{1-\eta}) = O_p(T^{1-\eta})$$

by Markov's inequality, such that

$$T^{-1} \sum_{t=1}^{\lfloor T\tau\rfloor-1} (z_t^*)^2 = T^{-1} \sum_{t=1}^{\lfloor T\tau\rfloor-1} (x_t^*)^2 + o_{p^*}(1) \left[1 + T^{-1} \sum_{t=1}^{T} (x_t^*)^2 \right]^{1/2} \tag{C.25}$$

again by Markov's inequality. Second,

$$\sum_{t=1}^{\lfloor T\tau\rfloor-1} |(x_t^*)^2 - (\xi_t^*)^2| \leq \sum_{t=1}^{\lfloor T\tau\rfloor-1} (x_t^* - \xi_t^*)^2 + 2 \left[\sum_{t=1}^{\lfloor T\tau\rfloor-1} (x_t^* - \xi_t^*)^2 \right]^{1/2} \left[\sum_{t=1}^{\lfloor T\tau\rfloor-1} (\xi_t^*)^2 \right]^{1/2},$$

where

$$\mathrm{E}^* \sum_{t=1}^{\lfloor T\tau\rfloor-1} (x_t^* - \xi_t^*)^2 \leq \sum_{t=1}^{T} \mathrm{E}^*(x_{t-1}^* - \xi_{t-1}^*)^2 = \sum_{t=1}^{T} (\hat{b}_i \hat{v}_{t-i-1} - b_i v_{t-i-1})^2$$

$$\leq 2 \sum_{t=1}^{T} \sum_{i=0}^{t-2} \hat{b}_i^2 (\hat{v}_{t-i-1} - v_{t-i-1})^2 + 2 \sum_{t=1}^{T} \sum_{i=0}^{t-2} (\hat{b}_i - b_i)^2 v_{t-i-1}^2$$

$$\leq 2T \max_{1\leq t\leq T} (\hat{v}_t - v_t)^2 \sum_{i=0}^{\infty} \hat{b}_i^2 + C \max_{1\leq t\leq T} |\hat{b}_i - b_i| \sum_{t=1}^{T} \sum_{i=0}^{t-2} \delta^i v_{t-i-1}^2$$

$$= o_p(T)$$

using Markov's inequality, such that

$$T^{-1} \sum_{t=1}^{\lfloor T\tau\rfloor-1} (x_t^*)^2 = T^{-1} \sum_{t=1}^{\lfloor T\tau\rfloor-1} (\xi_t^*)^2 + o_{p^*}(1) \left[1 + T^{-1} \sum_{t=1}^{T} (\xi_t^*)^2 \right]^{1/2} \tag{C.26}$$

again by Markov's inequality. Third,

$$\sum_{t=1}^{\lfloor T\tau\rfloor-1} \left[(\xi_t^*)^2 - \sum_{i=0}^{t-1} b_i^2 v_{t-i}^2 R_{t-i}^2 \right] = 2 \sum_{t=1}^{\lfloor T\tau\rfloor-1} \sum_{i=0}^{t-1} \sum_{j=i+1}^{t-1} b_i b_j v_{t-i} v_{t-j} R_{t-i} R_{t-j},$$

where the r.h.s., conditionally on the data, has expected square bounded by

$$4 \sum_{t=1}^{\lfloor T\tau\rfloor-1} \sum_{i=0}^{t-1} \sum_{j=i+1}^{t-1} |b_i||b_j| v_{t-i}^2 v_{t-j}^2 \sum_{s=1}^{\lfloor T\tau\rfloor-1} |b_{s-t+i}||b_{s-t+j}| \leq C \sum_{t=1}^{T} \sum_{i=0}^{t-1} \sum_{j=i+1}^{t-1} |b_i||b_j| v_{t-i}^2 v_{t-j}^2 = O_p(T)$$

by Markov's inequality, such that

$$T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}[(\xi_t^*)^2-\sum_{i=0}^{t-1}b_i^2v_{t-i}^2R_{t-i}^2]=T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1}b_i^2v_{t-i}^2R_{t-i}^2+o_{p^*}(1) \tag{C.27}$$

again by Markov's inequality. From (C.25)-(C.27) it follows that $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}(z_t^*)^2$ will converge to the limit asserted in part (c) if $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1}b_i^2v_{t-i}^2R_{t-i}^2$ converges to that same limit. We establish the latter convergence next.

From the Beveridge-Nelson decomposition $\sum_{i=0}^{t-1}b_i^2v_{t-i}^2R_{t-i}^2=\kappa^2v_t^2R_t^2+\Delta\tilde{v}_t$, where $\tilde{v}_t=\sum_{i=0}^{t-1}c_iv_{t-i}^2R_{t-i}^2$ for an appropriate exponentially decreasing sequence c_i, it follows that

$$\begin{aligned}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1}b_i^2v_{t-i}^2R_{t-i}^2&=\kappa^2\sum_{t=1}^{\lfloor T\tau\rfloor-1}v_t^2+\kappa^2\sum_{t=1}^{\lfloor T\tau\rfloor-1}v_t^2(R_t^2-1)+\tilde{v}_{\lfloor T\tau\rfloor-1}-\tilde{v}_0\\&=\kappa^2\sum_{t=1}^{\lfloor T\tau\rfloor-1}v_t^2+o_p(T)\end{aligned}$$

by Chebyshev's inequality for $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}v_t^2(R_t^2-1)$ and Markov's inequality for the quantity $\sum_{i=0}^{\lfloor T\tau\rfloor-1}|c_i|v_{\lfloor T\tau\rfloor-i-1}^2R_{\lfloor T\tau\rfloor-i-1}^2$. As $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor}v_t^2\xrightarrow{p}[M_v](\tau)$ by Lemma 3 and the limiting function is continuous, we can conclude that

$$T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1}b_i^2v_{t-i}^2R_{t-i}^2\xrightarrow{p}[M_v](\tau).$$

As, in addition, the functions on the l.h.s. and the r.h.s. of the previous convergence are increasing, the convergence is uniform in τ:

$$\sup_{\tau\in[0,1]}\left|\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1}b_i^2v_{t-i}^2R_{t-i}^2-[M_v](\tau)\right|\xrightarrow{p}0.$$

Finally, since convergence in probability to zero implies $\overset{p}{\Rightarrow}_p$-convergence to zero upon conditioning on the data, it holds that $T^{-1}\sum_{t=1}^{\lfloor T\tau\rfloor-1}\sum_{i=0}^{t-1}b_i^2v_{t-i}^2R_{t-i}^2\overset{p}{\Rightarrow}_p[M_v](\tau)$ in $\mathcal{D}$ conditionally on the data, which establishes part (c) by virtue also of the earlier discussion.

Finally, in part (d) the full-sample bootstrap residuals computed under the null hypothesis are $\hat{u}_t^*=u_t^*-T^{-1}\sum_{s=1}^{T}u_s^*$, such that

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau\rfloor}|(\hat{u}_t^*)^2-(u_t^*)^2|\le\frac{\lfloor T\tau\rfloor}{T^3}\left(\sum_{s=1}^{T}u_s^*\right)^2+\frac{2\sqrt{\lfloor T\tau\rfloor}}{T^2}\left|\sum_{s=1}^{T}u_s^*\right|\left[\sum_{t=1}^{\lfloor T\tau\rfloor}(u_t^*)^2\right]^{1/2}. \tag{C.28}$$

Here, first,

$$\mathrm{E}^*\left(\sum_{s=1}^{T}u_s^*\right)^2=\sum_{s=1}^{T}\hat{u}_s^2=T\hat{\sigma}_u^2(0,1)=O_p(T)$$

by (C.2), such that $\sum_{s=1}^{T} u_s^* = O_p^*(T^{1/2})$ by Chebyshev's inequality. Second,

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau \rfloor}(u_t^*)^2 - \frac{1}{T}\sum_{t=1}^{\lfloor T\tau \rfloor}\hat{u}_t^2 = \frac{1}{T}\sum_{t=1}^{\lfloor T\tau \rfloor}\hat{u}_t^2(R_t^2-1) = o_p^*(1)$$

again by Chebyshev's inequality:

$$\mathrm{E}^*\left(\sum_{t=1}^{\lfloor T\tau \rfloor}\hat{u}_t^2(R_t^2-1)\right)^2 = C\sum_{t=1}^{T}\hat{u}_t^4 \leq C\sum_{t=1}^{T}u_t^4 + C\sum_{t=1}^{T}(\hat{u}_t-u_t)^4 = O_p(T)$$

because u_t are uniformly L_4-bounded. Therefore,

$$\frac{1}{T}\sum_{t=1}^{\lfloor T\tau \rfloor}(u_t^*)^2 = \frac{1}{T}\sum_{t=1}^{\lfloor T\tau \rfloor}\hat{u}_t^2 + o_p(1) \xrightarrow{p} [M_u](\tau)$$

by (C.2). Returning to (3), it follows that $T^{-1}\sum_{t=1}^{\lfloor T\tau \rfloor}(\hat{u}_t^*)^2 = [M_u](\tau) + o_p^*(1)$, where the infinitesimal term is uniform in τ because the involved processes are increasing and $[M_u](\tau)$ is moreover continuous. The discussion of bootstrap residuals $\hat{u}_t^*$ computed over subsamples is similar. □

The proof of Lemma 7 will make use of the following estimates.

Lemma 9 *Under Assumptions 1.2 and 3:*

(a) $\max_{1\leq s\leq t\leq T}|\mathrm{E}^*(v_s^* x_t^*)| = O_p(1)\hat{v}_s^2$ *and* $\mathrm{E}^*(v_s^* x_t^*) = 0$ *for* $s > t$*;*

(b) $\max_{1\leq s,t\leq T}|\mathrm{E}^*(x_s^* x_t^*)| = O_p(1)\sum_{t=1}^{T}\hat{v}_t^2$.

Proof of Lemma 9. For $s > t$, the expectation in part (a) is zero by the conditional independence of v_s^* and x_t^*. For $s \leq t$ it holds that

$$\begin{aligned}|\mathrm{E}^*(v_s^* x_t^*)| &= \left|\hat{\phi}\sum_{i=0}^{t-s-1}[(\hat{A}(1))^{-1}+\hat{b}_i^*]\mathrm{E}^*(v_s^* x_{t-i-1}^*) + [(\hat{A}(1))^{-1}+\hat{b}_{t-s}]\hat{v}_s^2\right| \\ &\leq \hat{C}\left(|\hat{\phi}|\sum_{i=0}^{t-s-1}|\mathrm{E}^*(v_s^* x_{t-i-1}^*)| + \hat{v}_s^2\right)\end{aligned}$$

with $\hat{C} := |\hat{A}(1)|^{-1} + \sup_{i\geq 0}(|\hat{b}_i| + |\hat{b}_i^*|) = O_p(1)$. Thus, by recursive substitution, $|\mathrm{E}^*(v_s^* x_s^*)| \leq \hat{C}\hat{v}_s^2$ and $|\mathrm{E}^*(v_s^* x_{s+i}^*)| \leq \hat{C}(1+\hat{C}|\hat{\phi}|)^i\hat{v}_s^2$ for $i \geq 1$. These imply the estimate $|\mathrm{E}^*(v_s^* x_t^*)| \leq \hat{C}(1+\hat{C}|\hat{\phi}|)^T\hat{v}_s^2$ uniformly in $t = 1, ..., T$. As $\hat{\phi} = O_p(T^{-1})$ and $(1+\hat{C}|\hat{\phi}|)^T = O_p(1)$, part (a) follows.

Using the previous estimate, in part (b) we find that

$$\begin{aligned}|\mathrm{E}^*(x_s^* x_t^*)| &= \left|\sum_{i=0}^{t-1}\{(\hat{A}(1))^{-1}+\hat{b}_i^*\}\{\hat{\phi}\mathrm{E}^*(x_s^* x_{t-i-1}^*) + \mathrm{E}^*(x_s^* v_{t-i}^*)\}\right| \\ &\leq \hat{C}\left(|\hat{\phi}|\sum_{i=0}^{t-2}|\mathrm{E}^*(x_s^* x_{t-i-1}^*)| + \hat{C}(1+\hat{C}|\hat{\phi}|)^T\sum_{i=1}^{T}\hat{v}_i^2\right).\end{aligned}$$

Again by recursive substitution, $|\mathrm{E}^*(x_s^* x_1^*)| \leq \hat{C}^2(1+\hat{C}|\hat{\phi}|)^T\sum_{i=1}^{T}\hat{v}_i^2$ and $|\mathrm{E}^*(v_s^* x_t^*)| \leq \hat{C}^2(1+\hat{C}|\hat{\phi}|)^{T+t-1}\sum_{i=1}^{T}\hat{v}_i^2$. The estimate $\max_{1\leq s,t\leq T}|\mathrm{E}^*(x_s^* x_t^*)| \leq \hat{C}^2(1+\hat{C}|\hat{\phi}|)^{2T}\sum_{i=1}^{T}\hat{v}_i^2$ follows, and since $\hat{C}|\hat{\phi}| = O_p(T^{-1})$, also part (b). □

Proof of Lemma 7. In part (a), we write

$$T^{(1+\eta)/2}\left[N_T^*(\tau)-\tilde{N}_T^*(\tau)\right]=T^{(1+\eta)/2}[DN_{T1}^*(\tau)+DN_{T2}^*(\tau)+DN_{T3}^*(\tau)]$$

with

$$T^{(1+\eta)/2}DN_{T1}^*(\tau):=\hat{\phi}\sum_{t=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^j\sum_{i=0}^{t-j-2}\hat{b}_i x_{t-i-j-2}^*\right)u_t^*$$

$$T^{(1+\eta)/2}DN_{T2}^*(\tau):=\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^j\left(\sum_{i=0}^{t-j-2}\hat{b}_i v_{t-i-j-1}^*-\omega v_{t-j-1}^*\right)u_t^*$$

$$T^{(1+\eta)/2}DN_{T3}^*(\tau):=\sum_{t=1}^{\lfloor T\tau\rfloor}\zeta_{t-1}^*(u_t^*-\tilde{u}_t).$$

and notice that $T^{(1+\eta)/2}DN_{T1}^*(\tau)$ is a martingale conditional on the data, with conditional variance at 1 given by

$$\hat{\phi}^2\sum_{t=1}^{T}\left(\sum_{j,k=0}^{t-2}\varrho^{j+k}\sum_{i=0}^{t-j-2}\sum_{m=0}^{t-k-2}\hat{b}_i\hat{b}_m\mathrm{E}^*(x_{t-i-j-2}^*x_{t-m-k-2}^*)\right)\hat{u}_t^2$$
$$=O_p(1)\hat{\phi}^2\sum_{t=1}^{T}\hat{v}_t^2\sum_{t=1}^{T}\left(\sum_{j,k=0}^{t-2}\varrho^{j+k}\sum_{i=0}^{t-j-2}\sum_{m=0}^{t-k-2}|\hat{b}_i\hat{b}_m|\right)\hat{u}_t^2$$

by Lemma 9(b). Further, as $\hat{\phi}^2\sum_{t=1}^{T}\hat{v}_t^2=O_p(T^{-1})$, this conditional variance equals

$$O_p(T^{-1})\left(\sum_{i=0}^{T-2}|\hat{b}_i|\right)^2\left(\sum_{j=0}^{T-2}\varrho^j\right)^2\sum_{t=1}^{T}\hat{u}_t^2=O_p(T^{2\eta})=o_p(T^{1+\eta}).$$

Therefore, by Doob's martingale inequality, $\sup_{[0,1]}|DN_{T1}^*(\tau)|=o_p^*(1)$.

Since $\omega-\sum_{i=0}^{\infty}\hat{b}_i=A(1)^{-1}-(1-\sum_{i=1}^{p}\tilde{a}_i)^{-1}=o_p(1)$ and $\sup_{[0,1]}|\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^j v_{t-j-1}^*u_t^*|=O_p^*(T^{1/2+\eta/2})$ by Doob's martingale inequality, it will follow that $\sup_{[0,1]}|DN_{T2}^*(\tau)|=o_p^*(1)$ if we show that $\sup_{[0,1]}|D\tilde{N}_{T2}^*(\tau)|=o_p^*(1)$ for

$$T^{(1+\eta)/2}D\tilde{N}_{T2}^*(\tau):=\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^j\left(\sum_{i=0}^{t-j-2}\hat{b}_i v_{t-i-j-1}^*-\sum_{i=0}^{\infty}\hat{b}_i v_{t-j-1}^*\right)u_t^*.$$

Consider the decomposition

$$\sum_{j=0}^{t-2}\varrho^j\left(\sum_{i=0}^{t-j-2}\hat{b}_i v_{t-i-j-1}^*-\sum_{i=0}^{\infty}\hat{b}_i v_{t-j-1}^*\right)=-\sum_{i=0}^{t-2}\hat{b}_i^*v_{t-i-1}^*+(1-\varrho)\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\hat{b}_i^*v_{t-i-j-2}^*. \tag{C.29}$$

It holds that $\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{i=0}^{t-2}\hat{b}_i^*v_{t-i-1}^*u_t^*$ and $\sum_{t=1}^{\lfloor T\tau\rfloor}(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\hat{b}_i^*v_{t-i-j-2}^*)u_t^*$ are martingales conditional on the data with conditional variances at 1 equal respectively to

$$\sum_{t=1}^{T}\left[\mathrm{E}^*\left(\sum_{i=0}^{t-2}\hat{b}_i^* v_{t-i-1}^*\right)^2\right]\hat{u}_t^2 = \sum_{t=1}^{T}\sum_{i=0}^{t-2}(\hat{b}_i^*)^2\hat{v}_{t-i-1}^2\hat{u}_t^2$$
$$= (1+o_p(1))\sum_{t=1}^{T}\sum_{i=0}^{t-2}(\hat{b}_i^*)^2 v_{t-i-1}^2 u_t^2 = O_p(T)$$

and

$$\begin{aligned}\sum_{t=1}^{T}\left[\mathrm{E}^*\left(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\hat{b}_i^* v_{t-i-j-2}^*\right)^2\right]\hat{u}_t^2 &= \sum_{t=1}^{T}\left[\mathrm{E}^*\left(\sum_{s=1}^{t-2}v_s^*\sum_{j=0}^{t-s-2}\varrho^j\hat{b}_{t-s-j-2}^*\right)^2\right]\hat{u}_t^2 \\ &= \sum_{t=1}^{T}\sum_{s=1}^{t-2}\hat{v}_s^2\left(\sum_{j=0}^{t-s-2}\varrho^j\hat{b}_{t-s-j-2}^*\right)^2\hat{u}_t^2 \\ &= O_p(1)\sum_{t=1}^{T}\left(\sum_{s=1}^{t-2}\hat{v}_s^2\varrho^{2(t-s)}\right)\hat{u}_t^2 \\ &= O_p(1)\sum_{t=1}^{T}\left(\sum_{s=1}^{t-2}v_s^2\varrho^{2(t-s)}\right)u_t^2 \\ &= O_p(T^{1+\eta}),\end{aligned} \tag{C.30}$$

using the estimate $\left|\sum_{j=0}^{t-s-2}\varrho^j\hat{b}_{t-s-j-2}\right| \le C\sum_{j=0}^{t-s-2}\varrho^j\delta^{t-s-j-2} = \frac{C}{\rho-\delta}(\varrho^{t-s-1}-\delta^{t-s-1}) = O(\varrho^{t-s})$ a.s. ($\delta\in(0,1)$), and Markov's inequality. Therefore, by Doob's martingale inequality, it holds that

$$\sup_{[0,1]}\left|\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{i=0}^{t-2}\hat{b}_i^* v_{t-i-1}^* u_t^*\right| = O_p^*(T^{1/2}) \text{ and } \sup_{[0,1]}\left|\sum_{t=1}^{\lfloor T\tau\rfloor}(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\hat{b}_i^* v_{t-i-j-2}^*)u_t^*\right| = O_p^*(T^{(1+\eta)/2})$$

. As $1-\varrho = O(T^{-\eta})$, by combining the previous conclusions it follows that indeed $\sup_{[0,1]}|D\tilde{N}_{T2}^*(\tau)| = o_p^*(1)$.

Finally, also $T^{(1+\eta)/2}D\tilde{N}_{T3}^*(\tau)$ conditionally on the data is a martingale and its conditional variance at 1 is given by

$$\begin{aligned}\sum_{t=1}^{T}[\mathrm{E}^*(\zeta_{t-1}^*)^2](\hat{u}_t-u_t)^2 &\le \max_{1\le t\le T}(\hat{u}_t-u_t)^2\sum_{t=1}^{T}\mathrm{E}^*(\zeta_{t-1}^*)^2 \\ &= o_p(1)\sum_{t=1}^{T}\sum_{j=0}^{t-2}\varrho^{2j}\hat{v}_{t-j-1}^2 \\ &= o_p(1)\sum_{t=1}^{T}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 + o_p(T^{1+\eta}) = o_p(T^{1+\eta})\end{aligned}$$

by (C.17) and by Markov's inequality for $\sum_{t=1}^{T}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 = O_p(T^\eta)$. Therefore, by Doob's martingale inequality, $\sup_{[0,1]}|D\tilde{N}_{T3}^*(\tau)| = o_p^*(1)$. Part (a) follows by combining the previous results.

In part (b), let $\tilde{\zeta}_t := \omega \sum_{j=0}^{t-1} \varrho^j \tilde{v}_{t-j}$, $(\tilde{u}_t, \tilde{v}_t) := (u_t, v_t) R_t$. Consider first

$$\begin{aligned}
\mathrm{E}^* \sum_{t=1}^{T} (\zeta_{t-1}^* - \tilde{\zeta}_{t-1})^2 u_t^2 &= \sum_{t=1}^{T} \sum_{j=0}^{t-2} \varrho^{2j} (\hat{v}_{t-j-1} - v_{t-j-1})^2 u_t^2 \\
&\leq \max_{1 \leq t \leq T} |\hat{v}_t - v_t|^2 \sum_{j=0}^{T-2} \varrho^{2j} \sum_{t=1}^{T} u_t^2 = o_p(T^{1+\eta}).
\end{aligned}$$

Hence, by Markov's inequality, $T^{-1-\eta} \sum_{t=1}^{T} (\zeta_{t-1}^* - \tilde{\zeta}_{t-1})^2 u_t^2 = o_p^*(1)$. As a result,

$$\begin{aligned}
\left| \sum_{t=1}^{\lfloor T\tau \rfloor} [(\zeta_{t-1}^*)^2 - (\tilde{\zeta}_{t-1})^2] u_t^2 \right| &\leq \sum_{t=1}^{T} (\zeta_{t-1}^* - \tilde{\zeta}_{t-1})^2 u_t^2 + 2 \sqrt{\sum_{t=1}^{T} (\zeta_{t-1}^* - \tilde{\zeta}_{t-1})^2 u_t^2} \sqrt{\sum_{t=1}^{T} \tilde{\zeta}_{t-1}^2 u_t^2} \\
&= o_p^*(T^{1+\eta}) + 2 o_p^*(T^{1+\eta}) \sqrt{\check{V}_T(1)}
\end{aligned}$$

with $\check{V}_T(\tau) := T^{-1-\eta} \sum_{t=1}^{\lfloor T\tau \rfloor} \tilde{\zeta}_{t-1}^2 u_t^2$, such that

$$\tilde{V}_T^*(\tau) = \omega^2 \check{V}_T(\tau) + o_p^*(1) \left(1 + \check{V}_T(1)\right) \tag{C.31}$$

pointwise.

Next, it holds that $\mathrm{P}^*(\max_{1 \leq t \leq T} |v_t| \leq T^{1/3}) = \mathbb{I}\{\max_{1 \leq t \leq T} |v_t| \leq T^{1/3}\} \xrightarrow{p} 0$ because $\max_{1 \leq t \leq T} |v_t| = o_p(T^{1/3})$. Then, with $\check{v}_t = \mathbb{I}\{|v_t| \leq T^{1/3}\} v_t$, the decomposition

$$\begin{aligned}
T^{1+\eta} \check{V}_T(\tau) &= \sum_{t=1}^{\lfloor T\tau \rfloor} \sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2 u_t^2 + \sum_{t=1}^{\lfloor T\tau \rfloor} \sum_{j,k=0}^{t-2} \varrho^{j+k} \mathbb{I}_{j \neq k} \tilde{v}_{t-j-1} \tilde{v}_{t-k-1} u_t^2 \\
&= \sum_{t=1}^{\lfloor T\tau \rfloor} \sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2 u_t^2 + DV_{1T}(\tau) + 2DV_{2T}(\tau) + o_p^*(T^{1+\eta}) \qquad \text{(C.32)}
\end{aligned}$$

holds, where DV_{1T} and DV_{2T} are square-integrable under Assumption 3 and defined as follows. On the one hand, $DV_{1T}(\tau) := \sum_{s=1}^{\lfloor T\tau \rfloor - 1} \check{v}_s^2 (R_s^2 - 1) \sum_{t=s+1}^{\lfloor T\tau \rfloor} \varrho^{2(t-s-1)} u_t^2$ has

$$\begin{aligned}
\mathrm{E}^* \left(DV_{1T}(\tau)\right)^2 &\leq \mathrm{Var}(R_1^2) \sum_{s=1}^{T-1} \check{v}_s^4 \left[\sum_{t=s+1}^{T} \varrho^{2(t-s-1)} u_t^2 \right]^2 \\
&\leq C \sum_{s=1}^{T-1} v_s^4 \left\{ \left[\sum_{t=s+1}^{T} \varrho^{2(t-s-1)} (u_t^2 - \sigma_{ut}^2) \right]^2 + \max_{1 \leq t \leq T} \sigma_{ut}^4 \left[\sum_{t=s+1}^{T} \varrho^{2(t-s-1)} \right]^2 \right\} \\
&\leq [O_p(T) + O_p(T^{2\eta})] \sum_{s=1}^{T-1} v_s^4
\end{aligned}$$

by Lemma 2(d). Therefore, $\mathrm{E}^* \left(DV_{1T}(\tau)\right)^2 = O_p(T^2) + O_p(T^{2\eta+1}) = o_p(T^{2+2\eta})$ such that $T^{-1-\eta} DV_{1T}(\tau) =$

$o_p^*(1)$ by Chebyshev's inequality. On the other hand,

$$DV_{2T}(\tau) := \sum_{t=1}^{\lfloor T\tau \rfloor} \sum_{j=0}^{t-2} \sum_{k=j+1}^{t-2} \varrho^{j+k} R_{t-j-1} \check{v}_{t-j-1} R_{t-k-1} \check{v}_{t-k-1} u_t^2$$
$$= \sum_{s=1}^{\lfloor T\tau \rfloor -1} R_s \check{v}_s \sum_{r=s+1}^{\lfloor T\tau \rfloor -1} \varrho^{r-s} R_r \check{v}_r \sum_{t=r+1}^{\lfloor T\tau \rfloor} \varrho^{2(t-r-1)} u_t^2$$

has

$$\mathrm{E}^* \left(DV_{2T}(\tau)\right)^2 \leq \sum_{s=1}^{T-1} v_s^2 \sum_{r=s+1}^{T-1} \varrho^{2(r-s)} v_r^2 \left(\sum_{t=r+1}^{T} \varrho^{2(t-r-1)} u_t^2 \right)^2$$
$$= O_p(T) \sum_{s=1}^{T-1} v_s^2 \sum_{r=s+1}^{T-1} \varrho^{2(r-s)} v_r^2 + O(1) \sum_{s=1}^{T-1} v_s^2 \sum_{r=s+1}^{T-1} \varrho^{2(r-s)} v_r^2 \left(\sum_{t=1}^{T-r} \varrho^{2(t-1)} \right)^2$$

by Lemma 2(d) and further

$$\mathrm{E}^* \left(DV_{2T}(\tau)\right)^2 = O_p(T^{2+\eta}) + O(T^{2\eta}) \sum_{s=1}^{T-1} v_s^2 \sum_{r=s+1}^{T-1} \varrho^{2(r-s)} v_r^2$$
$$= O_p(T^{2+\eta}) + O_p(T^{1+3\eta}) = o_p(T^{2+2\eta})$$

using Markov's inequality, such that $T^{-1-\eta} DV_{2T}(\tau) = o_p^*(1)$ pointwise. Combining the previous results establishes the pointwise expansion $\check{V}_T(\tau) = T^{-1-\eta} \sum_{t=1}^{\lfloor T\tau \rfloor} \sum_{j=0}^{t-2} \varrho^{2j} v_{t-j-1}^2 u_t^2 + o_p^*(1)$ and, in view of (C.31), also point (b).

Part (c) follows by combining part (b) with Lemma 2(f,g). □

Proof of Lemma 8. In view of Lemma 7(a), to proof part (a) it is sufficient to show that $\tilde{N}_T^*(\tau) \overset{w}{\Rightarrow}_p \frac{\omega}{\sqrt{2a}} \int_0^\tau \sqrt{[M_v(s)]'[M_u(s)]'} \mathrm{d}B(s)$. We accomplish this by means of a Skorokhod representation on a probability space with a product structure.

Similarly to Lemma 3.5(ii) of Magdalinos (2020), the conditional Lindeberg condition $\sum_{t=2}^T \mathrm{E}_{t-1}^* \{\varsigma_t^2 \mathbb{I}_{|\varsigma_t|>1/n}\} = o_{p^*}(1)$ holds for $\varsigma_t := T^{-(1+\eta)/2} \zeta_{t-1}^* \tilde{u}_t$ and all $n \in \mathbb{N}$. In fact, as

$$\mathrm{E}\left[\mathrm{E}^*(\zeta_{t-1}^*)^4\right] = \mathrm{E}\left[\sum_{j=0}^{t-2} \varrho^{4j} v_{t-j}^4 \mathrm{E}(R_{t-j}^4) + 6 \sum_{j=0}^{t-2} \sum_{i=j+1}^{t-2} \varrho^{2(i+j)} v_{t-i-1}^2 v_{t-1-j}^2\right]$$
$$\leq C \sup_{1\leq t\leq T} \mathrm{E} v_t^4 \left[\sum_{j=0}^{T-2} \varrho^{4j} + 6 \sum_{j=0}^{T-2} \sum_{i=j+1}^{T-2} \varrho^{2(i+j)}\right] = O(T^{2\eta}),$$

it follows that $\max_{t\le T}|\zeta_{t-1}^*| \le [\sum_{t=1}^T (\zeta_{t-1}^*)^4]^{1/4} = O_p^*(T^{1/4+\eta/2})$ by Markov's inequality, such that

$$\begin{aligned}\sum_{t=2}^T \mathrm{E}_{t-1}^*(\varsigma_t^2 \mathbb{I}\{|\varsigma_t| > 1/n\}) &\le \sum_{t=2}^T \mathrm{E}_{t-1}^*(\varsigma_t^2)\mathbb{I}\{|\zeta_{t-1}^*| > \frac{T^{-\frac{1}{8}}T^{\frac{1+\eta}{2}}}{n}\} \\ &+ \sum_{t=2}^T \mathrm{E}_{t-1}^*(\varsigma_t^2 \mathbb{I}\{|R_t| > T^{1/16}\}) \\ &+ \sum_{t=2}^T \mathrm{E}_{t-1}^*(\varsigma_t^2 \mathbb{I}\{|u_t| > T^{1/8}\}) = o_{p^*}(1)\end{aligned}$$

because

$$\begin{aligned}\mathrm{P}^*\left(\sum_{t=2}^T \varsigma_t^2 \mathbb{I}\{|\zeta_{t-1}^*| > \frac{T^{-\frac{1}{8}}T^{\frac{1+\eta}{2}}}{n}\} = 0\right) &\le 1 - \mathrm{P}^*\left(\max_{t\le T}|\zeta_{t-1}^*| > \frac{T^{-\frac{1}{8}}T^{\frac{1+\eta}{2}}}{n}\right) \\ &= 1 - o_p(1),\end{aligned}$$

$$\begin{aligned}\mathrm{E}\left[\mathrm{E}^*\sum_{t=2}^T \mathrm{E}_{t-1}^*(\varsigma_t^2\mathbb{I}\{|R_t| > T^{1/16}\})\right] &= T^{-1-\eta}\sum_{t=2}^T \mathrm{E}\left[\{\mathrm{E}^*(\zeta_{t-1}^*)^2\}u_t^2R_t^2\mathbb{I}\{|R_t| > T^{1/16}\}\right] \\ &= T^{-1-\eta}\sum_{t=2}^T\sum_{j=0}^{t-2}\varrho^{2j}\mathrm{E}(v_{t-j-1}^2 u_t^2 R_t^2 \mathbb{I}\{|R_t| > T^{1/16}\}) \\ &\le T^{-1-\eta}\sum_{t=2}^T\sum_{j=0}^{t-2}\varrho^{2j}\sqrt{\mathrm{E}v_{t-j-1}^4}\sqrt{\mathrm{E}u_t^4\mathrm{E}[R_1^4\mathbb{I}\{|R_1| > T^{1/8}\}]} \\ &= o(T^{-1-\eta})\sum_{t=2}^T\sum_{j=0}^{t-2}\varrho^{2j} = o(1),\end{aligned}$$

and similarly

$$\begin{aligned}\mathrm{E}\left[\mathrm{E}^*\sum_{t=2}^T \mathrm{E}_{t-1}^*(\varsigma_t^2\mathbb{I}\{|u_t| > T^{1/8}\})\right] &\le T^{-1-\eta}\sum_{t=2}^T\sum_{j=0}^{t-2}\varrho^{2j}\sqrt{\mathrm{E}v_{t-j-1}^4}\sqrt{\mathrm{E}R_1^4\mathrm{E}[u_t^4\mathbb{I}\{|u_t| > T^{1/8}\}]} \\ &\le o(T^{-1-\eta})\sum_{t=2}^T\sum_{j=0}^{t-2}\varrho^{2j} = o(1)\end{aligned}$$

using the uniform integrability of u_t^4 (inherited from the uniformly L_4-bounded and stationary sequence $\boldsymbol{\psi}_t$ because $\mathbf{H}$ is bounded).

For $X_T = (x_0, ..., x_{T-1})$, $Y_T := (y_1, ..., y_T)$ and $R_T^* := (R_1, ..., R_T)$, write $\varsigma_t = \varsigma_t(X_T, Y_T, R_T^*)$ for the measurable transformation defining ς_t, and similarly, $\tilde{V}_T^* = \tilde{V}_T^*(X_T, Y_T, R_T^*)$. Fix the measurable functions functions $g_{Tn} : \mathbb{R}^{3T} \to \mathbb{R}$ as $g_{Tn}(x, y, R_T^*) = \sum_{t=2}^T \mathrm{E}_{t-1}^*\{\varsigma_t^2(x, y, R_T^*)\mathbb{I}_{|\varsigma_t(x,y,R_T^*)|>1/n}\}$ such that, by the independence of (X_T, Y_T) and R_T^*, it holds that $g_{Tn}(X_T, Y_T, R_T^*) = \sum_{t=2}^T \mathrm{E}_{t-1}^*\{\varsigma_t^2\mathbb{I}_{|\varsigma_t|>1/n}\}$ a.s. Introduce also $\delta_T : \mathbb{R}^{3T} \to \mathbb{R}$ by $\delta_T(x, y, R_T^*) := \sup_{[0,1]}|\tilde{V}_T^*(x, y, R_T^*)(\tau) - \frac{\omega^2}{2a}\int_0^\tau [M_u(s)]'[M_v(s)]'\mathrm{d}s|$ such that $\delta_T(X_T, Y_T, R_T^*) = o_p^*(1)$ by Lemma 7(d). Then the $\mathbb{R}^\infty$-valued function

$$\gamma_T(X_T, Y_T, R_T^*) := (\delta_T(X_T, Y_T, R_T^*), g_{T1}(X_T, Y_T, R_T^*), g_{T2}(X_T, Y_T, R_T^*), ...)$$

satisfies $\gamma_T(X_T, Y_T, R_T^*) = o_p^*(1)$ in the sense that $d(\gamma_T(X_T, Y_T, R_T^*), 0^\infty) = o_p^*(1)$ for the Frechet metric d and the zero sequence $0^\infty \in \mathbb{R}^\infty$. Equivalently,

$$\mathrm{E}^* f(d(\gamma_T(X_T, Y_T, R_T^*), 0^\infty)) \xrightarrow{p} f(0)$$

for every continuous and bounded $f : [0, \infty) \to \mathbb{R}$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a countable collection of continuous and bounded functions $[0, \infty) \to \mathbb{R}$ such that for any $w_T = w_T(R_T^*)$ the convergence $\mathrm{E}f_k(w_T) \dot{\to} f_k(0)$ for all functions in this collection is equivalent to $w_T \xrightarrow{p} 0$ (the expectation and the latter convergence are w.r.t. the distribution of R_T^*). Define on the support of (X_T, Y_T) the measurable deterministic functions $h_{Tk}(\cdot, \cdot) = \mathrm{E}f_k(d(\gamma_T(\cdot, \cdot, R_T^*), 0^\infty))$ (the expectation is w.r.t. the distribution of R_T^*), such that $h_{Tk}(X_T, Y_T) = \mathrm{E}^* f_k(d(\gamma_T(X_T, Y_T, R_T^*), 0^\infty))$ a.s., then

$$\chi_T(X_T, Y_T) := (h_{T1}(X_T, Y_T), h_{T2}(X_T, Y_T), ...) \xrightarrow{p} (f_1(0), f_2(0), ...)$$

in $\mathbb{R}^\infty$. By extended Skorokhod coupling (Corollary 5.12 of Kallenberg, 1997), there exist a probability space and random elements $(\tilde{X}_T, \tilde{Y}_T) \overset{d}{=} (X_T, Y_T)$ such that $\chi_T(\tilde{X}_T, \tilde{Y}_T) \xrightarrow{a.s.} (f_1(0), f_2(0), ...)$. On an extension of this probability space, define the i.i.d. $R_t^* \overset{d}{=} R_1$ and $\tilde{R}_T^* := (\tilde{R}_1, ..., \tilde{R}_T)$. Choose an almost certain event $\mathcal{A}$ in the factor space of $(\tilde{X}_T, \tilde{Y}_T)$ such that, for every fixed $\omega \in \mathcal{A}$ and every $k \in \mathbb{N}$,

$$\mathrm{E}f_k(d(\gamma_T(\tilde{X}_T(\omega), \tilde{Y}_T(\omega), \tilde{R}_T^*), 0^\infty)) = h_{Tk}(\tilde{X}_T(\omega), \tilde{Y}_T(\omega)) \to f_k(0),$$

where the expectation is w.r.t. the distribution of $\tilde{R}_T = R_T$. Then, due to the choice of f_k, it follows that $d(\gamma_T(\tilde{X}_T(\omega), \tilde{Y}_T(\omega), \tilde{R}_T^*), 0^\infty) \xrightarrow{p} 0$ for every fixed $\omega \in \mathcal{A}$. Equivalently, $\gamma_T(\tilde{X}_T(\omega), \tilde{Y}_T(\omega), \tilde{R}_T^*) \xrightarrow{p} 0^\infty$ for every $\omega \in \mathcal{A}$. A component-wise reading of this convergence shows that, for every fixed $\omega \in \mathcal{A}$, the conditions (predictable variance + Lindeberg) of a martingale invariance principle apply to $\tilde{N}_T^*(\tau)$ (redefined on the Skorokhod representation space) and regarded, upon fixing $\omega \in \mathcal{A}$, as a transformation of $\tilde{R}_T$ alone. Specifically, $\tilde{N}_T^*(\tau)$ on the Skorokhod representation space weakly converges to a continuous Gaussian martingale with variance $\frac{\omega^2}{2a}\int_0^\tau [M_u(s)]'[M_v(s)]'\mathrm{d}s$ for every $\omega \in \mathcal{A}$, and therefore, almost surely. Thus, on a general probability space $\tilde{N}_T^*(\tau)$ converges to the same (nonrandom) limit weakly in probability.

We now turn to the proof of part (b). The steps are analogous to those in Lemma 7. We show that, first, $\sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}^*)^2 = \hat{\omega}^2 \sum_{t=1}^{\lfloor T\tau\rfloor}(\zeta_{t-1}^*)^2 + o_p^*(T^{-1-\eta})$ pointwise, next, $\sum_{t=1}^{\lfloor T\tau\rfloor}(\zeta_{t-1}^*)^2 = \sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 + o_p^*(T^{-1-\eta})$ pointwise, and last, $T^{-1-\eta}\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2 \overset{p}{\Rightarrow}_p \frac{1}{2a}M_v(\tau)$.

To accomplish the first step, for $\tilde{z}_t := \sum_{j=1}^{t-2}\varrho^{2j}\sum_{i=0}^{t-j-2}\hat{b}_i v_{t-j-i-1}^*$ we find that

$$\begin{aligned}
\mathrm{E}^*\sum_{t=1}^{T}(z_{t-1}^* - \tilde{z}_{t-1})^2 &= \hat{\phi}^2\sum_{t=1}^{T}\mathrm{E}^*\left(\sum_{j=1}^{t-2}\varrho^{2j}\sum_{i=0}^{t-j-2}\hat{b}_i x_{t-j-i-2}^*\right)^2 \\
&= \hat{\phi}^2\sum_{t=1}^{T}\sum_{j,k=1}^{t-2}\varrho^{2(j+k)}\sum_{i=0}^{t-j-2}\sum_{m=0}^{t-k-2}\hat{b}_i\hat{b}_m\mathrm{E}^*(x_{t-j-i-2}x_{t-k-m-2}^*) \\
&= O_p(T^{-2})\sum_{t=1}^{T}\hat{v}_t^2\left(\sum_{j=1}^{T}\varrho^{2j}\right)^2\left(\sum_{i=0}^{\infty}|\hat{b}_i|\right)^2 = O_p(T^{2\eta-1})
\end{aligned}$$

using Lemma 9(b), such that $T^{-1-\eta}\sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}^*-\tilde{z}_t)^2=o_p^*(1)$. Hence,

$$\left|\sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}^*)^2-\sum_{t=1}^{\lfloor T\tau\rfloor}(\tilde{z}_{t-1}^*)^2\right|\le\sum_{t=1}^{T}(z_{t-1}^*-\tilde{z}_t)^2+2\sqrt{\sum_{t=1}^{T}(\tilde{z}_{t-1}^*)^2}\sqrt{\sum_{t=1}^{T}(z_{t-1}^*-\tilde{z}_t)^2} \tag{C.33}$$
$$=o_p^*(T^{1+\eta})\left(1+T^{-1-\eta}\sum_{t=1}^{T}(\tilde{z}_{t-1}^*)^2\right).$$

Similarly, by using the decomposition

$$\tilde{z}_{t-1}^*-\hat{\omega}\zeta_{t-1}^*=-\sum_{i=0}^{t-2}\hat{b}_i^*v_{t-i-1}^*+(1-\varrho)\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\hat{b}_i^*v_{t-i-j-2}^*$$

with

$$\mathrm{E}^*\sum_{t=1}^{T}(\sum_{i=0}^{t-2}\hat{b}_i^*v_{t-i-1}^*)^2=\sum_{t=1}^{T}\sum_{i=0}^{t-2}(\hat{b}_i^*)^2\hat{v}_{t-i-1}^2\le\sum_{t=1}^{T}\hat{v}_t^2\sum_{i=0}^{\infty}(\hat{b}_i^*)^2=O_p(T)$$

and $\mathrm{E}^*\sum_{t=1}^{T}(\sum_{j=0}^{t-3}\varrho^j\sum_{i=0}^{t-j-3}\hat{b}_i^*v_{t-i-j-2}^*)^2=O_p(T^{1+\eta})$, the latter by formally substituting $\hat{u}_t^2$ with 1 in (C.30), we can conclude that $\mathrm{E}^*\sum_{t=1}^{T}(\tilde{z}_{t-1}^*-\hat{\omega}\zeta_{t-1}^*)^2=o_p(T^{1+\eta})$ and $\sum_{t=1}^{T}(\tilde{z}_{t-1}^*-\hat{\omega}\zeta_{t-1}^*)^2=o_p^*(T^{-1-\eta})$. Therefore,

$$\left|\sum_{t=1}^{\lfloor T\tau\rfloor}(\tilde{z}_{t-1}^*)^2-\hat{\omega}^2\sum_{t=1}^{\lfloor T\tau\rfloor}(\zeta_{t-1}^*)^2\right|\le\sum_{t=1}^{T}(\tilde{z}_{t-1}^*-\hat{\omega}\zeta_{t-1}^*)^2+2|\hat{\omega}|\sqrt{\sum_{t=1}^{T}(\zeta_{t-1}^*)^2}\sqrt{\sum_{t=1}^{T}(\tilde{z}_{t-1}^*-\hat{\omega}\zeta_{t-1}^*)^2}.$$
$$=o_p^*(T^{1+\eta})\left(1+T^{-1-\eta}\sum_{t=1}^{T}(\zeta_{t-1}^*)^2\right).$$

As it will be shown next that $T^{-1-\eta}\sum_{t=1}^{T}(\zeta_{t-1}^*)^2=O_p(1)$, from the previous estimate and (C.33) it follows that $\sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}^*)^2=\hat{\omega}^2\sum_{t=1}^{\lfloor T\tau\rfloor}(\zeta_{t-1}^*)^2+o_p^*(T^{1+\eta})=\omega^2\sum_{t=1}^{\lfloor T\tau\rfloor}(\zeta_{t-1}^*)^2+o_p^*(T^{1+\eta})$.

At the second step, by formally substituting $(T,\tilde{v}_t,u_t)$ with $(\lfloor T\tau\rfloor,v_t^*,1)$ in the discussion of eq. (C.32), it can be concluded that $\sum_{t=1}^{\lfloor T\tau\rfloor}(\zeta_{t-1}^*)^2=\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}\hat{v}_{t-j-1}^2+o_p^*(T^{1+\eta})$. Then $\sum_{t=1}^{\lfloor T\tau\rfloor}(\zeta_{t-1}^*)^2=\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2+o_p^*(T^{1+\eta})$ by (C.17).

Finally, in the third step,

$$\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2=\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}\sigma_{v,t-j-1}^2+O(T^{2\eta})$$
$$=\sum_{t=1}^{\lfloor T\tau\rfloor}\left(\sum_{j=0}^{t-2}\varrho^{2j}\right)\sigma_{vt}^2+O(T^{2\eta})=\frac{T^\eta}{2a}\sum_{t=1}^{\lfloor T\tau\rfloor}\sigma_{vt}^2+O(T^{2\eta})$$

by formally substituting (T,σ_{ut}^2) with $(\lfloor T\tau\rfloor,1)$ in the proof of Lemma 2(d). The pointwise convergence $T^{-1-\eta}\sum_{t=1}^{\lfloor T\tau\rfloor}\sum_{j=0}^{t-2}\varrho^{2j}v_{t-j-1}^2\xrightarrow{p}\frac{1}{2a}[M_v](\tau)$ is now immediate. In conjunction with steps one and two it yields $T^{-1-\eta}\sum_{t=1}^{\lfloor T\tau\rfloor}(z_{t-1}^*)\xrightarrow{p}\frac{\omega^2}{2a}[M_v](\tau)$. As the involved processes are increasing and the limit function is continuous, the convergence is in fact uniform.

The proof of part (c) is a matter of routine and we omit it for brevity. □

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D Additional Monte Carlo Simulations

D.1 Full Sample Test Results

Our baseline DGP for all simulation results for the single predictor case below is the same as the one that was used in Section 5, i.e.,

$$y_t = \beta x_{t-1} + u_t, \tag{D.1}$$

where x_t satisfies the additive component model

$$x_t = \rho x_{t-1} + w_t, \tag{D.2}$$

$$w_t = \psi w_{t-1} + v_t. \tag{D.3}$$

The autoregressive process characterising the dynamics of the putative predictor, x_t, in (D.3) was initialised at $x_0 = 0$. Results are reported for a range of values of the autoregressive parameter ρ in (D.2) that cover stationary, persistent, and mildly explosive predictors; i.e., we consider $\rho = 1 - c/T$ with $c \in \{-5, -2.5, 0, 2.5, 5, 10, 25, 50, 75, 100, 125, 150, 200, 250\}$. The specific generation mechanism of the innovation vector $(u_t, v_t)'$ used to generated the artificial data for each specific DGP that will be considered is characterized in each Table, and four values for the innovations' correlation are considered, $\phi \in \{-0.95, -0.90, -0.50, 0\}$. For all cases results are reported for samples of length $T = 250$ and $T = 1000$.

We first report the empirical rejections frequencies at the 1%, 5% and 10% significance levels for the tests based on the DGPs reported in the text, i.e.,

- **DGP with iid predictor innovations (DGP1):** Data is generate from (D.1) - (D.3) with $\psi = 0$ and results are presented in Tables D.1 - D.4.

- **DGP ARCH with Leverage Effects (DGP2):** This DGP is designed to be such that the regularity conditions needed for the validity of the RWB when x_t is weakly persistent are violated. This DGP is a well known model where the conditional variance of the innovations $(u_t, v_t)'$ follows a stationary ARCH model with leverage effects and is of the form

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \boldsymbol{\psi}_t \tag{D.4}$$

with

$$\boldsymbol{\psi}_t = \begin{pmatrix} a_t \\ e_t \end{pmatrix} = \begin{pmatrix} \varepsilon_{1t}\sqrt{1 + \frac{1}{2}a_{t-1}^2 \mathbb{I}_{\{a_{t-1}<0\}}} \\ \varepsilon_{2t} \end{pmatrix}$$

and $(\varepsilon_{1t}, \varepsilon_{2t})' \sim NIID(\mathbf{0}, \mathbf{I_2})$. The AR parameter ρ is again set equal to $1 - c/T$.

DGP2 satisfies our assumptions of finite fourth moments of $\boldsymbol{\psi}_t$ and martingale approximability of $\boldsymbol{\psi}_t\boldsymbol{\psi}_t'$ (with $\epsilon = 0$). However, and crucially, the quadratic variation of $M_{\xi u}$ depends on,

$$\begin{aligned} h_{11}^2 h_{21}^2 b_1 b_2 \,\mathrm{E}(a_t^2 a_{t-1} a_{t-2}) &= \rho^3\, \mathrm{E}(a_t^2 a_{t-1} a_{t-2}) \\ &= \frac{\rho^3}{8}\,\mathrm{E}\,|\varepsilon_1|^3\,\mathrm{E}\Big\{|a_1|\Big[\sqrt{\Big(1 + \frac{1}{2}a_1^2\Big)^3} - 1\Big]\Big\} > 0. \end{aligned} \tag{D.5}$$

See detailed discussion in Section 5.1.1. Results are presented in Table D.5.

- **DGP with multiple predictors:** The DGP considered is as in Xu and Guo (2020); that is,

$$y_t = \alpha + \mathbf{x}'_{t-1}\boldsymbol{\beta} + u_t, \qquad t = 1, \ldots, T, \tag{D.6}$$
$$\mathbf{x}_t = \boldsymbol{\rho}\mathbf{x}_{t-1} + \mathbf{v}_t, \qquad t = 0, \ldots, T \tag{D.7}$$

where $\mathbf{x}_t := (x_{1,t}, ..., x_{K,t})'$ is a $K \times 1$ vector of predictor variables, $\boldsymbol{\beta}$ is a $K \times 1$ vector of parameters, $\alpha = 0.25$, $\boldsymbol{\rho}$ is a $K \times K$ diagonal matrix with common diagonal element ρ, i.e., $\boldsymbol{\rho} := \mathsf{diag}(\rho, ..., \rho)$, and $(u_t, \mathbf{v}'_t)' \sim NIID(\mathbf{0}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_u^2 & \sigma_{u,v_1} & 0 & \cdots & 0 \\ \sigma_{u,v_1} & \sigma_{v_1}^2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_{v_2}^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{v_K}^2 \end{pmatrix} \tag{D.8}$$

with $\sigma_u^2 = 0.037$, $\sigma_{u,v_1} = -0.035$, $\sigma_{v_1}^2 = ... = \sigma_{v_K}^2 = 0.045$. For the autoregressive parameter we consider $\rho = 1 - c/T$ with $c \in \{-5, 2.5, 0, 2.5, 5, 10, 25, 50, 75, 100, 125, 150, 200, 250\}$; and we consider for the number of predictors, $K \in \{1, 3, 5, 10\}$. For more detail see section 5.2 of the main text. Results are presented in Table D.6.

In addition, we report further results based on the following DGPs:

- **DGP with positively and negatively autocorrelated predictor innovations:** To evaluate the impact on the test statistics when the autoregressive process of the predictor displays short-run dependence we generate data from (D.1) - (D.3) with $\psi \neq 0$. The two cases considered are:
  - **DGP3**: Positively autocorrelated predictor innovations ($\psi = 0.5$); see Tables D.7 - D.10.
  - **DGP4**: Negatively autocorrelated predictor innovations ($\psi = -0.5$); see Tables D.11 - D.14.

- **DGP with Unconditional Heteroskedasticity:** To evaluate the impact of unconditional heteroskedasticity a contemporaneous one-time break of equal magnitude in the variances of u_t and v_t is considered. Specifically, defining the variance of $(u_t, v_t)'$ as

$$\boldsymbol{\Sigma}_t = \begin{bmatrix} \sigma_{ut}^2 & \phi\sigma_{ut}\sigma_{vt} \\ \phi\sigma_{vt}\sigma_{ut} & \sigma_{vt}^2 \end{bmatrix}$$

in DGP5 the simulation design considers an upward change in variance such that $\sigma_{ut}^2 = \sigma_{vt}^2 = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 4\mathbb{I}(t > \lfloor 0.5T \rfloor)$, and in DGP6 a downward change in variance is imposed, i.e., $\sigma_{ut}^2 = \sigma_{vt}^2 = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + \frac{1}{4}\mathbb{I}(t > \lfloor 0.5T \rfloor)$. Notice, therefore, that in both DGP5 and DGP6 the correlation between u_t and v_t does not display a break and is equal to ϕ throughout the sample. These experiments allow us to examine the impact of unconditional heteroskedasticity, both in isolation and in its interaction with ϕ, on the finite sample size of the tests. In both DGPs change in variance of a larger magnitude than we might expect to see in practice is imposed, but this serves to illustrate how the tests behave in the presence of a large change in unconditional volatility.

Hence, the two cases for which we provide results for are:

- **DGP5**: The innovations are characterised by an upward change in the unconditional variance; see Tables D.15 - D.18.
- **DGP6**: The innovations are characterised by a downward change in the unconditional variance; see Tables D.19 - D.22.

- **DGP with Conditional Heteroskedasticity - GARCH(1,1):** A further important feature of financial data is conditional heteroskedasticity. Hence, to evaluate the impact of this feature on the tests performance innovations $(u_t, v_t)'$ are generated to exhibit time-varying conditional second-order moments according to the design,

$$(u_t, v_t)' = \begin{bmatrix} \sigma_{1t} & 0 \\ 0 & \sigma_{2t} \end{bmatrix} \boldsymbol{\eta}_t; \quad E(\boldsymbol{\eta}_t) = \mathbf{0}, \quad E(\boldsymbol{\eta}_t \boldsymbol{\eta}_t') =: \boldsymbol{\Omega}_\phi = \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}$$

where $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})'$ is an i.i.d. vector drawn from either a bivariate Gaussian distribution or a (heavy-tailed) bivariate Student-t distribution with 5 degrees of freedom. The innovations' covariance matrix $\boldsymbol{\Omega}_\phi$ depends on the contemporaneous correlation coefficient ϕ, $\phi \in \{-0.95, -0.90, -0.50, 0\}$. The conditional variances $\left\{\sigma_{it}^2\right\}$ are driven by (normalised) stationary GARCH(1,1) processes $\sigma_{it}^2 = (1-\alpha-\beta)+\alpha e_{i,t-1}^2+\beta\sigma_{i,t-1}^2$, $i = 1, 2$ with $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$, such that $E\left(u_t^2\right) = E\left(v_t^2\right) = 1$. We consider $(\alpha, \beta) = (0.1, 0.85)$.

Hence, the two cases considered are:

- **DGP7**: GARCH(1,1) with Normal Innovations; see Tables D.23 - D.26.
- **DGP8**: GARCH(1,1) with Student-t distributed innovations with 5 degrees of freedom; see Tables D.27 - D.30.

- **DGP with Conditional Heteroskedasticity - GoGARCH(1,1):** In addition to the GARCH we also consider a GoGARCH characterisation of the conditional second moments of the innovations. Specifically, innovation vector $(u_t, v_t)'$ is generated as,

$$(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t, \tag{D.9}$$

where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z}$ is a 2×2 non-singular matrix, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2, i=1,2 are GARCH processes generated as $\sigma_{it}^2 = (1 - \alpha - \beta) + \alpha e_{i,t-1}^2 + \beta\sigma_{i,t-1}^2$, $i = 1, 2$ with $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$, such that $E\left(u_t^2\right) = E\left(v_t^2\right) = 1$. We consider $(\alpha, \beta) = (0.1, 0.85)$. Moreover, $\boldsymbol{\varepsilon}_t$ is either a vector of Gaussian innovations, $\boldsymbol{\varepsilon}_t \sim NIID(\mathbf{0}, diag(1, 1))$, or drawn from a bivariate Student-t distribution with 5 degrees of freedom, $\boldsymbol{\varepsilon}_t \sim iidt_5(\mathbf{0}, diag(1, 1))$. The unconditional covariance matrix of $(u_t, v_t)'$, $\boldsymbol{\Sigma}$, is $\boldsymbol{\Sigma} = \mathbf{Z}\mathbf{Z}' = \begin{bmatrix} 1 & \phi \\ \phi & 1 \end{bmatrix}$; see, for instance, Boswijk and van der Weide (2011) for further details on the GoGARCH model.

Thus, also for the GoGARCH two cases are considered:

- **DGP9**: GoGARCH(1,1) with Normal innovations; see Tables D.31 - D.34.

- **DGP10**: GoGARCH(1,1) with Student-t distributed innovations with 5 degrees of freedom; see Tables D.35 - D.38.

- **DGP with Conditional Heteroskedasticity - Stochastic Volatility:** Finally, we also evaluate the tests when the innovations are generated from an autoregressive (AR) stochastic volatility process. The innovations $(u_t, v_t)'$ follow from a first-order AR stochastic volatility process as $(u_t = e_{1t}\texttt{exp}(h_{1t}), v_t = e_{2t}\texttt{exp}(h_{2t}))'$, and

$$h_{it} = \lambda h_{i,t-1} + 0.5\xi_{it} \tag{D.10}$$

with $(\xi_{it}, e_{it})' \sim NIID(0, diag(\sigma_\xi^2, 1))$, independent across i = 1,2. Results are reported for $(\lambda, \sigma_\xi)' = (0.951, 0.314)'$.

  - **DGP 11**: Stochastic Volatility; see Tables D.39 - D.42.

Table D.1: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.95; \quad -0.95 \quad 1]$.

	1%				5%				10%			
c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - T=250												
-5	0.010	0.000	0.001	0.000	0.046	0.004	0.004	0.003	0.097	0.011	0.013	0.012
-2.5	0.006	0.000	0.000	0.000	0.045	0.000	0.000	0.001	0.111	0.002	0.002	0.002
0	0.014	0.000	0.000	0.000	0.041	0.001	0.001	0.001	0.065	0.002	0.002	0.003
2.5	0.021	0.001	0.001	0.001	0.062	0.005	0.005	0.005	0.099	0.012	0.012	0.011
5	0.023	0.002	0.001	0.001	0.068	0.010	0.011	0.010	0.113	0.026	0.026	0.025
10	0.020	0.003	0.003	0.002	0.064	0.019	0.019	0.018	0.115	0.043	0.044	0.042
25	0.017	0.006	0.006	0.005	0.057	0.029	0.030	0.028	0.108	0.059	0.059	0.058
50	0.012	0.007	0.006	0.006	0.056	0.034	0.036	0.035	0.105	0.072	0.074	0.071
75	0.011	0.007	0.007	0.007	0.056	0.037	0.038	0.037	0.105	0.078	0.082	0.080
100	0.011	0.007	0.008	0.008	0.054	0.038	0.040	0.038	0.108	0.083	0.087	0.084
125	0.011	0.007	0.008	0.007	0.054	0.039	0.042	0.041	0.109	0.089	0.091	0.087
150	0.011	0.007	0.008	0.008	0.055	0.043	0.046	0.042	0.107	0.090	0.094	0.090
200	0.010	0.008	0.009	0.009	0.054	0.046	0.048	0.045	0.108	0.092	0.097	0.094
250	0.011	0.010	0.011	0.009	0.054	0.048	0.051	0.048	0.110	0.099	0.101	0.098
Right-sided tests - $T = 250$												
-5	0.011	0.016	0.020	0.017	0.046	0.074	0.080	0.073	0.092	0.151	0.155	0.150
-2.5	0.010	0.016	0.018	0.017	0.041	0.094	0.097	0.093	0.088	0.240	0.241	0.238
0	0.011	0.022	0.025	0.023	0.053	0.105	0.114	0.110	0.112	0.225	0.231	0.228
2.5	0.014	0.022	0.027	0.023	0.064	0.112	0.116	0.115	0.124	0.226	0.233	0.228
5	0.013	0.023	0.026	0.023	0.062	0.107	0.116	0.112	0.128	0.208	0.215	0.211
10	0.014	0.022	0.025	0.024	0.062	0.097	0.102	0.099	0.120	0.181	0.186	0.184
25	0.012	0.017	0.019	0.017	0.057	0.078	0.084	0.080	0.110	0.147	0.150	0.148
50	0.011	0.013	0.016	0.015	0.052	0.067	0.072	0.067	0.108	0.135	0.139	0.136
75	0.011	0.014	0.015	0.014	0.053	0.064	0.068	0.065	0.105	0.125	0.129	0.126
100	0.011	0.012	0.016	0.014	0.053	0.061	0.065	0.062	0.105	0.119	0.124	0.119
125	0.011	0.012	0.014	0.013	0.052	0.060	0.063	0.060	0.103	0.116	0.120	0.116
150	0.011	0.012	0.013	0.013	0.053	0.056	0.060	0.059	0.103	0.111	0.115	0.111
200	0.010	0.012	0.013	0.011	0.050	0.054	0.056	0.053	0.103	0.109	0.112	0.109
250	0.010	0.011	0.013	0.010	0.051	0.051	0.055	0.053	0.103	0.103	0.107	0.102
Two-sided tests - $T = 250$												
-5	0.010	0.008	0.012	0.011	0.048	0.038	0.044	0.039	0.095	0.075	0.083	0.077
-2.5	0.009	0.008	0.010	0.009	0.038	0.040	0.048	0.044	0.083	0.094	0.098	0.094
0	0.010	0.011	0.013	0.011	0.047	0.051	0.057	0.053	0.095	0.105	0.115	0.110
2.5	0.012	0.012	0.015	0.013	0.053	0.058	0.062	0.060	0.107	0.116	0.121	0.120
5	0.012	0.012	0.014	0.012	0.054	0.058	0.063	0.060	0.111	0.118	0.127	0.121
10	0.012	0.012	0.015	0.013	0.055	0.060	0.066	0.060	0.109	0.115	0.122	0.118
25	0.011	0.010	0.014	0.012	0.056	0.056	0.060	0.058	0.105	0.109	0.114	0.109
50	0.011	0.010	0.012	0.011	0.051	0.051	0.054	0.052	0.102	0.101	0.108	0.102
75	0.011	0.010	0.012	0.011	0.049	0.047	0.052	0.049	0.102	0.100	0.106	0.102
100	0.009	0.010	0.011	0.010	0.049	0.048	0.052	0.050	0.101	0.099	0.105	0.100
125	0.010	0.010	0.011	0.011	0.050	0.049	0.053	0.051	0.101	0.100	0.105	0.101
150	0.009	0.009	0.011	0.010	0.051	0.049	0.054	0.052	0.102	0.099	0.106	0.101
200	0.009	0.010	0.012	0.010	0.050	0.048	0.054	0.050	0.101	0.099	0.104	0.098
250	0.010	0.010	0.012	0.011	0.049	0.048	0.053	0.050	0.101	0.100	0.105	0.101

	1%				5%				10%			
c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.045	0.002	0.003	0.003	0.094	0.010	0.011	0.010
-2.5	0.005	0.000	0.000	0.000	0.046	0.000	0.000	0.000	0.109	0.001	0.001	0.001
0	0.015	0.000	0.000	0.000	0.042	0.001	0.001	0.001	0.068	0.003	0.003	0.003
2.5	0.024	0.001	0.001	0.001	0.060	0.006	0.006	0.006	0.097	0.015	0.016	0.016
5	0.023	0.002	0.002	0.002	0.068	0.013	0.014	0.013	0.109	0.028	0.028	0.028
10	0.021	0.004	0.004	0.004	0.064	0.022	0.021	0.021	0.113	0.044	0.044	0.044
25	0.016	0.007	0.007	0.008	0.061	0.030	0.030	0.031	0.107	0.064	0.064	0.063
50	0.014	0.008	0.008	0.008	0.057	0.035	0.035	0.035	0.108	0.072	0.072	0.071
75	0.012	0.008	0.008	0.008	0.055	0.039	0.039	0.038	0.107	0.078	0.080	0.078
100	0.013	0.008	0.007	0.008	0.055	0.040	0.040	0.040	0.105	0.082	0.081	0.081
125	0.013	0.008	0.008	0.008	0.054	0.041	0.042	0.041	0.104	0.082	0.084	0.082
150	0.011	0.009	0.008	0.008	0.053	0.043	0.043	0.043	0.105	0.085	0.085	0.083
200	0.012	0.009	0.008	0.008	0.053	0.044	0.044	0.042	0.105	0.088	0.088	0.088
250	0.012	0.009	0.009	0.009	0.053	0.044	0.044	0.044	0.106	0.090	0.091	0.089
Right-sided tests - $T = 1000$												
-5	0.007	0.012	0.014	0.013	0.039	0.064	0.065	0.064	0.086	0.139	0.140	0.137
-2.5	0.008	0.018	0.017	0.016	0.041	0.092	0.092	0.091	0.086	0.229	0.226	0.225
0	0.010	0.019	0.020	0.019	0.050	0.103	0.104	0.102	0.105	0.223	0.222	0.223
2.5	0.010	0.020	0.021	0.020	0.059	0.107	0.108	0.107	0.117	0.218	0.218	0.218
5	0.010	0.020	0.020	0.019	0.059	0.105	0.106	0.106	0.121	0.205	0.207	0.205
10	0.010	0.020	0.020	0.018	0.059	0.097	0.098	0.098	0.116	0.181	0.182	0.181
25	0.011	0.016	0.017	0.016	0.055	0.081	0.082	0.081	0.108	0.154	0.154	0.151
50	0.010	0.015	0.015	0.015	0.051	0.070	0.071	0.070	0.104	0.133	0.133	0.133
75	0.011	0.013	0.013	0.013	0.051	0.067	0.068	0.067	0.102	0.126	0.127	0.126
100	0.010	0.013	0.013	0.012	0.051	0.065	0.066	0.065	0.104	0.123	0.124	0.123
125	0.009	0.012	0.012	0.011	0.051	0.063	0.063	0.063	0.103	0.121	0.122	0.121
150	0.009	0.012	0.011	0.011	0.051	0.061	0.061	0.061	0.103	0.121	0.121	0.120
200	0.008	0.012	0.011	0.010	0.050	0.059	0.060	0.059	0.106	0.119	0.120	0.119
250	0.010	0.011	0.011	0.010	0.050	0.059	0.059	0.059	0.105	0.116	0.116	0.116
Two-sided tests - $T = 1000$												
-5	0.007	0.006	0.007	0.006	0.040	0.030	0.032	0.031	0.087	0.066	0.067	0.066
-2.5	0.007	0.008	0.008	0.008	0.037	0.042	0.043	0.042	0.080	0.091	0.092	0.091
0	0.008	0.009	0.009	0.009	0.041	0.050	0.050	0.049	0.090	0.103	0.105	0.103
2.5	0.008	0.011	0.011	0.009	0.050	0.056	0.057	0.058	0.101	0.112	0.114	0.113
5	0.009	0.011	0.010	0.009	0.052	0.056	0.058	0.058	0.108	0.118	0.120	0.119
10	0.009	0.011	0.011	0.011	0.055	0.061	0.063	0.062	0.108	0.117	0.119	0.119
25	0.011	0.013	0.012	0.012	0.052	0.056	0.057	0.056	0.105	0.111	0.112	0.112
50	0.011	0.012	0.012	0.011	0.051	0.053	0.054	0.053	0.102	0.104	0.106	0.105
75	0.012	0.011	0.011	0.011	0.051	0.052	0.054	0.052	0.105	0.106	0.107	0.105
100	0.010	0.010	0.010	0.010	0.051	0.051	0.053	0.052	0.103	0.104	0.106	0.105
125	0.010	0.009	0.010	0.009	0.050	0.051	0.052	0.052	0.106	0.104	0.105	0.104
150	0.009	0.010	0.010	0.009	0.050	0.052	0.053	0.051	0.103	0.102	0.105	0.103
200	0.009	0.010	0.009	0.009	0.052	0.051	0.052	0.051	0.103	0.102	0.104	0.102
250	0.011	0.010	0.011	0.009	0.050	0.051	0.051	0.051	0.102	0.102	0.103	0.103

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4.

