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On the Tits alternative for cyclically presented groups with length-four positive relators

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Abstract. We investigate the Tits alternative for cyclically presented groups with length-four positive relators in terms of a system of congruences (A), (B), (C) in the defining parameters, introduced by Bogley and Parker. Except for the case when (B) holds and neither (A) nor (C) hold, we show that the Tits alternative is satisfied; in the remaining case, we show that the Tits alternative is satisfied when the number of generators of the cyclic presentation is at most 20.

1 Introduction

The *cyclically presented group* $G_n(w)$ is the group defined by the *cyclic presentation*

$$P_n(w) = \langle x_0, \dots, x_{n-1} \mid w, \theta(w), \dots, \theta^{n-1}(w) \rangle,$$

where $w(x_0, \dots, x_{n-1})$ is a cyclically reduced word in the free group F_n of rank $n \geq 1$ with generators $x_0, \dots, x_{n-1}$ and $\theta\colon F_n \to F_n$ is the *shift automorphism* given by $\theta(x_i) = x_{i+1}$ for each $0 \leq i < n$ (subscripts mod n). In this article, we study cyclically presented groups $G_n(w)$, where w is a positive word of length four; that is, we study the groups $G_n(j,k,l)$ defined by the presentations

$$P_n(j,k,l) = \langle x_0, \dots, x_{n-1} \mid x_i x_{i+j} x_{i+k} x_{i+l} \ (0 \leq i < n) \rangle$$

($0 \leq j,k,l < n$, subscripts mod n, $n \geq 1$). These were first investigated by Bogley and Parker in [3] in terms of a system of congruences (A), (B), (C) and so-called primary and secondary divisors d, γ (defined below). They classify the finite groups $G_n(j,k,l)$ and (with two unresolved cases) classify the aspherical presentations $P_n(j,k,l)$. Here we investigate whether the Tits alternative is satisfied; that is, whether each group $G = G_n(j,k,l)$ either contains a non-abelian free subgroup or has a solvable subgroup of finite index. In many cases where we show G contains a non-abelian free subgroup, we show that G satisfies the

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stronger properties of being *large* (that is, it has a finite index subgroup that maps onto the free group of rank 2) or of being *SQ-universal* (that is, every countable group embeds in a quotient of G). Similar studies have been carried out for cyclically presented groups with positive relators of length three ([12, 19], with one infinite family of groups unresolved) and with non-positive relators of length three ([7], with precisely two groups unresolved). Largeness and the Tits alternative have been investigated for other classes of cyclically presented groups in [4, 24].

We prove the following, which shows that the Tits alternative is satisfied, except possibly in the case when both the primary and secondary divisors are equal to one and (B) holds and neither (A) nor (C) hold. We will write A, B, C as T or F according to whether the conditions are true or false.

Theorem A. *Let* $n \geq 1$, $0 \leq j, k, l < n$, $d = \gcd(n, j, k, l)$,

$$\gamma = \gcd(n, k-2j, l-2k+j, k-2l, j+l),$$

let $G = G_n(j,k,l)$, *and if* $l \equiv j+k \bmod n$, *set* $p = j$, *and if* $j \equiv l+k \bmod n$, *set* $p = -l$. *Suppose that if* $d = \gamma = 1$, *then* $(\mathrm{A},\mathrm{B},\mathrm{C}) \neq (\mathrm{F},\mathrm{T},\mathrm{F})$.

(a) *If* $d > 1$ *or* $\gamma > 1$, *then* G *is large.*

(b) *If* $d = \gamma = 1$, *then one of the following holds:*

  (i) $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{F},\mathrm{F},\mathrm{F})$ *or* $(\mathrm{T},\mathrm{F},\mathrm{F})$, *in which case* G *contains a non-abelian free subgroup;*

  (ii) $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{F},\mathrm{F},\mathrm{T})$, *in which case* $G \cong \mathbb{Z}_4$ *if* $(n,p)=1$ *and* $(n,2k)=1$, *and* G *is large otherwise;*

  (iii) $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{F},\mathrm{T},\mathrm{T})$, *in which case* $G \cong \mathbb{Z}_4$*;*

  (iv) $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{T},\mathrm{F},\mathrm{T})$, *in which case* G *is infinite and solvable if* $n = 2$ *and large otherwise;*

  (v) $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{T},\mathrm{T},\mathrm{F})$, *in which case* G *is finite and solvable;*

  (vi) $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{T},\mathrm{T},\mathrm{T})$ *and either* $n = 1$, *in which case* $G \cong \mathbb{Z}_4$, *or* $n = 4$, *in which case* $G \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

In particular, the Tits alternative is satisfied.

The existence of an unresolved case in terms of the system of congruences is consistent with the current state of knowledge for the Tits alternative for cyclically presented groups with length-three positive relators, where the Tits alternative is known to be satisfied except for the case where the congruence conditions (A), (B), (C), (D) of [12] take truth values F, F, F, T, respectively (see [12, 19] for further results concerning that unresolved case).

