

Pseudo-Boolean Functions for Optimal Z-Complementary Code Sets with Flexible Lengths

Palash Sarkar, Sudhan Majhi, and Zilong Liu

Abstract—This paper aims to construct optimal Z-complementary code set (ZCCS) with non-power-of-two (NPT) lengths to enable interference-free multicarrier code-division multiple access (MC-CDMA) systems. The existing ZCCSs with NPT lengths, which are constructed from generalized Boolean functions (GBFs), are sub-optimal only with respect to the set size upper bound. For the first time in the literature, we advocate the use of pseudo-Boolean functions (PBFs) (each of which transforms a number of binary variables to a real number as a natural generalization of GBF) for direct constructions of optimal ZCCSs with NPT lengths.

Index Terms—Multicarrier code-division multiple access (MC-CDMA), generalized Boolean function (GBF), pseudo-Boolean function (PBF), Z-complementary code set (ZCCS), zero correlation zone (ZCZ)

I. INTRODUCTION

MULTICARRIER code-division multiple access (MC-CDMA) has been one of the most widely adopted wireless techniques in many communication systems/standards owing to its efficient fast Fourier transform (FFT) based implementation, resilience against intersymbol interference, and high spectral efficiency [1]. That being said, MC-CDMA may suffer from multiple-access interference (MAI) [2] and multipath interference (MPI) [3]. A promising way to address both MAI and MPI is to adopt proper spreading codes, such as complete complementary codes (CCC) [4] and Z-complementary code sets (ZCCSs) [5]. This paper focuses on efficient construction of ZCCSs with a new tool, called pseudo-Boolean functions (PBFs), to enable interference-free quasi-synchronous MC-CDMA systems.

In 2007, Z-complementary pairs (ZCPs) were introduced by Fan *et al.* [6] to overcome the limitation on the lengths of Golay complementary pairs (GCPs) [7], [8]. A ZCP refers to a pair of sequences of same length N having zero aperiodic auto-correlation sums for all time shifts τ satisfying $0 < |\tau| < Z$, where Z is called zero-correlation zone (ZCZ) width. When $Z = N$, the resultant sequence pair reduces to a GCP. In the literature, there are direct constructions of GCPs and ZCPs with the aid of generalized Boolean functions (GBFs) [9]–[11]. The idea of ZCPs introduced in [6] was generalized to ZCCS by Feng *et al.* in [12]. A ZCCS refers to a set of K codes, each of which consists of M constituent sequences of identical length L , having ideal aperiodic auto- and cross-correlation properties inside the ZCZ width and

TABLE I
COMPARISON OF THE PROPOSED CONSTRUCTION WITH [5], [21]–[24], [27]

ZCCS	Method	Length (N)	$\lfloor \frac{N}{Z} \rfloor$	Constraints	Optimality
[5]	Direct	2^m	≥ 2	$m \geq 2$	Optimal
[23]	Direct	2^m	≥ 2	$m \geq 2$	Optimal
[24]	Direct	$2^m + 2$	$= 1$	$m \geq 4$	Sub-optimal
[21]	Direct	2^m	≥ 2	$m \geq 2$	Optimal
[22]	Direct	2^m	≥ 2	$m \geq 2$	Optimal
[22]	Direct	$2^m + 2^h$	≥ 1	$m > 0, 0 < h \leq m$	Non-optimal
[27]	Indirect	L	≥ 2	$L \geq 1$	Optimal
<i>Theorem 1</i>	Direct	$p2^m$	≥ 2	$m \geq 2$, prime p	Optimal

satisfying the theoretical upper bound: $K \leq M \lfloor N/Z \rfloor$ [13]. When $Z = N$, the set is called a mutually orthogonal Golay complementary sets (MOGCSs) [4], which refers to collection of complementary codes (CCs) [14]–[16] with ideal cross-correlation properties. A set of CCCs is known as a MOGCSs with the equality $K = M$ [17]–[20]. The theoretical upper bound shows that an optimal ZCCS has larger set size as compared to CCCs provided $\lfloor \frac{N}{Z} \rfloor \geq 2$. Recently, several GBFs based constructions of optimal ZCCSs with power-of-two lengths have been reported in [5], [21]–[23]. In the recent literature, two direct constructions of ZCCSs with NPT lengths can be found in [24] and [22], which produces sub-optimal ZCCS with $\lfloor \frac{N}{Z} \rfloor = 1$ and non-optimal ZCCSs for NPT lengths with $\lfloor \frac{N}{Z} \rfloor < 1$, respectively. To the best of our knowledge, the construction of optimal ZCCSs of NPT lengths with $\lfloor \frac{N}{Z} \rfloor \geq 2$, based on GBFs remains open. Other methods which are dependent on the existence of special sequences, known as indirect constructions [11], to construct ZCCSs can be found in [25]–[27]. The indirect constructions heavily rely on a series of sequence operations which may not be feasible for rapid hardware generation, especially, when the sequence lengths are large [5].