Table D.2: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.90; \quad -0.90 \quad 1]$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 250$												
-5	0.009	0.001	0.001	0.000	0.049	0.004	0.005	0.004	0.097	0.014	0.016	0.013
-2.5	0.008	0.000	0.000	0.000	0.047	0.000	0.001	0.001	0.111	0.002	0.002	0.002
0	0.012	0.000	0.000	0.000	0.039	0.001	0.001	0.001	0.063	0.004	0.004	0.004
2.5	0.019	0.001	0.001	0.001	0.059	0.005	0.005	0.005	0.096	0.014	0.014	0.013
5	0.022	0.002	0.001	0.002	0.066	0.010	0.011	0.010	0.112	0.027	0.027	0.027
10	0.018	0.003	0.003	0.003	0.063	0.019	0.020	0.020	0.111	0.044	0.045	0.044
25	0.015	0.006	0.006	0.006	0.055	0.030	0.032	0.030	0.107	0.061	0.061	0.060
50	0.011	0.006	0.007	0.006	0.054	0.033	0.036	0.034	0.105	0.072	0.074	0.072
75	0.009	0.007	0.007	0.007	0.054	0.038	0.038	0.037	0.105	0.078	0.081	0.080
100	0.010	0.008	0.009	0.008	0.050	0.037	0.040	0.039	0.106	0.084	0.086	0.084
125	0.011	0.008	0.008	0.007	0.053	0.041	0.043	0.041	0.107	0.087	0.090	0.088
150	0.011	0.008	0.009	0.009	0.054	0.043	0.045	0.044	0.106	0.089	0.092	0.089
200	0.011	0.009	0.010	0.009	0.054	0.046	0.049	0.047	0.107	0.093	0.095	0.092
250	0.011	0.009	0.011	0.010	0.055	0.048	0.051	0.048	0.106	0.096	0.099	0.097
Right-sided tests - $T = 250$												
-5	0.010	0.015	0.019	0.017	0.044	0.074	0.079	0.073	0.093	0.150	0.155	0.149
-2.5	0.010	0.016	0.019	0.017	0.042	0.093	0.100	0.094	0.091	0.234	0.238	0.234
0	0.011	0.021	0.025	0.022	0.054	0.104	0.112	0.108	0.113	0.225	0.231	0.226
2.5	0.013	0.023	0.026	0.024	0.063	0.110	0.114	0.112	0.125	0.218	0.227	0.221
5	0.013	0.024	0.026	0.024	0.062	0.106	0.114	0.109	0.128	0.202	0.208	0.204
10	0.014	0.022	0.026	0.023	0.061	0.094	0.100	0.096	0.121	0.178	0.183	0.180
25	0.011	0.017	0.019	0.017	0.058	0.078	0.082	0.079	0.111	0.146	0.150	0.148
50	0.011	0.014	0.016	0.014	0.053	0.067	0.071	0.068	0.107	0.132	0.136	0.132
75	0.010	0.014	0.017	0.014	0.052	0.064	0.067	0.064	0.104	0.125	0.130	0.127
100	0.010	0.013	0.014	0.014	0.052	0.062	0.065	0.062	0.104	0.119	0.124	0.121
125	0.010	0.012	0.014	0.012	0.053	0.059	0.062	0.059	0.103	0.114	0.118	0.113
150	0.010	0.012	0.014	0.012	0.052	0.056	0.059	0.058	0.101	0.110	0.113	0.111
200	0.010	0.012	0.013	0.011	0.051	0.055	0.058	0.055	0.104	0.109	0.111	0.107
250	0.011	0.011	0.012	0.011	0.048	0.049	0.053	0.051	0.104	0.105	0.109	0.106
Two-sided tests - $T = 250$												
-5	0.010	0.008	0.011	0.010	0.045	0.036	0.044	0.038	0.096	0.077	0.084	0.077
-2.5	0.009	0.008	0.010	0.010	0.037	0.042	0.047	0.044	0.084	0.095	0.100	0.095
0	0.010	0.011	0.013	0.011	0.048	0.052	0.057	0.053	0.095	0.104	0.113	0.109
2.5	0.011	0.011	0.015	0.014	0.055	0.059	0.062	0.061	0.107	0.113	0.119	0.117
5	0.012	0.012	0.015	0.013	0.054	0.059	0.064	0.060	0.107	0.115	0.125	0.119
10	0.012	0.012	0.016	0.013	0.055	0.057	0.065	0.061	0.108	0.112	0.121	0.116
25	0.010	0.011	0.013	0.012	0.054	0.055	0.060	0.057	0.105	0.107	0.114	0.109
50	0.010	0.010	0.011	0.011	0.049	0.050	0.054	0.052	0.102	0.103	0.107	0.102
75	0.010	0.010	0.011	0.011	0.048	0.049	0.053	0.049	0.102	0.099	0.105	0.101
100	0.010	0.010	0.011	0.010	0.049	0.048	0.052	0.050	0.100	0.099	0.105	0.101
125	0.009	0.008	0.011	0.010	0.052	0.050	0.054	0.052	0.101	0.098	0.105	0.101
150	0.008	0.009	0.010	0.010	0.049	0.050	0.054	0.052	0.102	0.099	0.104	0.102
200	0.009	0.009	0.011	0.010	0.049	0.048	0.052	0.050	0.104	0.102	0.107	0.102
250	0.009	0.010	0.012	0.010	0.049	0.049	0.054	0.049	0.101	0.097	0.103	0.099

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.047	0.003	0.003	0.003	0.095	0.013	0.013	0.013
-2.5	0.006	0.000	0.000	0.000	0.048	0.000	0.000	0.000	0.109	0.001	0.001	0.001
0	0.013	0.000	0.000	0.000	0.040	0.002	0.002	0.002	0.066	0.004	0.004	0.004
2.5	0.021	0.001	0.001	0.001	0.058	0.007	0.007	0.007	0.095	0.018	0.017	0.018
5	0.022	0.003	0.003	0.003	0.065	0.014	0.014	0.014	0.106	0.030	0.031	0.030
10	0.019	0.004	0.004	0.004	0.063	0.023	0.022	0.022	0.110	0.047	0.045	0.045
25	0.016	0.008	0.008	0.008	0.059	0.031	0.031	0.031	0.110	0.063	0.063	0.063
50	0.014	0.008	0.008	0.008	0.056	0.036	0.036	0.036	0.108	0.076	0.076	0.075
75	0.012	0.008	0.008	0.008	0.054	0.039	0.040	0.039	0.108	0.080	0.081	0.080
100	0.013	0.008	0.008	0.008	0.054	0.040	0.041	0.040	0.107	0.083	0.083	0.082
125	0.013	0.008	0.008	0.008	0.055	0.044	0.043	0.043	0.105	0.085	0.084	0.083
150	0.012	0.008	0.008	0.008	0.055	0.045	0.044	0.044	0.103	0.085	0.086	0.084
200	0.012	0.008	0.008	0.008	0.054	0.045	0.046	0.046	0.104	0.088	0.089	0.088
250	0.011	0.009	0.009	0.009	0.052	0.046	0.046	0.046	0.105	0.090	0.091	0.091
Right-sided tests - $T = 1000$												
-5	0.008	0.012	0.013	0.013	0.041	0.066	0.067	0.064	0.085	0.140	0.141	0.140
-2.5	0.009	0.017	0.017	0.016	0.040	0.092	0.091	0.090	0.087	0.225	0.225	0.224
0	0.010	0.020	0.020	0.019	0.051	0.098	0.101	0.100	0.103	0.219	0.216	0.217
2.5	0.010	0.020	0.021	0.020	0.058	0.103	0.106	0.105	0.118	0.212	0.213	0.212
5	0.011	0.020	0.020	0.019	0.059	0.100	0.102	0.102	0.118	0.198	0.198	0.197
10	0.010	0.018	0.018	0.017	0.059	0.094	0.095	0.094	0.116	0.173	0.175	0.175
25	0.010	0.015	0.015	0.015	0.055	0.080	0.080	0.078	0.110	0.149	0.151	0.149
50	0.011	0.014	0.014	0.014	0.051	0.069	0.070	0.069	0.103	0.133	0.133	0.132
75	0.010	0.013	0.013	0.013	0.052	0.066	0.066	0.066	0.101	0.126	0.125	0.123
100	0.009	0.012	0.012	0.012	0.053	0.065	0.064	0.064	0.102	0.121	0.122	0.122
125	0.009	0.011	0.012	0.011	0.051	0.060	0.062	0.062	0.102	0.122	0.122	0.120
150	0.009	0.011	0.011	0.011	0.051	0.061	0.060	0.060	0.103	0.119	0.120	0.119
200	0.009	0.011	0.011	0.010	0.053	0.059	0.061	0.061	0.103	0.118	0.118	0.119
250	0.009	0.011	0.011	0.010	0.051	0.058	0.059	0.060	0.103	0.115	0.115	0.115
Two-sided tests - $T = 1000$												
-5	0.008	0.006	0.007	0.006	0.041	0.032	0.034	0.033	0.087	0.068	0.070	0.067
-2.5	0.007	0.008	0.008	0.008	0.037	0.040	0.043	0.041	0.081	0.090	0.091	0.090
0	0.009	0.011	0.010	0.010	0.043	0.048	0.049	0.048	0.087	0.100	0.102	0.101
2.5	0.008	0.011	0.010	0.010	0.050	0.056	0.057	0.057	0.098	0.109	0.113	0.112
5	0.010	0.011	0.011	0.010	0.051	0.058	0.059	0.058	0.104	0.117	0.116	0.116
10	0.009	0.010	0.010	0.010	0.054	0.059	0.061	0.060	0.105	0.115	0.117	0.116
25	0.010	0.011	0.011	0.010	0.052	0.056	0.056	0.055	0.105	0.110	0.111	0.109
50	0.010	0.011	0.012	0.011	0.051	0.053	0.054	0.052	0.102	0.106	0.106	0.105
75	0.010	0.011	0.011	0.011	0.051	0.053	0.053	0.053	0.102	0.105	0.106	0.104
100	0.010	0.011	0.010	0.010	0.052	0.052	0.052	0.053	0.103	0.103	0.105	0.104
125	0.010	0.010	0.010	0.010	0.051	0.052	0.053	0.053	0.103	0.103	0.105	0.104
150	0.009	0.010	0.010	0.009	0.051	0.052	0.054	0.052	0.104	0.105	0.105	0.104
200	0.009	0.010	0.009	0.009	0.049	0.050	0.053	0.051	0.106	0.105	0.107	0.107
250	0.010	0.009	0.009	0.009	0.050	0.050	0.051	0.050	0.105	0.104	0.105	0.106

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4.

Table D.3: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.50; \quad -0.50 \quad 1]$.

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - T=250												
-5	0.010	0.002	0.005	0.002	0.053	0.019	0.024	0.019	0.105	0.046	0.052	0.047
-2.5	0.010	0.000	0.001	0.000	0.050	0.006	0.007	0.005	0.102	0.016	0.017	0.016
0	0.006	0.000	0.001	0.001	0.030	0.005	0.006	0.006	0.059	0.017	0.017	0.017
2.5	0.009	0.002	0.002	0.002	0.045	0.016	0.017	0.016	0.086	0.037	0.039	0.037
5	0.012	0.004	0.004	0.004	0.050	0.023	0.024	0.023	0.095	0.051	0.052	0.050
10	0.012	0.006	0.006	0.006	0.052	0.030	0.031	0.030	0.099	0.062	0.064	0.063
25	0.011	0.007	0.007	0.006	0.050	0.036	0.038	0.036	0.101	0.076	0.079	0.077
50	0.010	0.006	0.007	0.007	0.048	0.039	0.040	0.039	0.100	0.083	0.087	0.085
75	0.008	0.007	0.007	0.006	0.050	0.042	0.045	0.043	0.097	0.082	0.085	0.083
100	0.009	0.007	0.008	0.007	0.049	0.043	0.045	0.043	0.097	0.085	0.088	0.087
125	0.010	0.008	0.009	0.008	0.051	0.044	0.046	0.045	0.096	0.086	0.088	0.089
150	0.010	0.008	0.009	0.009	0.051	0.046	0.048	0.047	0.097	0.089	0.091	0.089
200	0.010	0.019	0.010	0.010	0.052	0.047	0.050	0.048	0.102	0.093	0.096	0.095
250	0.010	0.010	0.012	0.010	0.053	0.049	0.052	0.051	0.103	0.098	0.100	0.097
Right-sided tests - $T = 250$												
-5	0.009	0.016	0.020	0.015	0.046	0.072	0.079	0.072	0.097	0.144	0.152	0.143
-2.5	0.012	0.020	0.026	0.020	0.053	0.101	0.107	0.101	0.107	0.196	0.203	0.197
0	0.013	0.020	0.022	0.019	0.061	0.097	0.102	0.096	0.121	0.191	0.197	0.190
2.5	0.014	0.019	0.022	0.020	0.061	0.090	0.096	0.090	0.119	0.171	0.175	0.174
5	0.013	0.018	0.020	0.019	0.061	0.081	0.087	0.085	0.113	0.158	0.162	0.158
10	0.012	0.017	0.018	0.018	0.056	0.074	0.078	0.076	0.111	0.142	0.147	0.145
25	0.012	0.014	0.016	0.016	0.053	0.065	0.069	0.067	0.110	0.131	0.134	0.131
50	0.010	0.013	0.014	0.013	0.054	0.063	0.066	0.064	0.108	0.120	0.124	0.122
75	0.009	0.012	0.013	0.011	0.055	0.060	0.065	0.062	0.107	0.116	0.121	0.119
100	0.009	0.011	0.012	0.011	0.054	0.059	0.063	0.061	0.109	0.116	0.119	0.117
125	0.010	0.010	0.012	0.011	0.055	0.059	0.061	0.059	0.109	0.112	0.118	0.114
150	0.010	0.011	0.012	0.010	0.055	0.057	0.061	0.058	0.107	0.111	0.115	0.112
200	0.009	0.010	0.010	0.010	0.053	0.054	0.057	0.054	0.105	0.107	0.111	0.108
250	0.009	0.009	0.010	0.009	0.051	0.052	0.055	0.051	0.105	0.105	0.110	0.107
Two-sided tests - $T = 250$												
-5	0.009	0.009	0.015	0.009	0.048	0.043	0.055	0.042	0.098	0.089	0.102	0.090
-2.5	0.010	0.011	0.014	0.011	0.049	0.050	0.059	0.051	0.101	0.106	0.113	0.106
0	0.011	0.010	0.012	0.012	0.048	0.051	0.057	0.053	0.098	0.100	0.108	0.102
2.5	0.012	0.011	0.013	0.013	0.052	0.054	0.059	0.055	0.101	0.105	0.113	0.106
5	0.011	0.012	0.014	0.013	0.052	0.055	0.059	0.057	0.103	0.105	0.110	0.107
10	0.013	0.011	0.013	0.013	0.052	0.053	0.057	0.055	0.104	0.104	0.110	0.106
25	0.011	0.011	0.013	0.012	0.049	0.050	0.054	0.052	0.101	0.102	0.107	0.103
50	0.009	0.008	0.011	0.008	0.049	0.050	0.054	0.050	0.101	0.100	0.106	0.104
75	0.008	0.008	0.010	0.008	0.049	0.049	0.054	0.052	0.104	0.102	0.110	0.105
100	0.008	0.009	0.011	0.010	0.050	0.048	0.054	0.052	0.103	0.103	0.108	0.104
125	0.008	0.009	0.011	0.009	0.051	0.050	0.055	0.050	0.105	0.103	0.107	0.104
150	0.008	0.010	0.011	0.009	0.051	0.049	0.054	0.052	0.105	0.104	0.109	0.105
200	0.009	0.010	0.012	0.010	0.049	0.049	0.055	0.052	0.104	0.100	0.107	0.102
250	0.009	0.011	0.013	0.010	0.049	0.049	0.053	0.050	0.103	0.099	0.107	0.102

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.010	0.002	0.002	0.002	0.050	0.018	0.018	0.018	0.097	0.045	0.045	0.044
-2.5	0.008	0.001	0.000	0.000	0.048	0.004	0.004	0.004	0.099	0.014	0.015	0.014
0	0.008	0.001	0.001	0.001	0.032	0.009	0.008	0.008	0.064	0.020	0.020	0.020
2.5	0.013	0.002	0.003	0.002	0.049	0.018	0.018	0.018	0.091	0.042	0.042	0.042
5	0.013	0.004	0.003	0.003	0.054	0.024	0.024	0.025	0.101	0.055	0.055	0.054
10	0.012	0.006	0.005	0.005	0.053	0.031	0.031	0.031	0.104	0.067	0.066	0.066
25	0.012	0.008	0.007	0.008	0.053	0.039	0.039	0.039	0.105	0.079	0.079	0.078
50	0.013	0.011	0.009	0.008	0.051	0.042	0.042	0.043	0.102	0.085	0.085	0.085
75	0.013	0.011	0.010	0.010	0.053	0.045	0.045	0.045	0.103	0.087	0.088	0.087
100	0.014	0.011	0.011	0.011	0.052	0.047	0.046	0.045	0.101	0.089	0.089	0.088
125	0.013	0.011	0.011	0.011	0.051	0.045	0.046	0.045	0.101	0.090	0.091	0.090
150	0.013	0.011	0.011	0.012	0.052	0.046	0.046	0.046	0.103	0.093	0.094	0.092
200	0.013	0.011	0.011	0.011	0.053	0.047	0.046	0.046	0.104	0.095	0.096	0.095
250	0.013	0.012	0.012	0.012	0.052	0.048	0.047	0.046	0.104	0.095	0.095	0.095
Right-sided tests - $T = 1000$												
-5	0.008	0.013	0.015	0.013	0.047	0.070	0.072	0.070	0.097	0.139	0.141	0.139
-2.5	0.009	0.018	0.016	0.016	0.047	0.095	0.096	0.094	0.102	0.191	0.193	0.191
0	0.011	0.018	0.019	0.018	0.056	0.091	0.090	0.091	0.116	0.189	0.185	0.185
2.5	0.012	0.019	0.020	0.019	0.058	0.087	0.087	0.086	0.114	0.167	0.168	0.167
5	0.011	0.018	0.018	0.018	0.057	0.082	0.081	0.080	0.111	0.154	0.155	0.155
10	0.011	0.017	0.015	0.016	0.053	0.073	0.074	0.074	0.106	0.139	0.140	0.139
25	0.009	0.013	0.013	0.012	0.052	0.064	0.065	0.063	0.101	0.122	0.125	0.124
50	0.011	0.012	0.012	0.012	0.049	0.057	0.059	0.058	0.097	0.115	0.115	0.114
75	0.010	0.012	0.012	0.012	0.049	0.056	0.057	0.057	0.096	0.112	0.112	0.111
100	0.011	0.012	0.012	0.011	0.049	0.056	0.059	0.057	0.101	0.110	0.111	0.111
125	0.010	0.012	0.012	0.011	0.049	0.054	0.056	0.055	0.101	0.109	0.110	0.111
150	0.011	0.012	0.012	0.012	0.050	0.055	0.055	0.054	0.100	0.109	0.109	0.110
200	0.011	0.012	0.012	0.012	0.050	0.054	0.054	0.054	0.098	0.110	0.109	0.109
250	0.011	0.012	0.013	0.012	0.050	0.053	0.054	0.055	0.102	0.108	0.108	0.108
Two-sided tests - $T = 1000$												
-5	0.008	0.008	0.008	0.007	0.047	0.042	0.043	0.041	0.098	0.088	0.090	0.089
-2.5	0.008	0.008	0.008	0.007	0.043	0.047	0.047	0.045	0.093	0.097	0.100	0.098
0	0.009	0.010	0.010	0.010	0.045	0.049	0.050	0.048	0.094	0.099	0.099	0.099
2.5	0.010	0.011	0.011	0.010	0.050	0.052	0.053	0.052	0.100	0.105	0.105	0.104
5	0.010	0.011	0.011	0.010	0.050	0.053	0.054	0.053	0.100	0.105	0.105	0.105
10	0.010	0.011	0.010	0.010	0.050	0.052	0.054	0.054	0.101	0.104	0.105	0.105
25	0.010	0.011	0.011	0.011	0.049	0.050	0.052	0.051	0.102	0.102	0.104	0.102
50	0.011	0.010	0.011	0.011	0.054	0.052	0.053	0.053	0.101	0.099	0.100	0.101
75	0.012	0.012	0.012	0.012	0.053	0.053	0.054	0.053	0.100	0.101	0.102	0.102
100	0.013	0.013	0.013	0.013	0.052	0.052	0.053	0.052	0.102	0.103	0.105	0.103
125	0.012	0.012	0.012	0.013	0.052	0.052	0.052	0.053	0.102	0.099	0.102	0.100
150	0.011	0.012	0.012	0.012	0.052	0.052	0.053	0.054	0.100	0.099	0.102	0.100
200	0.012	0.012	0.012	0.012	0.052	0.052	0.053	0.053	0.102	0.100	0.100	0.099
250	0.012	0.011	0.011	0.012	0.050	0.050	0.051	0.051	0.102	0.099	0.101	0.101

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4.

Table D.4: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad 0; \quad 0 \quad 1]$.

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.011	0.011	0.017	0.010	0.052	0.051 1	0.060	0.052	0.102	0.102	0.111	0.102
-2.5	0.011	0.011	0.015	0.010	0.050	0.050	0.058	0.051	0.103	0.103	0.111	0.102
0	0.011	0.011	0.013	0.011	0.049	0.049	0.056	0.050	0.098	0.100	0.100	0.097
2.5	0.009	0.009	0.012	0.010	0.051	0.050	0.053	0.050	0.099	0.099	0.101	0.099
5	0.010	0.010	0.011	0.010	0.052	0.051	0.053	0.050	0.098	0.098	0.099	0.097
10	0.010	0.011	0.012	0.011	0.051	0.051	0.054	0.052	0.102	0.100	0.102	0.101
25	0.010	0.010	0.011	0.010	0.050	0.050	0.052	0.051	0.098	0.096	0.098	0.099
50	0.009	0.009	0.010	0.009	0.048	0.048	0.051	0.049	0.100	0.098	0.101	0.101
75	0.009	0.009	0.010	0.010	0.047	0.046	0.050	0.048	0.099	0.096	0.099	0.098
100	0.009	0.010	0.010	0.011	0.048	0.048	0.051	0.048	0.097	0.095	0.098	0.098
125	0.010	0.010	0.011	0.011	0.048	0.046	0.049	0.048	0.096	0.093	0.097	0.096
150	0.009	0.010	0.011	0.011	0.047	0.047	0.048	0.048	0.095	0.092	0.096	0.095
200	0.009	0.009	0.011	0.010	0.047	0.047	0.049	0.047	0.096	0.093	0.095	0.095
250	0.009	0.011	0.011	0.010	0.049	0.048	0.051	0.048	0.096	0.094	0.098	0.095
Right-sided tests - $T = 250$												
-5	0.012	0.011	0.017	0.0118	0.052	0.051	0.060	0.050	0.101	0.101	0.110	0.100
-2.5	0.010	0.011	0.015	0.0100	0.053	0.052	0.058	0.052	0.097	0.100	0.105	0.097
0	0.010	0.012	0.014	0.0111	0.051	0.051	0.055	0.049	0.098	0.100	0.103	0.098
2.5	0.010	0.011	0.012	0.0103	0.052	0.052	0.053	0.052	0.102	0.101	0.103	0.103
5	0.011	0.011	0.013	0.0106	0.052	0.051	0.053	0.051	0.104	0.101	0.105	0.102
10	0.011	0.010	0.011	0.0096	0.051	0.051	0.054	0.051	0.102	0.101	0.104	0.104
25	0.012	0.011	0.013	0.0116	0.051	0.052	0.054	0.052	0.104	0.103	0.105	0.104
50	0.011	0.010	0.012	0.0109	0.056	0.054	0.057	0.056	0.103	0.103	0.106	0.103
75	0.010	0.010	0.010	0.0100	0.054	0.055	0.058	0.054	0.104	0.102	0.108	0.104
100	0.010	0.010	0.012	0.0104	0.054	0.052	0.056	0.055	0.106	0.104	0.106	0.104
125	0.009	0.011	0.012	0.0114	0.053	0.052	0.056	0.053	0.104	0.103	0.106	0.103
150	0.010	0.010	0.012	0.0111	0.051	0.052	0.054	0.052	0.104	0.101	0.105	0.102
200	0.010	0.010	0.011	0.0103	0.051	0.051	0.055	0.052	0.102	0.099	0.103	0.102
250	0.009	0.010	0.011	0.011	0.053	0.051	0.054	0.054	0.100	0.098	0.102	0.100
Two-sided tests - $T = 250$												
-5	0.010	0.011	0.020	0.011	0.050	0.052	0.065	0.051	0.102	0.101	0.119	0.102
-2.5	0.011	0.011	0.019	0.011	0.052	0.052	0.066	0.052	0.101	0.101	0.116	0.102
0	0.009	0.010	0.015	0.011	0.050	0.051	0.058	0.051	0.098	0.100	0.111	0.099
2.5	0.010	0.010	0.012	0.010	0.050	0.050	0.057	0.052	0.100	0.101	0.106	0.102
5	0.010	0.010	0.012	0.011	0.049	0.051	0.055	0.052	0.102	0.100	0.106	0.101
10	0.012	0.012	0.013	0.012	0.049	0.048	0.052	0.050	0.100	0.101	0.108	0.103
25	0.011	0.011	0.013	0.013	0.053	0.053	0.057	0.055	0.102	0.100	0.106	0.103
50	0.010	0.010	0.011	0.010	0.052	0.051	0.056	0.052	0.103	0.101	0.108	0.104
75	0.009	0.010	0.011	0.009	0.052	0.051	0.056	0.053	0.102	0.101	0.108	0.102
100	0.008	0.009	0.011	0.010	0.050	0.049	0.053	0.052	0.101	0.100	0.107	0.102
125	0.009	0.010	0.012	0.011	0.050	0.049	0.054	0.050	0.100	0.097	0.105	0.101
150	0.009	0.010	0.012	0.010	0.049	0.051	0.056	0.052	0.099	0.097	0.102	0.100
200	0.009	0.009	0.012	0.010	0.052	0.053	0.056	0.054	0.098	0.099	0.104	0.099
250	0.010	0.010	0.012	0.010	0.049	0.051	0.054	0.051	0.101	0.100	0.105	0.101

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.009	0.009	0.009	0.009	0.051	0.050	0.051	0.051	0.097	0.096	0.098	0.096
-2.5	0.010	0.009	0.010	0.009	0.050	0.050	0.050	0.050	0.097	0.096	0.096	0.095
0	0.011	0.011	0.011	0.010	0.053	0.052	0.052	0.053	0.102	0.102	0.103	0.103
2.5	0.012	0.012	0.011	0.012	0.055	0.055	0.054	0.053	0.102	0.103	0.103	0.103
5	0.012	0.011	0.011	0.011	0.052	0.051	0.053	0.051	0.105	0.106	0.105	0.105
10	0.010	0.011	0.010	0.011	0.051	0.052	0.052	0.052	0.104	0.103	0.104	0.104
25	0.011	0.011	0.011	0.011	0.054	0.053	0.053	0.052	0.104	0.104	0.104	0.103
50	0.012	0.011	0.011	0.011	0.053	0.052	0.053	0.053	0.105	0.104	0.106	0.104
75	0.012	0.011	0.011	0.011	0.053	0.051	0.051	0.051	0.103	0.102	0.104	0.102
100	0.012	0.011	0.011	0.011	0.052	0.052	0.052	0.051	0.103	0.104	0.104	0.103
125	0.012	0.011	0.012	0.012	0.051	0.051	0.052	0.051	0.103	0.103	0.103	0.102
150	0.011	0.012	0.012	0.012	0.052	0.051	0.053	0.052	0.103	0.102	0.103	0.102
200	0.011	0.011	0.011	0.011	0.053	0.052	0.053	0.053	0.100	0.100	0.101	0.100
250	0.011	0.011	0.011	0.012	0.051	0.051	0.051	0.052	0.099	0.099	0.100	0.098
Right-sided tests - $T = 1000$												
-5	0.011	0.011	0.012	0.010	0.048	0.048	0.049	0.048	0.100	0.099	0.103	0.099
-2.5	0.010	0.010	0.010	0.009	0.048	0.048	0.049	0.046	0.096	0.096	0.097	0.095
0	0.009	0.009	0.009	0.009	0.050	0.049	0.048	0.048	0.099	0.100	0.101	0.100
2.5	0.009	0.010	0.010	0.009	0.051	0.051	0.052	0.050	0.102	0.103	0.102	0.101
5	0.009	0.009	0.009	0.009	0.052	0.053	0.053	0.053	0.102	0.103	0.103	0.103
10	0.010	0.010	0.010	0.009	0.052	0.052	0.053	0.052	0.100	0.101	0.102	0.100
25	0.010	0.010	0.010	0.010	0.050	0.049	0.050	0.050	0.096	0.097	0.098	0.097
50	0.010	0.010	0.010	0.009	0.049	0.048	0.050	0.049	0.096	0.096	0.096	0.096
75	0.011	0.010	0.011	0.010	0.050	0.049	0.049	0.049	0.098	0.098	0.098	0.097
100	0.011	0.011	0.012	0.011	0.049	0.047	0.049	0.048	0.100	0.099	0.099	0.099
125	0.011	0.010	0.011	0.010	0.049	0.048	0.050	0.049	0.100	0.098	0.100	0.101
150	0.011	0.010	0.011	0.011	0.052	0.050	0.051	0.050	0.100	0.100	0.101	0.101
200	0.010	0.010	0.010	0.009	0.053	0.051	0.052	0.052	0.102	0.101	0.102	0.101
250	0.009	0.009	0.009	0.010	0.052	0.051	0.053	0.053	0.103	0.102	0.102	0.102
Two-sided tests - $T = 1000$												
-5	0.009	0.010	0.011	0.009	0.048	0.049	0.052	0.049	0.098	0.097	0.099	0.099
-2.5	0.011	0.010	0.011	0.010	0.050	0.050	0.051	0.049	0.098	0.097	0.099	0.096
0	0.010	0.010	0.010	0.010	0.050	0.050	0.050	0.049	0.100	0.100	0.100	0.101
2.5	0.010	0.011	0.010	0.010	0.050	0.051	0.051	0.050	0.102	0.105	0.105	0.103
5	0.010	0.011	0.010	0.010	0.052	0.050	0.050	0.051	0.104	0.104	0.106	0.104
10	0.011	0.010	0.010	0.010	0.051	0.050	0.051	0.050	0.104	0.103	0.104	0.104
25	0.011	0.010	0.010	0.011	0.052	0.053	0.052	0.050	0.103	0.100	0.103	0.103
50	0.012	0.012	0.012	0.011	0.052	0.051	0.051	0.051	0.103	0.100	0.103	0.103
75	0.011	0.011	0.012	0.012	0.052	0.052	0.052	0.052	0.102	0.099	0.101	0.100
100	0.012	0.012	0.012	0.012	0.052	0.051	0.052	0.053	0.098	0.098	0.101	0.099
125	0.011	0.012	0.011	0.011	0.051	0.051	0.051	0.052	0.100	0.100	0.102	0.100
150	0.011	0.012	0.011	0.011	0.052	0.051	0.051	0.052	0.103	0.102	0.104	0.102
200	0.013	0.011	0.012	0.011	0.051	0.050	0.051	0.051	0.105	0.103	0.106	0.105
250	0.011	0.011	0.011	0.011	0.050	0.050	0.051	0.050	0.103	0.102	0.104	0.105

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4.

Table D.5: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP2 (ARCH with Leverage Effects):** $\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 \\ \rho x_{t-1} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \boldsymbol{\psi}_t$ with $\boldsymbol{\psi}_t = (a_t; e_t)' = (\varepsilon_{1t}\sqrt{1 + \frac{1}{2}a_{t-1}^2 \mathbb{I}_{\{a_{t-1}<0\}}}; \varepsilon_{2t})'$ and $(\varepsilon_{1t}, \varepsilon_{2t})' \sim NIID(0, \mathbf{I}_2)$.

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
		1%				5%				10%		
					Left-sided tests - $T = 250$							
5	0.011	0.022	0.024	0.024	0.062	0.099	0.106	0.107	0.127	0.195	0.203	0.205
10	0.013	0.019	0.023	0.024	0.059	0.088	0.096	0.099	0.122	0.174	0.181	0.184
25	0.012	0.015	0.018	0.022	0.059	0.076	0.081	0.092	0.116	0.144	0.152	0.162
50	0.012	0.015	0.016	0.023	0.058	0.067	0.074	0.089	0.117	0.131	0.137	0.158
75	0.012	0.014	0.016	0.025	0.060	0.062	0.069	0.090	0.118	0.124	0.131	0.157
100	0.011	0.014	0.015	0.026	0.060	0.061	0.067	0.089	0.115	0.120	0.125	0.154
125	0.012	0.013	0.016	0.027	0.059	0.060	0.066	0.088	0.117	0.116	0.120	0.151
150	0.013	0.014	0.016	0.026	0.059	0.058	0.063	0.085	0.113	0.111	0.119	0.149
200	0.011	0.012	0.015	0.024	0.057	0.056	0.060	0.082	0.111	0.108	0.113	0.146
250	0.011	0.012	0.013	0.023	0.053	0.053	0.057	0.078	0.109	0.106	0.110	0.138
					Right-sided tests - $T = 250$							
5	0.017	0.002	0.002	0.003	0.058	0.014	0.015	0.015	0.101	0.035	0.035	0.036
10	0.016	0.004	0.004	0.004	0.059	0.021	0.022	0.024	0.110	0.050	0.050	0.053
25	0.013	0.006	0.006	0.010	0.061	0.030	0.030	0.039	0.111	0.065	0.066	0.078
50	0.016	0.007	0.008	0.015	0.062	0.037	0.038	0.053	0.109	0.071	0.073	0.092
75	0.015	0.008	0.009	0.019	0.061	0.039	0.041	0.061	0.111	0.074	0.078	0.104
100	0.016	0.008	0.010	0.022	0.059	0.037	0.040	0.065	0.111	0.078	0.080	0.109
125	0.015	0.009	0.009	0.024	0.057	0.039	0.041	0.067	0.110	0.081	0.083	0.114
150	0.014	0.008	0.010	0.025	0.059	0.040	0.042	0.071	0.109	0.082	0.085	0.117
200	0.013	0.010	0.011	0.025	0.056	0.041	0.045	0.072	0.106	0.087	0.091	0.118
250	0.012	0.011	0.011	0.025	0.054	0.043	0.047	0.075	0.102	0.088	0.093	0.119
					Two-sided tests - $T = 250$							
5	0.010	0.012	0.015	0.014	0.053	0.055	0.065	0.063	0.106	0.112	0.122	0.122
10	0.011	0.012	0.014	0.014	0.051	0.053	0.060	0.062	0.108	0.109	0.117	0.124
25	0.011	0.011	0.013	0.017	0.056	0.052	0.057	0.069	0.112	0.105	0.111	0.131
50	0.014	0.012	0.014	0.022	0.059	0.050	0.056	0.081	0.117	0.103	0.112	0.142
75	0.015	0.011	0.013	0.027	0.061	0.051	0.057	0.090	0.118	0.101	0.110	0.150
100	0.015	0.011	0.014	0.030	0.063	0.053	0.058	0.095	0.118	0.098	0.107	0.154
125	0.014	0.012	0.014	0.032	0.062	0.051	0.059	0.098	0.116	0.098	0.107	0.155
150	0.014	0.011	0.014	0.033	0.060	0.051	0.057	0.097	0.115	0.096	0.105	0.156
200	0.012	0.011	0.014	0.033	0.058	0.050	0.056	0.096	0.110	0.096	0.104	0.154
250	0.013	0.011	0.014	0.029	0.053	0.051	0.054	0.092	0.106	0.096	0.104	0.153

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
		1%				5%				10%		
					Left-sided tests - $T = 1000$							
5	0.012	0.021	0.021	0.021	0.060	0.097	0.101	0.102	0.124	0.191	0.193	0.194
10	0.012	0.019	0.020	0.020	0.059	0.090	0.092	0.093	0.118	0.170	0.173	0.177
25	0.012	0.017	0.016	0.019	0.055	0.076	0.075	0.080	0.109	0.146	0.146	0.152
50	0.011	0.014	0.015	0.018	0.053	0.066	0.068	0.074	0.107	0.128	0.130	0.140
75	0.011	0.013	0.014	0.017	0.055	0.064	0.066	0.075	0.105	0.120	0.121	0.136
100	0.011	0.013	0.013	0.018	0.057	0.061	0.063	0.076	0.107	0.118	0.118	0.137
125	0.012	0.012	0.013	0.018	0.058	0.061	0.062	0.078	0.109	0.116	0.118	0.139
150	0.012	0.012	0.012	0.019	0.059	0.061	0.063	0.080	0.114	0.119	0.119	0.142
200	0.012	0.012	0.012	0.022	0.061	0.060	0.063	0.085	0.116	0.115	0.118	0.150
250	0.012	0.013	0.013	0.023	0.062	0.062	0.063	0.088	0.120	0.116	0.118	0.151
					Right-sided tests - $T = 1000$							
5	0.015	0.002	0.002	0.002	0.057	0.013	0.013	0.013	0.107	0.034	0.034	0.035
10	0.014	0.003	0.003	0.004	0.057	0.021	0.020	0.021	0.109	0.047	0.047	0.049
25	0.012	0.006	0.006	0.007	0.057	0.030	0.030	0.034	0.110	0.069	0.068	0.073
50	0.014	0.008	0.007	0.011	0.058	0.037	0.037	0.045	0.110	0.075	0.075	0.084
75	0.015	0.008	0.008	0.013	0.061	0.041	0.040	0.052	0.108	0.079	0.079	0.094
100	0.016	0.009	0.009	0.015	0.060	0.041	0.041	0.055	0.109	0.082	0.081	0.099
125	0.016	0.010	0.009	0.018	0.061	0.042	0.042	0.058	0.111	0.081	0.082	0.105
150	0.017	0.009	0.009	0.021	0.060	0.042	0.042	0.062	0.111	0.083	0.084	0.109
200	0.017	0.009	0.010	0.023	0.062	0.042	0.042	0.067	0.112	0.083	0.084	0.112
250	0.018	0.010	0.010	0.026	0.062	0.042	0.044	0.071	0.111	0.085	0.086	0.119
					Two-sided tests - $T = 1000$							
5	0.010	0.011	0.011	0.011	0.051	0.056	0.058	0.058	0.103	0.109	0.114	0.114
10	0.010	0.012	0.011	0.012	0.053	0.056	0.058	0.059	0.103	0.108	0.112	0.115
25	0.012	0.011	0.012	0.013	0.053	0.053	0.055	0.059	0.105	0.104	0.106	0.113
50	0.012	0.012	0.012	0.014	0.054	0.051	0.052	0.062	0.106	0.102	0.105	0.119
75	0.014	0.011	0.011	0.017	0.059	0.052	0.054	0.069	0.115	0.103	0.106	0.127
100	0.014	0.012	0.011	0.019	0.061	0.055	0.054	0.074	0.115	0.102	0.104	0.130
125	0.015	0.011	0.011	0.021	0.064	0.054	0.055	0.079	0.115	0.102	0.104	0.136
150	0.016	0.011	0.012	0.024	0.064	0.054	0.056	0.084	0.119	0.101	0.105	0.142
200	0.016	0.010	0.011	0.027	0.068	0.053	0.056	0.089	0.122	0.104	0.105	0.152
250	0.016	0.011	0.011	0.030	0.068	0.052	0.055	0.097	0.124	0.102	0.106	0.160

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (12) and (14) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4.

Table D.6: Empirical rejection frequencies of Wald-type IVX-based tests for predictability in a multiple predictive regression context with $K \in \{1, 3, 5, 10\}$ predictors, for sample sizes $T = 250$ and $T = 1000$.

		$T = 250$											
		$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}	$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}	$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}
K	c	1%				5%				10%			
1	-5	0.009	0.008	0.011	0.009	0.048	0.037	0.045	0.040	0.098	0.080	0.088	0.080
	-2.5	0.009	0.010	0.012	0.010	0.042	0.045	0.051	0.048	0.098	0.106	0.113	0.108
	0	0.011	0.011	0.013	0.012	0.050	0.053	0.059	0.056	0.102	0.108	0.115	0.114
	2.5	0.012	0.012	0.015	0.013	0.054	0.058	0.062	0.059	0.106	0.112	0.119	0.115
	5	0.013	0.012	0.015	0.013	0.056	0.058	0.064	0.060	0.108	0.113	0.119	0.115
	10	0.013	0.013	0.015	0.013	0.054	0.056	0.061	0.056	0.103	0.110	0.116	0.113
	25	0.011	0.012	0.014	0.012	0.055	0.057	0.060	0.057	0.105	0.104	0.112	0.108
	50	0.012	0.013	0.015	0.014	0.054	0.055	0.059	0.056	0.107	0.104	0.111	0.106
	75	0.013	0.012	0.015	0.014	0.055	0.055	0.060	0.056	0.105	0.103	0.108	0.103
	100	0.012	0.012	0.015	0.012	0.055	0.054	0.059	0.055	0.106	0.102	0.109	0.104
	125	0.012	0.011	0.014	0.013	0.055	0.054	0.058	0.054	0.104	0.103	0.109	0.104
	150	0.011	0.011	0.014	0.012	0.053	0.051	0.056	0.052	0.106	0.103	0.111	0.105
	200	0.010	0.010	0.013	0.012	0.052	0.051	0.056	0.052	0.105	0.102	0.108	0.103
	250	0.009	0.011	0.012	0.011	0.051	0.049	0.055	0.051	0.106	0.103	0.109	0.103
3	-5	0.020	0.135	0.171	0.148	0.085	0.352	0.385	0.366	0.158	0.494	0.521	0.507
	-2.5	0.023	0.052	0.067	0.054	0.097	0.176	0.193	0.177	0.177	0.284	0.301	0.283
	0	0.016	0.027	0.035	0.027	0.075	0.105	0.117	0.104	0.134	0.184	0.196	0.183
	2.5	0.014	0.020	0.028	0.022	0.067	0.086	0.103	0.090	0.122	0.157	0.174	0.161
	5	0.014	0.018	0.025	0.021	0.059	0.077	0.095	0.083	0.118	0.145	0.166	0.151
	10	0.013	0.016	0.024	0.018	0.054	0.066	0.083	0.071	0.109	0.129	0.152	0.137
	25	0.011	0.012	0.019	0.014	0.052	0.061	0.075	0.066	0.104	0.112	0.131	0.120
	50	0.011	0.012	0.018	0.014	0.053	0.057	0.070	0.061	0.104	0.110	0.131	0.115
	75	0.011	0.012	0.018	0.014	0.053	0.053	0.069	0.058	0.105	0.107	0.130	0.114
	100	0.010	0.012	0.018	0.014	0.051	0.053	0.069	0.057	0.107	0.106	0.129	0.113
	125	0.011	0.012	0.018	0.014	0.052	0.054	0.070	0.058	0.107	0.105	0.128	0.113
	150	0.011	0.013	0.018	0.013	0.052	0.054	0.069	0.058	0.107	0.107	0.128	0.114
	200	0.010	0.011	0.017	0.014	0.052	0.055	0.071	0.059	0.109	0.107	0.130	0.114
	250	0.009	0.011	0.018	0.013	0.053	0.055	0.071	0.060	0.107	0.108	0.127	0.114
5	-5	0.018	0.167	0.225	0.184	0.074	0.402	0.466	0.421	0.138	0.558	0.606	0.573
	-2.5	0.022	0.087	0.117	0.089	0.091	0.239	0.281	0.241	0.160	0.372	0.405	0.374
	0	0.020	0.050	0.067	0.050	0.082	0.157	0.186	0.156	0.152	0.258	0.289	0.254
	2.5	0.017	0.036	0.053	0.039	0.069	0.120	0.156	0.129	0.132	0.208	0.246	0.215
	5	0.014	0.028	0.046	0.033	0.063	0.105	0.138	0.116	0.124	0.183	0.223	0.195
	10	0.013	0.022	0.040	0.028	0.062	0.086	0.120	0.098	0.114	0.157	0.197	0.171
	25	0.012	0.017	0.029	0.021	0.053	0.067	0.100	0.080	0.110	0.129	0.167	0.141
	50	0.011	0.014	0.025	0.017	0.052	0.059	0.089	0.069	0.107	0.118	0.155	0.130
	75	0.011	0.013	0.024	0.017	0.051	0.055	0.085	0.063	0.104	0.110	0.149	0.122
	100	0.010	0.013	0.022	0.015	0.049	0.053	0.082	0.062	0.103	0.106	0.145	0.119
	125	0.009	0.011	0.022	0.013	0.049	0.053	0.080	0.062	0.102	0.105	0.142	0.118
	150	0.008	0.011	0.020	0.013	0.046	0.052	0.078	0.061	0.101	0.106	0.142	0.117
	200	0.006	0.009	0.020	0.012	0.047	0.051	0.079	0.060	0.098	0.103	0.139	0.114
	250	0.007	0.010	0.019	0.012	0.044	0.049	0.077	0.058	0.100	0.104	0.142	0.116
10	-5	0.011	0.243	0.396	0.298	0.058	0.513	0.635	0.559	0.114	0.658	0.754	0.691
	-2.5	0.016	0.169	0.280	0.195	0.072	0.398	0.505	0.425	0.144	0.542	0.631	0.558
	0	0.020	0.114	0.208	0.132	0.087	0.306	0.406	0.324	0.168	0.443	0.531	0.456
	2.5	0.016	0.078	0.162	0.098	0.075	0.238	0.342	0.262	0.144	0.360	0.459	0.384
	5	0.013	0.061	0.133	0.078	0.067	0.191	0.301	0.225	0.130	0.303	0.415	0.337
	10	0.012	0.041	0.100	0.058	0.060	0.141	0.244	0.175	0.115	0.238	0.356	0.278
	25	0.011	0.022	0.064	0.034	0.050	0.089	0.174	0.118	0.104	0.162	0.263	0.201
	50	0.010	0.016	0.048	0.025	0.048	0.067	0.142	0.091	0.099	0.129	0.223	0.164
	75	0.009	0.014	0.044	0.022	0.046	0.060	0.129	0.081	0.100	0.118	0.204	0.146
	100	0.009	0.013	0.043	0.020	0.046	0.056	0.120	0.077	0.097	0.110	0.197	0.139
	125	0.008	0.011	0.041	0.020	0.043	0.053	0.117	0.074	0.093	0.104	0.191	0.134
	150	0.007	0.012	0.039	0.019	0.042	0.052	0.116	0.071	0.092	0.101	0.191	0.131
	200	0.006	0.010	0.036	0.017	0.039	0.049	0.116	0.070	0.091	0.101	0.190	0.132
	250	0.005	0.010	0.035	0.016	0.036	0.050	0.116	0.072	0.094	0.104	0.193	0.133

		$T = 1000$											
		$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}	$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}	$W_{zx}^{*,\text{RWB}}$	$W_{zx}^{*,\text{FRWB}}$	W_{zx}^{EW}	W_{zx}
K	c	1%				5%				10%			
1	-5	0.009	0.008	0.008	0.008	0.043	0.034	0.036	0.035	0.094	0.075	0.077	0.074
	-2.5	0.008	0.008	0.009	0.009	0.038	0.042	0.043	0.043	0.088	0.098	0.098	0.097
	0	0.008	0.010	0.010	0.009	0.045	0.051	0.051	0.050	0.093	0.105	0.106	0.105
	2.5	0.009	0.010	0.011	0.011	0.051	0.054	0.056	0.056	0.103	0.114	0.114	0.113
	5	0.010	0.011	0.010	0.010	0.054	0.059	0.060	0.058	0.107	0.117	0.117	0.117
	10	0.012	0.012	0.014	0.013	0.055	0.060	0.060	0.059	0.108	0.118	0.119	0.117
	25	0.012	0.013	0.013	0.013	0.056	0.059	0.059	0.059	0.105	0.113	0.113	0.112
	50	0.011	0.011	0.011	0.011	0.055	0.059	0.059	0.058	0.105	0.108	0.109	0.107
	75	0.012	0.011	0.012	0.012	0.056	0.057	0.058	0.058	0.104	0.106	0.107	0.106
	100	0.011	0.012	0.012	0.012	0.056	0.056	0.057	0.058	0.103	0.104	0.105	0.105
	125	0.012	0.012	0.012	0.012	0.056	0.056	0.057	0.056	0.104	0.105	0.106	0.105
	150	0.011	0.012	0.012	0.011	0.055	0.053	0.055	0.055	0.104	0.105	0.106	0.104
	200	0.011	0.012	0.012	0.011	0.054	0.054	0.053	0.052	0.104	0.106	0.106	0.104
	250	0.011	0.011	0.011	0.011	0.053	0.052	0.053	0.053	0.104	0.105	0.106	0.104
3	-5	0.020	0.126	0.131	0.126	0.083	0.346	0.354	0.346	0.151	0.493	0.498	0.492
	-2.5	0.024	0.048	0.049	0.047	0.092	0.162	0.159	0.155	0.167	0.262	0.264	0.260
	0	0.016	0.026	0.027	0.025	0.071	0.096	0.096	0.095	0.130	0.168	0.168	0.166
	2.5	0.013	0.021	0.022	0.020	0.063	0.081	0.084	0.083	0.117	0.149	0.153	0.149
	5	0.013	0.019	0.020	0.018	0.060	0.076	0.079	0.077	0.114	0.144	0.149	0.145
	10	0.013	0.016	0.018	0.017	0.057	0.072	0.078	0.075	0.111	0.135	0.140	0.136
	25	0.011	0.014	0.015	0.014	0.056	0.064	0.067	0.065	0.107	0.120	0.126	0.123
	50	0.011	0.012	0.013	0.012	0.053	0.058	0.061	0.059	0.103	0.112	0.115	0.113
	75	0.010	0.011	0.012	0.010	0.050	0.055	0.058	0.055	0.103	0.107	0.112	0.109
	100	0.010	0.011	0.011	0.010	0.048	0.052	0.054	0.051	0.100	0.104	0.107	0.105
	125	0.010	0.010	0.011	0.010	0.048	0.049	0.054	0.050	0.098	0.100	0.106	0.103
	150	0.009	0.010	0.010	0.010	0.047	0.049	0.052	0.049	0.098	0.098	0.104	0.102
	200	0.009	0.009	0.009	0.010	0.046	0.048	0.051	0.048	0.097	0.099	0.103	0.098
	250	0.009	0.010	0.010	0.010	0.046	0.048	0.050	0.048	0.097	0.096	0.101	0.099
5	-5	0.015	0.162	0.176	0.164	0.074	0.398	0.408	0.403	0.143	0.549	0.558	0.548
	-2.5	0.021	0.080	0.082	0.075	0.091	0.237	0.238	0.230	0.170	0.360	0.360	0.352
	0	0.019	0.044	0.046	0.043	0.085	0.152	0.154	0.148	0.155	0.253	0.252	0.246
	2.5	0.017	0.032	0.034	0.031	0.069	0.118	0.126	0.117	0.131	0.206	0.209	0.203
	5	0.014	0.027	0.029	0.027	0.063	0.104	0.110	0.105	0.122	0.184	0.192	0.185
	10	0.012	0.021	0.024	0.021	0.058	0.089	0.096	0.092	0.115	0.161	0.171	0.164
	25	0.012	0.016	0.019	0.017	0.052	0.069	0.077	0.071	0.108	0.134	0.143	0.138
	50	0.011	0.014	0.016	0.015	0.049	0.057	0.064	0.059	0.103	0.118	0.126	0.120
	75	0.011	0.013	0.014	0.013	0.050	0.055	0.062	0.057	0.102	0.112	0.120	0.114
	100	0.009	0.010	0.013	0.011	0.051	0.056	0.060	0.057	0.101	0.107	0.115	0.109
	125	0.009	0.011	0.012	0.011	0.052	0.055	0.060	0.057	0.100	0.104	0.112	0.107
	150	0.009	0.010	0.012	0.010	0.051	0.055	0.059	0.056	0.100	0.104	0.111	0.107
	200	0.009	0.011	0.012	0.011	0.051	0.052	0.057	0.054	0.098	0.102	0.110	0.104
	250	0.010	0.010	0.012	0.010	0.051	0.052	0.058	0.054	0.097	0.100	0.108	0.103
10	-5	0.014	0.244	0.265	0.245	0.060	0.502	0.526	0.501	0.118	0.650	0.666	0.647
	-2.5	0.019	0.166	0.177	0.154	0.076	0.384	0.394	0.371	0.149	0.523	0.531	0.509
	0	0.023	0.112	0.120	0.103	0.091	0.295	0.300	0.280	0.169	0.426	0.431	0.407
	2.5	0.019	0.079	0.090	0.075	0.078	0.229	0.244	0.224	0.146	0.349	0.363	0.346
	5	0.016	0.059	0.071	0.061	0.068	0.188	0.211	0.191	0.133	0.301	0.318	0.304
	10	0.014	0.044	0.053	0.044	0.061	0.147	0.166	0.151	0.119	0.242	0.268	0.249
	25	0.013	0.027	0.033	0.027	0.057	0.101	0.119	0.108	0.111	0.175	0.201	0.184
	50	0.012	0.020	0.025	0.020	0.057	0.081	0.096	0.085	0.108	0.144	0.167	0.152
	75	0.012	0.016	0.022	0.017	0.055	0.071	0.085	0.077	0.108	0.129	0.152	0.136
	100	0.011	0.015	0.021	0.016	0.055	0.066	0.080	0.071	0.106	0.122	0.143	0.129
	125	0.011	0.014	0.018	0.016	0.055	0.061	0.077	0.067	0.105	0.120	0.140	0.123
	150	0.011	0.013	0.017	0.014	0.052	0.058	0.075	0.064	0.103	0.115	0.134	0.120
	200	0.010	0.012	0.016	0.013	0.053	0.057	0.070	0.063	0.104	0.111	0.130	0.119
	250	0.009	0.011	0.015	0.012	0.051	0.055	0.070	0.061	0.103	0.109	0.130	0.116

Note: W_{zx} and W_{zx}^{EW} are the Wald-type IVX-based statistics discussed in Remark 9 of the main text, and $W_{zx}^{*,RWB}$ and $W_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) versions of W_{zx} computed as described in Algorithms 1 and 2 of Section 4 of the main text.

Table D.7: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP3 (Positive Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.95; \quad -0.95 \quad 1]$.

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.009	0.001	0.001	0.000	0.045	0.003	0.005	0.004	0.097	0.013	0.014	0.013
-2.5	0.006	0.000	0.000	0.000	0.042	0.000	0.001	0.001	0.105	0.002	0.002	0.002
0	0.013	0.000	0.000	0.000	0.040	0.001	0.001	0.001	0.064	0.002	0.002	0.002
2.5	0.021	0.000	0.001	0.001	0.059	0.005	0.005	0.004	0.094	0.012	0.012	0.011
5	0.023	0.001	0.001	0.001	0.068	0.010	0.011	0.010	0.110	0.025	0.026	0.024
10	0.019	0.003	0.003	0.002	0.065	0.019	0.019	0.018	0.113	0.041	0.042	0.041
25	0.016	0.006	0.006	0.005	0.056	0.027	0.029	0.027	0.107	0.056	0.056	0.057
50	0.015	0.007	0.007	0.007	0.055	0.032	0.033	0.032	0.105	0.068	0.070	0.068
75	0.012	0.007	0.007	0.007	0.055	0.036	0.038	0.036	0.106	0.073	0.076	0.073
100	0.011	0.006	0.007	0.007	0.056	0.038	0.040	0.038	0.105	0.077	0.079	0.076
125	0.011	0.007	0.007	0.007	0.055	0.039	0.040	0.038	0.103	0.079	0.080	0.079
150	0.011	0.008	0.008	0.007	0.055	0.038	0.040	0.038	0.104	0.080	0.083	0.082
200	0.011	0.007	0.009	0.008	0.054	0.038	0.040	0.040	0.108	0.085	0.087	0.085
250	0.011	0.007	0.008	0.007	0.053	0.040	0.042	0.041	0.107	0.087	0.091	0.087
Right-sided tests - $T = 250$												
-5	0.011	0.018	0.022	0.019	0.045	0.075	0.082	0.077	0.090	0.155	0.161	0.155
-2.5	0.009	0.017	0.019	0.018	0.044	0.102	0.109	0.104	0.093	0.256	0.257	0.258
0	0.012	0.021	0.027	0.024	0.056	0.113	0.122	0.117	0.115	0.242	0.250	0.244
2.5	0.013	0.023	0.030	0.026	0.064	0.118	0.124	0.120	0.128	0.234	0.244	0.239
5	0.013	0.024	0.029	0.026	0.065	0.114	0.121	0.115	0.128	0.217	0.225	0.219
10	0.013	0.024	0.027	0.024	0.063	0.101	0.110	0.105	0.123	0.189	0.194	0.192
25	0.012	0.020	0.022	0.021	0.058	0.083	0.089	0.086	0.109	0.153	0.160	0.156
50	0.010	0.016	0.018	0.016	0.055	0.072	0.076	0.074	0.106	0.136	0.140	0.138
75	0.010	0.015	0.018	0.015	0.053	0.068	0.071	0.070	0.104	0.132	0.137	0.135
100	0.010	0.015	0.016	0.015	0.053	0.066	0.069	0.067	0.105	0.129	0.131	0.129
125	0.012	0.014	0.016	0.014	0.052	0.064	0.069	0.066	0.103	0.126	0.129	0.127
150	0.011	0.014	0.015	0.014	0.053	0.064	0.068	0.065	0.103	0.122	0.127	0.123
200	0.011	0.014	0.016	0.014	0.051	0.060	0.066	0.062	0.102	0.120	0.123	0.120
250	0.011	0.012	0.014	0.013	0.054	0.060	0.063	0.060	0.103	0.115	0.120	0.116
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.013	0.012	0.046	0.040	0.046	0.041	0.092	0.078	0.087	0.081
-2.5	0.009	0.009	0.011	0.010	0.040	0.047	0.054	0.049	0.087	0.102	0.110	0.104
0	0.011	0.012	0.014	0.013	0.048	0.054	0.060	0.057	0.098	0.112	0.123	0.117
2.5	0.011	0.011	0.015	0.014	0.054	0.059	0.066	0.062	0.109	0.123	0.129	0.124
5	0.012	0.013	0.015	0.013	0.056	0.062	0.070	0.063	0.110	0.123	0.132	0.125
10	0.010	0.012	0.016	0.013	0.054	0.062	0.069	0.062	0.110	0.120	0.129	0.123
25	0.010	0.011	0.013	0.012	0.055	0.059	0.065	0.060	0.107	0.110	0.117	0.113
50	0.010	0.010	0.012	0.012	0.054	0.054	0.060	0.056	0.102	0.103	0.109	0.106
75	0.009	0.009	0.012	0.011	0.052	0.052	0.058	0.054	0.103	0.102	0.109	0.106
100	0.010	0.011	0.012	0.011	0.051	0.050	0.057	0.052	0.103	0.102	0.109	0.105
125	0.012	0.011	0.012	0.012	0.051	0.051	0.056	0.051	0.102	0.104	0.109	0.104
150	0.012	0.010	0.012	0.012	0.050	0.049	0.054	0.050	0.105	0.101	0.107	0.103
200	0.010	0.010	0.012	0.011	0.049	0.049	0.054	0.049	0.102	0.099	0.106	0.102
250	0.009	0.010	0.011	0.011	0.051	0.050	0.053	0.051	0.104	0.099	0.105	0.101

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.045	0.003	0.003	0.003	0.093	0.011	0.011	0.011
-2.5	0.005	0.000	0.000	0.000	0.044	0.000	0.000	0.000	0.108	0.001	0.001	0.001
0	0.015	0.000	0.000	0.000	0.042	0.001	0.001	0.001	0.068	0.003	0.003	0.004
2.5	0.023	0.001	0.001	0.001	0.059	0.006	0.006	0.006	0.096	0.015	0.016	0.015
5	0.024	0.002	0.002	0.002	0.068	0.013	0.014	0.013	0.108	0.027	0.027	0.028
10	0.021	0.003	0.004	0.004	0.064	0.021	0.022	0.022	0.114	0.044	0.043	0.043
25	0.016	0.007	0.007	0.008	0.061	0.031	0.030	0.029	0.108	0.063	0.064	0.063
50	0.014	0.007	0.008	0.007	0.058	0.035	0.035	0.034	0.107	0.073	0.074	0.072
75	0.013	0.008	0.008	0.008	0.057	0.039	0.038	0.038	0.105	0.078	0.078	0.078
100	0.013	0.008	0.008	0.008	0.054	0.040	0.040	0.040	0.106	0.079	0.080	0.079
125	0.012	0.008	0.008	0.008	0.052	0.040	0.040	0.038	0.106	0.082	0.082	0.081
150	0.012	0.009	0.009	0.008	0.052	0.040	0.042	0.039	0.106	0.083	0.083	0.081
200	0.012	0.008	0.008	0.008	0.052	0.041	0.041	0.041	0.104	0.084	0.085	0.084
250	0.012	0.009	0.008	0.008	0.052	0.043	0.042	0.041	0.104	0.085	0.085	0.084
Right-sided tests - $T = 1000$												
-5	0.007	0.013	0.015	0.014	0.040	0.066	0.068	0.066	0.084	0.143	0.144	0.140
-2.5	0.008	0.018	0.017	0.016	0.041	0.099	0.099	0.098	0.087	0.241	0.238	0.238
0	0.009	0.020	0.021	0.019	0.052	0.108	0.109	0.107	0.109	0.230	0.232	0.231
2.5	0.011	0.022	0.022	0.021	0.060	0.111	0.113	0.112	0.119	0.222	0.225	0.225
5	0.011	0.020	0.022	0.021	0.062	0.112	0.111	0.111	0.123	0.209	0.209	0.209
10	0.011	0.020	0.019	0.018	0.062	0.099	0.101	0.100	0.118	0.184	0.185	0.184
25	0.011	0.017	0.017	0.017	0.055	0.083	0.083	0.083	0.110	0.152	0.155	0.154
50	0.010	0.015	0.014	0.014	0.052	0.071	0.072	0.070	0.105	0.139	0.139	0.140
75	0.010	0.014	0.014	0.014	0.050	0.066	0.066	0.065	0.103	0.129	0.130	0.129
100	0.010	0.014	0.014	0.013	0.051	0.063	0.062	0.062	0.102	0.126	0.125	0.125
125	0.009	0.013	0.013	0.013	0.048	0.061	0.062	0.061	0.098	0.121	0.121	0.120
150	0.009	0.013	0.013	0.012	0.050	0.063	0.063	0.063	0.099	0.118	0.116	0.117
200	0.010	0.011	0.012	0.011	0.053	0.062	0.062	0.062	0.099	0.116	0.117	0.116
250	0.009	0.012	0.011	0.010	0.053	0.061	0.061	0.061	0.101	0.115	0.116	0.116
Two-sided tests - $T = 1000$												
-5	0.006	0.007	0.007	0.007	0.040	0.033	0.033	0.033	0.086	0.069	0.071	0.069
-2.5	0.007	0.008	0.008	0.008	0.037	0.042	0.045	0.044	0.082	0.096	0.099	0.098
0	0.008	0.010	0.010	0.009	0.044	0.052	0.053	0.051	0.091	0.110	0.110	0.108
2.5	0.009	0.010	0.011	0.010	0.052	0.061	0.060	0.059	0.104	0.117	0.120	0.117
5	0.009	0.010	0.011	0.010	0.054	0.061	0.062	0.061	0.109	0.124	0.125	0.124
10	0.010	0.011	0.011	0.011	0.058	0.062	0.065	0.064	0.110	0.120	0.123	0.122
25	0.011	0.011	0.011	0.011	0.053	0.059	0.059	0.058	0.107	0.112	0.113	0.112
50	0.011	0.012	0.011	0.011	0.052	0.055	0.055	0.054	0.100	0.108	0.107	0.104
75	0.010	0.011	0.010	0.010	0.051	0.052	0.053	0.053	0.102	0.102	0.104	0.103
100	0.010	0.010	0.010	0.010	0.050	0.051	0.052	0.051	0.100	0.101	0.102	0.102
125	0.009	0.010	0.010	0.010	0.051	0.051	0.053	0.051	0.098	0.101	0.102	0.100
150	0.010	0.011	0.011	0.010	0.051	0.050	0.051	0.050	0.101	0.103	0.105	0.102
200	0.011	0.011	0.010	0.010	0.051	0.051	0.051	0.051	0.101	0.101	0.103	0.102
250	0.010	0.011	0.011	0.010	0.051	0.052	0.053	0.052	0.103	0.103	0.103	0.102

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.8: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP3 (Positive Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0,\ \rho = 1 - c/T, \psi = 0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.90; \quad -0.90 \quad 1]$.

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.009	0.000	0.001	0.000	0.046	0.005	0.006	0.005	0.097	0.015	0.017	0.014
-2.5	0.007	0.000	0.000	0.000	0.045	0.001	0.001	0.001	0.106	0.002	0.002	0.002
0	0.011	0.000	0.000	0.000	0.039	0.001	0.001	0.001	0.062	0.003	0.003	0.003
2.5	0.019	0.001	0.001	0.001	0.056	0.005	0.005	0.005	0.093	0.013	0.013	0.013
5	0.022	0.001	0.002	0.001	0.065	0.011	0.011	0.010	0.108	0.027	0.028	0.026
10	0.018	0.003	0.003	0.003	0.064	0.018	0.020	0.019	0.114	0.042	0.043	0.043
25	0.015	0.006	0.006	0.005	0.056	0.028	0.029	0.028	0.105	0.058	0.060	0.058
50	0.014	0.007	0.007	0.006	0.053	0.032	0.035	0.032	0.104	0.067	0.068	0.068
75	0.013	0.007	0.008	0.007	0.054	0.036	0.037	0.035	0.105	0.073	0.075	0.072
100	0.011	0.006	0.007	0.007	0.053	0.037	0.040	0.038	0.104	0.075	0.076	0.075
125	0.010	0.007	0.007	0.006	0.053	0.039	0.040	0.039	0.103	0.079	0.080	0.077
150	0.010	0.007	0.007	0.007	0.052	0.038	0.040	0.038	0.105	0.081	0.082	0.080
200	0.011	0.007	0.009	0.008	0.054	0.039	0.041	0.040	0.104	0.083	0.085	0.084
250	0.011	0.007	0.008	0.007	0.052	0.040	0.043	0.041	0.107	0.086	0.090	0.088
Right-sided tests - $T = 250$												
-5	0.010	0.016	0.021	0.019	0.044	0.076	0.084	0.077	0.092	0.153	0.161	0.154
-2.5	0.010	0.017	0.020	0.018	0.043	0.106	0.112	0.106	0.095	0.249	0.254	0.251
0	0.012	0.022	0.027	0.024	0.055	0.111	0.117	0.114	0.117	0.238	0.246	0.241
2.5	0.014	0.024	0.029	0.024	0.063	0.113	0.120	0.118	0.127	0.232	0.237	0.232
5	0.014	0.025	0.028	0.025	0.063	0.110	0.115	0.112	0.127	0.211	0.219	0.213
10	0.014	0.023	0.026	0.024	0.062	0.099	0.105	0.101	0.121	0.184	0.189	0.187
25	0.011	0.018	0.022	0.020	0.057	0.081	0.086	0.083	0.110	0.154	0.157	0.153
50	0.011	0.015	0.017	0.015	0.054	0.073	0.078	0.076	0.106	0.136	0.141	0.137
75	0.011	0.014	0.017	0.015	0.055	0.067	0.072	0.070	0.106	0.131	0.135	0.133
100	0.010	0.014	0.017	0.015	0.054	0.067	0.070	0.067	0.105	0.125	0.129	0.127
125	0.011	0.014	0.017	0.015	0.054	0.065	0.070	0.065	0.105	0.124	0.127	0.125
150	0.010	0.014	0.017	0.014	0.053	0.063	0.065	0.063	0.103	0.121	0.126	0.123
200	0.011	0.014	0.015	0.013	0.053	0.060	0.063	0.062	0.102	0.119	0.123	0.120
250	0.010	0.012	0.014	0.012	0.054	0.058	0.062	0.059	0.104	0.116	0.118	0.113
Two-sided tests - $T = 250$												
-5	0.010	0.010	0.013	0.011	0.045	0.039	0.046	0.041	0.094	0.080	0.089	0.081
-2.5	0.009	0.010	0.012	0.011	0.040	0.045	0.051	0.048	0.088	0.104	0.112	0.106
0	0.010	0.011	0.014	0.012	0.048	0.053	0.060	0.055	0.097	0.110	0.118	0.115
2.5	0.012	0.013	0.016	0.014	0.054	0.058	0.065	0.061	0.108	0.118	0.125	0.123
5	0.012	0.012	0.016	0.014	0.055	0.060	0.067	0.063	0.108	0.120	0.126	0.122
10	0.012	0.013	0.015	0.014	0.054	0.061	0.066	0.061	0.107	0.117	0.125	0.120
25	0.010	0.011	0.014	0.011	0.056	0.058	0.063	0.061	0.104	0.107	0.115	0.111
50	0.010	0.010	0.012	0.011	0.054	0.054	0.059	0.056	0.104	0.105	0.113	0.108
75	0.010	0.010	0.011	0.010	0.052	0.052	0.057	0.053	0.104	0.102	0.110	0.105
100	0.010	0.009	0.012	0.010	0.048	0.050	0.055	0.051	0.106	0.105	0.110	0.105
125	0.010	0.009	0.012	0.011	0.049	0.049	0.055	0.050	0.105	0.104	0.110	0.104
150	0.011	0.010	0.012	0.012	0.050	0.049	0.054	0.050	0.100	0.100	0.105	0.101
200	0.010	0.009	0.012	0.011	0.050	0.048	0.052	0.050	0.101	0.099	0.104	0.101
250	0.010	0.009	0.011	0.010	0.052	0.048	0.054	0.052	0.102	0.097	0.105	0.101

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.047	0.003	0.003	0.003	0.096	0.013	0.014	0.014
-2.5	0.006	0.000	0.000	0.000	0.047	0.000	0.000	0.000	0.108	0.001	0.001	0.001
0	0.013	0.000	0.000	0.000	0.040	0.002	0.002	0.001	0.066	0.004	0.004	0.004
2.5	0.021	0.001	0.001	0.001	0.058	0.007	0.007	0.007	0.093	0.017	0.017	0.017
5	0.022	0.003	0.002	0.002	0.064	0.014	0.014	0.014	0.105	0.030	0.031	0.030
10	0.019	0.004	0.004	0.004	0.063	0.022	0.022	0.021	0.109	0.046	0.045	0.046
25	0.016	0.007	0.007	0.007	0.060	0.031	0.032	0.031	0.108	0.066	0.064	0.064
50	0.014	0.007	0.008	0.008	0.059	0.037	0.037	0.036	0.108	0.073	0.075	0.073
75	0.013	0.008	0.008	0.007	0.057	0.038	0.039	0.038	0.107	0.080	0.080	0.080
100	0.012	0.008	0.009	0.009	0.056	0.040	0.040	0.039	0.106	0.082	0.082	0.081
125	0.013	0.008	0.008	0.009	0.055	0.041	0.041	0.040	0.106	0.083	0.084	0.083
150	0.012	0.009	0.009	0.009	0.053	0.040	0.041	0.040	0.105	0.085	0.085	0.085
200	0.012	0.009	0.009	0.009	0.053	0.041	0.041	0.041	0.104	0.087	0.086	0.086
250	0.012	0.009	0.009	0.009	0.052	0.043	0.043	0.042	0.105	0.088	0.088	0.087
Right-sided tests - $T = 1000$												
-5	0.007	0.013	0.015	0.014	0.041	0.069	0.069	0.066	0.084	0.145	0.144	0.143
-2.5	0.008	0.016	0.017	0.016	0.041	0.098	0.098	0.097	0.091	0.236	0.235	0.236
0	0.010	0.021	0.021	0.020	0.051	0.104	0.105	0.102	0.107	0.228	0.225	0.225
2.5	0.011	0.022	0.022	0.022	0.059	0.111	0.111	0.109	0.120	0.216	0.218	0.216
5	0.011	0.022	0.021	0.020	0.062	0.105	0.108	0.105	0.119	0.203	0.202	0.201
10	0.010	0.018	0.018	0.017	0.061	0.099	0.099	0.098	0.116	0.179	0.178	0.177
25	0.010	0.015	0.015	0.015	0.055	0.083	0.083	0.082	0.109	0.150	0.151	0.150
50	0.010	0.014	0.014	0.014	0.051	0.069	0.071	0.071	0.104	0.136	0.138	0.136
75	0.010	0.013	0.014	0.014	0.051	0.065	0.066	0.066	0.099	0.128	0.128	0.129
100	0.011	0.013	0.014	0.013	0.051	0.061	0.061	0.061	0.099	0.122	0.125	0.123
125	0.011	0.013	0.013	0.013	0.051	0.060	0.061	0.061	0.099	0.119	0.120	0.120
150	0.010	0.012	0.013	0.012	0.049	0.059	0.061	0.061	0.098	0.116	0.117	0.117
200	0.010	0.012	0.012	0.012	0.050	0.060	0.060	0.060	0.100	0.116	0.116	0.117
250	0.009	0.012	0.012	0.011	0.050	0.060	0.059	0.059	0.100	0.114	0.116	0.117
Two-sided tests- $T = 1000$												
-5	0.007	0.006	0.007	0.006	0.041	0.034	0.035	0.034	0.087	0.071	0.072	0.070
-2.5	0.007	0.009	0.009	0.008	0.038	0.044	0.045	0.043	0.084	0.096	0.099	0.098
0	0.009	0.010	0.011	0.011	0.044	0.051	0.052	0.051	0.091	0.105	0.106	0.104
2.5	0.009	0.011	0.010	0.010	0.051	0.059	0.060	0.059	0.102	0.117	0.117	0.116
5	0.010	0.010	0.011	0.010	0.052	0.058	0.061	0.061	0.106	0.119	0.122	0.119
10	0.010	0.011	0.011	0.010	0.056	0.060	0.062	0.063	0.109	0.118	0.121	0.120
25	0.010	0.011	0.011	0.010	0.053	0.057	0.057	0.057	0.105	0.113	0.115	0.113
50	0.010	0.011	0.011	0.011	0.051	0.053	0.055	0.054	0.103	0.105	0.107	0.106
75	0.010	0.011	0.010	0.010	0.051	0.053	0.054	0.053	0.101	0.104	0.105	0.104
100	0.010	0.011	0.010	0.010	0.052	0.051	0.054	0.053	0.099	0.101	0.102	0.100
125	0.010	0.010	0.010	0.010	0.052	0.052	0.054	0.053	0.100	0.101	0.102	0.102
150	0.009	0.010	0.011	0.010	0.052	0.052	0.052	0.051	0.100	0.100	0.102	0.101
200	0.010	0.010	0.010	0.010	0.051	0.051	0.052	0.051	0.100	0.100	0.101	0.101
250	0.010	0.010	0.010	0.009	0.050	0.052	0.052	0.051	0.101	0.102	0.102	0.101

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.9: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP3 (Positive Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.50; \quad -0.50 \quad 1]$.