The case $d = \gamma = 1$ and $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{F},\mathrm{T},\mathrm{F})$ remains unresolved in general; we show that the Tits alternative also is satisfied in this case when $n \leq 20$.

Theorem B. *Suppose* $d = \gamma = 1$ *and* $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{F},\mathrm{T},\mathrm{F})$.

(a) *If* $n \leq 6$, *then* $G_n(j,k,l)$ *is finite.*

(b) *If* $7 \leq n \leq 20$, *then* $G_n(j,k,l)$ *is SQ-universal.*

2 Preliminaries

We first define the congruences (A), (B), (C) alluded to earlier (throughout this article, congruences are to be taken mod n, unless otherwise stated):

(A) $2k \equiv 0$ or $2j \equiv 2l$,

(B) $k \equiv 2j$ or $k \equiv 2l$ or $j + l \equiv 2k$ or $j + l \equiv 0$,

(C) $l \equiv j + k$ or $j \equiv l + k$.

Note that if (A) and (C) hold, then both congruences of (A) hold and both congruences of (C) hold; if, in addition, (B) holds, then all congruences of (B) hold. When (C) holds, it is convenient to set

$$p = \begin{cases} j & \text{if } l \equiv j + k, \\ -l & \text{if } j \equiv l + k. \end{cases} \tag{2.1}$$

Then, in the case $l = j + k$, we have $G_n(j,k,l) = G_n(x_0x_px_kx_{k+p})$, and in the case $j = l + k$, we have

$$G_n(j,k,l) = G_n(x_0x_{k-p}x_kx_{-p}) = G_n(x_px_kx_{k+p}x_0) = G_n(x_0x_px_kx_{k+p}),$$

by cyclically permuting the relators. Therefore, in each case,

$$G_n(j,k,l) = G_n(x_0x_px_kx_{k+p}).$$

As in [3], we define the *primary divisor* $d = \gcd(n, j, k, l)$ and the *secondary divisor*

$$\gamma = \gcd(n, k - 2j, l - 2k + j, k - 2l, j + l).$$

The shift automorphism θ of $G_n(w)$ satisfies $\theta^n = 1$, and the resulting $\mathbb{Z}_n$-action on $G_n(w)$ determines the *shift extension* $E_n(w) = G_n(w) \rtimes_\theta \mathbb{Z}_n$, which admits a presentation $E_n(W) = \langle a, x \mid a^n, W(x,a)\rangle$, where $W(x,a)$ is obtained from w

by rewriting it in terms of the substitutions $x_i = a^i x a^{-i}$ (see, for example, [17, Theorem 4]). In particular, the shift extension of $G_n(j,k,l)$ is the group

$$E_n(j,k,l) = \langle a, x \mid a^n, xa^j xa^{k-j} xa^{l-k} xa^{-l} \rangle. \tag{2.2}$$

In proving largeness and SQ-universality, we will use the following properties freely (see [21]). Every large group is SQ-universal. A group that maps homomorphically onto a large group (resp. SQ-universal group) is large (resp. SQ-universal) and if H is a finite-index subgroup of a group G, then H is large (resp. SQ-universal) if and only if G is large (resp. SQ-universal), so, in particular, $G_n(j,k,l)$ is large (resp. SQ-universal) if and only if $E_n(j,k,l)$ is large (resp. SQ-universal). A free product $H * K$ (with H, K non-trivial) is large if and only if either H and K have non-trivial finite homomorphic images $\bar{H}, \bar{K}$ such that $(|\bar{H}|, |\bar{K}|) \neq (2,2)$ or either H or K is large.

We first prove largeness when either the primary or secondary divisor is greater than one.

Lemma 2.1. *If the primary divisor $d > 1$, then $G_n(j,k,l)$ is large.*

Proof. The cyclically presented group $G = G_n(j,k,l) = G_n(x_0 x_j x_k x_l)$ splits as a free product of d copies of the cyclically presented group

$$H = G_{n/d}(x_0 x_{j/d} x_{k/d} x_{l/d})$$

(see [11]). There is an epimorphism ϕ of H onto $\mathbb{Z}_4 = \langle x \mid x^4 \rangle$ given by $\phi(x_i) = x$ for each $0 \leq i < n$. Therefore, there is an epimorphism of G onto the free product of d copies of $\mathbb{Z}_4$, and hence G is large. □

Lemma 2.2. *If the secondary divisor $\gamma > 1$, then $G_n(j,k,l)$ is large.*

Proof. Introducing the generator $u = xa^j$ and eliminating x shows that the shift extension $E_n(j,k,l)$, given at (2.2), has the alternative presentation

$$E_n(j,k,l) = \langle a, u \mid a^n, u^2 a^{k-2j} u a^{l-k-j} u a^{-l-j} \rangle.$$

