Semigroup Forum
https://doi.org/10.1007/s00233-022-10256-8

SHORT NOTE

Orientation-preserving and orientation-reversing mappings: a new description

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Received: 8 December 2020 / Accepted: 15 January 2022

Abstract
We characterise the respective semigroups of mappings that preserve, or that preserve or reverse orientation of a finite cycle, in terms of their actions on oriented triples and oriented quadruples. This leads to a proof that the latter semigroup coincides with the semigroup of all mappings that preserve intersections of chords on the corresponding circle.

Keywords Orientation-preserving · Transformation semigroup

1 Orientation-preserving and orientation-reversing mappings on a cycle

This section presents definitions and some known results; it is based mainly on Catarino and Higgins [1] and also on McAlister [4]. Let $[n]$ denote the set $\{0, 1, \dots, n-1\}$. Consider a sequence $S = (a_0, a_1, \dots, a_{t-1})$ drawn from $[n]$. A *cyclic variant* of S is a sequence $(a_{i+1}, a_{i+2}, \dots, a_i)$, where $0 \le i < t$, and subscripts are taken modulo t. We say S is *cyclic* if there is at most one subscript i such that $a_i > a_{i+1}$ (subscripts taken modulo t). Equivalently, S is cyclic if and only if at least one of its cyclic variants is non-decreasing $a_{i+1} \le a_{i+2} \le \cdots \le a_i$. We say S is *anti-cyclic* if there is at most one subscript i such that $a_i < a_{i+1}$. Equivalently, S is anti-cyclic if and only if at least one of its cyclic variants is non-increasing $a_{i+1} \ge a_{i+2} \ge \cdots \ge a_i$. We shall say that S is *oriented* if S is cyclic or S is anti-cyclic. Orientation, that is, being cyclic or anti-cyclic, is a property inherited by subsequences of an oriented sequence. We say that S is *uniquely oriented* if S is cyclic and not anti-cyclic or anti-cyclic and not cyclic. Let $|S|$

Communicated by Victoria Gould.

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Published online: 19 February 2022

denote the number of distinct elements of S, that is, $|S| = |\{a_0, \ldots, a_{t-1}\}|$. It is easy to see that if $|S| \le 2$ then S is either both cyclic and anti-cyclic (for example, $S = (0, 1)$) or neither (for example, $S = (0, 1, 0, 1)$). As we will show below, non-trivial and useful examples of oriented sequences are those with $|S| = 3$ and $|S| = 4$.

We introduce an equivalence relation $\sim$ on the set of uniquely oriented sequences whereby $S \sim T$ if S and T have the same orientation. For example, if S' is a cyclic variant of S, then $S \sim S'$.

We write $\mathcal{T}_n$ for the full transformation semigroup on $[n]$. For $\alpha \in \mathcal{T}_n$ and a sequence $S = (a_0, a_1, \ldots, a_{t-1})$ with entries drawn from $[n]$ we shall write $S\alpha$ for the sequence $(a_0\alpha, a_1\alpha, \ldots, a_{t-1}\alpha)$.

Definition 1 A mapping $\alpha \in \mathcal{T}_n$ is *orientation-preserving* (resp. *orientation-reversing*) if the sequence $(0, 1, \ldots, n-1)\alpha$ is cyclic (resp. anti-cyclic); these names are justified by Lemma 2. The collection of all orientation-preserving (resp. orientation-reversing) mappings in $\mathcal{T}_n$ is denoted by $\mathcal{OP}_n$ (resp. $\mathcal{OR}_n$), while $\mathcal{P}_n$ is defined as $\mathcal{P}_n = \mathcal{OP}_n \cup \mathcal{OR}_n$.

If S is a sequence, we denote the reversed sequence by S^R.

Lemma 2 *Let $\alpha \in \mathcal{OP}_n$ (resp. $\alpha \in \mathcal{OR}_n$) and let S be an oriented sequence such that $|S\alpha| \ge 3$. Then $S\alpha \sim S$ (resp. $S\alpha \sim S^R$).*

Since a sequence S is cyclic (resp. anti-cyclic) if and only if S^R is anti-cyclic (resp. cyclic), it follows that we may equally define $\mathcal{OP}_n$ as the set of all members of $\mathcal{T}_n$ that map anti-cyclic sequences to anti-cyclic sequences.

