Network formation with heterogeneous players*

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Abstract

This paper studies a connections model of network formation in which players are heterogeneous with respect to values as well as the costs of forming links. We start by showing that value heterogeneity is important in determining the level of connectedness of a network, while cost heterogeneity is important in shaping both the level of connectedness as well as the architecture of individual components in a network.

We then explore the role of cost heterogeneity in a society which is divided into distinct groups and intra-group links are cheaper as compared to inter-group links. Here we find that inter-connected stars with locally central players are socially efficient as well as dynamically stable.
We interpret our results as saying that centrality, center-sponsorship and small diameter are robust features of networks.

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1 Introduction

The role of social and economic networks in shaping individual behavior and aggregate phenomena has received increasing attention in recent years.[1] This work has motivated research into the processes through which networks emerge and is the primary motivation for developing a theory of network formation. In this paper, our interest is in the role of ex-ante asymmetries among the players in shaping the architecture of networks.[2]

We will focus on the connections model of network formation.[3] In this model there is a set of players who each gains from accessing other players. Player 1 can access player 2 directly by forming a link; this link also allows player 1 access to other players that player 2 is accessing on his own. We shall consider the version of this model in which links can be formed by individuals independently (they are one-sided), but flow of benefits is two-sided and frictionless.[4] Bala and Goyal (2000a) show that if a player's payoffs are increasing in the number of other players accessed and decreasing in the number of links formed, then a strict Nash network is either a center-sponsored star (a network in which one player (the center) forms links with all the other players) or the empty network (with no links). They also show that the dynamics induced by myopic best-response individual learning converge to the strict Nash networks. In this paper we will examine the impact of ex-ante player heterogeneity on these findings.

We start with a general model of heterogeneous players: the costs to player i of a link with player j as well as the benefits of such a link are allowed to depend on both i and j. In addition, we assume that the length of the path does not matter in defining the benefits (there is no decay). We first consider a particular form of cost heterogeneity: for any player i the costs of forming links with every other player

[1] There is a large body of work on this subject. See e.g., Burt (1992) on careers of professional managers, Montgomery (1991) on wage inequality in labour markets, Granovetter (1974) on flow of job information, and Coleman on diffusion of medical drugs (1966).

[2] Ex-ante asymmetries arise quite naturally in different contexts. For instance, in the context of information networks it is often the case that some individuals are more interested in particular issues (such as computer software) and therefore better informed which makes them more valuable at contacts. Similarly, individuals differ in communication and social skills and it seems natural that forming links is cheaper for some individuals as compared to others. Finally, individuals can often be classified into distinct groups (based on geographical or cultural reasons) and forming links within a group is easier and cheaper as compared to forming links across groups.

[3] This model has been extensively studied in the literature; see e.g., Bala and Goyal (2000a, 2000b), Dutta and Jackson (2000), Falk and Kosfeld (2003), Goyal (1993), Haller and Sarangi (2001), Jackson and Wolinsky (1996), Johnson and Gilles (2000), McBride (2002) and Watts (2000, 2001). We discuss this work after presenting the model and our results.

[4] Examples of this include telephone calls in which people exchange information and investments in personal relationship which creates a social tie yielding value to both partners.

are c_i but we allow this cost to vary across players. In this setting we find that if benefits are homogeneous then a strict equilibrium is either an empty network or a center-sponsored star. By contrast, if values are heterogeneous then partially connected networks can also arise, though each (non-singleton) component constitutes a center-sponsored star (Proposition 3.1). This results suggests that heterogeneity in benefits is important in determining the level of connectedness of a network. We then move to a model with general cost heterogeneity where costs of forming links vary across individuals and in addition for the same individual the costs of forming links are sensitive to the identity of the potential partner. In this setting we obtain the following equivalence result: a (strict) equilibrium network is minimal and conversely every minimal network is a (strict) equilibrium for suitable costs and benefits. We also find that this equivalence obtains even if benefits are restricted to be homogeneous (Proposition 3.2). Figure 1 illustrates the set of minimal network architectures in a society composed of four players. Taken together these results suggest that cost heterogeneity is important in shaping the level of connectedness of networks as well as the architecture of individual components. These results also clarify the role of different forms of cost heterogeneity and in particular imply that the 'everything is possible' nature of our equivalence result is closely related to cost heterogeneity which arises when the costs of linking vary for the same player.

This leads us to develop an *insider-outsider model* where the society is composed of distinct groups. The cost of forming a link between two players is (weakly) increasing in the distance between the groups to which the two players belong. Thus, the distance among groups may be interpreted as the degree of heterogeneity across players.

In this setting, we obtain *three* main results. Our *first* result provides a complete characterization of strict Nash equilibrium networks. This result shows that an equilibrium network is either a single center-sponsored star or a collection of center-sponsored stars or a generalized center-sponsored star (Proposition 4.1). The generalized center-sponsored star architecture has a central player i: if we start at player i and move along a path with players $i_1, i_2, i_3, ..., i_n$, then the link between players i and i_1 is formed by player i, the link between player i_1 and i_2 is formed by player i_1, and so on. Furthermore, the group to which player i belongs constitutes a center-sponsored star and all other groups are completely fragmented (between every pair of members of such a group there is a member of another group). Figure 3 depicts all the strict Nash architectures in a society composed of two groups. Our *second* result is about the dynamics of individual learning. In the two-groups case we show that a dynamic process based on individual myopic best responses converges to a minimal curb set and also provide a complete characterization of the minimal curb sets. We find that a minimal curb set is either a strict Nash equilibrium identified above or is a set of networks in which each group constitutes a center-sponsored star and there exists a single link between the center-sponsored stars (Propositions 5.2). The process cycles within this set as the player forming the single link between the groups is indifferent between linking with any of the players in the other group. This characterization of the minimal curb sets highlights how local centrality can arise in the long run, a pattern which was not picked up by the static model. This local centrality feature is also interesting from the viewpoint of social efficiency as the following remarks indicate.

Our *third* result is about efficient networks. In the insider-outsider model, it is clear that an efficient network must minimize the number of outsider links since they are costlier as compared to insider links. Thus in a society with 2 groups an efficient connected network has each group entirely internally linked and 1 outside link (Proposition 4.2). By contrast, a (connected) strict equilibrium network is a generalized center-sponsored star, with $n - n_l$ outsider links (where n_l is the number of players in the core group). If there are 2 groups and 50 players in each group then an efficient network has 98 insider links and 1 outsider link, while a strict equilibrium network has 49 insider links and 50 outsider links! This may suggest that individual link formation can lead to a significant waste of resources. Our results on the dynamics however suggest that individual learning can mitigate this problem: the dynamics may converge to a limiting set in which every network consists of two center-sponsored stars linked with a single cross group link.

Our paper is a contribution to the theory of network formation. This is a very active area of research currently; see e.g., Aumann and Myerson (1989), Dutta, van den Nouweland and Tijs (1995), Kranton and Minehart (2000, 2001), Slikker and van den Nouweland (2001a, 2001b), and the other references in footnote 1. These papers and indeed most of the existing literature focuses on homogeneous player models. We first relate our results to the findings of Bala and Goyal (2000) reported earlier. Our results on general heterogeneity elaborate on the respective roles of values and costs of forming links in shaping network architecture. In particular, we show that value heterogeneity across players is crucial in determining the connectedness of a network, while differences in costs of linking across players is crucial in shaping the architecture of individual components as well as the connectedness of a network. Our results – static and dynamic – on the insider-outsider model show that the properties of centrality, center-sponsorship and small diameter, are robust in settings where linking costs are based on membership of groups.

We next discuss two recent papers which also allow for heterogeneous players, Johnson and Gilles (2000) and McBride (2002). Johnson and Gilles (2001) consider two-sided link formation while we study one-sided link formation. There are also other differences in terms of the model specification, such as the role of decay and the insider-outsider formulation that we use. These differences lead to very different results. Johnson and Gilles find that if cost of linking is low as compared to the potential benefit, locally complete networks (where a player is always connected to at least one of his direct neighbors and belongs to a complete subnetwork), are more likely to arise in equilibrium. This is in contrast to our findings which show that a (non-empty) strict equilibrium network is a center-sponsored star (or its variants) for all costs of forming links. McBride (2002) focuses on value heterogeneity and partial information about network structure. In the present paper, we start by showing that value heterogeneity is important for connectedness but is not crucial for the architecture of the components in a network. This leads us to focus on the role of cost heterogeneity in shaping network architecture.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 presents results on equilibrium networks under general cost and value heterogeneity. Section 4 analyzes an insider-outsider model. Section 5 studies the dynamics while section 6 concludes.

2 The Model

Let $N = \{1, ..., n\}$ be a set of players and let i and j be typical members of this set. We shall assume throughout that the number of players $n \geq 3$. Each player is assumed to possess some information of value to himself and to other players. He can augment his information by communicating with other people; this communication takes resources, time and effort and is made possible via *pair-wise* links.

A strategy of player $i \in N$ is a (row) vector $g_i = (g_{i,1}, ..., g_{i,i-1}, g_{i,i+1}, ..., g_{i,n})$ where $g_{i,j} \in \{0, 1\}$ for each $j \in N \diagdown \{i\}$. We say that player i has a link with j if $g_{i,j} = 1$. A link between player i and j can allow for either one-way (asymmetric) or two-way (symmetric) flow of information. We assume throughout the paper that a link $g_{i,j} = 1$ allows both players to access each other's information. The set of strategies of player i is denoted by $\mathcal{G}_i$. Throughout the paper we restrict our attention to pure strategies. Since player i has the option of forming or not forming a link with each player of the remaining $n-1$ players, the number of strategies of player i is clearly $|\mathcal{G}_i| = 2^{n-1}$. The set $\mathcal{G} = \mathcal{G}_1 \times ... \times \mathcal{G}_n$ is the space of pure strategies of all the players.