It is noted that the MAI in MC-CDMA system can be mitigated using zero-correlation properties of a ZCCS provided that all the received multiuser signals are roughly synchronous within the ZCZ width [19]. In addition to their applications in MC-CDMA [18], [19], [27], ZCCSs have also been employed as optimal training sequences in multiple-input multiple-output (MIMO) communications [28], [29]. The limitation on the set size of CCCs and the unavailability of optimal ZCCSs with NPT lengths using direct constructions in the existing literature are a major motivation of this work. Specifically, for the first time in the literature, we propose to use PBFs for direct construction of optimal ZCCS of lengths $p2^m$, where p is a prime number and m is a positive integer. A PBF [30] refers to an arbitrary mapping of the set of binary m -tuples to

Palash Sarkar and Sudhan Majhi are with the Department of EE, IIT Patna, India, e-mail: { palash.pma15, smajhi }@iitp.ac.in.

Zilong Liu is with the School of CSEE, University of Essex, UK, e-mail:zilong.liu@essex.ac.uk.

real numbers. Being a natural generalization of GBFs, PBFs are also suitable for rapid hardware generation of sequences. A detailed comparison of the proposed construction with [5], [21]–[24], [27] is given in TABLE I.

II. PRELIMINARY

In this section, we present some basic definitions and lemmas to be used in the proposed construction. Let $\mathbf{y}_1 = (y_{1,0}, y_{1,1}, \dots, y_{1,N-1})$ and $\mathbf{y}_2 = (y_{2,0}, y_{2,1}, \dots, y_{2,N-1})$ denote a pair of sequences with complex components. For an integer τ , define [5]

$$\theta(\mathbf{y}_1, \mathbf{y}_2)(\tau) = \begin{cases} \sum_{i=0}^{N-1-\tau} y_{1,i+\tau} y_{2,i}^*, & 0 \leq \tau < N, \\ \sum_{i=0}^{N+\tau-1} y_{1,i} y_{2,i-\tau}^*, & -N < \tau < 0, \\ 0, & \text{otherwise,} \end{cases} \quad (1)$$

The functions $\theta(\mathbf{y}_1, \mathbf{y}_2)$ and $\theta(\mathbf{y}_1, \mathbf{y}_1)$ are called the aperiodic cross-correlation function (ACCF) between \mathbf{y}_1 and \mathbf{y}_2 , and the aperiodic auto-correlation function (AACF) of \mathbf{y}_1 , respectively. Let $\mathcal{S} = \{\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{K-1}\}$ be a set of K codes or ordered sets defined as

$$\mathcal{S}_\mu = (\mathbf{s}_0^\mu, \mathbf{s}_1^\mu, \dots, \mathbf{s}_{M-1}^\mu), \quad (2)$$

where \mathbf{s}_ν^μ ($0 \leq \nu \leq M-1, 0 \leq \mu \leq K-1$) is the ν -th element which we assume is a complex-valued sequence of length N in \mathcal{S}_μ . For $\mathcal{S}_{\mu_1}, \mathcal{S}_{\mu_2} \in \mathcal{S}$ ($0 \leq \mu_1, \mu_2 \leq K-1$), the ACCF between \mathcal{S}_{μ_1} and \mathcal{S}_{μ_2} is defined as

$$\theta(\mathcal{S}_{\mu_1}, \mathcal{S}_{\mu_2})(\tau) = \sum_{\nu=0}^{M-1} \theta(\mathbf{s}_\nu^{\mu_1}, \mathbf{s}_\nu^{\mu_2})(\tau). \quad (3)$$

Definition 1 ([5]): Code set \mathcal{S} is called a ZCCS if

$$\theta(\mathcal{S}_{\mu_1}, \mathcal{S}_{\mu_2})(\tau) = \begin{cases} MN, & \tau = 0, \mu_1 = \mu_2, \\ 0, & 0 < |\tau| < Z, \mu_1 = \mu_2, \\ 0, & |\tau| < Z, \mu_1 \neq \mu_2, \end{cases} \quad (4)$$

where Z is called ZCZ width. We denote a ZCCS with the parameters K, N, M , and Z by the notation (K, Z) -ZCCS $_M^N$. For $K = M$ and $Z = N$, a (K, Z) -ZCCS $_M^N$ is called a set of CCCs and we denote it by (K, K, N) -CCC.