That $\mathcal{OP}_n$ is a semigroup is now easily proved (Lemma 2.1 of [1]). However $\mathcal{OR}_n$ is not a semigroup: $\mathcal{OR}_n \cdot \mathcal{OR}_n = \mathcal{OP}_n$, $\mathcal{OR}_n \cdot \mathcal{OP}_n = \mathcal{OP}_n \cdot \mathcal{OR}_n = \mathcal{OR}_n$, and $\mathcal{OP}_n \cap \mathcal{OR}_n = \{\alpha \in \mathcal{P}_n : |\mathrm{im}(\alpha)| \le 2\}$. It follows that $\mathcal{P}_n$ is a semigroup.

2 Describing $\mathcal{OP}_n$ and $\mathcal{OR}_n$ with oriented triples

This result was stated but not proved in [3,Proposition 1.1], so here it is proved for the first time.

Theorem 3 *Let $\alpha \in \mathcal{T}_n$. Then $\alpha \in \mathcal{OP}_n$ (resp. $\alpha \in \mathcal{OR}_n$) if and only if for every triple $S = (i, j, k)$ of members of $[n]$, $S\alpha$ has the same (resp. the opposite) orientation as S.*

Proof ($\Rightarrow$) Follows from inheritance of orientation by subsequences.

($\Leftarrow$) First, assume $\alpha \notin \mathcal{OP}_n$. We find a cyclic triple (r, s, t) such that $(r\alpha, s\alpha, t\alpha)$ is not cyclic. Since α is not orientation-preserving there exist distinct integers i, j such that $i\alpha > (i+1)\alpha$ and $j\alpha > (j+1)\alpha$.

Case 1 $i\alpha \ne j\alpha$. Without loss we assume that $i\alpha > j\alpha$. Put $r = i, s = j, t = j+1$. Then (r, s, t) is cyclic but since $i\alpha > j\alpha > (j+1)\alpha$, $(r\alpha, s\alpha, t\alpha)$ is anti-cyclic (and not cyclic as the three entries are pairwise distinct).

Case 2 $(i+1)\alpha \ne (j+1)\alpha$. Without loss we assume that $(i+1)\alpha > (j+1)\alpha$. Put $r = i, s = i+1, t = j+1$. Then (r, s, t) is cyclic but since $i\alpha > (i+1)\alpha > (j+1)\alpha$, $(r\alpha, s\alpha, t\alpha)$ is anti-cyclic.

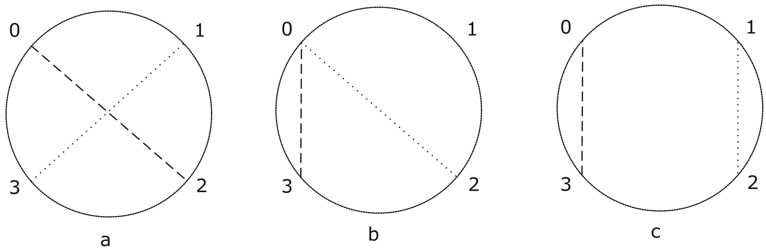

Fig. 1 Diagram **a** shows two intersecting chords 13 and 02 as a dotted line and dashed line. If we apply a mapping $0 \mapsto 0$, $1 \mapsto 0$, $2 \mapsto 3$, $3 \mapsto 2$, the rearranged chords still intersect, see diagram **b**. If, instead, we apply a mapping $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto 3$, $3 \mapsto 2$, the rearranged chords do not intersect, see diagram **c**

Case 3 $i\alpha = j\alpha$ and $(i+1)\alpha = (j+1)\alpha$. Since $|\mathrm{im}\alpha| \geq 3$ there exists k such that $k\alpha \notin \{i\alpha, (i+1)\alpha\}$. It follows that the members of the list $i, i+1, k, j, j+1$ are pairwise distinct and, by interchanging the symbols i and j if necessary, we may assume that $(i, i+1, k, j, j+1)$ is cyclic.