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.010	0.003	0.005	0.002	0.053	0.020	0.024	0.020	0.104	0.047	0.053	0.048
-2.5	0.010	0.000	0.001	0.000	0.050	0.006	0.007	0.005	0.101	0.017	0.017	0.017
0	0.005	0.001	0.001	0.001	0.030	0.006	0.006	0.005	0.061	0.017	0.017	0.017
2.5	0.010	0.002	0.002	0.002	0.043	0.015	0.016	0.015	0.086	0.035	0.036	0.036
5	0.012	0.004	0.004	0.004	0.048	0.022	0.023	0.023	0.097	0.049	0.050	0.049
10	0.012	0.006	0.006	0.005	0.052	0.029	0.030	0.030	0.098	0.062	0.063	0.061
25	0.011	0.007	0.008	0.007	0.051	0.037	0.038	0.037	0.100	0.074	0.076	0.074
50	0.011	0.007	0.008	0.007	0.050	0.038	0.039	0.037	0.099	0.081	0.082	0.082
75	0.009	0.008	0.008	0.007	0.048	0.036	0.038	0.038	0.097	0.082	0.085	0.082
100	0.009	0.007	0.008	0.007	0.050	0.039	0.041	0.040	0.099	0.083	0.086	0.085
125	0.008	0.006	0.007	0.006	0.049	0.040	0.042	0.040	0.097	0.084	0.086	0.085
150	0.008	0.006	0.007	0.006	0.048	0.040	0.043	0.042	0.096	0.084	0.087	0.085
200	0.009	0.008	0.008	0.007	0.048	0.043	0.044	0.043	0.098	0.088	0.089	0.087
250	0.011	0.009	0.009	0.008	0.051	0.045	0.046	0.045	0.097	0.085	0.088	0.089
Right-sided tests - $T = 250$												
-5	0.010	0.016	0.021	0.016	0.046	0.072	0.081	0.072	0.097	0.144	0.154	0.146
-2.5	0.012	0.022	0.028	0.022	0.053	0.104	0.110	0.106	0.107	0.201	0.208	0.201
0	0.014	0.023	0.025	0.022	0.061	0.100	0.104	0.100	0.123	0.195	0.203	0.195
2.5	0.013	0.021	0.024	0.022	0.060	0.094	0.100	0.094	0.118	0.174	0.178	0.175
5	0.014	0.019	0.021	0.019	0.061	0.084	0.088	0.086	0.115	0.159	0.163	0.158
10	0.014	0.018	0.019	0.018	0.056	0.077	0.079	0.077	0.109	0.147	0.149	0.146
25	0.011	0.015	0.016	0.015	0.054	0.069	0.071	0.068	0.102	0.129	0.134	0.130
50	0.011	0.013	0.015	0.014	0.052	0.061	0.066	0.064	0.106	0.123	0.127	0.124
75	0.010	0.013	0.014	0.014	0.052	0.061	0.064	0.062	0.107	0.120	0.124	0.123
100	0.011	0.013	0.014	0.013	0.054	0.063	0.066	0.064	0.106	0.119	0.123	0.119
125	0.010	0.013	0.014	0.012	0.055	0.061	0.067	0.064	0.107	0.118	0.122	0.120
150	0.010	0.012	0.013	0.012	0.055	0.061	0.065	0.061	0.108	0.118	0.121	0.118
200	0.010	0.011	0.012	0.011	0.055	0.059	0.062	0.060	0.110	0.117	0.120	0.117
250	0.009	0.011	0.012	0.011	0.056	0.058	0.061	0.059	0.107	0.114	0.118	0.114
Two-sided tests - $T = 250$												
-5	0.010	0.008	0.016	0.010	0.048	0.044	0.056	0.045	0.098	0.093	0.105	0.092
-2.5	0.011	0.012	0.016	0.012	0.048	0.054	0.062	0.053	0.100	0.109	0.118	0.111
0	0.012	0.012	0.015	0.013	0.049	0.054	0.058	0.053	0.097	0.107	0.110	0.106
2.5	0.012	0.012	0.014	0.013	0.051	0.053	0.058	0.056	0.104	0.108	0.115	0.109
5	0.012	0.013	0.014	0.014	0.053	0.056	0.061	0.059	0.103	0.107	0.112	0.108
10	0.013	0.012	0.014	0.013	0.051	0.053	0.058	0.055	0.102	0.105	0.110	0.106
25	0.011	0.011	0.013	0.012	0.052	0.052	0.056	0.055	0.102	0.104	0.109	0.105
50	0.010	0.009	0.011	0.010	0.050	0.050	0.055	0.052	0.097	0.099	0.105	0.100
75	0.009	0.010	0.010	0.010	0.049	0.051	0.056	0.051	0.097	0.096	0.102	0.100
100	0.008	0.009	0.011	0.009	0.048	0.050	0.054	0.050	0.103	0.101	0.107	0.104
125	0.009	0.008	0.011	0.010	0.049	0.047	0.053	0.050	0.101	0.101	0.109	0.104
150	0.008	0.008	0.009	0.009	0.051	0.049	0.054	0.051	0.102	0.101	0.108	0.103
200	0.008	0.009	0.011	0.009	0.050	0.048	0.053	0.052	0.103	0.102	0.106	0.103
250	0.008	0.009	0.011	0.009	0.050	0.049	0.055	0.050	0.105	0.102	0.107	0.104

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.009	0.002	0.003	0.002	0.050	0.018	0.018	0.019	0.098	0.045	0.045	0.044
-2.5	0.009	0.000	0.000	0.000	0.047	0.004	0.004	0.004	0.098	0.015	0.015	0.016
0	0.007	0.001	0.001	0.001	0.032	0.008	0.008	0.009	0.065	0.021	0.021	0.021
2.5	0.013	0.002	0.002	0.002	0.048	0.018	0.018	0.018	0.091	0.043	0.043	0.043
5	0.013	0.004	0.003	0.003	0.054	0.024	0.025	0.024	0.101	0.055	0.055	0.054
10	0.013	0.006	0.005	0.005	0.054	0.030	0.030	0.030	0.105	0.066	0.067	0.067
25	0.011	0.008	0.008	0.007	0.056	0.040	0.040	0.040	0.104	0.077	0.078	0.078
50	0.012	0.009	0.009	0.009	0.053	0.042	0.041	0.042	0.104	0.084	0.084	0.084
75	0.013	0.010	0.009	0.009	0.053	0.044	0.045	0.044	0.105	0.088	0.088	0.087
100	0.013	0.010	0.010	0.010	0.052	0.045	0.045	0.045	0.104	0.086	0.087	0.088
125	0.013	0.012	0.011	0.011	0.052	0.044	0.044	0.044	0.103	0.089	0.090	0.091
150	0.014	0.011	0.012	0.011	0.051	0.046	0.045	0.045	0.102	0.091	0.092	0.091
200	0.012	0.011	0.011	0.011	0.051	0.046	0.046	0.046	0.101	0.093	0.093	0.092
250	0.012	0.011	0.010	0.010	0.051	0.045	0.046	0.045	0.105	0.094	0.094	0.095
Right-sided tests - $T = 1000$												
-5	0.009	0.014	0.015	0.013	0.046	0.072	0.073	0.072	0.097	0.141	0.143	0.141
-2.5	0.009	0.018	0.017	0.016	0.049	0.098	0.099	0.097	0.102	0.195	0.195	0.193
0	0.012	0.019	0.020	0.019	0.058	0.095	0.094	0.093	0.115	0.193	0.192	0.191
2.5	0.013	0.021	0.020	0.020	0.059	0.088	0.089	0.088	0.115	0.168	0.169	0.169
5	0.012	0.020	0.019	0.018	0.056	0.083	0.083	0.082	0.109	0.156	0.156	0.156
10	0.010	0.016	0.016	0.016	0.054	0.076	0.074	0.073	0.107	0.139	0.140	0.140
25	0.009	0.014	0.014	0.013	0.051	0.064	0.064	0.063	0.101	0.125	0.124	0.124
50	0.011	0.014	0.014	0.013	0.048	0.058	0.058	0.059	0.097	0.117	0.117	0.116
75	0.011	0.013	0.013	0.013	0.049	0.055	0.057	0.057	0.098	0.113	0.113	0.113
100	0.010	0.011	0.012	0.011	0.049	0.056	0.057	0.057	0.098	0.109	0.111	0.110
125	0.010	0.013	0.013	0.012	0.050	0.055	0.055	0.056	0.096	0.109	0.110	0.109
150	0.011	0.013	0.013	0.013	0.050	0.056	0.057	0.056	0.097	0.109	0.107	0.108
200	0.011	0.012	0.012	0.012	0.048	0.054	0.055	0.055	0.099	0.107	0.108	0.108
250	0.012	0.013	0.012	0.012	0.050	0.055	0.054	0.054	0.101	0.109	0.109	0.109
Two-sided tests- $T = 1000$												
-5	0.009	0.008	0.008	0.007	0.045	0.042	0.044	0.041	0.096	0.089	0.091	0.091
-2.5	0.008	0.008	0.008	0.008	0.045	0.048	0.048	0.047	0.095	0.101	0.103	0.101
0	0.010	0.010	0.010	0.010	0.045	0.050	0.051	0.049	0.096	0.102	0.103	0.102
2.5	0.011	0.011	0.011	0.011	0.051	0.052	0.054	0.053	0.101	0.107	0.107	0.106
5	0.012	0.012	0.011	0.011	0.051	0.054	0.056	0.055	0.103	0.106	0.108	0.106
10	0.010	0.011	0.010	0.010	0.051	0.055	0.055	0.053	0.101	0.105	0.104	0.103
25	0.009	0.011	0.010	0.010	0.048	0.050	0.051	0.050	0.101	0.102	0.104	0.102
50	0.011	0.011	0.011	0.011	0.051	0.051	0.052	0.052	0.099	0.099	0.099	0.100
75	0.011	0.012	0.011	0.011	0.053	0.052	0.052	0.053	0.100	0.099	0.101	0.101
100	0.012	0.012	0.011	0.012	0.053	0.053	0.055	0.054	0.102	0.101	0.101	0.102
125	0.011	0.012	0.012	0.012	0.052	0.052	0.053	0.053	0.101	0.098	0.099	0.100
150	0.012	0.011	0.012	0.012	0.052	0.052	0.052	0.052	0.100	0.101	0.102	0.101
200	0.012	0.012	0.013	0.012	0.052	0.053	0.054	0.053	0.099	0.099	0.101	0.100
250	0.011	0.012	0.012	0.012	0.054	0.052	0.055	0.054	0.099	0.099	0.100	0.099

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.10: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP3 (Positive Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad 0; \quad 0 \quad 1]$.

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.010	0.011	0.017	0.010	0.053	0.051	0.060	0.052	0.102	0.101	0.110	0.102
-2.5	0.011	0.011	0.015	0.011	0.052	0.052	0.059	0.050	0.104	0.103	0.111	0.103
0	0.010	0.010	0.013	0.011	0.049	0.049	0.054	0.049	0.100	0.100	0.104	0.099
2.5	0.009	0.009	0.010	0.010	0.051	0.049	0.053	0.051	0.100	0.099	0.102	0.100
5	0.010	0.010	0.011	0.010	0.051	0.050	0.054	0.050	0.098	0.095	0.099	0.096
10	0.011	0.012	0.012	0.011	0.051	0.051	0.054	0.051	0.099	0.099	0.102	0.101
25	0.011	0.010	0.012	0.010	0.050	0.052	0.052	0.051	0.100	0.099	0.102	0.101
50	0.010	0.009	0.011	0.010	0.051	0.052	0.053	0.052	0.097	0.097	0.099	0.097
75	0.009	0.009	0.011	0.009	0.049	0.048	0.051	0.049	0.098	0.098	0.102	0.099
100	0.009	0.009	0.010	0.010	0.049	0.047	0.049	0.048	0.098	0.098	0.101	0.098
125	0.009	0.009	0.010	0.009	0.049	0.046	0.050	0.048	0.097	0.095	0.097	0.098
150	0.009	0.008	0.009	0.009	0.048	0.047	0.050	0.048	0.097	0.095	0.097	0.097
200	0.009	0.010	0.010	0.010	0.048	0.046	0.049	0.046	0.098	0.096	0.098	0.097
250	0.010	0.010	0.011	0.011	0.048	0.047	0.049	0.048	0.095	0.095	0.097	0.096
Right-sided tests - $T = 250$												
-5	0.012	0.012	0.017	0.012	0.051	0.051	0.060	0.050	0.101	0.101	0.110	0.099
-2.5	0.010	0.011	0.015	0.010	0.053	0.052	0.058	0.051	0.099	0.099	0.104	0.097
0	0.011	0.012	0.014	0.011	0.050	0.050	0.053	0.050	0.098	0.100	0.105	0.099
2.5	0.011	0.011	0.012	0.011	0.052	0.051	0.054	0.052	0.102	0.101	0.104	0.102
5	0.011	0.012	0.012	0.011	0.052	0.050	0.053	0.051	0.102	0.101	0.105	0.102
10	0.011	0.011	0.011	0.011	0.052	0.050	0.052	0.051	0.104	0.103	0.104	0.103
25	0.011	0.011	0.012	0.011	0.052	0.052	0.055	0.051	0.102	0.100	0.104	0.102
50	0.011	0.011	0.013	0.012	0.052	0.052	0.055	0.053	0.107	0.104	0.108	0.107
75	0.010	0.012	0.012	0.011	0.055	0.052	0.056	0.054	0.106	0.104	0.109	0.105
100	0.011	0.011	0.011	0.011	0.055	0.054	0.058	0.054	0.105	0.106	0.107	0.105
125	0.011	0.010	0.011	0.010	0.055	0.055	0.058	0.055	0.104	0.103	0.108	0.104
150	0.009	0.010	0.011	0.010	0.054	0.055	0.057	0.055	0.104	0.104	0.108	0.104
200	0.010	0.010	0.011	0.011	0.055	0.054	0.058	0.055	0.107	0.105	0.108	0.105
250	0.011	0.010	0.012	0.011	0.053	0.053	0.056	0.053	0.103	0.101	0.106	0.103
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.020	0.012	0.051	0.051	0.065	0.052	0.102	0.102	0.120	0.103
-2.5	0.011	0.012	0.018	0.011	0.051	0.051	0.067	0.052	0.101	0.103	0.116	0.101
0	0.011	0.011	0.015	0.010	0.051	0.053	0.059	0.052	0.096	0.098	0.107	0.098
2.5	0.010	0.010	0.012	0.010	0.050	0.053	0.057	0.051	0.100	0.101	0.107	0.102
5	0.011	0.010	0.013	0.011	0.050	0.051	0.055	0.052	0.100	0.101	0.107	0.101
10	0.011	0.011	0.013	0.011	0.050	0.049	0.053	0.052	0.101	0.100	0.106	0.102
25	0.012	0.011	0.013	0.012	0.052	0.052	0.056	0.053	0.102	0.102	0.107	0.102
50	0.011	0.011	0.013	0.011	0.051	0.050	0.056	0.054	0.104	0.104	0.108	0.105
75	0.010	0.009	0.012	0.010	0.052	0.049	0.055	0.053	0.103	0.100	0.106	0.104
100	0.010	0.010	0.011	0.010	0.050	0.049	0.053	0.052	0.103	0.100	0.107	0.102
125	0.009	0.008	0.011	0.010	0.050	0.049	0.053	0.051	0.104	0.101	0.108	0.103
150	0.008	0.008	0.010	0.009	0.050	0.051	0.055	0.052	0.102	0.102	0.106	0.102
200	0.009	0.009	0.010	0.009	0.051	0.048	0.054	0.052	0.103	0.099	0.106	0.101
250	0.009	0.010	0.012	0.011	0.049	0.049	0.054	0.050	0.100	0.099	0.105	0.101

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.009	0.009	0.009	0.009	0.049	0.050	0.051	0.051	0.097	0.097	0.098	0.096
-2.5	0.010	0.009	0.011	0.009	0.049	0.050	0.048	0.048	0.096	0.097	0.097	0.095
0	0.012	0.011	0.012	0.011	0.052	0.052	0.052	0.053	0.104	0.104	0.103	0.103
2.5	0.011	0.011	0.012	0.012	0.054	0.054	0.053	0.052	0.102	0.103	0.102	0.103
5	0.012	0.012	0.011	0.011	0.052	0.051	0.051	0.051	0.105	0.105	0.105	0.105
10	0.011	0.010	0.011	0.010	0.053	0.052	0.052	0.052	0.104	0.103	0.104	0.105
25	0.012	0.013	0.012	0.011	0.054	0.052	0.053	0.053	0.103	0.104	0.104	0.104
50	0.011	0.012	0.011	0.011	0.051	0.052	0.052	0.051	0.104	0.106	0.106	0.105
75	0.011	0.011	0.011	0.011	0.052	0.054	0.053	0.052	0.105	0.105	0.106	0.104
100	0.011	0.012	0.011	0.011	0.052	0.052	0.052	0.052	0.105	0.105	0.106	0.106
125	0.012	0.010	0.011	0.011	0.053	0.053	0.052	0.052	0.104	0.103	0.102	0.102
150	0.011	0.011	0.012	0.012	0.053	0.051	0.053	0.053	0.104	0.101	0.102	0.102
200	0.012	0.011	0.011	0.011	0.053	0.051	0.052	0.051	0.103	0.101	0.103	0.101
250	0.011	0.011	0.011	0.011	0.052	0.051	0.051	0.051	0.101	0.102	0.102	0.101
Right-sided tests - $T = 1000$												
-5	0.011	0.011	0.012	0.010	0.049	0.049	0.049	0.048	0.100	0.102	0.103	0.099
-2.5	0.010	0.010	0.011	0.009	0.048	0.048	0.049	0.047	0.096	0.095	0.097	0.094
0	0.008	0.009	0.009	0.009	0.049	0.049	0.048	0.047	0.099	0.100	0.101	0.099
2.5	0.009	0.010	0.010	0.009	0.051	0.051	0.051	0.049	0.102	0.101	0.101	0.100
5	0.010	0.010	0.010	0.009	0.053	0.052	0.051	0.052	0.103	0.103	0.103	0.102
10	0.011	0.010	0.010	0.010	0.052	0.050	0.052	0.051	0.099	0.099	0.101	0.099
25	0.011	0.011	0.011	0.011	0.049	0.049	0.048	0.048	0.097	0.095	0.095	0.096
50	0.011	0.011	0.011	0.011	0.050	0.050	0.050	0.049	0.094	0.093	0.094	0.093
75	0.010	0.010	0.011	0.011	0.049	0.049	0.049	0.049	0.097	0.095	0.097	0.096
100	0.011	0.011	0.011	0.011	0.049	0.051	0.050	0.049	0.099	0.099	0.101	0.100
125	0.011	0.011	0.011	0.011	0.051	0.051	0.052	0.051	0.102	0.101	0.102	0.103
150	0.011	0.011	0.011	0.011	0.051	0.051	0.052	0.051	0.102	0.103	0.102	0.102
200	0.011	0.011	0.012	0.012	0.053	0.052	0.052	0.052	0.101	0.100	0.101	0.100
250	0.010	0.011	0.011	0.010	0.053	0.052	0.054	0.053	0.100	0.100	0.100	0.100
Two-sided tests- $T = 1000$												
-5	0.010	0.009	0.011	0.009	0.048	0.049	0.052	0.049	0.098	0.097	0.100	0.099
-2.5	0.010	0.010	0.011	0.010	0.050	0.050	0.051	0.049	0.095	0.098	0.098	0.096
0	0.010	0.011	0.011	0.010	0.048	0.050	0.053	0.049	0.098	0.099	0.100	0.100
2.5	0.012	0.011	0.011	0.011	0.052	0.051	0.051	0.049	0.102	0.105	0.104	0.102
5	0.011	0.011	0.011	0.010	0.050	0.050	0.049	0.051	0.105	0.103	0.103	0.103
10	0.011	0.012	0.011	0.011	0.051	0.050	0.050	0.049	0.105	0.102	0.103	0.102
25	0.011	0.011	0.011	0.011	0.052	0.052	0.053	0.052	0.100	0.100	0.101	0.101
50	0.012	0.012	0.011	0.011	0.052	0.051	0.053	0.052	0.101	0.101	0.101	0.100
75	0.012	0.013	0.012	0.012	0.052	0.051	0.052	0.051	0.101	0.103	0.103	0.101
100	0.012	0.012	0.012	0.012	0.052	0.050	0.051	0.051	0.100	0.103	0.102	0.101
125	0.012	0.012	0.012	0.012	0.055	0.053	0.054	0.053	0.102	0.102	0.103	0.103
150	0.012	0.012	0.012	0.011	0.053	0.054	0.053	0.054	0.105	0.102	0.104	0.104
200	0.012	0.012	0.011	0.012	0.054	0.052	0.054	0.055	0.104	0.102	0.104	0.103
250	0.012	0.012	0.012	0.012	0.052	0.053	0.053	0.053	0.103	0.104	0.105	0.104

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.11: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP4 (Negative Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = -0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.95; \quad -0.95 \quad 1]$.

	1%				5%				10%			
c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.008	0.000	0.001	0.000	0.047	0.004	0.005	0.003	0.096	0.013	0.014	0.013
-2.5	0.006	0.000	0.000	0.000	0.042	0.000	0.001	0.001	0.107	0.002	0.002	0.002
0	0.013	0.000	0.000	0.000	0.039	0.001	0.001	0.001	0.063	0.002	0.003	0.003
2.5	0.021	0.001	0.001	0.001	0.060	0.004	0.005	0.005	0.095	0.013	0.013	0.012
5	0.023	0.001	0.001	0.001	0.066	0.011	0.011	0.010	0.112	0.027	0.028	0.026
10	0.018	0.003	0.003	0.003	0.063	0.020	0.021	0.019	0.115	0.044	0.045	0.045
25	0.014	0.005	0.005	0.005	0.058	0.029	0.030	0.031	0.108	0.064	0.065	0.063
50	0.010	0.006	0.006	0.006	0.055	0.037	0.038	0.035	0.106	0.076	0.078	0.079
75	0.010	0.006	0.008	0.006	0.053	0.038	0.040	0.040	0.109	0.085	0.089	0.085
100	0.010	0.007	0.008	0.007	0.054	0.043	0.045	0.043	0.110	0.092	0.095	0.093
125	0.011	0.009	0.010	0.009	0.055	0.046	0.049	0.047	0.109	0.095	0.098	0.096
150	0.011	0.009	0.011	0.010	0.055	0.047	0.051	0.049	0.108	0.099	0.100	0.099
200	0.011	0.011	0.012	0.011	0.055	0.052	0.054	0.051	0.106	0.100	0.103	0.102
250	0.011	0.011	0.013	0.012	0.054	0.054	0.056	0.054	0.108	0.104	0.108	0.107
Right-sided tests - $T = 250$												
-5	0.010	0.016	0.021	0.019	0.045	0.077	0.084	0.077	0.090	0.157	0.162	0.156
-2.5	0.010	0.018	0.021	0.019	0.044	0.105	0.110	0.106	0.094	0.263	0.265	0.262
0	0.014	0.024	0.027	0.025	0.063	0.118	0.125	0.120	0.126	0.251	0.252	0.249
2.5	0.016	0.025	0.027	0.025	0.070	0.117	0.125	0.122	0.136	0.232	0.238	0.236
5	0.015	0.024	0.027	0.024	0.068	0.111	0.116	0.113	0.134	0.209	0.216	0.210
10	0.015	0.021	0.026	0.023	0.064	0.095	0.100	0.096	0.123	0.177	0.183	0.180
25	0.013	0.016	0.018	0.017	0.057	0.073	0.078	0.074	0.109	0.141	0.144	0.141
50	0.011	0.012	0.014	0.012	0.053	0.063	0.065	0.061	0.110	0.125	0.130	0.128
75	0.009	0.011	0.013	0.011	0.054	0.058	0.061	0.058	0.107	0.115	0.121	0.117
100	0.010	0.011	0.011	0.011	0.052	0.053	0.056	0.054	0.107	0.112	0.116	0.113
125	0.010	0.011	0.011	0.011	0.050	0.051	0.054	0.051	0.105	0.109	0.113	0.110
150	0.010	0.010	0.012	0.011	0.050	0.049	0.054	0.049	0.105	0.104	0.108	0.105
200	0.010	0.010	0.010	0.010	0.053	0.049	0.054	0.049	0.105	0.098	0.102	0.099
250	0.011	0.009	0.010	0.009	0.052	0.049	0.051	0.049	0.103	0.096	0.099	0.096
Two-sided tests - $T = 250$												
-5	0.010	0.009	0.012	0.011	0.046	0.041	0.047	0.043	0.092	0.081	0.088	0.081
-2.5	0.009	0.009	0.010	0.010	0.040	0.048	0.053	0.049	0.087	0.105	0.111	0.107
0	0.012	0.012	0.015	0.012	0.054	0.056	0.063	0.059	0.105	0.119	0.125	0.121
2.5	0.014	0.012	0.015	0.014	0.057	0.062	0.066	0.062	0.114	0.121	0.130	0.126
5	0.013	0.013	0.017	0.013	0.056	0.060	0.066	0.060	0.113	0.119	0.127	0.123
10	0.012	0.012	0.014	0.012	0.056	0.057	0.063	0.059	0.108	0.113	0.121	0.115
25	0.010	0.010	0.012	0.011	0.052	0.051	0.056	0.054	0.105	0.100	0.108	0.104
50	0.008	0.008	0.010	0.008	0.049	0.047	0.052	0.049	0.101	0.098	0.103	0.097
75	0.009	0.009	0.010	0.008	0.050	0.045	0.052	0.050	0.102	0.096	0.102	0.099
100	0.009	0.009	0.011	0.009	0.050	0.047	0.051	0.049	0.101	0.096	0.101	0.097
125	0.010	0.009	0.012	0.010	0.049	0.048	0.053	0.048	0.101	0.096	0.103	0.098
150	0.008	0.010	0.012	0.009	0.050	0.048	0.052	0.049	0.100	0.098	0.104	0.098
200	0.010	0.010	0.012	0.010	0.051	0.049	0.054	0.051	0.105	0.101	0.109	0.100
250	0.010	0.010	0.011	0.010	0.052	0.048	0.055	0.050	0.104	0.101	0.107	0.103

	1%				5%				10%			
c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.045	0.003	0.003	0.003	0.094	0.011	0.011	0.011
-2.5	0.005	0.000	0.000	0.000	0.046	0.000	0.000	0.000	0.108	0.001	0.001	0.001
0	0.016	0.000	0.000	0.000	0.042	0.001	0.001	0.001	0.068	0.004	0.003	0.004
2.5	0.024	0.001	0.001	0.001	0.059	0.006	0.006	0.006	0.096	0.016	0.016	0.016
5	0.023	0.002	0.002	0.002	0.066	0.014	0.014	0.013	0.109	0.029	0.029	0.028
10	0.020	0.004	0.004	0.004	0.065	0.021	0.021	0.020	0.112	0.046	0.046	0.045
25	0.017	0.007	0.008	0.008	0.061	0.031	0.031	0.030	0.108	0.064	0.065	0.064
50	0.014	0.008	0.009	0.009	0.057	0.036	0.036	0.036	0.109	0.076	0.076	0.075
75	0.014	0.009	0.008	0.008	0.056	0.039	0.040	0.040	0.106	0.081	0.084	0.082
100	0.013	0.008	0.008	0.008	0.053	0.041	0.042	0.042	0.109	0.085	0.085	0.084
125	0.014	0.009	0.008	0.009	0.053	0.044	0.043	0.043	0.108	0.089	0.088	0.088
150	0.012	0.009	0.009	0.009	0.055	0.044	0.045	0.044	0.107	0.088	0.089	0.089
200	0.012	0.009	0.009	0.009	0.054	0.046	0.047	0.045	0.107	0.090	0.090	0.091
250	0.012	0.010	0.010	0.010	0.053	0.046	0.046	0.046	0.106	0.093	0.095	0.094
Right-sided tests - $T = 1000$												
-5	0.007	0.013	0.014	0.014	0.039	0.066	0.068	0.066	0.084	0.143	0.144	0.140
-2.5	0.008	0.019	0.019	0.018	0.042	0.100	0.098	0.098	0.088	0.244	0.241	0.241
0	0.011	0.021	0.022	0.020	0.052	0.111	0.112	0.110	0.110	0.236	0.237	0.235
2.5	0.011	0.023	0.023	0.022	0.060	0.110	0.114	0.111	0.120	0.224	0.226	0.226
5	0.011	0.021	0.022	0.020	0.062	0.108	0.110	0.108	0.124	0.208	0.209	0.209
10	0.012	0.021	0.019	0.019	0.061	0.098	0.099	0.098	0.117	0.184	0.184	0.183
25	0.011	0.017	0.017	0.017	0.056	0.080	0.081	0.080	0.111	0.151	0.152	0.152
50	0.011	0.014	0.014	0.015	0.052	0.070	0.068	0.069	0.105	0.134	0.134	0.134
75	0.009	0.013	0.012	0.012	0.053	0.066	0.066	0.064	0.105	0.127	0.126	0.126
100	0.010	0.013	0.012	0.012	0.051	0.063	0.064	0.064	0.103	0.121	0.124	0.122
125	0.010	0.012	0.012	0.012	0.052	0.061	0.061	0.061	0.102	0.119	0.120	0.120
150	0.010	0.013	0.011	0.011	0.052	0.057	0.058	0.058	0.104	0.117	0.117	0.116
200	0.010	0.011	0.011	0.011	0.051	0.057	0.058	0.057	0.103	0.113	0.114	0.113
250	0.010	0.011	0.010	0.011	0.051	0.056	0.056	0.054	0.103	0.113	0.113	0.112
Two-sided tests- $T = 1000$												
-5	0.007	0.007	0.007	0.007	0.040	0.032	0.034	0.032	0.085	0.069	0.070	0.069
-2.5	0.007	0.009	0.009	0.009	0.039	0.046	0.046	0.045	0.083	0.098	0.098	0.098
0	0.008	0.010	0.011	0.011	0.045	0.053	0.053	0.053	0.093	0.111	0.113	0.111
2.5	0.009	0.012	0.011	0.011	0.052	0.060	0.061	0.060	0.102	0.116	0.120	0.117
5	0.010	0.011	0.012	0.010	0.055	0.060	0.061	0.061	0.109	0.122	0.124	0.121
10	0.010	0.012	0.011	0.011	0.056	0.062	0.063	0.063	0.107	0.118	0.120	0.118
25	0.011	0.013	0.012	0.012	0.053	0.054	0.056	0.056	0.104	0.111	0.112	0.110
50	0.011	0.012	0.012	0.012	0.052	0.054	0.056	0.054	0.102	0.104	0.104	0.105
75	0.010	0.011	0.011	0.011	0.053	0.053	0.054	0.052	0.104	0.105	0.106	0.104
100	0.010	0.010	0.011	0.010	0.052	0.053	0.054	0.052	0.104	0.105	0.105	0.106
125	0.010	0.011	0.011	0.010	0.050	0.052	0.053	0.051	0.105	0.103	0.104	0.104
150	0.011	0.011	0.011	0.010	0.052	0.052	0.053	0.050	0.103	0.101	0.103	0.102
200	0.011	0.011	0.011	0.010	0.049	0.050	0.052	0.050	0.103	0.103	0.104	0.102
250	0.012	0.012	0.011	0.011	0.050	0.050	0.050	0.050	0.102	0.101	0.102	0.100

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual Wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.12: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP4 (Negative Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = -0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.90; \quad -0.90 \quad 1]$.

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
Left-sided tests - $T = 250$												
-5	0.009	0.001	0.001	0.000	0.048	0.005	0.005	0.005	0.098	0.015	0.016	0.014
-2.5	0.007	0.000	0.000	0.000	0.045	0.000	0.001	0.001	0.106	0.002	0.002	0.002
0	0.012	0.000	0.000	0.000	0.037	0.001	0.001	0.001	0.060	0.003	0.004	0.004
2.5	0.019	0.001	0.001	0.001	0.056	0.005	0.005	0.005	0.096	0.014	0.014	0.013
5	0.020	0.002	0.002	0.002	0.066	0.011	0.011	0.011	0.112	0.028	0.029	0.029
10	0.017	0.003	0.003	0.003	0.062	0.021	0.021	0.019	0.111	0.046	0.048	0.047
25	0.012	0.006	0.006	0.005	0.056	0.030	0.032	0.031	0.106	0.066	0.067	0.066
50	0.011	0.006	0.007	0.006	0.054	0.036	0.038	0.036	0.106	0.078	0.080	0.078
75	0.010	0.006	0.008	0.006	0.054	0.039	0.041	0.040	0.108	0.086	0.091	0.086
100	0.010	0.008	0.008	0.007	0.054	0.044	0.047	0.044	0.109	0.091	0.096	0.093
125	0.011	0.009	0.010	0.009	0.055	0.047	0.050	0.047	0.110	0.098	0.100	0.098
150	0.012	0.010	0.012	0.010	0.055	0.049	0.052	0.047	0.109	0.098	0.102	0.101
200	0.013	0.011	0.013	0.012	0.054	0.050	0.053	0.052	0.108	0.101	0.105	0.103
250	0.012	0.012	0.014	0.012	0.053	0.053	0.056	0.055	0.110	0.106	0.111	0.108
Right-sided tests - $T = 250$												
-5	0.009	0.017	0.021	0.018	0.044	0.076	0.084	0.078	0.090	0.155	0.161	0.154
-2.5	0.011	0.019	0.022	0.018	0.045	0.108	0.113	0.107	0.095	0.255	0.258	0.255
0	0.013	0.024	0.028	0.025	0.061	0.116	0.122	0.118	0.128	0.244	0.247	0.245
2.5	0.016	0.025	0.028	0.025	0.071	0.115	0.121	0.117	0.135	0.227	0.231	0.228
5	0.016	0.024	0.027	0.024	0.068	0.107	0.114	0.110	0.133	0.201	0.209	0.203
10	0.014	0.021	0.026	0.022	0.063	0.093	0.098	0.092	0.122	0.172	0.178	0.175
25	0.011	0.016	0.018	0.017	0.056	0.072	0.077	0.073	0.111	0.140	0.144	0.143
50	0.010	0.012	0.013	0.011	0.054	0.064	0.067	0.064	0.107	0.125	0.127	0.125
75	0.010	0.010	0.013	0.011	0.054	0.059	0.063	0.060	0.107	0.116	0.118	0.116
100	0.010	0.011	0.012	0.011	0.051	0.055	0.057	0.055	0.107	0.112	0.116	0.113
125	0.011	0.011	0.012	0.010	0.050	0.050	0.053	0.051	0.107	0.109	0.112	0.108
150	0.011	0.011	0.012	0.010	0.049	0.049	0.053	0.049	0.106	0.105	0.108	0.104
200	0.010	0.009	0.011	0.010	0.051	0.049	0.052	0.048	0.105	0.101	0.103	0.101
250	0.011	0.010	0.011	0.009	0.053	0.050	0.051	0.049	0.105	0.097	0.101	0.099
Two-sided tests - $T = 250$												
-5	0.008	0.009	0.012	0.009	0.045	0.041	0.047	0.041	0.093	0.080	0.090	0.083
-2.5	0.010	0.010	0.011	0.010	0.040	0.048	0.052	0.049	0.089	0.107	0.114	0.108
0	0.011	0.012	0.015	0.012	0.051	0.056	0.062	0.059	0.107	0.117	0.123	0.119
2.5	0.014	0.012	0.016	0.014	0.060	0.061	0.067	0.064	0.113	0.120	0.126	0.122
5	0.012	0.013	0.016	0.014	0.056	0.058	0.065	0.062	0.113	0.117	0.125	0.121
10	0.011	0.012	0.015	0.012	0.055	0.057	0.062	0.059	0.106	0.112	0.119	0.112
25	0.010	0.010	0.012	0.011	0.051	0.051	0.056	0.054	0.103	0.102	0.109	0.104
50	0.009	0.007	0.010	0.009	0.049	0.049	0.053	0.049	0.101	0.099	0.105	0.100
75	0.008	0.008	0.009	0.008	0.050	0.046	0.052	0.048	0.101	0.100	0.104	0.099
100	0.009	0.009	0.011	0.009	0.049	0.047	0.052	0.050	0.101	0.097	0.103	0.099
125	0.009	0.009	0.011	0.010	0.049	0.049	0.052	0.049	0.101	0.096	0.103	0.098
150	0.009	0.009	0.011	0.009	0.051	0.048	0.052	0.050	0.102	0.098	0.104	0.096
200	0.010	0.010	0.012	0.011	0.050	0.049	0.054	0.050	0.103	0.099	0.105	0.100
250	0.010	0.010	0.013	0.011	0.050	0.049	0.054	0.051	0.104	0.101	0.107	0.104

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.046	0.003	0.003	0.003	0.095	0.013	0.014	0.014
-2.5	0.004	0.000	0.000	0.000	0.048	0.000	0.000	0.000	0.110	0.001	0.001	0.001
0	0.014	0.000	0.000	0.000	0.040	0.002	0.002	0.002	0.066	0.004	0.004	0.004
2.5	0.022	0.001	0.001	0.001	0.058	0.007	0.007	0.007	0.096	0.017	0.018	0.018
5	0.022	0.003	0.003	0.003	0.064	0.014	0.015	0.014	0.107	0.031	0.031	0.031
10	0.019	0.005	0.005	0.005	0.063	0.021	0.022	0.021	0.109	0.047	0.047	0.046
25	0.016	0.007	0.008	0.008	0.058	0.031	0.032	0.031	0.109	0.064	0.065	0.065
50	0.014	0.008	0.008	0.008	0.056	0.039	0.039	0.038	0.109	0.078	0.079	0.077
75	0.014	0.009	0.008	0.007	0.057	0.042	0.042	0.041	0.105	0.082	0.084	0.083
100	0.013	0.009	0.008	0.008	0.055	0.043	0.044	0.044	0.107	0.084	0.085	0.085
125	0.012	0.009	0.009	0.008	0.054	0.045	0.045	0.045	0.107	0.089	0.089	0.089
150	0.013	0.009	0.009	0.009	0.056	0.045	0.046	0.045	0.107	0.091	0.092	0.091
200	0.012	0.009	0.010	0.010	0.053	0.046	0.046	0.046	0.106	0.093	0.094	0.092
250	0.012	0.010	0.010	0.010	0.053	0.046	0.046	0.046	0.108	0.096	0.097	0.095
Right-sided tests - $T = 1000$												
-5	0.008	0.014	0.014	0.013	0.040	0.069	0.069	0.066	0.084	0.143	0.144	0.143
-2.5	0.009	0.018	0.018	0.017	0.041	0.100	0.100	0.099	0.091	0.239	0.237	0.238
0	0.011	0.022	0.022	0.021	0.052	0.107	0.107	0.104	0.109	0.233	0.229	0.230
2.5	0.011	0.023	0.023	0.021	0.061	0.107	0.109	0.109	0.120	0.219	0.221	0.221
5	0.012	0.022	0.022	0.020	0.061	0.105	0.106	0.105	0.122	0.202	0.202	0.201
10	0.011	0.019	0.019	0.018	0.061	0.095	0.096	0.095	0.116	0.179	0.178	0.176
25	0.010	0.016	0.016	0.016	0.056	0.078	0.080	0.079	0.109	0.149	0.150	0.148
50	0.010	0.014	0.014	0.014	0.052	0.068	0.067	0.067	0.104	0.131	0.133	0.133
75	0.010	0.012	0.012	0.013	0.051	0.062	0.065	0.064	0.104	0.125	0.128	0.125
100	0.011	0.012	0.012	0.012	0.050	0.062	0.064	0.063	0.103	0.120	0.122	0.121
125	0.009	0.012	0.012	0.012	0.052	0.060	0.061	0.060	0.101	0.118	0.118	0.117
150	0.010	0.011	0.012	0.011	0.052	0.060	0.060	0.060	0.101	0.114	0.116	0.115
200	0.009	0.011	0.012	0.010	0.052	0.058	0.059	0.058	0.103	0.111	0.112	0.112
250	0.010	0.011	0.011	0.011	0.051	0.055	0.056	0.056	0.107	0.113	0.114	0.114
Two-sided tests- $T = 1000$												
-5	0.007	0.007	0.007	0.007	0.041	0.034	0.035	0.034	0.086	0.071	0.072	0.070
-2.5	0.007	0.009	0.009	0.009	0.038	0.045	0.046	0.045	0.085	0.098	0.100	0.099
0	0.009	0.010	0.011	0.011	0.045	0.052	0.054	0.053	0.093	0.108	0.109	0.106
2.5	0.009	0.010	0.012	0.011	0.052	0.060	0.059	0.059	0.102	0.115	0.116	0.116
5	0.010	0.012	0.012	0.011	0.054	0.059	0.060	0.060	0.105	0.118	0.121	0.119
10	0.010	0.011	0.011	0.011	0.054	0.061	0.062	0.061	0.106	0.117	0.118	0.116
25	0.010	0.012	0.012	0.011	0.051	0.055	0.055	0.055	0.104	0.109	0.111	0.109
50	0.012	0.011	0.011	0.011	0.052	0.054	0.056	0.054	0.103	0.105	0.106	0.105
75	0.010	0.011	0.011	0.011	0.052	0.053	0.054	0.053	0.105	0.105	0.106	0.105
100	0.010	0.010	0.010	0.009	0.052	0.053	0.053	0.052	0.105	0.105	0.108	0.107
125	0.010	0.011	0.011	0.010	0.053	0.052	0.054	0.052	0.106	0.104	0.107	0.105
150	0.011	0.010	0.010	0.010	0.052	0.054	0.054	0.052	0.106	0.105	0.106	0.105
200	0.010	0.010	0.010	0.009	0.050	0.050	0.051	0.049	0.104	0.102	0.105	0.104
250	0.010	0.011	0.011	0.011	0.051	0.050	0.052	0.051	0.103	0.101	0.102	0.101

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.13: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP4 (Negative Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = -0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.50; \quad -0.50 \quad 1]$.

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.010	0.003	0.005	0.002	0.052	0.020	0.024	0.019	0.105	0.048	0.053	0.048
-2.5	0.010	0.000	0.001	0.000	0.050	0.006	0.007	0.005	0.099	0.017	0.018	0.016
0	0.005	0.001	0.001	0.000	0.030	0.006	0.007	0.006	0.060	0.019	0.020	0.019
2.5	0.009	0.002	0.002	0.002	0.043	0.016	0.018	0.017	0.086	0.040	0.040	0.039
5	0.011	0.004	0.005	0.004	0.050	0.024	0.024	0.024	0.095	0.052	0.055	0.053
10	0.012	0.006	0.007	0.007	0.052	0.031	0.032	0.031	0.100	0.063	0.065	0.065
25	0.010	0.007	0.007	0.007	0.052	0.038	0.040	0.039	0.102	0.080	0.081	0.080
50	0.010	0.007	0.009	0.007	0.051	0.041	0.044	0.043	0.102	0.087	0.089	0.087
75	0.010	0.008	0.009	0.008	0.053	0.047	0.048	0.047	0.101	0.089	0.092	0.091
100	0.010	0.008	0.010	0.009	0.053	0.047	0.050	0.049	0.102	0.093	0.095	0.094
125	0.010	0.009	0.011	0.010	0.055	0.050	0.053	0.051	0.104	0.097	0.101	0.099
150	0.010	0.010	0.012	0.010	0.053	0.051	0.053	0.052	0.105	0.099	0.103	0.103
200	0.011	0.011	0.013	0.011	0.055	0.053	0.056	0.054	0.106	0.102	0.106	0.105
250	0.012	0.012	0.013	0.012	0.056	0.056	0.058	0.057	0.106	0.104	0.108	0.107
Right-sided tests - $T = 250$												
-5	0.009	0.016	0.021	0.016	0.045	0.072	0.081	0.073	0.097	0.144	0.154	0.146
-2.5	0.012	0.023	0.029	0.022	0.054	0.106	0.111	0.105	0.107	0.202	0.211	0.203
0	0.013	0.021	0.024	0.021	0.067	0.101	0.107	0.100	0.126	0.197	0.204	0.197
2.5	0.014	0.020	0.023	0.022	0.065	0.092	0.098	0.093	0.125	0.170	0.177	0.173
5	0.013	0.019	0.021	0.020	0.062	0.082	0.086	0.084	0.119	0.159	0.161	0.158
10	0.012	0.016	0.018	0.017	0.057	0.075	0.078	0.075	0.112	0.139	0.144	0.141
25	0.011	0.014	0.016	0.014	0.056	0.065	0.069	0.067	0.107	0.123	0.129	0.124
50	0.010	0.011	0.013	0.011	0.054	0.058	0.062	0.061	0.108	0.117	0.121	0.117
75	0.009	0.011	0.012	0.011	0.053	0.056	0.062	0.057	0.108	0.113	0.118	0.112
100	0.010	0.011	0.012	0.011	0.051	0.052	0.056	0.053	0.105	0.107	0.112	0.107
125	0.010	0.010	0.011	0.011	0.051	0.054	0.056	0.052	0.105	0.103	0.108	0.105
150	0.010	0.010	0.011	0.010	0.051	0.051	0.055	0.052	0.106	0.103	0.107	0.103
200	0.010	0.010	0.010	0.009	0.050	0.050	0.052	0.049	0.106	0.103	0.106	0.105
250	0.010	0.009	0.011	0.009	0.049	0.049	0.051	0.048	0.102	0.099	0.102	0.099
Two-sided tests - $T = 250$												
-5	0.009	0.009	0.016	0.009	0.048	0.044	0.057	0.044	0.097	0.090	0.105	0.092
-2.5	0.011	0.011	0.015	0.011	0.049	0.054	0.062	0.054	0.100	0.112	0.118	0.110
0	0.012	0.012	0.013	0.012	0.051	0.054	0.061	0.055	0.101	0.107	0.114	0.106
2.5	0.012	0.012	0.014	0.013	0.051	0.055	0.060	0.056	0.103	0.107	0.115	0.110
5	0.012	0.012	0.013	0.012	0.054	0.054	0.059	0.056	0.104	0.106	0.111	0.107
10	0.012	0.010	0.013	0.013	0.053	0.051	0.057	0.054	0.102	0.104	0.110	0.106
25	0.010	0.011	0.013	0.010	0.050	0.049	0.054	0.051	0.105	0.103	0.109	0.106
50	0.008	0.009	0.009	0.009	0.050	0.050	0.054	0.050	0.103	0.099	0.106	0.104
75	0.009	0.010	0.012	0.009	0.049	0.048	0.052	0.051	0.105	0.103	0.110	0.104
100	0.008	0.009	0.011	0.010	0.050	0.047	0.053	0.051	0.103	0.099	0.106	0.102
125	0.010	0.010	0.011	0.010	0.051	0.048	0.052	0.049	0.105	0.101	0.109	0.102
150	0.009	0.010	0.012	0.010	0.049	0.049	0.053	0.050	0.104	0.102	0.108	0.104
200	0.010	0.010	0.012	0.009	0.052	0.052	0.057	0.053	0.105	0.101	0.108	0.103
250	0.010	0.010	0.012	0.010	0.053	0.053	0.058	0.053	0.104	0.103	0.109	0.105

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.009	0.002	0.002	0.002	0.049	0.018	0.018	0.019	0.097	0.044	0.045	0.044
-2.5	0.009	0.001	0.001	0.000	0.050	0.004	0.004	0.004	0.098	0.015	0.015	0.015
0	0.008	0.001	0.000	0.000	0.033	0.008	0.008	0.008	0.065	0.021	0.021	0.020
2.5	0.012	0.002	0.003	0.002	0.048	0.018	0.018	0.018	0.092	0.043	0.044	0.043
5	0.013	0.004	0.003	0.003	0.054	0.025	0.024	0.025	0.100	0.056	0.056	0.057
10	0.014	0.006	0.005	0.005	0.054	0.031	0.030	0.030	0.103	0.067	0.067	0.067
25	0.013	0.008	0.008	0.008	0.054	0.039	0.039	0.038	0.105	0.080	0.080	0.080
50	0.014	0.010	0.010	0.010	0.054	0.043	0.045	0.044	0.104	0.085	0.086	0.085
75	0.014	0.011	0.011	0.011	0.053	0.044	0.046	0.046	0.102	0.090	0.090	0.089
100	0.014	0.011	0.012	0.012	0.053	0.047	0.047	0.047	0.104	0.093	0.092	0.092
125	0.013	0.012	0.011	0.011	0.054	0.048	0.047	0.047	0.105	0.094	0.095	0.094
150	0.013	0.011	0.011	0.011	0.054	0.048	0.049	0.048	0.104	0.096	0.096	0.094
200	0.012	0.011	0.011	0.010	0.055	0.051	0.051	0.050	0.103	0.096	0.097	0.096
250	0.012	0.011	0.010	0.011	0.054	0.050	0.051	0.050	0.104	0.098	0.098	0.098
Right-sided tests - $T = 1000$												
-5	0.009	0.014	0.015	0.013	0.045	0.072	0.073	0.073	0.097	0.141	0.143	0.141
-2.5	0.008	0.019	0.018	0.016	0.050	0.099	0.100	0.098	0.104	0.198	0.197	0.195
0	0.012	0.021	0.020	0.019	0.059	0.095	0.096	0.095	0.118	0.190	0.191	0.189
2.5	0.013	0.020	0.020	0.020	0.059	0.089	0.089	0.089	0.117	0.169	0.170	0.169
5	0.013	0.019	0.019	0.018	0.057	0.081	0.081	0.080	0.110	0.154	0.155	0.154
10	0.011	0.016	0.015	0.015	0.054	0.074	0.075	0.074	0.106	0.140	0.141	0.139
25	0.010	0.013	0.013	0.012	0.053	0.065	0.066	0.065	0.102	0.120	0.122	0.121
50	0.010	0.012	0.012	0.012	0.050	0.059	0.058	0.058	0.099	0.114	0.115	0.116
75	0.010	0.011	0.012	0.012	0.050	0.056	0.057	0.057	0.102	0.114	0.115	0.114
100	0.010	0.012	0.012	0.012	0.050	0.056	0.055	0.056	0.101	0.112	0.113	0.113
125	0.010	0.011	0.011	0.011	0.051	0.056	0.055	0.056	0.101	0.111	0.112	0.112
150	0.010	0.011	0.011	0.011	0.051	0.055	0.055	0.055	0.104	0.111	0.110	0.110
200	0.010	0.011	0.011	0.011	0.052	0.055	0.055	0.055	0.104	0.111	0.110	0.110
250	0.010	0.010	0.011	0.010	0.052	0.056	0.056	0.055	0.106	0.111	0.111	0.110
Two-sided tests- $T = 1000$												
-5	0.009	0.007	0.009	0.007	0.047	0.042	0.043	0.042	0.098	0.089	0.091	0.091
-2.5	0.007	0.008	0.009	0.008	0.046	0.049	0.049	0.047	0.096	0.102	0.104	0.102
0	0.009	0.011	0.010	0.011	0.047	0.051	0.052	0.051	0.097	0.102	0.104	0.103
2.5	0.011	0.010	0.011	0.010	0.050	0.054	0.056	0.053	0.100	0.106	0.107	0.106
5	0.010	0.012	0.011	0.011	0.052	0.055	0.055	0.054	0.098	0.104	0.105	0.104
10	0.010	0.011	0.011	0.011	0.052	0.054	0.054	0.054	0.100	0.104	0.105	0.104
25	0.010	0.011	0.011	0.011	0.051	0.051	0.053	0.051	0.101	0.103	0.105	0.104
50	0.011	0.011	0.012	0.011	0.053	0.052	0.052	0.052	0.101	0.102	0.102	0.102
75	0.011	0.012	0.013	0.012	0.053	0.051	0.053	0.052	0.103	0.100	0.102	0.103
100	0.012	0.013	0.012	0.013	0.053	0.051	0.052	0.052	0.102	0.101	0.102	0.103
125	0.013	0.012	0.012	0.012	0.053	0.052	0.052	0.053	0.103	0.102	0.102	0.103
150	0.012	0.012	0.012	0.012	0.053	0.052	0.052	0.052	0.103	0.102	0.104	0.103
200	0.010	0.010	0.011	0.010	0.052	0.050	0.052	0.052	0.103	0.105	0.106	0.105
250	0.010	0.011	0.010	0.010	0.049	0.047	0.049	0.049	0.104	0.105	0.107	0.105

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.14: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP4 (Negative Autocorrelation):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = -0.5$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad 0; \quad 0 \quad 1]$.

	1%				5%				10%			
c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.011	0.011	0.017	0.010	0.051	0.051	0.059	0.052	0.101	0.103	0.111	0.102
-2.5	0.011	0.012	0.015	0.010	0.051	0.052	0.059	0.049	0.103	0.104	0.110	0.103
0	0.011	0.011	0.014	0.010	0.050	0.051	0.055	0.050	0.099	0.100	0.102	0.100
2.5	0.010	0.010	0.012	0.010	0.049	0.048	0.053	0.051	0.099	0.098	0.101	0.099
5	0.010	0.010	0.012	0.011	0.050	0.051	0.053	0.050	0.100	0.098	0.101	0.099
10	0.011	0.011	0.013	0.011	0.052	0.050	0.053	0.051	0.103	0.102	0.105	0.105
25	0.010	0.010	0.011	0.010	0.049	0.049	0.052	0.051	0.100	0.099	0.102	0.101
50	0.011	0.009	0.010	0.010	0.048	0.046	0.049	0.049	0.098	0.098	0.100	0.099
75	0.010	0.009	0.011	0.011	0.049	0.047	0.050	0.048	0.097	0.094	0.097	0.095
100	0.009	0.009	0.010	0.010	0.048	0.048	0.050	0.048	0.097	0.093	0.097	0.096
125	0.009	0.009	0.011	0.010	0.050	0.049	0.052	0.049	0.095	0.093	0.097	0.094
150	0.009	0.010	0.011	0.011	0.048	0.049	0.052	0.049	0.097	0.096	0.098	0.095
200	0.009	0.009	0.011	0.010	0.048	0.048	0.050	0.048	0.099	0.097	0.099	0.098
250	0.009	0.010	0.011	0.010	0.050	0.049	0.053	0.051	0.101	0.097	0.103	0.098
Right-sided tests - $T = 250$												
-5	0.011	0.012	0.016	0.012	0.052	0.050	0.060	0.051	0.102	0.103	0.110	0.100
-2.5	0.011	0.010	0.016	0.010	0.052	0.051	0.057	0.051	0.102	0.101	0.107	0.100
0	0.012	0.011	0.015	0.011	0.050	0.051	0.055	0.049	0.097	0.096	0.101	0.097
2.5	0.010	0.010	0.012	0.010	0.052	0.052	0.053	0.051	0.103	0.101	0.103	0.102
5	0.011	0.010	0.012	0.010	0.050	0.050	0.052	0.050	0.103	0.102	0.105	0.103
10	0.010	0.011	0.011	0.011	0.052	0.052	0.054	0.052	0.100	0.099	0.102	0.101
25	0.011	0.013	0.014	0.012	0.051	0.049	0.053	0.053	0.103	0.101	0.105	0.103
50	0.010	0.011	0.012	0.011	0.051	0.050	0.054	0.053	0.103	0.101	0.106	0.101
75	0.011	0.010	0.011	0.011	0.051	0.051	0.053	0.051	0.101	0.100	0.104	0.101
100	0.010	0.010	0.011	0.010	0.049	0.050	0.053	0.051	0.101	0.099	0.103	0.100
125	0.010	0.010	0.012	0.011	0.051	0.050	0.052	0.051	0.101	0.099	0.101	0.099
150	0.010	0.010	0.012	0.011	0.050	0.050	0.052	0.050	0.099	0.098	0.101	0.098
200	0.011	0.011	0.011	0.011	0.048	0.048	0.051	0.050	0.099	0.097	0.100	0.098
250	0.010	0.010	0.011	0.010	0.048	0.048	0.051	0.048	0.100	0.096	0.100	0.097
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.020	0.012	0.051	0.050	0.066	0.051	0.102	0.101	0.119	0.103
-2.5	0.011	0.011	0.018	0.011	0.051	0.053	0.066	0.051	0.101	0.101	0.116	0.100
0	0.011	0.011	0.015	0.011	0.050	0.051	0.058	0.050	0.097	0.099	0.109	0.099
2.5	0.009	0.010	0.012	0.010	0.051	0.051	0.058	0.053	0.099	0.099	0.106	0.102
5	0.011	0.010	0.012	0.011	0.050	0.052	0.057	0.052	0.099	0.100	0.105	0.100
10	0.011	0.011	0.014	0.011	0.050	0.048	0.053	0.051	0.103	0.100	0.107	0.104
25	0.012	0.011	0.012	0.013	0.051	0.049	0.055	0.052	0.101	0.100	0.105	0.104
50	0.010	0.009	0.012	0.011	0.049	0.050	0.055	0.051	0.099	0.096	0.103	0.101
75	0.009	0.011	0.012	0.011	0.049	0.049	0.055	0.052	0.098	0.097	0.103	0.099
100	0.009	0.010	0.011	0.011	0.048	0.048	0.054	0.053	0.098	0.096	0.102	0.099
125	0.009	0.010	0.011	0.010	0.049	0.049	0.054	0.051	0.099	0.098	0.104	0.100
150	0.010	0.010	0.012	0.010	0.048	0.047	0.051	0.051	0.097	0.097	0.104	0.099
200	0.009	0.010	0.012	0.010	0.047	0.049	0.052	0.050	0.096	0.094	0.101	0.098
250	0.010	0.011	0.012	0.010	0.049	0.050	0.053	0.050	0.098	0.097	0.103	0.099

	1%				5%				10%			
c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.009	0.009	0.009	0.009	0.050	0.050	0.051	0.051	0.096	0.096	0.098	0.097
-2.5	0.009	0.009	0.010	0.009	0.050	0.050	0.051	0.050	0.096	0.096	0.096	0.096
0	0.010	0.010	0.011	0.010	0.052	0.053	0.053	0.053	0.102	0.103	0.103	0.102
2.5	0.012	0.011	0.011	0.011	0.053	0.053	0.054	0.053	0.105	0.104	0.105	0.105
5	0.012	0.011	0.010	0.011	0.051	0.053	0.052	0.051	0.106	0.105	0.105	0.105
10	0.011	0.011	0.010	0.010	0.051	0.050	0.052	0.052	0.103	0.104	0.104	0.104
25	0.011	0.012	0.011	0.010	0.054	0.052	0.052	0.053	0.103	0.104	0.104	0.103
50	0.012	0.011	0.011	0.011	0.054	0.052	0.053	0.053	0.102	0.102	0.102	0.102
75	0.011	0.012	0.012	0.012	0.053	0.053	0.053	0.052	0.102	0.102	0.102	0.103
100	0.012	0.013	0.012	0.012	0.053	0.052	0.052	0.051	0.103	0.102	0.103	0.102
125	0.012	0.012	0.012	0.013	0.051	0.051	0.052	0.052	0.102	0.103	0.103	0.103
150	0.012	0.012	0.011	0.012	0.052	0.052	0.051	0.052	0.101	0.102	0.102	0.101
200	0.012	0.011	0.011	0.011	0.051	0.051	0.051	0.051	0.102	0.102	0.101	0.100
250	0.012	0.012	0.011	0.012	0.050	0.051	0.051	0.050	0.103	0.104	0.104	0.102
Right-sided tests - $T = 1000$												
-5	0.011	0.011	0.012	0.010	0.047	0.049	0.049	0.048	0.101	0.100	0.103	0.100
-2.5	0.009	0.009	0.011	0.009	0.048	0.048	0.049	0.047	0.096	0.095	0.098	0.095
0	0.009	0.010	0.010	0.009	0.048	0.049	0.048	0.048	0.098	0.099	0.100	0.098
2.5	0.009	0.010	0.010	0.009	0.051	0.050	0.051	0.049	0.102	0.099	0.102	0.102
5	0.009	0.009	0.009	0.009	0.053	0.054	0.054	0.053	0.101	0.101	0.103	0.102
10	0.010	0.009	0.009	0.009	0.052	0.051	0.053	0.052	0.101	0.101	0.102	0.101
25	0.010	0.009	0.009	0.009	0.050	0.049	0.050	0.050	0.098	0.097	0.098	0.098
50	0.010	0.009	0.009	0.009	0.050	0.049	0.051	0.051	0.097	0.096	0.097	0.097
75	0.010	0.011	0.010	0.010	0.048	0.048	0.049	0.048	0.099	0.099	0.099	0.099
100	0.010	0.011	0.011	0.010	0.048	0.048	0.048	0.048	0.101	0.099	0.100	0.101
125	0.011	0.010	0.010	0.010	0.049	0.050	0.049	0.048	0.101	0.100	0.102	0.101
150	0.009	0.010	0.010	0.009	0.050	0.049	0.050	0.049	0.100	0.100	0.101	0.101
200	0.010	0.010	0.010	0.010	0.050	0.049	0.050	0.049	0.102	0.101	0.102	0.101
250	0.010	0.010	0.010	0.010	0.049	0.048	0.049	0.049	0.101	0.100	0.101	0.100
Two-sided tests- $T = 1000$												
-5	0.010	0.010	0.011	0.009	0.047	0.048	0.052	0.049	0.098	0.098	0.100	0.099
-2.5	0.010	0.010	0.010	0.009	0.050	0.049	0.051	0.049	0.096	0.099	0.100	0.097
0	0.010	0.011	0.010	0.010	0.048	0.048	0.050	0.048	0.101	0.102	0.101	0.101
2.5	0.010	0.011	0.011	0.011	0.051	0.052	0.052	0.051	0.103	0.103	0.105	0.102
5	0.010	0.010	0.011	0.010	0.050	0.051	0.051	0.051	0.105	0.105	0.105	0.104
10	0.010	0.010	0.010	0.010	0.051	0.050	0.051	0.050	0.103	0.102	0.104	0.104
25	0.012	0.010	0.010	0.010	0.052	0.052	0.052	0.051	0.103	0.101	0.102	0.103
50	0.012	0.012	0.011	0.012	0.052	0.051	0.052	0.052	0.103	0.102	0.103	0.104
75	0.012	0.012	0.012	0.011	0.053	0.052	0.052	0.052	0.101	0.101	0.102	0.100
100	0.011	0.011	0.012	0.012	0.051	0.050	0.052	0.051	0.100	0.098	0.099	0.099
125	0.011	0.012	0.011	0.011	0.051	0.050	0.051	0.050	0.100	0.100	0.101	0.100
150	0.012	0.012	0.012	0.011	0.050	0.051	0.052	0.050	0.100	0.100	0.101	0.100
200	0.012	0.011	0.012	0.012	0.049	0.048	0.050	0.049	0.100	0.099	0.101	0.100
250	0.011	0.012	0.012	0.012	0.049	0.047	0.048	0.048	0.099	0.099	0.100	0.099

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.15: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP5 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T$ and $\psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = [\sigma^2_{ut} \quad -0.95\sigma_{ut}\sigma_{vt}; \quad -0.95\sigma_{ut}\sigma_{vt} \quad \sigma^2_{vt}]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.006	0.001	0.000	0.001	0.040	0.006	0.004	0.011	0.089	0.017	0.013	0.030
-2.5	0.006	0.000	0.000	0.000	0.033	0.002	0.001	0.003	0.082	0.005	0.004	0.008
0	0.014	0.000	0.000	0.001	0.051	0.003	0.003	0.005	0.090	0.008	0.007	0.011
2.5	0.017	0.001	0.001	0.002	0.059	0.005	0.005	0.009	0.102	0.014	0.012	0.018
5	0.019	0.001	0.001	0.002	0.067	0.008	0.008	0.014	0.112	0.022	0.018	0.029
10	0.018	0.002	0.002	0.005	0.067	0.013	0.012	0.024	0.114	0.034	0.033	0.050
25	0.017	0.004	0.004	0.011	0.058	0.025	0.024	0.041	0.110	0.052	0.054	0.075
50	0.014	0.006	0.007	0.015	0.055	0.033	0.034	0.051	0.105	0.066	0.067	0.092
75	0.013	0.006	0.007	0.016	0.055	0.034	0.037	0.057	0.106	0.073	0.076	0.100
100	0.012	0.006	0.008	0.017	0.056	0.039	0.041	0.062	0.106	0.078	0.083	0.109
125	0.011	0.007	0.008	0.018	0.057	0.040	0.043	0.066	0.107	0.084	0.087	0.113
150	0.010	0.007	0.008	0.018	0.057	0.043	0.045	0.068	0.109	0.088	0.091	0.118
200	0.011	0.008	0.009	0.020	0.055	0.046	0.049	0.072	0.109	0.091	0.094	0.122
250	0.011	0.009	0.010	0.021	0.057	0.049	0.052	0.075	0.109	0.093	0.098	0.128
Right-sided tests - $T = 250$												
-5	0.009	0.015	0.011	0.028	0.043	0.072	0.054	0.103	0.088	0.143	0.113	0.182
-2.5	0.008	0.017	0.022	0.031	0.046	0.102	0.086	0.133	0.092	0.244	0.186	0.282
0	0.010	0.020	0.027	0.037	0.054	0.107	0.117	0.150	0.113	0.223	0.223	0.273
2.5	0.011	0.020	0.025	0.037	0.058	0.106	0.119	0.151	0.123	0.218	0.227	0.274
5	0.010	0.021	0.025	0.037	0.056	0.101	0.113	0.149	0.121	0.203	0.211	0.253
10	0.011	0.021	0.025	0.038	0.058	0.0923	0.100	0.132	0.113	0.183	0.189	0.229
25	0.012	0.018	0.022	0.034	0.059	0.082	0.088	0.116	0.114	0.152	0.158	0.195
50	0.011	0.015	0.018	0.032	0.056	0.071	0.076	0.107	0.112	0.135	0.142	0.177
75	0.012	0.015	0.019	0.030	0.056	0.066	0.073	0.101	0.107	0.128	0.133	0.167
100	0.011	0.014	0.016	0.029	0.055	0.063	0.070	0.097	0.107	0.120	0.126	0.158
125	0.012	0.014	0.017	0.028	0.055	0.060	0.066	0.092	0.106	0.116	0.122	0.153
150	0.012	0.014	0.017	0.028	0.053	0.058	0.063	0.089	0.107	0.113	0.120	0.150
200	0.012	0.012	0.015	0.027	0.054	0.056	0.060	0.087	0.107	0.108	0.114	0.146
250	0.012	0.011	0.014	0.025	0.054	0.054	0.060	0.085	0.108	0.105	0.109	0.142
Two-sided tests - $T = 250$												
-5	0.009	0.008	0.007	0.016	0.045	0.037	0.029	0.064	0.092	0.076	0.058	0.114
-2.5	0.007	0.008	0.012	0.017	0.042	0.044	0.048	0.069	0.085	0.102	0.087	0.136
0	0.008	0.009	0.014	0.020	0.048	0.052	0.064	0.083	0.099	0.109	0.120	0.155
2.5	0.009	0.010	0.014	0.021	0.047	0.052	0.062	0.087	0.103	0.110	0.124	0.160
5	0.009	0.010	0.014	0.022	0.048	0.053	0.059	0.087	0.103	0.109	0.121	0.163
10	0.009	0.011	0.012	0.024	0.054	0.056	0.061	0.087	0.102	0.105	0.112	0.156
25	0.012	0.010	0.012	0.025	0.053	0.055	0.060	0.093	0.109	0.106	0.112	0.157
50	0.011	0.011	0.013	0.027	0.053	0.053	0.059	0.091	0.108	0.103	0.110	0.158
75	0.012	0.011	0.013	0.027	0.053	0.052	0.058	0.092	0.106	0.101	0.109	0.158
100	0.011	0.011	0.013	0.028	0.053	0.051	0.057	0.091	0.107	0.100	0.111	0.159
125	0.010	0.010	0.013	0.028	0.054	0.050	0.056	0.091	0.107	0.100	0.109	0.158
150	0.010	0.010	0.014	0.029	0.054	0.050	0.055	0.091	0.106	0.099	0.108	0.157
200	0.010	0.010	0.014	0.027	0.053	0.049	0.057	0.092	0.107	0.100	0.109	0.159
250	0.011	0.010	0.014	0.027	0.053	0.050	0.057	0.095	0.110	0.102	0.112	0.160

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.002	0.045	0.007	0.005	0.014	0.091	0.022	0.016	0.035
-2.5	0.005	0.000	0.000	0.001	0.039	0.002	0.002	0.003	0.088	0.005	0.003	0.009
0	0.014	0.000	0.000	0.001	0.059	0.004	0.003	0.006	0.100	0.007	0.007	0.011
2.5	0.020	0.001	0.000	0.002	0.071	0.005	0.004	0.010	0.118	0.015	0.014	0.022
5	0.024	0.001	0.001	0.003	0.075	0.009	0.008	0.015	0.123	0.023	0.020	0.034
10	0.021	0.002	0.002	0.006	0.070	0.016	0.015	0.026	0.121	0.037	0.033	0.055
25	0.016	0.005	0.005	0.011	0.063	0.024	0.023	0.044	0.115	0.056	0.054	0.080
50	0.013	0.006	0.006	0.014	0.060	0.030	0.031	0.056	0.114	0.070	0.071	0.100
75	0.011	0.006	0.006	0.014	0.058	0.035	0.034	0.062	0.113	0.078	0.078	0.109
100	0.011	0.006	0.006	0.016	0.059	0.037	0.037	0.065	0.110	0.081	0.080	0.112
125	0.012	0.007	0.006	0.018	0.057	0.038	0.038	0.065	0.109	0.083	0.082	0.113
150	0.012	0.008	0.007	0.018	0.056	0.040	0.039	0.067	0.107	0.085	0.084	0.116
200	0.011	0.008	0.008	0.020	0.055	0.042	0.043	0.069	0.104	0.085	0.086	0.117
250	0.012	0.007	0.009	0.021	0.055	0.044	0.044	0.070	0.105	0.088	0.088	0.120
Right-sided tests - $T = 1000$												
-5	0.010	0.015	0.010	0.028	0.046	0.073	0.052	0.106	0.093	0.145	0.113	0.188
-2.5	0.006	0.015	0.015	0.025	0.037	0.093	0.076	0.119	0.080	0.238	0.171	0.275
0	0.006	0.017	0.020	0.033	0.046	0.100	0.103	0.138	0.098	0.216	0.209	0.267
2.5	0.008	0.019	0.023	0.036	0.056	0.107	0.111	0.145	0.114	0.208	0.208	0.256
5	0.010	0.020	0.022	0.035	0.054	0.106	0.109	0.143	0.117	0.197	0.195	0.240
10	0.011	0.022	0.022	0.036	0.056	0.095	0.096	0.131	0.113	0.177	0.175	0.220
25	0.012	0.018	0.019	0.034	0.056	0.080	0.081	0.112	0.104	0.153	0.151	0.193
50	0.011	0.015	0.016	0.032	0.054	0.073	0.073	0.103	0.101	0.136	0.137	0.176
75	0.010	0.015	0.015	0.030	0.053	0.069	0.070	0.101	0.104	0.131	0.132	0.168
100	0.009	0.014	0.014	0.029	0.052	0.067	0.067	0.098	0.105	0.126	0.128	0.162
125	0.010	0.013	0.013	0.029	0.052	0.064	0.065	0.095	0.102	0.123	0.124	0.159
150	0.010	0.013	0.012	0.029	0.053	0.063	0.063	0.094	0.103	0.121	0.122	0.157
200	0.010	0.012	0.012	0.028	0.053	0.061	0.062	0.091	0.100	0.116	0.118	0.153
250	0.011	0.012	0.012	0.027	0.053	0.060	0.061	0.091	0.100	0.113	0.115	0.150
Two-sided tests- $T = 1000$												
-5	0.009	0.007	0.006	0.016	0.049	0.039	0.026	0.066	0.099	0.079	0.056	0.120
-2.5	0.005	0.007	0.008	0.012	0.033	0.040	0.039	0.061	0.073	0.095	0.077	0.122
0	0.005	0.007	0.008	0.015	0.041	0.049	0.054	0.075	0.086	0.105	0.106	0.144
2.5	0.006	0.010	0.011	0.017	0.046	0.055	0.059	0.085	0.098	0.112	0.115	0.155
5	0.008	0.011	0.013	0.021	0.046	0.054	0.057	0.092	0.104	0.116	0.117	0.158
10	0.010	0.013	0.012	0.024	0.051	0.056	0.058	0.091	0.104	0.111	0.111	0.156
25	0.011	0.012	0.012	0.027	0.053	0.056	0.056	0.091	0.101	0.103	0.104	0.156
50	0.010	0.010	0.010	0.027	0.052	0.054	0.054	0.092	0.101	0.102	0.103	0.160
75	0.011	0.011	0.011	0.026	0.051	0.051	0.053	0.093	0.103	0.103	0.104	0.163
100	0.011	0.011	0.010	0.025	0.053	0.050	0.053	0.093	0.103	0.101	0.104	0.163
125	0.011	0.011	0.011	0.026	0.052	0.052	0.053	0.095	0.104	0.103	0.103	0.160
150	0.011	0.010	0.011	0.026	0.054	0.051	0.053	0.094	0.103	0.101	0.103	0.161
200	0.012	0.011	0.010	0.026	0.054	0.051	0.053	0.097	0.105	0.103	0.104	0.160
250	0.011	0.010	0.011	0.027	0.054	0.053	0.054	0.095	0.105	0.104	0.105	0.162