The secondary divisor γ divides each of $n, k-2j, l-k-j, -l-j$, so by adjoining the relator a^γ, the group $E_n(j,k,l)$ maps onto $\langle a, u \mid a^\gamma, u^4 \rangle \cong \mathbb{Z}_4 * \mathbb{Z}_\gamma$. Therefore, $E_n(j,k,l)$ is large if $\gamma > 1$. □

Thus we may assume $d = \gamma = 1$. In Section 3, we build on prior results to show that the Tits alternative is satisfied in the cases where (A, B, C) = (F, F, F), (F, T, T), (T, F, F), (T, F, T), (T, T, F) or (T, T, T). In Section 4, we consider the case (F, F, T) and give the proof of Theorem A. In Section 5, we classify the groups $G_n(j,k,l)$ that have infinite abelianisation, and observe that if (B) holds and $\gamma = 1$, then the abelianisation is finite. We use this result in Section 6 where we consider the Tits alternative for the case (F, T, F) for $n \leq 20$ and prove Theorem B.

3 The cases (F, F, F), (F, T, T), (T, F, F), (T, F, T), (T, T, F), (T, T, T)

Lemma 3.1. *Suppose* (A, B, C) $=$ (T, T, T)*,* $d = \gamma = 1$*, and let* $G = G_n(j,k,l)$*. Then either* $n = 1$*, in which case* $G \cong \mathbb{Z}_4$*, or* $n = 4$*, in which case* $G \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$*.*

Proof. Since (A), (B), (C) all hold, all congruences of (A), all congruences of (B) and all congruences of (C) hold. Therefore, $2k \equiv 0$. Suppose first that $k \equiv 0$. Then $l \equiv j$ and $l \equiv -j$, so either $j \equiv k \equiv l \equiv 0$, in which case $d = 1$ implies $n = 1$ and then $G \cong \mathbb{Z}_4$, or n is even and $j \equiv l \equiv n/2$, in which case $1 = \gamma = n$, a contradiction. Suppose then $k \not\equiv 0$. Then $2k \equiv 0$ implies n is even and $k \equiv n/2$. Therefore, $j \equiv -l \equiv \pm n/4$, in which case $d = 1$ implies $n = 4$, so $G_n(j,k,l) = G_4(\pm 1, 2, \mp 1) \cong G_4(1,2,3)$ (by negating subscripts if necessary) which is the group $\langle x_0, x_1, x_2, x_3 \mid x_0x_1x_2x_3 \rangle \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. □

Theorem 7.2 of [3], together with the following technical proposition, deals with the case (F, T, T).

Proposition 3.2. *Suppose* (B) *and* (C) *hold, and let* p *be as defined at* (2.1)*. Then* $\gamma = 1$ *if and only if* $(n, 2k) = 1$ *and* $(n, p) = 1$*.*

Proof. By interchanging the roles of j, l, it suffices to consider the case $l \equiv j + k$. Then $\gamma = (n, k - 2j, k + 2j)$, which divides $(n, 2k)$, so if $(n, 2k) = 1$, we have $\gamma = 1$. For the converse, suppose $\gamma = 1$. Then, by checking each of the congruences in (B) in turn, we see $\gamma = (n, 2k) = (n, 4j)$, and hence $(n, 2k) = 1$ and $(n, j) = 1$. □

Corollary 3.3 (to [3, Theorem 7.2]). *Suppose* (A, B, C) $=$ (F, T, T)*. Then the following are equivalent:*

(a) $E_n(j,k,l) \cong \mathbb{Z}_{4n}$*;*

(b) $G \cong \mathbb{Z}_4$*;*

(c) G *is finite;*

(d) $\gamma = 1$.

Theorem 8.1 of [3] deals with the case (T, T, F).

Theorem 3.4 ([3, Theorem 8.1 (b), (c)]). *Suppose* $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{T},\mathrm{T},\mathrm{F})$ *and* $\gamma = 1$. *Then* $G_n(j,k,l)$ *is finite and solvable.*

We now turn to the cases (T, F, F), (F, F, F), (T, F, T). Recall that the *deficiency* of a presentation $P = \langle X \mid R \rangle$ is defined as $\mathrm{def}(P) = |X| - |R|$, and the *deficiency* of a group G, $\mathrm{def}(G)$, is the maximum of the deficiencies of all finite presentations defining G.

