

How to Construct Mutually Orthogonal Complementary Sets with Non-Power-of-Two Lengths?

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Abstract—Mutually orthogonal complementary sets (MOCSs) have received significant research attention in recent years due to their wide applications in communications and radar. Existing MOCSs which are constructed based on generalized Boolean functions (GBFs) mostly have lengths of power-of-two. How to construct MOCSs with non-power-of-two lengths whilst having large set sizes is a largely open problem. With the aid of GBFs, in this paper, we present new constructions of such MOCSs and show that the maximal achievable set size is $1/2$ of the flock size of an MOCS.

Index Terms—Golay complementary set (GCS), mutually orthogonal complementary set (MOCS), complete complementary code (CCC), generalized Boolean function.

I. INTRODUCTION

The concept of Golay complementary pair (GCP), where the sums of aperiodic autocorrelations of two sequences are zero everywhere except at the in-phase position, was introduced by Marcel J. E. Golay in his design of infrared multislit spectrometry [1]. In 1972, Tseng and Liu generalized the concept to Golay complementary sets (GCSs) and mutually orthogonal complementary sets (MOCSs) [2]. Specifically, each GCS is composed of two or more sequences whose aperiodic autocorrelation sums display “impulse-like” correlation properties as GCP; MOCSs may be viewed as a collection of GCSs in which any two distinct GCSs are mutually orthogonal in terms of their zero cross-correlation sums for all the time-shifts. In 1988, Suehiro and Hatori proposed complete complementary codes (CCC) whose set size (denoted by M) achieves the upper bound of MOCSs (i.e., $M \leq N$, where N refers to the number of constituent sequences in an MOCS, called the flock size) [3]. Due to the ideal auto- and cross-correlation properties, MOCSs (and CCCs) have been adopted as a key component for next generation multicarrier code division multiple access (MC-CDMA) systems [4]–[8]. Other

applications include optimal channel estimation in multiple-input and multiple-output (MIMO) frequency-selective channels [9], MIMO radar [10], [11], cell search in orthogonal frequency division multiplexing (OFDM) systems [12], and data hiding [13], et al.

A primary approach to construct MOCSs is based on generalized Boolean functions (GBFs). This was initiated by the landmark work of Davis and Jedwab in [14] which shows that 2^h -ary GCPs lie in certain second-order cosets of the first-order Reed-Muller (RM) codes. Subsequently, Paterson generalized Davis and Jedwab’s methods to connect q -ary (for even q) GCPs with GBFs of degree 2 [15]. In the literature, GCPs obtained on GBFs given in [14], [15] are sometimes called the Golay-Davis-Jedwab (GDJ) pairs [16]–[18]. For GCSs, further constructions of GCSs based on GBFs of high degrees have been proposed in [19]–[22]. In [23], [24], connections of CCCs, MOCSs and cosets of the first-order RM codes have been proposed.

Existing complementary sequences constructed based on GBFs are mostly limited to lengths with powers of two. Recently, some research attempts have been made by Chen for GCSs with more flexible lengths [25]–[27]. GCSs with flexible lengths are more suitable for practical OFDM systems where the number of data subcarriers is various. With the aid of GBFs, however, how to construct MOCSs with non-power-of-two lengths whilst having large set size remains largely open, to the best of our knowledge. Although some MOCS constructions based on a series of sequence operations (e.g., reversals, negations, interleaving, and concatenations) have been proposed in [2], they are unable to give rise to MOCSs with large set sizes. Moreover, these sequence operations may not be friendly for efficient hardware generations. By contrast, the sequences from GBFs have algebraic structures and hence could have efficient encoding algorithms. In [28]–[30], Das *et al.* presented methods by using paraunitary (PU) matrices to construct MOCSs and CCCs with various lengths. PU matrices are powerful tools but 1) the available MOCSs heavily depend on the existence of PU matrix kernels and 2) the generations of MOCSs require multiple blocks in serial to carry out complicated PU matrix multiplications. Therefore, the MOCSs based on PU matrices and those based on GBFs might be different.

In this paper, we attack the aforementioned open problem by using GBFs. Such constructions are attractive due to their small storage requirement as well as fast generation capability [31]. For given flock size N , we define $r \triangleq N/M$ as the

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performance ratio which is upper bounded by one. In the case of power-of-two lengths, $r = 1$ can be achieved by CCC; For non-power-of-two lengths, it is intriguing to know the maximum achievable r . By constructing MOCSs with lengths $2^\theta + 2^\gamma$ (where $\theta \neq \gamma$ are positive integers)¹, it is found that $r = 1/2$ is achievable.

The remainder of the paper is organized as follow. In Section II, we give the definitions and some notations that will be used throughout this paper. Then we present the several constructions of GCSs and MOCSs with flexible lengths based on GBFs in Section III. Finally, we summarize our paper in Section IV.

II. PRELIMINARIES AND DEFINITIONS

A. Correlation Functions

For two L -length sequences $\mathbf{c} = (c_0, c_1, \dots, c_{L-1})$ and $\mathbf{d} = (d_0, d_1, \dots, d_{L-1})$ over $\mathbb{Z}_q = \{0, 1, \dots, q-1\}$. The *aperiodic cross-correlation function* of \mathbf{c} and \mathbf{d} at time-shift u is defined as

$$\rho(\mathbf{c}, \mathbf{d}; u) = \begin{cases} \sum_{k=0}^{L-1-u} \xi^{c_{k+u}-d_k}, & 0 \leq u \leq L-1 \\ \sum_{k=0}^{L-1+u} \xi^{c_k-d_{k-u}}, & -L+1 \leq u < 0 \end{cases}$$

where $\xi = e^{2\pi\sqrt{-1}/q}$. When $\mathbf{d} = \mathbf{c}$, the correlation $\rho(\mathbf{c}; u) \triangleq \rho(\mathbf{c}, \mathbf{c}; u)$ refers to the *aperiodic autocorrelation function* of \mathbf{c} .

B. Golay Complementary Set and Mutually Orthogonal Complementary Set

Consider a set of M sequence sets $\mathcal{C} = \{C^p, p = 0, 1, \dots, M-1\}$ where each sequence set C^p contains N sequences $C^p = \{c_0^p, c_1^p, \dots, c_{N-1}^p\}$ and each sequence has length L .

Definition 1: The set \mathcal{C} is called a **mutually orthogonal complementary set** (MOCS), denoted by (M, N, L) -MOCS, if

$$\rho(C^{p_1}, C^{p_2}; u) = \sum_{n=0}^{N-1} \rho(c_n^{p_1}, c_n^{p_2}; u) = \begin{cases} NL, & u = 0, p_1 = p_2 \\ 0, & 0 < |u| < L, p_1 = p_2 \\ 0, & |u| < L, p_1 \neq p_2 \end{cases} \quad (1)$$

where M is the set size, N the flock size, and L the sequence length. In the above definition, each sequence set C^p is called a *Golay Complementary Set*, denoted by (N, L) -GCS. In addition, any two distinctive GCSs in the set \mathcal{C} are said to be *mutually orthogonal*.

Lemma 2: For a (M, N, L) -MOCS, the upper bound on set size satisfies the inequality

$$M \leq N. \quad (2)$$

When $M = N$, it is called a *Complete Complementary Code* (CCC) [32].

¹See Corollary 5 in Section III for details.