A strategy profile $g = (g_1, ..., g_n)$ can be represented as a directed network. Let $g \in \mathcal{G}$. We use $g - g_{i,j}$ to refer to the network obtained when a link $g_{i,j} = 1$ is deleted from g. To describe information flows, it is useful to define the closure of g : this is a non-directed network denoted $\overline{g}$ =cl(g) , and define by $\overline{g}_{i,j} = \max\{g_{i,j}, g_{j,i}\}$ for each i and j in N.[5] Pictorially, the closure of a network simply means replacing every directed edge of g by a non-directed one. We say there is a path in g between i and j if either $\overline{g}_{i,j} = 1$ or there exist players $j_1, ..., j_m$ distinct from each other and i and j such that $\left\{\overline{g}_{i,j_1} = ... = \overline{g}_{j_m,j} = 1\right\}$. We write $i \overset{\overline{g}}{\longleftrightarrow} j$ to indicate a path between i and j in g. Furthermore, a path between i and j is said to be $i-$*oriented* if either $g_{i,j} = 1$ or there is a sequence of distinct players $i_1, i_2, ..., i_n$ with the property

[5]Note that $\overline{g}_{i,j} = \overline{g}_{j,i}$ so that the order of players is irrelevant.

that: $\{g_{i,i_1} = g_{i_1,i_2} = 1, ..., g_{i_n,j} = 1\}$. Define $N^d(i;g) = \{k \in N \,|g_{i,k} = 1\}$ as the set of players with whom i maintains a link and let $\mu_i^d(g) = \left|N^d(i;g)\right|$ be the cardinality of the set. The set $N(i;\overline{g}) = \left\{k \in N \,\middle|\, i \overset{\overline{g}}{\longleftrightarrow} k\right\} \cup \{i\}$ consists of players that i observes in g, while $\mu_i(g) = |N(i;\overline{g})|$ is its cardinality.

Given a network g, we define a component as a set $C(g) \subset N$ such that $\forall i,j \in C(g)$ there exists a path between them and there does not exist a path between $\forall i \in C(g)$ and an player $k \in N \setminus C(g)$. Given a network g, let $\#C(g)$ be the number of components in g. A network g is said to be minimal if $\#C(g) < \#C(g - g_{i,j})$, for any $g_{i,j} = 1$. Moreover a network g is said to be connected if it is composed by only one component, i.e. $\#C(g) = 1$. If this component is minimal, then g is said to be minimally connected. It follows that each link in a minimally connected network is critical in the way that it is enough to delete it, ceteris paribus, to induce some degree of social isolation in the society. Finally, a network g is partially connected if it is neither empty nor connected.

We note that center-sponsored star, g^{css}, is a network architecture in which one player forms links with each of the other $(n-1)$ players and there are no other links.

To complete the definition of a normal-form game of network formation, we specify the payoffs. Let $V_{i,j}$ denote the benefits that player i derives from accessing player j. Similarly, let $c_{i,j}$ denote the cost for player i of forming a link with player j. The payoff to player i in a network g can be written as follows:

$$\Pi_i(g) = \sum_{j \in N(i;\overline{g})} V_{i,j} - \sum_{j \in N^d(i;g)} c_{i,j} \tag{1}$$

We shall assume that $c_{i,j} > 0$ and $V_{i,j} > 0$ for all $i,j \in N$.

Given a network $g \in \mathcal{G}$, let g_{-i} denote the network obtained when all of player i's links are removed. Note that the network g_{-i} can be regarded as the strategy profile where i chooses not to form a link with anyone. The network g can be written as $g = g_i \otimes g_{-i}$ where the '$\otimes$' indicates that g is formed as the union of the links in g_i and g_{-i}. The strategy g_i is said to be a *best response* of player i to g_{-i} if:

$$\Pi_i(g_i \otimes g_{-i}) \geqslant \Pi_i(g_i' \otimes g_{-i}) \text{ for all } g_i' \in \mathcal{G}_i. \tag{2}$$

The set of all of player $i's$ best responses to g_{-i} is denoted by $\mathcal{BR}_i(g_{-i})$. Furthermore, a network $g = (g_1, ..., g_n)$ is said to be a *Nash network* if $g_i \in \mathcal{BR}_i(g_{-i})$ for each i, i.e. players are playing a Nash equilibrium. If a player has multiple best responses to the equilibrium strategies of the other players then this could make the network less stable as the player can switch to a payoff equivalent strategy. This switching possibility in non-strict Nash networks has been exploited and has been shown to be important in refining the set of equilibrium networks in earlier work (see e.g., Bala and Goyal (2000)). So we will focus on strict Nash equilibria in the present paper. A *strict* Nash equilibrium is a Nash equilibrium where each player gets a strictly higher payoff from his current strategy than he would with any other alternative strategy.

We define the social welfare of a network g as the sum of payoffs of all players. Formally, given a network g, its welfare, $W : \mathcal{G} \to R$, can be stated as follows:

$$W(g) = \sum_{i=1}^{n} \Pi_i(g) \text{ for } g \in \mathcal{G}. \tag{3}$$

A network is said to be efficient if $W(g) \geqslant W(g')$ for any $g' \in \mathcal{G}$. Hence, an efficient architecture can be seen as the one that minimize the cost of providing a certain amount of information to the players.

3 General Heterogeneity

In this section we shall study the scope of individual incentives in restricting network architectures in a setting of general costs and value heterogeneity. Our results will clarify that value heterogeneity is important in determining the connectedness of a network while heterogeneity in costs matters both for the level of connectedness as well as for the architecture of individual components of a network.

We start with a consideration of a setting in which players may differ in their costs of forming links but the costs of forming links for an individual are independent of the potential partner. Our first result establishes an equivalence between the set of center-sponsored star networks and the set of (strict) equilibrium networks if values are homogeneous. On the other hand, if values are allowed to vary freely then we find an equivalence between the set of minimal networks in which non-singleton components are center-sponsored stars and the set of (strict) equilibrium networks.

Proposition 3.1 *Let payoffs satisfy (1) and suppose $c_{i,j} = c_i$, $\forall j \in N$. If $V_{i,j} = V, \forall i, j \in N$, then a strict equilibrium is either empty or a center-sponsored star; conversely any such network is a strict equilibrium for some $\{c_i, V\}$. If values vary freely then a strict equilibrium is either empty or a minimal network in which every (non-singleton) component is a center-sponsored star; conversely any such network is a strict equilibrium for some $\{c_i, V_{i,j}\}$.*

Proof: We note first that any equilibrium network is minimal; this follows from the no decay assumption. We next show that if $c_{i,j} = c_i$, $\forall j \in N$ then any non-singleton component $C(g)$ in a strict equilibrium network g must be a center-sponsored star.

If there are two players in this component then the claim is obviously true. So let us consider a component with 3 or more players. Without loss of generality there is a pair of players i and j such that $g_{i,j} = 1$. We note that player i cannot access any other player k via this link with player j. If there were such a player then since $c_{i,j} = c_i$, $\forall j \in N$, player i would be indifferent between linking with j and k and g would not be a strict equilibrium. We next note that no such player k forms a link with i. If k formed a link with i then k would in turn be indifferent between linking with i and j. Combining these observations it follows that player i must be forming links with all players in the component and so it constitutes a center-sponsored star. We next take up the cases of homogenous and heterogeneous values, respectively.

First consider the case of homogeneous values. Suppose g is a non-empty (strict) equilibrium network. We will show that it is connected. Let $C_1(g)$ be a non-singleton component in g and let $j \notin C_1(g)$. From above it follows that there exists a player $i \in C_1(g)$ who is central and sponsors all links in $C_1(g)$. Since g is a strict equilibrium this implies that $c_i < V$ The marginal payoff to forming a link with j is at least V, and so player i can increase his payoff by forming an additional link, contradicting the hypothesis that g is an equilibrium. Thus g is connected and we have proved that if values are homogeneous then an equilibrium network is either empty or a center-sponsored star. We now take up the converse case. The empty network is a (strict) equilibrium if $c_i > V$ for all i, while a center sponsored star with i at the center is a (strict) equilibrium if $c_i < V$.

Second we consider the case of heterogeneous values. From the above arguments it follows that any component in a non-empty (strict) equilibrium network must be a center sponsored star. We now prove the converse. Fix some minimal network g in which every (non-singleton) component is a center sponsored star. Let there be m components in this network, $C_1(g), ..., C_m(g)$. Let $i \in C_1(g)$ be the central player in $C_1(g)$. For any link $g_{i,j} = 1$, set $c_i < V_{i,j}$, while for every component $C_k(g)$, $k = 2, ..., m$, set $\sum_{j \in C_k(g)} V_{x,j} < c_x$,for all $x \in C_1(g)$. It follows that the links of i are optimal while no additional links are profitable for any player $x \in C_1(g)$. Since $C_1(g)$ was arbitrary, the proof follows. $\square$

The above result illustrates the role of value heterogeneity in defining the level of connectedness of networks: homogeneous values ensure connectedness of networks, while heterogeneity can generate partially connected networks. We next note that $c_i = c$ is a special case of the above result. This tells us that the results on equilibrium networks with homogeneous costs and values obtained in Bala and Goyal (2000) can in fact be generalized to allow for heterogeneity in costs of forming links across individuals. Is this also true if costs of forming links are different for the same individual, depending on the potential partner? The following proposition shows that matters are quite complicated in this case.

Proposition 3.2 *Let payoffs satisfy (1) and suppose costs vary freely. Then a strict equilibrium is minimal; conversely, any minimal network is a strict equilibrium for some* $\{c_{i,j}, V_{i,j}\}$.

Proof: Minimality follows directly from the no decay assumption. We now prove the converse. Fix some minimal network g. We set the costs and values as follows: $V_{i,j} = V$, $\forall i, j \in N$ and for any link $g_{i,j} = 1$, let the corresponding cost $c_{i,j} = \epsilon < V$, while for any link $g_{i,j} = 0$, set the corresponding cost $c_{i,j} > (n-1)\,V$. The proof follows. $\square$

Figure 1 illustrates the set of minimal network architectures for a society composed of four players. In this figure a filled circle on a link next to a player indicates that this player has formed the link and pays for the link. This result shows that if costs of forming links for an individual vary across partners and costs of forming links are different for different players then strategic interaction imposes no restrictions on network architecture. We also note that the proof of the second part of the result actually uses homogeneous values to support arbitrary minimal networks. This shows that, in case of general cost heterogeneity, the level of value heterogeneity plays no important role in determining network architecture. We summarize our analysis of the general heterogeneity model in the following table.

Costs \ Values	**Homogeneous**	**Heterogeneous**
Homogeneous	g^e, g^{css}	g^e, minimal networks in which every non-singleton component is a center-sponsored star
$\mathbf{c_{ij}=c_i}$	g^e, g^{css}	g^e, minimal networks in which every non-singleton component is a center-sponsored star
Heterogeneous	Minimal networks	Minimal networks

Table 1

This table tells us that value heterogeneity is important in determining the level of connectedness of networks. We also observe that cost heterogeneity is important in shaping both the level of connectedness as well as the architecture of individual components. Finally, this table also highlights the significance of different forms of cost heterogeneity in shaping networks. In particular it implies that the 'everything is possible' nature of our equivalence result is closely related to cost heterogeneity which arises when the costs of linking vary for the same player.

4 An insider-outsider model

In this section we consider a society in which individuals are divided into pre-specified groups, and the costs of forming links within the groups is lower as compared to costs of forming links across groups. This leads to a model in which costs of linking are partner specific. Our analysis yields a characterization of strict Nash and efficient networks.