(1) Suppose that $k\alpha > i\alpha$. Put $r = k, s = i, t = i+1$. Then (r, s, t) is cyclic but since $k\alpha > i\alpha > (i+1)\alpha$, $(r\alpha, s\alpha, t\alpha)$ is anti-cyclic.
(2) Suppose $i\alpha > k\alpha > (i+1)\alpha$. Put $r = i, s = k, t = j+1$. Then (r, s, t) is cyclic but $(r\alpha, s\alpha, t\alpha)$ is anti-cyclic.
(3) Suppose $(i+1)\alpha > k\alpha$. Put $r = i, s = i+1, t = k$. Then (r, s, t) is cyclic but since $i\alpha > (i+1)\alpha > k\alpha$, $(r\alpha, s\alpha, t\alpha)$ is anti-cyclic.

This completes a proof of the reverse implication in the case where $\alpha \notin \mathcal{OP}_n$. For the case where $\alpha \notin \mathcal{OR}_n$ let γ be the permutation of order reversal: $i\gamma = n-1-i$ $(0 \leq i \leq n-1)$. Then $\gamma \in \mathcal{OR}_n$ so that $\alpha\gamma \notin \mathcal{OP}_n$ as otherwise $\alpha\gamma^2 = \alpha \in \mathcal{OR}_n$. We then apply the previous argument to conclude that there exist three pairwise distinct integers forming a cyclic triple (r, s, t) such that $(r\alpha\gamma, s\alpha\gamma, t\alpha\gamma)$ is anti-cyclic. But then $(r\alpha\gamma^2, s\alpha\gamma^2, t\alpha\gamma^2) = (r\alpha, s\alpha, t\alpha)$ is cyclic, and noting this completes the proof. □

3 Describing $\mathcal{P}_n$ with oriented quadruples and with chord intersections

We introduce a new characterization of $\mathcal{P}_n$ in terms of the preservation of a single geometric property. This stems from the study of *Gauss diagrams* in knot theory, which consist of a circle and chords. The paper [2] first described Gauss diagrams satisfying an important property known as being *realizable*, and now we know that one can describe realizable Gauss diagrams by only using the information stating, for every pair of chords, whether or not they intersect [5].

Definition 4 Consider the integers $0, 1, \ldots, n-1$ positioned clockwise around a circle. A mapping $\alpha \in \mathcal{T}_n$ has the *chord property* if whenever the chords ac and bd intersect $(a, b, c, d \in [n])$ then so do the chords $a\alpha c\alpha$ and $b\alpha d\alpha$. (See Figure 1.)

This definition makes allowance for one point chords. If there is repetition among the a, b, c, d due to the chords having a common endpoint, then chord intersection is preserved by any mapping α by virtue of the image of that common endpoint. We shall denote the clockwise arc of the circle R that runs from a to c as $\overrightarrow{ac}$.

Lemma 5 *Let $a, b, c, d \in [n]$ be four points on the circumference of a circle on which the members of $[n]$ are placed clockwise. The chords ac and bd meet if and only if the quadruple $S = (a, b, c, d)$ is oriented.*

Proof Suppose the chords ac and bd meet. Suppose that our four points are distinct. If b lies on the arc $\overrightarrow{ac}$ then d lies on $\overrightarrow{ca}$ and (a, b, c, d) is oriented clockwise on the circle, whence (a, b, c, d) is cyclic. Alternatively b lies on the arc $\overrightarrow{ca}$, and d does not, in which case d lies on $\overrightarrow{ac}$ and (a, d, c, b) is cyclic. But then $(a, d, c, b)^R = (b, c, d, a) \sim (a, b, c, d)$ is anti-cyclic. In either event, S is oriented.

On the other hand, suppose that S has a repeated entry. If $a = c$, then since ac meets bd, it follows that $a = b = c$ or $a = d = c$, in which case S is oriented; the same conclusion follows if $b = d$. Otherwise two cyclically adjacent entries in S, for instance a and b, are equal, in which case it also follows that S is oriented.

Conversely, if (a, b, c, d) is cyclic, then b lies on $\overrightarrow{ac}$ and d lies on $\overrightarrow{ca}$ so that the chords ac and bd meet. On the other hand if (a, b, c, d) is anti-cyclic then $(a, b, c, d)^R = (d, c, b, a)$ is cyclic and so the chord $db = bd$ meets the chord $ca = ac$, as required. □

Proposition 6 *A mapping $\alpha \in \mathcal{T}_n$ has the chord property if and only if for every oriented quadruple $S = (a, b, c, d)$, the image sequence $S\alpha$ is also oriented.*

Proof Suppose that α has the chord property and let $S = (a, b, c, d)$ be an oriented sequence. Let $S\alpha = (A, B, C, D)$. By Lemma 5, the chords ac and bd intersect, and hence by hypothesis so do the chords AC and BD. Again by Lemma 5 it follows that $S\alpha$ is oriented.