C. Generalized Boolean Functions (GBFs)

A GBF f is a function of m variables x_1, x_2, \dots, x_m which maps any binary vector (x_1, x_2, \dots, x_m) in \mathbb{Z}_2^m to a value in \mathbb{Z}_q [33]. We define a product of r variables, $x_{i_1}x_{i_2}\dots x_{i_r}$, to be a Boolean monomial of degree r . For example, $x_1x_2x_3$ is a monomial of degree 3. We list all the monomials as follows:

$$1, x_1, x_2, \dots, x_m, x_1x_2, x_1x_3, \dots, x_{m-1}x_m, \dots, x_1x_2\dots x_m.$$

For $q = 2$, any Boolean function f can be uniquely expressed as a linear combination of these 2^m monomials, where the coefficient of each monomial belongs to \mathbb{Z}_2 [34]. The resulting expression for f is called the *algebraic normal form*. Then, we can define a 2^m -length sequence $\mathbf{f} = (f_0, f_1, \dots, f_{2^m-1})$ for any GBF f . The i th element of \mathbf{f} is given by $f_i = f(i_1, i_2, \dots, i_m)$ where (i_1, i_2, \dots, i_m) is the binary representation of the integer $i = \sum_{j=1}^m i_j 2^{j-1}$. For example, for $f = x_1x_2 + 3$ with $m = 3$ and $q = 4$, we have the sequence $\mathbf{f} = (3, 3, 3, 0, 3, 3, 3, 0)$.

For a GBF f with m variables, the corresponding sequence \mathbf{f} defined above is of length 2^m . Considering the sequence of length $\neq 2^m$, we define the corresponding truncated sequence $\mathbf{f}^{(L)}$ of the GBF f by removing the last $2^m - L$ elements of the sequence \mathbf{f} [25]. That is, $\mathbf{f}^{(L)} = (f_0, f_1, \dots, f_{L-1})$ is a sequence of length L with $f_i = f(i_1, i_2, \dots, i_m)$ for $i = 0, 1, \dots, L-1$ and (i_1, i_2, \dots, i_m) is the binary representation of i . For example, if taking $m = 3$, and GBF $f = x_1x_2 + 1$, we have the corresponding truncated sequence $\mathbf{f}^{(6)} = (1, 1, 1, 0, 1, 1)$ and $\mathbf{f}^{(7)} = (1, 1, 1, 0, 1, 1, 1)$. For the sake of simplicity, the superscript of $\mathbf{f}^{(L)}$ is disregarded when the length L can be determined from the context.

III. CONSTRUCTIONS OF MOCSS WITH FLEXIBLE LENGTHS

In this section, we first provide a construction of GCSs based on GBF. Then we extend the construction to MOCSs of non-power-of-two lengths.

Theorem 3: For any positive integers m and k where $m \geq 2$ and $k \leq m$, we consider a partition of $\{1, 2, \dots, m-1\}$ denoted by I_1, I_2, \dots, I_k . Let π_α be a bijection from $\mathbb{N}_{m_\alpha} = \{1, 2, \dots, m_\alpha\}$ to I_α where m_α is the order of I_α for $\alpha = 1, 2, \dots, k$. Let the GBF

$$f = \frac{q}{2} \sum_{\alpha=1}^k \sum_{\beta=1}^{m_\alpha-1} x_{\pi_\alpha(\beta)} x_{\pi_\alpha(\beta+1)} + \sum_{l=1}^m g_l x_l + g_0, \quad (3)$$

where q is an even integer and $g_l \in \mathbb{Z}_q$ for $l = 0, 1, \dots, m$. For a positive integer t with $0 \leq t \leq m-1$, if $t \neq 0$, then there exist integers α' and β' such that $t = \sum_{\alpha=1}^{\alpha'-1} m_\alpha + \beta'$ where $1 \leq \beta' \leq m_{\alpha'}$. Note that if $t \leq m_1$, we let $\alpha' = 1$ and $\beta' = t$. When $t \neq 0$, we need an additional condition below.

$$\{\pi_1(1), \pi_1(2), \dots, \pi_1(m_1), \pi_2(1), \dots, \pi_2(m_2), \pi_3(1), \dots, \pi_{\alpha'}(1), \dots, \pi_{\alpha'}(\beta')\} = \{1, 2, \dots, t\}. \quad (4)$$

Then, $C = \{c_0, c_1, \dots, c_{2^{k+1}-1}\}$ is a $(2^{k+1}, 2^{m-1} + 2^t)$ -GCS where

$$c_n = \mathbf{f} + \frac{q}{2} \left(\sum_{\alpha=1}^k n_\alpha \mathbf{x}_{\pi_\alpha(1)} + n_{k+1} \mathbf{x}_m \right),$$

and $(n_1, n_2, \dots, n_{k+1})$ is the binary representations of n for $n = 0, 1, \dots, 2^{k+1} - 1$.

Proof: The proof is given in Appendix A. ■

Compared with [25] and [26], the proposed GCSs can exist more lengths by setting a generalized variable t in Theorem 3. In addition, the constructed GCSs from Theorem 3 can include the results in [25], [26]. For example, Theorem 3 includes [25, Th. 6] as a special case when $t = 0$. In addition, if we let $0 \leq t \leq m_1$, Theorem 3 can be reduced to [26]. Then we will illustrate the relationship between Theorem 3 and the results in [25], [26] via an example.

In the following example and from here onwards, we let $\pi(a_1, a_2, \dots, a_k)$ represent a mapping such that $\pi(1) = a_1$, $\pi(2) = a_2, \dots$, and $\pi(k) = a_k$.

Example 1: Taking $q = 4$, $m = 6$, and $k = 2$, we let $I_1 = \{1, 2\}$, $I_2 = \{3, 4, 5\}$, $\pi_1 = (1, 2)$, $\pi_2 = (3, 4, 5)$, and then the Boolean function is $f = 2x_1x_2 + 2x_3x_4 + 2x_4x_5$ by letting all g_i 's = 0 in (3). Let us consider the case $t = 2$, we have $\alpha' = 1$ and $\beta' = 2$. Since $\{\pi_1(1), \pi_1(2)\} = \{1, 2\}$ satisfying (4), the set $C = \{f + 2n_1x_1 + 2n_2x_3 + 2n_3x_6 : n_i \in \mathbb{Z}_2\}$ is an $(8, 36)$ -GCS from Theorem 3. The set C can also be constructed by [26]. Next, if we take $t = 3$, t can be expressed as $t = m_1 + 1$. Hence, we have $\alpha' = 2$ and $\beta' = 1$. The condition (4) still holds since $\{\pi_1(1), \pi_1(2), \pi_2(1)\} = \{1, 2, 3\}$. Therefore, we can obtain another $(8, 40)$ -GCS from Theorem 3. This $(8, 40)$ -GCS is listed as follows:

$$\begin{aligned} C &= \{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7\} \\ &= \{(0002000200022220000200022220000200020002), \\ &\quad (0200020002002022020002002022020002000200), \\ &\quad (0002222000020002000222202220222000022220), \\ &\quad (0200202202000200020020222022202202002022), \\ &\quad (0002000200022220000200022220000222202220), \\ &\quad (020002000200202202000200202202002022022), \\ &\quad (0002222000020002000222202220222022200002), \\ &\quad (0200202202000200020020222022202220220200)\}. \end{aligned} \quad (5)$$

In this case, [26] cannot construct this $(8, 40)$ -GCS. Furthermore, if taking $t = 0$, [25, Th. 6], [26, Th. 4] and Theorem 3 all give the same $(8, 33)$ -GCS.

Next, we construct multiple GCSs based on Theorem 3 with mutual orthogonal cross-correlation property, that is, a MOCS.