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (12) and (14) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.16: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP5 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T$ and $\psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = \left[\sigma_{ut}^2 \quad -0.9\sigma_{ut}\sigma_{vt}; \quad -0.9\sigma_{ut}\sigma_{vt} \quad \sigma_{vt}^2\right]$ and $\sigma_{ut}^2 = \sigma_{vt}^2 = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.007	0.000	0.000	0.002	0.040	0.007	0.005	0.013	0.090	0.019	0.013	0.032
-2.5	0.006	0.000	0.000	0.001	0.035	0.002	0.002	0.004	0.082	0.005	0.004	0.009
0	0.013	0.000	0.000	0.001	0.047	0.003	0.003	0.005	0.085	0.009	0.007	0.012
2.5	0.016	0.001	0.001	0.002	0.057	0.006	0.005	0.010	0.099	0.015	0.013	0.021
5	0.019	0.001	0.001	0.002	0.064	0.010	0.008	0.015	0.110	0.023	0.020	0.032
10	0.018	0.002	0.002	0.006	0.064	0.015	0.014	0.026	0.114	0.038	0.035	0.054
25	0.016	0.004	0.004	0.012	0.057	0.026	0.026	0.042	0.109	0.056	0.056	0.078
50	0.013	0.006	0.006	0.015	0.055	0.033	0.034	0.053	0.103	0.066	0.068	0.094
75	0.013	0.006	0.008	0.016	0.054	0.036	0.038	0.057	0.106	0.074	0.077	0.104
100	0.012	0.007	0.008	0.017	0.056	0.038	0.041	0.063	0.106	0.083	0.084	0.109
125	0.012	0.007	0.008	0.018	0.055	0.041	0.045	0.067	0.107	0.085	0.088	0.113
150	0.011	0.008	0.009	0.018	0.054	0.042	0.046	0.068	0.107	0.087	0.089	0.116
200	0.011	0.008	0.010	0.020	0.056	0.045	0.049	0.072	0.106	0.090	0.094	0.124
250	0.010	0.009	0.010	0.022	0.056	0.048	0.053	0.075	0.108	0.094	0.099	0.128
Right-sided tests - $T = 250$												
-5	0.007	0.015	0.010	0.027	0.044	0.072	0.051	0.103	0.089	0.143	0.113	0.181
-2.5	0.008	0.017	0.021	0.031	0.045	0.101	0.082	0.135	0.096	0.242	0.180	0.276
0	0.010	0.019	0.026	0.037	0.056	0.106	0.115	0.149	0.113	0.219	0.216	0.269
2.5	0.010	0.019	0.025	0.037	0.058	0.103	0.116	0.149	0.122	0.212	0.219	0.266
5	0.011	0.021	0.024	0.037	0.056	0.100	0.108	0.144	0.122	0.197	0.204	0.245
10	0.012	0.019	0.024	0.035	0.058	0.092	0.098	0.132	0.117	0.178	0.185	0.227
25	0.012	0.018	0.021	0.035	0.058	0.080	0.085	0.114	0.112	0.151	0.157	0.193
50	0.012	0.016	0.019	0.032	0.055	0.068	0.073	0.105	0.1108	0.134	0.138	0.173
75	0.011	0.015	0.018	0.029	0.053	0.064	0.068	0.098	0.108	0.125	0.131	0.163
100	0.012	0.014	0.018	0.029	0.054	0.060	0.066	0.094	0.106	0.120	0.125	0.157
125	0.012	0.013	0.017	0.028	0.053	0.059	0.062	0.091	0.107	0.114	0.120	0.153
150	0.012	0.013	0.017	0.028	0.054	0.057	0.061	0.087	0.105	0.113	0.117	0.150
200	0.011	0.012	0.015	0.027	0.054	0.055	0.060	0.087	0.106	0.109	0.114	0.145
250	0.011	0.012	0.014	0.025	0.055	0.052	0.058	0.082	0.107	0.105	0.110	0.141
Two-sided tests - $T = 250$												
-5	0.007	0.008	0.007	0.017	0.045	0.037	0.027	0.063	0.092	0.077	0.056	0.116
-2.5	0.007	0.009	0.012	0.017	0.041	0.044	0.046	0.070	0.089	0.103	0.084	0.139
0	0.008	0.010	0.014	0.021	0.047	0.051	0.062	0.083	0.098	0.109	0.118	0.154
2.5	0.008	0.010	0.013	0.020	0.049	0.054	0.062	0.087	0.102	0.108	0.121	0.158
5	0.009	0.010	0.014	0.021	0.048	0.053	0.061	0.088	0.105	0.109	0.116	0.159
10	0.010	0.010	0.012	0.024	0.051	0.055	0.061	0.088	0.105	0.108	0.112	0.157
25	0.011	0.010	0.013	0.025	0.055	0.055	0.059	0.092	0.106	0.104	0.110	0.156
50	0.011	0.010	0.013	0.028	0.055	0.053	0.059	0.090	0.105	0.100	0.108	0.158
75	0.010	0.010	0.013	0.029	0.053	0.053	0.059	0.091	0.104	0.099	0.106	0.155
100	0.011	0.010	0.013	0.029	0.053	0.050	0.058	0.091	0.105	0.099	0.107	0.156
125	0.011	0.010	0.014	0.030	0.053	0.050	0.057	0.092	0.106	0.099	0.107	0.158
150	0.010	0.011	0.014	0.029	0.053	0.051	0.058	0.091	0.105	0.098	0.107	0.155
200	0.010	0.012	0.014	0.028	0.053	0.050	0.057	0.093	0.107	0.100	0.109	0.159
250	0.010	0.011	0.014	0.028	0.053	0.049	0.058	0.096	0.108	0.101	0.111	0.157

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.010	0.001	0.000	0.002	0.045	0.008	0.005	0.016	0.094	0.025	0.017	0.038
-2.5	0.006	0.000	0.000	0.001	0.041	0.002	0.002	0.004	0.091	0.006	0.004	0.010
0	0.013	0.001	0.000	0.002	0.055	0.004	0.003	0.006	0.094	0.008	0.007	0.012
2.5	0.019	0.001	0.000	0.002	0.068	0.006	0.005	0.010	0.116	0.016	0.014	0.024
5	0.020	0.001	0.001	0.003	0.069	0.010	0.008	0.017	0.120	0.024	0.021	0.039
10	0.019	0.002	0.002	0.007	0.067	0.016	0.015	0.028	0.118	0.039	0.035	0.056
25	0.015	0.005	0.005	0.012	0.059	0.025	0.024	0.044	0.113	0.056	0.055	0.080
50	0.012	0.006	0.006	0.014	0.057	0.031	0.031	0.055	0.113	0.070	0.070	0.101
75	0.011	0.006	0.006	0.014	0.059	0.035	0.035	0.063	0.112	0.077	0.078	0.110
100	0.010	0.007	0.006	0.016	0.057	0.039	0.039	0.063	0.110	0.082	0.081	0.113
125	0.011	0.006	0.007	0.017	0.054	0.041	0.040	0.066	0.107	0.083	0.084	0.114
150	0.011	0.007	0.007	0.017	0.056	0.040	0.041	0.068	0.107	0.085	0.086	0.117
200	0.012	0.007	0.008	0.019	0.056	0.043	0.043	0.070	0.104	0.088	0.087	0.119
250	0.012	0.009	0.009	0.021	0.055	0.044	0.044	0.072	0.105	0.090	0.091	0.120
Right-sided tests - $T = 1000$												
-5	0.009	0.015	0.009	0.026	0.046	0.073	0.050	0.105	0.092	0.146	0.111	0.189
-2.5	0.005	0.013	0.015	0.025	0.038	0.095	0.074	0.123	0.082	0.230	0.165	0.266
0	0.007	0.017	0.018	0.030	0.046	0.098	0.101	0.136	0.101	0.212	0.204	0.260
2.5	0.008	0.020	0.021	0.035	0.054	0.103	0.106	0.144	0.113	0.204	0.201	0.252
5	0.010	0.020	0.021	0.035	0.055	0.103	0.103	0.138	0.115	0.189	0.191	0.236
10	0.012	0.020	0.021	0.035	0.055	0.092	0.093	0.127	0.110	0.174	0.173	0.216
25	0.011	0.017	0.019	0.033	0.054	0.078	0.077	0.110	0.103	0.147	0.148	0.190
50	0.011	0.016	0.017	0.032	0.053	0.070	0.071	0.101	0.103	0.134	0.136	0.171
75	0.011	0.015	0.015	0.029	0.053	0.067	0.068	0.101	0.106	0.130	0.131	0.167
100	0.010	0.014	0.014	0.029	0.054	0.066	0.067	0.099	0.104	0.125	0.126	0.160
125	0.010	0.012	0.012	0.028	0.054	0.065	0.067	0.096	0.105	0.123	0.124	0.157
150	0.011	0.013	0.012	0.028	0.054	0.062	0.065	0.093	0.102	0.120	0.122	0.154
200	0.009	0.013	0.012	0.028	0.055	0.063	0.063	0.093	0.103	0.116	0.118	0.150
250	0.010	0.013	0.012	0.027	0.054	0.060	0.061	0.092	0.102	0.113	0.114	0.147
Two-sided tests- $T = 1000$												
-5	0.009	0.008	0.006	0.016	0.049	0.038	0.025	0.068	0.098	0.080	0.055	0.121
-2.5	0.005	0.006	0.008	0.013	0.034	0.040	0.036	0.061	0.077	0.096	0.075	0.126
0	0.005	0.007	0.008	0.015	0.039	0.049	0.052	0.075	0.087	0.102	0.104	0.142
2.5	0.007	0.009	0.011	0.019	0.045	0.055	0.057	0.082	0.095	0.109	0.111	0.154
5	0.007	0.011	0.013	0.020	0.046	0.055	0.055	0.089	0.100	0.111	0.111	0.155
10	0.010	0.012	0.012	0.023	0.049	0.056	0.056	0.091	0.098	0.109	0.108	0.156
25	0.012	0.012	0.012	0.027	0.052	0.053	0.054	0.089	0.100	0.101	0.101	0.154
50	0.011	0.010	0.011	0.028	0.052	0.054	0.053	0.090	0.100	0.101	0.102	0.156
75	0.011	0.011	0.011	0.026	0.050	0.049	0.050	0.092	0.104	0.102	0.103	0.164
100	0.010	0.010	0.010	0.027	0.051	0.049	0.051	0.095	0.106	0.104	0.105	0.162
125	0.011	0.010	0.010	0.027	0.052	0.051	0.051	0.098	0.106	0.104	0.107	0.161
150	0.010	0.011	0.011	0.026	0.052	0.052	0.054	0.096	0.105	0.103	0.106	0.161
200	0.012	0.010	0.011	0.027	0.051	0.051	0.053	0.098	0.107	0.103	0.105	0.162
250	0.011	0.011	0.011	0.029	0.054	0.051	0.054	0.097	0.106	0.103	0.105	0.164

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.17: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP5 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T$ and $\psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = \left[\sigma_{ut}^2 \quad -0.5\sigma_{ut}\sigma_{vt}; \quad -0.5\sigma_{ut}\sigma_{vt} \quad \sigma_{vt}^2\right]$ and $\sigma_{ut}^2 = \sigma_{vt}^2 = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 250$												
-5	0.008	0.003	0.002	0.007	0.046	0.021	0.012	0.036	0.098	0.050	0.034	0.068
-2.5	0.008	0.001	0.001	0.002	0.040	0.010	0.004	0.013	0.086	0.022	0.014	0.029
0	0.008	0.002	0.001	0.003	0.038	0.011	0.010	0.017	0.075	0.026	0.022	0.036
2.5	0.010	0.002	0.002	0.005	0.046	0.018	0.014	0.026	0.090	0.042	0.035	0.054
5	0.012	0.003	0.003	0.008	0.051	0.022	0.020	0.035	0.098	0.050	0.045	0.069
10	0.012	0.005	0.004	0.010	0.054	0.027	0.026	0.045	0.103	0.062	0.060	0.085
25	0.011	0.007	0.008	0.014	0.053	0.034	0.035	0.058	0.105	0.075	0.075	0.102
50	0.010	0.007	0.008	0.016	0.050	0.039	0.041	0.062	0.105	0.082	0.084	0.115
75	0.010	0.007	0.008	0.018	0.054	0.042	0.045	0.068	0.103	0.084	0.088	0.117
100	0.010	0.008	0.010	0.019	0.055	0.044	0.048	0.070	0.101	0.086	0.090	0.120
125	0.011	0.009	0.010	0.022	0.052	0.044	0.047	0.072	0.101	0.088	0.092	0.120
150	0.011	0.009	0.011	0.022	0.052	0.045	0.048	0.070	0.101	0.091	0.094	0.122
200	0.011	0.010	0.013	0.022	0.052	0.046	0.050	0.072	0.100	0.092	0.095	0.125
250	0.011	0.011	0.012	0.022	0.052	0.046	0.051	0.076	0.105	0.097	0.102	0.132
Right-sided tests - $T = 250$												
-5	0.010	0.014	0.006	0.028	0.049	0.069	0.042	0.097	0.094	0.131	0.091	0.169
-2.5	0.011	0.020	0.015	0.031	0.052	0.100	0.061	0.126	0.110	0.193	0.132	0.221
0	0.011	0.019	0.025	0.033	0.057	0.094	0.086	0.126	0.117	0.183	0.167	0.218
2.5	0.011	0.017	0.020	0.033	0.057	0.086	0.088	0.124	0.116	0.165	0.163	0.208
5	0.012	0.019	0.021	0.033	0.056	0.080	0.082	0.116	0.113	0.151	0.153	0.195
10	0.012	0.018	0.020	0.033	0.055	0.070	0.074	0.106	0.109	0.143	0.146	0.184
25	0.011	0.014	0.018	0.030	0.055	0.067	0.070	0.100	0.108	0.129	0.132	0.168
50	0.011	0.014	0.015	0.027	0.054	0.063	0.067	0.095	0.106	0.117	0.122	0.156
75	0.011	0.013	0.016	0.027	0.052	0.055	0.061	0.089	0.105	0.113	0.116	0.152
100	0.011	0.012	0.015	0.027	0.052	0.055	0.059	0.086	0.103	0.111	0.115	0.148
125	0.010	0.011	0.013	0.026	0.051	0.055	0.058	0.086	0.105	0.108	0.114	0.145
150	0.010	0.011	0.013	0.026	0.053	0.055	0.059	0.085	0.105	0.107	0.112	0.144
200	0.010	0.011	0.013	0.025	0.053	0.054	0.059	0.085	0.108	0.108	0.112	0.142
250	0.009	0.011	0.013	0.023	0.053	0.053	0.058	0.084	0.106	0.103	0.108	0.142
Two-sided tests - $T = 250$												
-5	0.009	0.008	0.004	0.020	0.050	0.043	0.023	0.075	0.098	0.090	0.053	0.133
-2.5	0.010	0.010	0.008	0.020	0.046	0.053	0.035	0.076	0.100	0.109	0.065	0.139
0	0.010	0.010	0.013	0.021	0.047	0.051	0.051	0.081	0.097	0.104	0.096	0.143
2.5	0.009	0.010	0.012	0.022	0.048	0.051	0.054	0.085	0.101	0.102	0.102	0.150
5	0.009	0.009	0.012	0.023	0.049	0.052	0.052	0.084	0.100	0.102	0.102	0.151
10	0.011	0.010	0.013	0.027	0.049	0.052	0.052	0.086	0.100	0.099	0.100	0.151
25	0.010	0.011	0.013	0.026	0.050	0.049	0.053	0.090	0.102	0.100	0.105	0.157
50	0.011	0.010	0.014	0.027	0.049	0.048	0.053	0.093	0.103	0.101	0.108	0.157
75	0.009	0.009	0.012	0.026	0.050	0.050	0.056	0.092	0.104	0.097	0.106	0.157
100	0.009	0.010	0.012	0.027	0.052	0.050	0.057	0.092	0.104	0.099	0.107	0.156
125	0.010	0.010	0.013	0.027	0.053	0.050	0.060	0.092	0.102	0.098	0.105	0.158
150	0.010	0.010	0.013	0.027	0.051	0.050	0.058	0.092	0.104	0.098	0.107	0.155
200	0.010	0.011	0.014	0.028	0.050	0.050	0.057	0.093	0.104	0.098	0.109	0.157
250	0.010	0.012	0.015	0.027	0.050	0.048	0.055	0.092	0.107	0.099	0.109	0.160

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 1000$												
-5	0.010	0.003	0.001	0.008	0.050	0.024	0.012	0.038	0.101	0.053	0.033	0.076
-2.5	0.010	0.001	0.001	0.003	0.047	0.011	0.006	0.017	0.093	0.026	0.015	0.035
0	0.010	0.002	0.001	0.004	0.045	0.013	0.010	0.020	0.086	0.032	0.025	0.042
2.5	0.012	0.003	0.002	0.006	0.052	0.019	0.016	0.029	0.097	0.044	0.037	0.058
5	0.012	0.004	0.004	0.008	0.053	0.023	0.018	0.036	0.101	0.052	0.047	0.070
10	0.012	0.006	0.005	0.011	0.052	0.028	0.025	0.045	0.103	0.061	0.056	0.086
25	0.012	0.008	0.008	0.016	0.053	0.033	0.033	0.056	0.102	0.072	0.071	0.102
50	0.012	0.008	0.008	0.018	0.051	0.038	0.038	0.064	0.104	0.081	0.081	0.112
75	0.011	0.008	0.008	0.018	0.053	0.041	0.040	0.069	0.104	0.086	0.087	0.117
100	0.011	0.008	0.008	0.019	0.053	0.042	0.043	0.070	0.105	0.088	0.089	0.123
125	0.011	0.008	0.009	0.019	0.054	0.045	0.045	0.073	0.105	0.091	0.091	0.127
150	0.011	0.009	0.009	0.020	0.053	0.045	0.045	0.075	0.108	0.096	0.095	0.129
200	0.010	0.009	0.009	0.023	0.052	0.046	0.046	0.077	0.108	0.099	0.098	0.129
250	0.012	0.010	0.010	0.024	0.054	0.048	0.048	0.079	0.106	0.097	0.098	0.131
Right-sided tests - $T = 1000$												
-5	0.010	0.015	0.006	0.029	0.052	0.069	0.040	0.098	0.097	0.133	0.092	0.174
-2.5	0.007	0.018	0.011	0.027	0.046	0.096	0.052	0.118	0.101	0.187	0.120	0.216
0	0.008	0.017	0.015	0.030	0.049	0.087	0.074	0.117	0.106	0.172	0.148	0.206
2.5	0.009	0.017	0.017	0.029	0.052	0.083	0.077	0.112	0.106	0.159	0.151	0.197
5	0.009	0.017	0.016	0.029	0.050	0.078	0.073	0.110	0.105	0.150	0.145	0.189
10	0.010	0.015	0.014	0.029	0.051	0.073	0.073	0.102	0.103	0.137	0.133	0.177
25	0.012	0.016	0.016	0.028	0.049	0.063	0.063	0.093	0.098	0.122	0.122	0.160
50	0.011	0.016	0.015	0.029	0.050	0.061	0.061	0.089	0.097	0.117	0.117	0.153
75	0.011	0.013	0.014	0.027	0.053	0.060	0.060	0.088	0.098	0.115	0.115	0.151
100	0.011	0.013	0.013	0.026	0.052	0.060	0.060	0.091	0.102	0.114	0.115	0.149
125	0.010	0.012	0.012	0.025	0.052	0.059	0.060	0.089	0.102	0.113	0.115	0.149
150	0.010	0.012	0.011	0.025	0.053	0.058	0.061	0.088	0.104	0.113	0.115	0.146
200	0.010	0.011	0.011	0.025	0.054	0.059	0.060	0.088	0.103	0.109	0.111	0.146
250	0.010	0.011	0.011	0.027	0.053	0.059	0.059	0.088	0.104	0.111	0.110	0.145
Two-sided tests- $T = 1000$												
-5	0.010	0.008	0.003	0.021	0.053	0.046	0.021	0.079	0.104	0.093	0.052	0.137
-2.5	0.006	0.009	0.006	0.015	0.044	0.051	0.028	0.070	0.094	0.106	0.058	0.134
0	0.007	0.009	0.009	0.018	0.043	0.049	0.044	0.073	0.089	0.100	0.084	0.137
2.5	0.010	0.010	0.009	0.019	0.045	0.050	0.045	0.078	0.093	0.101	0.093	0.141
5	0.010	0.010	0.010	0.021	0.046	0.049	0.046	0.080	0.095	0.101	0.091	0.146
10	0.011	0.011	0.011	0.023	0.047	0.051	0.048	0.084	0.097	0.101	0.098	0.147
25	0.013	0.012	0.012	0.028	0.049	0.050	0.049	0.084	0.096	0.096	0.095	0.148
50	0.012	0.011	0.011	0.029	0.052	0.052	0.051	0.089	0.098	0.099	0.099	0.153
75	0.011	0.011	0.011	0.027	0.052	0.050	0.051	0.092	0.102	0.102	0.101	0.157
100	0.011	0.011	0.011	0.027	0.050	0.050	0.050	0.094	0.103	0.101	0.103	0.161
125	0.010	0.010	0.011	0.027	0.050	0.049	0.050	0.095	0.104	0.103	0.105	0.162
150	0.009	0.011	0.010	0.027	0.052	0.050	0.052	0.096	0.104	0.103	0.106	0.163
200	0.010	0.010	0.010	0.027	0.053	0.054	0.056	0.098	0.105	0.104	0.106	0.165
250	0.010	0.010	0.009	0.028	0.056	0.056	0.057	0.098	0.106	0.107	0.107	0.168

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.18: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP5 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = [\sigma^2_{ut} \quad 0; \quad 0 \quad \sigma^2_{vt}]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

	1%				5%				10%			
c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.009	0.009	0.004	0.019	0.050	0.048	0.026	0.076	0.102	0.101	0.064	0.131
-2.5	0.009	0.010	0.004	0.015	0.048	0.050	0.023	0.065	0.100	0.100	0.056	0.121
0	0.010	0.010	0.011	0.017	0.046	0.048	0.039	0.064	0.094	0.098	0.077	0.120
2.5	0.010	0.011	0.011	0.019	0.046	0.049	0.043	0.067	0.095	0.095	0.087	0.123
5	0.010	0.010	0.010	0.020	0.051	0.051	0.047	0.073	0.097	0.096	0.091	0.123
10	0.009	0.011	0.011	0.022	0.052	0.052	0.050	0.076	0.101	0.100	0.097	0.126
25	0.010	0.011	0.013	0.023	0.051	0.049	0.052	0.076	0.098	0.097	0.096	0.129
50	0.010	0.010	0.011	0.022	0.052	0.050	0.053	0.077	0.100	0.099	0.101	0.128
75	0.010	0.009	0.011	0.021	0.052	0.051	0.055	0.078	0.101	0.098	0.102	0.133
100	0.008	0.010	0.011	0.021	0.053	0.051	0.055	0.078	0.100	0.098	0.102	0.132
125	0.010	0.009	0.013	0.023	0.052	0.049	0.053	0.079	0.101	0.098	0.102	0.131
150	0.010	0.010	0.013	0.024	0.052	0.050	0.053	0.079	0.099	0.098	0.101	0.133
200	0.011	0.011	0.013	0.024	0.052	0.052	0.055	0.077	0.099	0.096	0.099	0.132
250	0.010	0.010	0.013	0.023	0.052	0.051	0.055	0.078	0.100	0.097	0.100	0.133
Right-sided tests - $T = 250$												
-5	0.013	0.012	0.003	0.023	0.055	0.053	0.030	0.077	0.106	0.104	0.066	0.133
-2.5	0.011	0.012	0.005	0.019	0.051	0.052	0.028	0.067	0.100	0.106	0.060	0.124
0	0.009	0.010	0.009	0.020	0.051	0.053	0.043	0.070	0.097	0.101	0.086	0.125
2.5	0.011	0.011	0.011	0.020	0.052	0.052	0.046	0.073	0.099	0.100	0.095	0.126
5	0.011	0.012	0.011	0.022	0.052	0.052	0.049	0.073	0.099	0.098	0.094	0.127
10	0.013	0.012	0.012	0.022	0.050	0.050	0.050	0.075	0.101	0.096	0.095	0.131
25	0.012	0.011	0.013	0.024	0.053	0.050	0.052	0.079	0.104	0.100	0.102	0.133
50	0.011	0.011	0.013	0.024	0.051	0.050	0.053	0.079	0.100	0.098	0.102	0.136
75	0.010	0.010	0.011	0.023	0.051	0.050	0.053	0.078	0.101	0.100	0.103	0.136
100	0.010	0.011	0.012	0.023	0.051	0.049	0.053	0.080	0.102	0.099	0.105	0.136
125	0.010	0.010	0.012	0.023	0.052	0.049	0.053	0.079	0.101	0.098	0.103	0.136
150	0.010	0.010	0.012	0.024	0.052	0.048	0.053	0.078	0.103	0.099	0.105	0.137
200	0.010	0.011	0.013	0.026	0.054	0.051	0.055	0.081	0.103	0.100	0.104	0.137
250	0.010	0.012	0.013	0.026	0.054	0.051	0.056	0.080	0.105	0.100	0.105	0.138
Two-sided tests - $T = 250$												
-5	0.011	0.010	0.004	0.025	0.054	0.051	0.025	0.086	0.103	0.100	0.056	0.153
-2.5	0.010	0.011	0.005	0.020	0.050	0.053	0.024	0.073	0.096	0.103	0.051	0.132
0	0.009	0.011	0.012	0.019	0.048	0.052	0.045	0.075	0.094	0.100	0.081	0.134
2.5	0.010	0.011	0.012	0.023	0.049	0.050	0.047	0.083	0.098	0.100	0.090	0.141
5	0.010	0.011	0.011	0.024	0.051	0.053	0.050	0.088	0.103	0.102	0.097	0.146
10	0.010	0.011	0.012	0.026	0.053	0.053	0.052	0.090	0.103	0.102	0.100	0.150
25	0.010	0.011	0.013	0.028	0.052	0.049	0.054	0.090	0.102	0.100	0.104	0.155
50	0.010	0.010	0.014	0.026	0.051	0.049	0.056	0.094	0.103	0.098	0.106	0.155
75	0.010	0.010	0.012	0.026	0.049	0.048	0.055	0.093	0.104	0.100	0.108	0.156
100	0.010	0.009	0.012	0.027	0.049	0.048	0.056	0.096	0.104	0.100	0.107	0.158
125	0.009	0.009	0.012	0.027	0.051	0.049	0.056	0.096	0.103	0.097	0.106	0.158
150	0.009	0.011	0.013	0.029	0.052	0.051	0.057	0.095	0.104	0.096	0.106	0.156
200	0.010	0.011	0.016	0.029	0.054	0.052	0.061	0.095	0.105	0.101	0.110	0.158
250	0.010	0.011	0.015	0.029	0.054	0.053	0.061	0.096	0.105	0.102	0.110	0.158

	1%				5%				10%			
c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.012	0.010	0.003	0.022	0.052	0.050	0.026	0.075	0.100	0.099	0.062	0.130
-2.5	0.010	0.011	0.003	0.019	0.050	0.053	0.026	0.067	0.101	0.103	0.058	0.121
0	0.010	0.012	0.010	0.019	0.048	0.052	0.041	0.069	0.099	0.102	0.084	0.123
2.5	0.010	0.011	0.009	0.018	0.047	0.048	0.044	0.070	0.099	0.101	0.089	0.127
5	0.011	0.010	0.009	0.019	0.049	0.049	0.045	0.071	0.099	0.101	0.091	0.129
10	0.011	0.011	0.010	0.020	0.048	0.050	0.046	0.071	0.098	0.099	0.092	0.128
25	0.011	0.012	0.012	0.023	0.049	0.050	0.048	0.072	0.096	0.096	0.094	0.130
50	0.011	0.011	0.010	0.023	0.052	0.052	0.052	0.078	0.101	0.101	0.099	0.132
75	0.011	0.010	0.010	0.023	0.053	0.051	0.052	0.080	0.103	0.100	0.101	0.132
100	0.012	0.011	0.011	0.023	0.052	0.051	0.052	0.079	0.102	0.102	0.101	0.136
125	0.011	0.012	0.011	0.023	0.052	0.052	0.052	0.080	0.101	0.101	0.101	0.138
150	0.011	0.011	0.011	0.023	0.052	0.052	0.051	0.079	0.103	0.102	0.103	0.138
200	0.010	0.011	0.011	0.024	0.054	0.051	0.053	0.081	0.104	0.101	0.102	0.138
250	0.010	0.010	0.010	0.025	0.053	0.053	0.053	0.080	0.101	0.101	0.101	0.137
Right-sided tests - $T = 1000$												
-5	0.010	0.010	0.003	0.021	0.053	0.052	0.025	0.079	0.105	0.104	0.063	0.133
-2.5	0.008	0.009	0.004	0.015	0.048	0.049	0.021	0.064	0.097	0.101	0.055	0.118
0	0.008	0.010	0.007	0.016	0.043	0.044	0.034	0.063	0.092	0.096	0.073	0.118
2.5	0.009	0.009	0.008	0.018	0.046	0.045	0.038	0.064	0.094	0.097	0.085	0.125
5	0.008	0.009	0.007	0.017	0.046	0.046	0.039	0.069	0.098	0.100	0.089	0.128
10	0.009	0.009	0.008	0.019	0.048	0.045	0.044	0.073	0.096	0.098	0.093	0.128
25	0.010	0.010	0.009	0.021	0.047	0.048	0.046	0.074	0.100	0.099	0.097	0.132
50	0.010	0.010	0.010	0.022	0.050	0.049	0.049	0.080	0.100	0.100	0.099	0.133
75	0.011	0.011	0.011	0.022	0.050	0.051	0.051	0.077	0.099	0.100	0.100	0.136
100	0.011	0.011	0.011	0.022	0.051	0.051	0.051	0.078	0.100	0.099	0.099	0.137
125	0.010	0.010	0.011	0.024	0.050	0.050	0.051	0.080	0.101	0.101	0.103	0.136
150	0.010	0.011	0.011	0.024	0.051	0.051	0.052	0.081	0.102	0.101	0.102	0.138
200	0.010	0.010	0.010	0.024	0.052	0.051	0.052	0.080	0.102	0.102	0.103	0.138
250	0.010	0.010	0.011	0.023	0.052	0.051	0.053	0.080	0.101	0.102	0.102	0.138
Two-sided tests- $T = 1000$												
-5	0.012	0.011	0.002	0.024	0.053	0.050	0.019	0.091	0.104	0.101	0.051	0.154
-2.5	0.010	0.011	0.003	0.018	0.048	0.053	0.020	0.074	0.098	0.101	0.047	0.131
0	0.008	0.011	0.010	0.019	0.045	0.049	0.038	0.073	0.091	0.096	0.075	0.132
2.5	0.009	0.012	0.010	0.021	0.046	0.047	0.040	0.076	0.094	0.093	0.081	0.134
5	0.012	0.011	0.011	0.022	0.044	0.045	0.039	0.077	0.094	0.094	0.084	0.140
10	0.011	0.010	0.010	0.023	0.046	0.047	0.042	0.081	0.094	0.096	0.090	0.144
25	0.011	0.010	0.010	0.026	0.050	0.049	0.048	0.086	0.096	0.097	0.094	0.146
50	0.011	0.011	0.011	0.027	0.050	0.049	0.048	0.092	0.101	0.102	0.100	0.158
75	0.012	0.012	0.011	0.028	0.051	0.052	0.050	0.093	0.102	0.102	0.103	0.157
100	0.012	0.012	0.012	0.028	0.051	0.051	0.052	0.094	0.103	0.102	0.103	0.157
125	0.011	0.012	0.012	0.029	0.052	0.052	0.053	0.094	0.102	0.102	0.103	0.160
150	0.011	0.011	0.012	0.028	0.053	0.051	0.053	0.096	0.103	0.103	0.102	0.160
200	0.011	0.010	0.011	0.027	0.053	0.053	0.054	0.096	0.104	0.103	0.105	0.161
250	0.009	0.010	0.010	0.028	0.052	0.052	0.053	0.098	0.104	0.104	0.105	0.160

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (12) and (14) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.19: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP6 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = [\sigma^2_{ut} \quad -0.95\sigma_{ut}\sigma_{vt}; \quad -0.95\sigma_{ut}\sigma_{vt} \quad \sigma^2_{vt}]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 1/4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.007	0.000	0.000	0.000	0.043	0.000	0.002	0.000	0.095	0.001	0.009	0.001
-2.5	0.008	0.000	0.000	0.000	0.071	0.000	0.000	0.000	0.152	0.000	0.000	0.000
0	0.005	0.000	0.000	0.000	0.014	0.000	0.000	0.000	0.024	0.000	0.000	0.000
2.5	0.018	0.000	0.000	0.000	0.048	0.001	0.001	0.001	0.080	0.006	0.005	0.006
5	0.028	0.001	0.001	0.001	0.066	0.010	0.008	0.012	0.107	0.024	0.021	0.027
10	0.025	0.003	0.003	0.006	0.069	0.020	0.019	0.027	0.111	0.042	0.041	0.054
25	0.017	0.006	0.007	0.012	0.062	0.029	0.031	0.046	0.110	0.062	0.062	0.082
50	0.013	0.007	0.008	0.014	0.057	0.036	0.038	0.056	0.111	0.072	0.075	0.103
75	0.013	0.006	0.008	0.016	0.055	0.037	0.039	0.061	0.113	0.080	0.083	0.113
100	0.011	0.007	0.008	0.018	0.057	0.040	0.043	0.068	0.109	0.084	0.088	0.114
125	0.012	0.008	0.008	0.019	0.055	0.043	0.046	0.068	0.110	0.087	0.090	0.117
150	0.012	0.008	0.009	0.020	0.057	0.045	0.047	0.070	0.109	0.092	0.094	0.119
200	0.013	0.009	0.011	0.021	0.059	0.049	0.052	0.074	0.107	0.092	0.097	0.124
250	0.011	0.009	0.011	0.022	0.060	0.051	0.056	0.077	0.107	0.096	0.101	0.126
Right-sided tests - $T = 250$												
-5	0.004	0.010	0.051	0.009	0.034	0.062	0.172	0.036	0.078	0.148	0.289	0.088
-2.5	0.008	0.016	0.038	0.023	0.035	0.078	0.214	0.089	0.071	0.213	0.441	0.205
0	0.009	0.019	0.029	0.032	0.055	0.104	0.138	0.142	0.1167	0.220	0.273	0.266
2.5	0.011	0.021	0.028	0.035	0.060	0.111	0.121	0.146	0.126	0.219	0.234	0.254
5	0.012	0.023	0.027	0.034	0.062	0.101	0.109	0.135	0.123	0.198	0.207	0.234
10	0.013	0.021	0.025	0.035	0.060	0.090	0.096	0.121	0.115	0.171	0.177	0.208
25	0.013	0.017	0.021	0.032	0.057	0.075	0.079	0.106	0.108	0.141	0.146	0.177
50	0.012	0.014	0.017	0.029	0.054	0.066	0.072	0.099	0.108	0.128	0.134	0.167
75	0.011	0.015	0.017	0.029	0.054	0.062	0.067	0.094	0.109	0.122	0.128	0.158
100	0.011	0.014	0.017	0.028	0.054	0.060	0.064	0.092	0.107	0.117	0.124	0.154
125	0.011	0.014	0.016	0.027	0.055	0.059	0.063	0.089	0.107	0.115	0.120	0.152
150	0.012	0.014	0.016	0.026	0.055	0.059	0.064	0.089	0.107	0.113	0.117	0.149
200	0.011	0.012	0.015	0.025	0.055	0.054	0.059	0.085	0.110	0.110	0.114	0.147
250	0.011	0.012	0.014	0.025	0.057	0.053	0.058	0.081	0.108	0.104	0.110	0.142
Two-sided tests - $T = 250$												
-5	0.004	0.005	0.033	0.006	0.033	0.028	0.103	0.017	0.077	0.062	0.174	0.036
-2.5	0.007	0.008	0.020	0.013	0.031	0.036	0.101	0.046	0.065	0.078	0.214	0.089
0	0.008	0.009	0.014	0.016	0.043	0.051	0.069	0.074	0.096	0.106	0.138	0.142
2.5	0.009	0.011	0.015	0.018	0.048	0.055	0.066	0.081	0.101	0.110	0.122	0.148
5	0.009	0.012	0.016	0.021	0.048	0.056	0.064	0.082	0.102	0.109	0.117	0.147
10	0.011	0.013	0.015	0.024	0.054	0.059	0.063	0.084	0.103	0.111	0.115	0.148
25	0.011	0.011	0.015	0.025	0.054	0.053	0.059	0.088	0.106	0.104	0.110	0.152
50	0.012	0.011	0.014	0.024	0.051	0.050	0.054	0.089	0.107	0.102	0.110	0.155
75	0.012	0.010	0.014	0.026	0.054	0.049	0.057	0.091	0.106	0.098	0.106	0.155
100	0.012	0.011	0.013	0.028	0.054	0.051	0.058	0.090	0.108	0.100	0.107	0.160
125	0.011	0.010	0.013	0.029	0.054	0.051	0.058	0.094	0.109	0.101	0.108	0.157
150	0.012	0.011	0.014	0.027	0.055	0.052	0.059	0.094	0.1081	0.103	0.111	0.159
200	0.010	0.011	0.014	0.029	0.055	0.053	0.060	0.095	0.110	0.103	0.111	0.158
250	0.011	0.011	0.015	0.027	0.056	0.054	0.061	0.098	0.113	0.103	0.114	0.157

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.009	0.000	0.000	0.000	0.043	0.001	0.005	0.000	0.093	0.004	0.010	0.001
-2.5	0.012	0.000	0.000	0.000	0.072	0.000	0.000	0.000	0.155	0.000	0.000	0.000
0	0.004	0.000	0.000	0.000	0.015	0.000	0.000	0.000	0.024	0.000	0.000	0.000
2.5	0.017	0.000	0.000	0.000	0.048	0.003	0.002	0.003	0.082	0.007	0.007	0.008
5	0.025	0.002	0.001	0.001	0.068	0.010	0.008	0.011	0.112	0.023	0.021	0.029
10	0.023	0.003	0.003	0.006	0.067	0.019	0.017	0.026	0.118	0.043	0.041	0.054
25	0.018	0.006	0.005	0.013	0.063	0.029	0.029	0.047	0.111	0.064	0.064	0.087
50	0.014	0.007	0.007	0.017	0.060	0.037	0.036	0.060	0.111	0.075	0.074	0.102
75	0.012	0.008	0.007	0.017	0.059	0.039	0.039	0.065	0.110	0.081	0.082	0.110
100	0.012	0.007	0.007	0.017	0.056	0.040	0.041	0.066	0.110	0.085	0.085	0.116
125	0.010	0.007	0.007	0.018	0.054	0.039	0.040	0.068	0.109	0.085	0.088	0.120
150	0.011	0.008	0.007	0.019	0.054	0.040	0.042	0.067	0.108	0.087	0.087	0.122
200	0.011	0.009	0.008	0.020	0.053	0.043	0.043	0.069	0.105	0.090	0.090	0.123
250	0.011	0.008	0.008	0.020	0.053	0.044	0.044	0.073	0.107	0.092	0.092	0.123
Right-sided tests - $T = 1000$												
-5	0.005	0.012	0.051	0.009	0.039	0.068	0.174	0.038	0.089	0.156	0.290	0.092
-2.5	0.005	0.012	0.031	0.017	0.027	0.072	0.208	0.080	0.060	0.197	0.434	0.187
0	0.008	0.016	0.021	0.025	0.043	0.095	0.118	0.126	0.100	0.208	0.257	0.254
2.5	0.009	0.019	0.022	0.028	0.052	0.100	0.109	0.131	0.110	0.209	0.217	0.248
5	0.009	0.020	0.022	0.032	0.054	0.095	0.097	0.124	0.108	0.190	0.189	0.225
10	0.011	0.020	0.020	0.031	0.052	0.086	0.086	0.114	0.107	0.165	0.164	0.198
25	0.011	0.016	0.016	0.027	0.050	0.074	0.075	0.102	0.102	0.137	0.139	0.174
50	0.010	0.014	0.015	0.027	0.051	0.068	0.067	0.093	0.097	0.123	0.124	0.158
75	0.009	0.012	0.013	0.026	0.049	0.064	0.062	0.093	0.099	0.119	0.120	0.152
100	0.009	0.012	0.013	0.025	0.049	0.059	0.060	0.088	0.098	0.117	0.120	0.151
125	0.010	0.013	0.013	0.025	0.049	0.057	0.058	0.087	0.098	0.115	0.117	0.151
150	0.010	0.013	0.013	0.025	0.047	0.056	0.056	0.085	0.096	0.114	0.115	0.148
200	0.011	0.012	0.013	0.025	0.047	0.054	0.055	0.083	0.099	0.110	0.113	0.144
250	0.010	0.012	0.013	0.024	0.049	0.055	0.056	0.085	0.098	0.107	0.109	0.143
Two-sided tests- $T = 1000$												
-5	0.005	0.006	0.031	0.006	0.039	0.033	0.102	0.018	0.090	0.068	0.179	0.038
-2.5	0.005	0.005	0.013	0.010	0.024	0.030	0.088	0.038	0.055	0.071	0.208	0.080
0	0.006	0.007	0.012	0.013	0.034	0.042	0.056	0.063	0.080	0.096	0.118	0.126
2.5	0.006	0.009	0.010	0.016	0.038	0.048	0.054	0.071	0.085	0.103	0.111	0.134
5	0.007	0.011	0.012	0.018	0.044	0.053	0.055	0.076	0.090	0.106	0.105	0.136
10	0.009	0.013	0.013	0.020	0.049	0.054	0.054	0.078	0.093	0.105	0.104	0.140
25	0.011	0.011	0.012	0.024	0.049	0.052	0.052	0.086	0.100	0.104	0.104	0.148
50	0.010	0.010	0.010	0.024	0.051	0.051	0.051	0.091	0.102	0.103	0.104	0.153
75	0.010	0.010	0.010	0.024	0.049	0.050	0.051	0.089	0.102	0.100	0.101	0.158
100	0.010	0.011	0.010	0.024	0.049	0.048	0.048	0.088	0.100	0.099	0.101	0.154
125	0.009	0.010	0.010	0.025	0.049	0.048	0.049	0.087	0.098	0.096	0.099	0.154
150	0.010	0.010	0.010	0.025	0.050	0.049	0.049	0.087	0.097	0.094	0.097	0.152
200	0.011	0.010	0.010	0.026	0.050	0.048	0.049	0.089	0.098	0.095	0.098	0.152
250	0.010	0.011	0.011	0.027	0.049	0.047	0.048	0.089	0.100	0.097	0.100	0.157

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (12) and (14) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.20: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP6 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = [\sigma^2_{ut} \quad -0.9\sigma_{ut}\sigma_{vt}; \quad -0.9\sigma_{ut}\sigma_{vt} \quad \sigma^2_{vt}]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 1/4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

	1%				5%				10%			
c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.007	0.000	0.001	0.000	0.044	0.001	0.005	0.000	0.095	0.003	0.014	0.001
-2.5	0.010	0.000	0.000	0.000	0.067	0.000	0.000	0.000	0.142	0.000	0.000	0.000
0	0.005	0.000	0.000	0.000	0.014	0.000	0.000	0.000	0.024	0.000	0.000	0.000
2.5	0.016	0.000	0.000	0.000	0.045	0.002	0.002	0.002	0.079	0.008	0.007	0.009
5	0.024	0.002	0.001	0.001	0.064	0.011	0.009	0.012	0.108	0.025	0.023	0.029
10	0.023	0.004	0.003	0.006	0.067	0.021	0.020	0.028	0.112	0.045	0.043	0.058
25	0.016	0.007	0.007	0.012	0.060	0.031	0.031	0.047	0.109	0.063	0.063	0.084
50	0.013	0.007	0.007	0.015	0.056	0.037	0.039	0.058	0.108	0.073	0.077	0.103
75	0.012	0.007	0.008	0.016	0.057	0.038	0.040	0.062	0.110	0.082	0.084	0.112
100	0.012	0.008	0.008	0.016	0.055	0.041	0.044	0.067	0.110	0.086	0.088	0.115
125	0.012	0.007	0.009	0.019	0.055	0.044	0.047	0.068	0.109	0.088	0.092	0.119
150	0.012	0.008	0.010	0.019	0.055	0.045	0.048	0.071	0.108	0.090	0.094	0.122
200	0.011	0.010	0.012	0.022	0.057	0.048	0.053	0.075	0.107	0.095	0.099	0.125
250	0.011	0.009	0.012	0.023	0.058	0.052	0.056	0.076	0.105	0.096	0.101	0.128
Right-sided tests - $T = 250$												
-5	0.004	0.010	0.064	0.008	0.033	0.063	0.188	0.034	0.076	0.154	0.302	0.092
-2.5	0.008	0.015	0.053	0.022	0.034	0.081	0.242	0.088	0.072	0.222	0.454	0.213
0	0.010	0.020	0.030	0.032	0.057	0.105	0.138	0.137	0.117	0.216	0.279	0.265
2.5	0.012	0.021	0.028	0.035	0.063	0.108	0.122	0.142	0.127	0.212	0.230	0.251
5	0.013	0.023	0.026	0.035	0.061	0.098	0.105	0.132	0.120	0.192	0.202	0.231
10	0.013	0.021	0.024	0.036	0.058	0.087	0.094	0.119	0.113	0.169	0.173	0.204
25	0.012	0.018	0.020	0.031	0.054	0.073	0.079	0.105	0.108	0.140	0.144	0.175
50	0.010	0.014	0.017	0.028	0.055	0.066	0.071	0.097	0.106	0.124	0.131	0.167
75	0.011	0.014	0.016	0.028	0.056	0.063	0.069	0.094	0.107	0.118	0.125	0.155
100	0.011	0.014	0.016	0.028	0.057	0.060	0.066	0.092	0.106	0.117	0.123	0.154
125	0.012	0.014	0.016	0.027	0.055	0.057	0.064	0.091	0.109	0.115	0.119	0.154
150	0.011	0.014	0.016	0.027	0.056	0.059	0.064	0.089	0.108	0.114	0.119	0.151
200	0.012	0.012	0.016	0.026	0.057	0.057	0.062	0.086	0.107	0.108	0.114	0.148
250	0.010	0.011	0.013	0.025	0.057	0.054	0.059	0.080	0.109	0.104	0.109	0.143
Two-sided tests - $T = 250$												
-5	0.004	0.005	0.039	0.005	0.033	0.028	0.120	0.017	0.075	0.064	0.193	0.034
-2.5	0.007	0.008	0.027	0.013	0.031	0.037	0.129	0.046	0.067	0.080	0.242	0.088
0	0.008	0.010	0.016	0.016	0.046	0.050	0.072	0.077	0.097	0.104	0.138	0.137
2.5	0.009	0.011	0.016	0.020	0.049	0.055	0.067	0.080	0.098	0.109	0.124	0.144
5	0.010	0.011	0.016	0.021	0.048	0.057	0.062	0.083	0.100	0.107	0.114	0.144
10	0.011	0.012	0.015	0.024	0.052	0.056	0.061	0.084	0.102	0.108	0.114	0.147
25	0.012	0.012	0.015	0.025	0.053	0.054	0.058	0.088	0.106	0.105	0.110	0.152
50	0.011	0.011	0.015	0.025	0.053	0.050	0.055	0.090	0.106	0.102	0.110	0.155
75	0.011	0.012	0.014	0.025	0.053	0.048	0.056	0.090	0.104	0.099	0.108	0.156
100	0.010	0.011	0.013	0.027	0.052	0.050	0.057	0.092	0.108	0.101	0.110	0.159
125	0.011	0.010	0.013	0.028	0.054	0.051	0.057	0.093	0.109	0.102	0.111	0.159
150	0.010	0.010	0.013	0.027	0.054	0.051	0.058	0.094	0.108	0.102	0.113	0.160
200	0.010	0.010	0.014	0.029	0.057	0.052	0.061	0.095	0.112	0.104	0.114	0.161
250	0.010	0.011	0.014	0.028	0.056	0.052	0.060	0.097	0.113	0.106	0.115	0.156

	1%				5%				10%			
c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.010	0.000	0.001	0.000	0.043	0.001	0.006	0.000	0.094	0.004	0.016	0.001
-2.5	0.012	0.000	0.000	0.000	0.069	0.000	0.000	0.000	0.142	0.000	0.000	0.000
0	0.004	0.000	0.000	0.000	0.014	0.000	0.000	0.000	0.025	0.000	0.000	0.000
2.5	0.015	0.000	0.000	0.000	0.046	0.003	0.002	0.004	0.082	0.009	0.008	0.011
5	0.022	0.001	0.001	0.002	0.063	0.011	0.009	0.013	0.111	0.025	0.024	0.031
10	0.020	0.004	0.003	0.007	0.064	0.019	0.018	0.028	0.117	0.044	0.041	0.056
25	0.017	0.006	0.005	0.013	0.060	0.031	0.029	0.049	0.112	0.065	0.065	0.089
50	0.014	0.007	0.006	0.017	0.058	0.037	0.036	0.058	0.111	0.077	0.076	0.105
75	0.012	0.007	0.007	0.016	0.058	0.039	0.039	0.064	0.110	0.083	0.081	0.113
100	0.012	0.008	0.007	0.016	0.056	0.041	0.041	0.067	0.110	0.085	0.086	0.116
125	0.012	0.008	0.007	0.018	0.056	0.042	0.042	0.067	0.108	0.087	0.086	0.119
150	0.010	0.008	0.007	0.018	0.054	0.042	0.042	0.070	0.107	0.088	0.088	0.121
200	0.011	0.008	0.008	0.019	0.053	0.042	0.043	0.070	0.106	0.089	0.091	0.125
250	0.011	0.008	0.009	0.019	0.055	0.044	0.044	0.073	0.106	0.093	0.093	0.125
Right-sided tests - $T = 1000$												
-5	0.005	0.013	0.061	0.009	0.040	0.071	0.190	0.040	0.088	0.162	0.305	0.096
-2.5	0.005	0.012	0.044	0.016	0.027	0.073	0.236	0.080	0.064	0.208	0.447	0.193
0	0.008	0.016	0.024	0.026	0.045	0.094	0.122	0.127	0.101	0.206	0.259	0.256
2.5	0.009	0.019	0.023	0.030	0.052	0.097	0.105	0.128	0.110	0.204	0.207	0.241
5	0.010	0.020	0.021	0.032	0.055	0.093	0.095	0.121	0.108	0.184	0.182	0.218
10	0.011	0.020	0.020	0.032	0.051	0.085	0.083	0.110	0.104	0.159	0.158	0.195
25	0.011	0.016	0.015	0.028	0.049	0.073	0.073	0.101	0.100	0.136	0.137	0.170
50	0.008	0.014	0.013	0.026	0.051	0.065	0.066	0.096	0.100	0.123	0.125	0.156
75	0.010	0.012	0.012	0.024	0.049	0.062	0.064	0.093	0.099	0.119	0.121	0.155
100	0.009	0.013	0.012	0.024	0.047	0.060	0.060	0.091	0.100	0.117	0.118	0.153
125	0.009	0.013	0.012	0.024	0.047	0.057	0.058	0.088	0.098	0.114	0.113	0.150
150	0.010	0.013	0.013	0.024	0.048	0.056	0.057	0.086	0.099	0.113	0.114	0.147
200	0.010	0.013	0.013	0.024	0.048	0.055	0.056	0.084	0.099	0.110	0.113	0.145
250	0.010	0.011	0.013	0.024	0.048	0.056	0.055	0.085	0.097	0.107	0.108	0.144
Two-sided tests- $T = 1000$												
-5	0.005	0.006	0.038	0.006	0.040	0.033	0.116	0.019	0.089	0.072	0.196	0.040
-2.5	0.004	0.006	0.020	0.010	0.024	0.030	0.117	0.037	0.058	0.072	0.236	0.080
0	0.005	0.008	0.012	0.014	0.037	0.044	0.059	0.063	0.081	0.093	0.122	0.127
2.5	0.006	0.008	0.010	0.016	0.040	0.050	0.055	0.070	0.088	0.101	0.107	0.132
5	0.008	0.011	0.012	0.018	0.045	0.052	0.053	0.075	0.090	0.103	0.104	0.134
10	0.011	0.013	0.013	0.021	0.047	0.053	0.053	0.079	0.094	0.103	0.101	0.138
25	0.012	0.012	0.012	0.024	0.049	0.053	0.052	0.083	0.098	0.102	0.103	0.149
50	0.010	0.010	0.011	0.025	0.049	0.053	0.053	0.090	0.101	0.102	0.103	0.154
75	0.010	0.011	0.011	0.024	0.048	0.049	0.050	0.088	0.102	0.101	0.103	0.157
100	0.010	0.010	0.011	0.025	0.048	0.048	0.048	0.089	0.100	0.100	0.101	0.158
125	0.009	0.010	0.010	0.025	0.049	0.047	0.048	0.088	0.097	0.098	0.100	0.155
150	0.010	0.010	0.010	0.024	0.048	0.046	0.048	0.086	0.098	0.099	0.099	0.156
200	0.010	0.010	0.011	0.025	0.048	0.048	0.049	0.088	0.098	0.096	0.099	0.155
250	0.011	0.010	0.011	0.025	0.048	0.047	0.049	0.089	0.098	0.097	0.099	0.158

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (12) and (14) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.21: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP6 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = [\sigma^2_{ut} \quad -0.5\sigma_{ut}\sigma_{vt}; \quad -0.5\sigma_{ut}\sigma_{vt} \quad \sigma^2_{vt}]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 1/4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 250$												
-5	0.009	0.001	0.054	0.000	0.045	0.011	0.096	0.004	0.097	0.030	0.127	0.017
-2.5	0.009	0.000	0.020	0.000	0.045	0.002	0.033	0.002	0.101	0.009	0.044	0.008
0	0.004	0.000	0.003	0.000	0.018	0.004	0.013	0.006	0.040	0.012	0.022	0.016
2.5	0.007	0.002	0.002	0.003	0.039	0.014	0.015	0.017	0.076	0.034	0.034	0.041
5	0.010	0.003	0.003	0.005	0.050	0.022	0.022	0.029	0.093	0.051	0.050	0.064
10	0.013	0.006	0.006	0.010	0.053	0.031	0.030	0.044	0.100	0.065	0.065	0.082
25	0.013	0.008	0.009	0.015	0.053	0.037	0.039	0.056	0.102	0.077	0.078	0.103
50	0.011	0.008	0.008	0.018	0.052	0.040	0.042	0.063	0.104	0.085	0.087	0.115
75	0.011	0.009	0.010	0.019	0.051	0.040	0.043	0.069	0.105	0.090	0.093	0.120
100	0.011	0.008	0.010	0.020	0.052	0.043	0.047	0.071	0.104	0.090	0.096	0.124
125	0.011	0.009	0.011	0.022	0.051	0.044	0.047	0.073	0.107	0.094	0.097	0.127
150	0.010	0.008	0.0107	0.022	0.052	0.0468	0.049	0.074	0.105	0.095	0.100	0.130
200	0.011	0.011	0.0129	0.023	0.053	0.049	0.051	0.076	0.106	0.097	0.102	0.133
250	0.011	0.010	0.0135	0.023	0.054	0.047	0.053	0.078	0.105	0.097	0.102	0.133
Right-sided tests - $T = 250$												
-5	0.005	0.012	0.186	0.006	0.040	0.073	0.287	0.038	0.087	0.146	0.356	0.097
-2.5	0.006	0.017	0.188	0.019	0.037	0.101	0.288	0.097	0.088	0.207	0.357	0.193
0	0.012	0.021	0.057	0.032	0.062	0.097	0.162	0.127	0.127	0.197	0.261	0.230
2.5	0.011	0.020	0.024	0.031	0.059	0.089	0.094	0.114	0.118	0.171	0.177	0.199
5	0.012	0.018	0.021	0.028	0.057	0.079	0.083	0.104	0.112	0.153	0.156	0.181
10	0.011	0.016	0.017	0.027	0.054	0.073	0.076	0.096	0.105	0.134	0.137	0.166
25	0.011	0.013	0.016	0.027	0.051	0.059	0.063	0.086	0.099	0.118	0.123	0.153
50	0.011	0.013	0.015	0.025	0.051	0.055	0.061	0.084	0.100	0.110	0.113	0.146
75	0.010	0.012	0.014	0.024	0.052	0.057	0.061	0.086	0.102	0.110	0.115	0.146
100	0.010	0.012	0.014	0.025	0.052	0.054	0.060	0.088	0.105	0.111	0.117	0.149
125	0.010	0.012	0.014	0.025	0.051	0.054	0.059	0.086	0.108	0.108	0.114	0.149
150	0.011	0.011	0.014	0.026	0.052	0.054	0.058	0.085	0.105	0.106	0.111	0.147
200	0.011	0.012	0.015	0.026	0.054	0.054	0.058	0.084	0.108	0.105	0.110	0.143
250	0.011	0.012	0.015	0.024	0.055	0.054	0.059	0.083	0.106	0.104	0.110	0.139
Two-sided tests - $T = 250$												
-5	0.004	0.006	0.203	0.003	0.039	0.039	0.306	0.016	0.087	0.085	0.382	0.042
-2.5	0.005	0.008	0.177	0.010	0.033	0.048	0.265	0.046	0.077	0.103	0.321	0.099
0	0.009	0.010	0.044	0.019	0.047	0.052	0.108	0.074	0.096	0.100	0.175	0.133
2.5	0.009	0.010	0.015	0.019	0.045	0.051	0.060	0.073	0.096	0.102	0.109	0.131
5	0.009	0.010	0.012	0.020	0.046	0.051	0.053	0.073	0.096	0.101	0.105	0.134
10	0.009	0.011	0.012	0.023	0.050	0.053	0.057	0.081	0.100	0.103	0.106	0.139
25	0.012	0.011	0.012	0.026	0.050	0.051	0.056	0.084	0.100	0.095	0.101	0.142
50	0.011	0.010	0.012	0.026	0.052	0.050	0.055	0.089	0.101	0.096	0.103	0.146
75	0.010	0.010	0.012	0.025	0.051	0.048	0.054	0.091	0.101	0.096	0.103	0.155
100	0.010	0.009	0.012	0.027	0.049	0.048	0.054	0.089	0.103	0.098	0.107	0.159
125	0.010	0.010	0.012	0.027	0.051	0.050	0.056	0.087	0.100	0.097	0.106	0.160
150	0.010	0.009	0.013	0.028	0.053	0.052	0.059	0.091	0.102	0.098	0.107	0.159
200	0.010	0.011	0.015	0.030	0.054	0.052	0.060	0.095	0.105	0.101	0.109	0.160
250	0.011	0.012	0.017	0.030	0.053	0.052	0.058	0.095	0.107	0.101	0.112	0.161

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
	1%				5%				10%			
Left-sided tests - $T = 1000$												
-5	0.010	0.001	0.053	0.000	0.048	0.013	0.100	0.004	0.098	0.030	0.136	0.017
-2.5	0.008	0.000	0.026	0.000	0.048	0.003	0.042	0.002	0.100	0.007	0.055	0.006
0	0.004	0.000	0.004	0.001	0.020	0.004	0.014	0.006	0.043	0.013	0.024	0.016
2.5	0.010	0.002	0.002	0.003	0.041	0.015	0.016	0.022	0.081	0.040	0.037	0.048
5	0.012	0.004	0.003	0.006	0.049	0.025	0.022	0.033	0.095	0.053	0.050	0.064
10	0.013	0.005	0.005	0.011	0.052	0.030	0.028	0.042	0.100	0.064	0.060	0.082
25	0.012	0.008	0.007	0.016	0.055	0.039	0.038	0.059	0.107	0.080	0.078	0.108
50	0.011	0.008	0.008	0.019	0.054	0.044	0.044	0.067	0.105	0.088	0.087	0.116
75	0.011	0.009	0.009	0.020	0.054	0.044	0.045	0.070	0.105	0.089	0.090	0.122
100	0.011	0.009	0.008	0.020	0.053	0.046	0.046	0.073	0.105	0.092	0.092	0.124
125	0.010	0.009	0.009	0.020	0.053	0.046	0.046	0.074	0.105	0.095	0.094	0.125
150	0.011	0.009	0.008	0.021	0.054	0.047	0.048	0.075	0.104	0.094	0.095	0.128
200	0.011	0.009	0.009	0.021	0.055	0.048	0.049	0.078	0.104	0.096	0.098	0.127
250	0.011	0.010	0.009	0.022	0.053	0.049	0.050	0.076	0.105	0.096	0.096	0.130
Right-sided tests - $T = 1000$												
-5	0.008	0.016	0.195	0.006	0.048	0.078	0.307	0.044	0.093	0.152	0.380	0.099
-2.5	0.006	0.019	0.227	0.018	0.038	0.098	0.323	0.092	0.082	0.204	0.389	0.189
0	0.009	0.018	0.054	0.029	0.057	0.094	0.149	0.122	0.122	0.192	0.252	0.229
2.5	0.010	0.018	0.021	0.029	0.054	0.085	0.087	0.107	0.111	0.165	0.165	0.190
5	0.011	0.017	0.017	0.027	0.052	0.079	0.076	0.098	0.103	0.147	0.143	0.173
10	0.010	0.015	0.014	0.027	0.052	0.069	0.066	0.094	0.100	0.134	0.130	0.162
25	0.010	0.014	0.013	0.026	0.050	0.062	0.060	0.086	0.098	0.120	0.120	0.152
50	0.009	0.012	0.012	0.026	0.049	0.058	0.058	0.086	0.099	0.112	0.112	0.147
75	0.009	0.011	0.012	0.024	0.049	0.055	0.057	0.085	0.101	0.110	0.112	0.145
100	0.010	0.012	0.012	0.023	0.049	0.056	0.056	0.084	0.099	0.109	0.109	0.143
125	0.010	0.012	0.012	0.024	0.048	0.055	0.055	0.085	0.099	0.108	0.108	0.142
150	0.009	0.012	0.012	0.024	0.048	0.055	0.055	0.085	0.101	0.109	0.109	0.143
200	0.010	0.011	0.011	0.022	0.050	0.054	0.054	0.084	0.100	0.107	0.107	0.144
250	0.010	0.011	0.011	0.022	0.049	0.052	0.054	0.084	0.101	0.106	0.109	0.144
Two-sided tests- $T = 1000$												
-5	0.007	0.008	0.205	0.003	0.047	0.044	0.327	0.018	0.094	0.091	0.407	0.048
-2.5	0.005	0.008	0.225	0.009	0.034	0.047	0.309	0.045	0.074	0.102	0.365	0.095
0	0.007	0.009	0.038	0.015	0.044	0.047	0.103	0.068	0.092	0.099	0.162	0.129
2.5	0.009	0.010	0.013	0.018	0.043	0.052	0.054	0.071	0.091	0.100	0.103	0.128
5	0.009	0.011	0.011	0.018	0.048	0.052	0.051	0.073	0.095	0.102	0.098	0.131
10	0.010	0.011	0.010	0.020	0.048	0.053	0.050	0.077	0.094	0.098	0.094	0.136
25	0.010	0.011	0.009	0.024	0.050	0.052	0.051	0.084	0.100	0.099	0.098	0.145
50	0.010	0.010	0.010	0.025	0.052	0.050	0.051	0.089	0.102	0.102	0.101	0.153
75	0.010	0.010	0.009	0.026	0.051	0.050	0.051	0.089	0.102	0.100	0.102	0.155
100	0.009	0.010	0.010	0.026	0.050	0.049	0.050	0.091	0.102	0.100	0.102	0.157
125	0.010	0.009	0.010	0.025	0.050	0.050	0.050	0.090	0.101	0.100	0.101	0.159
150	0.011	0.009	0.011	0.026	0.051	0.050	0.051	0.091	0.102	0.100	0.103	0.160
200	0.010	0.010	0.011	0.026	0.049	0.049	0.051	0.091	0.102	0.102	0.103	0.162
250	0.011	0.011	0.011	0.026	0.048	0.050	0.051	0.092	0.100	0.103	0.104	0.160

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.22: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP6 (Unconditional Heteroskedasticity):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$, and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma_t})$, with $\mathbf{\Sigma}_t = \left[\sigma^2_{ut} \quad 0; \quad 0 \quad \sigma^2_{vt}\right]$ and $\sigma^2_{ut} = \sigma^2_{vt} = 1\mathbb{I}(t \leq \lfloor 0.5T \rfloor) + 1/4\mathbb{I}(t > \lfloor 0.5T \rfloor)$.

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.010	0.012	0.108	0.004	0.046	0.049	0.150	0.025	0.095	0.099	0.175	0.063
-2.5	0.008	0.012	0.027	0.010	0.040	0.051	0.045	0.046	0.086	0.099	0.059	0.093
0	0.010	0.011	0.032	0.017	0.048	0.047	0.070	0.061	0.092	0.094	0.114	0.110
2.5	0.009	0.009	0.013	0.014	0.045	0.047	0.050	0.061	0.093	0.096	0.097	0.113
5	0.010	0.010	0.011	0.016	0.045	0.046	0.048	0.063	0.095	0.096	0.094	0.115
10	0.011	0.010	0.012	0.019	0.048	0.048	0.049	0.065	0.093	0.094	0.094	0.117
25	0.012	0.011	0.012	0.022	0.050	0.050	0.052	0.073	0.097	0.096	0.098	0.124
50	0.011	0.010	0.011	0.021	0.049	0.047	0.051	0.074	0.098	0.098	0.101	0.131
75	0.010	0.010	0.011	0.023	0.050	0.047	0.052	0.074	0.098	0.097	0.101	0.132
100	0.011	0.011	0.012	0.023	0.050	0.048	0.052	0.075	0.100	0.096	0.102	0.133
125	0.010	0.010	0.013	0.024	0.052	0.050	0.053	0.076	0.102	0.100	0.104	0.139
150	0.010	0.012	0.013	0.024	0.051	0.050	0.054	0.078	0.105	0.100	0.106	0.137
200	0.011	0.011	0.013	0.026	0.053	0.052	0.056	0.080	0.106	0.101	0.106	0.139
250	0.011	0.012	0.014	0.027	0.057	0.055	0.058	0.081	0.107	0.101	0.106	0.136
Right-sided tests - $T = 250$												
-5	0.008	0.009	0.111	0.003	0.048	0.049	0.150	0.026	0.098	0.101	0.175	0.064
-2.5	0.007	0.012	0.031	0.009	0.039	0.049	0.048	0.043	0.086	0.101	0.063	0.092
0	0.011	0.011	0.034	0.017	0.047	0.049	0.075	0.062	0.099	0.098	0.119	0.118
2.5	0.009	0.010	0.014	0.016	0.046	0.049	0.050	0.063	0.093	0.099	0.098	0.115
5	0.009	0.010	0.010	0.016	0.047	0.049	0.049	0.064	0.096	0.097	0.096	0.118
10	0.011	0.011	0.011	0.018	0.047	0.048	0.049	0.066	0.096	0.097	0.096	0.123
25	0.012	0.010	0.013	0.021	0.052	0.051	0.053	0.070	0.096	0.095	0.097	0.122
50	0.011	0.011	0.013	0.024	0.051	0.049	0.052	0.074	0.098	0.095	0.099	0.130
75	0.011	0.010	0.012	0.023	0.051	0.051	0.053	0.077	0.101	0.099	0.102	0.132
100	0.010	0.010	0.013	0.023	0.051	0.050	0.055	0.078	0.103	0.102	0.105	0.132
125	0.010	0.011	0.013	0.024	0.052	0.051	0.056	0.079	0.102	0.100	0.104	0.135
150	0.011	0.011	0.014	0.025	0.053	0.052	0.055	0.078	0.104	0.099	0.104	0.138
200	0.011	0.012	0.015	0.026	0.054	0.053	0.056	0.079	0.104	0.101	0.106	0.137
250	0.013	0.013	0.015	0.026	0.053	0.050	0.055	0.081	0.104	0.100	0.104	0.134
Two-sided tests - $T = 250$												
-5	0.008	0.010	0.196	0.002	0.046	0.049	0.258	0.020	0.093	0.098	0.300	0.051
-2.5	0.007	0.013	0.050	0.010	0.036	0.052	0.075	0.044	0.077	0.099	0.092	0.089
0	0.009	0.011	0.052	0.018	0.049	0.049	0.101	0.068	0.094	0.097	0.145	0.123
2.5	0.010	0.009	0.015	0.017	0.043	0.048	0.053	0.067	0.091	0.096	0.100	0.123
5	0.010	0.010	0.012	0.019	0.046	0.047	0.048	0.069	0.091	0.095	0.096	0.126
10	0.013	0.012	0.013	0.022	0.049	0.049	0.051	0.077	0.094	0.096	0.098	0.131
25	0.012	0.012	0.014	0.026	0.053	0.052	0.055	0.087	0.100	0.100	0.105	0.143
50	0.012	0.011	0.014	0.027	0.051	0.048	0.055	0.087	0.100	0.097	0.103	0.148
75	0.010	0.011	0.013	0.028	0.050	0.049	0.055	0.089	0.100	0.097	0.105	0.151
100	0.010	0.010	0.014	0.028	0.053	0.050	0.056	0.093	0.101	0.099	0.106	0.153
125	0.011	0.010	0.014	0.028	0.054	0.053	0.059	0.095	0.103	0.100	0.109	0.155
150	0.010	0.010	0.014	0.030	0.054	0.054	0.060	0.096	0.103	0.101	0.108	0.156
200	0.009	0.011	0.016	0.032	0.056	0.055	0.063	0.099	0.107	0.104	0.111	0.159
250	0.011	0.012	0.016	0.032	0.055	0.054	0.062	0.101	0.110	0.105	0.113	0.162

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.008	0.009	0.156	0.002	0.046	0.049	0.197	0.023	0.097	0.098	0.221	0.063
-2.5	0.006	0.010	0.026	0.008	0.039	0.050	0.041	0.045	0.085	0.100	0.054	0.093
0	0.012	0.012	0.036	0.019	0.053	0.051	0.080	0.067	0.102	0.101	0.121	0.124
2.5	0.011	0.011	0.014	0.018	0.050	0.053	0.051	0.066	0.100	0.102	0.100	0.120
5	0.010	0.010	0.010	0.018	0.048	0.050	0.047	0.066	0.098	0.101	0.098	0.121
10	0.010	0.009	0.009	0.018	0.049	0.049	0.047	0.068	0.098	0.100	0.097	0.124
25	0.010	0.010	0.009	0.019	0.051	0.051	0.049	0.073	0.100	0.102	0.100	0.131
50	0.010	0.011	0.010	0.023	0.051	0.051	0.051	0.075	0.101	0.102	0.100	0.134
75	0.011	0.010	0.010	0.024	0.051	0.050	0.050	0.079	0.102	0.102	0.103	0.136
100	0.011	0.011	0.010	0.023	0.051	0.051	0.050	0.079	0.102	0.101	0.102	0.136
125	0.011	0.010	0.010	0.023	0.053	0.052	0.052	0.079	0.103	0.102	0.103	0.137
150	0.010	0.010	0.010	0.023	0.052	0.052	0.052	0.081	0.101	0.102	0.102	0.135
200	0.010	0.011	0.011	0.023	0.055	0.053	0.053	0.081	0.101	0.100	0.101	0.136
250	0.010	0.011	0.011	0.023	0.051	0.052	0.052	0.081	0.102	0.101	0.100	0.137
Right-sided tests - $T = 1000$												
-5	0.009	0.010	0.156	0.002	0.051	0.053	0.195	0.028	0.098	0.101	0.220	0.065
-2.5	0.007	0.012	0.026	0.009	0.039	0.051	0.041	0.045	0.087	0.101	0.055	0.093
0	0.011	0.011	0.036	0.016	0.050	0.049	0.076	0.064	0.098	0.101	0.121	0.120
2.5	0.009	0.010	0.011	0.015	0.047	0.051	0.049	0.062	0.097	0.101	0.097	0.119
5	0.010	0.011	0.009	0.017	0.048	0.048	0.046	0.063	0.094	0.098	0.094	0.117
10	0.012	0.011	0.010	0.020	0.049	0.049	0.047	0.066	0.097	0.095	0.094	0.122
25	0.011	0.010	0.010	0.021	0.052	0.049	0.048	0.073	0.098	0.099	0.097	0.127
50	0.010	0.010	0.010	0.021	0.049	0.048	0.048	0.074	0.099	0.096	0.097	0.133
75	0.011	0.011	0.011	0.020	0.047	0.046	0.047	0.075	0.097	0.098	0.097	0.134
100	0.011	0.011	0.011	0.020	0.047	0.046	0.046	0.077	0.098	0.098	0.098	0.133
125	0.010	0.011	0.011	0.021	0.046	0.045	0.046	0.077	0.101	0.099	0.100	0.133
150	0.011	0.012	0.011	0.021	0.047	0.046	0.047	0.075	0.101	0.099	0.099	0.133
200	0.010	0.011	0.011	0.021	0.046	0.046	0.048	0.078	0.101	0.099	0.099	0.132
250	0.010	0.010	0.011	0.022	0.048	0.049	0.049	0.078	0.099	0.100	0.099	0.134
Two-sided tests- $T = 1000$												
-5	0.007	0.010	0.286	0.001	0.047	0.049	0.353	0.019	0.097	0.101	0.392	0.051
-2.5	0.005	0.011	0.044	0.008	0.035	0.051	0.066	0.043	0.076	0.101	0.081	0.090
0	0.011	0.012	0.057	0.020	0.051	0.050	0.110	0.075	0.103	0.101	0.156	0.131
2.5	0.009	0.009	0.013	0.018	0.049	0.050	0.056	0.070	0.096	0.104	0.100	0.128
5	0.010	0.010	0.009	0.019	0.048	0.051	0.049	0.072	0.095	0.099	0.093	0.129
10	0.011	0.011	0.010	0.022	0.047	0.049	0.048	0.077	0.098	0.098	0.094	0.134
25	0.010	0.011	0.010	0.024	0.050	0.048	0.048	0.085	0.099	0.099	0.097	0.146
50	0.010	0.010	0.010	0.024	0.051	0.050	0.050	0.087	0.099	0.098	0.099	0.149
75	0.011	0.011	0.011	0.026	0.050	0.048	0.050	0.086	0.098	0.096	0.097	0.154
100	0.011	0.010	0.011	0.027	0.049	0.049	0.049	0.087	0.098	0.096	0.097	0.156
125	0.010	0.011	0.011	0.027	0.050	0.049	0.051	0.087	0.098	0.096	0.098	0.156
150	0.010	0.010	0.011	0.027	0.050	0.050	0.051	0.088	0.100	0.097	0.098	0.156
200	0.011	0.011	0.010	0.026	0.049	0.050	0.051	0.090	0.100	0.100	0.101	0.159
250	0.010	0.011	0.012	0.026	0.048	0.050	0.051	0.090	0.100	0.099	0.102	0.158

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (12) and (14) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.23: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP7 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\, \boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim NIID(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad -0.95; -0.95 \quad 1]$ and $\sigma^2_{it} = 0.05 + 0.1 e^2_{i,t-1} + 0.85\sigma^2_{i,t-1},\ i = 1, 2$.