Lemma 3.5. *Suppose* $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{T},\mathrm{F},\mathrm{F})$ *or* $(\mathrm{F},\mathrm{F},\mathrm{F})$. *Then* $G_n(j,k,l)$ *contains a non-abelian free subgroup.*

Proof. Since (B) and (C) are false, [3, Lemma 6.2] implies that the cyclic presentation $P = P_n(j,k,l)$ satisfies the C(4)-T(4) small cancellation condition and is combinatorially aspherical, and then, by [3, Lemma 6.1 (a)], the group $G_n(j,k,l)$ is torsion-free. As discussed in [3, Section 2] (see [2, Section 3], [8, 22]), P is therefore topologically aspherical (in the sense that the second homotopy group of the presentation complex of P is trivial) if no relator of P is a proper power or is conjugate to any other relator or its inverse. Now if a relator $x_i x_{i+j} x_{i+k} x_{i+l}$ is a proper power, then $k \equiv 0$ and $j \equiv l$, and hence (C) holds, a contradiction. Since the relators of $P_n(j,k,l)$ are positive words, no relator is conjugate to the inverse of another relator. If a relator $x_i x_{i+j} x_{i+k} x_{i+l}$ is conjugate to a relator $x_t x_{t+j} x_{t+k} x_{t+l}$ $(0 \le i,t < n, i \ne t)$, then $x_i x_{i+j} x_{i+k} x_{i+l}$ is freely equal to $x_{t+j} x_{t+k} x_{t+l} x_t$ or $x_{t+k} x_{t+l} x_t x_{t+j}$ or $x_{t+l} x_t x_{t+j} x_{t+k}$, and by equating subscripts (mod n), we see that (C) must hold, a contradiction. Therefore, P is topologically aspherical, and hence, by [23, page 478], $\mathrm{def}(G) = 0$.

By [9], a group defined by C(4)-T(4) presentation contains a non-abelian free subgroup unless it is isomorphic to one of 8 groups, each of which either contains non-trivial torsion or has positive deficiency. Therefore, $G_n(j,k,l)$ contains a non-abelian free subgroup, as required. □

Lemma 3.6. *Suppose* $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{T},\mathrm{F},\mathrm{T})$, *and let* $G = G_n(j,k,l)$. *If* $n = 2$, *then* $G \cong \langle a, b \mid a^2 = b^2 \rangle$, *which is infinite and solvable, and* G *is large otherwise.*

Proof. If $n \le 2$, then $n = 2$ and $(j,k,l) = (0,1,1)$ or $(1,1,0)$, so

$$G = \langle x_0, x_1 \mid x_0^2 x_1^2 \rangle = \langle a, b \mid a^2 = b^2 \rangle,$$

the fundamental group of the Klein bottle, which is infinite and solvable. So assume $n \ge 3$. Since (C) holds, by [3, Lemma 5.2], the shift extension $E = E_n(j,k,l)$ has a presentation

$$E = \langle a, z \mid a^n, z^2 a^{k-2p} z^2 a^{-k-2p} \rangle,$$

where p is as defined at (2.1). Since (A) holds, either $2k \equiv 0$ or $2j \equiv 2l$, and in the latter case, (C) then implies $2k \equiv 0$. Therefore, $E = \langle a, z \mid a^n, (z^2a^{k-2p})^2 \rangle$, which maps homomorphically onto the generalised triangle group

$$\Delta = \langle a, z \mid a^n, z^7, (z^2a^{k-2p})^2 \rangle.$$

Since (B) does not hold, we have $k - 2p \not\equiv 0$, so the group Δ, and hence E, is large by [1, Theorem B]. □

4 The case (F, F, T)

In this section, we prove the following.

Theorem 4.1. *Suppose* (A, B, C) = (F, F, T)*, and let p be as defined at* (2.1)*. If $(n, p) = 1$ and $(n, 2k) = 1$, then $G_n(j, k, l) \cong \mathbb{Z}_4$; otherwise, $G_n(j, k, l)$ is large.*

We prove this via the following three lemmas.

Lemma 4.2. *Suppose* (A, B, C) = (F, F, T)*, and let p be as defined at* (2.1)*. If $G = G_n(j, k, l)$ is not large, then one of the following holds:*

(a) $(n, p) = 1$ *and* $(n, 2k) = 1$*, in which case* $G \cong \mathbb{Z}_4$*;*

(b) $G \cong G_n(1, J, J + 1)$*, where* $(n, 4) = 2$ *and* $(n, J) = 1$*;*

(c) $G \cong G_n(J, 1, J + 1)$*, where* $(n, 4) = 2$ *and* $(n, J) = 2$.

Proof. Suppose $G_n(j, k, l)$ is not large. Since (C) holds, [3, Lemma 5.2] implies that $E = E_n(j, k, l)$ has a presentation of the form

$$E = \langle a, z \mid a^n, z^2a^{k-2p}z^2a^{-k-2p} \rangle.$$

If (n, k) is even, then E maps onto $\langle a, z \mid a^2, z^4 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_4$, which is large, a contradiction. Therefore, (n, k) is odd. If $(n, 4p) > 2$, then (by adjoining the relator z^2) E maps onto $\langle a, z \mid a^{(n,4p)}, z^2 \rangle \cong \mathbb{Z}_{(n,4p)} * \mathbb{Z}_2$, which is large, a contradiction. Therefore, $(n, 4p) \leq 2$. Also, for any $q \geq 1$, the group E maps onto $\Delta(q) = \langle a, z \mid a^{(n,2k)}, (z^2a^{k-2p})^2, z^q \rangle$. If $k - 2p \equiv 0 \bmod (n, 2k)$, then the group $\Delta(4) \cong \mathbb{Z}_{(n,2k)} * \mathbb{Z}_4$, which is large if $(n, 2k) > 1$. If $k - 2p \not\equiv 0 \bmod (n, 2k)$, then $\Delta(7)$ is large if $(n, 2k) > 2$ by [1, Theorem B]. Thus $(n, 2k) \leq 2$.