We call a (K, Z) -ZCCS $_M^N$ optimal if it achieves the equality in the theoretical upper-bound, given by $K \leq M \lfloor \frac{N}{Z} \rfloor$ [13].

A. Generalized Boolean Functions (GBFs)

Let x_0, x_1, \dots, x_{m-1} denote m variables which take values from \mathbb{Z}_2 . A monomial of degree i ($0 \leq i \leq m$) is defined as the product of any i distinct variables among x_0, x_1, \dots, x_{m-1} . Let us assume that \mathcal{A}_i denotes the set of all monomials of degree i , where

$$\mathcal{A}_i = \{x_0^{r_0} x_1^{r_1} \cdots x_{m-1}^{r_{m-1}} : r_0 + r_1 + \cdots + r_{m-1} = i, (r_0, r_1, \dots, r_{m-1}) \in \mathbb{Z}_2^m\}. \quad (5)$$

A function $f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q$ is said to be a GBF if it can uniquely be expressed as a linear combination of the monomials in \mathcal{A}_m , where the coefficient of each monomial is drawn from \mathbb{Z}_q , where \mathbb{Z}_q denotes the set of integers modulo q . The

highest degree monomial with non-zero coefficient present in the expression of f determine the order of f . As an example, $2x_0x_1 + x_1 + 1$ is a second order GBF of two variables x_0 and x_1 over \mathbb{Z}_3 . We denote the graph of a second-order GBF f by $G(f)$ [14]. It contains m vertices which are denoted by the m variables of f . The edges in the $G(f)$ are determined by the second-degree monomials present in the expression of f with non-zero coefficients. There is an edge of weight w between the vertices x_α and x_β of $G(f)$ if the expression of f contains the term $wx_\alpha x_\beta$. Let $\psi(f)$ denotes the complex-valued sequence corresponding to a GBF f and it is defined as [14], $\psi(f) = (\omega_q^{f_0}, \omega_q^{f_1}, \dots, \omega_q^{f_{2^m-1}})$, where ω_q denotes $\exp(2\pi\sqrt{-1}/q)$, $f_r = f(r_0, r_1, \dots, r_{m-1})$, $(r_0, r_1, \dots, r_{m-1})$ is the binary vector representation of integer r ($r = \sum_{\alpha=0}^{m-1} r_\alpha 2^\alpha$), and q denotes an even number, no

less than 2. We denote by $\bar{x} = 1 - x$ the binary complement of $x \in \{0, 1\}$. For any given GBF f in m variables, we denote the function $f(1 - x_0, 1 - x_1, \dots, 1 - x_{m-1})$ or $f(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{m-1})$ by \tilde{f} . Let $\mathcal{C} = (g_1, g_2, \dots, g_M)$ be an ordered set of M GBFs. We define the code $\psi(\mathcal{C})$ corresponding to \mathcal{C} as $\psi(\mathcal{C}) = (\psi(g_1), \psi(g_2), \dots, \psi(g_M))$.

Lemma 1: (Construction of CCC [4])

Let $f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q$ be a second-order GBF. Let us assume that $G(f)$ contains the vertices $x_{j_0}, x_{j_1}, \dots, x_{j_{k-1}}$ such a way that after performing a deletion operation on those vertices, the resulting graph reduces to a path. Let the edges in the path have identical weight of $\frac{q}{2}$ and $\mathbf{t} = (t_0, t_1, \dots, t_{k-1})$ be the binary representation of the integer t . Define the CC, C_t to be

$$\left\{ f + \frac{q}{2} \left((\mathbf{d} + \mathbf{t}) \cdot \mathbf{x} + dx_\gamma \right) : \mathbf{d} \in \{0, 1\}^k, d \in \{0, 1\} \right\}, \quad (6)$$

and \bar{C}_t to be

$$\left\{ \tilde{f} + \frac{q}{2} \left((\mathbf{d} + \mathbf{t}) \cdot \bar{\mathbf{x}} + \bar{d}x_\gamma \right) : \mathbf{d} \in \{0, 1\}^k, d \in \{0, 1\} \right\}, \quad (7)$$