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.009	0.000	0.001	0.001	0.048	0.003	0.004	0.006	0.095	0.013	0.013	0.016
-2.5	0.005	0.000	0.000	0.000	0.043	0.001	0.001	0.001	0.108	0.001	0.001	0.002
0	0.012	0.000	0.000	0.000	0.037	0.001	0.001	0.001	0.063	0.003	0.003	0.003
2.5	0.023	0.001	0.001	0.001	0.057	0.005	0.005	0.006	0.095	0.015	0.015	0.016
5	0.025	0.002	0.002	0.002	0.066	0.012	0.012	0.014	0.114	0.027	0.026	0.030
10	0.022	0.003	0.003	0.006	0.066	0.020	0.020	0.025	0.116	0.041	0.041	0.050
25	0.017	0.006	0.007	0.012	0.062	0.028	0.029	0.040	0.107	0.059	0.060	0.076
50	0.016	0.007	0.008	0.016	0.059	0.034	0.036	0.051	0.108	0.070	0.071	0.089
75	0.015	0.008	0.009	0.016	0.057	0.038	0.040	0.055	0.107	0.077	0.079	0.098
100	0.015	0.008	0.010	0.017	0.057	0.038	0.042	0.058	0.106	0.079	0.083	0.101
125	0.015	0.009	0.010	0.020	0.057	0.039	0.042	0.059	0.106	0.082	0.086	0.107
150	0.015	0.009	0.011	0.020	0.055	0.042	0.045	0.062	0.108	0.084	0.089	0.112
200	0.013	0.010	0.011	0.020	0.056	0.045	0.047	0.067	0.109	0.089	0.092	0.114
250	0.012	0.009	0.011	0.021	0.055	0.046	0.049	0.068	0.106	0.092	0.097	0.121
Right-sided tests - $T = 250$												
-5	0.008	0.016	0.023	0.019	0.044	0.072	0.083	0.074	0.086	0.150	0.164	0.146
-2.5	0.010	0.019	0.024	0.023	0.043	0.100	0.113	0.107	0.091	0.239	0.255	0.246
0	0.011	0.024	0.029	0.031	0.054	0.104	0.117	0.123	0.109	0.228	0.240	0.245
2.5	0.012	0.023	0.027	0.032	0.061	0.115	0.123	0.133	0.125	0.219	0.230	0.242
5	0.012	0.023	0.027	0.032	0.060	0.107	0.118	0.130	0.125	0.207	0.217	0.232
10	0.013	0.022	0.026	0.032	0.059	0.097	0.105	0.119	0.120	0.183	0.192	0.211
25	0.011	0.016	0.020	0.027	0.060	0.081	0.089	0.106	0.114	0.156	0.163	0.185
50	0.010	0.015	0.018	0.024	0.056	0.071	0.079	0.099	0.114	0.138	0.145	0.169
75	0.011	0.013	0.016	0.023	0.056	0.066	0.071	0.093	0.111	0.132	0.138	0.163
100	0.011	0.014	0.016	0.025	0.056	0.063	0.067	0.087	0.112	0.125	0.131	0.156
125	0.011	0.012	0.016	0.025	0.056	0.062	0.067	0.087	0.113	0.119	0.126	0.153
150	0.010	0.013	0.014	0.025	0.058	0.059	0.065	0.087	0.111	0.116	0.120	0.146
200	0.010	0.012	0.014	0.024	0.058	0.058	0.062	0.083	0.109	0.110	0.114	0.139
250	0.012	0.011	0.014	0.024	0.058	0.054	0.059	0.081	0.109	0.107	0.112	0.136
Two-sided tests - $T = 250$												
-5	0.007	0.008	0.013	0.011	0.044	0.038	0.047	0.042	0.088	0.074	0.087	0.080
-2.5	0.009	0.010	0.013	0.014	0.039	0.046	0.056	0.055	0.084	0.102	0.113	0.108
0	0.008	0.011	0.014	0.015	0.046	0.053	0.062	0.066	0.094	0.105	0.118	0.125
2.5	0.011	0.012	0.015	0.018	0.050	0.058	0.067	0.076	0.105	0.119	0.129	0.138
5	0.009	0.012	0.016	0.020	0.051	0.060	0.065	0.077	0.107	0.118	0.130	0.144
10	0.011	0.013	0.016	0.020	0.055	0.060	0.066	0.080	0.108	0.116	0.125	0.144
25	0.011	0.011	0.014	0.022	0.055	0.056	0.063	0.084	0.109	0.109	0.117	0.145
50	0.012	0.011	0.013	0.023	0.055	0.052	0.060	0.083	0.110	0.106	0.114	0.150
75	0.012	0.010	0.014	0.025	0.055	0.051	0.058	0.084	0.109	0.103	0.111	0.148
100	0.012	0.010	0.014	0.025	0.055	0.051	0.058	0.086	0.108	0.101	0.109	0.146
125	0.011	0.010	0.013	0.026	0.058	0.054	0.059	0.085	0.106	0.100	0.108	0.146
150	0.011	0.010	0.013	0.026	0.059	0.053	0.060	0.086	0.110	0.101	0.109	0.149
200	0.011	0.010	0.013	0.027	0.058	0.053	0.058	0.087	0.111	0.102	0.108	0.149
250	0.011	0.011	0.013	0.027	0.057	0.052	0.060	0.086	0.111	0.100	0.108	0.149

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.043	0.003	0.003	0.003	0.093	0.011	0.011	0.010
-2.5	0.007	0.000	0.000	0.000	0.044	0.000	0.000	0.000	0.109	0.001	0.001	0.001
0	0.014	0.000	0.000	0.000	0.041	0.001	0.001	0.002	0.067	0.003	0.003	0.004
2.5	0.023	0.001	0.001	0.001	0.058	0.006	0.005	0.006	0.099	0.015	0.014	0.015
5	0.025	0.002	0.002	0.003	0.067	0.012	0.011	0.013	0.114	0.029	0.029	0.031
10	0.021	0.004	0.004	0.005	0.067	0.019	0.019	0.024	0.113	0.045	0.046	0.053
25	0.018	0.006	0.006	0.010	0.063	0.029	0.029	0.040	0.111	0.061	0.062	0.076
50	0.015	0.007	0.007	0.014	0.059	0.032	0.033	0.049	0.111	0.071	0.072	0.093
75	0.013	0.007	0.007	0.016	0.058	0.034	0.034	0.055	0.111	0.077	0.078	0.102
100	0.013	0.008	0.007	0.017	0.056	0.037	0.038	0.060	0.108	0.081	0.080	0.105
125	0.014	0.008	0.008	0.019	0.055	0.038	0.039	0.061	0.107	0.081	0.081	0.108
150	0.013	0.008	0.008	0.020	0.056	0.040	0.039	0.063	0.105	0.082	0.083	0.112
200	0.013	0.010	0.010	0.021	0.055	0.042	0.042	0.068	0.104	0.086	0.086	0.116
250	0.013	0.010	0.010	0.022	0.054	0.043	0.044	0.069	0.104	0.087	0.086	0.118
Right-sided tests - $T = 1000$												
-5	0.006	0.013	0.013	0.014	0.041	0.066	0.069	0.065	0.086	0.147	0.152	0.144
-2.5	0.007	0.015	0.018	0.018	0.036	0.093	0.096	0.094	0.081	0.234	0.240	0.232
0	0.009	0.021	0.021	0.025	0.047	0.107	0.109	0.113	0.103	0.223	0.229	0.231
2.5	0.010	0.023	0.024	0.026	0.053	0.112	0.117	0.123	0.117	0.217	0.223	0.233
5	0.010	0.021	0.024	0.027	0.056	0.105	0.109	0.117	0.115	0.205	0.207	0.218
10	0.010	0.021	0.022	0.026	0.058	0.096	0.098	0.110	0.111	0.180	0.182	0.198
25	0.010	0.017	0.018	0.025	0.053	0.082	0.084	0.099	0.108	0.155	0.157	0.178
50	0.009	0.015	0.015	0.026	0.055	0.076	0.077	0.099	0.108	0.140	0.141	0.164
75	0.011	0.015	0.016	0.027	0.054	0.072	0.074	0.095	0.106	0.132	0.134	0.162
100	0.011	0.015	0.016	0.028	0.055	0.070	0.070	0.094	0.107	0.128	0.129	0.157
125	0.011	0.015	0.015	0.028	0.055	0.067	0.068	0.092	0.109	0.128	0.130	0.156
150	0.011	0.014	0.014	0.027	0.054	0.065	0.065	0.093	0.107	0.126	0.128	0.156
200	0.011	0.013	0.014	0.026	0.054	0.064	0.065	0.092	0.107	0.121	0.122	0.157
250	0.010	0.012	0.013	0.026	0.054	0.062	0.062	0.091	0.108	0.119	0.121	0.156
Two-sided tests- $T = 1000$												
-5	0.006	0.007	0.007	0.007	0.041	0.033	0.037	0.033	0.087	0.068	0.072	0.068
-2.5	0.006	0.008	0.008	0.009	0.031	0.041	0.044	0.043	0.074	0.093	0.096	0.095
0	0.007	0.010	0.011	0.013	0.039	0.053	0.057	0.057	0.087	0.106	0.110	0.115
2.5	0.009	0.011	0.013	0.014	0.044	0.054	0.058	0.063	0.095	0.117	0.122	0.129
5	0.009	0.012	0.013	0.015	0.048	0.057	0.062	0.069	0.098	0.116	0.120	0.130
10	0.009	0.012	0.013	0.017	0.050	0.057	0.060	0.073	0.103	0.115	0.118	0.134
25	0.010	0.011	0.011	0.020	0.051	0.058	0.059	0.081	0.104	0.111	0.113	0.139
50	0.010	0.010	0.010	0.023	0.053	0.056	0.057	0.085	0.105	0.108	0.110	0.148
75	0.012	0.011	0.011	0.025	0.053	0.054	0.055	0.088	0.105	0.105	0.108	0.150
100	0.011	0.011	0.011	0.026	0.054	0.053	0.054	0.089	0.107	0.106	0.107	0.154
125	0.011	0.011	0.012	0.027	0.055	0.054	0.056	0.091	0.108	0.105	0.107	0.153
150	0.011	0.012	0.012	0.027	0.055	0.053	0.055	0.090	0.105	0.104	0.104	0.156
200	0.012	0.012	0.012	0.030	0.053	0.052	0.053	0.092	0.108	0.104	0.107	0.160
250	0.012	0.011	0.011	0.030	0.054	0.053	0.053	0.093	0.107	0.105	0.106	0.160

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (12) and (14) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.24: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP7 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\,\boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim NIID(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad -0.9; -0.9 \quad 1]$ and $\sigma^2_{it} = 0.05 + 0.1e^2_{i,t-1} + 0.85\sigma^2_{i,t-1},\ i = 1, 2$.

	1%				5%				10%			
c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.008	0.000	0.001	0.001	0.048	0.005	0.006	0.007	0.098	0.015	0.017	0.018
-2.5	0.007	0.000	0.000	0.000	0.047	0.001	0.001	0.001	0.108	0.002	0.002	0.003
0	0.012	0.000	0.000	0.000	0.035	0.001	0.001	0.002	0.061	0.004	0.004	0.004
2.5	0.022	0.001	0.001	0.001	0.054	0.006	0.007	0.007	0.094	0.017	0.017	0.018
5	0.024	0.002	0.002	0.003	0.065	0.013	0.014	0.015	0.110	0.028	0.029	0.032
10	0.022	0.004	0.004	0.006	0.065	0.021	0.021	0.026	0.111	0.042	0.042	0.050
25	0.018	0.006	0.008	0.012	0.059	0.028	0.028	0.038	0.107	0.060	0.061	0.074
50	0.014	0.007	0.008	0.014	0.056	0.033	0.035	0.048	0.106	0.071	0.071	0.090
75	0.014	0.006	0.008	0.014	0.054	0.036	0.038	0.052	0.105	0.077	0.079	0.097
100	0.014	0.008	0.010	0.015	0.055	0.038	0.040	0.056	0.104	0.080	0.081	0.101
125	0.014	0.009	0.010	0.018	0.054	0.038	0.042	0.058	0.105	0.083	0.086	0.105
150	0.014	0.009	0.011	0.019	0.053	0.040	0.043	0.059	0.106	0.084	0.088	0.110
200	0.012	0.009	0.011	0.019	0.054	0.043	0.046	0.063	0.105	0.088	0.090	0.113
250	0.011	0.010	0.011	0.020	0.052	0.046	0.049	0.066	0.105	0.093	0.097	0.115
Right-sided tests - $T = 250$												
-5	0.008	0.015	0.022	0.017	0.045	0.074	0.086	0.075	0.087	0.153	0.167	0.147
-2.5	0.009	0.018	0.026	0.022	0.044	0.103	0.117	0.106	0.094	0.239	0.256	0.243
0	0.010	0.022	0.027	0.029	0.053	0.105	0.119	0.121	0.111	0.225	0.240	0.240
2.5	0.013	0.024	0.027	0.031	0.062	0.113	0.122	0.127	0.123	0.215	0.226	0.233
5	0.013	0.023	0.027	0.031	0.062	0.107	0.116	0.124	0.124	0.201	0.210	0.222
10	0.012	0.022	0.025	0.030	0.059	0.094	0.102	0.113	0.118	0.177	0.184	0.202
25	0.011	0.017	0.020	0.026	0.059	0.082	0.086	0.103	0.114	0.154	0.159	0.180
50	0.010	0.014	0.018	0.023	0.058	0.071	0.076	0.093	0.114	0.140	0.145	0.166
75	0.010	0.014	0.016	0.023	0.056	0.065	0.071	0.090	0.113	0.133	0.137	0.158
100	0.010	0.013	0.015	0.022	0.056	0.063	0.068	0.085	0.111	0.125	0.130	0.152
125	0.011	0.012	0.015	0.023	0.056	0.060	0.065	0.084	0.110	0.118	0.123	0.148
150	0.010	0.011	0.014	0.023	0.056	0.059	0.064	0.082	0.108	0.114	0.120	0.143
200	0.011	0.012	0.014	0.022	0.057	0.056	0.061	0.080	0.108	0.109	0.113	0.136
250	0.013	0.012	0.015	0.022	0.057	0.055	0.059	0.077	0.111	0.108	0.113	0.134
Two-sided tests - $T = 250$												
-5	0.008	0.008	0.013	0.010	0.046	0.038	0.049	0.045	0.091	0.078	0.092	0.081
-2.5	0.008	0.009	0.014	0.013	0.040	0.048	0.057	0.052	0.087	0.103	0.118	0.107
0	0.008	0.011	0.015	0.015	0.045	0.053	0.061	0.064	0.094	0.106	0.120	0.123
2.5	0.010	0.012	0.016	0.016	0.052	0.056	0.067	0.073	0.105	0.119	0.128	0.134
5	0.011	0.013	0.016	0.019	0.054	0.060	0.066	0.076	0.109	0.119	0.129	0.140
10	0.011	0.012	0.016	0.019	0.056	0.059	0.066	0.078	0.109	0.115	0.123	0.140
25	0.011	0.012	0.014	0.021	0.055	0.055	0.062	0.080	0.107	0.109	0.114	0.141
50	0.011	0.010	0.013	0.021	0.053	0.053	0.059	0.081	0.108	0.103	0.111	0.141
75	0.011	0.011	0.013	0.023	0.055	0.052	0.057	0.080	0.108	0.101	0.109	0.142
100	0.011	0.009	0.013	0.022	0.055	0.051	0.059	0.082	0.108	0.099	0.108	0.141
125	0.011	0.010	0.013	0.023	0.055	0.052	0.060	0.081	0.107	0.098	0.107	0.141
150	0.011	0.010	0.013	0.024	0.058	0.052	0.059	0.082	0.106	0.099	0.107	0.141
200	0.011	0.011	0.014	0.025	0.058	0.052	0.058	0.083	0.108	0.099	0.108	0.143
250	0.011	0.010	0.014	0.026	0.056	0.051	0.057	0.081	0.108	0.100	0.108	0.143

	1%				5%				10%			
c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.000	0.000	0.046	0.003	0.004	0.004	0.096	0.013	0.013	0.014
-2.5	0.007	0.000	0.000	0.000	0.048	0.000	0.000	0.001	0.109	0.001	0.001	0.001
0	0.013	0.000	0.000	0.000	0.037	0.001	0.001	0.002	0.064	0.004	0.004	0.004
2.5	0.022	0.001	0.001	0.002	0.056	0.006	0.006	0.007	0.096	0.016	0.017	0.018
5	0.023	0.002	0.002	0.003	0.066	0.012	0.012	0.015	0.109	0.030	0.030	0.033
10	0.020	0.004	0.003	0.005	0.064	0.020	0.020	0.024	0.111	0.048	0.047	0.052
25	0.016	0.006	0.005	0.010	0.060	0.029	0.030	0.040	0.111	0.063	0.062	0.074
50	0.014	0.007	0.007	0.014	0.060	0.036	0.036	0.050	0.109	0.071	0.072	0.091
75	0.013	0.007	0.008	0.015	0.057	0.036	0.036	0.055	0.106	0.077	0.078	0.098
100	0.012	0.008	0.007	0.016	0.056	0.039	0.039	0.059	0.106	0.080	0.081	0.105
125	0.013	0.008	0.008	0.018	0.055	0.038	0.039	0.061	0.106	0.082	0.083	0.107
150	0.014	0.008	0.008	0.019	0.055	0.041	0.040	0.062	0.105	0.083	0.084	0.110
200	0.013	0.009	0.010	0.021	0.056	0.042	0.043	0.065	0.106	0.085	0.086	0.113
250	0.013	0.009	0.010	0.022	0.053	0.043	0.043	0.066	0.105	0.088	0.089	0.117
Right-sided tests - $T = 1000$												
-5	0.007	0.012	0.013	0.013	0.039	0.068	0.071	0.066	0.088	0.147	0.153	0.144
-2.5	0.006	0.015	0.017	0.017	0.037	0.098	0.099	0.097	0.086	0.230	0.239	0.231
0	0.010	0.021	0.022	0.024	0.049	0.108	0.109	0.110	0.104	0.223	0.229	0.229
2.5	0.011	0.022	0.024	0.025	0.054	0.109	0.112	0.119	0.117	0.213	0.215	0.225
5	0.011	0.022	0.023	0.025	0.056	0.104	0.106	0.115	0.116	0.198	0.200	0.210
10	0.011	0.022	0.023	0.025	0.057	0.094	0.094	0.105	0.111	0.177	0.178	0.194
25	0.011	0.016	0.018	0.024	0.055	0.081	0.083	0.097	0.109	0.151	0.152	0.170
50	0.011	0.016	0.015	0.024	0.056	0.075	0.075	0.095	0.109	0.139	0.141	0.161
75	0.010	0.015	0.016	0.027	0.056	0.071	0.071	0.091	0.108	0.134	0.135	0.159
100	0.010	0.016	0.016	0.027	0.053	0.068	0.069	0.092	0.107	0.128	0.129	0.155
125	0.010	0.016	0.015	0.026	0.053	0.067	0.067	0.089	0.109	0.129	0.130	0.155
150	0.011	0.014	0.015	0.025	0.053	0.064	0.065	0.089	0.108	0.126	0.129	0.154
200	0.011	0.014	0.014	0.024	0.054	0.061	0.063	0.087	0.107	0.123	0.123	0.155
250	0.011	0.013	0.013	0.024	0.056	0.063	0.063	0.087	0.107	0.119	0.119	0.151
Two-sided tests- $T = 1000$												
-5	0.006	0.006	0.007	0.007	0.040	0.033	0.036	0.032	0.089	0.070	0.075	0.070
-2.5	0.006	0.008	0.008	0.008	0.032	0.040	0.044	0.042	0.079	0.096	0.100	0.097
0	0.007	0.011	0.011	0.013	0.041	0.053	0.055	0.056	0.088	0.108	0.110	0.112
2.5	0.010	0.013	0.013	0.013	0.044	0.055	0.059	0.062	0.096	0.115	0.118	0.126
5	0.010	0.012	0.013	0.015	0.048	0.057	0.059	0.067	0.101	0.114	0.118	0.130
10	0.010	0.012	0.013	0.017	0.053	0.059	0.061	0.071	0.101	0.113	0.114	0.129
25	0.010	0.011	0.011	0.019	0.052	0.057	0.058	0.077	0.106	0.110	0.113	0.136
50	0.011	0.010	0.010	0.020	0.053	0.054	0.056	0.083	0.107	0.109	0.111	0.145
75	0.011	0.010	0.010	0.022	0.056	0.054	0.055	0.083	0.107	0.106	0.108	0.146
100	0.011	0.012	0.012	0.024	0.056	0.054	0.055	0.086	0.107	0.106	0.108	0.151
125	0.011	0.012	0.012	0.025	0.056	0.055	0.055	0.085	0.106	0.104	0.106	0.150
150	0.012	0.011	0.012	0.025	0.055	0.052	0.055	0.086	0.106	0.104	0.105	0.151
200	0.013	0.013	0.013	0.027	0.053	0.053	0.055	0.088	0.108	0.104	0.105	0.152
250	0.013	0.012	0.013	0.027	0.054	0.053	0.055	0.090	0.107	0.105	0.106	0.152

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (12) and (14) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.25: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP7 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\,\boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim NIID(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad -0.5; -0.5 \quad 1]$ and $\sigma_{it}^2 = 0.05 + 0.1e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2,\ i = 1, 2.$

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.010	0.002	0.007	0.004	0.051	0.019	0.027	0.022	0.102	0.047	0.056	0.047
-2.5	0.008	0.001	0.002	0.001	0.049	0.005	0.008	0.008	0.098	0.016	0.021	0.018
0	0.007	0.001	0.002	0.001	0.029	0.007	0.010	0.008	0.058	0.018	0.021	0.021
2.5	0.012	0.003	0.003	0.003	0.044	0.019	0.020	0.020	0.084	0.038	0.039	0.039
5	0.014	0.004	0.005	0.006	0.050	0.024	0.026	0.025	0.095	0.051	0.051	0.053
10	0.014	0.006	0.007	0.008	0.053	0.030	0.032	0.034	0.101	0.064	0.066	0.068
25	0.013	0.007	0.008	0.010	0.053	0.038	0.040	0.044	0.101	0.074	0.077	0.080
50	0.010	0.007	0.008	0.010	0.052	0.039	0.043	0.046	0.099	0.079	0.081	0.086
75	0.010	0.008	0.009	0.010	0.052	0.043	0.046	0.049	0.098	0.083	0.085	0.090
100	0.011	0.009	0.010	0.011	0.052	0.043	0.046	0.049	0.098	0.084	0.087	0.093
125	0.011	0.009	0.010	0.011	0.050	0.043	0.047	0.051	0.098	0.084	0.088	0.094
150	0.010	0.009	0.009	0.011	0.051	0.044	0.047	0.053	0.099	0.089	0.092	0.095
200	0.010	0.009	0.011	0.012	0.050	0.045	0.050	0.054	0.099	0.091	0.094	0.099
250	0.010	0.010	0.011	0.012	0.051	0.047	0.051	0.055	0.100	0.092	0.095	0.102
Right-sided tests - $T = 250$												
-5	0.009	0.015	0.034	0.016	0.047	0.074	0.100	0.073	0.097	0.143	0.167	0.140
-2.5	0.013	0.023	0.043	0.023	0.054	0.103	0.124	0.104	0.111	0.203	0.220	0.200
0	0.013	0.020	0.032	0.021	0.059	0.099	0.113	0.097	0.122	0.194	0.204	0.193
2.5	0.012	0.018	0.025	0.021	0.062	0.092	0.101	0.094	0.120	0.171	0.182	0.173
5	0.012	0.019	0.023	0.020	0.061	0.083	0.091	0.087	0.116	0.159	0.167	0.164
10	0.011	0.015	0.018	0.018	0.057	0.075	0.082	0.079	0.112	0.143	0.150	0.150
25	0.010	0.014	0.017	0.015	0.052	0.065	0.070	0.071	0.110	0.132	0.136	0.137
50	0.010	0.011	0.014	0.014	0.052	0.061	0.067	0.066	0.108	0.121	0.125	0.127
75	0.008	0.010	0.012	0.013	0.054	0.060	0.064	0.067	0.107	0.119	0.122	0.125
100	0.008	0.010	0.011	0.014	0.053	0.057	0.061	0.063	0.109	0.114	0.119	0.121
125	0.009	0.010	0.011	0.012	0.053	0.055	0.059	0.064	0.109	0.113	0.116	0.119
150	0.009	0.010	0.012	0.012	0.054	0.055	0.060	0.063	0.108	0.111	0.114	0.118
200	0.009	0.011	0.012	0.013	0.055	0.056	0.058	0.061	0.107	0.108	0.109	0.115
250	0.012	0.011	0.013	0.015	0.055	0.054	0.058	0.062	0.106	0.104	0.107	0.114
Two-sided tests - $T = 250$												
-5	0.009	0.009	0.029	0.011	0.048	0.045	0.076	0.050	0.100	0.093	0.127	0.095
-2.5	0.011	0.012	0.031	0.012	0.049	0.054	0.082	0.054	0.099	0.106	0.132	0.111
0	0.010	0.011	0.022	0.012	0.048	0.051	0.068	0.054	0.096	0.105	0.122	0.105
2.5	0.010	0.011	0.017	0.012	0.051	0.054	0.066	0.058	0.105	0.109	0.121	0.115
5	0.010	0.011	0.015	0.013	0.053	0.052	0.062	0.059	0.104	0.107	0.117	0.112
10	0.011	0.011	0.014	0.014	0.052	0.053	0.061	0.060	0.103	0.105	0.114	0.112
25	0.011	0.010	0.012	0.014	0.051	0.051	0.058	0.058	0.101	0.102	0.110	0.115
50	0.009	0.009	0.011	0.013	0.050	0.049	0.055	0.058	0.103	0.099	0.109	0.112
75	0.009	0.008	0.011	0.012	0.048	0.048	0.055	0.057	0.104	0.102	0.110	0.116
100	0.008	0.008	0.011	0.011	0.051	0.047	0.052	0.057	0.103	0.100	0.107	0.112
125	0.008	0.009	0.012	0.012	0.050	0.050	0.057	0.060	0.103	0.098	0.105	0.114
150	0.009	0.010	0.012	0.012	0.050	0.050	0.055	0.060	0.104	0.098	0.107	0.116
200	0.009	0.010	0.012	0.013	0.052	0.048	0.058	0.063	0.104	0.101	0.109	0.115
250	0.010	0.011	0.013	0.014	0.051	0.050	0.057	0.062	0.104	0.101	0.109	0.116

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.010	0.001	0.002	0.003	0.052	0.019	0.022	0.019	0.101	0.048	0.050	0.048
-2.5	0.010	0.001	0.000	0.001	0.049	0.005	0.005	0.006	0.100	0.017	0.018	0.016
0	0.008	0.001	0.001	0.001	0.034	0.008	0.009	0.009	0.062	0.022	0.022	0.022
2.5	0.011	0.003	0.003	0.004	0.048	0.017	0.017	0.019	0.089	0.042	0.043	0.044
5	0.012	0.004	0.004	0.005	0.052	0.024	0.023	0.026	0.099	0.055	0.056	0.056
10	0.013	0.005	0.005	0.006	0.056	0.031	0.031	0.035	0.103	0.067	0.066	0.069
25	0.011	0.006	0.006	0.009	0.052	0.036	0.037	0.041	0.103	0.078	0.077	0.084
50	0.011	0.007	0.007	0.011	0.053	0.042	0.042	0.047	0.102	0.083	0.083	0.090
75	0.011	0.008	0.007	0.010	0.054	0.042	0.043	0.049	0.103	0.085	0.087	0.094
100	0.010	0.008	0.008	0.011	0.052	0.043	0.043	0.049	0.102	0.087	0.088	0.096
125	0.010	0.008	0.008	0.011	0.053	0.043	0.043	0.051	0.103	0.090	0.090	0.098
150	0.010	0.009	0.008	0.011	0.052	0.044	0.044	0.052	0.103	0.091	0.091	0.100
200	0.011	0.008	0.008	0.012	0.052	0.045	0.046	0.054	0.104	0.093	0.093	0.102
250	0.012	0.009	0.010	0.013	0.051	0.045	0.048	0.055	0.103	0.093	0.094	0.103
Right-sided tests - $T = 1000$												
-5	0.008	0.013	0.017	0.013	0.045	0.071	0.079	0.068	0.095	0.137	0.144	0.134
-2.5	0.008	0.018	0.024	0.017	0.051	0.097	0.108	0.094	0.106	0.200	0.205	0.196
0	0.013	0.020	0.025	0.020	0.060	0.096	0.101	0.095	0.119	0.192	0.196	0.192
2.5	0.012	0.020	0.023	0.021	0.059	0.090	0.092	0.091	0.118	0.169	0.169	0.170
5	0.012	0.019	0.020	0.018	0.059	0.084	0.086	0.086	0.112	0.157	0.157	0.158
10	0.012	0.018	0.019	0.019	0.054	0.074	0.076	0.077	0.108	0.141	0.142	0.146
25	0.012	0.015	0.015	0.015	0.054	0.069	0.069	0.071	0.104	0.128	0.127	0.130
50	0.011	0.014	0.014	0.014	0.054	0.064	0.065	0.069	0.103	0.122	0.121	0.125
75	0.011	0.013	0.014	0.015	0.055	0.063	0.063	0.068	0.102	0.118	0.118	0.124
100	0.011	0.014	0.014	0.015	0.054	0.062	0.062	0.065	0.104	0.117	0.117	0.123
125	0.011	0.013	0.013	0.015	0.053	0.061	0.061	0.067	0.108	0.116	0.118	0.124
150	0.013	0.013	0.013	0.016	0.054	0.061	0.061	0.068	0.106	0.117	0.117	0.126
200	0.011	0.013	0.014	0.016	0.054	0.058	0.059	0.064	0.106	0.114	0.114	0.122
250	0.012	0.014	0.014	0.016	0.054	0.058	0.059	0.064	0.105	0.112	0.112	0.121
Two-sided tests- $T = 1000$												
-5	0.008	0.007	0.010	0.008	0.046	0.041	0.052	0.043	0.099	0.089	0.100	0.087
-2.5	0.007	0.009	0.013	0.007	0.046	0.047	0.055	0.048	0.096	0.103	0.114	0.100
0	0.010	0.011	0.013	0.012	0.046	0.051	0.057	0.050	0.097	0.104	0.110	0.104
2.5	0.010	0.011	0.012	0.012	0.052	0.054	0.057	0.057	0.101	0.108	0.110	0.109
5	0.011	0.010	0.012	0.011	0.053	0.054	0.056	0.059	0.102	0.108	0.109	0.112
10	0.011	0.012	0.012	0.013	0.051	0.054	0.056	0.057	0.102	0.107	0.106	0.111
25	0.011	0.011	0.011	0.013	0.051	0.053	0.054	0.058	0.103	0.105	0.107	0.112
50	0.011	0.010	0.011	0.014	0.053	0.052	0.053	0.060	0.105	0.105	0.108	0.116
75	0.011	0.010	0.011	0.014	0.053	0.053	0.053	0.060	0.105	0.105	0.106	0.117
100	0.011	0.011	0.012	0.014	0.053	0.052	0.051	0.060	0.105	0.103	0.105	0.115
125	0.010	0.011	0.011	0.015	0.052	0.050	0.051	0.060	0.106	0.105	0.104	0.118
150	0.011	0.010	0.011	0.015	0.052	0.050	0.052	0.061	0.105	0.104	0.105	0.120
200	0.011	0.012	0.012	0.015	0.053	0.051	0.052	0.063	0.104	0.104	0.105	0.119
250	0.012	0.012	0.012	0.016	0.054	0.052	0.053	0.064	0.105	0.103	0.106	0.119

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.26: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP7 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0,\ \rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\,\boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim NIID(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad 0; 0 \quad 1]$ and $\sigma^2_{it} = 0.05 + 0.1e^2_{i,t-1} + 0.85\sigma^2_{i,t-1},\ i = 1, 2.$

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.011	0.011	0.025	0.012	0.050	0.049	0.070	0.052	0.101	0.101	0.122	0.098
-2.5	0.010	0.010	0.025	0.011	0.048	0.048	0.062	0.046	0.096	0.097	0.104	0.092
0	0.010	0.009	0.018	0.010	0.046	0.046	0.058	0.046	0.096	0.096	0.103	0.094
2.5	0.011	0.010	0.015	0.011	0.048	0.047	0.054	0.047	0.098	0.096	0.103	0.097
5	0.011	0.010	0.013	0.011	0.049	0.047	0.051	0.048	0.099	0.099	0.105	0.098
10	0.010	0.010	0.011	0.011	0.050	0.049	0.054	0.049	0.102	0.103	0.105	0.102
25	0.011	0.011	0.013	0.012	0.053	0.051	0.053	0.053	0.103	0.101	0.103	0.103
50	0.010	0.010	0.012	0.011	0.050	0.048	0.052	0.049	0.101	0.099	0.104	0.100
75	0.010	0.010	0.011	0.010	0.049	0.047	0.050	0.049	0.097	0.096	0.098	0.096
100	0.010	0.010	0.011	0.010	0.048	0.046	0.049	0.047	0.096	0.093	0.096	0.093
125	0.010	0.010	0.011	0.010	0.046	0.045	0.049	0.047	0.096	0.093	0.096	0.094
150	0.010	0.010	0.011	0.010	0.049	0.046	0.050	0.047	0.097	0.094	0.098	0.095
200	0.010	0.009	0.010	0.009	0.048	0.048	0.050	0.049	0.095	0.094	0.097	0.096
250	0.009	0.009	0.010	0.010	0.049	0.048	0.051	0.049	0.099	0.096	0.099	0.098
Right-sided tests - $T = 250$												
-5	0.011	0.011	0.026	0.013	0.049	0.050	0.070	0.050	0.099	0.098	0.116	0.098
-2.5	0.011	0.011	0.025	0.012	0.052	0.052	0.067	0.052	0.103	0.104	0.113	0.102
0	0.011	0.010	0.018	0.012	0.051	0.050	0.057	0.050	0.101	0.100	0.102	0.099
2.5	0.010	0.010	0.014	0.011	0.050	0.049	0.054	0.050	0.103	0.102	0.106	0.100
5	0.009	0.009	0.013	0.010	0.050	0.049	0.055	0.050	0.102	0.100	0.105	0.101
10	0.009	0.009	0.010	0.009	0.051	0.049	0.052	0.051	0.102	0.100	0.105	0.102
25	0.010	0.009	0.011	0.010	0.051	0.050	0.053	0.051	0.103	0.101	0.102	0.103
50	0.009	0.009	0.011	0.009	0.051	0.051	0.052	0.050	0.101	0.099	0.104	0.099
75	0.009	0.008	0.010	0.010	0.053	0.050	0.054	0.053	0.101	0.100	0.103	0.100
100	0.010	0.009	0.010	0.009	0.050	0.050	0.053	0.050	0.102	0.100	0.105	0.101
125	0.008	0.009	0.010	0.009	0.050	0.049	0.052	0.049	0.104	0.100	0.104	0.100
150	0.007	0.008	0.010	0.010	0.051	0.050	0.052	0.050	0.102	0.101	0.104	0.100
200	0.008	0.009	0.010	0.010	0.050	0.049	0.053	0.051	0.104	0.102	0.104	0.102
250	0.009	0.010	0.011	0.011	0.050	0.049	0.051	0.050	0.104	0.102	0.105	0.102
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.038	0.015	0.049	0.049	0.089	0.053	0.099	0.099	0.140	0.102
-2.5	0.011	0.011	0.037	0.012	0.048	0.048	0.082	0.050	0.097	0.101	0.130	0.097
0	0.011	0.010	0.025	0.011	0.047	0.048	0.068	0.048	0.095	0.095	0.115	0.095
2.5	0.011	0.011	0.017	0.012	0.047	0.048	0.060	0.049	0.098	0.096	0.108	0.097
5	0.010	0.010	0.014	0.011	0.050	0.047	0.056	0.050	0.097	0.095	0.106	0.098
10	0.010	0.009	0.012	0.010	0.049	0.047	0.053	0.052	0.100	0.098	0.106	0.100
25	0.011	0.011	0.012	0.012	0.049	0.047	0.053	0.050	0.102	0.100	0.107	0.103
50	0.009	0.009	0.011	0.010	0.051	0.048	0.056	0.052	0.100	0.098	0.104	0.099
75	0.008	0.008	0.010	0.010	0.047	0.046	0.052	0.048	0.100	0.097	0.104	0.101
100	0.009	0.008	0.010	0.009	0.046	0.046	0.051	0.048	0.096	0.095	0.101	0.098
125	0.009	0.008	0.010	0.009	0.047	0.046	0.050	0.047	0.096	0.094	0.100	0.096
150	0.008	0.008	0.011	0.009	0.046	0.046	0.051	0.048	0.099	0.097	0.102	0.097
200	0.008	0.008	0.011	0.009	0.050	0.049	0.054	0.050	0.098	0.097	0.103	0.100
250	0.008	0.009	0.011	0.010	0.049	0.049	0.054	0.051	0.098	0.096	0.103	0.099

c	$t^{*,RWB}_{zx}$ (1%)	$t^{*,FRWB}_{zx}$ (1%)	t^{EW}_{zx} (1%)	t_{zx} (1%)	$t^{*,RWB}_{zx}$ (5%)	$t^{*,FRWB}_{zx}$ (5%)	t^{EW}_{zx} (5%)	t_{zx} (5%)	$t^{*,RWB}_{zx}$ (10%)	$t^{*,FRWB}_{zx}$ (10%)	t^{EW}_{zx} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.010	0.001	0.002	0.003	0.052	0.019	0.022	0.019	0.101	0.048	0.050	0.048
-2.5	0.010	0.001	0.000	0.001	0.049	0.005	0.005	0.006	0.100	0.017	0.018	0.016
0	0.008	0.001	0.001	0.001	0.034	0.008	0.009	0.009	0.062	0.022	0.022	0.022
2.5	0.011	0.003	0.003	0.004	0.048	0.017	0.017	0.019	0.089	0.042	0.043	0.044
5	0.012	0.004	0.004	0.005	0.052	0.024	0.023	0.026	0.099	0.055	0.056	0.056
10	0.013	0.005	0.005	0.006	0.056	0.031	0.031	0.035	0.103	0.067	0.066	0.069
25	0.011	0.006	0.006	0.009	0.052	0.036	0.037	0.041	0.103	0.078	0.077	0.084
50	0.011	0.007	0.007	0.011	0.053	0.042	0.042	0.047	0.102	0.083	0.083	0.090
75	0.011	0.008	0.007	0.010	0.054	0.042	0.043	0.049	0.103	0.085	0.087	0.094
100	0.010	0.008	0.008	0.011	0.052	0.043	0.043	0.049	0.102	0.087	0.088	0.096
125	0.010	0.008	0.008	0.011	0.053	0.043	0.043	0.051	0.103	0.090	0.090	0.098
150	0.010	0.009	0.008	0.011	0.052	0.044	0.044	0.052	0.103	0.091	0.091	0.100
200	0.011	0.008	0.008	0.012	0.052	0.045	0.046	0.054	0.104	0.093	0.093	0.102
250	0.012	0.009	0.010	0.013	0.051	0.045	0.048	0.055	0.103	0.093	0.094	0.103
Right-sided tests - $T = 1000$												
-5	0.011	0.011	0.016	0.012	0.049	0.049	0.057	0.048	0.100	0.099	0.107	0.100
-2.5	0.010	0.010	0.018	0.010	0.050	0.050	0.059	0.050	0.095	0.095	0.107	0.095
0	0.011	0.010	0.014	0.011	0.048	0.051	0.054	0.048	0.101	0.103	0.104	0.099
2.5	0.008	0.008	0.011	0.008	0.049	0.049	0.052	0.048	0.103	0.104	0.105	0.101
5	0.009	0.009	0.009	0.009	0.050	0.050	0.050	0.050	0.100	0.101	0.103	0.101
10	0.008	0.010	0.010	0.010	0.052	0.051	0.051	0.051	0.101	0.100	0.101	0.099
25	0.011	0.011	0.011	0.011	0.051	0.050	0.050	0.050	0.100	0.100	0.101	0.100
50	0.012	0.011	0.011	0.012	0.052	0.051	0.052	0.052	0.102	0.101	0.103	0.101
75	0.012	0.011	0.011	0.011	0.053	0.052	0.053	0.052	0.101	0.101	0.102	0.101
100	0.011	0.011	0.012	0.011	0.052	0.052	0.053	0.052	0.100	0.098	0.100	0.099
125	0.012	0.011	0.011	0.011	0.052	0.053	0.052	0.052	0.100	0.101	0.101	0.099
150	0.011	0.012	0.011	0.011	0.053	0.053	0.052	0.053	0.102	0.102	0.102	0.100
200	0.011	0.011	0.011	0.011	0.052	0.051	0.053	0.051	0.104	0.104	0.105	0.102
250	0.011	0.010	0.011	0.010	0.052	0.050	0.051	0.051	0.106	0.105	0.106	0.105
Two-sided tests- $T = 1000$												
-5	0.011	0.011	0.017	0.012	0.049	0.049	0.065	0.049	0.099	0.099	0.116	0.099
-2.5	0.009	0.009	0.022	0.010	0.051	0.049	0.068	0.050	0.097	0.099	0.118	0.098
0	0.011	0.011	0.017	0.011	0.048	0.050	0.062	0.050	0.099	0.101	0.113	0.099
2.5	0.010	0.010	0.011	0.010	0.048	0.049	0.054	0.048	0.099	0.102	0.107	0.100
5	0.009	0.009	0.010	0.008	0.049	0.048	0.051	0.048	0.101	0.100	0.103	0.101
10	0.009	0.009	0.010	0.009	0.050	0.050	0.052	0.050	0.101	0.100	0.101	0.100
25	0.010	0.010	0.010	0.011	0.051	0.051	0.052	0.052	0.100	0.097	0.099	0.099
50	0.010	0.010	0.011	0.011	0.050	0.050	0.050	0.050	0.101	0.099	0.100	0.101
75	0.010	0.010	0.010	0.010	0.051	0.049	0.050	0.051	0.102	0.100	0.102	0.100
100	0.010	0.010	0.010	0.011	0.051	0.051	0.052	0.051	0.102	0.101	0.102	0.101
125	0.010	0.010	0.011	0.011	0.051	0.050	0.051	0.052	0.101	0.101	0.101	0.101
150	0.011	0.011	0.011	0.011	0.053	0.049	0.052	0.051	0.100	0.100	0.100	0.101
200	0.011	0.011	0.011	0.011	0.050	0.050	0.051	0.051	0.102	0.100	0.101	0.100
250	0.011	0.011	0.011	0.011	0.050	0.050	0.051	0.051	0.099	0.097	0.099	0.099

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (12) and (14) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.27: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP8 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\,\boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim iidt_5(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad -0.95; -0.95 \quad 1]$ and $\sigma^2_{it} = 0.05 + 0.1e^2_{i,t-1} + 0.85\sigma^2_{i,t-1},\ i = 1, 2$. $t_5(\mathbf{0}, \boldsymbol{\Omega})$ defines a mean zero Student-t distribution with 5 degrees of freedom and variance matrix $\boldsymbol{\Omega}$).

c	1% $t^{*,RWB}_{zx}$	1% $t^{*,FRWB}_{zx}$	1% t^{EW}_{zx}	1% t_{zx}	5% $t^{*,RWB}_{zx}$	5% $t^{*,FRWB}_{zx}$	5% t^{EW}_{zx}	5% t_{zx}	10% $t^{*,RWB}_{zx}$	10% $t^{*,FRWB}_{zx}$	10% t^{EW}_{zx}	10% t_{zx}
Left-sided tests - $T = 250$												
-5	0.007	0.001	0.001	0.007	0.047	0.005	0.005	0.018	0.096	0.016	0.015	0.032
-2.5	0.007	0.001	0.000	0.003	0.044	0.002	0.002	0.009	0.093	0.005	0.005	0.013
0	0.010	0.001	0.001	0.005	0.038	0.003	0.003	0.009	0.067	0.006	0.007	0.015
2.5	0.019	0.001	0.001	0.006	0.054	0.006	0.006	0.015	0.094	0.016	0.016	0.029
5	0.022	0.002	0.002	0.008	0.064	0.012	0.011	0.025	0.110	0.028	0.027	0.047
10	0.023	0.004	0.003	0.015	0.069	0.018	0.018	0.041	0.116	0.041	0.041	0.071
25	0.019	0.006	0.006	0.025	0.064	0.028	0.027	0.066	0.113	0.057	0.058	0.101
50	0.017	0.006	0.007	0.035	0.064	0.033	0.035	0.083	0.116	0.069	0.072	0.126
75	0.016	0.007	0.009	0.039	0.065	0.036	0.039	0.092	0.117	0.078	0.082	0.140
100	0.016	0.007	0.010	0.041	0.064	0.039	0.045	0.099	0.120	0.083	0.090	0.149
125	0.015	0.007	0.011	0.044	0.063	0.041	0.048	0.101	0.119	0.088	0.094	0.155
150	0.015	0.007	0.011	0.046	0.065	0.045	0.051	0.105	0.119	0.090	0.097	0.160
200	0.013	0.007	0.013	0.049	0.065	0.048	0.055	0.112	0.122	0.096	0.107	0.165
250	0.011	0.008	0.015	0.050	0.065	0.050	0.058	0.114	0.121	0.098	0.108	0.172
Right-sided tests - $T = 250$												
-5	0.008	0.014	0.029	0.038	0.044	0.084	0.106	0.106	0.083	0.173	0.188	0.180
-2.5	0.006	0.017	0.025	0.037	0.035	0.102	0.122	0.141	0.083	0.251	0.257	0.279
0	0.007	0.017	0.024	0.051	0.048	0.101	0.122	0.172	0.105	0.221	0.243	0.300
2.5	0.008	0.018	0.026	0.058	0.056	0.105	0.123	0.183	0.120	0.212	0.227	0.298
5	0.009	0.020	0.027	0.062	0.061	0.100	0.113	0.183	0.123	0.198	0.208	0.286
10	0.011	0.021	0.027	0.066	0.062	0.093	0.104	0.175	0.123	0.177	0.189	0.266
25	0.012	0.018	0.023	0.067	0.062	0.080	0.091	0.165	0.122	0.154	0.166	0.243
50	0.012	0.016	0.020	0.065	0.061	0.075	0.085	0.157	0.123	0.137	0.149	0.229
75	0.011	0.014	0.020	0.064	0.061	0.068	0.080	0.154	0.124	0.133	0.145	0.220
100	0.011	0.012	0.018	0.062	0.061	0.065	0.077	0.147	0.122	0.128	0.139	0.216
125	0.012	0.012	0.017	0.062	0.061	0.062	0.074	0.143	0.122	0.122	0.135	0.207
150	0.011	0.011	0.017	0.062	0.061	0.058	0.072	0.138	0.122	0.119	0.131	0.202
200	0.010	0.011	0.018	0.060	0.062	0.057	0.068	0.131	0.120	0.111	0.123	0.195
250	0.011	0.010	0.016	0.055	0.061	0.053	0.064	0.126	0.120	0.107	0.120	0.187
Two-sided tests - $T = 250$												
-5	0.007	0.007	0.019	0.029	0.043	0.041	0.062	0.081	0.085	0.088	0.112	0.124
-2.5	0.005	0.008	0.014	0.024	0.030	0.043	0.062	0.081	0.075	0.104	0.123	0.149
0	0.006	0.008	0.012	0.035	0.040	0.049	0.064	0.110	0.087	0.104	0.125	0.182
2.5	0.007	0.008	0.013	0.042	0.045	0.052	0.065	0.120	0.098	0.111	0.129	0.197
5	0.008	0.010	0.015	0.047	0.049	0.056	0.067	0.130	0.105	0.111	0.124	0.208
10	0.011	0.012	0.017	0.054	0.055	0.058	0.067	0.141	0.111	0.112	0.122	0.216
25	0.013	0.012	0.017	0.064	0.059	0.054	0.064	0.156	0.118	0.107	0.118	0.230
50	0.014	0.012	0.016	0.070	0.063	0.051	0.063	0.163	0.122	0.107	0.120	0.240
75	0.013	0.011	0.015	0.073	0.062	0.052	0.064	0.168	0.121	0.104	0.119	0.246
100	0.013	0.010	0.016	0.076	0.062	0.050	0.064	0.167	0.122	0.102	0.121	0.245
125	0.012	0.010	0.016	0.076	0.060	0.049	0.063	0.166	0.121	0.103	0.121	0.244
150	0.012	0.009	0.016	0.077	0.061	0.050	0.064	0.165	0.121	0.102	0.123	0.243
200	0.010	0.009	0.017	0.078	0.063	0.049	0.067	0.167	0.124	0.102	0.123	0.243
250	0.009	0.008	0.016	0.076	0.060	0.049	0.068	0.167	0.123	0.102	0.122	0.241

c	1% $t^{*,RWB}_{zx}$	1% $t^{*,FRWB}_{zx}$	1% t^{EW}_{zx}	1% t_{zx}	5% $t^{*,RWB}_{zx}$	5% $t^{*,FRWB}_{zx}$	5% t^{EW}_{zx}	5% t_{zx}	10% $t^{*,RWB}_{zx}$	10% $t^{*,FRWB}_{zx}$	10% t^{EW}_{zx}	10% t_{zx}
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.001	0.007	0.044	0.004	0.004	0.017	0.094	0.013	0.012	0.028
-2.5	0.007	0.000	0.000	0.003	0.047	0.001	0.001	0.007	0.103	0.003	0.003	0.011
0	0.008	0.000	0.000	0.004	0.033	0.002	0.002	0.009	0.060	0.005	0.005	0.015
2.5	0.016	0.001	0.001	0.006	0.051	0.005	0.004	0.017	0.091	0.013	0.012	0.030
5	0.022	0.001	0.001	0.011	0.066	0.010	0.009	0.030	0.108	0.023	0.020	0.055
10	0.023	0.003	0.002	0.022	0.070	0.015	0.014	0.055	0.117	0.037	0.036	0.088
25	0.023	0.004	0.004	0.048	0.071	0.024	0.023	0.094	0.120	0.054	0.053	0.133
50	0.021	0.006	0.006	0.064	0.068	0.030	0.030	0.118	0.117	0.064	0.064	0.164
75	0.021	0.007	0.007	0.075	0.068	0.034	0.034	0.132	0.116	0.070	0.070	0.178
100	0.020	0.008	0.009	0.082	0.066	0.038	0.037	0.142	0.114	0.075	0.076	0.185
125	0.019	0.009	0.009	0.088	0.064	0.040	0.040	0.149	0.115	0.075	0.078	0.190
150	0.020	0.010	0.010	0.093	0.066	0.040	0.041	0.151	0.115	0.078	0.079	0.197
200	0.019	0.010	0.011	0.096	0.066	0.043	0.044	0.159	0.115	0.082	0.084	0.207
250	0.018	0.011	0.012	0.101	0.064	0.045	0.047	0.166	0.113	0.086	0.087	0.214
Right-sided tests - $T = 1000$												
-5	0.005	0.015	0.054	0.033	0.028	0.083	0.132	0.092	0.070	0.175	0.221	0.156
-2.5	0.003	0.014	0.041	0.044	0.020	0.098	0.146	0.136	0.054	0.241	0.283	0.258
0	0.003	0.019	0.024	0.070	0.027	0.091	0.115	0.191	0.071	0.204	0.234	0.309
2.5	0.004	0.019	0.024	0.086	0.033	0.098	0.110	0.213	0.086	0.197	0.212	0.320
5	0.005	0.020	0.024	0.095	0.040	0.095	0.103	0.218	0.090	0.186	0.195	0.315
10	0.006	0.020	0.023	0.105	0.045	0.091	0.093	0.219	0.095	0.169	0.175	0.308
25	0.007	0.018	0.019	0.115	0.049	0.079	0.082	0.220	0.101	0.153	0.156	0.300
50	0.008	0.017	0.016	0.121	0.051	0.072	0.074	0.222	0.103	0.140	0.145	0.296
75	0.008	0.016	0.017	0.123	0.049	0.070	0.074	0.219	0.108	0.133	0.138	0.290
100	0.009	0.015	0.016	0.125	0.053	0.067	0.072	0.216	0.106	0.129	0.133	0.287
125	0.009	0.015	0.015	0.126	0.055	0.065	0.070	0.214	0.106	0.126	0.131	0.283
150	0.009	0.014	0.016	0.126	0.054	0.065	0.069	0.214	0.107	0.123	0.127	0.280
200	0.010	0.014	0.016	0.125	0.056	0.061	0.066	0.213	0.109	0.120	0.127	0.277
250	0.010	0.013	0.016	0.125	0.054	0.060	0.066	0.213	0.109	0.117	0.123	0.275
Two-sided tests- $T = 1000$												
-5	0.004	0.007	0.041	0.026	0.027	0.038	0.086	0.071	0.070	0.086	0.136	0.109
-2.5	0.002	0.007	0.025	0.032	0.015	0.038	0.081	0.084	0.046	0.097	0.147	0.143
0	0.003	0.010	0.013	0.054	0.021	0.043	0.060	0.126	0.053	0.091	0.117	0.200
2.5	0.003	0.010	0.014	0.066	0.024	0.049	0.058	0.153	0.064	0.102	0.114	0.230
5	0.004	0.011	0.014	0.076	0.030	0.051	0.058	0.171	0.072	0.105	0.111	0.248
10	0.007	0.011	0.013	0.095	0.036	0.053	0.056	0.194	0.084	0.105	0.108	0.274
25	0.011	0.011	0.012	0.125	0.049	0.054	0.056	0.233	0.102	0.104	0.105	0.314
50	0.013	0.011	0.011	0.146	0.055	0.053	0.056	0.260	0.109	0.101	0.104	0.340
75	0.014	0.011	0.012	0.158	0.059	0.053	0.056	0.273	0.112	0.104	0.108	0.352
100	0.014	0.011	0.013	0.168	0.060	0.052	0.058	0.282	0.113	0.102	0.109	0.359
125	0.015	0.012	0.013	0.173	0.061	0.055	0.059	0.286	0.116	0.104	0.110	0.363
150	0.016	0.012	0.014	0.176	0.061	0.055	0.061	0.289	0.116	0.105	0.111	0.365
200	0.015	0.012	0.014	0.181	0.062	0.056	0.062	0.294	0.118	0.103	0.111	0.372
250	0.016	0.012	0.014	0.184	0.065	0.057	0.062	0.300	0.116	0.102	0.113	0.379

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (12) and (14) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.28: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP8 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\,\boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim iidt_5(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad -0.9; -0.9 \quad 1]$ and $\sigma_{it}^2 = 0.05 + 0.1e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2$, $i = 1, 2$. $t_5(\mathbf{0}, \boldsymbol{\Omega})$ defines a mean zero Student-t distribution with 5 degrees of freedom and variance matrix $\boldsymbol{\Omega}$).

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 250$												
-5	0.007	0.001	0.001	0.007	0.046	0.006	0.007	0.019	0.097	0.017	0.017	0.035
-2.5	0.007	0.001	0.001	0.003	0.044	0.002	0.002	0.009	0.092	0.006	0.006	0.014
0	0.009	0.001	0.001	0.005	0.035	0.003	0.004	0.010	0.066	0.008	0.008	0.016
2.5	0.017	0.001	0.002	0.006	0.054	0.007	0.008	0.018	0.091	0.019	0.019	0.032
5	0.022	0.002	0.002	0.009	0.065	0.013	0.014	0.028	0.108	0.032	0.031	0.050
10	0.022	0.004	0.004	0.016	0.067	0.021	0.021	0.044	0.113	0.044	0.043	0.074
25	0.018	0.006	0.006	0.026	0.063	0.029	0.029	0.064	0.116	0.058	0.059	0.101
50	0.016	0.006	0.008	0.034	0.064	0.032	0.035	0.081	0.115	0.069	0.073	0.123
75	0.016	0.007	0.009	0.037	0.062	0.036	0.041	0.090	0.117	0.079	0.082	0.136
100	0.016	0.007	0.010	0.040	0.062	0.040	0.045	0.095	0.116	0.083	0.088	0.143
125	0.014	0.007	0.011	0.041	0.062	0.042	0.049	0.098	0.116	0.087	0.093	0.150
150	0.014	0.007	0.011	0.043	0.062	0.043	0.051	0.101	0.119	0.091	0.098	0.153
200	0.013	0.008	0.012	0.046	0.062	0.047	0.056	0.107	0.120	0.096	0.105	0.160
250	0.011	0.008	0.014	0.047	0.063	0.049	0.060	0.109	0.121	0.098	0.109	0.165
Right-sided tests - $T = 250$												
-5	0.009	0.015	0.034	0.037	0.045	0.086	0.110	0.107	0.087	0.173	0.188	0.178
-2.5	0.006	0.016	0.029	0.035	0.038	0.110	0.129	0.139	0.089	0.251	0.257	0.270
0	0.007	0.018	0.024	0.047	0.048	0.101	0.122	0.163	0.106	0.223	0.242	0.291
2.5	0.010	0.019	0.027	0.054	0.056	0.103	0.120	0.173	0.120	0.208	0.221	0.285
5	0.010	0.021	0.027	0.059	0.063	0.099	0.111	0.171	0.122	0.195	0.208	0.267
10	0.011	0.021	0.025	0.061	0.062	0.089	0.101	0.165	0.124	0.175	0.185	0.252
25	0.012	0.018	0.023	0.063	0.060	0.078	0.088	0.154	0.119	0.152	0.161	0.231
50	0.012	0.014	0.019	0.060	0.061	0.071	0.083	0.148	0.120	0.135	0.145	0.217
75	0.011	0.014	0.017	0.058	0.059	0.065	0.078	0.143	0.125	0.131	0.141	0.212
100	0.011	0.012	0.017	0.056	0.061	0.063	0.075	0.137	0.122	0.127	0.137	0.205
125	0.010	0.012	0.016	0.055	0.059	0.060	0.073	0.134	0.122	0.121	0.132	0.200
150	0.009	0.010	0.016	0.058	0.060	0.058	0.071	0.130	0.121	0.119	0.129	0.194
200	0.010	0.009	0.016	0.055	0.061	0.057	0.069	0.126	0.119	0.111	0.125	0.186
250	0.011	0.009	0.015	0.052	0.063	0.056	0.067	0.121	0.119	0.109	0.117	0.179
Two-sided tests - $T = 250$												
-5	0.008	0.007	0.025	0.029	0.044	0.042	0.068	0.080	0.088	0.091	0.117	0.126
-2.5	0.006	0.007	0.017	0.024	0.033	0.046	0.069	0.080	0.079	0.112	0.132	0.148
0	0.006	0.009	0.013	0.034	0.040	0.047	0.065	0.103	0.086	0.104	0.126	0.173
2.5	0.007	0.009	0.014	0.040	0.045	0.052	0.067	0.114	0.100	0.110	0.127	0.190
5	0.008	0.010	0.015	0.043	0.050	0.057	0.068	0.125	0.108	0.111	0.124	0.199
10	0.011	0.012	0.015	0.049	0.056	0.057	0.066	0.135	0.111	0.111	0.122	0.209
25	0.013	0.012	0.016	0.059	0.058	0.056	0.064	0.143	0.115	0.105	0.116	0.218
50	0.013	0.010	0.015	0.066	0.062	0.052	0.063	0.152	0.119	0.104	0.118	0.229
75	0.012	0.010	0.014	0.066	0.060	0.050	0.062	0.157	0.120	0.100	0.119	0.233
100	0.011	0.008	0.014	0.066	0.060	0.047	0.062	0.159	0.120	0.103	0.121	0.232
125	0.010	0.009	0.014	0.069	0.058	0.048	0.061	0.158	0.119	0.101	0.122	0.232
150	0.010	0.009	0.014	0.070	0.060	0.049	0.063	0.158	0.119	0.102	0.121	0.231
200	0.009	0.009	0.015	0.071	0.058	0.050	0.067	0.158	0.122	0.104	0.124	0.233
250	0.009	0.008	0.015	0.070	0.060	0.052	0.066	0.162	0.124	0.105	0.127	0.229

c	$t_{zx}^{*,RWB}$ (1%)	$t_{zx}^{*,FRWB}$ (1%)	t_{zx}^{EW} (1%)	t_{zx} (1%)	$t_{zx}^{*,RWB}$ (5%)	$t_{zx}^{*,FRWB}$ (5%)	t_{zx}^{EW} (5%)	t_{zx} (5%)	$t_{zx}^{*,RWB}$ (10%)	$t_{zx}^{*,FRWB}$ (10%)	t_{zx}^{EW} (10%)	t_{zx} (10%)
Left-sided tests - $T = 1000$												
-5	0.008	0.000	0.002	0.006	0.045	0.005	0.007	0.019	0.097	0.016	0.017	0.032
-2.5	0.009	0.000	0.000	0.003	0.047	0.001	0.001	0.008	0.103	0.003	0.003	0.013
0	0.007	0.001	0.001	0.004	0.030	0.002	0.003	0.010	0.060	0.006	0.006	0.015
2.5	0.014	0.001	0.001	0.006	0.049	0.005	0.005	0.018	0.088	0.016	0.016	0.036
5	0.018	0.002	0.001	0.011	0.062	0.010	0.009	0.034	0.105	0.026	0.024	0.059
10	0.022	0.003	0.003	0.023	0.065	0.016	0.014	0.056	0.112	0.040	0.038	0.089
25	0.019	0.005	0.005	0.044	0.068	0.026	0.023	0.092	0.114	0.054	0.053	0.130
50	0.020	0.007	0.006	0.062	0.064	0.032	0.032	0.115	0.115	0.066	0.065	0.156
75	0.019	0.007	0.007	0.071	0.065	0.036	0.036	0.126	0.114	0.071	0.071	0.169
100	0.019	0.009	0.009	0.079	0.066	0.038	0.039	0.135	0.114	0.076	0.076	0.176
125	0.018	0.009	0.010	0.084	0.064	0.040	0.041	0.139	0.113	0.078	0.080	0.182
150	0.018	0.010	0.011	0.086	0.065	0.042	0.042	0.143	0.115	0.080	0.081	0.189
200	0.018	0.011	0.012	0.089	0.063	0.043	0.045	0.150	0.113	0.083	0.085	0.196
250	0.017	0.010	0.012	0.092	0.064	0.045	0.048	0.155	0.112	0.083	0.086	0.206
Right-sided tests - $T = 1000$												
-5	0.005	0.015	0.054	0.031	0.030	0.085	0.127	0.091	0.071	0.174	0.208	0.156
-2.5	0.004	0.016	0.049	0.042	0.024	0.105	0.150	0.132	0.063	0.241	0.273	0.248
0	0.005	0.019	0.029	0.063	0.031	0.095	0.120	0.179	0.078	0.209	0.236	0.295
2.5	0.005	0.018	0.026	0.077	0.038	0.098	0.111	0.194	0.091	0.198	0.209	0.298
5	0.006	0.020	0.024	0.085	0.042	0.096	0.103	0.198	0.096	0.186	0.193	0.292
10	0.007	0.021	0.023	0.093	0.047	0.089	0.094	0.198	0.099	0.170	0.174	0.285
25	0.007	0.018	0.019	0.102	0.050	0.082	0.084	0.202	0.105	0.150	0.154	0.278
50	0.008	0.016	0.016	0.107	0.051	0.070	0.074	0.203	0.105	0.141	0.145	0.275
75	0.010	0.017	0.017	0.108	0.051	0.068	0.072	0.202	0.107	0.132	0.138	0.270
100	0.009	0.015	0.017	0.109	0.052	0.066	0.070	0.199	0.104	0.125	0.132	0.266
125	0.009	0.015	0.016	0.110	0.054	0.063	0.067	0.196	0.105	0.123	0.128	0.263
150	0.010	0.015	0.015	0.110	0.055	0.062	0.067	0.195	0.105	0.122	0.126	0.260
200	0.011	0.014	0.016	0.109	0.054	0.062	0.067	0.194	0.108	0.116	0.121	0.260
250	0.010	0.014	0.015	0.109	0.054	0.060	0.065	0.195	0.108	0.114	0.120	0.258
Two-sided tests- $T = 1000$												
-5	0.004	0.007	0.042	0.026	0.029	0.042	0.086	0.069	0.073	0.088	0.134	0.110
-2.5	0.002	0.008	0.033	0.029	0.020	0.044	0.089	0.081	0.052	0.105	0.151	0.140
0	0.004	0.010	0.017	0.048	0.023	0.046	0.066	0.114	0.059	0.096	0.123	0.188
2.5	0.004	0.011	0.015	0.060	0.028	0.049	0.060	0.141	0.068	0.104	0.116	0.212
5	0.005	0.011	0.014	0.070	0.031	0.053	0.059	0.158	0.075	0.105	0.112	0.231
10	0.008	0.012	0.013	0.084	0.039	0.054	0.058	0.177	0.085	0.105	0.108	0.254
25	0.011	0.011	0.012	0.110	0.049	0.052	0.054	0.214	0.101	0.107	0.107	0.294
50	0.013	0.012	0.012	0.129	0.055	0.051	0.054	0.239	0.109	0.102	0.106	0.317
75	0.014	0.011	0.012	0.140	0.060	0.052	0.056	0.255	0.114	0.104	0.108	0.327
100	0.014	0.011	0.011	0.149	0.061	0.053	0.057	0.261	0.114	0.103	0.109	0.334
125	0.015	0.011	0.013	0.153	0.061	0.054	0.058	0.266	0.116	0.103	0.108	0.335
150	0.015	0.012	0.013	0.160	0.060	0.056	0.061	0.267	0.117	0.103	0.109	0.338
200	0.016	0.012	0.015	0.163	0.063	0.056	0.061	0.270	0.115	0.104	0.112	0.345
250	0.014	0.013	0.015	0.163	0.064	0.055	0.061	0.276	0.117	0.105	0.114	0.349

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.29: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP8 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\,\boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim iid t_5(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad -0.5; -0.5 \quad 1]$ and $\sigma_{it}^2 = 0.05 + 0.1 e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2$, $i = 1, 2$. $t_5(\mathbf{0}, \boldsymbol{\Omega})$ defines a mean zero Student-t distribution with 5 degrees of freedom and variance matrix $\boldsymbol{\Omega}$).