We consider a society composed by m groups. Let $n_l = |N_l|$ be the size of group l, with $l = 1, 2, 3, ..., m$. The set of players is then $N \equiv \cup_{l=1}^{m} N_l$. We assume perfect symmetry in value across individuals and we normalize it to one, i.e. $V_{i,j} = 1$ for all $i, j \in N$.[6] To allow for cost heterogeneity we consider a spatial cost structure: groups can be ordered in a line according to some well defined characteristics. The distance between two groups can be interpreted as a measure of the heterogeneity that distinguishes them. Given two players $i \in N_l$ and $j \in N_k$, the cost of forming a link $g_{i,j}$, is:

$$c_{i,j} = c_{j,i} = f(|l - k|) \tag{4}$$

If i and j belong to the same group we let:

$$c_{i,j} = c_{j,i} = f(0) = c_L \tag{5}$$

We shall assume that $f(\cdot)$ is (weakly) increasing in its argument and $c_L > 0$. Let $N^{d,k}(i; g) = \{j \in N_k | g_{i,j} = 1\}$, for $k = 1, ..., m$,; then define $N^d(i; g) \equiv \cup_{k=1}^{m} N^{d,k}(i; g)$. Furthermore, let $\mu_i^{d,k}(g)$ be the cardinality of $N^{d,k}(i; g)$. In other words, $\mu_i^{d,k}(g)$ represents the number of links initiated by i with members of group k. Hence, given a network g and a player $i \in N_l$, the payoff function described by (1) can be rewritten as follows:

$$\Pi_i(g) = \mu_i(g) - \sum_{k=1}^{m} \mu_i^{d,k} f(|l - k|) \tag{6}$$

We note two interesting special cases of our specification.

[6] This normalization simplifies the statement of our results; on occasion this normalization can create some confusion between the notions of component value and component size. For instance, our statements relating costs of forming links with specific networks are clearly restrictions on component value and not on component size alone.

1. Homogeneous Players: This case arises when $f(0) = f(1) = ... = f(m-1) = c$. This implies that player i's payoff is the number of players he observes less the total cost of link formation. Clearly, the distinction between inside and outside links becomes irrelevant and we can consider that the whole society is composed of one group. In this case the payoff may be written as follows:

$$\Pi_i(g) = \mu_i(g) - \mu_i^d(g)\, c. \tag{7}$$

This is the linear payoff model presented in Bala and Goyal (2000).
2. Two-cost levels: The case of two-cost levels arises when we assume that $f(d) = c_H$, $\forall d \geqslant 1$, and $f(0) = c_L < c_H$. We can then write the cost structure as follows:

$$c_{i,j} = \begin{cases} c_L, & \text{if } i,j \in N_l \\ c_H, & \text{if } i \in N_l \text{ and } j \in N_k,\ l \neq k \end{cases} \tag{8}$$

In words, the cost of creating an outside link across groups, c_H, is higher than the cost of creating an inside link within a group, c_L. However, links formed with different external groups are equally costly.

We now develop some additional notation. Given a network g, we say that two players $i, i' \in N_l$ are internally linked if either $g_{i,i'} = 1$ or there exists a group of distinct players $\{i_1, i_2, ..i_k\}$ where $i_x \in N_l$ for any $x \in \{1, ..., k\}$ such that $\bar{g}_{i,i_1} = \bar{g}_{i_1,i_2} = ... = \bar{g}_{i_k,i'} = 1$. A group N_l is entirely internally linked if every pair of players $i, i' \in N_l$ is internally linked. Similarly, a pair of players i, i' is externally linked if $g_{i,i'} = 0$ and there exists a group of distinct players $\{j_1, j_2, ..j_k\}$ where $j_x \notin N_l$ for any $x \in \{1, ..., k\}$ such that $\bar{g}_{i,j_1} = \bar{g}_{j_1,j_2} = ... = \bar{g}_{j_k,i'} = 1$. A group N_l is entirely externally linked if every pair of players $i, i' \in N_l$ is externally linked. Moreover, a group which is linked but neither entirely internally linked nor entirely externally linked is referred to as a hybrid group. Finally, let the diameter of a non-singleton component $C(g)$ be defined as the length of the largest geodesic distance between any pair of players belonging to it, *i.e.* $D(C(g)) = \max_{i,j \in C(g)} d(i,j; C(g))$.[7] We now define some network architectures that arise in this model.

Definition 4.1 *A generalized center-sponsored star is a minimally connected network which satisfies the following conditions:*

(i) $\exists l$ *and* $\exists i \in N_l$ *such that* $g_{i,j} = 1, \forall j \in N_l \diagdown \{i\}$.

(ii) For any $j \in N$, $i \overset{g}{\longleftrightarrow} j$, *is an* $i-$*oriented path.*

(iii) Consider an $i-$*oriented path,* $i, i_1, i_2, ..., i_n$ *with* $\left\{g_{i,i_1} = ... = g_{i_{n-1},i_n} = 1\right\}$. *Let* $i_k \in N_{l_k}$, *then* $f(|l_k - l_{k+1}|) < f(|l_k - l_x|)$ *for* $x \in \{k+2, k+3, ..., n\}$.

[7] Given two players i and j in g, the geodesic distance, $d(i,j;g)$, is defined as the length of the shortest path between them.

We note that a generalized center-sponsored star will have the feature that along any path starting from the central player there can be at most m players. Thus the diameter of any such network is at most $2m$, which is independent of the size of the society and only depends on the number of groups. We shall use g^{gcs} to refer to any generalized center-sponsored star network. Figure 2 illustrates a generalized center-sponsored star. A network in which each group constitutes a distinct center-sponsored star and there are no links across groups has the unconnected center-sponsored stars architecture. We shall use g^{ucs} to refer to any network with this architecture.

Our first result provides a complete characterization of strict Nash networks in the insider-outsider model.

Proposition 4.1 *Suppose (4) and (6) hold. Assume that* $n_l \geqslant 2$, $\forall l = 1, ..., m$.

1. *If* $c_L > 1$ *then the empty network is the unique strict equilibrium.*
2. *Suppose* $c_L \in (0,1)$, *then there are three cases: (2a) if* $f(1) \in (c_L, 1)$, *then a strict equilibrium is a generalized center-sponsored star. (2b) If* $f(1) \in (1, \max[n_1, ..., n_m])$, *then a strict equilibrium does not exist. (2c) If* $f(1) > \max[n_1, ..., n_m]$, *then the only strict equilibrium is an unconnected center-sponsored stars.*

Proof: See the appendix.

Figure 3 illustrates the different strict Nash architectures for a society with two groups of two players each ($n_1 = n_2 = 2$). We note that strict equilibrium networks have very specific architectures and thus strictness is a useful refinement. The proof consists of a set of lemmas, which are stated and proved in the appendix. The *first* step of the proof shows that in every non-singleton component of a strict Nash network g there exists at least one inside link (Lemmas $1 - 3$). For simplicity assume that g is connected; then there is a path between any two players belonging to the same group, say $i, i' \in N_l$. There are two possible path configurations. First, the two players are directly linked and if this is the case the claim follows. Second, the two players access each other indirectly, through other players. In this case it is shown that the network must have the following pattern of links: $\{g_{j,i} = 1, ..., g_{k,i'} = 1\}$, with $j, k \notin N_l$. Next we note that the same property must also hold for $j, j' \in N_{l'}$: there exists a player $k' \notin N_{l'}$ who lies in the path between j and j' . Since the number of groups is finite and each group is composed of at least two players, an iteration of this argument shows that there will exist two players belonging to the same group who access each other via a direct link. The *second* step shows that if a group has an inside link then it has to be entirely internally linked and constitutes a center-sponsored star (Lemma 4). Here we use two arguments. One, we use network externality effects to argue that if two players of a group are directly linked then all members of this group must belong to the same component. Two, we use the switching argument to show that given an inside link, i.e. $g_{i,i'} = 1$ with $i, i' \in N_l$, i will bear all the links with members of his own group. Hence, group N_l is entirely internally linked and constitutes a center-sponsored star.

The *third* step in the proof shows that if a group is not entirely internally linked then it is entirely externally linked (Lemma 5). Consider a connected strict Nash network. Let N_l be the group highlighted in the previous step and let i be the center of this group. Consider a path between i and j, who is an end-player of the path. Suppose for simplicity that i and j have a direct link. If $g_{j,i} = 1$ then player j has a strict incentive to delete his link with player i and instead form a link with some player j' whom he accesses via player i and who belongs to his own group. Given our assumption $n_l \geq 2$, there exists such a player. Hence this cannot be an equilibrium network. A variant of this argument involving switching allows us to cover the case in which players i and j are indirectly linked. Given the $i-orientedness$ of each path, it is easy to see that along any path leading away from player i, there can be at most one player of any specific group. Hence it follows that if we take a pair of players in a group $l' \neq l$ there exists a path (since g is connected) and along this path there is no player of group l'. Thus all groups apart from l are entirely externally linked. This observation yields the property that the diameter of the network is less than or equal to $2m$. The *final* step in the proof consists of combining the above observations for different cost parameters.

We discuss some aspects of this characterization result. The *first* remark is about insider and outsider links. Our result shows that there is one group, the *core* group, which is entirely internally linked in the connected strict Nash network, while all other groups are entirely externally linked. In other words, the formation of local connections is not allowed in equilibrium (except for one group). This is an unexpected result and it suggests that incentives for link formation can completely undermine the structure that one might have expected: a set of local center-sponsored stars (corresponding to individual groups) linked with each other. *Two*, we note that the diameter of connected strict equilibrium networks is independent of the number of players, and depends only on the number of groups. Thus we expect strict equilibrium networks to have a relatively short diameter.

The *third* observation concerns the centrality and center-sponsorship properties. If the strict Nash network is connected, there is a player i such that all paths are oriented toward him. Hence, this player plays a particularly central role in the network. Furthermore, if the strict Nash network is non-empty but unconnected, then each component consists of members of one group and it has the center-sponsored star structure.

The *fourth* remark is related to the two special cases introduced in the specification of the insider-outsider model. An application of Proposition 4.1 to the homogeneous player example reveals that if $c > 1$ the only strict Nash network is the empty one, while if $c \in (0, 1)$ then the only strict Nash network is a center-sponsored star. This corresponds to the finding of Bala and Goyal (2000) for the linear model. Let's now turn to the two-cost levels case. When $c_H \in (c_L, 1)$, a strict Nash network has a generalized center-sponsored star architecture. More formally, there is an individual, say $i \in N_l$ which is the center of the whole network: each path in the network is oriented to him. Furthermore, group N_l is the only group to be entirely internally linked. Moreover, the members of all the remaining groups are passively linked with some members belonging to group N_l.[8] In particular, if all the remaining players are passively linked with player i, then the network is a center-sponsored star.