where $(\cdot) \cdot (\cdot)$ denotes the dot product between two real-valued vector (\cdot) and (\cdot) , γ is the label of either end vertex in the path, $\mathbf{x} = (x_{j_0}, x_{j_1}, \dots, x_{j_{k-1}})$, $\bar{\mathbf{x}} = (1 - x_{j_0}, 1 - x_{j_1}, \dots, 1 - x_{j_{k-1}})$, and $\mathbf{d} = (d_0, d_1, \dots, d_{k-1})$. Then $\{\psi(C_t), \psi^*(\bar{C}_t) : 0 \leq t < 2^k\}$ forms $(2^{k+1}, 2^{k+1}, 2^m)$ -CCC, where $\psi^*(\cdot)$ denotes the complex conjugate of $\psi(\cdot)$.

B. Pseudo-Boolean Functions (PBFs)

A function $F : \{0, 1\}^m \rightarrow \mathbb{R}$ is said to be a PBF if it can be uniquely expressed as a linear combination of monomials in \mathcal{A}_m with the coefficients drawn from \mathbb{R} , where \mathbb{R} denotes the set of real numbers. Therefore, PBFs are a natural generalization of GBFs [30]. As an example, $\frac{2}{3}x_0x_1 + x_0 + 1$ is a second-order PBF of two variables x_0 and x_1 but not a GBF. Let $f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q$ be a GBF of the variables x_0, x_1, \dots, x_{m-1} . Let us assume that p denotes a prime number and define the following PBFs with the help of the GBF f :

$$\begin{aligned} F^\lambda &= f + \frac{\lambda q}{p} (x_m + 2x_{m+1} + \cdots + 2^{s-1}x_{m+s-1}), \\ G^\lambda &= \tilde{f} + \frac{\lambda q}{p} (x_m + 2x_{m+1} + \cdots + 2^{s-1}x_{m+s-1}), \end{aligned} \quad (8)$$

where $s \in \mathbb{Z}^+$ which denotes the set of all positive integers, $2 \leq p < 2^{s+1}$, and $\lambda = 0, 1, \dots, p-1$. From (8), it is clear that F^λ and G^λ are PBFs of $m+s$ variables $x_0, x_1, \dots, x_{m+s-1}$. From (8), it can be observed that the PBFs F^λ and G^λ reduce to \mathbb{Z}_q -valued GBFs if p divides q .

III. PROPOSED CONSTRUCTION OF Z-COMPLEMENTARY CODE SET

In this section, we shall present our proposed construction of ZCCS using PBFs. To this end, we first present a lemma which will be used in our proposed construction.

Lemma 2: ([31]) Let λ and λ' be two non-negative integers, where $0 \leq \lambda \neq \lambda' < p$, p is a prime number as defined in

Section-II. Then $\sum_{\alpha=0}^{p-1} \omega_p^{(\lambda-\lambda')\alpha} = 0$.

For $0 \leq t < 2^k$ and $0 \leq \lambda < p$, we define the following sets of PBFs:

$$U_t^\lambda = \left\{ F^\lambda + \frac{q}{2} \left((\mathbf{d} + \mathbf{t}) \cdot \mathbf{x} + dx_\gamma \right) : \mathbf{d} \in \{0, 1\}^k, d \in \{0, 1\} \right\}, \quad (9)$$

and

$$V_t^\lambda = \left\{ G^\lambda + \frac{q}{2} \left((\mathbf{d} + \mathbf{t}) \cdot \bar{\mathbf{x}} + \bar{d}x_\gamma \right) : \mathbf{d} \in \{0, 1\}^k, d \in \{0, 1\} \right\}. \quad (10)$$