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
Left-sided tests - $T = 250$												
-5	0.008	0.003	0.009	0.013	0.048	0.019	0.027	0.036	0.102	0.046	0.050	0.064
-2.5	0.008	0.002	0.004	0.007	0.043	0.009	0.012	0.019	0.091	0.025	0.024	0.034
0	0.007	0.002	0.005	0.007	0.033	0.012	0.016	0.022	0.070	0.027	0.030	0.039
2.5	0.011	0.005	0.006	0.011	0.048	0.021	0.026	0.035	0.089	0.045	0.049	0.062
5	0.014	0.005	0.007	0.014	0.055	0.027	0.030	0.044	0.097	0.056	0.058	0.073
10	0.017	0.008	0.009	0.020	0.057	0.032	0.034	0.053	0.102	0.064	0.066	0.088
25	0.016	0.010	0.009	0.025	0.058	0.040	0.043	0.063	0.103	0.077	0.078	0.102
50	0.014	0.008	0.009	0.026	0.058	0.042	0.046	0.069	0.106	0.085	0.089	0.114
75	0.012	0.008	0.010	0.024	0.059	0.045	0.048	0.074	0.108	0.087	0.092	0.119
100	0.010	0.008	0.010	0.025	0.057	0.045	0.050	0.076	0.110	0.090	0.095	0.123
125	0.010	0.008	0.011	0.025	0.056	0.045	0.051	0.077	0.109	0.092	0.099	0.124
150	0.010	0.009	0.011	0.026	0.054	0.045	0.052	0.078	0.108	0.092	0.099	0.122
200	0.009	0.009	0.013	0.028	0.055	0.045	0.053	0.077	0.109	0.095	0.103	0.127
250	0.009	0.009	0.013	0.027	0.054	0.048	0.054	0.079	0.109	0.098	0.107	0.131
Right-sided tests - $T = 250$												
-5	0.010	0.018	0.039	0.033	0.048	0.078	0.087	0.096	0.099	0.151	0.146	0.160
-2.5	0.010	0.021	0.037	0.030	0.053	0.102	0.102	0.113	0.110	0.202	0.178	0.203
0	0.011	0.017	0.031	0.029	0.057	0.087	0.104	0.109	0.115	0.185	0.181	0.197
2.5	0.012	0.017	0.027	0.032	0.058	0.084	0.096	0.108	0.116	0.160	0.174	0.187
5	0.012	0.017	0.023	0.032	0.057	0.079	0.088	0.104	0.114	0.149	0.156	0.180
10	0.012	0.014	0.019	0.031	0.055	0.071	0.079	0.101	0.109	0.138	0.147	0.172
25	0.010	0.013	0.016	0.032	0.053	0.062	0.070	0.095	0.109	0.125	0.132	0.157
50	0.010	0.013	0.016	0.032	0.053	0.059	0.066	0.090	0.105	0.116	0.125	0.153
75	0.009	0.011	0.015	0.031	0.054	0.057	0.064	0.090	0.108	0.115	0.121	0.148
100	0.009	0.010	0.014	0.031	0.055	0.056	0.065	0.090	0.110	0.113	0.121	0.144
125	0.009	0.011	0.014	0.031	0.055	0.055	0.062	0.087	0.110	0.110	0.120	0.143
150	0.009	0.009	0.014	0.032	0.054	0.054	0.062	0.087	0.111	0.110	0.119	0.145
200	0.009	0.010	0.015	0.032	0.056	0.054	0.061	0.087	0.113	0.107	0.116	0.145
250	0.010	0.011	0.015	0.032	0.058	0.052	0.061	0.085	0.112	0.103	0.114	0.145
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.035	0.031	0.048	0.049	0.073	0.081	0.096	0.096	0.114	0.132
-2.5	0.008	0.012	0.030	0.020	0.047	0.056	0.071	0.075	0.097	0.111	0.114	0.132
0	0.009	0.010	0.024	0.024	0.044	0.049	0.069	0.076	0.093	0.098	0.120	0.130
2.5	0.011	0.012	0.021	0.028	0.049	0.052	0.068	0.083	0.101	0.104	0.122	0.144
5	0.012	0.012	0.019	0.031	0.055	0.052	0.065	0.089	0.105	0.105	0.119	0.148
10	0.015	0.012	0.015	0.034	0.056	0.054	0.062	0.093	0.108	0.104	0.113	0.154
25	0.014	0.010	0.013	0.037	0.057	0.050	0.060	0.100	0.106	0.103	0.113	0.158
50	0.012	0.010	0.014	0.039	0.056	0.048	0.057	0.102	0.111	0.101	0.112	0.159
75	0.010	0.009	0.014	0.037	0.054	0.048	0.056	0.104	0.111	0.101	0.113	0.163
100	0.009	0.009	0.012	0.037	0.054	0.048	0.059	0.105	0.110	0.101	0.115	0.166
125	0.009	0.008	0.013	0.038	0.052	0.047	0.061	0.104	0.109	0.098	0.113	0.165
150	0.009	0.009	0.014	0.038	0.052	0.049	0.060	0.101	0.109	0.099	0.114	0.165
200	0.008	0.009	0.015	0.040	0.053	0.049	0.062	0.100	0.109	0.098	0.114	0.164
250	0.008	0.010	0.014	0.040	0.052	0.049	0.062	0.102	0.110	0.099	0.115	0.164

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
Left-sided tests - $T = 1000$												
-5	0.010	0.003	0.011	0.013	0.050	0.021	0.030	0.035	0.103	0.051	0.053	0.059
-2.5	0.007	0.002	0.006	0.007	0.042	0.010	0.014	0.018	0.093	0.026	0.026	0.036
0	0.005	0.002	0.006	0.007	0.031	0.011	0.016	0.021	0.066	0.030	0.031	0.042
2.5	0.006	0.003	0.006	0.011	0.042	0.018	0.023	0.037	0.083	0.045	0.047	0.064
5	0.009	0.004	0.006	0.016	0.047	0.022	0.026	0.046	0.092	0.053	0.057	0.079
10	0.010	0.005	0.006	0.021	0.049	0.026	0.028	0.057	0.100	0.061	0.064	0.093
25	0.011	0.007	0.006	0.031	0.050	0.031	0.033	0.071	0.102	0.071	0.070	0.109
50	0.013	0.008	0.008	0.041	0.055	0.038	0.039	0.082	0.103	0.077	0.077	0.126
75	0.015	0.009	0.009	0.044	0.056	0.040	0.041	0.089	0.109	0.081	0.082	0.132
100	0.014	0.009	0.010	0.045	0.057	0.041	0.042	0.094	0.110	0.085	0.085	0.138
125	0.014	0.010	0.011	0.047	0.057	0.042	0.043	0.095	0.110	0.087	0.089	0.141
150	0.014	0.011	0.011	0.046	0.056	0.043	0.044	0.099	0.110	0.089	0.091	0.143
200	0.013	0.010	0.010	0.047	0.054	0.043	0.045	0.100	0.109	0.091	0.092	0.148
250	0.012	0.010	0.011	0.048	0.055	0.045	0.046	0.100	0.111	0.092	0.094	0.150
Right-sided tests - $T = 1000$												
-5	0.008	0.016	0.035	0.028	0.041	0.074	0.081	0.073	0.088	0.144	0.132	0.129
-2.5	0.010	0.020	0.045	0.030	0.049	0.100	0.102	0.098	0.095	0.191	0.162	0.176
0	0.010	0.020	0.040	0.034	0.052	0.098	0.106	0.108	0.106	0.185	0.176	0.190
2.5	0.011	0.020	0.034	0.038	0.053	0.089	0.097	0.112	0.106	0.163	0.169	0.184
5	0.011	0.019	0.029	0.039	0.055	0.081	0.094	0.110	0.103	0.154	0.161	0.177
10	0.011	0.017	0.022	0.041	0.053	0.077	0.085	0.112	0.104	0.145	0.150	0.174
25	0.011	0.014	0.015	0.046	0.054	0.068	0.073	0.111	0.106	0.134	0.136	0.176
50	0.011	0.014	0.014	0.048	0.053	0.065	0.069	0.111	0.105	0.124	0.127	0.171
75	0.010	0.013	0.013	0.049	0.053	0.061	0.064	0.113	0.104	0.118	0.123	0.171
100	0.011	0.012	0.013	0.050	0.052	0.060	0.064	0.112	0.105	0.116	0.119	0.167
125	0.010	0.013	0.013	0.049	0.052	0.057	0.062	0.110	0.106	0.116	0.120	0.168
150	0.010	0.013	0.013	0.049	0.053	0.056	0.060	0.112	0.103	0.113	0.116	0.168
200	0.010	0.013	0.014	0.049	0.052	0.056	0.059	0.110	0.103	0.109	0.113	0.167
250	0.009	0.013	0.015	0.048	0.052	0.055	0.057	0.110	0.105	0.107	0.112	0.165
Two-sided tests- $T = 1000$												
-5	0.008	0.010	0.037	0.029	0.041	0.046	0.074	0.068	0.085	0.095	0.112	0.108
-2.5	0.009	0.011	0.039	0.024	0.042	0.053	0.078	0.069	0.084	0.110	0.116	0.116
0	0.007	0.011	0.033	0.027	0.040	0.055	0.075	0.077	0.082	0.109	0.122	0.129
2.5	0.008	0.011	0.027	0.032	0.044	0.053	0.072	0.092	0.092	0.105	0.119	0.149
5	0.008	0.011	0.023	0.038	0.045	0.054	0.070	0.099	0.094	0.102	0.120	0.156
10	0.009	0.011	0.016	0.045	0.048	0.052	0.061	0.110	0.096	0.102	0.113	0.169
25	0.011	0.012	0.013	0.055	0.050	0.052	0.055	0.125	0.103	0.099	0.105	0.182
50	0.012	0.011	0.012	0.064	0.055	0.051	0.055	0.135	0.105	0.104	0.108	0.193
75	0.012	0.011	0.011	0.068	0.056	0.053	0.056	0.142	0.109	0.102	0.106	0.202
100	0.012	0.011	0.012	0.071	0.054	0.051	0.056	0.144	0.107	0.101	0.106	0.206
125	0.012	0.010	0.011	0.072	0.055	0.052	0.056	0.144	0.107	0.098	0.105	0.205
150	0.012	0.011	0.012	0.072	0.054	0.052	0.056	0.144	0.106	0.097	0.104	0.211
200	0.011	0.011	0.013	0.073	0.054	0.052	0.056	0.145	0.105	0.097	0.104	0.210
250	0.012	0.012	0.012	0.074	0.055	0.052	0.057	0.148	0.109	0.098	0.103	0.210

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.30: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP8 (GARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = [\sigma_{1t} \quad 0; 0 \quad \sigma_{2t}]\, \boldsymbol{\eta}_t$; $\boldsymbol{\eta}_t := (\eta_{1t}, \eta_{2t})' \sim iid t_5(\mathbf{0}, \boldsymbol{\Omega})$ with $\boldsymbol{\Omega} = [1 \quad 0; 0 \quad 1]$ and $\sigma_{it}^2 = 0.05 + 0.1 e_{i,t-1}^2 + 0.85 \sigma_{i,t-1}^2$, $i = 1, 2$. $t_5(\mathbf{0}, \boldsymbol{\Omega})$ defines a mean zero Student-t distribution with 5 degrees of freedom and variance matrix $\boldsymbol{\Omega}$).

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
Left-sided tests - $T = 250$												
-5	0.011	0.011	0.022	0.025	0.051	0.049	0.057	0.067	0.101	0.099	0.097	0.107
-2.5	0.012	0.012	0.021	0.019	0.049	0.048	0.047	0.056	0.093	0.095	0.084	0.102
0	0.012	0.012	0.022	0.018	0.046	0.048	0.056	0.057	0.093	0.095	0.093	0.104
2.5	0.012	0.012	0.019	0.021	0.051	0.049	0.057	0.062	0.097	0.097	0.103	0.111
5	0.012	0.012	0.016	0.021	0.052	0.050	0.057	0.066	0.099	0.101	0.105	0.114
10	0.013	0.011	0.013	0.021	0.055	0.051	0.055	0.068	0.103	0.100	0.106	0.115
25	0.014	0.012	0.014	0.023	0.053	0.051	0.057	0.067	0.106	0.102	0.107	0.117
50	0.010	0.009	0.013	0.021	0.053	0.049	0.055	0.069	0.106	0.099	0.107	0.117
75	0.009	0.009	0.012	0.020	0.054	0.050	0.055	0.068	0.108	0.101	0.109	0.120
100	0.010	0.008	0.012	0.020	0.053	0.051	0.055	0.068	0.107	0.101	0.107	0.120
125	0.009	0.009	0.013	0.019	0.053	0.049	0.055	0.067	0.107	0.099	0.106	0.119
150	0.010	0.009	0.013	0.020	0.051	0.049	0.055	0.068	0.105	0.097	0.105	0.117
200	0.010	0.010	0.013	0.021	0.050	0.047	0.053	0.069	0.105	0.098	0.105	0.120
250	0.008	0.010	0.014	0.022	0.051	0.048	0.054	0.070	0.107	0.098	0.108	0.120
Right-sided tests - $T = 250$												
-5	0.011	0.011	0.022	0.026	0.053	0.053	0.057	0.068	0.105	0.101	0.095	0.113
-2.5	0.009	0.010	0.021	0.017	0.053	0.054	0.049	0.061	0.100	0.102	0.086	0.109
0	0.010	0.010	0.016	0.020	0.050	0.051	0.055	0.061	0.096	0.099	0.093	0.109
2.5	0.010	0.010	0.018	0.022	0.051	0.051	0.058	0.065	0.101	0.096	0.103	0.112
5	0.011	0.010	0.016	0.022	0.051	0.050	0.056	0.066	0.103	0.099	0.105	0.115
10	0.011	0.009	0.013	0.021	0.054	0.051	0.057	0.066	0.104	0.100	0.106	0.117
25	0.011	0.010	0.012	0.021	0.053	0.050	0.057	0.066	0.101	0.099	0.105	0.117
50	0.011	0.009	0.012	0.024	0.052	0.052	0.058	0.068	0.103	0.102	0.107	0.119
75	0.011	0.011	0.014	0.024	0.054	0.051	0.056	0.068	0.102	0.098	0.107	0.122
100	0.010	0.011	0.014	0.026	0.055	0.052	0.058	0.071	0.108	0.101	0.107	0.123
125	0.010	0.011	0.014	0.026	0.055	0.054	0.059	0.072	0.108	0.104	0.109	0.124
150	0.010	0.010	0.015	0.025	0.057	0.054	0.062	0.073	0.111	0.106	0.112	0.125
200	0.010	0.010	0.014	0.026	0.058	0.054	0.061	0.074	0.111	0.103	0.112	0.127
250	0.009	0.010	0.013	0.026	0.056	0.053	0.059	0.074	0.115	0.105	0.114	0.127
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.034	0.037	0.051	0.052	0.074	0.084	0.102	0.103	0.114	0.135
-2.5	0.011	0.011	0.031	0.023	0.050	0.051	0.063	0.070	0.098	0.100	0.096	0.117
0	0.011	0.011	0.028	0.024	0.050	0.049	0.068	0.070	0.094	0.099	0.111	0.118
2.5	0.012	0.012	0.025	0.027	0.053	0.053	0.070	0.077	0.101	0.098	0.115	0.127
5	0.013	0.011	0.020	0.026	0.054	0.052	0.065	0.077	0.102	0.098	0.113	0.132
10	0.013	0.011	0.015	0.028	0.052	0.051	0.061	0.079	0.108	0.101	0.113	0.135
25	0.013	0.010	0.014	0.031	0.052	0.049	0.058	0.082	0.106	0.101	0.114	0.133
50	0.010	0.009	0.014	0.029	0.052	0.051	0.059	0.082	0.105	0.101	0.113	0.137
75	0.010	0.008	0.013	0.030	0.053	0.049	0.061	0.082	0.106	0.100	0.111	0.136
100	0.009	0.008	0.014	0.029	0.052	0.049	0.061	0.084	0.108	0.101	0.114	0.139
125	0.008	0.009	0.014	0.029	0.050	0.050	0.059	0.084	0.107	0.103	0.114	0.139
150	0.008	0.010	0.015	0.030	0.050	0.049	0.059	0.085	0.106	0.100	0.116	0.141
200	0.009	0.010	0.015	0.030	0.049	0.049	0.061	0.085	0.107	0.102	0.115	0.143
250	0.008	0.010	0.015	0.032	0.050	0.049	0.061	0.087	0.105	0.099	0.114	0.144

c	$t_{zx}^{*,RWB}$ 1%	$t_{zx}^{*,FRWB}$ 1%	t_{zx}^{EW} 1%	t_{zx} 1%	$t_{zx}^{*,RWB}$ 5%	$t_{zx}^{*,FRWB}$ 5%	t_{zx}^{EW} 5%	t_{zx} 5%	$t_{zx}^{*,RWB}$ 10%	$t_{zx}^{*,FRWB}$ 10%	t_{zx}^{EW} 10%	t_{zx} 10%
Left-sided tests - $T = 1000$												
-5	0.009	0.010	0.023	0.022	0.047	0.048	0.050	0.055	0.099	0.097	0.083	0.096
-2.5	0.009	0.009	0.016	0.017	0.047	0.046	0.040	0.051	0.095	0.099	0.072	0.095
0	0.007	0.007	0.018	0.016	0.047	0.048	0.051	0.056	0.093	0.095	0.088	0.100
2.5	0.009	0.009	0.017	0.019	0.047	0.045	0.055	0.061	0.096	0.097	0.102	0.104
5	0.010	0.009	0.016	0.020	0.049	0.047	0.057	0.060	0.099	0.096	0.103	0.106
10	0.009	0.009	0.014	0.022	0.046	0.046	0.054	0.061	0.097	0.095	0.102	0.107
25	0.010	0.010	0.011	0.025	0.048	0.048	0.051	0.068	0.096	0.093	0.098	0.109
50	0.010	0.011	0.010	0.028	0.048	0.047	0.049	0.071	0.099	0.098	0.101	0.116
75	0.011	0.010	0.011	0.029	0.050	0.049	0.050	0.074	0.101	0.098	0.101	0.118
100	0.011	0.010	0.012	0.031	0.051	0.050	0.053	0.074	0.100	0.096	0.100	0.117
125	0.010	0.010	0.010	0.030	0.051	0.050	0.051	0.075	0.101	0.097	0.099	0.120
150	0.009	0.009	0.010	0.030	0.049	0.050	0.051	0.075	0.101	0.097	0.099	0.122
200	0.009	0.009	0.010	0.030	0.050	0.047	0.051	0.077	0.101	0.098	0.101	0.123
250	0.009	0.008	0.009	0.030	0.050	0.050	0.052	0.078	0.102	0.098	0.101	0.124
Right-sided tests - $T = 1000$												
-5	0.010	0.010	0.024	0.022	0.046	0.048	0.049	0.053	0.092	0.095	0.083	0.094
-2.5	0.010	0.011	0.023	0.017	0.046	0.049	0.049	0.052	0.091	0.098	0.079	0.098
0	0.010	0.011	0.018	0.016	0.045	0.049	0.052	0.056	0.093	0.101	0.091	0.104
2.5	0.010	0.009	0.020	0.020	0.047	0.051	0.059	0.061	0.100	0.102	0.105	0.110
5	0.012	0.012	0.019	0.022	0.050	0.050	0.059	0.063	0.101	0.100	0.110	0.111
10	0.011	0.010	0.015	0.024	0.050	0.051	0.056	0.067	0.100	0.102	0.108	0.115
25	0.011	0.011	0.013	0.027	0.051	0.053	0.055	0.073	0.106	0.102	0.106	0.118
50	0.011	0.011	0.011	0.029	0.055	0.053	0.056	0.076	0.104	0.101	0.106	0.123
75	0.011	0.010	0.011	0.030	0.055	0.054	0.055	0.080	0.106	0.104	0.109	0.125
100	0.010	0.009	0.011	0.031	0.056	0.053	0.056	0.080	0.106	0.104	0.107	0.127
125	0.010	0.010	0.011	0.031	0.055	0.052	0.055	0.079	0.104	0.104	0.107	0.127
150	0.010	0.010	0.011	0.031	0.053	0.051	0.055	0.080	0.104	0.102	0.107	0.128
200	0.010	0.010	0.011	0.031	0.053	0.049	0.054	0.080	0.103	0.101	0.106	0.128
250	0.011	0.010	0.011	0.033	0.054	0.049	0.053	0.079	0.105	0.100	0.104	0.130
Two-sided tests- $T = 1000$												
-5	0.009	0.011	0.037	0.030	0.046	0.047	0.070	0.069	0.094	0.097	0.100	0.108
-2.5	0.008	0.010	0.030	0.023	0.045	0.047	0.061	0.061	0.092	0.095	0.089	0.103
0	0.008	0.010	0.028	0.019	0.043	0.046	0.061	0.063	0.091	0.094	0.103	0.112
2.5	0.010	0.009	0.026	0.027	0.046	0.046	0.066	0.073	0.093	0.096	0.114	0.122
5	0.011	0.010	0.025	0.030	0.047	0.047	0.067	0.076	0.097	0.097	0.116	0.123
10	0.011	0.010	0.016	0.032	0.049	0.047	0.061	0.079	0.098	0.096	0.110	0.127
25	0.012	0.010	0.013	0.037	0.050	0.048	0.054	0.088	0.098	0.100	0.105	0.141
50	0.010	0.010	0.011	0.039	0.052	0.049	0.054	0.097	0.102	0.100	0.105	0.147
75	0.009	0.009	0.010	0.040	0.052	0.051	0.056	0.099	0.104	0.101	0.105	0.154
100	0.009	0.008	0.010	0.042	0.051	0.049	0.054	0.099	0.105	0.102	0.109	0.153
125	0.009	0.009	0.010	0.043	0.051	0.050	0.054	0.099	0.105	0.101	0.106	0.155
150	0.010	0.008	0.010	0.044	0.051	0.049	0.054	0.099	0.104	0.100	0.106	0.155
200	0.009	0.009	0.010	0.044	0.049	0.049	0.054	0.101	0.102	0.098	0.105	0.157
250	0.008	0.009	0.010	0.044	0.048	0.046	0.052	0.103	0.104	0.100	0.105	0.157

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.31: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP9 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.95; -0.95 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2 are GARCH processes generated as $\sigma_{it}^2 = 0.05 + 0.1 e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2$, $i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim NIID(\mathbf{0}, \mathbf{I_2})$ where $\mathbf{I_2}$ is a 2×2 identity matrix.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.009	0.000	0.000	0.001	0.047	0.002	0.003	0.004	0.097	0.011	0.013	0.014
-2.5	0.005	0.000	0.000	0.000	0.047	0.000	0.000	0.000	0.109	0.001	0.001	0.001
0	0.012	0.000	0.000	0.000	0.040	0.001	0.001	0.001	0.066	0.003	0.003	0.003
2.5	0.022	0.001	0.001	0.001	0.058	0.007	0.007	0.006	0.095	0.013	0.013	0.014
5	0.024	0.003	0.002	0.003	0.066	0.011	0.011	0.012	0.108	0.026	0.025	0.027
10	0.021	0.003	0.003	0.005	0.067	0.018	0.018	0.021	0.113	0.042	0.042	0.045
25	0.016	0.005	0.006	0.008	0.061	0.029	0.030	0.036	0.109	0.058	0.059	0.067
50	0.014	0.006	0.008	0.010	0.057	0.033	0.035	0.043	0.104	0.071	0.072	0.081
75	0.013	0.008	0.008	0.011	0.055	0.036	0.038	0.049	0.103	0.075	0.077	0.087
100	0.012	0.008	0.009	0.012	0.053	0.039	0.040	0.051	0.102	0.076	0.080	0.091
125	0.011	0.008	0.009	0.012	0.053	0.041	0.043	0.051	0.101	0.079	0.082	0.093
150	0.011	0.009	0.009	0.013	0.053	0.042	0.045	0.054	0.101	0.082	0.083	0.096
200	0.011	0.010	0.011	0.015	0.053	0.044	0.046	0.055	0.100	0.085	0.089	0.101
250	0.011	0.010	0.011	0.014	0.052	0.044	0.048	0.057	0.099	0.087	0.092	0.105
Right-sided tests - $T = 250$												
-5	0.009	0.015	0.019	0.017	0.041	0.069	0.080	0.072	0.088	0.146	0.158	0.142
-2.5	0.009	0.016	0.022	0.022	0.044	0.100	0.109	0.104	0.089	0.245	0.254	0.246
0	0.011	0.022	0.026	0.026	0.055	0.107	0.120	0.119	0.110	0.231	0.241	0.242
2.5	0.014	0.024	0.028	0.029	0.061	0.110	0.122	0.123	0.124	0.220	0.232	0.237
5	0.013	0.024	0.030	0.029	0.067	0.110	0.118	0.125	0.126	0.204	0.211	0.219
10	0.014	0.023	0.027	0.030	0.065	0.101	0.109	0.118	0.125	0.184	0.192	0.202
25	0.013	0.020	0.022	0.026	0.062	0.085	0.091	0.099	0.115	0.154	0.158	0.171
50	0.012	0.016	0.019	0.023	0.058	0.072	0.078	0.089	0.114	0.138	0.143	0.155
75	0.011	0.014	0.017	0.021	0.058	0.069	0.073	0.083	0.111	0.129	0.132	0.147
100	0.011	0.014	0.015	0.020	0.055	0.064	0.068	0.080	0.112	0.123	0.129	0.143
125	0.011	0.013	0.014	0.020	0.052	0.059	0.063	0.074	0.109	0.120	0.124	0.141
150	0.010	0.011	0.014	0.018	0.053	0.058	0.061	0.073	0.108	0.115	0.122	0.135
200	0.010	0.011	0.013	0.017	0.053	0.054	0.059	0.071	0.108	0.110	0.115	0.130
250	0.011	0.010	0.012	0.017	0.055	0.052	0.056	0.069	0.107	0.106	0.110	0.125
Two-sided tests - $T = 250$												
-5	0.009	0.009	0.012	0.010	0.041	0.036	0.042	0.039	0.090	0.072	0.083	0.076
-2.5	0.008	0.009	0.011	0.011	0.039	0.045	0.054	0.051	0.083	0.100	0.109	0.104
0	0.010	0.011	0.014	0.014	0.046	0.053	0.061	0.060	0.093	0.108	0.121	0.120
2.5	0.011	0.013	0.016	0.016	0.053	0.061	0.070	0.071	0.106	0.116	0.128	0.129
5	0.011	0.013	0.017	0.017	0.058	0.063	0.072	0.076	0.112	0.122	0.129	0.137
10	0.012	0.013	0.017	0.019	0.059	0.063	0.069	0.073	0.112	0.119	0.127	0.139
25	0.012	0.013	0.016	0.019	0.057	0.059	0.065	0.076	0.111	0.113	0.120	0.134
50	0.011	0.011	0.014	0.019	0.055	0.055	0.060	0.072	0.109	0.105	0.113	0.132
75	0.012	0.011	0.014	0.018	0.052	0.050	0.056	0.070	0.109	0.104	0.110	0.132
100	0.011	0.011	0.014	0.018	0.048	0.047	0.053	0.067	0.106	0.102	0.108	0.131
125	0.010	0.011	0.013	0.017	0.050	0.046	0.053	0.069	0.102	0.099	0.106	0.125
150	0.010	0.010	0.012	0.017	0.052	0.047	0.053	0.070	0.101	0.099	0.106	0.126
200	0.010	0.010	0.013	0.017	0.052	0.049	0.053	0.069	0.104	0.097	0.104	0.126
250	0.010	0.010	0.012	0.018	0.052	0.049	0.054	0.069	0.104	0.097	0.104	0.126

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.009	0.000	0.000	0.000	0.042	0.003	0.003	0.004	0.093	0.012	0.012	0.011
-2.5	0.006	0.000	0.000	0.000	0.043	0.000	0.000	0.000	0.107	0.001	0.001	0.001
0	0.012	0.000	0.000	0.000	0.040	0.001	0.001	0.001	0.066	0.002	0.002	0.003
2.5	0.021	0.001	0.001	0.001	0.060	0.005	0.005	0.005	0.098	0.013	0.012	0.012
5	0.021	0.002	0.002	0.001	0.068	0.010	0.010	0.011	0.111	0.026	0.025	0.028
10	0.019	0.003	0.003	0.003	0.065	0.020	0.019	0.020	0.115	0.040	0.040	0.044
25	0.016	0.006	0.006	0.008	0.060	0.028	0.028	0.033	0.114	0.060	0.060	0.066
50	0.014	0.007	0.007	0.010	0.057	0.035	0.036	0.045	0.106	0.069	0.068	0.081
75	0.014	0.007	0.007	0.012	0.056	0.038	0.038	0.049	0.107	0.074	0.074	0.088
100	0.012	0.008	0.008	0.014	0.055	0.038	0.038	0.050	0.105	0.080	0.080	0.094
125	0.013	0.008	0.007	0.014	0.056	0.039	0.040	0.053	0.106	0.083	0.082	0.098
150	0.012	0.008	0.008	0.014	0.055	0.040	0.040	0.054	0.105	0.083	0.084	0.101
200	0.013	0.008	0.008	0.016	0.054	0.041	0.042	0.056	0.104	0.084	0.086	0.103
250	0.012	0.009	0.009	0.016	0.051	0.041	0.042	0.056	0.104	0.088	0.087	0.107
Right-sided tests - $T = 1000$												
-5	0.007	0.014	0.015	0.015	0.040	0.068	0.070	0.068	0.089	0.148	0.150	0.146
-2.5	0.010	0.019	0.020	0.021	0.042	0.095	0.095	0.094	0.085	0.242	0.244	0.238
0	0.008	0.021	0.020	0.022	0.054	0.113	0.115	0.114	0.109	0.229	0.233	0.233
2.5	0.010	0.023	0.022	0.024	0.060	0.116	0.119	0.122	0.123	0.226	0.228	0.231
5	0.011	0.021	0.021	0.024	0.059	0.111	0.115	0.118	0.123	0.213	0.213	0.220
10	0.011	0.020	0.021	0.024	0.058	0.099	0.101	0.108	0.115	0.190	0.191	0.198
25	0.011	0.018	0.018	0.022	0.056	0.083	0.084	0.093	0.108	0.154	0.155	0.166
50	0.010	0.016	0.016	0.022	0.055	0.075	0.077	0.088	0.108	0.144	0.145	0.158
75	0.010	0.015	0.015	0.021	0.055	0.071	0.072	0.086	0.109	0.138	0.136	0.151
100	0.011	0.014	0.014	0.020	0.055	0.070	0.070	0.083	0.108	0.131	0.132	0.148
125	0.011	0.015	0.015	0.021	0.054	0.066	0.067	0.081	0.106	0.125	0.128	0.145
150	0.012	0.014	0.015	0.022	0.053	0.064	0.064	0.080	0.105	0.124	0.124	0.140
200	0.013	0.014	0.015	0.021	0.052	0.061	0.063	0.078	0.102	0.118	0.120	0.138
250	0.012	0.014	0.014	0.019	0.053	0.060	0.061	0.075	0.104	0.117	0.117	0.135
Two-sided tests - $T = 1000$												
-5	0.007	0.008	0.008	0.009	0.040	0.033	0.034	0.034	0.091	0.070	0.072	0.071
-2.5	0.008	0.011	0.012	0.012	0.038	0.046	0.048	0.046	0.079	0.094	0.096	0.094
0	0.007	0.010	0.010	0.010	0.045	0.056	0.057	0.060	0.094	0.114	0.116	0.115
2.5	0.008	0.011	0.011	0.012	0.049	0.058	0.061	0.065	0.103	0.119	0.124	0.127
5	0.009	0.010	0.011	0.012	0.050	0.058	0.059	0.065	0.103	0.119	0.125	0.128
10	0.009	0.012	0.012	0.013	0.051	0.057	0.058	0.065	0.103	0.118	0.120	0.128
25	0.010	0.011	0.011	0.015	0.054	0.056	0.059	0.069	0.104	0.110	0.111	0.126
50	0.010	0.011	0.010	0.017	0.053	0.059	0.058	0.073	0.108	0.111	0.112	0.132
75	0.010	0.010	0.009	0.017	0.054	0.056	0.057	0.074	0.109	0.108	0.110	0.135
100	0.009	0.010	0.009	0.018	0.056	0.054	0.057	0.076	0.108	0.107	0.108	0.133
125	0.010	0.009	0.009	0.019	0.054	0.055	0.056	0.076	0.106	0.104	0.107	0.134
150	0.010	0.009	0.009	0.019	0.053	0.054	0.055	0.075	0.104	0.103	0.104	0.134
200	0.010	0.010	0.010	0.020	0.054	0.051	0.053	0.074	0.104	0.102	0.104	0.134
250	0.012	0.011	0.011	0.021	0.053	0.052	0.053	0.075	0.103	0.103	0.103	0.131

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.32: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP9 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.9; -0.9 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2 are GARCH processes generated as $\sigma_{it}^2 = 0.05 + 0.1e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2$, $i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim NIID(\mathbf{0}, \mathbf{I_2})$ where $\mathbf{I_2}$ is a 2×2 identity matrix.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests												
-5	0.008	0.000	0.001	0.001	0.048	0.003	0.005	0.005	0.098	0.013	0.015	0.016
-2.5	0.007	0.000	0.000	0.000	0.046	0.000	0.000	0.001	0.110	0.001	0.001	0.002
0	0.011	0.000	0.000	0.000	0.039	0.001	0.001	0.001	0.064	0.003	0.004	0.003
2.5	0.020	0.001	0.001	0.001	0.055	0.007	0.007	0.007	0.092	0.015	0.014	0.016
5	0.023	0.003	0.003	0.003	0.063	0.011	0.012	0.013	0.106	0.028	0.027	0.030
10	0.019	0.004	0.004	0.005	0.064	0.018	0.019	0.023	0.110	0.044	0.044	0.049
25	0.016	0.006	0.006	0.008	0.060	0.029	0.030	0.035	0.108	0.062	0.061	0.068
50	0.013	0.006	0.007	0.010	0.056	0.035	0.036	0.043	0.103	0.071	0.072	0.081
75	0.012	0.007	0.009	0.010	0.055	0.037	0.038	0.048	0.103	0.076	0.077	0.087
100	0.011	0.008	0.009	0.011	0.053	0.038	0.041	0.049	0.101	0.079	0.081	0.090
125	0.012	0.007	0.009	0.012	0.054	0.039	0.042	0.051	0.100	0.079	0.082	0.093
150	0.011	0.008	0.009	0.012	0.052	0.041	0.044	0.052	0.098	0.081	0.084	0.096
200	0.012	0.009	0.011	0.014	0.052	0.042	0.047	0.054	0.099	0.086	0.088	0.099
250	0.011	0.010	0.010	0.014	0.051	0.045	0.048	0.055	0.098	0.089	0.092	0.104
Right-sided tests												
-5	0.009	0.015	0.020	0.018	0.043	0.074	0.084	0.076	0.091	0.150	0.162	0.146
-2.5	0.010	0.018	0.023	0.020	0.044	0.102	0.111	0.105	0.094	0.244	0.253	0.243
0	0.011	0.023	0.026	0.026	0.053	0.107	0.116	0.115	0.111	0.232	0.240	0.239
2.5	0.013	0.023	0.028	0.029	0.063	0.110	0.122	0.122	0.123	0.217	0.226	0.231
5	0.013	0.024	0.028	0.030	0.067	0.107	0.114	0.119	0.127	0.200	0.207	0.215
10	0.014	0.022	0.026	0.028	0.065	0.099	0.106	0.114	0.121	0.179	0.187	0.195
25	0.011	0.017	0.020	0.025	0.060	0.082	0.087	0.096	0.115	0.151	0.157	0.167
50	0.011	0.015	0.017	0.022	0.058	0.071	0.077	0.086	0.111	0.136	0.141	0.152
75	0.010	0.013	0.016	0.020	0.056	0.068	0.073	0.080	0.109	0.126	0.133	0.145
100	0.010	0.013	0.015	0.019	0.053	0.063	0.068	0.078	0.110	0.123	0.127	0.140
125	0.010	0.012	0.014	0.018	0.053	0.058	0.064	0.074	0.110	0.119	0.123	0.138
150	0.009	0.011	0.013	0.017	0.054	0.057	0.061	0.072	0.108	0.112	0.119	0.133
200	0.009	0.011	0.012	0.017	0.053	0.054	0.058	0.069	0.107	0.110	0.114	0.127
250	0.012	0.011	0.012	0.017	0.054	0.053	0.056	0.066	0.108	0.105	0.110	0.123
Two-sided tests												
-5	0.009	0.008	0.012	0.010	0.044	0.035	0.045	0.040	0.093	0.076	0.089	0.080
-2.5	0.009	0.009	0.012	0.011	0.041	0.044	0.053	0.049	0.085	0.100	0.112	0.106
0	0.010	0.011	0.014	0.015	0.045	0.053	0.061	0.059	0.093	0.106	0.117	0.117
2.5	0.010	0.013	0.017	0.016	0.054	0.060	0.066	0.069	0.104	0.117	0.129	0.129
5	0.011	0.013	0.016	0.016	0.057	0.062	0.070	0.073	0.110	0.118	0.125	0.132
10	0.012	0.013	0.017	0.018	0.058	0.063	0.068	0.073	0.111	0.117	0.125	0.137
25	0.012	0.012	0.014	0.018	0.055	0.058	0.063	0.073	0.110	0.111	0.117	0.131
50	0.011	0.011	0.013	0.017	0.055	0.053	0.059	0.070	0.109	0.105	0.113	0.130
75	0.010	0.009	0.013	0.017	0.051	0.050	0.055	0.067	0.107	0.103	0.111	0.128
100	0.010	0.010	0.013	0.017	0.050	0.046	0.051	0.065	0.105	0.101	0.109	0.127
125	0.010	0.010	0.013	0.016	0.049	0.045	0.051	0.066	0.103	0.098	0.106	0.125
150	0.010	0.011	0.012	0.016	0.052	0.047	0.053	0.068	0.103	0.100	0.106	0.124
200	0.011	0.010	0.012	0.016	0.050	0.048	0.052	0.066	0.102	0.098	0.105	0.123
250	0.010	0.010	0.012	0.017	0.051	0.048	0.054	0.068	0.103	0.096	0.104	0.122

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests												
-5	0.009	0.000	0.000	0.000	0.044	0.004	0.003	0.004	0.094	0.013	0.013	0.014
-2.5	0.006	0.000	0.000	0.000	0.046	0.000	0.000	0.001	0.105	0.001	0.001	0.001
0	0.012	0.000	0.000	0.000	0.036	0.001	0.001	0.001	0.064	0.003	0.003	0.004
2.5	0.020	0.001	0.001	0.001	0.059	0.005	0.005	0.006	0.097	0.014	0.014	0.015
5	0.020	0.002	0.002	0.002	0.068	0.011	0.011	0.012	0.109	0.028	0.028	0.029
10	0.017	0.003	0.003	0.004	0.063	0.020	0.020	0.022	0.115	0.044	0.043	0.047
25	0.015	0.005	0.006	0.008	0.058	0.028	0.028	0.034	0.111	0.062	0.062	0.068
50	0.013	0.007	0.007	0.010	0.058	0.036	0.036	0.045	0.107	0.070	0.070	0.082
75	0.012	0.007	0.007	0.012	0.056	0.038	0.038	0.047	0.105	0.076	0.076	0.089
100	0.012	0.007	0.007	0.012	0.056	0.037	0.038	0.049	0.106	0.079	0.080	0.093
125	0.012	0.007	0.007	0.013	0.054	0.040	0.040	0.052	0.105	0.082	0.083	0.097
150	0.013	0.008	0.007	0.013	0.054	0.040	0.041	0.052	0.104	0.083	0.083	0.099
200	0.011	0.008	0.007	0.015	0.052	0.041	0.041	0.054	0.103	0.086	0.086	0.102
250	0.011	0.008	0.008	0.016	0.048	0.042	0.041	0.054	0.102	0.087	0.088	0.103
Right-sided tests												
-5	0.008	0.014	0.015	0.015	0.043	0.070	0.073	0.070	0.092	0.149	0.154	0.146
-2.5	0.009	0.020	0.021	0.020	0.040	0.098	0.099	0.099	0.090	0.240	0.242	0.237
0	0.009	0.021	0.020	0.021	0.054	0.108	0.111	0.112	0.111	0.228	0.230	0.231
2.5	0.011	0.023	0.022	0.023	0.059	0.113	0.114	0.117	0.120	0.220	0.224	0.226
5	0.011	0.022	0.022	0.023	0.059	0.108	0.111	0.113	0.123	0.207	0.209	0.212
10	0.011	0.022	0.022	0.023	0.059	0.097	0.098	0.104	0.116	0.182	0.183	0.192
25	0.012	0.019	0.019	0.021	0.057	0.080	0.081	0.090	0.107	0.150	0.152	0.162
50	0.010	0.016	0.016	0.021	0.057	0.075	0.075	0.084	0.109	0.139	0.139	0.151
75	0.010	0.014	0.014	0.020	0.056	0.071	0.073	0.085	0.109	0.134	0.134	0.147
100	0.010	0.014	0.014	0.019	0.057	0.069	0.069	0.081	0.107	0.129	0.130	0.144
125	0.011	0.014	0.015	0.020	0.056	0.067	0.067	0.079	0.105	0.124	0.125	0.140
150	0.011	0.014	0.015	0.020	0.054	0.063	0.064	0.077	0.106	0.123	0.123	0.136
200	0.012	0.014	0.015	0.020	0.054	0.061	0.062	0.075	0.104	0.116	0.118	0.135
250	0.013	0.015	0.014	0.020	0.053	0.058	0.059	0.073	0.105	0.114	0.116	0.134
Two-sided tests												
-5	0.008	0.009	0.009	0.009	0.044	0.034	0.036	0.036	0.095	0.074	0.076	0.074
-2.5	0.008	0.011	0.011	0.011	0.036	0.045	0.046	0.044	0.084	0.099	0.100	0.099
0	0.008	0.010	0.010	0.011	0.044	0.053	0.055	0.057	0.092	0.110	0.112	0.113
2.5	0.009	0.011	0.011	0.011	0.048	0.059	0.060	0.063	0.103	0.117	0.120	0.123
5	0.009	0.011	0.011	0.012	0.049	0.058	0.060	0.063	0.103	0.117	0.121	0.125
10	0.009	0.012	0.012	0.013	0.050	0.056	0.057	0.065	0.102	0.116	0.118	0.126
25	0.011	0.011	0.011	0.014	0.054	0.055	0.057	0.069	0.103	0.107	0.109	0.124
50	0.010	0.011	0.010	0.016	0.053	0.055	0.057	0.071	0.108	0.109	0.111	0.129
75	0.011	0.009	0.010	0.017	0.055	0.055	0.057	0.073	0.109	0.108	0.111	0.132
100	0.010	0.009	0.010	0.017	0.055	0.055	0.058	0.074	0.107	0.106	0.108	0.130
125	0.010	0.009	0.009	0.018	0.055	0.055	0.056	0.074	0.106	0.106	0.106	0.131
150	0.011	0.010	0.010	0.018	0.055	0.054	0.055	0.073	0.105	0.102	0.105	0.129
200	0.010	0.010	0.010	0.019	0.054	0.054	0.054	0.072	0.104	0.101	0.102	0.129
250	0.011	0.011	0.011	0.020	0.053	0.053	0.054	0.072	0.100	0.099	0.100	0.126

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.33: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP9 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.5; -0.5 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2 are GARCH processes generated as $\sigma_{it}^2 = 0.05 + 0.1e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2,\ i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim NIID(\mathbf{0}, \mathbf{I_2})$ where $\mathbf{I_2}$ is a 2×2 identity matrix.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.009	0.002	0.005	0.003	0.050	0.017	0.022	0.019	0.101	0.042	0.051	0.044
-2.5	0.010	0.001	0.002	0.001	0.048	0.005	0.008	0.005	0.100	0.014	0.018	0.015
0	0.007	0.001	0.002	0.001	0.028	0.006	0.007	0.006	0.061	0.017	0.019	0.017
2.5	0.014	0.002	0.003	0.003	0.042	0.017	0.018	0.018	0.082	0.036	0.036	0.037
5	0.014	0.004	0.005	0.005	0.050	0.024	0.024	0.025	0.092	0.048	0.050	0.049
10	0.014	0.006	0.007	0.007	0.053	0.030	0.031	0.032	0.102	0.061	0.064	0.065
25	0.013	0.007	0.008	0.010	0.051	0.038	0.039	0.040	0.100	0.073	0.076	0.077
50	0.013	0.008	0.009	0.010	0.051	0.039	0.042	0.044	0.096	0.078	0.081	0.083
75	0.011	0.008	0.009	0.009	0.049	0.041	0.044	0.044	0.097	0.081	0.084	0.086
100	0.010	0.008	0.009	0.010	0.049	0.041	0.042	0.044	0.097	0.085	0.089	0.090
125	0.011	0.009	0.010	0.009	0.048	0.042	0.045	0.046	0.097	0.085	0.089	0.091
150	0.009	0.009	0.010	0.010	0.049	0.043	0.046	0.048	0.097	0.086	0.089	0.093
200	0.010	0.009	0.010	0.010	0.051	0.045	0.048	0.050	0.099	0.089	0.093	0.094
250	0.010	0.010	0.011	0.012	0.050	0.046	0.049	0.051	0.098	0.090	0.094	0.095
Right-sided tests - $T = 250$												
-5	0.010	0.015	0.033	0.018	0.049	0.075	0.098	0.072	0.095	0.144	0.163	0.141
-2.5	0.010	0.020	0.040	0.019	0.054	0.104	0.124	0.100	0.109	0.206	0.220	0.201
0	0.012	0.019	0.029	0.021	0.060	0.098	0.110	0.100	0.123	0.199	0.213	0.200
2.5	0.012	0.018	0.024	0.020	0.063	0.092	0.100	0.097	0.121	0.173	0.181	0.175
5	0.012	0.018	0.022	0.020	0.059	0.085	0.091	0.087	0.117	0.160	0.167	0.163
10	0.011	0.017	0.019	0.018	0.057	0.074	0.081	0.079	0.109	0.145	0.149	0.146
25	0.010	0.013	0.016	0.014	0.054	0.065	0.071	0.071	0.108	0.131	0.136	0.135
50	0.009	0.011	0.013	0.012	0.053	0.060	0.064	0.065	0.107	0.122	0.128	0.127
75	0.008	0.010	0.012	0.011	0.052	0.058	0.063	0.064	0.108	0.116	0.121	0.122
100	0.008	0.009	0.011	0.011	0.050	0.056	0.061	0.062	0.107	0.115	0.119	0.120
125	0.008	0.010	0.012	0.012	0.052	0.056	0.060	0.060	0.105	0.112	0.118	0.118
150	0.008	0.010	0.012	0.011	0.052	0.055	0.059	0.060	0.107	0.109	0.114	0.113
200	0.010	0.010	0.011	0.012	0.054	0.052	0.056	0.059	0.105	0.106	0.109	0.111
250	0.010	0.011	0.013	0.013	0.051	0.053	0.056	0.056	0.106	0.104	0.107	0.109
Two-sided tests - $T = 250$												
-5	0.010	0.009	0.025	0.012	0.049	0.044	0.068	0.046	0.099	0.090	0.120	0.091
-2.5	0.009	0.010	0.026	0.010	0.049	0.051	0.075	0.052	0.097	0.109	0.132	0.106
0	0.009	0.009	0.019	0.011	0.047	0.052	0.067	0.055	0.098	0.104	0.117	0.106
2.5	0.009	0.010	0.015	0.012	0.052	0.056	0.066	0.057	0.105	0.110	0.118	0.115
5	0.010	0.010	0.014	0.012	0.052	0.056	0.064	0.059	0.103	0.106	0.115	0.112
10	0.011	0.011	0.015	0.013	0.052	0.052	0.058	0.058	0.102	0.104	0.112	0.110
25	0.011	0.010	0.013	0.013	0.051	0.050	0.056	0.056	0.102	0.101	0.109	0.110
50	0.009	0.009	0.010	0.011	0.050	0.048	0.055	0.055	0.102	0.098	0.106	0.109
75	0.009	0.008	0.010	0.010	0.050	0.047	0.053	0.056	0.100	0.098	0.107	0.108
100	0.008	0.008	0.010	0.010	0.047	0.046	0.051	0.054	0.099	0.096	0.104	0.105
125	0.008	0.008	0.010	0.010	0.046	0.044	0.048	0.053	0.102	0.097	0.105	0.107
150	0.009	0.009	0.010	0.011	0.045	0.046	0.050	0.054	0.102	0.098	0.105	0.108
200	0.010	0.010	0.012	0.012	0.050	0.049	0.055	0.055	0.102	0.096	0.104	0.108
250	0.009	0.011	0.013	0.013	0.050	0.048	0.054	0.058	0.101	0.098	0.105	0.106

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.009	0.002	0.004	0.004	0.051	0.017	0.021	0.018	0.102	0.043	0.049	0.044
-2.5	0.008	0.000	0.001	0.001	0.047	0.005	0.006	0.005	0.099	0.014	0.016	0.014
0	0.006	0.001	0.001	0.001	0.029	0.006	0.006	0.007	0.059	0.017	0.018	0.017
2.5	0.010	0.002	0.003	0.002	0.044	0.015	0.015	0.015	0.086	0.037	0.038	0.038
5	0.011	0.003	0.004	0.004	0.049	0.021	0.022	0.021	0.096	0.047	0.049	0.050
10	0.012	0.005	0.005	0.006	0.049	0.026	0.027	0.028	0.101	0.060	0.061	0.063
25	0.011	0.006	0.007	0.008	0.050	0.035	0.035	0.036	0.100	0.073	0.073	0.077
50	0.011	0.008	0.008	0.009	0.049	0.039	0.039	0.042	0.099	0.079	0.080	0.084
75	0.013	0.009	0.010	0.011	0.052	0.042	0.043	0.046	0.100	0.082	0.083	0.089
100	0.012	0.009	0.010	0.011	0.051	0.043	0.044	0.047	0.101	0.089	0.089	0.091
125	0.012	0.010	0.010	0.011	0.052	0.043	0.044	0.047	0.101	0.091	0.091	0.094
150	0.011	0.010	0.010	0.012	0.052	0.044	0.046	0.048	0.103	0.091	0.092	0.097
200	0.010	0.008	0.010	0.011	0.053	0.045	0.047	0.051	0.101	0.090	0.093	0.096
250	0.009	0.008	0.009	0.010	0.052	0.048	0.048	0.051	0.102	0.093	0.094	0.099
Right-sided tests - $T = 1000$												
-5	0.009	0.014	0.017	0.015	0.047	0.073	0.079	0.071	0.098	0.141	0.149	0.138
-2.5	0.010	0.020	0.024	0.019	0.054	0.106	0.113	0.105	0.113	0.204	0.210	0.202
0	0.010	0.018	0.019	0.018	0.058	0.101	0.104	0.097	0.123	0.199	0.205	0.196
2.5	0.011	0.020	0.020	0.018	0.061	0.093	0.093	0.093	0.119	0.174	0.178	0.175
5	0.011	0.019	0.019	0.019	0.057	0.084	0.084	0.082	0.114	0.160	0.163	0.164
10	0.013	0.018	0.017	0.018	0.054	0.075	0.075	0.076	0.107	0.145	0.144	0.144
25	0.013	0.016	0.016	0.016	0.050	0.063	0.065	0.067	0.101	0.127	0.128	0.128
50	0.012	0.014	0.014	0.015	0.053	0.062	0.062	0.066	0.104	0.121	0.121	0.124
75	0.010	0.012	0.012	0.014	0.055	0.061	0.064	0.065	0.107	0.119	0.121	0.125
100	0.010	0.012	0.013	0.013	0.056	0.061	0.062	0.065	0.105	0.117	0.118	0.121
125	0.011	0.013	0.013	0.014	0.055	0.061	0.061	0.065	0.106	0.117	0.118	0.119
150	0.012	0.013	0.013	0.014	0.054	0.060	0.060	0.064	0.104	0.114	0.115	0.119
200	0.012	0.012	0.014	0.015	0.054	0.056	0.058	0.062	0.102	0.112	0.112	0.117
250	0.012	0.013	0.013	0.015	0.053	0.057	0.059	0.061	0.103	0.112	0.111	0.116
Two-sided tests - $T = 1000$												
-5	0.009	0.008	0.010	0.008	0.048	0.043	0.051	0.045	0.098	0.090	0.098	0.088
-2.5	0.009	0.009	0.012	0.009	0.050	0.052	0.060	0.050	0.103	0.111	0.119	0.109
0	0.008	0.009	0.011	0.009	0.045	0.050	0.054	0.051	0.097	0.109	0.112	0.105
2.5	0.010	0.012	0.012	0.011	0.052	0.054	0.057	0.055	0.105	0.109	0.112	0.111
5	0.010	0.012	0.013	0.012	0.050	0.054	0.056	0.055	0.100	0.106	0.108	0.106
10	0.010	0.012	0.012	0.011	0.051	0.054	0.054	0.055	0.100	0.104	0.104	0.108
25	0.011	0.012	0.013	0.014	0.051	0.052	0.053	0.054	0.098	0.100	0.101	0.106
50	0.011	0.011	0.011	0.012	0.052	0.053	0.055	0.058	0.102	0.102	0.104	0.111
75	0.011	0.010	0.010	0.012	0.053	0.051	0.053	0.059	0.104	0.103	0.107	0.111
100	0.010	0.010	0.011	0.012	0.054	0.052	0.054	0.058	0.105	0.103	0.104	0.111
125	0.010	0.009	0.010	0.012	0.052	0.053	0.054	0.059	0.105	0.103	0.104	0.114
150	0.011	0.011	0.010	0.012	0.051	0.051	0.053	0.058	0.104	0.101	0.104	0.111
200	0.011	0.010	0.011	0.014	0.053	0.051	0.053	0.059	0.102	0.101	0.102	0.110
250	0.011	0.011	0.011	0.013	0.053	0.050	0.052	0.060	0.104	0.102	0.104	0.110

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.34: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP9 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad 0; 0 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2 are GARCH processes generated as $\sigma_{it}^2 = 0.05 + 0.1e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2,\ i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim NIID(\mathbf{0}, \mathbf{I_2})$ where $\mathbf{I_2}$ is a 2×2 identity matrix.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.010	0.011	0.025	0.012	0.049	0.049	0.070	0.052	0.101	0.101	0.122	0.098
-2.5	0.010	0.010	0.025	0.011	0.049	0.048	0.062	0.046	0.098	0.096	0.104	0.092
0	0.009	0.010	0.018	0.010	0.045	0.047	0.058	0.046	0.094	0.097	0.103	0.094
2.5	0.010	0.010	0.015	0.011	0.048	0.047	0.054	0.047	0.097	0.097	0.103	0.097
5	0.010	0.010	0.013	0.011	0.048	0.048	0.051	0.048	0.100	0.099	0.105	0.098
10	0.010	0.010	0.011	0.011	0.051	0.050	0.054	0.049	0.103	0.102	0.105	0.102
25	0.011	0.010	0.013	0.012	0.053	0.051	0.053	0.053	0.102	0.100	0.103	0.103
50	0.011	0.010	0.012	0.011	0.049	0.049	0.052	0.049	0.101	0.100	0.104	0.100
75	0.010	0.010	0.011	0.010	0.048	0.046	0.050	0.049	0.097	0.095	0.098	0.096
100	0.010	0.009	0.011	0.010	0.048	0.046	0.049	0.047	0.096	0.093	0.096	0.093
125	0.009	0.010	0.011	0.010	0.047	0.046	0.049	0.047	0.096	0.095	0.096	0.094
150	0.009	0.009	0.011	0.010	0.047	0.047	0.050	0.047	0.096	0.094	0.098	0.095
200	0.009	0.009	0.010	0.009	0.050	0.048	0.050	0.049	0.096	0.094	0.097	0.096
250	0.009	0.009	0.010	0.010	0.049	0.048	0.051	0.049	0.098	0.093	0.099	0.098
Right-sided tests - $T = 250$												
-5	0.011	0.011	0.026	0.013	0.051	0.049	0.070	0.050	0.098	0.100	0.116	0.098
-2.5	0.012	0.011	0.025	0.012	0.053	0.053	0.067	0.052	0.105	0.104	0.113	0.102
0	0.011	0.009	0.018	0.012	0.050	0.050	0.057	0.050	0.100	0.100	0.102	0.099
2.5	0.009	0.009	0.014	0.011	0.051	0.048	0.054	0.050	0.103	0.101	0.106	0.100
5	0.009	0.009	0.013	0.010	0.051	0.049	0.055	0.050	0.102	0.101	0.105	0.101
10	0.008	0.008	0.010	0.009	0.050	0.047	0.052	0.051	0.102	0.100	0.105	0.102
25	0.009	0.010	0.011	0.010	0.052	0.051	0.053	0.051	0.101	0.099	0.102	0.103
50	0.009	0.009	0.011	0.009	0.052	0.050	0.052	0.050	0.102	0.099	0.104	0.099
75	0.009	0.009	0.010	0.010	0.053	0.051	0.054	0.053	0.102	0.099	0.103	0.100
100	0.008	0.009	0.010	0.009	0.050	0.048	0.053	0.050	0.101	0.102	0.105	0.101
125	0.008	0.009	0.010	0.009	0.049	0.049	0.052	0.049	0.104	0.101	0.104	0.100
150	0.008	0.008	0.010	0.010	0.051	0.049	0.052	0.050	0.103	0.101	0.104	0.100
200	0.008	0.009	0.010	0.010	0.050	0.049	0.053	0.051	0.105	0.101	0.104	0.102
250	0.009	0.009	0.011	0.011	0.049	0.050	0.051	0.050	0.105	0.101	0.105	0.102
Two-sided tests - $T = 250$												
-5	0.011	0.011	0.038	0.015	0.049	0.050	0.089	0.053	0.100	0.099	0.140	0.102
-2.5	0.010	0.011	0.037	0.012	0.048	0.048	0.082	0.050	0.098	0.100	0.130	0.097
0	0.010	0.010	0.025	0.011	0.047	0.047	0.068	0.048	0.094	0.096	0.115	0.095
2.5	0.011	0.011	0.017	0.012	0.047	0.047	0.060	0.049	0.099	0.096	0.108	0.097
5	0.010	0.010	0.014	0.011	0.049	0.046	0.056	0.050	0.097	0.096	0.106	0.098
10	0.010	0.010	0.012	0.010	0.049	0.047	0.053	0.052	0.100	0.097	0.106	0.100
25	0.011	0.010	0.012	0.012	0.050	0.048	0.053	0.050	0.104	0.101	0.107	0.103
50	0.009	0.010	0.011	0.010	0.050	0.049	0.056	0.052	0.100	0.098	0.104	0.099
75	0.008	0.009	0.010	0.010	0.048	0.047	0.052	0.048	0.100	0.095	0.104	0.101
100	0.008	0.009	0.010	0.009	0.047	0.046	0.051	0.048	0.098	0.095	0.101	0.098
125	0.008	0.009	0.010	0.009	0.047	0.044	0.050	0.047	0.096	0.093	0.100	0.096
150	0.008	0.009	0.011	0.009	0.046	0.045	0.051	0.048	0.098	0.096	0.102	0.097
200	0.008	0.009	0.011	0.009	0.050	0.051	0.054	0.050	0.100	0.097	0.103	0.100
250	0.009	0.009	0.011	0.010	0.048	0.048	0.054	0.051	0.099	0.097	0.103	0.099

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.011	0.010	0.013	0.010	0.051	0.051	0.059	0.051	0.102	0.101	0.110	0.099
-2.5	0.010	0.011	0.018	0.011	0.048	0.048	0.059	0.048	0.100	0.096	0.105	0.096
0	0.011	0.011	0.015	0.012	0.051	0.052	0.059	0.051	0.099	0.102	0.109	0.098
2.5	0.010	0.011	0.013	0.011	0.053	0.053	0.055	0.052	0.101	0.101	0.101	0.101
5	0.010	0.010	0.010	0.010	0.052	0.051	0.053	0.051	0.105	0.102	0.105	0.103
10	0.009	0.010	0.010	0.010	0.049	0.049	0.051	0.048	0.101	0.101	0.103	0.101
25	0.011	0.010	0.010	0.010	0.048	0.049	0.049	0.049	0.100	0.100	0.099	0.100
50	0.010	0.009	0.009	0.009	0.050	0.049	0.049	0.049	0.102	0.100	0.101	0.098
75	0.010	0.010	0.010	0.010	0.048	0.048	0.049	0.048	0.098	0.098	0.099	0.097
100	0.010	0.011	0.010	0.010	0.049	0.048	0.049	0.048	0.096	0.095	0.098	0.094
125	0.010	0.010	0.010	0.010	0.049	0.047	0.049	0.049	0.099	0.097	0.096	0.096
150	0.010	0.010	0.010	0.010	0.047	0.048	0.049	0.048	0.098	0.099	0.099	0.096
200	0.010	0.009	0.010	0.010	0.050	0.047	0.049	0.049	0.099	0.099	0.100	0.097
250	0.010	0.010	0.010	0.010	0.048	0.049	0.048	0.048	0.102	0.100	0.101	0.101
Right-sided tests - $T = 1000$												
-5	0.011	0.011	0.016	0.012	0.049	0.050	0.057	0.048	0.099	0.099	0.107	0.100
-2.5	0.011	0.010	0.018	0.010	0.050	0.050	0.059	0.050	0.095	0.096	0.107	0.095
0	0.010	0.010	0.014	0.011	0.048	0.051	0.054	0.048	0.101	0.102	0.104	0.099
2.5	0.008	0.008	0.011	0.008	0.049	0.049	0.052	0.048	0.103	0.104	0.105	0.101
5	0.008	0.009	0.009	0.009	0.051	0.050	0.050	0.050	0.101	0.102	0.103	0.101
10	0.008	0.010	0.010	0.010	0.052	0.051	0.051	0.051	0.102	0.100	0.101	0.099
25	0.011	0.010	0.011	0.011	0.051	0.049	0.050	0.050	0.099	0.101	0.101	0.100
50	0.012	0.011	0.011	0.012	0.052	0.051	0.052	0.052	0.102	0.101	0.103	0.101
75	0.012	0.011	0.011	0.011	0.054	0.052	0.053	0.052	0.101	0.101	0.102	0.101
100	0.011	0.011	0.012	0.011	0.053	0.052	0.053	0.052	0.100	0.099	0.100	0.099
125	0.012	0.011	0.011	0.011	0.052	0.052	0.052	0.052	0.101	0.101	0.101	0.099
150	0.012	0.011	0.011	0.011	0.053	0.052	0.052	0.053	0.101	0.103	0.102	0.100
200	0.011	0.011	0.011	0.011	0.053	0.051	0.053	0.051	0.103	0.104	0.105	0.102
250	0.011	0.010	0.011	0.010	0.051	0.050	0.051	0.051	0.105	0.105	0.106	0.105
Two-sided tests - $T = 1000$												
-5	0.012	0.011	0.017	0.012	0.049	0.050	0.065	0.049	0.100	0.100	0.116	0.099
-2.5	0.009	0.010	0.022	0.010	0.050	0.049	0.068	0.050	0.098	0.100	0.118	0.098
0	0.011	0.012	0.017	0.011	0.049	0.049	0.062	0.050	0.099	0.101	0.113	0.099
2.5	0.010	0.009	0.011	0.010	0.048	0.049	0.054	0.048	0.101	0.102	0.107	0.100
5	0.009	0.009	0.010	0.008	0.049	0.049	0.051	0.048	0.102	0.101	0.103	0.101
10	0.009	0.009	0.010	0.009	0.049	0.050	0.052	0.050	0.101	0.100	0.101	0.100
25	0.010	0.010	0.010	0.011	0.051	0.051	0.052	0.052	0.099	0.098	0.099	0.099
50	0.011	0.009	0.011	0.011	0.050	0.049	0.050	0.050	0.101	0.099	0.100	0.101
75	0.010	0.009	0.010	0.010	0.050	0.049	0.050	0.051	0.102	0.101	0.102	0.100
100	0.010	0.010	0.010	0.011	0.051	0.050	0.052	0.051	0.102	0.100	0.102	0.101
125	0.010	0.010	0.011	0.011	0.051	0.050	0.051	0.052	0.101	0.099	0.101	0.101
150	0.011	0.011	0.011	0.011	0.052	0.051	0.052	0.051	0.100	0.100	0.100	0.101
200	0.011	0.011	0.011	0.011	0.050	0.050	0.051	0.051	0.102	0.099	0.101	0.100
250	0.011	0.011	0.011	0.011	0.050	0.049	0.051	0.051	0.099	0.098	0.099	0.099

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.35: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP10 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.95; -0.95 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2 are GARCH processes generated as $\sigma_{it}^2 = 0.05 + 0.1e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2$, $i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim iidt_5(\mathbf{0}, \mathbf{I_2})$ where $t_5(\mathbf{0}, \mathbf{I_2})$ defines a mean zero Student-t distribution with 5 degrees of freedom and an 2×2 identity variance matrix.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.007	0.001	0.000	0.006	0.045	0.004	0.004	0.018	0.095	0.013	0.012	0.031
-2.5	0.007	0.000	0.000	0.003	0.041	0.002	0.002	0.008	0.098	0.005	0.004	0.014
0	0.010	0.001	0.001	0.005	0.042	0.003	0.004	0.010	0.073	0.007	0.007	0.017
2.5	0.019	0.001	0.001	0.006	0.060	0.007	0.007	0.014	0.100	0.014	0.013	0.026
5	0.024	0.002	0.001	0.008	0.068	0.012	0.011	0.023	0.115	0.023	0.022	0.041
10	0.024	0.002	0.002	0.012	0.069	0.016	0.016	0.037	0.118	0.038	0.036	0.066
25	0.019	0.003	0.004	0.022	0.068	0.025	0.028	0.062	0.116	0.055	0.057	0.100
50	0.018	0.005	0.006	0.030	0.066	0.032	0.035	0.078	0.113	0.068	0.070	0.119
75	0.016	0.006	0.008	0.034	0.063	0.037	0.039	0.086	0.115	0.074	0.079	0.133
100	0.015	0.007	0.009	0.037	0.062	0.039	0.043	0.093	0.113	0.078	0.084	0.141
125	0.013	0.008	0.011	0.040	0.060	0.041	0.045	0.096	0.112	0.081	0.088	0.146
150	0.014	0.008	0.012	0.044	0.061	0.042	0.049	0.099	0.113	0.085	0.091	0.152
200	0.012	0.008	0.012	0.046	0.062	0.046	0.054	0.106	0.116	0.091	0.099	0.159
250	0.011	0.008	0.013	0.048	0.059	0.046	0.055	0.110	0.118	0.097	0.107	0.165
Right-sided tests - $T = 250$												
-5	0.009	0.013	0.022	0.038	0.042	0.078	0.092	0.110	0.085	0.170	0.176	0.184
-2.5	0.007	0.016	0.023	0.039	0.037	0.104	0.112	0.145	0.083	0.258	0.244	0.292
0	0.009	0.020	0.025	0.052	0.047	0.101	0.117	0.173	0.104	0.222	0.236	0.303
2.5	0.009	0.020	0.025	0.060	0.052	0.099	0.115	0.180	0.115	0.210	0.224	0.298
5	0.010	0.021	0.026	0.063	0.058	0.100	0.114	0.177	0.119	0.193	0.209	0.283
10	0.011	0.021	0.027	0.065	0.061	0.094	0.105	0.172	0.122	0.178	0.189	0.261
25	0.012	0.018	0.023	0.067	0.062	0.084	0.093	0.160	0.121	0.154	0.166	0.237
50	0.009	0.013	0.018	0.065	0.059	0.071	0.081	0.150	0.118	0.140	0.151	0.224
75	0.010	0.012	0.017	0.062	0.059	0.065	0.078	0.145	0.120	0.132	0.142	0.213
100	0.009	0.011	0.017	0.060	0.057	0.060	0.074	0.141	0.121	0.128	0.139	0.210
125	0.009	0.011	0.018	0.059	0.058	0.059	0.069	0.137	0.120	0.122	0.135	0.205
150	0.010	0.010	0.017	0.059	0.059	0.058	0.069	0.135	0.120	0.118	0.131	0.202
200	0.010	0.010	0.017	0.058	0.062	0.057	0.068	0.129	0.120	0.112	0.124	0.193
250	0.012	0.011	0.017	0.054	0.063	0.056	0.066	0.126	0.118	0.107	0.119	0.184
Two-sided tests - $T = 250$												
-5	0.009	0.006	0.013	0.030	0.042	0.037	0.049	0.079	0.086	0.081	0.096	0.128
-2.5	0.006	0.009	0.013	0.027	0.033	0.044	0.055	0.085	0.075	0.104	0.114	0.153
0	0.008	0.011	0.015	0.036	0.041	0.051	0.062	0.112	0.088	0.103	0.121	0.183
2.5	0.009	0.011	0.015	0.042	0.045	0.054	0.065	0.119	0.096	0.106	0.122	0.194
5	0.010	0.011	0.015	0.048	0.050	0.056	0.066	0.126	0.102	0.109	0.124	0.199
10	0.011	0.011	0.015	0.052	0.054	0.057	0.065	0.138	0.110	0.108	0.121	0.208
25	0.012	0.011	0.014	0.061	0.057	0.054	0.062	0.148	0.116	0.108	0.120	0.222
50	0.011	0.009	0.012	0.065	0.061	0.050	0.063	0.157	0.117	0.102	0.115	0.228
75	0.011	0.008	0.013	0.067	0.058	0.050	0.062	0.156	0.117	0.101	0.117	0.231
100	0.011	0.009	0.014	0.069	0.056	0.048	0.061	0.156	0.114	0.099	0.117	0.234
125	0.010	0.008	0.014	0.072	0.058	0.048	0.061	0.158	0.114	0.100	0.114	0.232
150	0.010	0.010	0.015	0.074	0.059	0.049	0.066	0.160	0.117	0.099	0.117	0.234
200	0.010	0.009	0.017	0.077	0.060	0.049	0.067	0.164	0.121	0.102	0.122	0.235
250	0.009	0.009	0.017	0.074	0.058	0.051	0.071	0.163	0.119	0.101	0.121	0.237