Our *fifth* remark is on the assumption that there are at least two members in each group. If we relax this assumption and allow for some groups to have only one member then two substantial changes occur. The first change is that there may exist more than one entirely internally linked group while the second change is that the non-existence of equilibrium may be avoided. The following example illustrates these points. Consider a society composed by three groups, where group N_1 and N_3 consist of three players and group N_2 has only one player. Let g be a connected network depicted in Figure 4. When $f(1) \in (c_L, 1)$, g is strict Nash. We note that in g all groups are entirely internally linked. Now, suppose that $f(1) = 1 + \epsilon$, where ϵ is positive and small enough. Again, the network g is strict Nash. However, if we assume that group N_2 consists of more than one player, a standard switching argument leads to the non-existence result.

We now turn to the issue of efficiency. We first introduce some new terminology that will be used in the proposition below. Let g^{mc} refer to a minimally connected network with each group N_l forming a minimally connected component with $n_l - 1$ inside links respectively and with $(m - 1)$ outside links of distance one. Finally, a partially connected network with each group generating a minimally connected component will be denoted as g^{pc}_m.

We start with an example which illustrates the role of group size in shaping efficient networks. For simplicity, we consider the two-cost levels case. Let the society be composed by three groups where group N_1 is small while groups N_2 and N_3 are large. Suppose now that $c_L \in (0, 1)$ and $c_H < 2n_2n_3$, then an efficient network must have the three groups internally linked and group N_2 and N_3 connected by one outside link. However, if $c_H \in (2n_1(n_2 + n_3), 2n_2n_3)$ then it is socially efficient to leave group N_1 isolated. Therefore, the efficient network is one in which the three groups are linked internally and where group N_2 and N_3 are connected by one outside link while group N_1 is left out. Clearly, if $c_L \in (0, 1)$ and $c_H < 2n_1(n_2 + n_3)$ the efficient network is minimally connected with $m - 1$ outside links, while if $c_H > 2n_2n_3$ then only a

[8] We say that an agent i is passively linked with an agent j if $g_{j,i} = 1$ and $g_{i,j} = 0$.

partially connected network where the three groups are linked internally is efficient. Finally for $c_L > \max\{n_1, n_2, n_3\}$ and c_H sufficiently high the only efficient network is the empty one. This variety in efficient networks arises due to the differences in sizes of the different groups. To keep matters simple, in what follows we therefore restrict attention to the case of equal size groups.

The following result provides a complete characterization of efficient networks for the case of equal group sizes. Let $n_l = \overline{n}$ for all $l = 1, 2, .., m$; moreover, we define $c_1 = m\overline{n}^2$ and $c_2 = [m\overline{n}(m\overline{n} - 1) - (m\overline{n} - m)c_L]/(m - 1)$.

Proposition 4.2 *Suppose (4) and (6) hold. In addition suppose that* $n_l = \overline{n}$, $\forall$ $l = 1, 2, .., m$.

1) *Suppose* $c_L \in (0, \overline{n})$. *If* $f(1) \in (c_L, c_1)$ *the network* g^{mc} *is uniquely efficient, while if* $f(1) > c_1$ *then the network* g^{pc}_m *is uniquely efficient.*

2) *Suppose* $c_L \in (\overline{n}, m\overline{n})$. *If* $f(1) \in (c_L, c_2)$ *then the network* g^{mc} *is uniquely efficient, while if* $f(1) > c_2$ *then the empty network is uniquely efficient.*

3) *If* $c_L > m\overline{n}$ *then the empty network is uniquely efficient.*

Proof: See the appendix.

Figure 5 illustrates two (non-empty) efficient architectures for a society composed of three groups and three players each. The proof is presented in the appendix. We briefly sketch the main steps here. An efficient network is minimal; this follows from the no-decay assumption. When c_L is high enough the empty network is efficient, while if c_L is relatively low it is beneficial for the society to have each group internally linked. Considerations on $f(1)$ allow us to divide this cost space into two sub-spaces: for $f(1)$ high enough the society is better-off leaving each group isolated by the others, yielding the network g^{pc}_m, while if $f(1)$ is not so high then the connected network arises. However, only a connected network with a minimal number of outside links $(m - 1)$ and all of 'length' one, is efficient. This yields us g^{mc}.[9]

We have showed that if g^{mc} is efficient the corresponding set of strict Nash networks does not contain any architectures compatible with the efficient one. This conflict persists until the level of $f(1)$ is such that any outside link is not beneficial both from an individual and social point of view. When this is the case, the heterogeneity introduced in the model becomes irrelevant and our problem degenerates in a sum of independent homogeneous problems leading to unconnected center-sponsored stars

[9] Each minimally connected network produces the same gross social welfare but different minimally connected networks will have a different total cost depending on the allocation of links.

networks. It follows that the trade-off between efficiency and stability fades in this case.

The conflict between efficient and equilibrium connected architectures arises out of a misallocation of links: too many outside links are set-up in order to obtain connectedness. Consider a connected network g and pick two players belonging to a group different from the core group, then if g is strict Nash, they will access each other via a sequence of outside links. This does not allow network participants to minimize the costs of connecting with each other and this lowers social welfare.

We now make two remarks about how this conflict between equilibrium and efficiency can be mitigated. One possibility arises within the static framework if we relax the assumption, used in the characterization of strict Nash networks, that each group is composed of at least two players. Consider a society composed by three groups where groups N_1 and N_3 consist of three individuals each while group N_2 consists of a single individual. Suppose $f(1) \in (c_L, 1)$. The network depicted in Figure 4 is strict Nash. Moreover, this network satisfies all the necessary conditions for a connected network to be efficient: the allocation of links is optimal from a societal point of view. In general, the presence of a single player between two heterogeneous groups composed by at least two individuals mitigates substantially the conflict between the notion of efficiency and strategic stability. The second remark is about the stability of the network in which each group constitutes a center-sponsored star and there is a single link across the groups. For example, consider a society composed by two groups of the same size ($n_1 = n_2$) and let g be a network where each group constitutes a center-sponsored star and a player i belonging to group 1 forms a link with a player j belonging to group 2. We note that this network is efficient for $c_L < 1$ and $c_H < n_1$. We note that this network is not a strict Nash network but that it is stable in the following sense: player i has a strict incentive to retain each of the within group links, and the moving around of the cross group link has no effect on the attractiveness of the within group links. This suggests a form of local stability. We now turn to a dynamic analysis to explore the scope of this argument.

5 Dynamics of network formation

In this section we shall examine a dynamic model of network formation based on myopic best response decision making by individuals. Our results establish that the dynamic process always converges and provide a characterization of networks that arise in the long run.

For a given set A, let $\Delta(A)$ denote the set of probability distributions on A. We suppose that for each agent i there exists a number $p_i \in (0,1)$ and a function $\phi_i : \mathcal{G} \to \Delta(\mathcal{G}_i)$ where ϕ_i satisfies, for all $g \in \mathcal{G} : \phi_i(g) \in \text{Interior } \Delta(\mathcal{BR}_i(g_{-i}))$. For $\hat{g}_i$

in the support of $\phi_i(g)$, the notation $\phi_i(g)(\hat{g})$ denotes the probability assigned to $\hat{g}_i$ by the probability measure $\phi_i(g)$. If the network at time $t \geq 1$ is $g' = g'_i \otimes g'_{-i}$, the strategy of agent i at time $t+1$ is assumed to be given by:

$$g_i^{t+1} = \begin{cases} \hat{g}_i \in \text{support } \phi_i(g), & \text{with probability } p_i \times \phi_i(g)(\hat{g}_i) \\ g'_i, & \text{with probability } 1 - p_i \end{cases}$$

We assume that the choice of inertia as well as the randomization over best responses by different agents is independent across agents. This implies that our decision rules induce a transition matrix T mapping the state space $\mathcal{G}$ to the set of all probability distributions $\Delta(\mathcal{G})$ on $\mathcal{G}$. Let $\{X_t\}$ be the stationary Markow chain starting from the initial network $g \in \mathcal{G}$ with the above transition matrix.

We note that there is an equivalence between the set of absorbing states and the set of strict Nash equilibria of the static game studied in sections above. However, our results on strict equilibrium networks reveal that in the insider-outsider model there are certain parameter values for which there exist no strict Nash networks. Thus the study of the dynamics is interesting from two points of view: one, they tell us whether the learning by individuals converges to a definite network and two, they tell us what happens when there does not exist a limiting state.

We will use the notion of curb sets in our analysis. A strategy profile set, $\tilde{\mathcal{G}} \subseteq \mathcal{G}$ is closed under rational behavior (*curb*) if $\mathcal{BR}(g) \subseteq \tilde{\mathcal{G}}$ for any $g \in \tilde{\mathcal{G}}$. A curb set $\tilde{\mathcal{G}}$ is minimal if there not exist a proper subset which is a curb set. It is well known that any game with a compact strategy set and payoffs that are continuous with respect to the strategies of players has at least one minimal curb set. However, a game may have several such sets, each of them containing networks with different architectures. Our results will prove convergence of the dynamic process to a minimal curb set and also completely characterize these minimal curb sets.

We first study the case where for each player the costs of forming links are independent of the potential partner. The proposition below summarizes our analysis for this case.

Proposition 5.1 *Let payoffs satisfy (1). Suppose that for any player i, $c_{i,j} = c_i$, and $V_{j,i} = 1$ for all i and j. Let player x be such that $c_x = \min\{c_j\}_{j \in N}$. If $c_x < 1$ then the dynamic process converges to a center-sponsored star and if $c_x > 1$ then the dynamic process converges to the empty network, with probability 1.*

Proof (Sketch): The proof of this result uses arguments analogous to Theorem 4.1 in Bala and Goyal (2001) and we provide a brief sketch only. We will focus on the case $c_i < 1$. The case where $c_i > 1$ is analogous and omitted. The first step shows that starting from any initial network the dynamics transit to a minimally connected network with positive probability. This follows from noting that a transition path in which players move in sequence one at a time and player x moves last has positive probability. The second step in the argument starts from a minimally connected network and uses a combination of mis-coordination and agglomeration arguments as in Bala and Goyal (2000) to show that there is a positive probability of reaching a center-sponsored star. The only difference is that the agglomeration argument is applied not to an arbitrary player, but to a player such that $c_j < V$. These arguments show that starting from any initial network there is a positive probability of transition to a center-sponsored star with some player j (where $c_j < 1$) as the center of the star. The proof then follows from standard results in the theory of Markov chains. $\square$

This result makes two points. The first point is that the dynamics of the model are well behaved: they converge to strict Nash networks which have a unique architecture and always exist. This is line with the result presented by Bala and Goyal (2000) for homogeneous players. Hence, whenever the costs of linking of each player are independent of the potential partner, cost heterogeneity does not affect the process of agglomeration and miscoordination through which players learn about the network. The second point is about the identity of central players: heterogeneity restricts the players who can be central players. For example, if $c_i < 1$ while $c_j \geq 1$ for all the remaining players then the only strict Nash network is a center-sponsored star with i being the center.