If $(n, 2k) = 1$, then $(n, 4p) = 1$, so $(n, p) = 1$, in which case $G_n(j, k, l) \cong \mathbb{Z}_4$ by [3, Theorem 7.2], giving case (a). Thus we may assume $(n, 2k) = 2$, so also $(n, 4p) = 2$, $(n, k) = 1$; in particular, $(n, 4) = 2$. As discussed in Section 2, since

(C) holds, $G = G_n(j,k,l) = G_n(x_0x_px_kx_{k+p})$; then, since $(n,k) = 1$, we have $G \cong G_n(x_0x_Jx_1x_{J+1})$, where $J = pk^{-1} \bmod n$ (see [3, Section 3]). Then

$$(n, J) = (n, pk^{-1}) = (n, p) = 1 \text{ or } 2,$$

the latter case giving case (c). If $(n, J) = 1$, then

$$G_n(x_0x_Jx_1x_{J+1}) \cong G_n(x_0x_1x_{J^{-1}}x_{J^{-1}+1}),$$

which, after replacing J^{-1} by J, gives case (b). □

We deal with cases (b), (c) of Lemma 4.2 in Lemmas 4.3, 4.4, respectively.

Lemma 4.3. *Suppose $n \geq 4$ is even and J is odd. Then $G_n(1, J, J+1)$ is large.*

Proof. Let $G = G_n(1, J, J+1)$. Then

$$G = \langle x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1} \mid y_i = x_ix_{i+1},\ y_iy_{i+J} = 1\ (0 \leq i < n)\rangle.$$

Therefore, we have $y_0 = y_J^{-1} = y_{2J} = y_{3J}^{-1} = \cdots = y_{(n-2)J} = y_{(n-1)J}^{-1}$; that is, $y_i = y_0^{(-1)^i}$ (since J is odd), and so

$$\begin{aligned}
G &= \langle x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1} \mid y_i = x_ix_{i+1},\ y_iy_{i+J} = 1, \\
&\qquad\qquad y_i = y_0^{(-1)^i}\ (0 \leq i < n)\rangle \\
&= \langle x_0, \dots, x_{n-1}, y \mid y^{(-1)^i} = x_ix_{i+1}\ (0 \leq i < n)\rangle \\
&\qquad \text{(by eliminating } y_1, \dots, y_{n-1} \text{ and writing } y = y_0) \\
&= \langle x_0, \dots, x_{n-1}, y \mid x_{2u}x_{2u+1} = y,\ x_{2u+1}x_{2u+2} = y^{-1}\ (0 \leq u < n/2)\rangle \\
&= \langle x_0, \dots, x_{n-1}, y \mid x_{2u}x_{2u+1} = y,\ x_{2u+1} = y^{-1}x_{2u+2}^{-1}\ (0 \leq u < n/2)\rangle \\
&= \langle x_0, x_2 \dots, x_{n-2}, y \mid x_{2u}y^{-1}x_{2u+2}^{-1} = y\ (0 \leq u < n/2)\rangle \\
&\qquad \text{(by eliminating } x_1, x_3, \dots, x_{n-1}) \\
&= \langle x_0, x_2 \dots, x_{n-2}, y \mid x_{2u+2} = y^{-1}x_{2u}y^{-1}\ (0 \leq u < n/2)\rangle.
\end{aligned}$$

Eliminating $x_{n-2}, x_{n-4}, \dots, x_2$ in turn and writing $x = x_0$ then gives

$$G = \langle x, y \mid x = y^{-n/2}xy^{-n/2}\rangle.$$

By adjoining the relator $y^{n/2}$, the group G maps onto $\langle x, y \mid y^{n/2}\rangle \cong \mathbb{Z} * \mathbb{Z}_{n/2}$ which is large, since $n \geq 4$. □

Lemma 4.4. *Suppose $n \geq 4$, $(n,4)=2$, $(n,J)=2$. Then $G_n(J,1,J+1)$ is large.*

Proof. Let $n=2m$, $J=2q$, where $m \geq 3$ is odd, $(m,q)=1$, and suppose that $G=G_n(J,1,J+1)$. Then

$$\begin{aligned} G &= \langle x_0,\ldots,x_{2m-1} \mid x_i x_{i+2q} x_{i+1} x_{i+2q+1}\ (0 \leq i < 2m)\rangle \\ &= \langle x_0,\ldots,x_{2m-1}, y_0,\ldots,y_{2m-1} \mid y_i y_{i+1} = 1, \\ &\qquad\qquad y_i = x_i x_{i+2q}\ (0 \leq i < 2m)\rangle. \end{aligned}$$