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.007	0.000	0.000	0.006	0.041	0.004	0.002	0.015	0.092	0.011	0.010	0.025
-2.5	0.005	0.000	0.000	0.003	0.044	0.001	0.001	0.007	0.098	0.004	0.003	0.011
0	0.009	0.000	0.000	0.004	0.036	0.002	0.002	0.008	0.064	0.005	0.005	0.014
2.5	0.019	0.001	0.001	0.006	0.057	0.005	0.005	0.016	0.094	0.014	0.013	0.028
5	0.027	0.001	0.002	0.011	0.068	0.009	0.009	0.030	0.111	0.023	0.022	0.050
10	0.029	0.003	0.002	0.022	0.071	0.016	0.014	0.053	0.116	0.036	0.034	0.081
25	0.024	0.004	0.004	0.045	0.070	0.023	0.022	0.090	0.118	0.049	0.047	0.128
50	0.021	0.006	0.006	0.063	0.064	0.029	0.028	0.115	0.113	0.060	0.060	0.158
75	0.020	0.007	0.007	0.072	0.063	0.033	0.034	0.129	0.112	0.066	0.066	0.173
100	0.019	0.007	0.007	0.080	0.064	0.037	0.037	0.137	0.110	0.069	0.070	0.182
125	0.019	0.008	0.008	0.087	0.065	0.038	0.040	0.144	0.114	0.073	0.075	0.188
150	0.018	0.009	0.009	0.091	0.064	0.040	0.042	0.149	0.116	0.077	0.079	0.193
200	0.018	0.008	0.010	0.098	0.066	0.042	0.044	0.158	0.115	0.081	0.085	0.202
250	0.018	0.009	0.010	0.100	0.066	0.044	0.046	0.164	0.118	0.085	0.087	0.210
Right-sided tests - $T = 1000$												
-5	0.004	0.013	0.042	0.039	0.030	0.081	0.129	0.099	0.067	0.184	0.225	0.166
-2.5	0.003	0.014	0.027	0.049	0.018	0.096	0.134	0.149	0.052	0.248	0.289	0.281
0	0.003	0.016	0.020	0.078	0.027	0.090	0.112	0.203	0.068	0.207	0.233	0.330
2.5	0.004	0.019	0.022	0.094	0.033	0.098	0.109	0.223	0.084	0.198	0.215	0.337
5	0.005	0.019	0.022	0.104	0.039	0.095	0.102	0.228	0.093	0.191	0.198	0.333
10	0.006	0.020	0.022	0.110	0.043	0.089	0.095	0.230	0.094	0.172	0.179	0.317
25	0.007	0.018	0.019	0.121	0.046	0.082	0.083	0.229	0.103	0.154	0.158	0.306
50	0.008	0.016	0.017	0.127	0.049	0.073	0.078	0.227	0.104	0.139	0.144	0.300
75	0.009	0.015	0.016	0.129	0.052	0.069	0.074	0.226	0.108	0.134	0.140	0.293
100	0.009	0.015	0.015	0.130	0.054	0.067	0.072	0.223	0.110	0.133	0.140	0.289
125	0.010	0.014	0.016	0.133	0.056	0.068	0.071	0.223	0.112	0.131	0.137	0.287
150	0.011	0.014	0.016	0.133	0.055	0.067	0.071	0.224	0.113	0.129	0.135	0.289
200	0.011	0.014	0.016	0.131	0.056	0.064	0.070	0.223	0.115	0.126	0.133	0.285
250	0.011	0.014	0.017	0.131	0.057	0.063	0.066	0.221	0.118	0.124	0.129	0.282
Two-sided tests - $T = 1000$												
-5	0.003	0.006	0.032	0.029	0.028	0.038	0.078	0.074	0.069	0.084	0.131	0.115
-2.5	0.001	0.007	0.013	0.036	0.014	0.037	0.067	0.093	0.045	0.096	0.135	0.156
0	0.002	0.008	0.010	0.057	0.021	0.042	0.054	0.135	0.053	0.090	0.114	0.211
2.5	0.003	0.010	0.011	0.074	0.026	0.049	0.058	0.163	0.064	0.100	0.113	0.239
5	0.004	0.010	0.012	0.085	0.029	0.051	0.058	0.179	0.073	0.103	0.111	0.258
10	0.006	0.010	0.012	0.099	0.038	0.054	0.055	0.203	0.086	0.105	0.109	0.283
25	0.011	0.012	0.012	0.129	0.047	0.053	0.055	0.238	0.097	0.103	0.105	0.319
50	0.013	0.010	0.011	0.150	0.054	0.052	0.056	0.261	0.104	0.103	0.106	0.342
75	0.014	0.011	0.011	0.163	0.057	0.051	0.054	0.273	0.109	0.102	0.108	0.356
100	0.015	0.011	0.013	0.171	0.058	0.052	0.056	0.285	0.114	0.103	0.110	0.361
125	0.016	0.011	0.013	0.176	0.058	0.054	0.057	0.291	0.116	0.106	0.111	0.367
150	0.016	0.012	0.014	0.183	0.060	0.054	0.058	0.296	0.116	0.106	0.113	0.373
200	0.015	0.013	0.014	0.185	0.063	0.054	0.061	0.302	0.121	0.105	0.114	0.381
250	0.014	0.012	0.013	0.189	0.064	0.055	0.062	0.305	0.119	0.106	0.113	0.385

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.36: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP10 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.9; -0.9 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2 are GARCH processes generated as $\sigma_{it}^2 = 0.05 + 0.1 e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2$, $i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim iidt_5(\mathbf{0}, \mathbf{I_2})$ where $t_5(\mathbf{0}, \mathbf{I_2})$ defines a mean zero Student-t distribution with 5 degrees of freedom and an 2×2 identity variance matrix.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests												
-5	0.007	0.001	0.000	0.007	0.045	0.004	0.005	0.018	0.094	0.016	0.015	0.032
-2.5	0.007	0.001	0.000	0.003	0.043	0.003	0.002	0.008	0.095	0.006	0.005	0.015
0	0.009	0.001	0.001	0.004	0.039	0.003	0.004	0.011	0.070	0.007	0.007	0.017
2.5	0.018	0.001	0.001	0.006	0.059	0.006	0.007	0.015	0.099	0.015	0.015	0.028
5	0.023	0.002	0.002	0.008	0.068	0.012	0.011	0.023	0.112	0.026	0.024	0.043
10	0.022	0.003	0.002	0.013	0.069	0.018	0.017	0.037	0.117	0.038	0.037	0.067
25	0.018	0.004	0.004	0.023	0.067	0.027	0.028	0.061	0.115	0.057	0.056	0.098
50	0.017	0.006	0.007	0.029	0.063	0.033	0.035	0.077	0.114	0.069	0.071	0.118
75	0.015	0.007	0.008	0.034	0.062	0.036	0.041	0.085	0.114	0.076	0.079	0.132
100	0.015	0.006	0.009	0.036	0.061	0.039	0.043	0.092	0.114	0.080	0.084	0.139
125	0.013	0.008	0.010	0.039	0.061	0.041	0.046	0.094	0.111	0.083	0.088	0.143
150	0.012	0.008	0.011	0.042	0.061	0.042	0.048	0.097	0.113	0.086	0.092	0.150
200	0.011	0.008	0.012	0.045	0.061	0.046	0.055	0.106	0.114	0.092	0.100	0.156
250	0.012	0.008	0.014	0.046	0.060	0.048	0.055	0.107	0.117	0.098	0.107	0.163
Right-sided tests												
-5	0.009	0.015	0.027	0.040	0.043	0.083	0.096	0.113	0.089	0.172	0.181	0.189
-2.5	0.007	0.018	0.025	0.038	0.038	0.103	0.114	0.144	0.085	0.253	0.247	0.284
0	0.009	0.021	0.025	0.051	0.048	0.102	0.119	0.168	0.106	0.221	0.235	0.296
2.5	0.009	0.020	0.026	0.059	0.055	0.102	0.115	0.175	0.114	0.206	0.221	0.290
5	0.010	0.020	0.026	0.062	0.059	0.099	0.112	0.173	0.119	0.194	0.203	0.278
10	0.011	0.020	0.027	0.065	0.062	0.095	0.106	0.166	0.121	0.174	0.186	0.254
25	0.011	0.018	0.022	0.064	0.063	0.082	0.091	0.156	0.121	0.153	0.163	0.231
50	0.010	0.015	0.019	0.061	0.060	0.071	0.080	0.145	0.119	0.138	0.150	0.220
75	0.008	0.012	0.017	0.059	0.058	0.067	0.076	0.138	0.120	0.129	0.142	0.209
100	0.009	0.011	0.017	0.058	0.056	0.060	0.074	0.136	0.120	0.124	0.136	0.205
125	0.009	0.011	0.017	0.058	0.056	0.059	0.071	0.134	0.119	0.120	0.134	0.200
150	0.009	0.010	0.017	0.057	0.058	0.058	0.069	0.131	0.121	0.119	0.131	0.197
200	0.010	0.011	0.017	0.056	0.060	0.058	0.069	0.126	0.119	0.111	0.124	0.190
250	0.012	0.012	0.017	0.055	0.063	0.056	0.066	0.123	0.119	0.107	0.119	0.181
Two-sided tests												
-5	0.008	0.009	0.017	0.031	0.043	0.040	0.054	0.082	0.089	0.087	0.100	0.131
-2.5	0.006	0.009	0.014	0.027	0.035	0.046	0.059	0.084	0.077	0.105	0.116	0.153
0	0.008	0.011	0.015	0.035	0.040	0.051	0.062	0.107	0.090	0.104	0.123	0.179
2.5	0.009	0.011	0.014	0.042	0.047	0.055	0.064	0.119	0.096	0.106	0.122	0.190
5	0.009	0.012	0.014	0.045	0.050	0.056	0.064	0.124	0.104	0.109	0.123	0.196
10	0.011	0.011	0.015	0.051	0.055	0.057	0.065	0.133	0.110	0.112	0.123	0.203
25	0.011	0.011	0.014	0.058	0.059	0.054	0.064	0.145	0.115	0.107	0.119	0.217
50	0.012	0.009	0.013	0.062	0.061	0.051	0.063	0.152	0.117	0.103	0.115	0.222
75	0.011	0.008	0.013	0.065	0.059	0.048	0.061	0.151	0.114	0.101	0.117	0.223
100	0.010	0.010	0.013	0.066	0.056	0.047	0.060	0.152	0.118	0.099	0.116	0.228
125	0.010	0.009	0.014	0.069	0.058	0.048	0.062	0.154	0.116	0.098	0.116	0.229
150	0.010	0.009	0.015	0.072	0.058	0.047	0.065	0.158	0.116	0.100	0.117	0.228
200	0.009	0.009	0.016	0.074	0.059	0.049	0.069	0.161	0.119	0.104	0.123	0.231
250	0.009	0.009	0.017	0.072	0.059	0.050	0.069	0.157	0.119	0.103	0.122	0.229

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.007	0.000	0.000	0.006	0.042	0.004	0.003	0.016	0.092	0.012	0.011	0.027
-2.5	0.006	0.000	0.000	0.004	0.044	0.001	0.001	0.007	0.097	0.004	0.003	0.011
0	0.008	0.000	0.000	0.004	0.035	0.002	0.002	0.008	0.064	0.006	0.006	0.014
2.5	0.019	0.001	0.001	0.006	0.055	0.005	0.005	0.016	0.093	0.015	0.014	0.030
5	0.026	0.001	0.002	0.011	0.067	0.010	0.009	0.031	0.110	0.025	0.023	0.051
10	0.029	0.003	0.002	0.022	0.070	0.017	0.014	0.052	0.115	0.038	0.035	0.083
25	0.024	0.005	0.004	0.044	0.068	0.023	0.023	0.088	0.118	0.051	0.049	0.126
50	0.021	0.006	0.006	0.061	0.064	0.029	0.029	0.112	0.113	0.061	0.060	0.155
75	0.019	0.007	0.007	0.071	0.062	0.033	0.035	0.126	0.113	0.065	0.065	0.173
100	0.020	0.008	0.008	0.078	0.065	0.037	0.037	0.135	0.111	0.071	0.072	0.182
125	0.020	0.008	0.008	0.085	0.066	0.038	0.039	0.142	0.113	0.075	0.075	0.185
150	0.019	0.009	0.009	0.090	0.066	0.040	0.041	0.146	0.116	0.077	0.079	0.190
200	0.018	0.009	0.010	0.094	0.065	0.043	0.044	0.155	0.115	0.083	0.085	0.197
250	0.018	0.010	0.010	0.099	0.065	0.044	0.047	0.161	0.116	0.084	0.087	0.206
Right-sided tests - $T = 1000$												
-5	0.004	0.015	0.049	0.038	0.032	0.084	0.136	0.098	0.072	0.187	0.231	0.166
-2.5	0.003	0.014	0.032	0.048	0.020	0.099	0.141	0.144	0.053	0.247	0.286	0.278
0	0.005	0.016	0.023	0.073	0.030	0.091	0.115	0.197	0.072	0.206	0.234	0.322
2.5	0.004	0.019	0.023	0.091	0.036	0.098	0.109	0.219	0.085	0.196	0.211	0.332
5	0.005	0.019	0.022	0.100	0.039	0.094	0.102	0.219	0.093	0.189	0.196	0.326
10	0.006	0.020	0.021	0.105	0.043	0.089	0.093	0.224	0.097	0.172	0.177	0.310
25	0.007	0.018	0.018	0.115	0.045	0.081	0.083	0.222	0.101	0.151	0.156	0.300
50	0.008	0.015	0.018	0.121	0.048	0.071	0.076	0.220	0.103	0.138	0.142	0.294
75	0.010	0.014	0.016	0.124	0.052	0.070	0.073	0.219	0.106	0.132	0.138	0.285
100	0.010	0.014	0.015	0.127	0.055	0.067	0.071	0.217	0.108	0.133	0.138	0.283
125	0.010	0.015	0.016	0.128	0.056	0.066	0.070	0.216	0.111	0.130	0.135	0.282
150	0.010	0.014	0.016	0.129	0.055	0.065	0.070	0.217	0.114	0.128	0.135	0.280
200	0.012	0.014	0.015	0.126	0.056	0.064	0.069	0.217	0.114	0.127	0.132	0.280
250	0.010	0.013	0.016	0.127	0.056	0.061	0.066	0.216	0.115	0.123	0.129	0.277
Two-sided tests - $T = 1000$												
-5	0.003	0.007	0.036	0.029	0.030	0.040	0.086	0.076	0.073	0.089	0.139	0.114
-2.5	0.002	0.006	0.018	0.035	0.016	0.039	0.073	0.091	0.046	0.097	0.142	0.151
0	0.003	0.008	0.011	0.055	0.021	0.042	0.054	0.131	0.057	0.093	0.117	0.206
2.5	0.003	0.010	0.012	0.070	0.026	0.049	0.057	0.157	0.068	0.101	0.114	0.235
5	0.004	0.011	0.012	0.080	0.030	0.052	0.058	0.173	0.075	0.104	0.111	0.250
10	0.006	0.011	0.013	0.096	0.038	0.052	0.054	0.196	0.086	0.106	0.107	0.276
25	0.011	0.012	0.012	0.125	0.047	0.051	0.054	0.231	0.096	0.103	0.106	0.310
50	0.013	0.010	0.010	0.146	0.054	0.051	0.055	0.252	0.103	0.099	0.105	0.331
75	0.015	0.011	0.011	0.156	0.057	0.052	0.055	0.265	0.108	0.103	0.108	0.345
100	0.015	0.011	0.012	0.162	0.056	0.052	0.055	0.279	0.113	0.102	0.108	0.352
125	0.015	0.012	0.012	0.168	0.056	0.053	0.056	0.282	0.117	0.104	0.110	0.357
150	0.016	0.012	0.014	0.174	0.058	0.051	0.058	0.288	0.116	0.105	0.111	0.363
200	0.016	0.013	0.014	0.180	0.061	0.053	0.059	0.292	0.119	0.105	0.112	0.372
250	0.014	0.012	0.013	0.182	0.064	0.054	0.060	0.294	0.119	0.106	0.113	0.376

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.37: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP10 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad -0.5; -0.5 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma^2_{1t}, \sigma^2_{2t})$, σ^2_{it} are GARCH processes generated as $\sigma^2_{it} = 0.05 + 0.1e^2_{i,t-1} + 0.85\sigma^2_{i,t-1}$, $i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim iidt_5(\mathbf{0}, \mathbf{I_2})$ where $t_5(\mathbf{0}, \mathbf{I_2})$ defines a mean zero Student-t distribution with 5 degrees of freedom and an 2×2 identity variance matrix.

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests												
-5	0.010	0.002	0.006	0.012	0.050	0.015	0.020	0.031	0.097	0.039	0.040	0.056
-2.5	0.010	0.001	0.002	0.006	0.044	0.008	0.007	0.016	0.087	0.018	0.016	0.026
0	0.008	0.001	0.003	0.007	0.034	0.009	0.010	0.017	0.069	0.020	0.023	0.030
2.5	0.012	0.003	0.003	0.009	0.050	0.015	0.017	0.026	0.092	0.033	0.034	0.051
5	0.014	0.004	0.004	0.011	0.056	0.020	0.021	0.036	0.101	0.044	0.044	0.064
10	0.016	0.005	0.005	0.016	0.059	0.026	0.026	0.046	0.105	0.058	0.058	0.081
25	0.016	0.006	0.006	0.022	0.060	0.035	0.036	0.063	0.108	0.068	0.072	0.102
50	0.014	0.007	0.008	0.026	0.061	0.037	0.040	0.072	0.111	0.079	0.081	0.117
75	0.014	0.007	0.009	0.027	0.060	0.040	0.044	0.077	0.111	0.082	0.088	0.122
100	0.013	0.007	0.010	0.030	0.059	0.043	0.048	0.078	0.109	0.086	0.090	0.126
125	0.011	0.008	0.011	0.031	0.059	0.044	0.050	0.081	0.110	0.089	0.094	0.130
150	0.011	0.008	0.011	0.032	0.059	0.046	0.052	0.085	0.111	0.091	0.095	0.131
200	0.010	0.008	0.013	0.033	0.058	0.047	0.054	0.088	0.113	0.096	0.102	0.138
250	0.010	0.010	0.013	0.034	0.057	0.046	0.056	0.089	0.112	0.097	0.106	0.142
Right-sided tests												
-5	0.009	0.017	0.037	0.035	0.047	0.081	0.096	0.101	0.094	0.160	0.168	0.170
-2.5	0.009	0.020	0.042	0.036	0.051	0.109	0.117	0.128	0.105	0.221	0.205	0.228
0	0.010	0.018	0.032	0.039	0.056	0.098	0.111	0.132	0.116	0.198	0.204	0.233
2.5	0.010	0.018	0.028	0.042	0.057	0.090	0.104	0.130	0.116	0.178	0.185	0.220
5	0.011	0.019	0.025	0.041	0.059	0.087	0.097	0.132	0.117	0.166	0.175	0.209
10	0.011	0.017	0.021	0.042	0.061	0.081	0.088	0.124	0.114	0.153	0.162	0.201
25	0.010	0.015	0.018	0.042	0.058	0.072	0.080	0.116	0.113	0.135	0.144	0.185
50	0.011	0.012	0.016	0.039	0.056	0.064	0.072	0.111	0.110	0.123	0.133	0.177
75	0.011	0.011	0.015	0.040	0.055	0.059	0.068	0.106	0.110	0.120	0.129	0.172
100	0.009	0.011	0.017	0.041	0.057	0.058	0.067	0.105	0.113	0.116	0.127	0.166
125	0.009	0.011	0.016	0.041	0.057	0.058	0.068	0.105	0.116	0.115	0.126	0.163
150	0.010	0.010	0.017	0.040	0.057	0.057	0.067	0.104	0.115	0.115	0.126	0.162
200	0.011	0.011	0.016	0.041	0.060	0.056	0.067	0.101	0.117	0.112	0.121	0.161
250	0.011	0.010	0.017	0.040	0.059	0.055	0.065	0.097	0.117	0.109	0.120	0.156
Two-sided tests												
-5	0.009	0.009	0.032	0.032	0.047	0.047	0.072	0.081	0.096	0.095	0.116	0.132
-2.5	0.008	0.010	0.030	0.024	0.046	0.060	0.077	0.084	0.096	0.115	0.124	0.144
0	0.008	0.009	0.022	0.028	0.048	0.050	0.072	0.088	0.095	0.107	0.122	0.149
2.5	0.009	0.010	0.018	0.031	0.049	0.052	0.066	0.093	0.100	0.105	0.121	0.156
5	0.010	0.011	0.015	0.034	0.051	0.055	0.065	0.099	0.106	0.107	0.117	0.167
10	0.011	0.011	0.014	0.038	0.056	0.053	0.062	0.106	0.109	0.106	0.115	0.170
25	0.011	0.009	0.013	0.041	0.058	0.052	0.059	0.113	0.109	0.107	0.116	0.179
50	0.013	0.009	0.013	0.045	0.055	0.048	0.058	0.115	0.110	0.101	0.112	0.183
75	0.011	0.009	0.013	0.045	0.056	0.049	0.059	0.117	0.111	0.100	0.113	0.183
100	0.010	0.008	0.014	0.046	0.058	0.048	0.061	0.118	0.111	0.100	0.115	0.184
125	0.010	0.008	0.014	0.048	0.055	0.047	0.061	0.120	0.113	0.101	0.117	0.185
150	0.008	0.009	0.014	0.049	0.055	0.047	0.062	0.121	0.115	0.102	0.119	0.189
200	0.008	0.010	0.016	0.051	0.054	0.048	0.062	0.122	0.116	0.103	0.121	0.189
250	0.009	0.010	0.016	0.052	0.055	0.050	0.065	0.123	0.116	0.101	0.121	0.186

c	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}	$t^{*,RWB}_{zx}$	$t^{*,FRWB}_{zx}$	t^{EW}_{zx}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.009	0.001	0.005	0.011	0.048	0.012	0.017	0.028	0.098	0.030	0.034	0.045
-2.5	0.007	0.000	0.001	0.005	0.043	0.004	0.005	0.012	0.092	0.011	0.013	0.022
0	0.007	0.001	0.002	0.005	0.029	0.005	0.007	0.013	0.062	0.013	0.014	0.025
2.5	0.011	0.001	0.002	0.008	0.044	0.012	0.012	0.026	0.086	0.027	0.026	0.047
5	0.016	0.002	0.002	0.013	0.052	0.017	0.016	0.038	0.101	0.036	0.036	0.066
10	0.019	0.003	0.003	0.021	0.058	0.022	0.021	0.056	0.106	0.048	0.047	0.089
25	0.017	0.005	0.004	0.036	0.057	0.028	0.026	0.079	0.107	0.061	0.059	0.117
50	0.017	0.006	0.006	0.045	0.057	0.033	0.033	0.093	0.105	0.068	0.068	0.139
75	0.015	0.007	0.007	0.051	0.057	0.036	0.036	0.105	0.106	0.074	0.074	0.149
100	0.015	0.008	0.007	0.055	0.059	0.037	0.038	0.111	0.107	0.079	0.079	0.157
125	0.015	0.007	0.008	0.058	0.059	0.040	0.041	0.117	0.110	0.082	0.082	0.161
150	0.015	0.008	0.009	0.060	0.060	0.041	0.043	0.120	0.110	0.084	0.084	0.166
200	0.014	0.010	0.010	0.063	0.062	0.044	0.046	0.127	0.110	0.085	0.088	0.174
250	0.014	0.009	0.010	0.068	0.062	0.045	0.048	0.129	0.111	0.089	0.091	0.176
Right-sided tests - $T = 1000$												
-5	0.007	0.017	0.054	0.033	0.038	0.084	0.115	0.090	0.083	0.170	0.186	0.155
-2.5	0.007	0.020	0.051	0.040	0.038	0.110	0.141	0.130	0.083	0.223	0.234	0.231
0	0.007	0.020	0.037	0.053	0.041	0.099	0.123	0.152	0.095	0.205	0.219	0.256
2.5	0.009	0.020	0.031	0.063	0.047	0.095	0.110	0.156	0.099	0.186	0.196	0.257
5	0.010	0.020	0.026	0.063	0.046	0.088	0.097	0.161	0.101	0.173	0.181	0.247
10	0.010	0.018	0.021	0.066	0.047	0.080	0.084	0.162	0.101	0.157	0.164	0.239
25	0.010	0.016	0.017	0.071	0.048	0.071	0.073	0.159	0.099	0.138	0.143	0.238
50	0.008	0.014	0.015	0.076	0.049	0.066	0.069	0.160	0.102	0.127	0.132	0.229
75	0.010	0.014	0.014	0.079	0.050	0.063	0.066	0.158	0.104	0.125	0.130	0.227
100	0.009	0.014	0.015	0.081	0.052	0.063	0.065	0.157	0.105	0.124	0.128	0.226
125	0.010	0.013	0.014	0.081	0.054	0.059	0.065	0.157	0.106	0.123	0.126	0.226
150	0.010	0.014	0.014	0.083	0.054	0.060	0.066	0.159	0.108	0.122	0.127	0.227
200	0.011	0.012	0.014	0.082	0.055	0.060	0.065	0.158	0.110	0.120	0.127	0.224
250	0.011	0.013	0.014	0.084	0.054	0.061	0.066	0.161	0.111	0.119	0.124	0.222
Two-sided tests - $T = 1000$												
-5	0.006	0.009	0.047	0.029	0.038	0.046	0.090	0.077	0.083	0.095	0.132	0.117
-2.5	0.005	0.008	0.039	0.028	0.031	0.055	0.090	0.082	0.071	0.113	0.146	0.142
0	0.006	0.010	0.026	0.040	0.031	0.051	0.077	0.103	0.070	0.103	0.129	0.165
2.5	0.007	0.012	0.020	0.050	0.037	0.053	0.066	0.117	0.078	0.106	0.121	0.181
5	0.008	0.013	0.016	0.056	0.039	0.053	0.060	0.129	0.083	0.105	0.113	0.198
10	0.011	0.012	0.014	0.063	0.043	0.052	0.055	0.141	0.090	0.102	0.105	0.218
25	0.011	0.010	0.011	0.080	0.047	0.052	0.053	0.165	0.096	0.098	0.099	0.239
50	0.012	0.009	0.010	0.092	0.052	0.049	0.051	0.181	0.101	0.098	0.102	0.253
75	0.013	0.010	0.011	0.099	0.053	0.049	0.052	0.193	0.104	0.098	0.102	0.263
100	0.012	0.010	0.011	0.104	0.054	0.050	0.053	0.200	0.107	0.098	0.103	0.268
125	0.013	0.010	0.013	0.109	0.056	0.050	0.053	0.203	0.110	0.099	0.106	0.273
150	0.013	0.011	0.013	0.111	0.056	0.050	0.053	0.205	0.110	0.101	0.109	0.279
200	0.013	0.011	0.013	0.115	0.056	0.051	0.055	0.210	0.115	0.104	0.110	0.284
250	0.014	0.010	0.013	0.118	0.055	0.051	0.054	0.213	0.114	0.107	0.113	0.290

Note: t_{zx} and t^{EW}_{zx} correspond to the statistics presented in (12) and (14) of the main text, and $t^{*,RWB}_{zx}$ and $t^{*,FRWB}_{zx}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.38: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP10 (GoGARCH(1,1)):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' = \mathbf{Z}\mathbf{H}_t^{1/2}\boldsymbol{\varepsilon}_t = \mathbf{Z}\mathbf{e}_t$, where $\mathbf{e}_t = (e_{1t}, e_{2t})'$, $\mathbf{Z} = [1 \quad 0; 0 \quad 1]^{1/2}$, $\mathbf{H}_t = diag(\sigma_{1t}^2, \sigma_{2t}^2)$, σ_{it}^2 are GARCH processes generated as $\sigma_{it}^2 = 0.05 + 0.1e_{i,t-1}^2 + 0.85\sigma_{i,t-1}^2$, $i = 1, 2$ and $\boldsymbol{\varepsilon}_t \sim iidt_5(\mathbf{0}, \mathbf{I_2})$ where $t_5(\mathbf{0}, \mathbf{I_2})$ defines a mean zero Student-t distribution with 5 degrees of freedom and an 2×2 identity variance matrix.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests												
-5	0.011	0.011	0.022	0.025	0.051	0.050	0.057	0.067	0.099	0.098	0.097	0.107
-2.5	0.013	0.012	0.021	0.019	0.048	0.048	0.047	0.056	0.094	0.096	0.084	0.102
0	0.012	0.012	0.022	0.018	0.047	0.049	0.056	0.057	0.093	0.095	0.093	0.104
2.5	0.013	0.012	0.019	0.021	0.053	0.049	0.057	0.062	0.098	0.096	0.103	0.111
5	0.013	0.012	0.016	0.021	0.053	0.050	0.057	0.066	0.100	0.099	0.105	0.114
10	0.013	0.012	0.013	0.021	0.055	0.051	0.055	0.068	0.103	0.101	0.106	0.115
25	0.013	0.012	0.014	0.023	0.052	0.052	0.057	0.067	0.107	0.103	0.107	0.117
50	0.011	0.010	0.013	0.021	0.053	0.050	0.055	0.069	0.107	0.101	0.107	0.117
75	0.010	0.008	0.012	0.020	0.054	0.050	0.055	0.068	0.109	0.102	0.109	0.120
100	0.009	0.009	0.012	0.020	0.054	0.050	0.055	0.068	0.108	0.101	0.107	0.120
125	0.008	0.009	0.013	0.019	0.052	0.047	0.055	0.067	0.106	0.100	0.106	0.119
150	0.009	0.009	0.013	0.020	0.050	0.048	0.055	0.068	0.105	0.096	0.105	0.117
200	0.009	0.010	0.013	0.021	0.050	0.047	0.053	0.069	0.105	0.098	0.105	0.120
250	0.009	0.010	0.014	0.022	0.050	0.047	0.054	0.070	0.107	0.100	0.108	0.120
Right-sided tests												
-5	0.010	0.011	0.022	0.026	0.055	0.054	0.057	0.068	0.104	0.103	0.095	0.113
-2.5	0.010	0.009	0.021	0.017	0.053	0.052	0.049	0.061	0.101	0.102	0.086	0.109
0	0.010	0.010	0.016	0.020	0.051	0.050	0.055	0.061	0.096	0.100	0.093	0.109
2.5	0.010	0.011	0.018	0.022	0.052	0.049	0.058	0.065	0.099	0.096	0.103	0.112
5	0.011	0.011	0.016	0.022	0.051	0.050	0.056	0.066	0.103	0.099	0.105	0.115
10	0.011	0.010	0.013	0.021	0.052	0.049	0.057	0.066	0.103	0.100	0.106	0.117
25	0.010	0.010	0.012	0.021	0.053	0.051	0.057	0.066	0.101	0.097	0.105	0.117
50	0.010	0.009	0.012	0.024	0.051	0.051	0.058	0.068	0.103	0.101	0.107	0.119
75	0.011	0.010	0.014	0.024	0.053	0.052	0.056	0.068	0.105	0.100	0.107	0.122
100	0.011	0.010	0.014	0.026	0.054	0.053	0.058	0.071	0.105	0.101	0.107	0.123
125	0.010	0.010	0.014	0.026	0.054	0.053	0.059	0.072	0.109	0.104	0.109	0.124
150	0.011	0.010	0.015	0.025	0.056	0.054	0.062	0.073	0.112	0.106	0.112	0.125
200	0.010	0.011	0.014	0.026	0.057	0.055	0.061	0.074	0.113	0.104	0.112	0.127
250	0.010	0.011	0.013	0.026	0.057	0.054	0.059	0.074	0.115	0.105	0.114	0.127
Two-sided tests												
-5	0.011	0.010	0.034	0.037	0.051	0.053	0.074	0.084	0.102	0.103	0.114	0.135
-2.5	0.011	0.011	0.031	0.023	0.049	0.052	0.063	0.070	0.097	0.100	0.096	0.117
0	0.011	0.011	0.028	0.024	0.049	0.049	0.068	0.070	0.095	0.098	0.111	0.118
2.5	0.012	0.011	0.025	0.027	0.054	0.052	0.070	0.077	0.101	0.099	0.115	0.127
5	0.013	0.012	0.020	0.026	0.053	0.051	0.065	0.077	0.102	0.100	0.113	0.132
10	0.012	0.010	0.015	0.028	0.053	0.049	0.061	0.079	0.106	0.100	0.113	0.135
25	0.013	0.011	0.014	0.031	0.053	0.049	0.058	0.082	0.107	0.102	0.114	0.133
50	0.011	0.010	0.014	0.029	0.052	0.051	0.059	0.082	0.105	0.101	0.113	0.137
75	0.009	0.008	0.013	0.030	0.053	0.049	0.061	0.082	0.106	0.101	0.111	0.136
100	0.009	0.009	0.014	0.029	0.052	0.050	0.061	0.084	0.109	0.102	0.114	0.139
125	0.009	0.009	0.014	0.029	0.050	0.051	0.059	0.084	0.107	0.101	0.114	0.139
150	0.009	0.009	0.015	0.030	0.049	0.050	0.059	0.085	0.106	0.101	0.116	0.141
200	0.007	0.011	0.015	0.030	0.050	0.049	0.061	0.085	0.106	0.100	0.115	0.143
250	0.008	0.011	0.015	0.032	0.050	0.047	0.061	0.087	0.106	0.100	0.114	0.144

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.010	0.010	0.023	0.022	0.047	0.050	0.050	0.055	0.097	0.098	0.083	0.096
-2.5	0.009	0.009	0.016	0.017	0.048	0.046	0.040	0.051	0.094	0.101	0.072	0.095
0	0.008	0.008	0.018	0.016	0.046	0.048	0.051	0.056	0.091	0.096	0.088	0.100
2.5	0.009	0.010	0.017	0.019	0.047	0.046	0.055	0.061	0.096	0.095	0.102	0.104
5	0.009	0.010	0.016	0.020	0.049	0.048	0.057	0.060	0.098	0.095	0.103	0.106
10	0.009	0.009	0.014	0.022	0.049	0.047	0.054	0.061	0.097	0.096	0.102	0.107
25	0.010	0.010	0.011	0.025	0.049	0.048	0.051	0.068	0.096	0.094	0.098	0.109
50	0.010	0.010	0.010	0.028	0.047	0.048	0.049	0.071	0.098	0.096	0.101	0.116
75	0.010	0.010	0.011	0.029	0.049	0.047	0.050	0.074	0.101	0.097	0.101	0.118
100	0.010	0.011	0.012	0.031	0.050	0.050	0.053	0.074	0.101	0.099	0.100	0.117
125	0.010	0.010	0.010	0.030	0.049	0.050	0.051	0.075	0.101	0.097	0.099	0.120
150	0.010	0.009	0.010	0.030	0.050	0.049	0.051	0.075	0.101	0.098	0.099	0.122
200	0.009	0.009	0.010	0.030	0.050	0.048	0.051	0.077	0.101	0.098	0.101	0.123
250	0.009	0.008	0.009	0.030	0.049	0.049	0.052	0.078	0.101	0.099	0.101	0.124
Right-sided tests - $T = 1000$												
-5	0.010	0.011	0.024	0.022	0.046	0.046	0.049	0.053	0.092	0.096	0.083	0.094
-2.5	0.010	0.010	0.023	0.017	0.046	0.051	0.049	0.052	0.091	0.098	0.079	0.098
0	0.009	0.010	0.018	0.016	0.046	0.048	0.052	0.056	0.093	0.100	0.091	0.104
2.5	0.010	0.010	0.020	0.020	0.047	0.048	0.059	0.061	0.099	0.100	0.105	0.110
5	0.011	0.011	0.019	0.022	0.049	0.050	0.059	0.063	0.099	0.099	0.110	0.111
10	0.011	0.010	0.015	0.024	0.051	0.050	0.056	0.067	0.100	0.099	0.108	0.115
25	0.012	0.011	0.013	0.027	0.051	0.052	0.055	0.073	0.106	0.101	0.106	0.118
50	0.010	0.010	0.011	0.029	0.054	0.053	0.056	0.076	0.105	0.104	0.106	0.123
75	0.010	0.010	0.011	0.030	0.054	0.053	0.055	0.080	0.105	0.105	0.109	0.125
100	0.010	0.010	0.011	0.031	0.055	0.053	0.056	0.080	0.106	0.103	0.107	0.127
125	0.010	0.011	0.011	0.031	0.055	0.052	0.055	0.079	0.103	0.103	0.107	0.127
150	0.011	0.011	0.011	0.031	0.055	0.052	0.055	0.080	0.104	0.102	0.107	0.128
200	0.011	0.010	0.011	0.031	0.053	0.050	0.054	0.080	0.103	0.100	0.106	0.128
250	0.011	0.010	0.011	0.033	0.052	0.049	0.053	0.079	0.104	0.102	0.104	0.130
Two-sided tests - $T = 1000$												
-5	0.008	0.010	0.037	0.030	0.047	0.048	0.070	0.069	0.093	0.097	0.100	0.108
-2.5	0.010	0.010	0.030	0.023	0.045	0.046	0.061	0.061	0.092	0.097	0.089	0.103
0	0.007	0.009	0.028	0.019	0.044	0.046	0.061	0.063	0.089	0.096	0.103	0.112
2.5	0.010	0.009	0.026	0.027	0.045	0.047	0.066	0.073	0.094	0.094	0.114	0.122
5	0.011	0.009	0.025	0.030	0.047	0.047	0.067	0.076	0.097	0.097	0.116	0.123
10	0.011	0.010	0.016	0.032	0.049	0.047	0.061	0.079	0.099	0.097	0.110	0.127
25	0.010	0.011	0.013	0.037	0.050	0.048	0.054	0.088	0.099	0.099	0.105	0.141
50	0.011	0.010	0.011	0.039	0.051	0.049	0.054	0.097	0.101	0.100	0.105	0.147
75	0.009	0.009	0.010	0.040	0.052	0.050	0.056	0.099	0.102	0.101	0.105	0.154
100	0.008	0.010	0.010	0.042	0.050	0.049	0.054	0.099	0.105	0.100	0.109	0.153
125	0.009	0.009	0.010	0.043	0.052	0.050	0.054	0.099	0.105	0.102	0.106	0.155
150	0.008	0.010	0.010	0.044	0.051	0.049	0.054	0.099	0.103	0.101	0.106	0.155
200	0.010	0.008	0.010	0.044	0.050	0.049	0.054	0.101	0.103	0.098	0.105	0.157
250	0.009	0.008	0.010	0.044	0.048	0.047	0.052	0.103	0.102	0.098	0.105	0.157

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.39: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP11 (Stochastic Volatility):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)'$ follow from a first-order AR stochastic volatility process as $(u_t = e_{1t}\texttt{exp}(h_{1t}), v_t = e_{2t}\texttt{exp}(h_{2t}))'$ with $h_{it} = \lambda h_{i,t-1} + 0.5\xi_{it},\ i = 1,2$ and $(\xi_{it}, e_{it})' \sim NIID(0, diag(\sigma_\xi^2, 1))$, independent across i = 1,2. Results are reported for $(\lambda, \sigma_\xi) = (0.951, 0.314)$ and $(e_{1t}, e_{2t})' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} = [1 \quad -0.95; -0.95 \quad 1]$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.009	0.000	0.002	0.002	0.050	0.005	0.009	0.010	0.099	0.018	0.024	0.022
-2.5	0.009	0.000	0.000	0.001	0.049	0.001	0.001	0.002	0.103	0.004	0.004	0.006
0	0.008	0.000	0.000	0.000	0.035	0.001	0.002	0.002	0.064	0.005	0.007	0.006
2.5	0.016	0.001	0.002	0.001	0.055	0.007	0.009	0.009	0.095	0.019	0.020	0.021
5	0.019	0.002	0.002	0.003	0.061	0.014	0.014	0.016	0.107	0.032	0.032	0.032
10	0.017	0.004	0.003	0.005	0.062	0.021	0.021	0.023	0.109	0.046	0.047	0.048
25	0.014	0.005	0.006	0.008	0.060	0.031	0.034	0.034	0.111	0.062	0.064	0.066
50	0.013	0.007	0.007	0.009	0.061	0.038	0.040	0.038	0.112	0.075	0.078	0.076
75	0.013	0.008	0.009	0.010	0.062	0.039	0.043	0.042	0.112	0.082	0.085	0.082
100	0.013	0.008	0.010	0.011	0.058	0.041	0.044	0.043	0.115	0.086	0.091	0.084
125	0.013	0.009	0.010	0.011	0.056	0.042	0.046	0.045	0.113	0.088	0.093	0.084
150	0.013	0.009	0.011	0.011	0.057	0.045	0.048	0.045	0.113	0.090	0.095	0.088
200	0.011	0.010	0.011	0.011	0.056	0.047	0.051	0.047	0.111	0.094	0.100	0.091
250	0.012	0.010	0.013	0.011	0.057	0.049	0.053	0.047	0.115	0.100	0.105	0.094
Right-sided tests - $T = 250$												
-5	0.011	0.015	0.041	0.019	0.045	0.083	0.117	0.078	0.094	0.160	0.199	0.149
-2.5	0.015	0.021	0.046	0.020	0.055	0.115	0.157	0.109	0.113	0.251	0.284	0.235
0	0.012	0.021	0.037	0.017	0.062	0.116	0.143	0.103	0.127	0.243	0.265	0.229
2.5	0.013	0.022	0.034	0.017	0.067	0.113	0.135	0.102	0.130	0.221	0.241	0.211
5	0.014	0.021	0.031	0.018	0.064	0.106	0.121	0.096	0.125	0.200	0.216	0.190
10	0.013	0.021	0.026	0.016	0.059	0.089	0.103	0.083	0.118	0.174	0.188	0.166
25	0.012	0.017	0.022	0.014	0.055	0.076	0.083	0.068	0.106	0.146	0.157	0.135
50	0.011	0.017	0.019	0.012	0.057	0.071	0.076	0.063	0.109	0.131	0.137	0.124
75	0.010	0.014	0.018	0.011	0.057	0.067	0.074	0.060	0.110	0.126	0.133	0.121
100	0.011	0.014	0.016	0.013	0.055	0.062	0.070	0.057	0.112	0.121	0.129	0.115
125	0.011	0.013	0.016	0.014	0.054	0.060	0.066	0.056	0.110	0.119	0.126	0.110
150	0.010	0.012	0.015	0.014	0.054	0.058	0.065	0.053	0.109	0.114	0.120	0.106
200	0.010	0.012	0.014	0.013	0.054	0.056	0.062	0.053	0.108	0.107	0.115	0.103
250	0.011	0.011	0.013	0.013	0.054	0.054	0.058	0.051	0.106	0.105	0.110	0.100
Two-sided tests - $T = 250$												
-5	0.010	0.009	0.029	0.013	0.045	0.039	0.079	0.044	0.096	0.086	0.126	0.088
-2.5	0.012	0.012	0.031	0.011	0.051	0.049	0.091	0.052	0.103	0.116	0.158	0.111
0	0.009	0.010	0.022	0.008	0.050	0.055	0.081	0.050	0.101	0.117	0.145	0.105
2.5	0.010	0.010	0.020	0.010	0.052	0.059	0.079	0.053	0.111	0.119	0.143	0.112
5	0.012	0.011	0.018	0.010	0.054	0.058	0.073	0.053	0.109	0.118	0.135	0.111
10	0.012	0.012	0.017	0.011	0.054	0.057	0.066	0.053	0.104	0.109	0.124	0.106
25	0.011	0.011	0.015	0.012	0.055	0.055	0.062	0.051	0.103	0.104	0.117	0.101
50	0.010	0.010	0.013	0.012	0.054	0.052	0.062	0.052	0.111	0.108	0.116	0.101
75	0.011	0.011	0.014	0.012	0.055	0.053	0.060	0.052	0.111	0.105	0.117	0.102
100	0.011	0.011	0.014	0.014	0.057	0.054	0.060	0.051	0.108	0.103	0.114	0.100
125	0.011	0.011	0.014	0.014	0.054	0.051	0.059	0.050	0.110	0.101	0.111	0.100
150	0.011	0.011	0.014	0.014	0.054	0.052	0.059	0.052	0.109	0.101	0.114	0.098
200	0.012	0.012	0.015	0.014	0.052	0.052	0.059	0.053	0.109	0.102	0.113	0.099
250	0.011	0.011	0.015	0.013	0.053	0.051	0.058	0.053	0.108	0.101	0.111	0.098

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.010	0.001	0.001	0.002	0.046	0.008	0.009	0.009	0.095	0.021	0.023	0.023
-2.5	0.008	0.000	0.000	0.000	0.050	0.001	0.001	0.001	0.105	0.004	0.004	0.004
0	0.010	0.000	0.000	0.000	0.035	0.002	0.002	0.003	0.062	0.006	0.006	0.008
2.5	0.018	0.001	0.001	0.001	0.054	0.010	0.010	0.012	0.094	0.023	0.024	0.025
5	0.020	0.002	0.003	0.004	0.059	0.017	0.017	0.019	0.105	0.036	0.037	0.038
10	0.018	0.004	0.005	0.006	0.063	0.024	0.024	0.027	0.107	0.050	0.050	0.054
25	0.014	0.006	0.007	0.009	0.057	0.031	0.030	0.033	0.111	0.065	0.065	0.068
50	0.012	0.007	0.007	0.010	0.057	0.035	0.036	0.039	0.107	0.073	0.073	0.076
75	0.012	0.008	0.008	0.010	0.055	0.037	0.037	0.039	0.108	0.077	0.077	0.077
100	0.012	0.008	0.008	0.010	0.053	0.038	0.039	0.040	0.106	0.078	0.080	0.082
125	0.011	0.007	0.007	0.010	0.053	0.040	0.041	0.042	0.103	0.079	0.080	0.082
150	0.012	0.007	0.008	0.010	0.054	0.041	0.042	0.044	0.104	0.081	0.081	0.082
200	0.011	0.008	0.008	0.010	0.054	0.043	0.043	0.044	0.104	0.085	0.085	0.086
250	0.011	0.008	0.008	0.010	0.054	0.043	0.045	0.046	0.104	0.086	0.085	0.087
Right-sided tests - $T = 1000$												
-5	0.009	0.014	0.020	0.017	0.044	0.072	0.087	0.072	0.094	0.154	0.167	0.147
-2.5	0.009	0.019	0.023	0.015	0.052	0.104	0.121	0.099	0.101	0.232	0.244	0.224
0	0.014	0.022	0.027	0.019	0.061	0.113	0.121	0.105	0.122	0.231	0.237	0.222
2.5	0.015	0.026	0.028	0.020	0.067	0.115	0.121	0.107	0.129	0.214	0.221	0.206
5	0.017	0.027	0.030	0.021	0.066	0.106	0.112	0.097	0.126	0.196	0.203	0.193
10	0.015	0.024	0.026	0.019	0.062	0.095	0.097	0.084	0.118	0.180	0.185	0.171
25	0.013	0.019	0.021	0.015	0.057	0.081	0.083	0.072	0.108	0.153	0.155	0.141
50	0.011	0.015	0.017	0.013	0.053	0.073	0.076	0.063	0.106	0.139	0.141	0.128
75	0.010	0.014	0.014	0.010	0.054	0.067	0.070	0.061	0.104	0.133	0.135	0.123
100	0.010	0.013	0.014	0.010	0.054	0.066	0.067	0.057	0.104	0.128	0.130	0.118
125	0.010	0.013	0.014	0.010	0.052	0.063	0.064	0.056	0.104	0.124	0.126	0.116
150	0.009	0.014	0.014	0.010	0.053	0.061	0.064	0.056	0.105	0.123	0.124	0.114
200	0.010	0.012	0.013	0.010	0.051	0.060	0.061	0.054	0.106	0.119	0.121	0.113
250	0.009	0.012	0.012	0.011	0.052	0.059	0.060	0.053	0.105	0.117	0.119	0.111
Two-sided tests - $T = 1000$												
-5	0.009	0.008	0.012	0.010	0.045	0.038	0.048	0.041	0.096	0.080	0.095	0.082
-2.5	0.008	0.008	0.013	0.008	0.046	0.049	0.057	0.047	0.093	0.105	0.123	0.100
0	0.010	0.012	0.014	0.011	0.049	0.054	0.064	0.052	0.100	0.116	0.124	0.108
2.5	0.012	0.014	0.017	0.011	0.056	0.061	0.068	0.058	0.110	0.123	0.131	0.119
5	0.013	0.016	0.018	0.011	0.057	0.064	0.069	0.060	0.109	0.123	0.129	0.116
10	0.013	0.016	0.018	0.013	0.058	0.061	0.064	0.059	0.109	0.118	0.121	0.111
25	0.013	0.014	0.015	0.013	0.052	0.058	0.059	0.052	0.105	0.110	0.113	0.106
50	0.012	0.013	0.012	0.012	0.051	0.053	0.053	0.051	0.105	0.107	0.112	0.101
75	0.011	0.011	0.012	0.012	0.051	0.052	0.053	0.052	0.105	0.103	0.107	0.100
100	0.009	0.011	0.012	0.011	0.052	0.052	0.053	0.050	0.105	0.103	0.106	0.097
125	0.009	0.011	0.011	0.010	0.051	0.051	0.052	0.050	0.102	0.102	0.105	0.098
150	0.009	0.010	0.010	0.010	0.052	0.051	0.053	0.051	0.104	0.102	0.106	0.099
200	0.009	0.010	0.010	0.011	0.054	0.051	0.052	0.051	0.103	0.103	0.104	0.099
250	0.010	0.009	0.010	0.011	0.052	0.050	0.054	0.048	0.105	0.101	0.105	0.099

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.40: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP11 (Stochastic Volatility):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)'$ follow from a first-order AR stochastic volatility process as $(u_t = e_{1t}\mathtt{exp}(h_{1t}), v_t = e_{2t}\mathtt{exp}(h_{2t}))'$ with $h_{it} = \lambda h_{i,t-1} + 0.5\xi_{it},\ i = 1, 2.$ and $(\xi_{it}, e_{it})' \sim NIID(0, diag(\sigma_\xi^2, 1))$, independent across i = 1,2. Results are reported for $(\lambda, \sigma_\xi) = (0.951, 0.314)$ and $(e_{1t}, e_{2t})' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} = [1 \quad -0.9; -0.9 \quad 1]$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.009	0.000	0.003	0.002	0.050	0.006	0.012	0.012	0.100	0.021	0.030	0.025
-2.5	0.009	0.000	0.001	0.001	0.050	0.001	0.002	0.002	0.106	0.004	0.006	0.006
0	0.008	0.000	0.000	0.001	0.034	0.002	0.002	0.002	0.064	0.007	0.008	0.007
2.5	0.015	0.001	0.002	0.002	0.051	0.008	0.010	0.010	0.092	0.022	0.023	0.023
5	0.018	0.003	0.003	0.003	0.058	0.014	0.016	0.017	0.104	0.033	0.035	0.034
10	0.016	0.003	0.004	0.006	0.060	0.021	0.023	0.025	0.108	0.047	0.048	0.049
25	0.014	0.005	0.006	0.008	0.059	0.032	0.034	0.033	0.110	0.065	0.066	0.067
50	0.014	0.007	0.007	0.009	0.059	0.038	0.040	0.039	0.112	0.077	0.079	0.077
75	0.012	0.008	0.009	0.010	0.060	0.039	0.042	0.042	0.113	0.082	0.085	0.082
100	0.014	0.008	0.010	0.011	0.059	0.041	0.044	0.043	0.116	0.089	0.093	0.085
125	0.013	0.009	0.010	0.011	0.059	0.042	0.045	0.044	0.113	0.092	0.094	0.088
150	0.013	0.008	0.011	0.011	0.058	0.043	0.047	0.045	0.115	0.096	0.098	0.090
200	0.011	0.010	0.011	0.012	0.057	0.046	0.051	0.047	0.113	0.096	0.100	0.093
250	0.011	0.010	0.013	0.011	0.056	0.048	0.053	0.049	0.114	0.099	0.105	0.097
Right-sided tests - $T = 250$												
-5	0.009	0.014	0.042	0.019	0.044	0.077	0.114	0.075	0.091	0.157	0.198	0.145
-2.5	0.012	0.018	0.050	0.019	0.056	0.117	0.160	0.110	0.113	0.249	0.282	0.233
0	0.012	0.020	0.040	0.019	0.065	0.115	0.144	0.105	0.129	0.242	0.263	0.224
2.5	0.014	0.021	0.036	0.018	0.067	0.115	0.134	0.104	0.132	0.216	0.239	0.206
5	0.013	0.020	0.031	0.017	0.064	0.104	0.118	0.096	0.128	0.196	0.210	0.184
10	0.012	0.020	0.025	0.017	0.061	0.090	0.101	0.083	0.116	0.170	0.181	0.161
25	0.012	0.017	0.021	0.014	0.054	0.074	0.081	0.066	0.105	0.140	0.151	0.132
50	0.010	0.015	0.018	0.012	0.057	0.068	0.075	0.063	0.107	0.130	0.135	0.120
75	0.010	0.014	0.016	0.011	0.056	0.065	0.071	0.061	0.108	0.125	0.131	0.118
100	0.010	0.013	0.016	0.012	0.054	0.062	0.069	0.058	0.110	0.121	0.127	0.114
125	0.011	0.013	0.016	0.013	0.054	0.058	0.064	0.054	0.108	0.118	0.125	0.108
150	0.010	0.012	0.015	0.012	0.054	0.057	0.062	0.052	0.108	0.113	0.119	0.106
200	0.010	0.011	0.014	0.013	0.054	0.055	0.060	0.052	0.106	0.107	0.113	0.103
250	0.010	0.011	0.013	0.012	0.054	0.053	0.058	0.052	0.107	0.104	0.109	0.100
Two-sided tests - $T = 250$												
-5	0.008	0.008	0.033	0.012	0.044	0.038	0.081	0.045	0.093	0.083	0.126	0.087
-2.5	0.010	0.010	0.032	0.010	0.051	0.052	0.093	0.053	0.104	0.117	0.162	0.112
0	0.009	0.010	0.024	0.009	0.053	0.058	0.083	0.054	0.104	0.117	0.146	0.107
2.5	0.010	0.010	0.022	0.010	0.056	0.059	0.080	0.056	0.111	0.121	0.143	0.113
5	0.011	0.011	0.019	0.010	0.055	0.058	0.071	0.055	0.109	0.117	0.135	0.113
10	0.012	0.012	0.017	0.011	0.052	0.055	0.066	0.052	0.105	0.110	0.124	0.107
25	0.011	0.012	0.015	0.012	0.053	0.054	0.061	0.053	0.102	0.106	0.115	0.099
50	0.011	0.011	0.013	0.012	0.054	0.055	0.061	0.050	0.110	0.106	0.115	0.101
75	0.011	0.010	0.013	0.012	0.053	0.053	0.059	0.051	0.110	0.103	0.114	0.103
100	0.010	0.011	0.014	0.014	0.052	0.050	0.057	0.051	0.109	0.102	0.112	0.101
125	0.011	0.012	0.015	0.014	0.053	0.051	0.058	0.051	0.106	0.100	0.109	0.098
150	0.011	0.010	0.015	0.014	0.053	0.051	0.058	0.052	0.107	0.100	0.109	0.097
200	0.011	0.012	0.015	0.013	0.052	0.051	0.056	0.053	0.108	0.101	0.112	0.100
250	0.011	0.012	0.015	0.014	0.053	0.050	0.057	0.053	0.109	0.101	0.111	0.100

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.011	0.001	0.002	0.002	0.049	0.009	0.010	0.011	0.097	0.024	0.027	0.026
-2.5	0.008	0.000	0.000	0.000	0.050	0.001	0.002	0.002	0.105	0.004	0.005	0.005
0	0.010	0.000	0.000	0.000	0.034	0.003	0.003	0.003	0.062	0.007	0.008	0.009
2.5	0.016	0.001	0.001	0.001	0.053	0.011	0.012	0.012	0.093	0.025	0.025	0.027
5	0.018	0.003	0.003	0.004	0.058	0.018	0.017	0.020	0.104	0.037	0.038	0.040
10	0.016	0.005	0.005	0.007	0.062	0.025	0.025	0.027	0.106	0.052	0.052	0.057
25	0.014	0.007	0.007	0.009	0.058	0.032	0.031	0.034	0.110	0.067	0.068	0.070
50	0.013	0.008	0.007	0.010	0.056	0.035	0.036	0.039	0.109	0.075	0.076	0.077
75	0.012	0.007	0.008	0.010	0.055	0.036	0.037	0.041	0.107	0.077	0.077	0.078
100	0.012	0.007	0.008	0.010	0.055	0.039	0.040	0.042	0.106	0.080	0.080	0.081
125	0.011	0.008	0.008	0.010	0.055	0.040	0.042	0.043	0.103	0.080	0.082	0.082
150	0.011	0.008	0.008	0.010	0.055	0.041	0.043	0.043	0.104	0.082	0.083	0.085
200	0.010	0.007	0.008	0.009	0.055	0.044	0.044	0.046	0.103	0.085	0.086	0.085
250	0.011	0.008	0.008	0.010	0.056	0.044	0.044	0.047	0.103	0.088	0.086	0.088
Right-sided tests - $T = 1000$												
-5	0.010	0.013	0.020	0.016	0.047	0.079	0.089	0.077	0.097	0.156	0.168	0.146
-2.5	0.010	0.017	0.024	0.016	0.051	0.107	0.123	0.100	0.104	0.229	0.243	0.220
0	0.013	0.023	0.026	0.020	0.058	0.111	0.121	0.103	0.122	0.226	0.234	0.216
2.5	0.014	0.026	0.028	0.020	0.067	0.111	0.118	0.103	0.125	0.208	0.217	0.198
5	0.016	0.025	0.028	0.020	0.065	0.103	0.108	0.095	0.124	0.194	0.198	0.186
10	0.015	0.023	0.026	0.018	0.060	0.093	0.095	0.085	0.116	0.175	0.181	0.166
25	0.013	0.018	0.019	0.013	0.055	0.079	0.082	0.071	0.110	0.150	0.154	0.140
50	0.011	0.016	0.015	0.012	0.053	0.070	0.073	0.061	0.106	0.137	0.139	0.125
75	0.009	0.014	0.014	0.010	0.052	0.067	0.068	0.059	0.103	0.130	0.130	0.120
100	0.010	0.014	0.014	0.010	0.052	0.064	0.065	0.058	0.103	0.125	0.127	0.117
125	0.011	0.013	0.014	0.011	0.052	0.062	0.064	0.055	0.102	0.122	0.124	0.113
150	0.010	0.013	0.014	0.010	0.051	0.061	0.062	0.054	0.102	0.120	0.122	0.112
200	0.010	0.013	0.013	0.011	0.053	0.059	0.061	0.054	0.102	0.116	0.118	0.111
250	0.010	0.012	0.012	0.011	0.052	0.059	0.060	0.054	0.103	0.116	0.118	0.110
Two-sided tests - $T = 1000$												
-5	0.009	0.007	0.012	0.010	0.048	0.041	0.052	0.043	0.100	0.086	0.099	0.088
-2.5	0.008	0.009	0.014	0.009	0.045	0.050	0.063	0.046	0.093	0.108	0.125	0.102
0	0.009	0.011	0.014	0.010	0.048	0.057	0.063	0.051	0.099	0.112	0.124	0.107
2.5	0.011	0.014	0.017	0.011	0.056	0.060	0.068	0.058	0.107	0.122	0.130	0.115
5	0.013	0.014	0.018	0.012	0.056	0.062	0.069	0.058	0.110	0.120	0.125	0.115
10	0.014	0.015	0.018	0.013	0.056	0.060	0.063	0.057	0.107	0.115	0.120	0.112
25	0.013	0.014	0.014	0.013	0.052	0.055	0.057	0.054	0.104	0.110	0.114	0.105
50	0.011	0.012	0.012	0.012	0.052	0.054	0.056	0.052	0.103	0.105	0.109	0.100
75	0.011	0.011	0.011	0.012	0.053	0.052	0.054	0.050	0.104	0.103	0.105	0.100
100	0.011	0.011	0.011	0.011	0.051	0.051	0.054	0.050	0.103	0.103	0.105	0.099
125	0.010	0.010	0.011	0.010	0.052	0.052	0.054	0.051	0.104	0.103	0.105	0.098
150	0.009	0.009	0.010	0.010	0.053	0.050	0.053	0.051	0.105	0.102	0.105	0.097
200	0.009	0.009	0.010	0.011	0.052	0.050	0.052	0.049	0.106	0.103	0.105	0.100
250	0.010	0.009	0.010	0.012	0.051	0.049	0.052	0.049	0.105	0.102	0.104	0.100

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.41: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP11 (Stochastic Volatility):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)'$ follow from a first-order AR stochastic volatility process as $(u_t = e_{1t}\texttt{exp}(h_{1t}), v_t = e_{2t}\texttt{exp}(h_{2t}))'$ with $h_{it} = \lambda h_{i,t-1} + 0.5\xi_{it},\ i = 1, 2.$ and $(\xi_{it}, e_{it})' \sim NIID(0, diag(\sigma_\xi^2, 1))$, independent across i = 1,2. Results are reported for $(\lambda, \sigma_\xi) = (0.951, 0.314)$ and $(e_{1t}, e_{2t})' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} = [1 \quad -0.5; -0.5 \quad 1]$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.225	0.003	0.015	0.005	0.235	0.020	0.037	0.026	0.247	0.051	0.067	0.053
-2.5	0.010	0.001	0.007	0.002	0.047	0.008	0.016	0.011	0.097	0.022	0.030	0.024
0	0.006	0.001	0.004	0.002	0.032	0.010	0.015	0.012	0.063	0.024	0.030	0.026
2.5	0.010	0.003	0.005	0.004	0.042	0.018	0.022	0.020	0.085	0.041	0.048	0.043
5	0.011	0.004	0.006	0.005	0.049	0.024	0.028	0.026	0.094	0.054	0.057	0.057
10	0.010	0.005	0.007	0.008	0.053	0.032	0.035	0.032	0.100	0.067	0.070	0.067
25	0.011	0.008	0.009	0.009	0.053	0.039	0.041	0.041	0.107	0.080	0.084	0.080
50	0.011	0.009	0.010	0.011	0.054	0.043	0.047	0.045	0.109	0.091	0.094	0.090
75	0.010	0.009	0.010	0.011	0.055	0.045	0.050	0.046	0.111	0.093	0.097	0.093
100	0.011	0.008	0.011	0.011	0.055	0.047	0.052	0.048	0.109	0.095	0.101	0.095
125	0.010	0.008	0.011	0.010	0.055	0.047	0.052	0.049	0.112	0.100	0.104	0.096
150	0.011	0.009	0.011	0.011	0.053	0.046	0.052	0.048	0.112	0.100	0.104	0.098
200	0.010	0.010	0.011	0.010	0.056	0.048	0.054	0.050	0.114	0.103	0.108	0.102
250	0.010	0.009	0.012	0.011	0.054	0.051	0.056	0.054	0.114	0.103	0.110	0.102
Right-sided tests - $T = 250$												
-5	0.009	0.016	0.048	0.019	0.049	0.074	0.109	0.069	0.095	0.141	0.169	0.130
-2.5	0.013	0.022	0.055	0.022	0.059	0.107	0.131	0.100	0.115	0.199	0.214	0.190
0	0.014	0.022	0.045	0.021	0.069	0.100	0.124	0.096	0.131	0.193	0.208	0.185
2.5	0.014	0.020	0.036	0.020	0.066	0.093	0.107	0.089	0.127	0.169	0.184	0.164
5	0.012	0.018	0.028	0.018	0.060	0.083	0.096	0.080	0.120	0.154	0.169	0.151
10	0.011	0.015	0.020	0.016	0.058	0.073	0.083	0.071	0.111	0.141	0.148	0.136
25	0.010	0.013	0.017	0.012	0.050	0.061	0.066	0.058	0.101	0.118	0.128	0.117
50	0.010	0.012	0.015	0.013	0.052	0.058	0.062	0.054	0.098	0.108	0.114	0.106
75	0.011	0.013	0.015	0.013	0.051	0.054	0.060	0.052	0.098	0.106	0.112	0.102
100	0.011	0.012	0.014	0.012	0.051	0.056	0.061	0.052	0.099	0.107	0.112	0.103
125	0.010	0.010	0.013	0.012	0.050	0.053	0.060	0.052	0.102	0.109	0.112	0.103
150	0.009	0.011	0.013	0.011	0.050	0.052	0.059	0.053	0.104	0.107	0.113	0.102
200	0.008	0.009	0.011	0.011	0.053	0.054	0.060	0.052	0.106	0.105	0.111	0.101
250	0.009	0.010	0.012	0.011	0.051	0.051	0.056	0.050	0.107	0.103	0.110	0.100
Two-sided tests - $T = 250$												
-5	0.008	0.008	0.047	0.016	0.049	0.049	0.098	0.053	0.095	0.092	0.146	0.095
-2.5	0.011	0.011	0.045	0.013	0.053	0.060	0.100	0.057	0.105	0.114	0.148	0.111
0	0.011	0.012	0.033	0.012	0.054	0.056	0.087	0.056	0.106	0.111	0.139	0.108
2.5	0.010	0.011	0.027	0.013	0.054	0.055	0.076	0.057	0.108	0.111	0.129	0.109
5	0.011	0.011	0.020	0.012	0.053	0.053	0.069	0.057	0.106	0.108	0.124	0.107
10	0.009	0.009	0.014	0.011	0.053	0.050	0.061	0.055	0.106	0.104	0.118	0.103
25	0.010	0.011	0.014	0.012	0.052	0.049	0.055	0.051	0.100	0.097	0.107	0.099
50	0.011	0.011	0.014	0.012	0.049	0.048	0.054	0.051	0.102	0.100	0.109	0.099
75	0.011	0.011	0.013	0.013	0.052	0.049	0.056	0.052	0.104	0.100	0.110	0.098
100	0.010	0.010	0.013	0.013	0.052	0.048	0.056	0.052	0.104	0.101	0.112	0.100
125	0.008	0.009	0.013	0.013	0.051	0.049	0.056	0.052	0.104	0.100	0.112	0.101
150	0.007	0.009	0.011	0.011	0.049	0.049	0.056	0.053	0.104	0.098	0.111	0.101
200	0.007	0.008	0.012	0.012	0.047	0.048	0.056	0.052	0.105	0.101	0.115	0.102
250	0.008	0.009	0.013	0.013	0.051	0.051	0.059	0.051	0.105	0.101	0.112	0.104

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.011	0.004	0.009	0.005	0.055	0.026	0.036	0.028	0.106	0.059	0.070	0.057
-2.5	0.011	0.002	0.005	0.002	0.051	0.011	0.016	0.011	0.101	0.026	0.032	0.027
0	0.009	0.001	0.003	0.003	0.036	0.013	0.015	0.014	0.067	0.031	0.034	0.031
2.5	0.012	0.004	0.006	0.006	0.048	0.024	0.026	0.026	0.091	0.050	0.051	0.052
5	0.013	0.006	0.005	0.007	0.053	0.030	0.032	0.031	0.096	0.061	0.063	0.062
10	0.013	0.006	0.007	0.008	0.053	0.035	0.036	0.037	0.103	0.069	0.070	0.071
25	0.011	0.008	0.008	0.010	0.057	0.039	0.040	0.043	0.102	0.080	0.081	0.079
50	0.011	0.008	0.008	0.010	0.053	0.042	0.044	0.046	0.103	0.084	0.086	0.083
75	0.011	0.008	0.008	0.010	0.053	0.045	0.045	0.046	0.103	0.089	0.089	0.087
100	0.010	0.009	0.008	0.009	0.055	0.046	0.048	0.047	0.102	0.088	0.091	0.089
125	0.010	0.008	0.008	0.009	0.054	0.047	0.047	0.047	0.104	0.090	0.092	0.090
150	0.010	0.008	0.008	0.010	0.053	0.046	0.047	0.046	0.104	0.092	0.094	0.092
200	0.010	0.008	0.008	0.009	0.053	0.047	0.048	0.047	0.105	0.094	0.095	0.094
250	0.009	0.008	0.009	0.010	0.053	0.047	0.048	0.048	0.104	0.096	0.097	0.095
Right-sided tests - $T = 1000$												
-5	0.008	0.014	0.029	0.016	0.046	0.067	0.088	0.067	0.095	0.136	0.157	0.127
-2.5	0.010	0.018	0.040	0.018	0.052	0.096	0.116	0.092	0.104	0.187	0.201	0.181
0	0.013	0.019	0.029	0.017	0.059	0.095	0.109	0.089	0.124	0.181	0.193	0.178
2.5	0.013	0.019	0.024	0.018	0.062	0.087	0.093	0.082	0.114	0.158	0.165	0.152
5	0.013	0.018	0.021	0.017	0.057	0.080	0.086	0.075	0.111	0.145	0.151	0.142
10	0.012	0.017	0.017	0.015	0.055	0.071	0.074	0.067	0.106	0.137	0.139	0.133
25	0.011	0.014	0.015	0.012	0.052	0.062	0.065	0.060	0.101	0.123	0.126	0.122
50	0.010	0.013	0.013	0.010	0.050	0.060	0.061	0.055	0.103	0.117	0.120	0.113
75	0.008	0.011	0.011	0.009	0.050	0.058	0.061	0.054	0.099	0.114	0.116	0.107
100	0.008	0.011	0.011	0.009	0.051	0.058	0.060	0.054	0.099	0.112	0.114	0.106
125	0.009	0.011	0.011	0.009	0.051	0.058	0.059	0.053	0.100	0.111	0.114	0.107
150	0.010	0.012	0.011	0.010	0.052	0.058	0.059	0.052	0.102	0.113	0.113	0.106
200	0.010	0.012	0.013	0.010	0.049	0.055	0.057	0.050	0.103	0.110	0.113	0.106
250	0.011	0.012	0.012	0.011	0.049	0.053	0.055	0.051	0.103	0.109	0.111	0.105
Two-sided tests - $T = 1000$												
-5	0.009	0.008	0.024	0.011	0.048	0.045	0.074	0.046	0.099	0.093	0.124	0.094
-2.5	0.009	0.010	0.030	0.011	0.047	0.052	0.082	0.050	0.097	0.106	0.132	0.103
0	0.010	0.010	0.021	0.010	0.049	0.052	0.065	0.051	0.100	0.108	0.125	0.103
2.5	0.012	0.011	0.018	0.012	0.053	0.056	0.062	0.055	0.105	0.111	0.118	0.108
5	0.010	0.012	0.015	0.012	0.055	0.054	0.059	0.055	0.105	0.109	0.118	0.106
10	0.012	0.012	0.013	0.013	0.055	0.054	0.056	0.056	0.104	0.107	0.110	0.104
25	0.011	0.011	0.011	0.012	0.052	0.051	0.054	0.053	0.104	0.103	0.105	0.102
50	0.011	0.010	0.010	0.011	0.052	0.050	0.052	0.050	0.101	0.102	0.105	0.100
75	0.010	0.010	0.009	0.011	0.051	0.050	0.052	0.049	0.103	0.103	0.106	0.100
100	0.009	0.009	0.010	0.010	0.049	0.048	0.051	0.050	0.105	0.105	0.107	0.101
125	0.009	0.009	0.010	0.010	0.050	0.049	0.051	0.050	0.105	0.104	0.107	0.100
150	0.010	0.009	0.010	0.010	0.050	0.049	0.049	0.049	0.104	0.102	0.105	0.098
200	0.010	0.010	0.010	0.010	0.049	0.049	0.051	0.050	0.102	0.100	0.105	0.097
250	0.010	0.010	0.011	0.011	0.050	0.048	0.050	0.050	0.102	0.099	0.103	0.099

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

Table D.42: Empirical rejection frequencies of one-sided (left and right tail) and two-sided predictability tests, for sample sizes $T = 250$ and $T = 1000$. **DGP11 (Stochastic Volatility):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)'$ follow from a first-order AR stochastic volatility process as $(u_t = e_{1t}\texttt{exp}(h_{1t}), v_t = e_{2t}\texttt{exp}(h_{2t}))'$ with $h_{it} = \lambda h_{i,t-1} + 0.5\xi_{it},\ i = 1, 2.$ and $(\xi_{it}, e_{it})' \sim NIID(0, diag(\sigma_\xi^2, 1))$, independent across i = 1,2. Results are reported for $(\lambda, \sigma_\xi) = (0.951, 0.314)$ and $(e_{1t}, e_{2t})' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$ with $\mathbf{\Sigma} = [1 \quad 0; 0 \quad 1]$.