Theorem 4: We follow the same notations given in Theorem 3 with f defined in (3). For any positive integer t with $m_1 \leq t \leq m - 1$, we can obtain integers α' and β' such that $t = \sum_{\alpha=1}^{\alpha'-1} m_\alpha + \beta'$ where $\alpha' \geq 1$ and $1 \leq \beta' \leq m_{\alpha'}$. **Note that $\alpha' = 1$ and $\beta' = m_1$ if $t = m_1$.** Let k' be the largest integer satisfying $\sum_{\alpha=1}^{k'} m_\alpha \leq t$. If

$$\begin{aligned} &\{\pi_1(1), \pi_1(2), \dots, \pi_1(m_1), \pi_2(1), \dots, \pi_2(m_2), \\ &\quad \pi_3(1), \dots, \pi_{\alpha'}(1), \dots, \pi_{\alpha'}(\beta')\} = \{1, 2, \dots, t\}, \end{aligned} \quad (6)$$

then the set $\mathcal{C} = \{C^0, C^1, \dots, C^{2^{k'}-1}\}$ where $C^p = \{c_0^p, c_1^p, \dots, c_{2^{k'}-1}^p\}$,

$$c_n^p = f + \frac{q}{2} \left(\sum_{\alpha=1}^k n_\alpha x_{\pi_\alpha(1)} + n_{k+1} x_m + \sum_{\alpha=1}^{k'} p_\alpha x_{\pi_\alpha(m_\alpha)} \right),$$

for $n = 0, 1, \dots, 2^{k'} - 1$ and $p = 0, 1, \dots, 2^{k'} - 1$, $(n_1, n_2, \dots, n_{k+1})$ and $(p_1, p_2, \dots, p_{k'})$ are the binary representations of n and p , respectively, is a $(2^{k'}, 2^{k'+1}, 2^{m-1} + 2^t)$ -MOCS.

Proof: The proof is given in Appendix B. ■

Please note that $k' = \alpha'$ if $\beta' = m_{\alpha'}$ and $k' = \alpha' - 1$ if $\beta' < m_{\alpha'}$.

Example 2: Following the same parameters q, m, k , permutations π_1, π_2 and Boolean function f given in Example 1, we obtain an $(8, 40)$ -GCS $C^0 = C$ in (5). According to Theorem 4, we have $k' = 1$ since $m_1 = 2 \leq t$ and $m_1 + m_2 = 2 + 3 > t$ when t is taken as 3. As a result, we can obtain a $(2, 8, 40)$ -MOCS from Theorem 4. In addition to C^0 , another constituent GCS is $C' = \{f + 2n_1x_1 + 2n_2x_3 + 2n_3x_6 + 2x_2 : n_i \in \mathbb{Z}_2\}$ and given by

$$\begin{aligned} C^1 &= \{c_0^1, c_1^1, c_2^1, c_3^1, c_4^1, c_5^1, c_6^1, c_7^1\} \\ &= \{(0002000200022220000200022220000200020002), \\ &\quad (0200020002002022020002002022020002000200), \\ &\quad (0002222000020002000222202220222000022220), \\ &\quad (0200202202000200020020222022202202002022), \\ &\quad (0002000200022220000200022220000222202220), \\ &\quad (020002000200202202000200202202002022022), \\ &\quad (0002222000020002000222202220222022200002), \\ &\quad (0200202202000200020020222022202220220200)\}. \end{aligned}$$

Similarly, for $t = 4$, we can obtain a $(2, 8, 48)$ -MOCS.

From Theorem 4, we can obtain a $(2^{k'}, 2^{k'+1}, 2^{m-1} + 2^t)$ -MOCS where $m_1 \leq t \leq m - 1$. Additionally taking $t = m - 1$, the constructed MOCSs are reduced to CCCs of length 2^m . Theorem 4 provides MOCSs of more lengths. Next, we discuss the set size of the constructed MOCSs with length $2^{m-1} + 2^t$. Let M_o denote the optimal set size of the constructed MOCS, i.e., $M_o = 2^{k'+1}$. For $m_{k-1} \leq t < m - 1$, we only have set size $2^{k'} = 2^{\alpha'-1} = 2^{k-1}$. This means that the set size is $2^{k-1}/M_o = 1/4$ which is one fourth of the optimal set size. Therefore, we propose a new corollary for expanding the set size of MOCSs.

Corollary 5: We follow the same notations given in Theorem 3. If $t = \sum_{\alpha=1}^{k-1} m_\alpha + \beta'$ for some positive integer $\beta' \leq m_k$, we let

$$y = x_m x_{\pi_k(\beta')} + (1 \oplus x_m) x_{\pi_k(m_k)}.$$

Then, $\mathcal{C} = \{C^0, C^1, \dots, C^{2^{k'}-1}\}$ is a $(2^k, 2^{k+1}, 2^{m-1} + 2^t)$ -MOCS where $C^p = \{c_0^p, c_1^p, \dots, c_{2^{k'}-1}^p\}$ and

$$\begin{aligned} c_n^p &= f \\ &+ \frac{q}{2} \left(\sum_{\alpha=1}^k n_\alpha x_{\pi_\alpha(1)} + n_{k+1} x_m + \sum_{\alpha=1}^{k-1} p_\alpha x_{\pi_\alpha(m_\alpha)} + p_k y \right), \end{aligned}$$

for $n = 0, 1, \dots, 2^{k'} - 1$ and $p = 0, 1, \dots, 2^k - 1$. Note that $(n_1, n_2, \dots, n_{k+1})$ and (p_1, p_2, \dots, p_k) are the binary representations of n and p , respectively.

Proof: The proof is given in Appendix C. ■

From Corollary 5, the set size of MOCSs is 2^k . This means that the ratio of the set size over the flock size is $M/N = 1/2$.

TABLE I
THE EXISTENCE OF MOCS FOR VARIOUS LENGTHS

Length \ Set size	4	8	12	16	20	24	32	36	40	48	64
4 [24]	✓	✓	-	✓	-	-	✓	-	-	-	✓
4 (Th. 4 and Cor. 5)	-	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
8 [24]	-	✓	-	✓	-	-	✓	-	-	-	✓
8 (Th. 4 and Cor. 5)	-	-	-	✓	-	✓	✓	-	✓	✓	✓
16 [24]	-	-	-	✓	-	-	✓	-	-	-	✓
16 (Th. 4 and Cor. 5)	-	-	-	-	-	-	✓	-	-	✓	✓