Conversely, suppose that the image of every oriented quadruple under α is oriented and let chords ac and bd meet. By Lemma 5 (a, b, c, d) is oriented and so by hypothesis so is (A, B, C, D). Again by Lemma 5, it follows that chords AC and BD meet, whence we conclude that α has the chord property. □

Theorem 7 (a) *$\alpha \in \mathcal{P}_n$ if and only if for every oriented quadruple $S = (a, b, c, d)$, $S\alpha$ is also oriented.*

(b) *α has the chord property if and only if $\alpha \in \mathcal{P}_n$.*

Proof (b) follows from Proposition 6 together with (a).

(a) ($\Rightarrow$) Follows from inheritance of orientation by subsequences.

(a) ($\Leftarrow$) We again argue the contrapositive, like in Theorem 3, so suppose that $\alpha \notin \mathcal{P}_n$. Let $m = \min(\mathrm{im}(\alpha))$. Choose i such that $i\alpha = m$ but $i\alpha < (i + 1)\alpha$. Consider the cyclic sequence $I : i + 1, i + 2, \ldots, i + n - 2$ and let j be the first listed member of I such that $j\alpha > (j + 1)\alpha$; since $\alpha \notin \mathcal{OP}_n$, j occurs in I, as there are at least 2 integers t such that $t\alpha > (t + 1)\alpha$. Similarly let $M = \max(\mathrm{im}(\alpha))$. Choose i' such that $i'\alpha = M$ but $i'\alpha > (i' + 1)\alpha$. Consider the cyclic sequence $I' : i' + 1, i' + 2, \ldots, i' + n - 2$ and let j' be the first listed member of I' such that $j'\alpha < (j' + 1)\alpha$; since $\alpha \notin \mathcal{OR}_n$, j' occurs in I' as there are at least 2 integers t such that $t\alpha < (t + 1)\alpha$. Note that $(i + 1)\alpha \leq j\alpha$ and $(i' + 1)\alpha \geq j'\alpha$.

Case 1 $(i+1)\alpha = j\alpha$ or $(i'+1)\alpha = j'\alpha$. First suppose $(i+1)\alpha = j\alpha$, whence by definition of j and the assumption that $(i+1)\alpha = j\alpha$ it follows that

$$m = i\alpha < (i+1)\alpha = \cdots = j\alpha > (j+1)\alpha. \tag{3.1}$$

Consider the subsequence J of I given by $J : j+1, j+2, \ldots, i+n-2$. Since $\alpha \notin \mathcal{OR}_n$, there exists k in J such that $k\alpha < (k+1)\alpha$, (for otherwise the first such k is $k = i+n-1$, but then $k+1 = i$, contradicting $(k+1)\alpha > k\alpha \geq i\alpha$). By choosing the first listed such k in J we have by (3.1) that $j\alpha > k\alpha$ and so

$$m = i\alpha < (i+1)\alpha = j\alpha > k\alpha < (k+1)\alpha > i\alpha = m. \tag{3.2}$$

From (3.2) we infer that $S = (i, i+1, k, k+1)$ is a cyclic quadruple of pairwise distinct integers but, by (3.2), $S\alpha$ is not oriented.

Similarly if $(i'+1)\alpha = j'\alpha$ the dual argument obtained by reversing all inequalities and interchanging $\mathcal{OP}_n$ and $\mathcal{OR}_n$ throughout yields a cyclic sequence $S' = (i', i'+1, k', k'+1)$ such that $S'\alpha$ is not oriented as

$$M = i'\alpha > (i'+1)\alpha = j'\alpha < k'\alpha > (k'+1)\alpha < i'\alpha = M.$$

Case 2: $(i+1)\alpha < j\alpha$ and $(i'+1)\alpha > j'\alpha$. From the given inequalities we now have

$$m = i\alpha < (i+1)\alpha < M = i'\alpha > (i'+1)\alpha > m = i\alpha,$$

which implies that $T = (i, i+1, i', i'+1)$ is a cyclic sequence of pairwise distinct integers such that $T\alpha = (i\alpha, (i+1)\alpha, i'\alpha, (i'+1)\alpha)$ is not oriented. □

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