Let us assume that $f^{\mathbf{d}, \mathbf{t}, d} = f + \frac{q}{2} \left((\mathbf{d} + \mathbf{t}) \cdot \mathbf{x} + dx_\gamma \right)$, $g^{\mathbf{d}, \mathbf{t}, d} = \tilde{f} + \frac{q}{2} \left((\mathbf{d} + \mathbf{t}) \cdot \bar{\mathbf{x}} + \bar{d}x_\gamma \right)$, in Lemma 1. We also assume $F^{\mathbf{d}, \mathbf{t}, d, \lambda} = F^\lambda + \frac{q}{2} \left((\mathbf{d} + \mathbf{t}) \cdot \mathbf{x} + dx_\gamma \right)$, in (9), and $G^{\mathbf{d}, \mathbf{t}, d, \lambda} = G^\lambda + \frac{q}{2} \left((\mathbf{d} + \mathbf{t}) \cdot \bar{\mathbf{x}} + \bar{d}x_\gamma \right)$, in (10). As per our assumption, for any choice of $\mathbf{d}, \mathbf{t} \in \{0, 1\}^k$, and $d \in \{0, 1\}$, the functions $f^{\mathbf{d}, \mathbf{t}, d}$ and $g^{\mathbf{d}, \mathbf{t}, d}$ are \mathbb{Z}_q -valued GBFs of m variables. For any choice of $\mathbf{d}, \mathbf{t} \in \{0, 1\}^k$, $d \in \{0, 1\}$, and $\lambda \in \{0, 1, \dots, p-1\}$, the functions $F^{\mathbf{d}, \mathbf{t}, d, \lambda}$ and $G^{\mathbf{d}, \mathbf{t}, d, \lambda}$ are PBFs of $m+s$ variables. We define $\psi(F^{\mathbf{d}, \mathbf{t}, d, \lambda})$, the complex-valued sequence corresponding to $F^{\mathbf{d}, \mathbf{t}, d, \lambda}$, as

$$\psi(F^{\mathbf{d}, \mathbf{t}, d, \lambda}) = (\omega_q^{F_0^{\mathbf{d}, \mathbf{t}, d, \lambda}}, \omega_q^{F_1^{\mathbf{d}, \mathbf{t}, d, \lambda}}, \dots, \omega_q^{F_{2^{m+s}-1}^{\mathbf{d}, \mathbf{t}, d, \lambda}}), \quad (11)$$

where $F_{r'}^{\mathbf{d}, \mathbf{t}, d, \lambda} = F^{\mathbf{d}, \mathbf{t}, d, \lambda}(r_0, r_1, \dots, r_{m+s-1})$, $r' = m+s-1$

$\sum_{\alpha=0} r_\alpha 2^\alpha$. The r' -th component of $\psi(F^{\mathbf{d}, \mathbf{t}, d, \lambda})$ is given by

$$\begin{aligned} F_{r'}^{\mathbf{d}, \mathbf{t}, d, \lambda} &= \omega_q^{f^{\mathbf{d}, \mathbf{t}, d}(r_0, r_1, \dots, r_{m-1}) + \frac{q\lambda}{p}(r_m + 2r_{m+1} + \dots + 2^{s-1}r_{m+s-1})} \\ &= \omega_q^{f_r^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(r_m + 2r_{m+1} + \dots + 2^{s-1}r_{m+s-1})}. \end{aligned} \quad (12)$$

From (12), it can be observed that $\omega_q^{F_{r'}^{\mathbf{d}, \mathbf{t}, d, \lambda}}$ is a root of the polynomial: $z^\delta - 1$, where $\delta = \text{lcm}(p, q)$, denotes a positive integer given by the least common multiple (lcm) of p and q . Therefore, the components of $\psi(F^{\mathbf{d}, \mathbf{t}, d, \lambda})$ are given by the roots of the polynomial: $z^\delta - 1$. From (11) and (12), we have

$$\begin{aligned} \psi(F^{\mathbf{d}, \mathbf{t}, d, \lambda}) &= \underbrace{(\omega_q^{f_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(0)}, \omega_q^{f_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(0)}, \dots, \omega_q^{f_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(0)})}_{\omega_q^{f_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \omega_q^{f_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots, \omega_q^{f_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots} \\ &\quad \underbrace{(\omega_q^{f_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(2^s-1)}, \omega_q^{f_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(2^s-1)}, \dots, \omega_q^{f_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(2^s-1)})}_{\omega_q^{f_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \omega_q^{f_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots, \omega_q^{f_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots} \end{aligned} \quad (13)$$

Let us also define $\psi_{2^{m+s}-p2^m}(F^{\mathbf{d}, \mathbf{t}, d, \lambda})$ which is defined to be obtained from $\psi(F^{\mathbf{d}, \mathbf{t}, d, \lambda})$ by removing its last $2^{m+s} - p2^m$ components.

$$\begin{aligned} &\psi_{2^{m+s}-p2^m}(F^{\mathbf{d}, \mathbf{t}, d, \lambda}) \\ &= \underbrace{(\omega_q^{f_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(0)}, \omega_q^{f_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(0)}, \dots, \omega_q^{f_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(0)})}_{\omega_q^{f_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \omega_q^{f_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots, \omega_q^{f_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots} \\ &\quad \underbrace{(\omega_q^{f_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(p-1)}, \omega_q^{f_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(p-1)}, \dots, \omega_q^{f_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(p-1)})}_{\omega_q^{f_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \omega_q^{f_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots, \omega_q^{f_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots} \end{aligned} \quad (14)$$