Then $y_i = y_0^{(-1)^i}$ for each $0 \leq i < 2m$, so eliminating $y_1,\ldots,y_{2m-1}$ and writing $y=y_0$, we have

$$\begin{aligned} G &= \langle x_0,\ldots,x_{2m-1}, y \mid y^{(-1)^i} = x_i x_{i+2q}\ (0 \leq i < 2m)\rangle \\ &= \langle x_0,\ldots,x_{2m-1}, y \mid y = x_{2u} x_{2u+2q}, \\ &\qquad\qquad y^{-1} = x_{2u+1} x_{2(u+q)+1}\ (0 \leq u < m)\rangle \\ &= \langle a_0,\ldots,a_{m-1}, b_0,\ldots,b_{m-1}, y \mid y = a_u a_{u+q}, \\ &\qquad\qquad y^{-1} = b_u b_{u+q}\ (0 \leq u < m)\rangle \end{aligned}$$

by writing $a_u = x_{2u}$ and $b_u = x_{2u+1}$ $(0 \leq u < m)$, where subscripts are now taken mod m. For each $0 \leq u < m$, multiplying the subscripts by q^{-1} mod m and setting $v = uq^{-1}$ mod m gives

$$G = \langle a_0,\ldots,a_{m-1}, b_0,\ldots,b_{m-1}, y \mid y = a_v a_{v+1},\ y^{-1} = b_v b_{v+1}\ (0 \leq v < m)\rangle.$$

Eliminating $a_{m-1}, a_{m-2},\ldots,a_1$ and $b_{m-1}, b_{m-2},\ldots,b_1$ in turn and writing $a=a_0$, $b=b_0^{-1}$ then gives

$$\begin{aligned} G &= \langle a,b,y \mid a = y^{-(m-1)/2} a^{-1} y^{(m+1)/2},\ b = y^{-(m-1)/2} b^{-1} y^{(m+1)/2}\rangle \\ &= \langle a,b,y \mid a y^{(m-1)/2} a = y^{(m+1)/2},\ b y^{(m-1)/2} b = y^{(m+1)/2}\rangle \\ &= \langle a,b,y \mid (a y^{(m-1)/2})^2 = y^m,\ (b y^{(m-1)/2})^2 = y^m\rangle, \end{aligned}$$

which (by adjoining relators ab^{-1}, y^m and a^7) maps onto

$$Q = \langle a,y \mid (a y^{(m-1)/2})^2, y^m, a^7\rangle,$$

which is large for all odd $m \geq 3$ by [1, Theorem B]. □

Theorem 4.1 then follows from Lemmas 4.2, 4.3, 4.4. We are now in a position to prove Theorem A.

Proof of Theorem A. If $d>1$ or $\gamma>1$, then G is large by Lemmas 2.1, 2.2, so assume $d=\gamma=1$. Then parts (b) (i)–(vi) follow from Lemma 3.5, Theorem 4.1, Corollary 3.3, Lemma 3.6, Theorem 3.4, Lemma 3.1, respectively. □

5 Abelianisations

Here, we prove the following theorem, which classifies the groups $G_n(j,k,l)$ whose abelianisations are infinite.

Theorem 5.1. *Suppose* $d = 1$. *The abelianisation* $G_n(j,k,l)^{\mathrm{ab}}$ *is infinite if and only if* n *is even and* $j+k+l$ *is even.*

Proof. The abelianisation of a cyclically presented group $G_n(w)$ is infinite if and only if $f(\zeta) = 0$ for some $\zeta^n = 1$, where $f(t) = \sum_{i=0}^{n-1} a_i t^i$, where a_i is the exponent sum of x_i in w (see, for example, [16, page 77]). For the groups $G = G_n(j,k,l)$, we have $f(t) = 1 + t^j + t^k + t^l$, and so G^{ab} is infinite if and only if $1 + \zeta^j + \zeta^k + \zeta^l = 0$ for some $\zeta^n = 1$. If n is even and $j+k+l$ is even, then (since $d = 1$) precisely one of j,k,l is even, and so $\zeta = -1$ satisfies these conditions.