TABLE II
THE EXISTENCE OF EVEN-LENGTH (M, N) -MOCS

(M, N) \ Length	4	6	8	10	12	16	18	20	24	32	34	36	40	M/N
Construction	(4, 4)	-	(4, 4)	-	-	(4, 4)	-	-	-	(4, 4)	-	-	-	1
[23]	-	-	(8, 8)	-	-	(8, 8)	-	-	-	(8, 8)	-	-	-	1
[24]	-	-	-	-	-	(16, 16)	-	-	-	(16, 16)	-	-	-	1
	-	-	-	-	-	-	-	-	-	(32, 32)	-	-	-	1
Theorem 4	(2, 4)	-	(2, 4)	-	-	(2, 4)	-	-	-	(2, 4)	-	-	-	1/2
	-	(2, 8)	-	(2, 8)	(2, 8)	-	(2, 8)	(2, 8)	(2, 8)	-	(2, 8)	(2, 8)	(2, 8)	1/4
	-	-	-	(2, 16)	-	-	(2, 16)	(2, 16)	-	-	(2, 16)	(2, 16)	(2, 16)	1/8
	-	-	-	-	-	-	(2, 32)	-	-	-	(2, 32)	(2, 32)	-	1/16
	-	-	-	-	-	-	-	-	-	-	(2, 64)	-	-	1/32
	-	-	(4, 8)	-	-	(4, 8)	-	-	-	(4, 8)	-	-	-	1/2
	-	-	-	-	(4, 16)	-	-	(4, 16)	(4, 16)	-	-	(4, 16)	(4, 16)	1/4
	-	-	-	-	-	-	-	(4, 32)	-	-	-	(4, 32)	(4, 32)	1/8
	-	-	-	-	-	-	(8, 16)	-	-	-	-	(4, 64)	-	1/16
	-	-	-	-	-	-	-	-	-	(8, 16)	-	-	-	1/2
	-	-	-	-	-	-	-	-	(8, 32)	-	-	-	(8, 32)	1/4
	-	-	-	-	-	-	-	-	-	-	-	-	(8, 64)	1/8
-	-	-	-	-	-	-	-	-	-	(16, 32)	-	-	1/2	
Corollary 5	(2, 4)	(2, 4)	(2, 4)	(2, 4)	(2, 4)	(2, 4)	(2, 4)	(2, 4)	(2, 4)	(2, 4)	(2, 4)	(2, 4)	(2, 4)	1/2
	-	-	(4, 8)	-	(4, 8)	(4, 8)	-	(4, 8)	(4, 8)	(4, 8)	-	(4, 8)	(4, 8)	1/2
	-	-	-	-	-	(8, 16)	-	-	(8, 16)	(8, 16)	-	-	(8, 16)	1/2
	-	-	-	-	-	-	-	-	-	(16, 32)	-	-	-	1/2

Table I shows the existences of MOCSs of set size 4, 8 and 16 for various lengths. Compared with CCCs from [24], Theorem 4 and Corollary 5 can construct MOCSs of non-power-of-two lengths. For example, it can be observed that the MOCSs of set size 4 are available for lengths 18, 20, 24 between 2^4 and 2^5 .

Example 3: Let us consider the same parameter setting in Example 2. Following the construction in Corollary 5, let $C^2 = \{\mathbf{f} + 2n_1\mathbf{x}_1 + 2n_2\mathbf{x}_3 + 2n_3\mathbf{x}_6 + 2\mathbf{y} : n_i \in \mathbb{Z}_2\}$ and $C^3 = \{\mathbf{f} + 2n_1\mathbf{x}_1 + 2n_2\mathbf{x}_3 + 2n_3\mathbf{x}_6 + 2\mathbf{x}_2 + 2\mathbf{y} : n_i \in \mathbb{Z}_2\}$ where $\mathbf{y} = \mathbf{x}_6\mathbf{x}_{\pi_2(1)} + (\mathbf{1} \oplus \mathbf{x}_6)\mathbf{x}_{\pi_2(3)} = \mathbf{x}_6\mathbf{x}_3 + (\mathbf{1} \oplus \mathbf{x}_6)\mathbf{x}_5$. Then $\{C^0, C^1, C^2, C^3\}$ forms a $(4, 8, 48)$ -MOCS.

Example 4: Taking $q = 2$, $m = 6$ and $k = 3$, we let $I_1 = \{1\}$, $I_2 = \{2, 4\}$, $I_3 = \{3, 5\}$, $\pi_1 = (1)$, $\pi_2 = (4, 2)$, $\pi_3 = (3, 5)$, and then the Boolean function is $f = x_4x_2 + x_3x_5$ by letting $g_l = 0$ for all l in (3). Let us consider the case $t = 4$, we have $\alpha' = 3$ and $\beta' = 1$ since $t = 4 = m_1 + m_2 + 1$. Due to $\{\pi_1(1), \pi_2(1), \pi_2(2), \pi_3(1)\} = \{1, 4, 2, 3\}$, the set $\mathcal{C} = \{C^0, C^1, \dots, C^7\}$ is an $(8, 16, 48)$ -MOCS from Corollary 5 where $C^p = \{\mathbf{f} + n_1\mathbf{x}_1 + n_2\mathbf{x}_4 + n_3\mathbf{x}_3 + n_4\mathbf{x}_6 + p_1\mathbf{x}_1 + p_2\mathbf{x}_2 + p_3(\mathbf{x}_6\mathbf{x}_3 + (\mathbf{1} \oplus \mathbf{x}_6)\mathbf{x}_5) : n_i \in \mathbb{Z}_2\}$ and (p_1, p_2, p_3) is the binary representation of p .

In Table II, MOCSs of lengths between 4 and 40 are listed. Theorem 4 and Corollary 5 can construct MOCSs of non-power-of-two lengths but the set sizes can not be equal to the individual flock sizes. The ratio of the set size over the flock size is $M/N = 1/2$ for the MOCSs obtained by Corollary 5.

IV. CONCLUSION

In this paper, we have presented new constructions of GCSs and MOCSs based on GBFs with flexible lengths. First, a construction of GCSs with flexible lengths is proposed in Theorem 3, which includes the results in [25], [26] as special cases. In addition, new constructions of MOCSs have been proposed in Theorem 4 and Corollary 5. The resultant MOCSs can be obtained directly from GBFs without using other tedious sequence operations.

The proposed MOCSs have lengths of the form $2^{m-1} + 2^t$ where $t > 0$. Possible future research includes the study of MOCSs with more available lengths, e.g., $t = 0$. In Corollary 5, the ratio of set size over flock size (i.e., M/N) is $1/2$ only. It would be interesting to know that whether CCCs (i.e., $M/N = 1$) with non-power-of-two lengths can be constructed by GBFs.

APPENDIX A PROOF OF THEOREM 3

Before starting the proof of Theorem 3, we introduce several lemmas which can be regarded as crucial tools to prove our main theorems. We follow the same notations as given in Theorem 3 and define the common notations as follows. Let two nonnegative integers $i, j < 2^m$ have binary representations (i_1, i_2, \dots, i_m) and (j_1, j_2, \dots, j_m) , respectively.

Lemma 6: [25, Lemma 2] For two integers i and j with $0 \leq i < j < 2^m$ and $m \geq 2$, if $i_s = j_s$ for $s = 1, 2, \dots, t$ for a positive integer t , then $j \geq i + 2^t$.

Lemma 7: [25, Lemma 3] For an integer i with $2^{m-1} \leq i \leq 2^{m-1} + 2^t - 1$ where $1 \leq t \leq m-1$ and $m \geq 2$, if i' is an integer with binary representation $(i_1, i_2, \dots, i_{r-1}, 1 - i_r, i_{r+1}, \dots, i_m)$ and $r \leq t$, then we have $2^{m-1} \leq i' \leq 2^{m-1} + 2^t - 1$.

Lemma 8: [35, Lemma 4] Following the same definitions and notations given in Theorem 3. If $i_{\pi_\alpha(1)} \neq j_{\pi_\alpha(1)}$ for some $\alpha \in \{1, 2, \dots, k\}$, then for any sequences $\mathbf{c}_n = (c_{n,0}, c_{n,1}, \dots, c_{n,L-1}) \in C$, there exists $\mathbf{c}'_n = (c'_{n,0}, c'_{n,1}, \dots, c'_{n,L-1}) = \mathbf{c}_n + (q/2)\mathbf{x}_1 \in C$ such that $\xi^{c_{n,j}-c_{n,i}} + \xi^{c'_{n,j}+c'_{n,i}} = 0$. Similarly if $i_m \neq j_m$, then for any sequence $\mathbf{c}_n \in C$, there exists $\mathbf{c}'_n = \mathbf{c}_n + (q/2)\mathbf{x}_m \in C$ such that $\xi^{c_{n,j}-c_{n,i}} + \xi^{c'_{n,j}+c'_{n,i}} = 0$.