Similarly, we can also obtain $\psi_{2^{m+s}-p2^m}(G^{\mathbf{d}, \mathbf{t}, d, \lambda})$ as

$$\begin{aligned} &\psi_{2^{m+s}-p2^m}(G^{\mathbf{d}, \mathbf{t}, d, \lambda}) \\ &= \underbrace{(\omega_q^{g_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(0)}, \omega_q^{g_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(0)}, \dots, \omega_q^{g_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(0)})}_{\omega_q^{g_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \omega_q^{g_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots, \omega_q^{g_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots} \\ &\quad \underbrace{(\omega_q^{g_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(p-1)}, \omega_q^{g_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(p-1)}, \dots, \omega_q^{g_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(p-1)})}_{\omega_q^{g_0^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \omega_q^{g_1^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots, \omega_q^{g_{2^m-1}^{\mathbf{d}, \mathbf{t}, d}} \omega_p^{\lambda(1)}, \dots} \end{aligned} \quad (15)$$

Theorem 1: Let $f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q^m$ be a GBF as defined in Lemma 1. Then the set of codes

$$\left\{ \psi_{2^{m+s}-p2^m}(U_t^\lambda), \psi_{2^{m+s}-p2^m}^*(V_t^\lambda) : 0 \leq t < 2^k, 0 \leq \lambda < p \right\}, \quad (16)$$

forms $(p2^{k+1}, 2^m)$ -ZCCS $_{p2^{k+1}}^{p2^m}$.

Proof: In (13), (14), and (15), each of the parentheses below a complex-valued sequence contains 2^m components of the complex-valued sequence. It can be observed that the 2^m components in the i -th parentheses of $\psi_{2^{m+s}-p2^m}(F^{\mathbf{d}, \mathbf{t}, d, \lambda})$ and $\psi_{2^{m+s}-p2^m}(G^{\mathbf{d}, \mathbf{t}, d, \lambda})$ represent the complex-valued sequences $\omega_p^{\lambda(i-1)}\psi(f^{\mathbf{d}, \mathbf{t}})$ and $\omega_p^{\lambda(i-1)}\psi(g^{\mathbf{d}, \mathbf{t}})$, respectively, where $i = 1, 2, \dots, p$. Using (9), (14), Lemma 1, and Lemma 2, the ACCF between $\psi_{2^{m+s}-p2^m}(U_t^\lambda)$ and $\psi_{2^{m+s}-p2^m}(U_{t'}^{\lambda'})$ for $\tau = 0$ can be derived as follows:

$$\begin{aligned} &\theta(\psi_{2^{m+s}-p2^m}(U_t^\lambda), \psi_{2^{m+s}-p2^m}(U_{t'}^{\lambda'}))(0) \\ &= \sum_{\mathbf{d}, d} \theta(\psi_{2^{m+s}-p2^m}(F^{\mathbf{d}, \mathbf{t}, d, \lambda}), \psi_{2^{m+s}-p2^m}(F^{\mathbf{d}, \mathbf{t}', d, \lambda'}))(0) \\ &= \sum_{\mathbf{d}, d} \theta(\psi(f^{\mathbf{d}, \mathbf{t}, d}), \psi(f^{\mathbf{d}, \mathbf{t}', d}))(0) \sum_{\alpha=0}^{p-1} \omega_p^{(\lambda-\lambda')\alpha} \\ &= \theta(\psi(C_t), \psi(C_{t'}))(0) \sum_{\alpha=0}^{p-1} \omega_p^{(\lambda-\lambda')\alpha} \\ &= \begin{cases} p2^{m+k+1}, & t = t', \lambda = \lambda', \\ 0, & t = t', \lambda \neq \lambda', \\ 0, & t \neq t', \lambda = \lambda', \\ 0, & t \neq t', \lambda \neq \lambda'. \end{cases} \end{aligned} \quad (17)$$