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 250$												
-5	0.011	0.010	0.036	0.014	0.052	0.050	0.080	0.051	0.103	0.103	0.124	0.098
-2.5	0.011	0.011	0.028	0.012	0.051	0.050	0.063	0.051	0.100	0.100	0.103	0.097
0	0.011	0.011	0.024	0.011	0.051	0.051	0.061	0.050	0.099	0.097	0.101	0.096
2.5	0.010	0.009	0.017	0.010	0.050	0.047	0.058	0.049	0.099	0.097	0.106	0.095
5	0.009	0.009	0.014	0.010	0.051	0.048	0.057	0.049	0.099	0.096	0.108	0.094
10	0.010	0.010	0.012	0.011	0.051	0.049	0.055	0.049	0.098	0.099	0.106	0.096
25	0.011	0.010	0.012	0.011	0.051	0.051	0.055	0.052	0.102	0.102	0.110	0.099
50	0.011	0.010	0.012	0.011	0.052	0.049	0.055	0.051	0.106	0.104	0.111	0.101
75	0.009	0.009	0.011	0.011	0.053	0.052	0.057	0.052	0.108	0.105	0.110	0.104
100	0.009	0.009	0.011	0.011	0.054	0.052	0.058	0.053	0.108	0.104	0.109	0.103
125	0.009	0.009	0.012	0.011	0.053	0.052	0.058	0.054	0.110	0.105	0.113	0.104
150	0.009	0.010	0.013	0.012	0.054	0.051	0.058	0.052	0.112	0.106	0.111	0.105
200	0.009	0.011	0.013	0.012	0.052	0.051	0.057	0.051	0.112	0.106	0.112	0.107
250	0.009	0.010	0.013	0.012	0.055	0.052	0.057	0.054	0.111	0.105	0.110	0.106
Right-sided tests - $T = 250$												
-5	0.010	0.010	0.035	0.014	0.051	0.051	0.074	0.051	0.100	0.101	0.116	0.097
-2.5	0.010	0.010	0.027	0.012	0.052	0.051	0.061	0.050	0.104	0.102	0.098	0.101
0	0.011	0.011	0.023	0.011	0.056	0.054	0.062	0.053	0.107	0.104	0.107	0.102
2.5	0.012	0.012	0.020	0.012	0.055	0.055	0.065	0.055	0.110	0.105	0.114	0.106
5	0.011	0.012	0.016	0.013	0.058	0.056	0.062	0.054	0.107	0.103	0.112	0.106
10	0.011	0.010	0.014	0.012	0.056	0.054	0.060	0.053	0.107	0.105	0.111	0.104
25	0.010	0.009	0.011	0.010	0.053	0.052	0.056	0.053	0.106	0.105	0.111	0.102
50	0.010	0.010	0.011	0.010	0.051	0.051	0.055	0.053	0.103	0.101	0.106	0.099
75	0.010	0.008	0.011	0.010	0.051	0.050	0.056	0.052	0.103	0.100	0.105	0.097
100	0.010	0.009	0.012	0.010	0.051	0.050	0.054	0.054	0.101	0.098	0.104	0.097
125	0.009	0.010	0.012	0.011	0.051	0.048	0.054	0.050	0.103	0.098	0.105	0.096
150	0.009	0.009	0.012	0.011	0.051	0.048	0.054	0.051	0.103	0.099	0.106	0.097
200	0.008	0.009	0.012	0.010	0.051	0.051	0.055	0.053	0.105	0.099	0.104	0.098
250	0.008	0.010	0.013	0.011	0.050	0.050	0.054	0.050	0.103	0.099	0.106	0.099
Two-sided tests - $T = 250$												
-5	0.009	0.009	0.057	0.017	0.051	0.051	0.106	0.060	0.101	0.100	0.154	0.103
-2.5	0.009	0.010	0.043	0.013	0.051	0.053	0.087	0.055	0.102	0.101	0.124	0.101
0	0.011	0.011	0.035	0.013	0.053	0.052	0.077	0.054	0.104	0.105	0.123	0.103
2.5	0.010	0.011	0.023	0.012	0.054	0.053	0.071	0.055	0.105	0.103	0.123	0.104
5	0.010	0.010	0.017	0.013	0.054	0.053	0.066	0.055	0.107	0.104	0.119	0.103
10	0.010	0.009	0.014	0.013	0.053	0.050	0.058	0.056	0.104	0.103	0.115	0.102
25	0.010	0.009	0.012	0.012	0.051	0.048	0.058	0.053	0.103	0.103	0.112	0.104
50	0.010	0.010	0.013	0.012	0.050	0.048	0.055	0.053	0.102	0.101	0.110	0.104
75	0.009	0.010	0.012	0.011	0.048	0.049	0.056	0.053	0.104	0.101	0.113	0.104
100	0.009	0.009	0.012	0.011	0.048	0.049	0.058	0.054	0.105	0.102	0.112	0.107
125	0.008	0.008	0.012	0.011	0.050	0.047	0.056	0.054	0.103	0.100	0.111	0.104
150	0.007	0.009	0.013	0.012	0.048	0.048	0.057	0.053	0.103	0.099	0.112	0.102
200	0.008	0.009	0.013	0.013	0.049	0.049	0.058	0.054	0.104	0.100	0.112	0.104
250	0.007	0.009	0.013	0.014	0.049	0.051	0.060	0.053	0.104	0.100	0.112	0.103

c	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}	$t_{zx}^{*,RWB}$	$t_{zx}^{*,FRWB}$	t_{zx}^{EW}	t_{zx}
Left-sided tests - $T = 1000$												
-5	0.010	0.011	0.025	0.012	0.053	0.055	0.071	0.052	0.105	0.104	0.122	0.103
-2.5	0.010	0.012	0.028	0.012	0.051	0.052	0.066	0.050	0.102	0.104	0.111	0.101
0	0.012	0.012	0.022	0.013	0.053	0.053	0.063	0.051	0.104	0.105	0.107	0.101
2.5	0.013	0.013	0.016	0.013	0.054	0.052	0.058	0.053	0.100	0.101	0.105	0.099
5	0.012	0.012	0.015	0.012	0.052	0.052	0.055	0.051	0.099	0.097	0.102	0.096
10	0.012	0.012	0.012	0.013	0.053	0.053	0.054	0.051	0.099	0.100	0.102	0.096
25	0.012	0.012	0.012	0.013	0.051	0.053	0.055	0.051	0.101	0.099	0.100	0.097
50	0.012	0.011	0.012	0.013	0.050	0.051	0.051	0.049	0.099	0.099	0.099	0.095
75	0.012	0.010	0.011	0.012	0.051	0.051	0.052	0.050	0.100	0.100	0.102	0.095
100	0.011	0.011	0.011	0.012	0.053	0.053	0.052	0.051	0.102	0.100	0.101	0.097
125	0.012	0.010	0.010	0.012	0.054	0.051	0.053	0.052	0.102	0.101	0.102	0.097
150	0.011	0.011	0.010	0.012	0.052	0.052	0.052	0.051	0.103	0.101	0.103	0.098
200	0.011	0.010	0.011	0.011	0.051	0.051	0.052	0.051	0.101	0.102	0.103	0.097
250	0.010	0.010	0.011	0.010	0.051	0.051	0.052	0.050	0.100	0.100	0.102	0.098
Right-sided tests - $T = 1000$												
-5	0.009	0.009	0.025	0.012	0.050	0.049	0.070	0.050	0.099	0.099	0.117	0.094
-2.5	0.011	0.011	0.026	0.012	0.050	0.050	0.063	0.050	0.097	0.096	0.109	0.095
0	0.010	0.010	0.019	0.010	0.047	0.047	0.056	0.046	0.097	0.100	0.101	0.095
2.5	0.011	0.011	0.014	0.011	0.050	0.049	0.056	0.048	0.099	0.099	0.104	0.097
5	0.010	0.010	0.013	0.011	0.049	0.050	0.052	0.049	0.100	0.098	0.104	0.097
10	0.010	0.010	0.011	0.011	0.049	0.050	0.050	0.048	0.100	0.101	0.102	0.099
25	0.010	0.009	0.010	0.011	0.049	0.048	0.049	0.047	0.102	0.100	0.100	0.097
50	0.010	0.010	0.009	0.010	0.050	0.049	0.050	0.048	0.098	0.098	0.100	0.096
75	0.010	0.010	0.011	0.010	0.050	0.049	0.049	0.048	0.099	0.098	0.099	0.097
100	0.010	0.010	0.010	0.010	0.049	0.048	0.049	0.046	0.099	0.098	0.100	0.099
125	0.009	0.009	0.011	0.011	0.050	0.047	0.050	0.049	0.099	0.099	0.099	0.096
150	0.010	0.009	0.010	0.011	0.050	0.049	0.050	0.049	0.098	0.098	0.098	0.095
200	0.009	0.010	0.011	0.010	0.049	0.050	0.051	0.049	0.099	0.098	0.100	0.096
250	0.010	0.011	0.011	0.011	0.050	0.050	0.051	0.048	0.100	0.096	0.099	0.098
Two-sided tests - $T = 1000$												
-5	0.009	0.010	0.034	0.013	0.051	0.052	0.089	0.055	0.102	0.105	0.141	0.102
-2.5	0.011	0.012	0.039	0.013	0.051	0.051	0.082	0.051	0.101	0.102	0.129	0.100
0	0.012	0.013	0.029	0.013	0.053	0.053	0.071	0.052	0.099	0.100	0.119	0.097
2.5	0.012	0.012	0.020	0.013	0.052	0.050	0.063	0.052	0.102	0.101	0.113	0.101
5	0.012	0.013	0.016	0.014	0.052	0.051	0.057	0.053	0.100	0.101	0.108	0.099
10	0.012	0.013	0.013	0.014	0.052	0.051	0.053	0.052	0.101	0.100	0.104	0.099
25	0.011	0.010	0.011	0.012	0.049	0.051	0.052	0.051	0.102	0.100	0.104	0.098
50	0.011	0.011	0.011	0.014	0.050	0.052	0.053	0.050	0.099	0.099	0.101	0.097
75	0.010	0.010	0.011	0.012	0.051	0.052	0.051	0.052	0.101	0.100	0.101	0.097
100	0.010	0.009	0.011	0.013	0.050	0.049	0.053	0.051	0.100	0.099	0.101	0.097
125	0.010	0.009	0.011	0.012	0.051	0.050	0.051	0.050	0.102	0.100	0.103	0.100
150	0.010	0.010	0.011	0.012	0.050	0.050	0.052	0.049	0.101	0.100	0.102	0.100
200	0.010	0.010	0.011	0.012	0.050	0.050	0.050	0.050	0.099	0.102	0.103	0.100
250	0.010	0.010	0.011	0.012	0.052	0.051	0.052	0.053	0.102	0.099	0.103	0.098

Note: t_{zx} and t_{zx}^{EW} correspond to the statistics presented in (12) and (14) of the main text, and $t_{zx}^{*,RWB}$ and $t_{zx}^{*,FRWB}$ are the corresponding residual wild bootstrap (RWB) and fixed regressor wild bootstrap (FRWB) implementations of (12) computed as described in Algorithms 1 and 2 of Section 4 in the main text.

D.2 Subsample Test Results

In this section we provide simulation results for the subsample IVX-based predictability tests introduced in section 3.2. Specifically, we consider upper-tail tests based on the maximum *forward recursive* statistic

$$\mathcal{T}_U^F := \max_{\tau_L \leq \tau \leq 1} \{t_{zx}(0, \tau)\}; \tag{D.11}$$

and on the maximum of the *backward recursive* statistics

$$\mathcal{T}_U^B := \max_{0 \leq \tau \leq \tau_U} \{t_{zx}(\tau, 1)\}; \tag{D.12}$$

and on the maximum of the *rolling* statistics

$$\mathcal{T}_U^R := \max_{0 \leq \tau \leq 1-\Delta\tau} \{t_{zx}(\tau, \tau + \Delta\tau)\}. \tag{D.13}$$

The results we present pertain to $\tau_L = 1/3$, $\tau_U = 2/3$ and $\Delta\tau = 1/3$.

We also present results for the corresponding left-tailed versions of these tests, which are based on the minimum across the sequences, denoted $\mathcal{T}_L^k$, $k = F, B, R$, together with two-tailed versions of these tests based on the maximum taken over sequences of $(t_{zx}(0, \tau))^2$, $(t_{zx}(\tau, 1))^2$ and $(t_{zx}(\tau, \tau + \Delta\tau))^2$ statistics, denoted $\mathcal{T}_2^k$, with $k = F, B, R$, respectively. The reported subsample IVX-based tests are computed using either the residual wild bootstrap (RWB) or the fixed regressor wild bootstrap $(FRWB)$ algorithms described in Algorithms 1 and 2, respectively, in section 4. In the Tables and Figures below, we denote the corresponding bootstrap based tests by $\mathcal{T}_s^{k,M}$ with: $s = 2, L, U$ indicating two-sided, lower-tail and upper-tail tests, respectively; $k = F, B, R$ indicating tests formed from the forward recursive, backward recursive and rolling sequences, respectively; and $M = RWB, FRWB$ indicating whether the RWB or FRWB was used. Additionally, the subscript "*one-sided*" in Figures D.1–D.3 is used to indicate that the (discontinuous) power function graphed is that of the left-sided version of the test for $b < 0$ and the right-sided version of the test for $b > 0$.

For comparative purposes we also report results for the two-sided two-stage least-squares (2SLS) forward recursive, backward recursive and rolling tests proposed in Demetrescu *et al.* (2022), which we denote $\mathcal{T}_2^{k,2SLS}$, with the superscript $k = F, B, R$ again denoting the forward recursive, backward recursive and rolling implementations of these tests, respectively.

Table D.43: Empirical rejection frequencies of rolling predictability test statistics, for sample sizes $T = 250$ and $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.95; \quad -0.95 \quad 1]$.

	1%							5%							10%						
c	$\mathcal{T}_2^{R,2SLS}$	$\mathcal{T}_2^{R,FRWB}$	$\mathcal{T}_2^{R,RWB}$	$\mathcal{T}_L^{R,FRWB}$	$\mathcal{T}_L^{R,RWB}$	$\mathcal{T}_U^{R,FRWB}$	$\mathcal{T}_U^{R,RWB}$	$\mathcal{T}_2^{R,2SLS}$	$\mathcal{T}_2^{R,FRWB}$	$\mathcal{T}_2^{R,RWB}$	$\mathcal{T}_L^{R,FRWB}$	$\mathcal{T}L^{R,RWB}$	$\mathcal{T}_U^{R,FRWB}$	$\mathcal{T}_U^{R,RWB}$	$\mathcal{T}_2^{R,2SLS}$	$\mathcal{T}_2^{R,FRWB}$	$\mathcal{T}_2^{R,RWB}$	$\mathcal{T}_L^{R,FRWB}$	$\mathcal{T}_L^{R,RWB}$	$\mathcal{T}_U^{R,FRWB}$	$\mathcal{T}_U^{R,RWB}$
											$T = 250$										
-5	0.015	0.004	0.005	0.000	0.008	0.007	0.005	0.066	0.016	0.036	0.000	0.051	0.031	0.036	0.122	0.032	0.084	0.000	0.099	0.055	0.084
-2.5	0.015	0.007	0.006	0.000	0.005	0.013	0.006	0.062	0.029	0.029	0.000	0.036	0.055	0.029	0.122	0.056	0.061	0.000	0.082	0.104	0.062
0	0.010	0.010	0.007	0.000	0.006	0.021	0.007	0.055	0.050	0.041	0.000	0.026	0.094	0.042	0.109	0.095	0.085	0.001	0.055	0.176	0.086
2.5	0.011	0.012	0.009	0.000	0.010	0.024	0.010	0.051	0.058	0.048	0.000	0.041	0.110	0.049	0.105	0.112	0.097	0.001	0.080	0.209	0.098
5	0.012	0.013	0.010	0.000	0.011	0.024	0.010	0.054	0.059	0.050	0.001	0.049	0.116	0.051	0.104	0.118	0.105	0.004	0.094	0.220	0.107
10	0.012	0.013	0.010	0.001	0.013	0.027	0.011	0.057	0.062	0.054	0.003	0.060	0.117	0.055	0.109	0.119	0.108	0.007	0.108	0.218	0.110
25	0.012	0.013	0.010	0.001	0.014	0.025	0.010	0.060	0.064	0.057	0.008	0.060	0.116	0.058	0.118	0.124	0.113	0.018	0.111	0.219	0.116
50	0.013	0.013	0.010	0.003	0.014	0.023	0.010	0.062	0.064	0.059	0.013	0.063	0.110	0.061	0.119	0.123	0.119	0.033	0.115	0.203	0.123
75	0.012	0.012	0.010	0.003	0.014	0.022	0.011	0.060	0.062	0.058	0.018	0.062	0.102	0.060	0.120	0.121	0.123	0.042	0.120	0.189	0.127
100	0.012	0.012	0.010	0.004	0.013	0.021	0.011	0.058	0.059	0.058	0.023	0.063	0.097	0.061	0.117	0.118	0.121	0.050	0.121	0.176	0.128
125	0.012	0.012	0.010	0.005	0.013	0.020	0.011	0.055	0.056	0.057	0.026	0.063	0.086	0.062	0.111	0.114	0.120	0.057	0.121	0.167	0.128
150	0.012	0.012	0.011	0.005	0.013	0.018	0.011	0.054	0.055	0.055	0.029	0.063	0.082	0.061	0.110	0.109	0.118	0.062	0.121	0.157	0.128
200	0.012	0.012	0.012	0.006	0.011	0.016	0.013	0.052	0.052	0.056	0.036	0.062	0.074	0.060	0.110	0.110	0.120	0.075	0.120	0.141	0.126
250	0.011	0.011	0.011	0.008	0.010	0.015	0.014	0.054	0.054	0.058	0.041	0.059	0.065	0.063	0.107	0.107	0.117	0.084	0.120	0.125	0.127
											$T = 1000$										
-5	0.018	0.004	0.005	0.000	0.016	0.007	0.005	0.062	0.016	0.034	0.000	0.057	0.028	0.034	0.123	0.027	0.078	0.000	0.111	0.047	0.078
-2.5	0.017	0.009	0.007	0.000	0.008	0.014	0.007	0.069	0.031	0.029	0.000	0.041	0.055	0.029	0.137	0.058	0.060	0.000	0.085	0.099	0.060
0	0.013	0.014	0.009	0.000	0.008	0.026	0.009	0.063	0.056	0.043	0.000	0.035	0.110	0.043	0.131	0.111	0.096	0.000	0.061	0.184	0.098
2.5	0.009	0.012	0.008	0.000	0.012	0.023	0.009	0.052	0.061	0.049	0.000	0.047	0.126	0.050	0.113	0.126	0.106	0.002	0.085	0.222	0.109
5	0.010	0.011	0.008	0.000	0.012	0.025	0.008	0.057	0.065	0.051	0.001	0.057	0.129	0.052	0.119	0.135	0.109	0.004	0.106	0.237	0.112
10	0.012	0.014	0.010	0.000	0.016	0.025	0.010	0.066	0.071	0.054	0.003	0.063	0.129	0.056	0.125	0.133	0.111	0.009	0.115	0.237	0.114
25	0.014	0.016	0.012	0.001	0.014	0.027	0.012	0.065	0.068	0.053	0.008	0.061	0.126	0.056	0.127	0.133	0.115	0.023	0.113	0.238	0.116
50	0.016	0.016	0.014	0.002	0.013	0.027	0.014	0.066	0.068	0.057	0.012	0.061	0.116	0.059	0.123	0.126	0.111	0.029	0.111	0.223	0.114
75	0.017	0.018	0.013	0.003	0.013	0.027	0.012	0.066	0.068	0.059	0.017	0.055	0.114	0.060	0.127	0.129	0.112	0.033	0.108	0.208	0.116
100	0.016	0.016	0.015	0.004	0.013	0.028	0.016	0.066	0.067	0.057	0.019	0.054	0.106	0.059	0.124	0.127	0.114	0.040	0.104	0.196	0.115
125	0.018	0.016	0.013	0.004	0.014	0.024	0.013	0.065	0.064	0.060	0.022	0.053	0.103	0.059	0.122	0.124	0.116	0.046	0.103	0.192	0.120
150	0.016	0.017	0.015	0.005	0.012	0.024	0.014	0.063	0.063	0.062	0.024	0.052	0.102	0.062	0.121	0.123	0.115	0.048	0.102	0.190	0.120
200	0.015	0.016	0.014	0.005	0.013	0.021	0.014	0.060	0.061	0.058	0.024	0.057	0.095	0.062	0.117	0.118	0.115	0.055	0.101	0.176	0.121
250	0.015	0.015	0.014	0.005	0.011	0.023	0.014	0.060	0.060	0.060	0.028	0.055	0.090	0.064	0.117	0.116	0.112	0.062	0.101	0.167	0.118

Notes: $\mathcal{T}_2^{R,2SLS}$, denotes the two-tailed two-stage least squares based rolling test for predictability, and $\mathcal{T}_2^{R,M}$, $\mathcal{T}_L^{R,M}$, and $\mathcal{T}_U^{R,M}$, $M = RWB, FRWB$ denote, respectively, the two-tailed, lower-tail and upper-tail residual wild bootstrap ($M = RWB$) and fixed regressor wild bootstrap ($M = FRWB$) based rolling predictability test statistics. For computation of the forward and backward recursive statistics we set $\Delta\tau = 1/3$.

Table D.44: Empirical rejection frequencies of rolling predictability test statistics, for sample sizes $T = 250$ and $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.50; \quad -0.50 \quad 1]$.

	1%							5%							10%						
c	$\mathcal{T}_2^{R,2SLS}$	$\mathcal{T}_2^{R,FRWB}$	$\mathcal{T}_2^{R,RWB}$	$\mathcal{T}_L^{R,FRWB}$	$\mathcal{T}_L^{R,RWB}$	$\mathcal{T}_U^{R,FRWB}$	$\mathcal{T}_U^{R,RWB}$	$\mathcal{T}_2^{R,2SLS}$	$\mathcal{T}_2^{R,FRWB}$	$\mathcal{T}_2^{R,RWB}$	$\mathcal{T}_L^{R,FRWB}$	$\mathcal{T}L^{R,RWB}$	$\mathcal{T}_U^{R,FRWB}$	$\mathcal{T}_U^{R,RWB}$	$\mathcal{T}_2^{R,2SLS}$	$\mathcal{T}_2^{R,FRWB}$	$\mathcal{T}_2^{R,RWB}$	$\mathcal{T}_L^{R,FRWB}$	$\mathcal{T}_L^{R,RWB}$	$\mathcal{T}_U^{R,FRWB}$	$\mathcal{T}_U^{R,RWB}$
											$T = 250$										
-5	0.013	0.003	0.007	0.001	0.007	0.007	0.006	0.061	0.018	0.042	0.003	0.044	0.034	0.045	0.119	0.038	0.089	0.008	0.087	0.066	0.089
-2.5	0.010	0.007	0.006	0.000	0.005	0.012	0.007	0.052	0.028	0.033	0.001	0.026	0.059	0.034	0.116	0.062	0.071	0.003	0.065	0.107	0.074
0	0.013	0.009	0.008	0.000	0.004	0.018	0.010	0.055	0.047	0.039	0.002	0.026	0.088	0.042	0.114	0.093	0.083	0.005	0.054	0.163	0.091
2.5	0.010	0.013	0.009	0.001	0.005	0.024	0.010	0.058	0.056	0.048	0.005	0.034	0.101	0.053	0.117	0.106	0.095	0.012	0.074	0.195	0.104
5	0.010	0.013	0.010	0.002	0.007	0.027	0.011	0.055	0.058	0.053	0.008	0.042	0.103	0.053	0.106	0.108	0.100	0.017	0.080	0.198	0.104
10	0.012	0.014	0.011	0.003	0.009	0.024	0.012	0.053	0.058	0.056	0.011	0.046	0.102	0.055	0.104	0.115	0.105	0.025	0.092	0.193	0.110
25	0.011	0.013	0.010	0.004	0.012	0.021	0.010	0.054	0.055	0.050	0.018	0.051	0.091	0.055	0.105	0.109	0.103	0.042	0.097	0.173	0.110
50	0.011	0.010	0.009	0.005	0.010	0.016	0.011	0.052	0.053	0.049	0.022	0.047	0.078	0.054	0.095	0.099	0.097	0.054	0.101	0.160	0.107
75	0.012	0.011	0.009	0.006	0.010	0.014	0.010	0.053	0.053	0.050	0.027	0.046	0.074	0.054	0.099	0.100	0.102	0.058	0.095	0.148	0.114
100	0.011	0.012	0.008	0.008	0.013	0.014	0.008	0.050	0.050	0.047	0.030	0.048	0.070	0.053	0.101	0.101	0.107	0.061	0.098	0.137	0.114
125	0.008	0.008	0.007	0.008	0.011	0.016	0.006	0.049	0.049	0.047	0.033	0.047	0.065	0.051	0.096	0.097	0.103	0.069	0.098	0.134	0.110
150	0.009	0.009	0.007	0.008	0.008	0.014	0.009	0.046	0.046	0.046	0.038	0.048	0.060	0.049	0.097	0.099	0.100	0.071	0.100	0.130	0.111
200	0.010	0.010	0.009	0.008	0.011	0.012	0.008	0.047	0.046	0.044	0.040	0.047	0.054	0.048	0.094	0.094	0.096	0.082	0.102	0.119	0.108
250	0.010	0.010	0.008	0.010	0.010	0.012	0.007	0.049	0.050	0.045	0.043	0.047	0.056	0.055	0.098	0.099	0.095	0.088	0.106	0.101	0.103
											$T = 1000$										
-5	0.013	0.001	0.007	0.000	0.010	0.005	0.008	0.054	0.011	0.041	0.001	0.053	0.021	0.040	0.114	0.024	0.090	0.003	0.106	0.045	0.089
-2.5	0.011	0.004	0.005	0.000	0.009	0.008	0.005	0.055	0.022	0.030	0.002	0.040	0.041	0.029	0.113	0.043	0.062	0.003	0.083	0.083	0.064
0	0.011	0.009	0.007	0.001	0.007	0.016	0.007	0.049	0.039	0.035	0.002	0.034	0.071	0.037	0.105	0.078	0.078	0.008	0.067	0.151	0.084
2.5	0.014	0.012	0.013	0.001	0.009	0.022	0.012	0.055	0.049	0.051	0.006	0.048	0.089	0.051	0.107	0.096	0.097	0.016	0.088	0.177	0.100
5	0.014	0.012	0.010	0.001	0.013	0.024	0.012	0.053	0.052	0.055	0.010	0.049	0.098	0.054	0.109	0.107	0.105	0.024	0.098	0.180	0.105
10	0.013	0.013	0.010	0.002	0.014	0.024	0.010	0.059	0.060	0.057	0.013	0.057	0.100	0.055	0.112	0.113	0.111	0.028	0.100	0.182	0.111
25	0.012	0.012	0.007	0.004	0.014	0.020	0.007	0.054	0.054	0.051	0.019	0.052	0.103	0.054	0.115	0.118	0.111	0.040	0.108	0.183	0.114
50	0.011	0.013	0.011	0.005	0.012	0.020	0.009	0.058	0.059	0.054	0.023	0.055	0.090	0.056	0.111	0.115	0.111	0.054	0.109	0.166	0.112
75	0.014	0.014	0.013	0.006	0.014	0.018	0.011	0.057	0.056	0.057	0.033	0.054	0.083	0.056	0.110	0.112	0.110	0.062	0.104	0.159	0.113
100	0.014	0.014	0.014	0.007	0.012	0.018	0.015	0.055	0.056	0.055	0.033	0.054	0.076	0.053	0.104	0.106	0.108	0.067	0.105	0.153	0.108
125	0.013	0.014	0.015	0.007	0.012	0.016	0.012	0.055	0.055	0.053	0.034	0.057	0.074	0.053	0.107	0.105	0.105	0.074	0.106	0.144	0.108
150	0.011	0.011	0.012	0.007	0.013	0.017	0.011	0.053	0.053	0.052	0.039	0.056	0.071	0.052	0.108	0.111	0.110	0.078	0.109	0.136	0.111
200	0.011	0.010	0.012	0.007	0.012	0.016	0.010	0.053	0.053	0.054	0.040	0.057	0.071	0.056	0.109	0.111	0.111	0.084	0.111	0.131	0.106
250	0.010	0.011	0.009	0.008	0.010	0.014	0.012	0.052	0.052	0.054	0.040	0.059	0.073	0.055	0.111	0.111	0.113	0.081	0.114	0.129	0.111

See notes under Table D.43.

Table D.45: Empirical rejection frequencies of forward and backward recursive predictability test statistics, for sample size $T = 250$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.50; \quad -0.50 \quad 1]$.

	Forward Recursive							*Backward Recursive*						
c	$\mathcal{T}_2^{F,2SLS}$	$\mathcal{T}_2^{F,FRWB}$	$\mathcal{T}_2^{F,RWB}$	$\mathcal{T}_L^{F,FRWB}$	$\mathcal{T}_L^{F,RWB}$	$\mathcal{T}_U^{F,FRWB}$	$\mathcal{T}_U^{F,RWB}$	$\mathcal{T}_2^{B,2SLS}$	$\mathcal{T}_2^{B,FRWB}$	$\mathcal{T}_2^{B,RWB}$	$\mathcal{T}_L^{B,FRWB}$	$\mathcal{T}_L^{B,RWB}$	$\mathcal{T}_U^{B,FRWB}$	$\mathcal{T}_U^{B,RWB}$
				1%							**1%**			
-5	0.014	0.006	0.011	0.001	0.006	0.012	0.012	0.010	0.004	0.010	0.000	0.010	0.008	0.011
-2.5	0.013	0.010	0.012	0.002	0.007	0.018	0.013	0.010	0.005	0.006	0.000	0.004	0.011	0.007
0	0.013	0.013	0.011	0.004	0.006	0.022	0.014	0.011	0.011	0.010	0.001	0.003	0.019	0.013
2.5	0.012	0.013	0.010	0.006	0.011	0.020	0.013	0.009	0.009	0.009	0.003	0.008	0.013	0.011
5	0.012	0.013	0.011	0.008	0.013	0.018	0.013	0.011	0.010	0.009	0.003	0.011	0.016	0.010
10	0.011	0.013	0.009	0.008	0.012	0.017	0.014	0.010	0.010	0.009	0.008	0.012	0.013	0.009
25	0.013	0.011	0.008	0.011	0.012	0.016	0.009	0.011	0.010	0.010	0.008	0.010	0.010	0.009
50	0.013	0.014	0.009	0.013	0.012	0.015	0.011	0.012	0.011	0.010	0.009	0.011	0.009	0.010
75	0.012	0.014	0.010	0.013	0.010	0.013	0.012	0.015	0.013	0.012	0.011	0.011	0.011	0.011
100	0.011	0.014	0.011	0.013	0.010	0.013	0.012	0.014	0.012	0.011	0.011	0.011	0.014	0.011
125	0.012	0.016	0.010	0.013	0.008	0.014	0.011	0.016	0.014	0.012	0.010	0.011	0.012	0.011
150	0.013	0.014	0.010	0.012	0.010	0.014	0.013	0.013	0.013	0.011	0.010	0.010	0.013	0.010
200	0.012	0.013	0.011	0.013	0.009	0.012	0.011	0.012	0.011	0.010	0.010	0.009	0.011	0.010
250	0.014	0.014	0.010	0.012	0.011	0.013	0.012	0.010	0.009	0.009	0.011	0.010	0.010	0.009
				5%							**5%**			
-5	0.056	0.033	0.052	0.004	0.038	0.063	0.061	0.053	0.023	0.048	0.014	0.051	0.040	0.048
-2.5	0.062	0.043	0.057	0.006	0.038	0.072	0.062	0.049	0.026	0.033	0.004	0.023	0.049	0.040
0	0.053	0.052	0.052	0.019	0.036	0.079	0.066	0.051	0.043	0.047	0.014	0.029	0.076	0.058
2.5	0.051	0.052	0.053	0.033	0.047	0.074	0.061	0.044	0.049	0.049	0.026	0.045	0.073	0.056
5	0.052	0.057	0.059	0.038	0.053	0.070	0.060	0.045	0.048	0.049	0.029	0.048	0.071	0.054
10	0.055	0.056	0.055	0.044	0.056	0.064	0.056	0.051	0.045	0.046	0.039	0.050	0.062	0.049
25	0.057	0.058	0.058	0.046	0.051	0.060	0.057	0.052	0.045	0.045	0.045	0.053	0.053	0.045
50	0.054	0.055	0.054	0.048	0.056	0.059	0.059	0.056	0.048	0.048	0.046	0.054	0.053	0.045
75	0.056	0.056	0.051	0.050	0.057	0.058	0.058	0.056	0.049	0.047	0.049	0.054	0.050	0.048
100	0.056	0.054	0.050	0.048	0.051	0.054	0.053	0.053	0.050	0.051	0.051	0.057	0.047	0.047
125	0.053	0.055	0.051	0.053	0.053	0.052	0.055	0.050	0.049	0.048	0.051	0.058	0.051	0.050
150	0.051	0.054	0.049	0.052	0.054	0.053	0.055	0.054	0.050	0.049	0.054	0.058	0.048	0.049
200	0.052	0.053	0.049	0.056	0.056	0.052	0.054	0.055	0.050	0.052	0.061	0.062	0.049	0.052
250	0.052	0.057	0.052	0.056	0.056	0.053	0.051	0.054	0.053	0.053	0.058	0.058	0.051	0.053
				10%							**10%**			
-5	0.108	0.068	0.104	0.014	0.093	0.131	0.118	0.106	0.052	0.096	0.036	0.103	0.083	0.092
-2.5	0.107	0.080	0.103	0.022	0.081	0.138	0.125	0.101	0.052	0.072	0.009	0.063	0.104	0.084
0	0.099	0.096	0.102	0.045	0.073	0.145	0.117	0.103	0.090	0.094	0.034	0.062	0.150	0.117
2.5	0.098	0.105	0.106	0.066	0.094	0.137	0.121	0.092	0.097	0.103	0.050	0.082	0.148	0.118
5	0.099	0.109	0.109	0.076	0.100	0.135	0.120	0.091	0.099	0.100	0.065	0.092	0.140	0.115
10	0.107	0.109	0.111	0.087	0.101	0.125	0.117	0.100	0.097	0.099	0.075	0.095	0.126	0.109
25	0.113	0.106	0.109	0.091	0.106	0.115	0.113	0.101	0.096	0.097	0.084	0.100	0.110	0.107
50	0.109	0.108	0.113	0.096	0.105	0.113	0.110	0.106	0.097	0.100	0.090	0.104	0.105	0.103
75	0.103	0.107	0.113	0.096	0.107	0.112	0.110	0.106	0.097	0.100	0.099	0.111	0.100	0.099
100	0.107	0.103	0.105	0.099	0.111	0.106	0.110	0.102	0.097	0.102	0.104	0.110	0.100	0.100
125	0.108	0.104	0.107	0.101	0.110	0.105	0.111	0.103	0.097	0.103	0.103	0.113	0.097	0.102
150	0.109	0.108	0.108	0.100	0.115	0.107	0.111	0.107	0.102	0.106	0.107	0.115	0.094	0.104
200	0.103	0.108	0.109	0.103	0.112	0.100	0.112	0.111	0.108	0.109	0.106	0.114	0.097	0.106
250	0.102	0.103	0.102	0.106	0.108	0.107	0.109	0.106	0.106	0.108	0.108	0.114	0.099	0.107

Notes: $\mathcal{T}_2^{k,2SLS}, k = F, B$, denote the two-tailed two-stage least squares based forward $(k = F)$ and backward $(k = B)$ recursive test statistics, and $\mathcal{T}_2^{k,M}$, $\mathcal{T}_L^{k,M}$, and $\mathcal{T}_U^{k,M}$, $k = F, B$ and $M = RWB, FRWB$ denote, respectively, the two-tailed, lower-tail and upper-tail residual wild bootstrap $(M = RWB)$ and fixed regressor wild bootstrap $(M = FRWB)$ based forward $(k = F)$ and backward $(k = B)$ recursive predictability test statistics. For computation of the forward and backward recursive statistics we set $\tau_L = 1/3$ and $\tau_U = 2/3$, respectively.

Table D.46: Empirical rejection frequencies of forward and backward recursive predictability test statistics, for sample size $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.50; \quad -0.50 \quad 1]$.

	Forward Recursive							*Backward Recursive*						
c	$\mathcal{T}_2^{F,2SLS}$	$\mathcal{T}_2^{F,FRWB}$	$\mathcal{T}_2^{F,RWB}$	$\mathcal{T}_L^{F,FRWB}$	$\mathcal{T}_L^{F,RWB}$	$\mathcal{T}_U^{F,FRWB}$	$\mathcal{T}_U^{F,RWB}$	$\mathcal{T}_2^{B,2SLS}$	$\mathcal{T}_2^{B,FRWB}$	$\mathcal{T}_2^{B,RWB}$	$\mathcal{T}_L^{B,FRWB}$	$\mathcal{T}_L^{B,RWB}$	$\mathcal{T}_U^{B,FRWB}$	$\mathcal{T}_U^{B,RWB}$
				1%							**1%**			
-5	0.014	0.011	0.012	0.012	0.007	0.009	0.008	0.011	0.010	0.014	0.001	0.003	0.004	0.001
-2.5	0.014	0.011	0.013	0.015	0.009	0.008	0.008	0.009	0.009	0.019	0.002	0.007	0.010	0.001
0	0.012	0.013	0.015	0.018	0.010	0.012	0.012	0.009	0.014	0.021	0.005	0.011	0.014	0.005
2.5	0.014	0.013	0.014	0.015	0.013	0.013	0.012	0.010	0.013	0.020	0.010	0.012	0.015	0.007
5	0.016	0.012	0.014	0.013	0.013	0.011	0.011	0.012	0.016	0.016	0.010	0.011	0.015	0.009
10	0.016	0.011	0.012	0.013	0.013	0.013	0.013	0.016	0.015	0.017	0.013	0.011	0.014	0.008
25	0.014	0.017	0.012	0.014	0.012	0.012	0.010	0.015	0.012	0.016	0.011	0.012	0.013	0.010
50	0.013	0.014	0.011	0.014	0.011	0.012	0.010	0.012	0.010	0.014	0.008	0.010	0.011	0.012
75	0.013	0.015	0.009	0.010	0.008	0.011	0.011	0.012	0.009	0.013	0.008	0.011	0.011	0.011
100	0.011	0.012	0.010	0.009	0.008	0.013	0.010	0.010	0.009	0.011	0.008	0.011	0.013	0.008
125	0.009	0.012	0.010	0.009	0.008	0.010	0.011	0.010	0.010	0.010	0.010	0.012	0.012	0.011
150	0.008	0.010	0.009	0.009	0.009	0.012	0.012	0.009	0.008	0.009	0.008	0.012	0.013	0.009
200	0.010	0.008	0.007	0.008	0.008	0.010	0.010	0.011	0.011	0.012	0.011	0.011	0.013	0.011
250	0.010	0.009	0.009	0.009	0.009	0.011	0.011	0.012	0.012	0.009	0.012	0.010	0.013	0.014
				5%							**5%**			
-5	0.052	0.047	0.054	0.058	0.054	0.045	0.043	0.055	0.032	0.056	0.008	0.015	0.030	0.010
-2.5	0.052	0.050	0.058	0.060	0.045	0.037	0.042	0.036	0.042	0.071	0.014	0.025	0.050	0.009
0	0.061	0.058	0.054	0.059	0.039	0.051	0.053	0.041	0.049	0.070	0.024	0.046	0.074	0.021
2.5	0.061	0.049	0.058	0.057	0.048	0.054	0.058	0.052	0.055	0.066	0.035	0.052	0.071	0.033
5	0.055	0.049	0.059	0.055	0.059	0.056	0.056	0.056	0.058	0.063	0.042	0.051	0.067	0.042
10	0.059	0.052	0.057	0.055	0.055	0.058	0.052	0.059	0.058	0.060	0.046	0.056	0.063	0.048
25	0.056	0.060	0.054	0.052	0.055	0.057	0.048	0.055	0.058	0.056	0.046	0.057	0.053	0.049
50	0.053	0.057	0.054	0.055	0.051	0.053	0.051	0.058	0.049	0.059	0.048	0.051	0.052	0.050
75	0.052	0.054	0.049	0.057	0.052	0.050	0.051	0.053	0.046	0.058	0.045	0.047	0.050	0.047
100	0.050	0.058	0.047	0.059	0.050	0.049	0.050	0.056	0.047	0.061	0.046	0.048	0.054	0.049
125	0.051	0.052	0.049	0.056	0.048	0.048	0.049	0.054	0.046	0.057	0.045	0.046	0.056	0.047
150	0.048	0.055	0.047	0.056	0.049	0.049	0.055	0.052	0.045	0.059	0.048	0.049	0.052	0.047
200	0.045	0.044	0.047	0.049	0.052	0.055	0.053	0.056	0.050	0.053	0.047	0.051	0.051	0.054
250	0.048	0.048	0.050	0.049	0.050	0.055	0.056	0.055	0.054	0.053	0.048	0.052	0.055	0.052
				10%							**10%**			
-5	0.103	0.093	0.104	0.107	0.112	0.091	0.086	0.097	0.064	0.117	0.022	0.039	0.069	0.032
-2.5	0.110	0.094	0.114	0.124	0.086	0.077	0.085	0.072	0.083	0.143	0.030	0.056	0.097	0.019
0	0.108	0.102	0.102	0.121	0.080	0.095	0.108	0.083	0.095	0.139	0.054	0.091	0.145	0.048
2.5	0.099	0.104	0.101	0.114	0.103	0.107	0.111	0.095	0.101	0.132	0.073	0.102	0.138	0.069
5	0.102	0.102	0.107	0.106	0.105	0.108	0.107	0.105	0.104	0.121	0.081	0.108	0.129	0.077
10	0.110	0.108	0.108	0.108	0.104	0.109	0.106	0.109	0.107	0.118	0.090	0.107	0.121	0.089
25	0.109	0.107	0.105	0.110	0.102	0.107	0.102	0.112	0.103	0.116	0.093	0.101	0.112	0.098
50	0.104	0.112	0.103	0.111	0.105	0.105	0.100	0.109	0.104	0.115	0.096	0.103	0.106	0.098
75	0.108	0.107	0.109	0.114	0.109	0.104	0.100	0.107	0.103	0.112	0.097	0.100	0.104	0.097
100	0.109	0.107	0.109	0.109	0.105	0.105	0.103	0.106	0.107	0.109	0.096	0.100	0.108	0.096
125	0.108	0.103	0.107	0.110	0.104	0.103	0.105	0.103	0.105	0.109	0.100	0.101	0.110	0.095
150	0.104	0.107	0.109	0.113	0.104	0.100	0.108	0.103	0.104	0.108	0.101	0.098	0.107	0.097
200	0.102	0.101	0.108	0.097	0.104	0.111	0.111	0.103	0.101	0.105	0.100	0.109	0.106	0.107
250	0.104	0.102	0.104	0.098	0.100	0.109	0.111	0.101	0.101	0.101	0.102	0.108	0.100	0.101

See notes under Table D.45.

Table D.47: Empirical rejection frequencies of forward and backward recursive predictability test statistics, for sample size $T = 250$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.95; \quad -0.95 \quad 1]$.

	Forward Recursive							*Backward Recursive*						
c	$\mathcal{T}_2^{F,2SLS}$	$\mathcal{T}_2^{F,FRWB}$	$\mathcal{T}_2^{F,RWB}$	$\mathcal{T}_L^{F,FRWB}$	$\mathcal{T}_L^{F,RWB}$	$\mathcal{T}_U^{F,FRWB}$	$\mathcal{T}_U^{F,RWB}$	$\mathcal{T}_2^{B,2SLS}$	$\mathcal{T}_2^{B,FRWB}$	$\mathcal{T}_2^{B,RWB}$	$\mathcal{T}_L^{B,FRWB}$	$\mathcal{T}_L^{B,RWB}$	$\mathcal{T}_U^{B,FRWB}$	$\mathcal{T}_U^{B,RWB}$
				1%							**1%**			
-5	0.016	0.004	0.011	0.000	0.010	0.009	0.011	0.012	0.002	0.005	0.000	0.008	0.003	0.004
-2.5	0.018	0.010	0.012	0.000	0.007	0.017	0.013	0.014	0.007	0.005	0.000	0.003	0.010	0.006
0	0.015	0.013	0.012	0.000	0.007	0.024	0.014	0.018	0.012	0.012	0.000	0.004	0.019	0.012
2.5	0.010	0.014	0.011	0.002	0.013	0.022	0.014	0.011	0.014	0.014	0.001	0.013	0.024	0.016
5	0.010	0.015	0.011	0.003	0.014	0.022	0.014	0.011	0.014	0.013	0.003	0.013	0.023	0.013
10	0.011	0.015	0.012	0.005	0.014	0.021	0.013	0.014	0.016	0.015	0.004	0.012	0.025	0.015
25	0.014	0.010	0.007	0.008	0.012	0.019	0.010	0.020	0.013	0.014	0.007	0.015	0.019	0.014
50	0.016	0.014	0.010	0.010	0.012	0.019	0.011	0.020	0.013	0.012	0.008	0.014	0.015	0.012
75	0.014	0.014	0.011	0.013	0.013	0.018	0.012	0.016	0.012	0.010	0.011	0.013	0.015	0.013
100	0.015	0.015	0.008	0.013	0.010	0.016	0.013	0.016	0.014	0.012	0.010	0.011	0.014	0.013
125	0.013	0.017	0.010	0.015	0.012	0.017	0.013	0.017	0.014	0.014	0.011	0.013	0.016	0.016
150	0.014	0.015	0.009	0.013	0.013	0.017	0.013	0.016	0.015	0.013	0.013	0.011	0.016	0.015
200	0.012	0.015	0.012	0.013	0.012	0.016	0.013	0.014	0.013	0.011	0.011	0.011	0.014	0.013
250	0.013	0.015	0.011	0.015	0.010	0.017	0.014	0.013	0.011	0.011	0.009	0.009	0.014	0.017
				5%							**5%**			
-5	0.065	0.019	0.059	0.000	0.047	0.036	0.059	0.049	0.008	0.031	0.001	0.040	0.017	0.031
-2.5	0.070	0.033	0.047	0.001	0.038	0.062	0.052	0.054	0.022	0.021	0.000	0.028	0.036	0.025
0	0.063	0.048	0.049	0.006	0.029	0.088	0.065	0.073	0.040	0.043	0.002	0.028	0.088	0.055
2.5	0.053	0.054	0.054	0.017	0.053	0.086	0.067	0.054	0.050	0.058	0.012	0.047	0.088	0.064
5	0.050	0.057	0.053	0.025	0.059	0.088	0.066	0.056	0.052	0.055	0.017	0.056	0.088	0.059
10	0.056	0.059	0.057	0.034	0.063	0.082	0.062	0.060	0.053	0.054	0.028	0.056	0.074	0.058
25	0.061	0.056	0.057	0.045	0.058	0.074	0.062	0.069	0.055	0.058	0.039	0.062	0.065	0.059
50	0.064	0.060	0.056	0.044	0.054	0.067	0.059	0.068	0.055	0.058	0.046	0.060	0.063	0.060
75	0.067	0.060	0.056	0.048	0.057	0.068	0.061	0.065	0.053	0.054	0.052	0.062	0.061	0.055
100	0.066	0.062	0.059	0.053	0.059	0.064	0.060	0.065	0.053	0.054	0.052	0.063	0.057	0.059
125	0.066	0.065	0.062	0.053	0.057	0.060	0.058	0.065	0.054	0.056	0.051	0.060	0.054	0.053
150	0.064	0.063	0.058	0.055	0.058	0.062	0.058	0.064	0.057	0.056	0.050	0.057	0.049	0.052
200	0.059	0.060	0.056	0.056	0.059	0.058	0.062	0.058	0.056	0.059	0.052	0.057	0.051	0.054
250	0.058	0.062	0.058	0.058	0.055	0.059	0.060	0.060	0.056	0.057	0.053	0.052	0.049	0.057
				10%							**10%**			
-5	0.120	0.036	0.116	0.002	0.099	0.077	0.117	0.098	0.018	0.080	0.004	0.098	0.044	0.079
-2.5	0.132	0.063	0.096	0.002	0.083	0.117	0.104	0.111	0.036	0.046	0.001	0.062	0.076	0.052
0	0.126	0.092	0.102	0.016	0.065	0.163	0.121	0.141	0.086	0.097	0.008	0.049	0.179	0.121
2.5	0.104	0.104	0.113	0.042	0.096	0.161	0.130	0.108	0.098	0.106	0.029	0.088	0.188	0.130
5	0.103	0.112	0.118	0.058	0.107	0.153	0.128	0.105	0.099	0.109	0.041	0.102	0.173	0.126
10	0.114	0.112	0.118	0.073	0.111	0.145	0.123	0.111	0.100	0.107	0.059	0.116	0.154	0.117
25	0.128	0.115	0.115	0.088	0.119	0.130	0.118	0.129	0.107	0.110	0.082	0.118	0.131	0.118
50	0.117	0.111	0.109	0.093	0.113	0.124	0.116	0.133	0.107	0.115	0.097	0.122	0.123	0.114
75	0.113	0.114	0.115	0.095	0.112	0.117	0.121	0.131	0.111	0.114	0.100	0.119	0.116	0.115
100	0.112	0.113	0.115	0.094	0.111	0.117	0.120	0.124	0.105	0.116	0.098	0.119	0.117	0.117
125	0.111	0.116	0.115	0.095	0.114	0.115	0.122	0.120	0.105	0.111	0.102	0.113	0.111	0.117
150	0.112	0.114	0.114	0.105	0.113	0.114	0.122	0.111	0.103	0.109	0.100	0.115	0.105	0.112
200	0.115	0.111	0.113	0.109	0.112	0.110	0.119	0.113	0.103	0.109	0.101	0.107	0.104	0.114
250	0.115	0.115	0.110	0.107	0.113	0.103	0.117	0.110	0.099	0.109	0.106	0.107	0.097	0.118

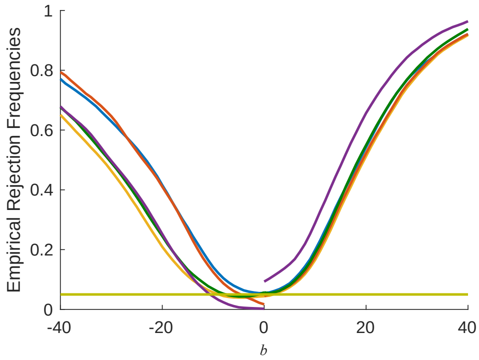
See notes under Table D.45.

Table D.48: Empirical rejection frequencies of forward and backward recursive predictability test statistics, for sample size $T = 1000$. **DGP1 (homoskedastic IID innovations):** $y_t = \beta x_{t-1} + u_t, x_t = \rho x_{t-1} + w_t$ and $w_t = \psi w_{t-1} + v_t$, where $\beta = 0$, $\rho = 1 - c/T, \psi = 0$ and $(u_t, v_t)' \sim NIID(\mathbf{0}, \mathbf{\Sigma})$, with $\mathbf{\Sigma} = [1 \quad -0.95; \quad -0.95 \quad 1]$.

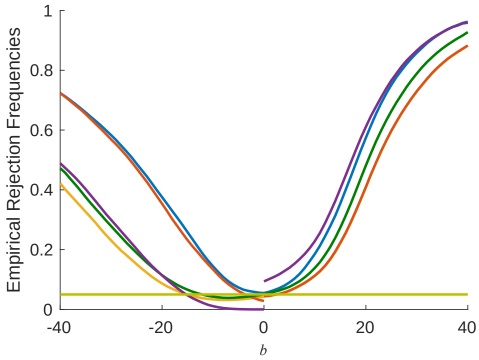
	Forward Recursive							*Backward Recursive*						
c	$\mathcal{T}_2^{F,2SLS}$	$\mathcal{T}_2^{F,FRWB}$	$\mathcal{T}_2^{F,RWB}$	$\mathcal{T}_L^{F,FRWB}$	$\mathcal{T}_L^{F,RWB}$	$\mathcal{T}_U^{F,FRWB}$	$\mathcal{T}_U^{F,RWB}$	$\mathcal{T}_2^{B,2SLS}$	$\mathcal{T}_2^{B,FRWB}$	$\mathcal{T}_2^{B,RWB}$	$\mathcal{T}_L^{B,FRWB}$	$\mathcal{T}_L^{B,RWB}$	$\mathcal{T}_U^{B,FRWB}$	$\mathcal{T}_U^{B,RWB}$
				1%							**1%**			
-5	0.016	0.004	0.010	0.000	0.010	0.008	0.010	0.013	0.002	0.006	0.000	0.009	0.004	0.005
-2.5	0.015	0.010	0.011	0.000	0.007	0.017	0.012	0.013	0.006	0.005	0.000	0.003	0.009	0.006
0	0.016	0.014	0.012	0.001	0.007	0.025	0.014	0.018	0.012	0.011	0.000	0.003	0.021	0.013
2.5	0.011	0.016	0.010	0.002	0.012	0.022	0.014	0.011	0.014	0.013	0.001	0.011	0.024	0.014
5	0.011	0.014	0.013	0.003	0.013	0.022	0.016	0.012	0.014	0.014	0.003	0.012	0.024	0.015
10	0.011	0.013	0.010	0.005	0.013	0.021	0.011	0.014	0.015	0.013	0.003	0.014	0.024	0.016
25	0.016	0.013	0.006	0.009	0.013	0.020	0.010	0.019	0.016	0.014	0.007	0.014	0.020	0.015
50	0.017	0.013	0.009	0.010	0.012	0.018	0.010	0.020	0.014	0.012	0.009	0.014	0.015	0.012
75	0.014	0.014	0.012	0.012	0.013	0.017	0.015	0.014	0.015	0.011	0.010	0.014	0.016	0.014
100	0.014	0.015	0.008	0.013	0.013	0.018	0.011	0.016	0.014	0.011	0.011	0.011	0.017	0.012
125	0.012	0.017	0.010	0.014	0.013	0.015	0.012	0.014	0.015	0.014	0.010	0.012	0.015	0.014
150	0.016	0.014	0.010	0.013	0.012	0.017	0.011	0.015	0.013	0.015	0.011	0.011	0.015	0.014
200	0.014	0.016	0.011	0.014	0.011	0.016	0.014	0.016	0.015	0.011	0.010	0.010	0.015	0.016
250	0.013	0.016	0.012	0.013	0.013	0.015	0.013	0.013	0.014	0.014	0.011	0.012	0.014	0.016
				5%							**5%**			
-5	0.066	0.019	0.060	0.000	0.047	0.035	0.061	0.048	0.009	0.031	0.001	0.041	0.017	0.030
-2.5	0.071	0.033	0.046	0.001	0.038	0.064	0.052	0.051	0.019	0.020	0.000	0.029	0.036	0.024
0	0.064	0.045	0.049	0.006	0.032	0.087	0.065	0.072	0.043	0.044	0.002	0.028	0.087	0.055
2.5	0.053	0.055	0.055	0.016	0.053	0.084	0.066	0.053	0.052	0.056	0.011	0.048	0.091	0.063
5	0.051	0.057	0.056	0.025	0.059	0.088	0.065	0.056	0.053	0.057	0.017	0.055	0.085	0.062
10	0.058	0.056	0.055	0.034	0.061	0.082	0.064	0.061	0.054	0.052	0.027	0.055	0.077	0.057
25	0.064	0.055	0.058	0.045	0.057	0.071	0.064	0.069	0.054	0.056	0.039	0.061	0.068	0.060
50	0.063	0.062	0.057	0.045	0.054	0.068	0.059	0.067	0.056	0.058	0.045	0.060	0.065	0.060
75	0.068	0.061	0.058	0.047	0.057	0.065	0.060	0.066	0.054	0.053	0.052	0.063	0.062	0.057
100	0.066	0.064	0.059	0.052	0.056	0.064	0.059	0.066	0.053	0.054	0.053	0.061	0.060	0.057
125	0.063	0.064	0.061	0.055	0.058	0.061	0.058	0.062	0.055	0.054	0.049	0.057	0.055	0.055
150	0.062	0.064	0.060	0.056	0.054	0.059	0.062	0.065	0.055	0.054	0.049	0.058	0.053	0.053
200	0.057	0.061	0.055	0.059	0.058	0.060	0.062	0.057	0.054	0.057	0.051	0.054	0.051	0.054
250	0.056	0.061	0.056	0.057	0.055	0.059	0.059	0.059	0.055	0.059	0.050	0.053	0.052	0.058
				10%							**10%**			
-5	0.120	0.036	0.115	0.001	0.100	0.077	0.116	0.097	0.018	0.079	0.005	0.096	0.045	0.081
-2.5	0.132	0.064	0.098	0.002	0.084	0.115	0.103	0.107	0.036	0.046	0.000	0.062	0.076	0.053
0	0.128	0.089	0.104	0.014	0.064	0.163	0.123	0.138	0.086	0.099	0.009	0.050	0.178	0.122
2.5	0.103	0.102	0.113	0.041	0.096	0.163	0.129	0.109	0.098	0.107	0.027	0.089	0.185	0.133
5	0.103	0.111	0.118	0.058	0.109	0.153	0.126	0.104	0.097	0.108	0.044	0.101	0.172	0.126
10	0.114	0.111	0.120	0.074	0.110	0.148	0.123	0.112	0.099	0.109	0.060	0.113	0.155	0.119
25	0.127	0.115	0.115	0.090	0.118	0.129	0.118	0.127	0.109	0.111	0.081	0.118	0.130	0.114
50	0.116	0.110	0.108	0.095	0.114	0.123	0.117	0.132	0.107	0.117	0.097	0.124	0.124	0.113
75	0.114	0.113	0.113	0.094	0.112	0.117	0.119	0.132	0.112	0.115	0.101	0.119	0.117	0.114
100	0.117	0.112	0.114	0.099	0.114	0.112	0.118	0.125	0.107	0.115	0.099	0.118	0.115	0.113
125	0.115	0.113	0.115	0.101	0.111	0.115	0.119	0.118	0.101	0.109	0.098	0.115	0.110	0.113
150	0.112	0.112	0.114	0.099	0.116	0.117	0.123	0.112	0.101	0.111	0.100	0.112	0.107	0.115
200	0.113	0.114	0.113	0.105	0.113	0.109	0.117	0.110	0.103	0.107	0.101	0.110	0.101	0.117
250	0.110	0.117	0.114	0.108	0.113	0.103	0.114	0.108	0.102	0.111	0.106	0.111	0.102	0.113

See notes under Table D.45.

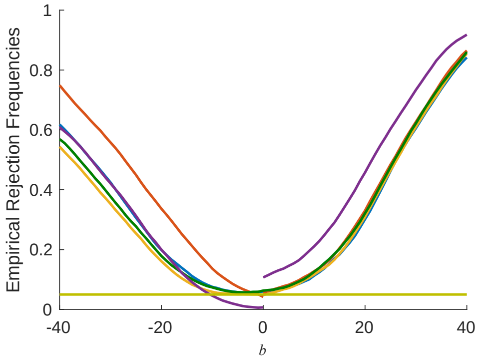
Figure D.1: Power plots for rolling two-sided and the one-sided tests for predictability. Data generated from DGP1 with $\phi = \{-0.95, -0.50\}$ and $T = 250$.



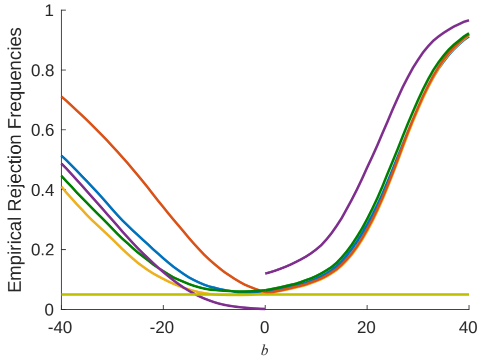
(a) $c = 0$, $\phi = -0.50$.



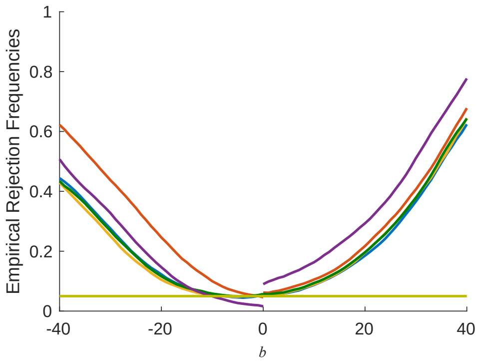
(b) $c = 0$, $\phi = -0.95$.



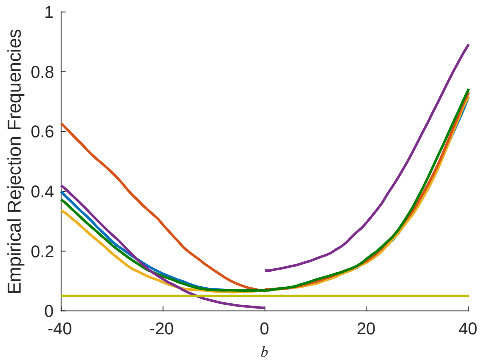
(c) $c = 10$, $\phi = -0.50$.



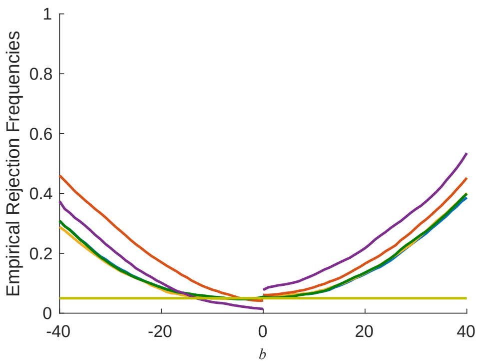
(d) $c = 10$, $\phi = -0.95$.



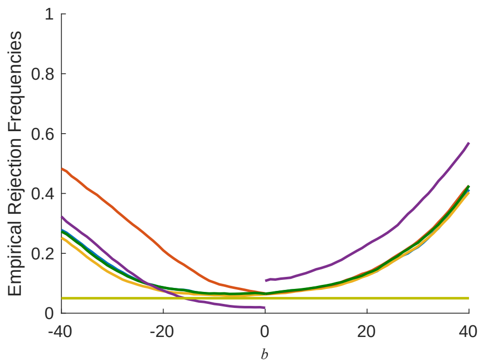
(e) $c = 25$, $\phi = -0.50$.



(f) $c = 25$, $\phi = -0.95$.



(g) $c = 50$, $\phi = -0.50$.

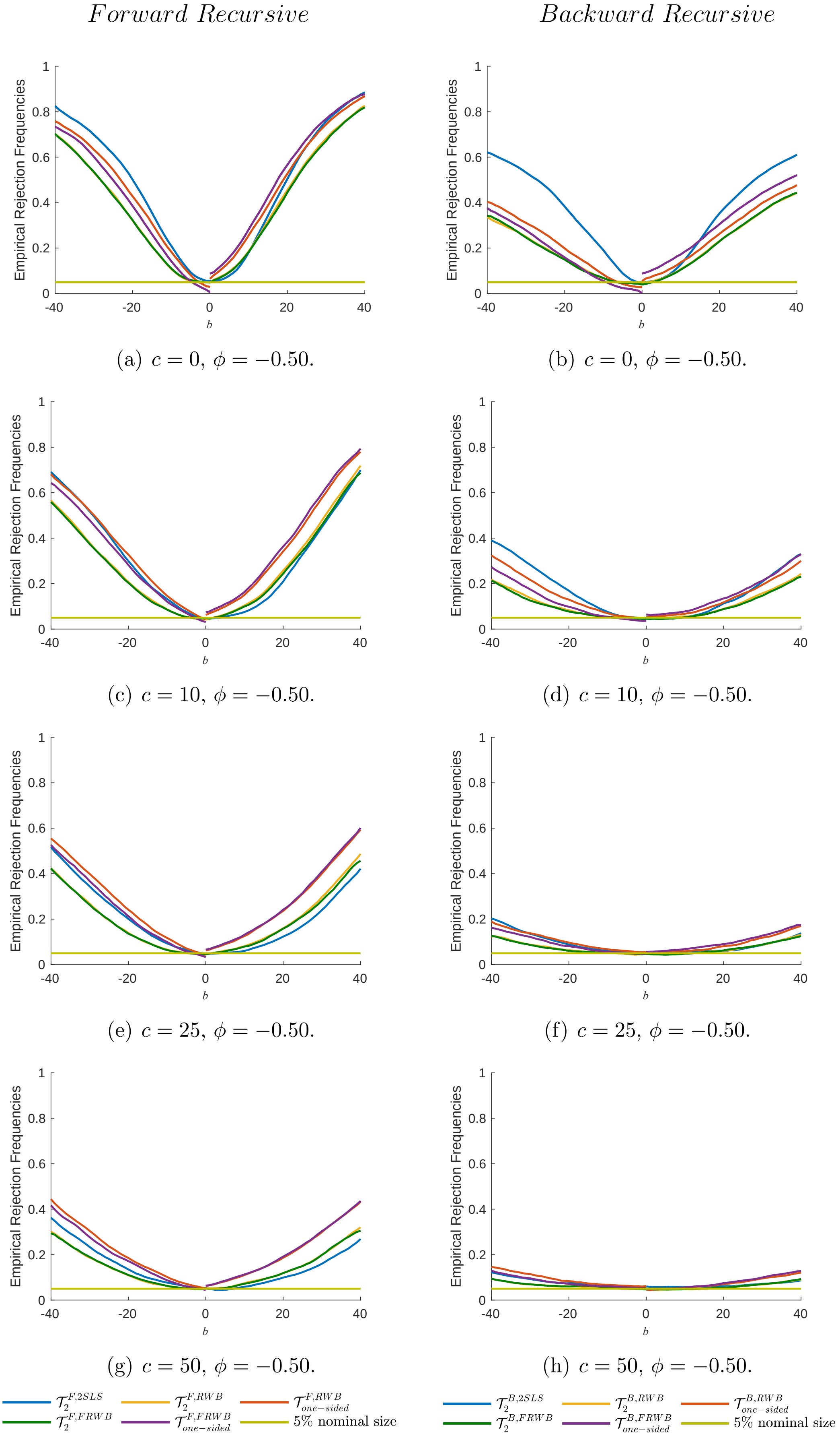


(h) $c = 50$, $\phi = -0.95$.

$\mathcal{T}_2^{R,2SLS}$ $\mathcal{T}_2^{R,RWB}$ $\mathcal{T}_{one-sided}^{R,RWB}$ $\mathcal{T}_2^{R,FRWB}$ $\mathcal{T}_{one-sided}^{R,FRWB}$ 5% nominal size

Note: The red and purple lines correspond to the rejection frequencies of the rolling one-sided RWB- and FRWB-based tests, respectively. For $b < 0$ the results for the one-sided tests correspond to the left-sided tests and for $b > 0$ to the right-sided tests.

Figure D.2: Power plots for forward and backward recursive two-sided and one-sided tests for predictability. Data generated from DGP1 with $\phi = -0.50$ and $T = 250$.

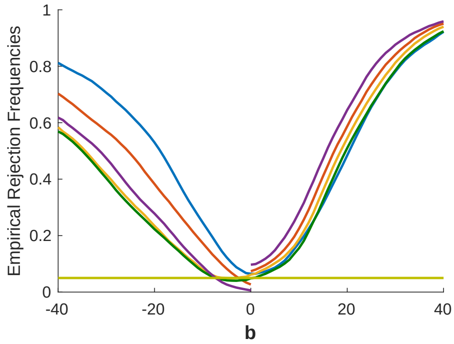


(a) $c = 0$, $\phi = -0.50$.
(b) $c = 0$, $\phi = -0.50$.
(c) $c = 10$, $\phi = -0.50$.
(d) $c = 10$, $\phi = -0.50$.
(e) $c = 25$, $\phi = -0.50$.
(f) $c = 25$, $\phi = -0.50$.
(g) $c = 50$, $\phi = -0.50$.
(h) $c = 50$, $\phi = -0.50$.

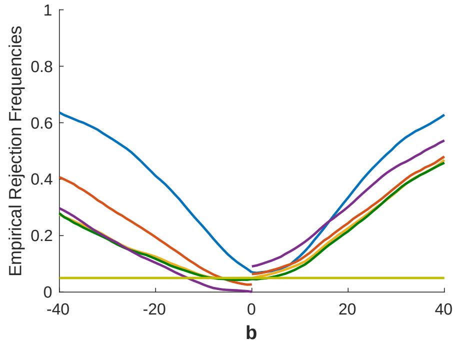
Note: The red and purple lines correspond to the rejection frequencies of the forward/backward recursive one-sided RWB- and FRWB-based tests, respectively. For $b < 0$ the results for the one-sided tests correspond to the left-sided tests and for $b > 0$ to the right-sided tests.

Figure D.3: Power plots for forward and backward recursive two-sided and one-sided tests for predictability. Data generated from DGP1 with $\phi = -0.95$ and $T = 250$.

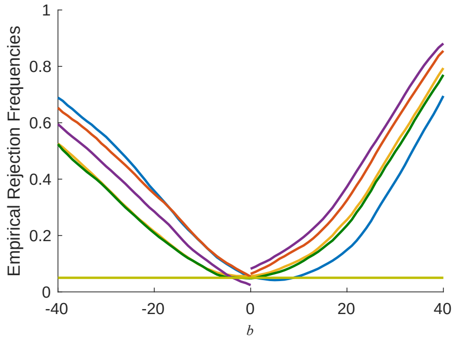
Forward Recursive | *Backward Recursive*



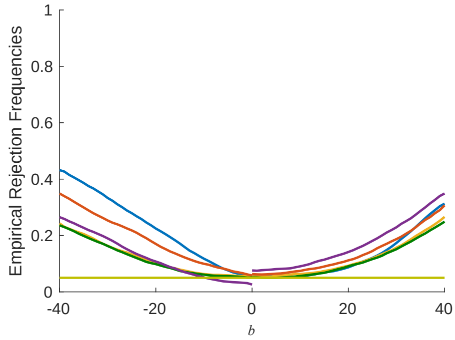
(a) $c = 0$, $\phi = -0.50$.



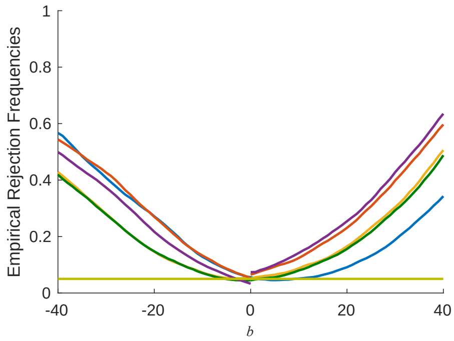
(b) $c = 0$, $\phi = -0.50$.



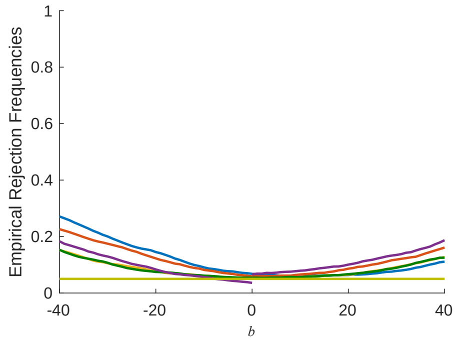
(c) $c = 10$, $\phi = -0.50$.



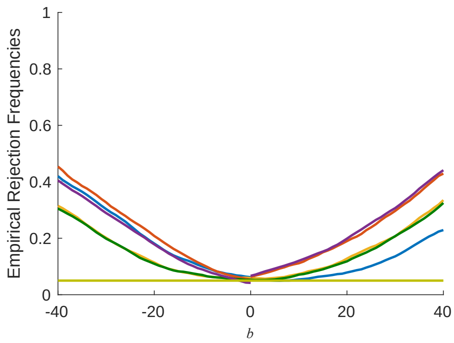
(d) $c = 10$, $\phi = -0.50$.



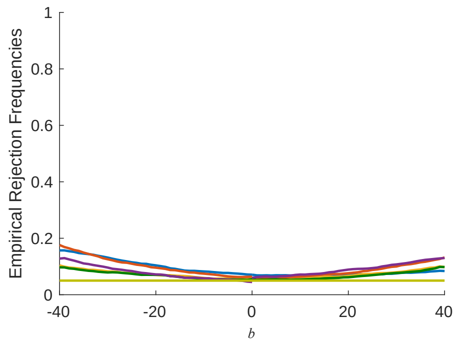
(e) $c = 25$, $\phi = -0.50$.



(f) $c = 25$, $\phi = -0.50$.



(g) $c = 50$, $\phi = -0.50$.



(h) $c = 50$, $\phi = -0.50$.

$\mathcal{T}_2^{F,2SLS}$ $\mathcal{T}_2^{F,RWB}$ $\mathcal{T}_{one-sided}^{F,RWB}$ $\mathcal{T}_2^{F,FRWB}$ $\mathcal{T}_{one-sided}^{F,FRWB}$ 5% nominal size

$\mathcal{T}_2^{B,2SLS}$ $\mathcal{T}_2^{B,RWB}$ $\mathcal{T}_{one-sided}^{B,RWB}$ $\mathcal{T}_2^{B,FRWB}$ $\mathcal{T}_{one-sided}^{B,FRWB}$ 5% nominal size

Note: The red and purple lines correspond to the rejection frequencies of the forward/backward recursive one-sided RWB- and FRWB-based tests, respectively. For $b < 0$ the results for the one-sided tests correspond to the left-sided tests and for $b > 0$ to the right-sided tests.