Lemma 9: [35, Lemma 5] Suppose $i_{\pi_\alpha(1)} = j_{\pi_\alpha(1)}$ for $\alpha = 1, 2, \dots, k$. Let us consider three conditions:

- (C1) $\hat{\alpha}$ is the largest integer satisfying $i_{\pi_\alpha(\hat{\alpha})} = j_{\pi_\alpha(\hat{\alpha})}$ for $\alpha = 1, \dots, \hat{\alpha} - 1$ and $\beta = 1, \dots, m_\alpha$.
- (C2) $\hat{\beta}$ is the smallest integer such that $i_{\pi_{\hat{\alpha}}(\hat{\beta})} \neq j_{\pi_{\hat{\alpha}}(\hat{\beta})}$.
- (C3) Let i' and j' be integers which differ from i and j , respectively, in only one position $\pi_{\hat{\alpha}}(\hat{\beta} - 1)$. That is, $i'_{\pi_{\hat{\alpha}}(\hat{\beta}-1)} = 1 - i_{\pi_{\hat{\alpha}}(\hat{\beta}-1)}$ and $j'_{\pi_{\hat{\alpha}}(\hat{\beta}-1)} = 1 - j_{\pi_{\hat{\alpha}}(\hat{\beta}-1)}$.

If the above conditions are all satisfied, we have

$$f_j - f_i - f_{j'} + f_{i'} \equiv q/2 \pmod{q}.$$

Proof of Theorem 3: Here we are going to prove that C is a GCS of size 2^{k+1} and of length $2^{m-1} + 2^t$. To this end, it is sufficient to show

$$\begin{aligned} \rho(C; u) &= \sum_{n=0}^{2^{k+1}-1} \rho(\mathbf{c}_n; u) \\ &= \sum_{n=0}^{2^{k+1}-1} \sum_{i=0}^{L-1-u} \xi^{c_{n,i+u}-c_{n,i}} \\ &= \sum_{i=0}^{L-1-u} \sum_{n=0}^{2^{k+1}-1} \xi^{c_{n,i+u}-c_{n,i}} = 0 \end{aligned} \quad (7)$$

for $0 < u < 2^{m-1} + 2^t - 1$. For any integer $i < 2^m + 2^t - 1 - u$ with binary representation (i_1, i_2, \dots, i_m) , we let $j = i + u$ with binary representation (j_1, j_2, \dots, j_m) . Then, we will show that (7) holds by considering the following four cases.

Case 1: If $i_{\pi_\alpha(1)} \neq j_{\pi_\alpha(1)}$ for some $\alpha \in \{1, 2, \dots, k\}$, according to Lemma 8, we can obtain

$$\sum_{n=0}^{2^{k+1}-1} \xi^{c_{n,j}-c_{n,i}} = 0.$$

Case 2: In this case, we assume $i_{\pi_\alpha(1)} = j_{\pi_\alpha(1)}$ for all $\alpha \in \{1, 2, \dots, k\}$ and $i_m = j_m = 0$. Here, there exist $\hat{\alpha}$ and $\hat{\beta}$ satisfying the conditions (C1) and (C2) in Lemma 9. Let i' and j' be integers fulfilling condition (C3) in Lemma 9. According to Lemma 9, we have $j' = i' + u$ and $f_j - f_i - f_{j'} + f_{i'} \equiv q/2 \pmod{q}$. Therefore,

$$\begin{aligned} c_{n,j} - c_{n,i} - c_{n,j'} + c_{n,i'} &= f_j - f_i - f_{j'} + f_{i'} \\ &\equiv q/2 \pmod{q}. \end{aligned}$$

Then the above equation implies

$$\xi^{c_{n,j}-c_{n,i}} + \xi^{c_{n,j'}-c_{n,i'}} = 0.$$

Case 3: In this case, suppose $i_{\pi_\alpha(1)} = j_{\pi_\alpha(1)}$ for all $\alpha \in \{1, 2, \dots, k\}$ and $i_m = j_m = 1$ which means $2^{m-1} \leq i, j \leq 2^{m-1} + 2^t - 1$. Here, there exist $\hat{\alpha}$ and $\hat{\beta}$ satisfying the conditions (C1) and (C2) in Lemma 9. Then, we will have $\pi_{\hat{\alpha}}(\hat{\beta}) \leq t$. Suppose not, we assume $\pi_{\hat{\alpha}}(\hat{\beta}) > t$. Hence, we obtain that $i_s = j_s$ for $s = 1, 2, \dots, t$ since $\{\pi_1(1), \dots, \pi_1(m_1), \pi_2(1), \dots, \pi_{\alpha'}(\beta')\} = \{1, 2, \dots, t\}$. According to Lemma 6, we have $j \geq i + 2^t \geq 2^{m-1} + 2^t$ which contradicts the assumption. Thus, we have $\pi_{\hat{\alpha}}(\hat{\beta}) \leq t$. Then let i' and j' be integers fulfilling condition (C3) in Lemma 9. Since $\pi_{\hat{\alpha}}(\hat{\beta}) \leq t$, we also have $\pi_{\hat{\alpha}}(\hat{\beta} - 1) \leq t$. According to Lemma 7, we obtain $2^{m-1} \leq i', j' \leq 2^{m-1} + 2^{\pi_{\hat{\alpha}}(\hat{\beta}-1)} - 1 \leq 2^{m-1} + 2^t - 1$. Therefore, for any integers i and j with $2^{m-1} \leq i, j \leq 2^{m-1} + 2^t - 1$, there exist integers $i', j' \leq 2^{m-1} + 2^t - 1$ and $j' = i' + u$. Therefore, we obtain

$$\xi^{c_{n,j}-c_{n,i}} + \xi^{c_{n,j'}-c_{n,i'}} = 0.$$

according to Lemma 9.

Case 4: If $i_m \neq j_m$, according to Lemma 8, we also have

$$\sum_{n=0}^{2^{k+1}-1} \xi^{c_{n,j}-c_{n,i}} = 0.$$

According to Case 1 to Case 4, we can confirm that C is a GCS of length $L = 2^{m-1} + 2^t$. ■

APPENDIX B PROOF OF THEOREM 4

Lemma 10: Let $\mathbf{x}_{n_1}, \mathbf{x}_{n_2}, \dots, \mathbf{x}_{n_k}$ be the sequences corresponding to Boolean functions $x_{n_1}, x_{n_2}, \dots, x_{n_k}$, respectively, where $n_1 < n_2 < \dots < n_k$. Let a binary sequence $\mathbf{d} = (d_0, d_1, \dots, d_{L-1}) = \mathbf{x}_{n_1} \oplus \mathbf{x}_{n_2} \oplus \dots \oplus \mathbf{x}_{n_k}$ be the sum of $\mathbf{x}_{n_1}, \mathbf{x}_{n_2}, \dots, \mathbf{x}_{n_k}$. If $2^{n_1} | L$, we can obtain that the Hamming weight of \mathbf{d} is $L/2$.