We now consider a society where an individual's cost of forming links can depend on the identity of the linked player. Matters are considerably more complicated here as the following example illustrates: consider the insider-outsider model with two groups A and B each with three players $\{1_A, 2_A, 3_A\}$ and $\{1_B, 2_B, 3_B\}$, respectively. Let $c_L < 1$ and $c_H < 3$. Consider a network in which players 1_A and 1_B respectively form the centers of their respective groups and sponsor the links as well. In addition suppose that player 1_A forms a link with player 1_B. In this network all the players except player 1_A have a unique best response while player 1_A has multiple best responses: he retains links with his own cohorts but can switch between players 1_B, 2_B and 3_B. It is to be noted that the best responses of the other players are insensitive to such changes by player 1_A. It is easily verified that this network along with the possible best responses of player 1_A constitutes a minimal curb set of the game. Moreover, once the dynamics enters this set it will cycle within this set forever. We also note that if $c_L < c_H < 1$ then in addition to this minimal curb set there are other minimal curb sets which correspond to the generalized center-sponsored stars (these are strict Nash networks in this parameter range). Our next result builds on these observations to obtain a convergence result for the two-groups setting. To state and prove this result we need to introduce some additional notation.

A network in which each group constitutes a center-sponsored star and a single player i of group l forms a link with some player j of group l', $l' \neq l$, is referred to as a connected center-sponsored stars network. Note that as player i varies his links across players of group l', distinct connected center-sponsored stars networks arise. We shall use $\mathcal{G}_i^{css}$ to denote the set of these networks. We shall use $\mathcal{G}^{css}$ to refer to sets with this property in general. A set of networks $\mathcal{G}^* \subseteq \mathcal{G}$ is a super tight curb set if $\mathcal{BR}(g) = \mathcal{G}^*$ for any $g \in \mathcal{G}^*$. This set may be considered a generalization of the strict Nash notion. To see this note that in a super tight curb set, for any $i \in N$, $\mathcal{BR}_i(g_{-i}) = \mathcal{G}_i^*$ for any $g_{-i} \in \mathcal{G}_{-i}^*$. In words, the best response set of a player i is invariant inside the set $\mathcal{G}^*$. We note that a strict Nash network constitutes a super tight curb set.

Proposition 5.2 *Suppose (4) and (6) hold and there are 2 groups. For generic values of* c_H *and* c_L*, the dynamic process converges to a super tight curb set,* $\mathcal{G}^*$*, with probability* 1*. (I) Suppose* $c_L \in (0,1)$*: if* $c_H \in (c_L, \max[n_1, n_2])$ *then* $\mathcal{G}^* \in \{g^{gcs}, \mathcal{G}^{ccs}\}$*, while if* $c_H > \max[n_1, n_2]$ *then* $\mathcal{G}^* = \{g^{ucs}\}$*. (II) Suppose* $c_L > 1$*: then* $\mathcal{G}^* = \{g^e\}$*, for all* $c_H > c_L$*.*

Proof: See the appendix.

The proof of this result consists of several steps. We start with the case $c_L \in (0,1)$ and $c_H \in (c_L, \max[n_1, n_2])$. The general argument is to show that starting from any initial network there is a positive probability of transiting to a super tight curb set. We sketch the argument for the case $c_L < 1$ and $c_H < 1$. In the following steps each of the transitions occurs with positive probability. The first step shows that starting from any network g_0, the process transits to a minimal network g_1. The second step shows that starting from a minimal network the process converges to a minimal network g_2 in which at least one of the two groups N_1 or N_2 is entirely internally linked. The third step shows that the process transits to a minimal network g_3 in which the internally linked group is a center-sponsored star. The fourth step takes up the different cases that can arise in g_3: in case g_3 consists of two group based center-sponsored stars then there is a transition to the set $\mathcal{G}^{ccs}$. In case g_3 consists of only one center-sponsored star the process transits to either a single g^{gcs} or to the set $\mathcal{G}^{ccs}$.

We next sketch the arguments for the case $c_L > 1$. The first step checks if some player has an incentive to form links. If not then we get all players to move and arrive at the empty network which is the unique strict Nash network in this range of parameters. If some player of (say) group 1 wishes to form links, then we can use an agglomeration argument to show that all players in this group will want to access this player in turn. We then work with a component which contains all members of the group. We now identify a player of this group who is maximally linked x and show that there is an agglomeration cum isolation process at work. A player will either move and link with this player x or be isolated in this process (we exploit $1 < c_L < c_H$ to show isolation). This process thus transits to a network in which a group 1 player is either a part of a periphery-sponsored star or is isolated. This part of the argument is quite complicated as we have to keep track of the members of the two groups along the transition process. The final set of arguments exploit mis-coordination among players to transit from a periphery-sponsored star to a network in which all members of the group are isolated.

The above result shows that myopic individual learning leads over time to a stable architecture of networks. Moreover, the long run (connected) network is either a single center-sponsored star or a generalized center-sponsored star or an interlinked star with locally central players. We would like to emphasize two aspects of the above result. One, it shows convergence to one of two specific architectural form, in all cases. This is a strong result given the very large number of possible network architectures. Two, it shows how dynamics can complement the static analysis nicely. In the static model we noted that for some parameter ranges no strict Nash networks exist and that there was a sharp conflict between strict Nash and efficient networks in some cases. The study of dynamics shows us that in the case where no strict Nash networks exist the process is still very well behaved and we can pin down precisely the long run outcomes. Moreover, we find that these long run outcomes are in fact much more efficient than the strict Nash outcomes. Thus dynamics may help resolve some of the tension between individual incentives and social efficiency as well.

6 Conclusion

We have studied a connections model of network formation in which players are heterogeneous with respect to benefits as well as the costs of forming links. We start by showing that value heterogeneity across players is crucial in determining the connectedness of a network, while differences in costs of linking across players are crucial in shaping both the level of connectedness as well as the architecture of individual components in a network. We then explore an insider-outsider model in which it is cheaper to form intra-group links as compared to inter-group links. Our main finding here is that interconnected stars with local central players are socially efficient as well as dynamically stable in such a setting. The results in our paper lead us to believe that centrality, center-sponsorship and small diameter are robust features of networks.

Appendix

Proof of Proposition 4.1: We recall some definitions that will be used in the proof. In a network g, a path between i and j is said to be $i-$*oriented* if either $g_{i,j} = 1$ or there is a sequence of distinct players $\{i_1, i_2, ..., i_n\}$ with the property that: $\{g_{i,i_1} = g_{i_1,i_2} = 1, ..., g_{i_n,j} = 1\}$. The proof consists of a sequence of steps, which are covered in the following lemmas.

Lemma 1: *Suppose g is a strict Nash network. If $g_{i,j} = 1$, where $i \in N_l$ and $j \in N_{l'}$, $l \neq l'$, then i does not access any player j' via the link $g_{i,j} = 1$ where $j' \in N_k$ and k is such that $|l - k| \leq |l - l'|$.*

Proof: Consider a strict Nash network g. Choose $i \in N_l$ and $j \in N_{l'}$, $l \neq l'$, such that $g_{i,j} = 1$. Let $j' \in N_k$ where k is such that $|l - k| \leq |l - l'|$. Suppose i accesses j' via the link $g_{i,j} = 1$. The spatial cost structure implies that i can do at least as well by deleting his link with j and forming a link with j'. This contradicts strict Nash. $\square$

Lemma 2: Suppose g is a strict Nash network. Assume $g_{i,j_0} = 1$, $i \in N_l$, $j \in N_{l_0}$, $l \neq l_0$ and let $\{j_0, j_1, ..., j_k\}$ where $j_x \in N_{l_x}$ for any $x \in \{0, ..., k\}$, be the set of players who agent i accesses via the link $g_{i,j_0} = 1$, then $g_{j',i} = 0$, $\forall j' \in N_k$ such that $|k - l| \geq |k - l_x|$ for some $x \in \{0, ..., k\}$.

Proof: Suppose $g_{j',i} = 1$. Since the cost of forming links is non-decreasing in the distance between players' groups, j' can do at least as well by deleting his link with i and forming a link with j_x. This contradicts strict Nash. $\square$

Lemma 3: *Suppose $n_l \geqslant 2$, $\forall\ l = 1, ..., m$. Suppose g is a strict Nash network, then in any non-singleton component there exists a pair of players who belong to the same group (this group will differ across components) and have a direct link.*

Proof(Sketch): Consider a non-singleton component $C(g)$. There exists $g_{i,j} = 1$, $i \in N_l$ and $j \in N \diagdown \{i\}$. Suppose that $j \in N_{l'}$, $l \neq l'$. We first note that, given $g_{i,j} = 1$, it must be true that $N_l \subset C(g)$. This follows by noting that the returns to a player $k \in N_l$ from linking with component $C(g)$ are strictly greater than the returns to player i, while the costs are strictly smaller (since k forms a link with i). Hence every player $k \in N_l$ must belong to $C(g)$. Therefore $i \in N_l$ must access every $i' \in N_l$ in g. There are two possibilities. One, i accesses i' via j. This violates Lemma 1. Two, i accesses i' via a player j', where $g_{j',i} = 1$. Given $g_{i,j} = 1$, Lemma 2 implies that the link $g_{j',i} = 1$ is sustainable in a strict Nash network, only if j' belongs to a group that is not accessed by i before the link $g_{j',i} = 1$ has been formed. Next note that, using the above argument, it follows that all members of j''s group must belong to $C(g)$. Suppose $j' \in N_x$. Then Lemmas 1 and 2 imply that j' accesses any $j'' \in N_x$ either by being directly linked, and if this is the case the proof trivially follows, or by being passively linked with some player $j''' \in N_y$, belonging to a group other than l. We can then repeat the same argument with respect to j'' and j'''. Since the number of groups is finite, we will eventually arrive at a point where two members of the same group are directly linked. The proof follows. $\square$

Lemma 4: *Assume $n_l \geqslant 2$, $\forall\ l = 1, ..., m$. Suppose g is a non-empty strict Nash network. If $g_{i,i'} = 1$, $i, i' \in N_l$, then $g_{i,i''} = 1$, $\forall i'' \in N_l \diagdown \{i\}$.*