Suppose then $\zeta^n = 1$, $f(\zeta) = 0$. Taking the complex conjugate gives $f(\bar{\zeta}) = 0$. Now $\zeta^n = 1$ implies $1 = |\zeta|^2 = \zeta\bar{\zeta}$, so $\bar{\zeta} = \zeta^{-1}$, so $f(\zeta^{-1}) = 0$. Thus

$$1 + \iota + \kappa + \lambda = 0, \tag{5.1}$$
$$1 + \iota^{-1} + \kappa^{-1} + \lambda^{-1} = 0,$$

where $\iota = \zeta^j$, $\kappa = \zeta^k$, $\lambda = \zeta^l$. Therefore,

$$\begin{aligned} 1 = \iota\iota^{-1} &= (-\kappa - \lambda - 1)(-\kappa^{-1} - \lambda^{-1} - 1) \\ &= 3 + \kappa\lambda^{-1} + \lambda\kappa^{-1} + \kappa + \kappa^{-1} + \lambda + \lambda^{-1} \end{aligned}$$

or equivalently $(\kappa + \lambda)(1 + \lambda)(1 + \kappa) = 0$. Similarly, $(\lambda + \iota)(1 + \iota)(1 + \lambda) = 0$ and $(\iota + \kappa)(1 + \kappa)(1 + \iota) = 0$. These three equations imply that at least one of ι, κ, λ is equal to -1, for otherwise $\iota = \kappa = \lambda = 0$, a contradiction. Then, by (5.1), we have $(\iota, \kappa, \lambda) = (-1, \kappa, -\kappa)$, $(\iota, -1, -\iota)$ or $(\iota, -\iota, -1)$.

Without loss of generality, we may assume $(\iota, \kappa, \lambda) = (-1, \kappa, -\kappa)$, and so, since $\iota = -1$, n is even. Then $\zeta^k = \kappa = -\lambda = \iota\lambda = \zeta^{j+l}$, so $\zeta^{j+l-k} = 1$, and hence $j + l - k \equiv 0 \bmod m$, where m is the order of ζ. Now $\zeta^j = -1$, so m is even, so $j + l - k$, and hence $j + k + l$, is even, as required. □

For use in Section 6, we record the following.

Corollary 5.2. *If* (B) *holds and* $d = \gamma = 1$, *then* $G_n(j,k,l)^{\mathrm{ab}}$ *is finite.*

Proof. If n is odd, then the result follows from Theorem 5.1, so assume n is even. If k and $(j+l)$ are both even, then γ is even, a contradiction; if k and $(j+l)$ are both odd, then (B) does not hold, a contradiction. Therefore, $k + (j+l)$ is odd, and the result follows from Theorem 5.1. □

6 The case (F, T, F)

The case (F, T, F) was observed in [3] to be the most complex case. We have been unable to determine if the Tits alternative is satisfied in this case for all n, so in this section, we report results of computations that show it is satisfied for $n \leq 20$.

The case $n \leq 6$ follows from the results of [3]. Specifically, in the case where $(\mathrm{A}, \mathrm{B}, \mathrm{C}) = (\mathrm{F}, \mathrm{T}, \mathrm{F})$, $d = \gamma = 1$ and $n \leq 6$, the group $G_n(j, k, l)$ is isomorphic to one of the following groups: $G_5(0, 1, 2)$ (which is finite and solvable of order 220), $G_6(0, 1, 2)$, $G_6(1, 4, 2)$, which are non-isomorphic, finite, non-solvable groups of order $2^7 \cdot 3^3 \cdot 7 \cdot 13^2 = 4088448$. These are the groups (I5), (I6′), (I6″) discussed in [3, Section 9]. This proves Theorem B (a), and so we may assume $n \geq 7$. The following lemma (compare [15, Corollary 14]) shows that, to prove $G_n(j, k, l)$ is SQ-universal, it suffices to prove that it is hyperbolic.

Lemma 6.1. *Let $n \geq 7$, $d = \gamma = 1$ and $(\mathrm{A}, \mathrm{B}, \mathrm{C}) = (\mathrm{F}, \mathrm{T}, \mathrm{F})$. If $G = G_n(j, k, l)$ is hyperbolic, then it is non-elementary hyperbolic, and hence SQ-universal.*

Proof. A torsion-free group is virtually $\mathbb{Z}$ if and only if it is isomorphic to $\mathbb{Z}$ (see, for example, [18, Lemma 3.2]), so any non-trivial, torsion-free, hyperbolic group with finite abelianisation is non-elementary hyperbolic, and hence SQ-universal by [10, 20]. Therefore, it suffices to show that G is non-trivial, torsion-free, with finite abelianisation. The group G has finite abelianisation by Corollary 5.2, and it is non-trivial since there is an epimorphism onto $\mathbb{Z}_4$ obtained by sending each x_i to some fixed generator of $\mathbb{Z}_4$.

Since $n \geq 7$ and $d = \gamma = 1$, the group G is not of type (I) or (U) of [3], and so [3, Theorem 9.2] implies that $P = P_n(j, k, l)$ is combinatorially aspherical. As in the proof of Lemma 3.5, since (C) does not hold, no relator of P is a proper power or is conjugate to any other relator or its inverse. Thus, P is topologically aspherical, and so (as discussed in the proof of Lemma 3.5) G is torsion-free, as required. □

It is likely to be a challenging problem to determine in general which of the groups $G_n(j, k, l)$ are hyperbolic (compare, for example, [6, 7], which consider hyperbolicity of cyclically presented groups with length-three relators). However, the automatic groups software KBMAG [14] can be used to show that groups $G_n(j, k, l)$ are hyperbolic in particular instances.