Proof: For any integers i , if $2^{n_1} | L$, then there exists an integer i' which is different from i in only one position n_1 , i.e., $(i'_1, i'_2, \dots, i'_m) = (i_1, i_2, \dots, i_{n_1-1}, 1 - i_{n_1}, i_{n_1+1}, \dots, i_m)$. Thus,

$$\begin{aligned} d_{i'} - d_i &= (i'_{n_1} \oplus i'_{n_2} \oplus \dots \oplus i'_{n_k}) - (i_{n_1} \oplus i_{n_2} \oplus \dots \oplus i_{n_k}) \\ &\equiv i'_{n_1} - i_{n_1} \equiv 1 \pmod{2}. \end{aligned}$$

Therefore, $d_{i'} = 1 - d_i$ which means there are half 0's and half 1's for the binary sequence \mathbf{d} . That is, the Hamming weight of \mathbf{d} is $L/2$. ■

Proof of Theorem 4: We will show that any two distinct sets C^e and C^p where $0 \leq e \neq p \leq 2^{k'} - 1$ satisfy the ideal cross-correlation property. Let $\mathbf{c}_n^e = (c_{n,0}^e, c_{n,1}^e, \dots, c_{n,L-1}^e) \in$

C^e and $\mathbf{c}_n^p = (c_{n,0}^p, c_{n,1}^p, \dots, c_{n,L-1}^p) \in C^p$ where $L = 2^{m-1} + 2^t$. For $0 \leq u < L-1$, we have to show

$$\begin{aligned} \rho(C^e, C^p; u) &= \sum_{n=0}^{2^{k+1}-1} \rho(\mathbf{c}_n^e, \mathbf{c}_n^p; u) \\ &= \sum_{n=0}^{2^{k+1}-1} \sum_{i=0}^{L-1-u} \xi^{c_{n,i}^e - c_{n,i}^p} \\ &= \sum_{i=0}^{L-1-u} \sum_{n=0}^{2^{k+1}-1} \xi^{c_{n,i}^e - c_{n,i}^p} = 0. \end{aligned} \quad (8)$$

For any integer $i < 2^{m-1} + 2^t - 1 - u$ with binary representation (i_1, i_2, \dots, i_m) , we let $j = i + u$ with binary representation (j_1, j_2, \dots, j_m) . Then, we will show that (8) holds by considering the following four cases.

Case 1: Suppose $i_{\pi_\alpha(1)} \neq j_{\pi_\alpha(1)}$ for some $\alpha \in \{1, 2, \dots, k\}$, according to Lemma 8, we can obtain

$$\sum_{n=0}^{2^{k+1}-1} \xi^{c_{n,j}^e - c_{n,i}^p} = 0.$$

Case 2: In this case, we have $i_{\pi_\alpha(1)} = j_{\pi_\alpha(1)}$ for all $\alpha \in \{1, 2, \dots, k\}$ and $i_m = j_m = 0$. Let $\hat{\alpha}, \hat{\beta}, i',$ and j' be given as those in Case 2 of the proof of Theorem 3. Similarly, according to Lemma 9, we have $f_j - f_i - f_{j'} + f_{i'} \equiv q/2 \pmod{q}$. Thus,

$$\xi^{c_{n,j}^e - c_{n,i}^p} + \xi^{c_{n,j'}^e - c_{n,i'}^p} = 0.$$

Case 3: In this case, we have $i_{\pi_\alpha(1)} = j_{\pi_\alpha(1)}$ for all $\alpha \in \{1, 2, \dots, k\}$ and $i_m = j_m = 1$. Let $\hat{\alpha}, \hat{\beta}, i',$ and j' be given as those in Case 3 of the proof of the Theorem 3. According to Lemma 9, we also have $f_j - f_i - f_{j'} + f_{i'} \equiv q/2 \pmod{q}$. Then,

$$\xi^{c_{n,j}^e - c_{n,i}^p} + \xi^{c_{n,j'}^e - c_{n,i'}^p} = 0.$$

Case 4: Suppose $i_m \neq j_m$, according to Lemma 8, we have

$$\sum_{n=0}^{2^{k+1}-1} \xi^{c_{n,j}^e - c_{n,i}^p} = 0.$$

Combining these four cases, we can obtain that equation (8) holds for $0 < u < L-1$. Now, it remains to show that for $u = 0$,

$$\begin{aligned} \rho(C^e, C^p; 0) &= \sum_{n=0}^{2^{k+1}-1} \rho(\mathbf{c}_n^e, \mathbf{c}_n^p; 0) \\ &= \sum_{n=0}^{2^{k+1}-1} \sum_{i=0}^{L-1} \xi^{c_{n,i}^e - c_{n,i}^p} = 0. \end{aligned}$$

For any integer $n < 2^{k+1}$, we have

$$\mathbf{c}_n^e - \mathbf{c}_n^p \equiv \frac{q}{2} \mathbf{d} \pmod{q}$$

where

$$\begin{aligned} \mathbf{d} &= (e_1 \oplus p_1) \mathbf{x}_{\pi_1(m_1)} \oplus (e_2 \oplus p_2) \mathbf{x}_{\pi_2(m_2)} \oplus \dots \\ &\quad \oplus (e_{k'} \oplus p_{k'}) \mathbf{x}_{\pi_{k'}(m_{k'})} \end{aligned}$$

where $(e_1, e_2, \dots, e_{k'})$ and $(p_1, p_2, \dots, p_{k'})$ are the binary representations of e and p , respectively. We have $\mathbf{d} \neq \mathbf{0}$

since $e \neq p$. Let s denote the minimum integer in the set $\{\pi_1(m_1), \pi_2(m_2), \dots, \pi_{k'}(m_{k'})\}$. Due to the condition (6), we have $\{\pi_1(m_1), \pi_2(m_2), \dots, \pi_{k'}(m_{k'})\} \subseteq \{1, 2, \dots, t\}$. Hence, we have $s \leq t$ implying $2^s | (2^{m-1} + 2^t)$. According to Lemma 10, we can obtain that the Hamming weight of \mathbf{d} is $L/2$. Hence, for $i = 0, 1, \dots, 2^{m-1} + 2^t - 1$, there are $L/2$ pairs $(c_{n,i}^e, c_{n,i}^p)$ such that $\xi^{c_{n,i}^e - c_{n,i}^p} = \xi^{q/2} = -1$ and $L/2$ pairs $(c_{n,i}^e, c_{n,i}^p)$ such that $\xi^{c_{n,i}^e - c_{n,i}^p} = \xi^0 = 1$, so we have

$$\rho(\mathbf{c}_n^e, \mathbf{c}_n^p; 0) = \sum_{i=0}^{L-1} \xi^{c_{n,i}^e - c_{n,i}^p} = 0$$

which completes the proof. \blacksquare

APPENDIX C

PROOF OF COROLLARY 5

Corollary 5 can be proved by following the similar logics in Theorem 4. Therefore, we only need to consider some other cases in this proof. We will show that any two distinct sets C^e and C^p where $0 \leq e \neq p \leq 2^k - 1$ satisfy the ideal cross-correlation property. For $0 < u < L-1$ where $L = 2^{m-1} + 2^t$, we also have to show that the equation (8) is satisfied. We also consider four cases as follows.