Proof: Consider a non-singleton component, $C(g)$. Given the argument in Lemma 3, if $g_{i,i'} = 1$, for $i, i' \in N_l$, then $N_l \subset C(g)$. We first note that, if $g_{i,i'} = 1$, then $g_{i'',i} = 0$, $\forall\ i'' \in N_l \diagdown \{i\}$. This follows from the standard switching argument: if $g_{i'',i} = 1$ then player i'' is indifferent between linking with i and i', and g is therefore not a strict Nash network. We now have two possible configurations. First, suppose that $N_l \equiv C(g)$. Then an application of the switching argument immediately implies that $g_{i,i''} = 1$, for all $i'' \in N_l$. Second, suppose $N_l \subsetneqq C(g)$. Since $C(g)$ is connected, there is a path between i and i'', and $d(i,i'') \geq 2$. Then there is some player $j \neq i''$ such that $\bar{g}_{i,j} = 1$. Suppose that $j \in N_l$. If $g_{i,j} = 1$ then a simple switching argument applies with regard to player i and this contradicts the hypothesis that g is strict Nash. If $g_{j,i} = 1$ then the switching argument applies to player j, who is indifferent between the link with i and the link with i'. This contradicts the hypothesis that g is strict Nash. Similar arguments can be used in the case that $j \notin N_l$ to complete the proof of this lemma. □

Lemma 5: *Assume $n_l \geqslant 2$, $\forall\ l = 1, ..., m$. Suppose g is a connected strict Nash network and let $i \in N_l$ be the player identified by Lemma 4. Then any path $i \overset{g}{\longleftrightarrow} j$, $\forall j \in N \diagdown \{i\}$, is $i-$oriented.*

Proof: Let g be a strict Nash network which is connected. Since g is minimal, every path starting at i ends with a well defined end-player. The proof proceeds by contradiction. Suppose there is a path ending with player j, which is not i-oriented. If $\bar{g}_{i,j} = 1$ and j is not i-oriented then $g_{j,i} = 1$. From Lemma 4 we infer then that $j \in N_{l'}$ where $l' \neq l$. Next, since $n_l \geq 2$, we can apply a switching argument for player j with respect to some $i' \in N_l$, and that contradicts the hypothesis that g is a strict Nash network.

Suppose next that $\bar{g}_{i,j} = 0$. Let $\{i_1, i_2, i_3, ..., i_n\}$, be the players on the path between i and j, with $\bar{g}_{i,i_1} = ... = \bar{g}_{i_n,j} = 1$. We first take up the case $g_{j,i_n} = 1$. Let $j \in N_x$; if $i_n \notin N_x$ then a simple switching argument with regard to player j and some member of group x implies that g is not a strict Nash network. If $i_n \in N_x$, there are two possibilities: (i) $g_{i_{n-1},i_n} = 1$ and (ii) $g_{i_n,i_{n-1}} = 1$. In the first case, player i_{n-1} is indifferent between a link with player i_n and a link with player j. This contradicts the hypothesis that g is a strict Nash network. In the second case, there are two sub-cases: suppose i_n and i_{n-1} belong to the same group; then a switching argument applies to player j, with respect to players i_n and i_{n-1}. If i_n and i_{n-1} belong to different groups then a switching argument applies to player i_n with regard to members of the group of i_{n-1} (given that $n_l \geq 2$, for all $l = 1, 2, ..., m$).

Consider finally the case $g_{i_n,j} = 1$. Let k be the first player along the path $\{i_1, i_2, ..., i_n\}$, such that $g_{k,k-1} = 1$. Let $i_{k-1} \in N_y$. Since $g_{k-2,k-1} = 1$ by hypothesis, Lemma 1 implies that $i_k, i_{k+1}, ..., i_n \notin N_y$. By hypothesis, $n_y \geq 2$, and so there is a player $p \in N_y$, $p \neq i_{k-1}$, and we know that $p \notin \{i_k, i_{k+1}, ..i, j\}$. This is true because otherwise i_{k-2}

can switch from i_{k-1} to p. Thus, $p \in N \backslash \{i_{k-1}, i_k, ..., i_n, j\}$. In this case however, a switching argument would apply to player i_k with regard to p. Hence g is a not a strict Nash network. This contradiction completes the proof of the lemma. □

Lemma 6: *Assume $n_l \geqslant 2$, $\forall\ l = 1, ..., m$. Suppose g is a connected strict Nash network. Then $D(g) \leq 2m$.*

Proof: This follows directly by Lemma 1, 3, 4 and 5 □

We now complete the proof of Proposition 4.1.

1. Consider a strict Nash network g and suppose $c_L > 1$. We claim that the only strict Nash network is the empty one. Suppose that there exists a non-singleton component $C(g)$. Using arguments from Lemma 3 it follows that if $i \in N_l$, and $g_{i,j} = 1$, then $N_l \subset C(g)$. If $N_l \equiv C(g)$, then it is easy to show by applying the switching argument that $C(g)$ is a center-sponsored star. However, this is impossible given the hypothesis that $c_L > 1$. If on the other hand, $C(g)$ contains players from more than one group then it follows that g is a connected network. Lemma 5 now implies that there is central player and that all paths are oriented towards this player. However, given that $f(1) \geq c_l > 1$, this is not sustainable in equilibrium. This contradicts the hypothesis that g is a strict Nash equilibrium. Hence the empty network is the only possible strict Nash network.

2a. Suppose $c_L \in (0, 1)$ and $f(1) \in (c_L, 1)$. Suppose g is a strict Nash network; given the parameter restrictions, it is immediate that g must be connected. Lemma 3 and Lemma 4 imply that g satisfies property (i). Since g is connected, Lemma 5 holds and that implies property (ii). Considering the restrictions imposed by Lemma 1, Property (iii) follows by verification.

2b. Suppose $c_L \in (0, 1)$ and $f(1) \in (1, \max[n_1, ..., n_m])$. Suppose g is a strict Nash network; first we note that it must be connected. Lemma 5 implies that g has a central player i, and that all paths are i-oriented. However, $f(1) > 1$, g cannot be sustained in equilibrium, leading to a contradiction. Hence, there does not exist a strict Nash network.

2c. Suppose $c_L \in (0, 1)$ and $f(1) > \max[n_1, ..., n_m]$. Consider a strict Nash network g. From Lemmas 3 and 4 it follows that either g has m components corresponding to each of the groups or it is connected. In the former case, Lemmas 3 and 4 imply that each of the components is a center-sponsored star. In the latter case, Lemma 5 implies that g has a central player and all the paths are oriented towards this player. But then the argument from Part 2b applies and such a network cannot arise in equilibrium given that $f(1) > \max[n_1, ..., n_m]$. □

Proof of Proposition 4.2: In this proposition we assume equal group size, *i.e.* $n_l = \overline{n}$ for any $l = 1, ..., m$. We first start with two observations: (a) The no-decay assumption implies that each non-singleton component part of an efficient architecture is minimal; (b) If g is efficient and non-empty then it is either minimally connected with $m-1$ outside links of 'length' one and $m\overline{n}-m$ inside links, or partially connected with each group generating a minimally connected component. This observation follows by the assumption of equal group size and by the definition of efficiency concept. If a link between two members of the same group is socially efficient, then, from a societal point of view, each group should be internally linked. Furthermore, the assumption of equal group sizes implies that each group internally linked contributes equally to the total social welfare produced by the network. It follows that if an outside link is social enhancing, then an efficient network should be minimally connected. Moreover, since the definition of efficiency requires the minimization of the total cost of information flow, a connected efficient network should have $m-1$ outside links of length one. Using these observations we compare three different architectures:

1) The social welfare from g^{mc}, is given by:

$$W\left(g^{mc}\right) = \left(m\overline{n}\right)^2 - m\left(\overline{n}-1\right)c_L - (m-1)\, f\left(1\right) \tag{9}$$

2) The social welfare from g^{pc}_m, is given by:

$$W\left(g^{pc}_m\right) = m\left(\overline{n}\right)^2 - m\left(\overline{n}-1\right)c_L \tag{10}$$

3) The social welfare from g^e is given by:

$$W\left(g^e\right) = m\overline{n} \tag{11}$$

First, we compare g^{pc}_m with g^e. It is easily checked that $W\left(g^{pc}_m\right) \geqslant W\left(g^e\right)$ if and only if $c_L \leq \overline{n}$.

Second, suppose $c_L \in (0, \overline{n}]$ and compare g^{mc} with g^{pc}_m. Simple computations show that $W\left(g^{mc}\right) \geqslant W\left(g^{pc}\right)$ if and only if $f\left(1\right) \leq m\overline{n}^2 = c_1$. It follows that given $c_L \in (0, \overline{n}]$ if $f\left(1\right) \in (c_L, c_1]$ the only efficient network is g^{mc}, while if $f\left(1\right) > c_1$ the only efficient network is g^{pc}_m. This proves part (1).

Third, suppose $c_L > \overline{n}$ and compare g^{mc} with g^e. Again, simple computations show that $W\left(g^{mc}\right) \geqslant W\left(g^e\right)$ if and only if $f\left(1\right) \leq \frac{m\overline{n}(m\overline{n}-1)-(m\overline{n}-m)c_L}{m-1} = c_2$. We note that c_2 is a decreasing function of c_L and attains the value $m\overline{n}$ when $c_L = m\overline{n}$. Suppose therefore that $c_L \in (\overline{n}, m\overline{n})$. If $f\left(1\right) \in (c_L, c_2]$ then g^{mc} is uniquely efficient, while if $f(1) > c_2$ then g^e is uniquely efficient. Finally, if $c_L \geqslant m\overline{n}$ then $c_2 \leq c_L$. Given our hypothesis that $f\left(1\right) > c_L$ it follows that empty network is uniquely efficient. This proves parts (2) and (3). $\square$

Proof of Proposition 5.2: (I). $c_L < 1$: The *first* step shows that from any initial network g_0 there is a positive probability of transiting to a minimal network g_1. Fix a network g_0. Number the players $1, 2, ..., n$. Consider this sequence of players moving one at a time starting with 1. We claim that after player n has moved the network g_1 is minimal. Suppose not and there is a cycle of players. In that case consider players in the cycle who initiate links. Within this set of players fix the player who moved last. Clearly, this player did not choose a best response, as deleting one of his links in the cycle would have increased his net payoffs. This contradiction completes the argument.