Using the isomorphisms amongst the family of groups $G_n(j, k, l)$ obtained in [3, Section 3], we wrote a computer program in GAP [13] to obtain a (potentially redundant) list of 4-tuples (n, j, k, l) that define all isomorphism classes of groups $G_n(j, k, l)$ with $n \leq 20$ for which $(\mathrm{A}, \mathrm{B}, \mathrm{C}) = (\mathrm{F}, \mathrm{T}, \mathrm{F})$. We then attempted to prove that the corresponding groups are hyperbolic using KBMAG. In the handful of cases where the computation was inconclusive, we proved largeness using Magma [5]. In this way, we obtain the following theorem, from which Theorem B (b) follows by an application of Lemma 6.1.

Theorem 6.2. *Let* $7 \leq n \leq 20$*, and suppose* $d = \gamma = 1$*,* $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{F},\mathrm{T},\mathrm{F})$*, and let* $G = G_n(j,k,l)$*. Then* G *is either hyperbolic or is isomorphic to one of the following groups, each of which is large:* $G_7(1,2,4)$*,* $G_8(0,1,2)$*,* $G_8(1,2,4)$*,* $G_{12}(1,2,4)$*,* $G_{12}(1,3,5)$*,* $G_{12}(1,8,4)$*,* $G_{20}(1,2,6)$*,* $G_{20}(1,5,9)$ *or* $G_{20}(1,12,6)$*.*

Proof. The program described above produced a list of 87 4-tuples (n,j,k,l). Except in the cases listed in the statement and the cases $(n,j,k,l) = (13,1,2,6)$, $(15,1,6,3)$, $(19,1,2,8)$, KBMAG proved the corresponding cyclically presented group to be hyperbolic. (In most cases, the computation completed quickly, but a few were computationally challenging, for example, $G_9(1,3,6)$, $G_{11}(1,2,4)$ and $G_{17}(1,2,6)$ for which KBMAG exhibited geodesic difference machines with 3367, 2839, 4183 states, respectively.) The groups $G_{13}(1,2,6)$, $G_{15}(1,6,3)$ and $G_{19}(1,2,8)$ have shift extensions

$$\langle y,t \mid t^{13}, y^3tyt^2\rangle, \quad \langle y,t \mid t^{15}, y^2tyt^{-1}yt^{-1}\rangle, \quad \langle y,t \mid t^{19}, y^3tyt^2\rangle,$$

respectively (after writing $y = xt$ and applying an automorphism of $\langle t \mid t^n\rangle$). Computations in KBMAG show that each of these shift extensions are hyperbolic, and hence the corresponding cyclically presented groups are hyperbolic.

For the remaining 9 groups, Magma's largeness functionality shows the existence of a finite index subgroup that maps onto the free group of rank 2, and so are large. The groups and the index of the subgroup produced are as follows: $G_7(1,2,4)$ (index 2), $G_8(0,1,2)$ (index 6), $G_8(1,2,4)$ (index 6), $G_{12}(1,2,4)$ (index 5), $G_{12}(1,3,5)$ (index 5), $G_{12}(1,8,4)$ (index 5), $G_{20}(1,2,6)$ (index 4), $G_{20}(1,5,9)$ (index 3), $G_{20}(1,2,6)$ (index 3). □

Corollary 6.3. *Let* $n \geq 1$*,* $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{F},\mathrm{T},\mathrm{F})$*,* $d = \gamma = 1$*, and suppose* $m \mid n$ *for some* $7 \leq m \leq 20$*. If* $2k \not\equiv 0$*,* $2j \not\equiv 2l$*,* $l \not\equiv j + k$ *and* $j \not\equiv l + k \bmod m$*, then* $G_n(j,k,l)$ *is SQ-universal.*

As mentioned in the introduction, the problem of the Tits alternative for cyclically presented groups G with length-three positive relators holds a comparable status, in that the Tits alternative is known to be satisfied, except for the case when $(n,6) = 2$ and the (A), (B), (C), (D) conditions of [12] are F, F, F, T, respectively. In this case, the group G is isomorphic to $G_n(x_0x_1x_{n/2-1})$, so precisely one one-parameter infinite family of groups remains unresolved. The situation is less clear cut in the case of positive length-four relators, where (if $d = \gamma = 1$ and

(A, B, C) $=$ (F, T, F)) there can be more than one group $G_n(j,k,l)$ (up to isomorphism) with the same value of n.

Based on the evidence provided by Theorem B (b), we conclude by posing the following conjecture.

Conjecture 6.4. *Let* $n \geq 7$, $(\mathrm{A},\mathrm{B},\mathrm{C}) = (\mathrm{F},\mathrm{T},\mathrm{F})$, $d = \gamma = 1$. *Then* $G_n(j,k,l)$ *is SQ-universal.*

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