Case 1: In this case, we have $i_{\pi_\alpha(1)} \neq j_{\pi_\alpha(1)}$ for some $\alpha \in \{1, 2, \dots, k\}$. Same as Case 1 of the proof of Theorem 4, we can obtain

$$\sum_{n=0}^{2^{k+1}-1} \xi^{c_{n,i+u}^e - c_{n,i}^p} = 0.$$

Case 2: In this case, we have $i_{\pi_\alpha(1)} = j_{\pi_\alpha(1)}$ for all $\alpha \in \{1, 2, \dots, k\}$ and $i_m = j_m = 0$. Let $\hat{\alpha}, \hat{\beta}, i',$ and j' be given as those in Case 2 of the proof of Theorem 3. Here, $y_i = i_m i_{\pi_k(\beta')} + (1 \oplus i_m) i_{\pi_k(m_k)} = i_{\pi_k(m_k)}$. Also, $y_{i'} = i'_{\pi_k(m_k)}$, $y_j = j_{\pi_k(m_k)}$, and $y_{j'} = j'_{\pi_k(m_k)}$. According to Lemma 9, we have $f_j - f_i - f_{j'} + f_{i'} \equiv q/2 \pmod{q}$. Then,

$$\begin{aligned} &c_{n,j}^e - c_{n,i}^p - c_{n,j'}^e + c_{n,i'}^p \\ &= \left(f_j + \frac{q}{2} \sum_{\alpha=1}^{k-1} e_\alpha j_{\pi_\alpha(m_\alpha)} + \frac{q}{2} e_k y_j \right) \\ &\quad - \left(f_i + \frac{q}{2} \sum_{\alpha=1}^{k-1} p_\alpha i_{\pi_\alpha(m_\alpha)} + \frac{q}{2} p_k y_i \right) \\ &\quad - \left(f_{j'} + \frac{q}{2} \sum_{\alpha=1}^{k-1} e_\alpha j'_{\pi_\alpha(m_\alpha)} + \frac{q}{2} e_k y_{j'} \right) \\ &\quad + \left(f_{i'} + \frac{q}{2} \sum_{\alpha=1}^{k-1} p_\alpha i'_{\pi_\alpha(m_\alpha)} + \frac{q}{2} p_k y_{i'} \right) \\ &= \left(f_j + \frac{q}{2} \sum_{\alpha=1}^{k-1} e_\alpha j_{\pi_\alpha(m_\alpha)} + \frac{q}{2} e_k j_{\pi_k(m_k)} \right) \\ &\quad - \left(f_i + \frac{q}{2} \sum_{\alpha=1}^{k-1} p_\alpha i_{\pi_\alpha(m_\alpha)} + \frac{q}{2} p_k i_{\pi_k(m_k)} \right) \\ &\quad - \left(f_{j'} + \frac{q}{2} \sum_{\alpha=1}^{k-1} e_\alpha j'_{\pi_\alpha(m_\alpha)} + \frac{q}{2} e_k j'_{\pi_k(m_k)} \right) \end{aligned}$$

$$\begin{aligned}
& + \left(f_{i'} + \frac{q}{2} \sum_{\alpha=1}^{k-1} p_{\alpha} i'_{\pi_{\alpha}(m_{\alpha})} + \frac{q}{2} p_k i'_{\pi_k(m_k)} \right) \\
& = f_j - f_i - f_{j'} + f_{i'} \\
& \equiv \frac{q}{2} \pmod{q}
\end{aligned}$$

since $i_{\pi_{\alpha}(m_{\alpha})} = i'_{\pi_{\alpha}(m_{\alpha})}$ and $j_{\pi_{\alpha}(m_{\alpha})} = j'_{\pi_{\alpha}(m_{\alpha})}$ for $\alpha = 1, 2, \dots, k$. This is because i and i' differ in only one position $\pi_{\hat{\alpha}(\hat{\beta}-1)}$ and $\hat{\beta} - 1 < m_{\hat{\alpha}}$; so do j and j' . Hence, we can obtain

$$\xi^{c_{n,i+u}^e - c_{n,i}^p} + \xi^{c_{n,i'+u}^e - c_{n,i'}^p} = 0.$$

Case 3: In this case, we have $i_{\pi_{\alpha}(1)} = j_{\pi_{\alpha}(1)}$ for all $\alpha \in \{1, 2, \dots, k\}$ and $i_m = j_m = 1$. Different to Case 2, we have $y_i = i_{\pi_k(\beta')}$, $y_{i'} = i'_{\pi_k(\beta')}$, $y_j = j_{\pi_k(\beta')}$, and $y_{j'} = j'_{\pi_k(\beta')}$ instead. Let $\hat{\alpha}, \hat{\beta}, i'$ and j' be given as those in Case 4 of the proof of Theorem 3. According to Lemma 9, we have $f_j - f_i - f_{j'} + f_{i'} \equiv q/2 \pmod{q}$. Thus,

$$\begin{aligned}
& c_{n,j}^e - c_{n,i}^p - c_{n,j'}^e + c_{n,i'}^p \\
& = \left(f_j + \frac{q}{2} \sum_{\alpha=1}^{k-1} e_{\alpha} j_{\pi_{\alpha}(m_{\alpha})} + \frac{q}{2} e_k j_{\pi_k(\beta')} \right) \\
& - \left(f_i + \frac{q}{2} \sum_{\alpha=1}^{k-1} p_{\alpha} i_{\pi_{\alpha}(m_{\alpha})} + \frac{q}{2} p_k i_{\pi_k(\beta')} \right) \\
& - \left(f_{j'} + \frac{q}{2} \sum_{\alpha=1}^{k-1} e_{\alpha} j'_{\pi_{\alpha}(m_{\alpha})} + \frac{q}{2} e_k j'_{\pi_k(\beta')} \right) \\
& + \left(f_{i'} + \frac{q}{2} \sum_{\alpha=1}^{k-1} p_{\alpha} i'_{\pi_{\alpha}(m_{\alpha})} + \frac{q}{2} p_k i'_{\pi_k(\beta')} \right) \\
& = f_j - f_i - f_{j'} + f_{i'} \\
& + \frac{q}{2} e'_{\alpha} \left(j_{\pi_k(\beta')} - j'_{\pi_k(\beta')} \right) - \frac{q}{2} p'_{\alpha} \left(i_{\pi_k(\beta')} - i'_{\pi_k(\beta')} \right) \quad (9)
\end{aligned}$$

where $i_{\pi_{\alpha}(m_{\alpha})} = i'_{\pi_{\alpha}(m_{\alpha})}$ and $j_{\pi_{\alpha}(m_{\alpha})} = j'_{\pi_{\alpha}(m_{\alpha})}$ for $\alpha = 1, 2, \dots, k$ as mentioned in Case 2. If $\pi_k(\beta') = \pi_k(m_k)$, then (9) can be further expressed as $c_{n,j}^e - c_{n,i}^p - c_{n,j'}^e + c_{n,i'}^p = f_j - f_i - f_{j'} + f_{i'} \equiv q/2 \pmod{q}$. In another case, if $\pi_k(\beta') \neq \pi_k(m_k)$ which implies $\beta' \neq m_k$, then we shall have $\pi_{\hat{\alpha}}(\hat{\beta}) < \pi_k(\beta' + 1)$. Otherwise, $i_s = j_s$ for $s = 1, 2, \dots, t$ since

$$\begin{aligned}
& \{\pi_1(1), \pi_1(2), \dots, \pi_1(m_1), \pi_2(1), \dots, \pi_2(m_2), \\
& \pi_3(1), \dots, \pi_k(1), \dots, \pi_k(\beta')\} = \{1, 2, \dots, t\}
\end{aligned}$$