The *second* step shows that starting from a minimal network g_1 there is a positive probability that the process transits to a minimal network g_2 in which there is at least one group with one internal link. We focus on the case where g_1 is connected and both groups are entirely externally linked. Then there exist players $i, i' \in N_1$ with $\bar{g}_{i,j} = 1$ where $j \in N_2$, and player i accesses i' via j. If $g_{i,j} = 1$ then (since $c_L < 1$) there exists a best response for player i in which he will disconnect from j and instead link with i' and this will yield a hybrid group. The other possibility is that $g_{j,i} = 1$. Since $n_l \geq 2$, for $l = 1, 2$, there is a player $j' \in N_2$ who is accessed by j. If this player is accessed via i the above argument leads to N_2 being a hybrid group. Since g_1 is minimal and we only let one player update at a time, the network g_2 must be minimal. The other possibility is that j' is accessed via some other player i''. In this case again variants of the above argument apply and the process transits to a network with one group having at least one internal link. Similar arguments apply if the initial network is not connected.

The *third* step shows that starting from a minimal network g_2 in which group N_1 is hybrid there is a positive probability that the process transits to a minimal network g_3 in which group N_1 is entirely internally linked. Let $\sigma_1(g)$ be the number of links between pairs of players in group N_1. By hypothesis, $\sigma_1(g) \in [1, n_1 - 1)$. Let $g_{i,i'} = 1$, for some pair of players $i, i' \in N_1$. We distinguish between two cases. The first case arises if players of N_1 are spread over more than one component. Pick some player $i \in N_1$ and get him to choose a best response. It is straightforward to verify that since $c_L < 1$ any best response of i, g'_i, has the property that he accesses all players in own group. Let $g' = g'_i \otimes g_{-i}$ be the new network. It follows that $\sigma_1(g') \geq \sigma_1(g) + 1$. We note that since g is a minimal network and g'_i is a best response, it follows that g' is a minimal network as well. The second case is one in which all members of group N_1 belong to a single component. Since N_1 is hybrid it follows that there exists a pair of players $x, y \in N_1$ such that $x \overset{\bar{g}}{\leftrightarrow} y$ contains only players belong to N_2. This implies in turn that there is at least one player $i'' \in N_1$ who is not internally linked with i. We will focus on the case where the path $i \overset{\bar{g}}{\leftrightarrow} i''$ contains only players $j_1, j_2, ..j_n \in N_2$.[10]

[10] It is possible that for instance player i' lies along this path; the arguments given below can be adapted to deal with this complication easily.

There are two sub-cases to consider.

(2a). If $g_{i,j_1} = 1$, then allow player i to play a best response. It follows from the hypothesis $c_L < 1$ that there is a best response in which player i will maintain all his current links with players in own group (since network is minimal); in addition in any best response, he will delete the link $g_{i,j_1} = 1$ and replace it with a link with some player of his own group along the path. We can suppose without loss of generality that the link $g_{i,i''} = 1$ is formed. Define $g' = g'_i \otimes g_{-i}$. It follows that $\sigma_1(g') > \sigma_1(g)$; again note that g' is a minimal network. A similar argument applies if $g_{i'',j_n} = 1$.

(2b). $g_{j_1,i} = g_{j_n,i''} = 1$: There are two possibilities here. (i). $j_1 \in N_2$ does not access any player $j' \in N_2$ via the link $g_{j_1,i} = 1$ and (ii) j_1 does access some $j' \in N_2$ via this link $g_{j_1,i} = 1$. We take these cases up in turn.

2b(i). We first allow player j_1 to choose a best response; he is indifferent between linking with i and i'. If he does not link with the component that contains i then we arrive a network in which i does not access i'' and we get i to choose a best response. This leads clearly to a network g' in which $\sigma_1(g') \geq \sigma_1(g) + 1$, and we are done. The other possibility is that j_1's best response g'_i involves a link with i's component and in that case let us suppose that he forms a link with i' and this yields a new network $g' = g'_i \otimes g_{-i}$. In the new network g', player i is indifferent between linking with i' or i''. Given g' let j_1 and i move simultaneously. There is a best response in which player i switches from i' to i'', while player j_1 switches from i' to i, yielding the network g''. We note that in g'', player i' will be isolated and that g'' will not be minimal. Now allow player i' to choose a best response. Any best response will involve a link with the component containing i and we can suppose without loss of generality that he forms a link with player i. We now get player j_1 to move and any best response will involve deletion of the link $g_{j_1,i}$. We have reached a minimal network g''' in which $\sigma_1(g''') \geq \sigma_1(g) + 1$.

2b(ii). Let j' be the first player of group N_2 along the path $j_1, i, i_1, ., i_n...$ in g. We first consider the case that $g_{j',i_n} = 1$. Allow players j_1 and j' to choose a best response. In any best response player j_1 will delete the link $g_{j_1,i} = 1$ and instead link with some player of his own group such as j'. Suppose this is the case. Similarly, in any best response player j' will delete the link $g_{j',i_n} = 1$ and instead link with someone of own group such as j_1. Denote by g' the resulting network. Now consider player i: in any best response he will want to form a link with someone such as i''. Allow player i to choose a best response. Finally, let player j_1 move and the resulting network g'' is minimal as well. It follows that $\sigma_1(g'') \geq \sigma_1(g) + 1$. Next we take up the case $g_{i_n,j'} = 1$. Let j_1 choose a best response. It follows that in any best response he will delete the link with i and switch to a player of own group such as j'. Denote the resulting network as g'. We now note that g' is minimal and in g', agent i_n observes i'' via the link $g_{i_n.j'}$. Thus, we are in case 2(a) above, and the argument follows. We have thus shown that starting from a minimal network g with N_1 as a hybrid group there exists a path which leads to a minimal network g' in which $\sigma_1(g') \geq \sigma_1(g) + 1$. Since the minimal network g is arbitrary we can repeat this step to arrive at a minimal network in which group 1 is entirely internally linked.

The *fourth* step shows that starting from network g_3 the process transits with positive probability to a network g_4 in which group N_1 is a center-sponsored star. Moreover, g_4 is minimal. Suppose that N_1 is entirely internally linked. Now assume that all players $j \in N_2$ exhibit inertia. We note that the process is analogous to a process with only homogenous players choosing links starting at a minimally connected network. So the arguments in Theorem 4.1 in Bala and Goyal (2000) can be applied to show that there exists a sequence of best responses leading to a network g' in which N_1 is a center-sponsored star.

We now complete the proof for $c_L \in (0, 1)$ and $c_H < \max\{n_1, n_2\}$: First suppose g_4 consists of two center-sponsored stars one for each group. If the network is connected and minimal then it must be the case that there is a single link between the two stars. If this network is Nash then it is easily verified that the process has entered a set of networks in which the player i initiating this single cross-group link is indifferent between forming this link with any of the players in the other star and the set of networks generated by this switching of links by the player i constitutes a super-tight curb set. Suppose the network is connected but not Nash. Since $c_L < 1$ and $c_H < \max\{n_1, n_2\}$, this must mean that there is a player $j \in N_l$, $l = 1, 2$ who wishes to delete the cross group link. Allow this player j to move. He will delete this link and retain any internal links he has in g_4 (since $c_L < 1$). Next choose the central player in the other group $N_{l'}$, with $l' \neq l$ and get him to choose a best response. From $c_L < 1$ and $c_H < \max\{n_1, n_2\}$ it follows that he will retain all his current links with own group members and in addition form a link with some player in n_1. We have now reached a Nash network and the first part of the argument can now be applied. We note that if g_4 contains two center-sponsored stars and the network is not connected then allowing any player in the smaller group to move will lead to a Nash network as above.

Second, we examine the case where g_4 has only one center-sponsored star and let it consist of N_1. Given that $c_L < 1$ we can assume that N_2 is connected as well. Here we have two possibilities. One, group N_2 is a hybrid group. Using the arguments presented in steps 3-4 it follows that there exists a sequence of best responses which leads to a network where group N_2 constitutes a center-sponsored star as well. We can then apply the arguments presented above. Two, suppose group N_2 is entirely externally linked. Then it has to be the case that g is minimally connected. If all the links have the appropriate orientation then g is a generalized center-sponsored star. Then if $c_H \in (0, 1)$ it follows that g is strict Nash and the proof follows. If $1 < c_H < \max\{n_1, n_2\}$ then g is not Nash. In particular, no outside links with isolated players are profitable. Let all $i' \in N_1$ move while all $j \in N_2$ exhibit inertia. Denote the resulting network by g'. Note that in g' group N_1 is a center-sponsored star, while each $j \in N_2$ is a singleton. Now have a player $j \in N_2$ move and any best response by him will yield a network with 2 center-sponsored stars. If in addition there is a single link across the groups initiated by j then we are done. Otherwise, get a player $i \in N_1$ to move and he will form a link with some $j \in N_2$ (because $c_H < \max\{n_1, n_2\}$. Finally, assume one of the links is not suitably oriented. Since group N_2 is entirely externally linked, and N_1 constitutes a center-sponsored star, there exists some player $j \in N_2$ who forms a link with some $i' \in N_1$. Let player j update. It follows from $c_L < 1$ and $c_L < c_H$ that the link to i' will be replaced by a link to some $j' \in N_2$. In the resulting minimal network group 2 is a hybrid group, while the architecture of group 1 is unchanged. We then apply arguments in steps 3-4 to arrive at two center-sponsored stars and the above arguments in this step to complete the proof.

We next complete the proof for $c_L \in (0, 1)$ and $c_H > \max\{n_1, n_2\}$: If there are two center-sponsored stars in g_4 then this network is a strict Nash network and we are done. If there is only one center-sponsored star consisting of group N_1 then consider the other group. Suppose as before that it is connected. If it is hybrid then we first use arguments in steps 3-4 to get this group to form a center-sponsored network and then follow with the arguments above. The other case is that this group is entirely externally linked. We get players from group 1 to move and since $c_H > 1$, they will all delete links with players in N_2. Now get a player in N_2 to move and this player will link with all players in own group. We have arrived at a network with two center-sponsored stars and we are done.

(II). $c_L > 1$: First, we note that the empty network is the unique strict Nash network in this parameter range. We will argue that there is a positive probability of transiting from any network g to the empty network g^e. The *first* step constructs a path of transition to a minimal network g_1. This is similar to what we did in step 1 in part (I) above. The *second* step checks if there is any player who wants to form a link. If the answer is *no* then we have all players move at the same time and they all delete any links they have and form no new links, which yields the empty network and we are done. If the answer is *yes* then we suppose that this player i belongs to N_1, without loss of generality. We then construct a path of transition such that the process reaches a minimal network in which players in N_1 are directly or indirectly connected. Here, we first get player i to move and let g' be the resulting network. We then choose a player $i' \in N_1$ who is not a member of the same component as i in g' to move and so on. The resulting network is denoted by g_2.

The *third* step is the main part of the proof: here we construct a path which leads to a network g_3 in which all members of N_1 are isolated. Let C_1 be the component that contains all members of N_1 and let $i_n \in N_1$ who has the maximum internal links.