Again, Using (9), (14), and *Lemma 1*, the ACCF between $\psi_{2^{m+s}-p2^m}(U_t^\lambda)$ and $\psi_{2^{m+s}-p2^m}(U_{t'}^{\lambda'})$ for $0 < |\tau| < 2^m$ can be derived as

$$\begin{aligned} & \theta(\psi_{2^{m+s}-p2^m}(U_t^\lambda), \psi_{2^{m+s}-p2^m}(U_{t'}^{\lambda'}))(\tau) \\ &= \theta(\psi(C_t), \psi(C_{t'}))(\tau) \sum_{\alpha=0}^{p-1} \omega_p^{(\lambda-\lambda')\alpha} \\ & \quad + \theta(\psi(C_t), \psi(C_{t'}))(\tau - 2^m) \sum_{\alpha=0}^{p-2} \omega_p^{\lambda(\alpha+1)-\lambda'\alpha}. \end{aligned} \quad (18)$$

From *Lemma 1*, we have

$$\theta(\psi(C_t), \psi(C_{t'}))(\tau) = 0, \quad 0 < |\tau| < 2^m. \quad (19)$$

From (18) and (19), we have

$$\theta(\psi_{2^{m+s}-p2^m}(U_t^\lambda), \psi_{2^{m+s}-p2^m}(U_{t'}^{\lambda'}))(\tau) = 0, \quad 0 < |\tau| < 2^m. \quad (20)$$

From (17) and (20), we have

$$\begin{aligned} & \theta(\psi_{2^{m+s}-p2^m}(U_t^\lambda), \psi_{2^{m+s}-p2^m}(U_{t'}^{\lambda'}))(\tau) \\ &= \begin{cases} p2^{m+k+1}, & t = t', \lambda = \lambda', \tau = 0, \\ 0, & t = t', \lambda \neq \lambda', 0 < |\tau| < 2^m, \\ 0, & t \neq t', \lambda = \lambda', 0 < |\tau| < 2^m, \\ 0, & t \neq t', \lambda \neq \lambda', 0 < |\tau| < 2^m. \end{cases} \end{aligned} \quad (21)$$

Similarly, it can be shown that

$$\begin{aligned} & \theta(\psi_{2^{m+s}-p2^m}^*(V_t^\lambda), \psi_{2^{m+s}-p2^m}^*(V_{t'}^{\lambda'}))(\tau) \\ &= \begin{cases} p2^{m+k+1}, & t = t', \lambda = \lambda', \tau = 0, \\ 0, & t = t', \lambda \neq \lambda', 0 < |\tau| < 2^m, \\ 0, & t \neq t', \lambda = \lambda', 0 < |\tau| < 2^m, \\ 0, & t \neq t', \lambda \neq \lambda', 0 < |\tau| < 2^m. \end{cases} \end{aligned} \quad (22)$$

From *Lemma 1*, (9), (10), (14), and (15), the ACCF between $\psi_{2^{m+s}-p2^m}(U_t^\lambda)$ and $\psi_{2^{m+s}-p2^m}^*(V_{t'}^{\lambda'})$ for $\tau = 0$ can be derived as

$$\begin{aligned} & \theta(\psi_{2^{m+s}-p2^m}(U_t^\lambda), \psi_{2^{m+s}-p2^m}^*(V_{t'}^{\lambda'}))(0) \\ &= \theta(\psi(C_t), \psi^*(\bar{C}_{t'}))(0) \sum_{\alpha=0}^{p-1} \omega_p^{(\lambda+\lambda')\alpha}. \end{aligned} \quad (23)$$

From *Lemma 1*, we have

$$\theta(\psi(C_t), \psi^*(\bar{C}_{t'}))(0) = 0. \quad (24)$$

From (23) and (24), we have

$$\theta(\psi_{2^{m+s}-p2^m}(U_t^\lambda), \psi_{2^{m+s}-p2^m}^*(V_{t'}^{\lambda'}))(0) = 0. \quad (25)$$

From *Lemma 1*, (9), (10), (14), (15), and (24), the ACCF between $\psi_{2^{m+s}-p2^m}(U_t^\lambda)$ and $\psi_{2^{m+s}-p2^m}^*(V_{t'}^{\lambda'})$ for $0 < |\tau| < 2^m$ can be derived as

$$\begin{aligned} & \theta(\psi_{2^{m+s}-p2^m}(U_t^\lambda), \psi_{2^{m+s}-p2^m}^*(V_{t'}^{\lambda'}))(\tau) \\ &= \theta(\psi(C_t), \psi^*(\bar{C}_{t'}))(\tau) \sum_{\alpha=0}^{p-1} \omega_p^{(\lambda+\lambda')\alpha} \\ & \quad + \theta(\psi(C_t), \psi^*(\bar{C}_{t'}))(\tau - 2^m) \sum_{\alpha=0}^{p-2} \omega_p^{\lambda(\alpha+1)+\lambda'\alpha} \\ &= 0. \end{aligned} \quad (26)$$