and $\pi_k(\beta') < \pi_{\hat{\alpha}}(\hat{\beta})$. According to Lemma 6, we have $j \geq i + 2^t \geq 2^{m-1} + 2^t$ which contradicts the assumption that $j < L = 2^{m-1} + 2^t$. Therefore, $\pi_{\hat{\alpha}}(\hat{\beta}) < \pi_k(\beta' + 1)$ implying $\pi_{\hat{\alpha}}(\hat{\beta} - 1) < \pi_k(\beta')$. Thus, we have $i_{\pi_k(\beta')} = i'_{\pi_k(\beta')}$ and $j_{\pi_k(\beta')} = j'_{\pi_k(\beta')}$. So we can obtain

$$\begin{aligned}
c_{n,j}^e - c_{n,i}^p - c_{n,j'}^e + c_{n,i'}^p & = f_j - f_i - f_{j'} + f_{i'} \\
& \equiv \frac{q}{2} \pmod{q}
\end{aligned}$$

implying

$$\xi^{c_{n,i+u}^e - c_{n,i}^p} + \xi^{c_{n,i'+u}^e - c_{n,i'}^p} = 0.$$

Case 4: In this case, we assume $i_m \neq j_m$. Same as Case 4 of the proof of Theorem 4, we have

$$\sum_{n=0}^{2^{k+1}-1} \xi^{c_{n,i+u}^e - c_{n,i}^p} = 0.$$

Combining these four cases, we can obtain that equation (8) holds for $0 < u < L - 1$. Similarly, it can also be obtained that equation (8) holds for $0 < u < -L + 1$. Now, it remains to show that

$$\begin{aligned}
\rho(C^e, C^p; 0) & = \sum_{n=0}^{2^{k+1}-1} \rho(\mathbf{c}_n^e, \mathbf{c}_n^p; 0) \\
& = \sum_{n=0}^{2^{k+1}-1} \sum_{i=0}^{L-1} \xi^{c_{n,i}^e - c_{n,i}^p} = 0.
\end{aligned}$$

For any integer $n < 2^{k+1}$, we have

$$\mathbf{c}_n^e - \mathbf{c}_n^p \equiv \frac{q}{2} \mathbf{d} \pmod{q}$$

where

$$\begin{aligned}
\mathbf{d} & = (e_1 \oplus p_1) \mathbf{x}_{\pi_1(m_1)} \oplus (e_2 \oplus p_2) \mathbf{x}_{\pi_2(m_2)} \oplus \dots \\
& \oplus (e_{k-1} \oplus p_{k-1}) \mathbf{x}_{\pi_{k-1}(m_{k-1})} \\
& \oplus (e_k \oplus p_k) (\mathbf{x}_m \mathbf{x}_{\pi_k(\beta')} + (\mathbf{1} \oplus \mathbf{x}_m) \mathbf{x}_{\pi_k(m_k)})
\end{aligned}$$

where (e_1, e_2, \dots, e_k) and (p_1, p_2, \dots, p_k) are the binary representations of e and p , respectively. We have $\mathbf{d} \neq \mathbf{0}$ since $e \neq p$. Here, we consider two cases to show that the Hamming weight of \mathbf{d} is $L/2$.

(i). For $e_k \oplus p_k = 0$, we have

$$\begin{aligned}
\mathbf{d} & = (e_1 \oplus p_1) \mathbf{x}_{\pi_1(m_1)} \oplus (e_2 \oplus p_2) \mathbf{x}_{\pi_2(m_2)} \oplus \dots \\
& \oplus (e_{k-1} \oplus p_{k-1}) \mathbf{x}_{\pi_{k-1}(m_{k-1})}.
\end{aligned}$$

Then, arguing as in Case 4 of the proof of Theorem 4, we obtain the same result that \mathbf{d} has Hamming weight $L/2$.

(ii). For $e_k \oplus p_k = 1$, we have

$$\begin{aligned}
\mathbf{d} & = (e_1 \oplus p_1) \mathbf{x}_{\pi_1(m_1)} \oplus (e_2 \oplus p_2) \mathbf{x}_{\pi_2(m_2)} \oplus \dots \\
& \oplus (e_{k-1} \oplus p_{k-1}) \mathbf{x}_{\pi_{k-1}(m_{k-1})} \\
& \oplus \mathbf{x}_m \mathbf{x}_{\pi_k(\beta')} \oplus (\mathbf{1} \oplus \mathbf{x}_m) \mathbf{x}_{\pi_k(m_k)}.
\end{aligned}$$

We express the sequence \mathbf{d} into two parts, i.e., $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2)$ where $\mathbf{d}_1 = (d_0, d_1, \dots, d_{2^m-1})$ and $\mathbf{d}_2 = (d_{2^m}, d_{2^m+1}, \dots, d_{2^m+2^t-1})$. For any positive integer $0 \leq i \leq 2^m - 1$, we have $i_m = 0$ implying

$$\begin{aligned}
\mathbf{d}_1 & = (e_1 \oplus p_1) \mathbf{x}_{\pi_1(m_1)} \oplus (e_2 \oplus p_2) \mathbf{x}_{\pi_2(m_2)} \oplus \dots \\
& \oplus (e_{\alpha'-1} \oplus p_{\alpha'-1}) \mathbf{x}_{\pi_{\alpha'-1}(m_{\alpha'-1})} \oplus \mathbf{x}_{\pi_k(m_k)}.
\end{aligned}$$

We have $\pi_{\alpha}(m_{\alpha}) \leq m - 1$ for $\alpha = 1, 2, \dots, k$, so $2^{\pi_{\alpha}(m_{\alpha})} | 2^{m-1}$. According to Lemma 10, it can be obtained that the Hamming weight of \mathbf{d}_1 is half length, i.e., 2^{m-2} .

For any positive integer i with $2^m \leq i \leq 2^m + 2^t - 1$, we have $i_m = 1$. Therefore, \mathbf{d}_2 can be expressed as

$$\begin{aligned}
\mathbf{d}_2 & = (e_1 \oplus p_1) \mathbf{x}_{\pi_1(m_1)} \oplus (e_2 \oplus p_2) \mathbf{x}_{\pi_2(m_2)} \oplus \dots \\
& \oplus (e_{k-1} \oplus p_{k-1}) \mathbf{x}_{\pi_{k-1}(m_{k-1})} \oplus \mathbf{x}_{\pi_k(\beta')}.
\end{aligned}$$

Since $\{\pi_1(m_1), \pi_2(m_2), \dots, \pi_{k-1}(m_{k-1}), \pi_k(\beta')\} \in \{1, 2, \dots, t\}$, we have $\pi_k(\beta') \leq t$ and $\pi_{\alpha}(m_{\alpha}) \leq t$ for

$\alpha = 1, 2, \dots, k - 1$. Thus, it can be obtained that the Hamming weight of d_2 is $2^t/2 = 2^{t-1}$ according to Lemma 10.

Hence, the Hamming weight of d is $2^{m-2} + 2^{t-1} = L/2$. There are $L/2$ pairs $(c_{n,i}^e, c_{n,i}^p)$ such that $\xi^{c_{n,i}^e - c_{n,i}^p} = \xi^{q/2} = -1$ and $L/2$ pairs $(c_{n,i}^e, c_{n,i}^p)$ such that $\xi^{c_{n,i}^e - c_{n,i}^p} = \xi^0 = 1$, so we have

$$\rho(c_n^e, c_n^p; 0) = \sum_{i=0}^{2^m-1} \xi^{c_{n,i}^e - c_{n,i}^p} = 0$$

which completes the proof.

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