Since g_2 is minimal it follows that for each path leading away from i_n, we can define a player who is furthest away from i_n and call him an end-player. Let $E_k(g_2)$ for $k = 1, 2$ be the set of end players belonging to groups 1 and 2, respectively. We first take up end-players $i \in E_1(g_2)$ who have initiated links. We let them move one at a time. If they have a best response in which they form no links then we allow them to delete their links and they become isolated. Note that they will not form links with any other component if they do not form a link with C_1. If they have a best response which involves forming links then surely they have a best response in which they form a single link with player i_n. Let them all link with i_n and continue this process so long as there is any end-player of group 1, who initiates links with some player other than i_n. This process thus leads to a network g' in which if an end-player belongs to group 1 then he either does not initiate a link or initiates a link with i_n. Moreover if $x \in N_1$ but $x \notin C_1(g')$ then x is isolated.

We now take an end-player $i \in E_1(g')$ who does not initiate a link. If there is some such player x then there exists $y \in N_2$ such that $g_{y,x} = 1$. Let player y move. Since $1 < c_L < c_H$, any best response of y must have $g_{y,x} = 0$, and player x will then be isolated. We repeat this step until all end-players not initiating a link have been isolated, and so all end-players in group 1 are initiating a link with i_n. Now consider an end-player $j \in N_2$ and look at the path $j \leftrightarrow i_n$. If there is no such player then we have arrived at a periphery-sponsored star and we can proceed to the last part of this argument. If there is such an end-player and he initiates the link then check whether this player wants to remain linked with this component. If not then allow the player to move and delink from the component, and the end-players as above. If this player wishes to remain linked with the component, then using arguments above we arrive at a network in which all players in group 2 are connected. Now define $j_n \in N_2$ the player who has the maximum internal links and it follows that player j has a best response in which he forms a link with player j_n. We now repeat the steps above but for end-players in group 2 and arrive at a network in which all end-players of group 2 are initiating links with j_n or isolated.

We have now defined two central players one for each group 1 and 2, respectively. We repeat the above argument in tandem to proceed with the agglomeration process with regard to each of the groups. This process leads to a network g''' with one the following structures: there is a single component which is an inter-linked periphery-sponsored star with members of N_1 forming one star and members of N_2 forming the other star, it is two distinct periphery-sponsored stars, it is one periphery-sponsored star with members of group 1 or group 2 and the other group has disintegrated or the network is empty. In the last case we are done. In the first three cases we use the following transition path: we number the periphery-players of a star from 1 to m, and get player 1 to switch his link from the center to a link with player 2 and player 2 to link with 3 and so on, until m links with 1. This leads to the central player becoming isolated. We now get all players in the circle to move and their unique best response is to delete their single link in the circle. We have thus reached a network in which no pair of players in the group are connected to each other. It is now easy to repeat the argument with the other group and we arrive at the empty network. □

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Figures:

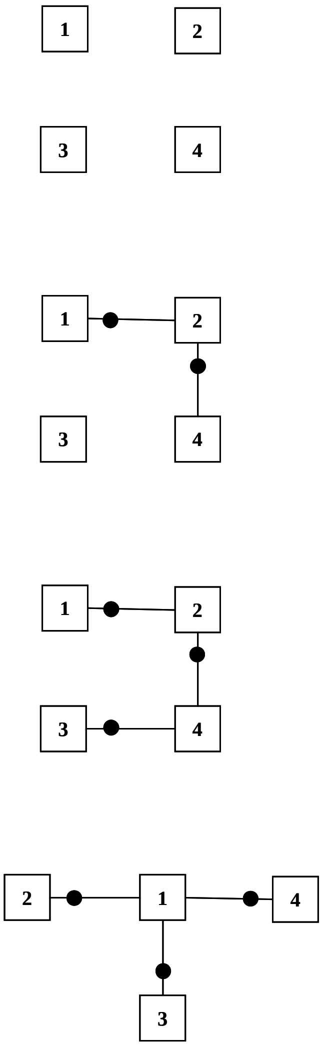


Figure 1: Minimal Network Architectures ($n = 4$)

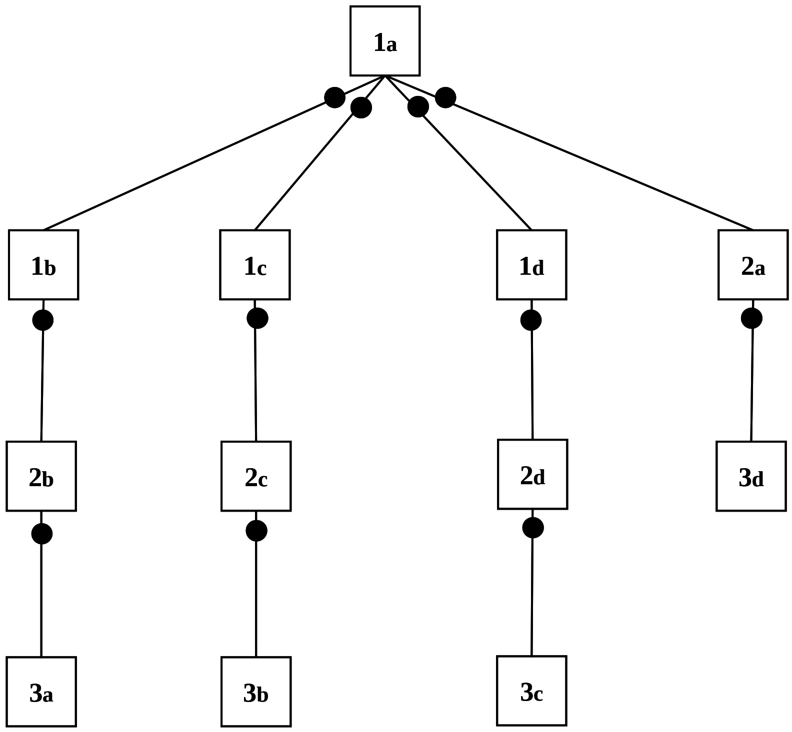
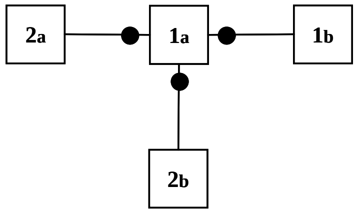
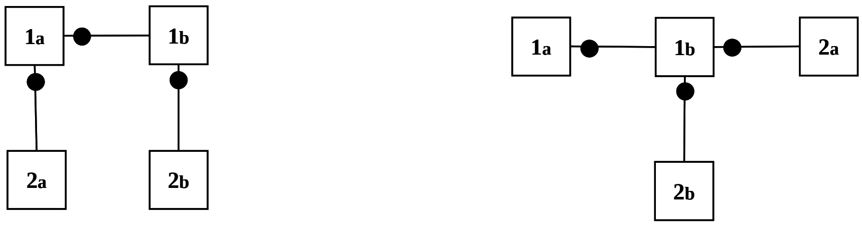


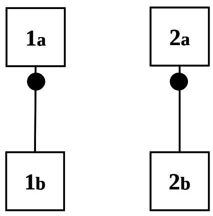
Figure 2: A Generalized Center-Sponsored Star Architecture
$\left(n_1 = n_2 = n_3 = 4\right)$



Center-Sponsored Star



Generalized Center-Sponsored
Stars



Unconnected Center-Sponsored
Stars

Figure 3: Strict Nash Architectures $(n_1 = n_2 = 2)$

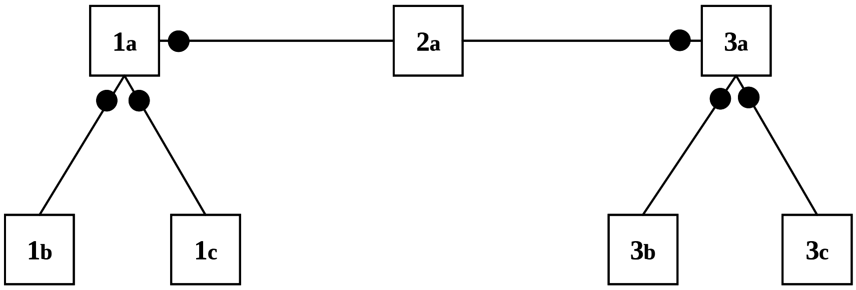


Figure 4: Single Intermediary Between Two Center-Sponsored Stars

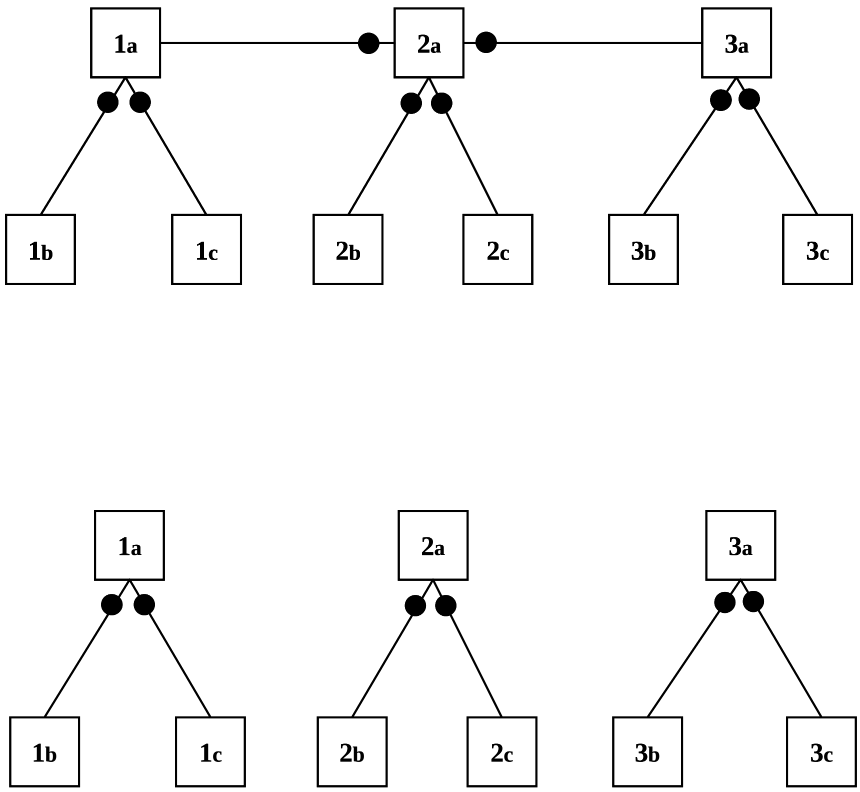


Figure 5: Two Efficient Architectures $(n_1 = n_2 = n_3 = 3)$