From (25) and (26), we have

$$\theta(\psi_{2^{m+s}-p2^m}(U_t^\lambda), \psi_{2^{m+s}-p2^m}^*(V_{t'}^{\lambda'}))(\tau) = 0, \quad |\tau| < 2^m. \quad (27)$$

The obtained results in (20), (22), and (27) show that the following set of codes

$$\left\{ \psi_{2^{m+s}-p2^m}(U_t^\lambda), \psi_{2^{m+s}-p2^m}^*(V_{t'}^{\lambda'}) : 0 \leq t < 2^k, 0 \leq \lambda < p \right\}$$

forms $(p2^{k+1}, 2^m)$ -ZCCS $_{2^{k+1}}^{p2^m}$. ■

The proposed $(p2^{k+1}, 2^m)$ -ZCCS $_{2^{k+1}}^{p2^m}$ is optimal as it satisfies the equality $K = M \lfloor \frac{N}{Z} \rfloor$.

Remark 1: For $p = 2$, $\delta = \text{lcm}(p, q) = q$, and the PBFs F^λ and G^λ become GBFs of $m+s$ variables over \mathbb{Z}_q . For the same value of p , from *Theorem 1*, we obtain $(2^{k+2}, 2^m)$ -ZCCS $_{2^{k+1}}^{2^{m+1}}$ which is optimal and the components of each codewords from a code in $(2^{k+2}, 2^m)$ -ZCCS $_{2^{k+1}}^{2^{m+1}}$ are drawn from the roots of the polynomial: $z^q - 1$. Therefore, the proposed construction also generates ZCCSs of length in the form of power-of-two over the ring \mathbb{Z}_q .

Let us illustrate the Theorem 1 with the following example:

Example 1: Let us assume that $q = 2$, $p = 3$, $m = 3$, $k = 1$ and $s = 2$. Let us take the GBF $f : \{0, 1\}^3 \rightarrow \mathbb{Z}_2$ as follows: $f = x_1x_2$, where $G(f|_{x_0=0})$ and $G(f|_{x_0=1})$ give a path with x_2 as one of the end vertices. From (8), we have

$$F^\lambda = x_1x_2 + \frac{2\lambda}{3}(x_3 + 2x_4), \quad G^\lambda = \bar{x}_1\bar{x}_2 + \frac{2\lambda}{3}(x_3 + 2x_4), \quad (28)$$

where $\lambda = 0, 1, 2$. From (9) and (10), we have

$$\begin{aligned} U_t^\lambda &= \{F^\lambda + d_0x_0 + t_0x_0 + dx_2 : d_0, d \in \{0, 1\}\} \\ V_t^\lambda &= \{G^\lambda + d_0\bar{x}_0 + t_0\bar{x}_0 + \bar{d}x_2 : d_0, d \in \{0, 1\}\}, \end{aligned} \quad (29)$$

where (t_0) is the binary vector representation of t . Therefore, $\{\psi_8(U_t^\lambda), \psi_8^*(V_t^{\lambda'}) : 0 \leq t \leq 1, 0 \leq \lambda \leq 2\}$ forms $(12, 8)$ -ZCCS $_4^{24}$ which also optimal. The components of each code word from a code in $(12, 8)$ -ZCCS $_4^{24}$ are drawn from the roots of the polynomial: $z^\delta - 1$, where $\delta = \text{lcm}(p, q) = \text{lcm}(2, 3) = 6$.

Remark 2: From (14) and (15), we see that $\psi_{2^{m+s}-p2^m}(U_t^\lambda)$ and $\psi_{2^{m+s}-p2^m}^*(V_{t'}^{\lambda'})$ can also be expressed as the concatenation of $\omega_p^{\lambda(i-1)}\psi(C_t)$ and $\omega_p^{-\lambda(i-1)}\psi^*(\bar{C}_{t'})$, respectively, where $i = 1, 2, \dots, p$. Therefore, the proposed PBF generators establish a link between the proposed direct construction and the indirect constructions of ZCCSs which are obtained by performing cocatenation operation on the CCCs from [4].

IV. CONCLUSIONS

In this paper, we have developed a direct construction of optimal ZCCS with NPT lengths. Unlike the current state-of-the-art which can only generate sub-optimal ZCCSs with NPT lengths, the novelty of this work stems from the use of PBFs.

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