Topology and its Applications 312 (2022) 108083

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Fractional Fibonacci groups with an odd number of generators ☆

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A R T I C L E I N F O

Article history:
Received 12 May 2021
Accepted 14 March 2022
Available online 22 March 2022

MSC:
20F05
57M05
57M60

Keywords:
Fibonacci group
Cyclically presented group
Orbifold
Manifold
Aspherical presentation

A B S T R A C T

The Fibonacci groups $F(n)$ are known to exhibit significantly different behaviour depending on the parity of n. We extend known results for $F(n)$ for odd n to the family of Fractional Fibonacci groups $F^{k/l}(n)$. We show that for odd n the group $F^{k/l}(n)$ is not the fundamental group of an orientable hyperbolic 3-orbifold of finite volume. We obtain results concerning the existence of torsion in the groups $F^{k/l}(n)$ (where n is odd) paying particular attention to the groups $F^k(n)$ and $F^{k/l}(3)$, and observe consequences concerning the asphericity of relative presentations of their shift extensions. We show that if $F^k(n)$ (where n is odd) and $F^{k/l}(3)$ are non-cyclic 3-manifold groups then they are isomorphic to the direct product of the quaternion group Q_8 and a finite cyclic group.

1. Introduction

The *Fibonacci groups*

$$F(n) = \langle x_0, \dots, x_{n-1} \mid x_i x_{i+1} = x_{i+2} \ (0 \leq i < n) \rangle$$

(subscripts mod n) were introduced by Conway in [11], and they have since been studied from both algebraic and topological perspectives. The *Fractional Fibonacci groups*

$$F^{k/l}(n) = \langle x_0, \dots, x_{n-1} \mid x_i^l x_{i+1}^k = x_{i+2}^l \ (0 \leq i < n) \rangle \tag{1}$$

where $k, l \neq 0$, $n \geq 1$, subscripts mod n, introduced in [39], generalise the Fibonacci groups $F(n) = F^{1/1}(n)$ and also the groups $F^k(n) = F^{k/1}(n)$ considered in [28,29]. For even $n \geq 6$ and coprime integers $k, l \geq 1$

☆ This work was supported by the Leverhulme Trust Research Project Grant RPG-2017-334.
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https://doi.org/10.1016/j.topol.2022.108083
0166-8641/

the groups $F^{k/l}(n)$ have been shown to be fundamental groups of 3-manifolds (see [16,19,18,9] for the case $k = l = 1$, see [29] for the case $l = 1$, and [39] for the case of coprime integers $k, l \geq 1$).

It is known that the Fibonacci groups $F(n)$ exhibit substantially different behaviour depending on the parity of n. For instance, if n is even then $F(n)$ is the fundamental group of a 3-manifold, namely an $n/2$-fold cyclic cover of S^3 branched over the figure eight knot (which is spherical if $n = 2, 4$, an affine Riemannian manifold if $n = 6$, and hyperbolic if $n \geq 8$) [16,19,18,9], whereas if $n \geq 3$ is odd then $F(n)$ is not the fundamental group of any hyperbolic 3-orbifold of finite volume [28, Theorem 3.1], and if $n \geq 9$ is odd then $F(n)$ is not the fundamental group of any 3-manifold [23, Theorem 3]. Moreover, if $n \geq 6$ is even then $F(n)$ is infinite and torsion-free by statements P(3), P(4) of [16] whereas if $n \geq 9$ is odd then $F(n)$ is infinite [20,31,26,10] and contains an element of order 2 by [2, Proposition 3.1] ($F(2), F(3), F(4), F(5), F(7)$ are finite groups).

In this article we consider the Fractional Fibonacci groups $F^{k/l}(n)$ when n is odd. In Section 2 we obtain some basic observations about the groups $F^{k/l}(n)$. In Section 3 we obtain a recurrence relation formula for the order $|F^{k/l}(n)^{\mathrm{ab}}|$ (Theorem 3.1) and consequences of it that will be used in later sections. In Section 4 we prove Theorem 4.1, which states that for odd n the group $F^{k/l}(n)$ is not the fundamental group of an orientable hyperbolic 3-orbifold of finite volume and in Corollary 4.2, we prove that if, in addition, k is odd then $F^{k/l}(n)$ is not the fundamental group of a hyperbolic 3-orbifold of finite volume. In Section 5 we consider torsion elements in $F^{k/l}(n)$ and introduce a word $w(n, k)$ that is the basis for much of this section. By a result of Bardakov and Vesnin [2], for odd $n \geq 9$, the word $w(n, 1)$ is an element of order 2 in $F(n)$ (Theorem 5.1) and this has consequences for the asphericity of the relative presentation of the shift extension of $F(n)$ (Corollary 5.2), and the result was the basis for the proof in [23] that $F(n)$ is not the fundamental group of a 3-manifold (Theorem 6.1). We develop extensions of these results to the general case $F^{k/l}(n)$ and apply them to the groups $F^k(n)$ and $F^{k/l}(3)$. In Theorem 5.3 we show that $w(n, k)^2 = 1$ in $F^{k/l}(n)$ and that $w(n, k)$ is a commutator. Corollary 5.4 shows that $w(n, k) = 1$ if and only if $F^k(n)$ is abelian, and Corollary 5.5 does the same for the group $F^{k/l}(3)$. Corollary 5.7 then shows that if $w(n, k) \neq 1$ then the relative presentation of the shift extension of $F^{k/l}(n)$ is not aspherical and Corollaries 5.8, 5.9 show that this relative presentation is not aspherical in the cases $l = 1$ and $n = 3$, respectively. In Section 6 we consider when $F^{k/l}(n)$ is a 3-manifold group and show that if $w(n, k) \neq 1$ then $F^{k/l}(n)$ is not a 2-generator, infinite, 3-manifold group (Lemma 6.3). In Theorem 6.2 we use this to prove that if $F^k(n)$ or $F^{k/l}(3)$ is a 3-manifold group then it is either a finite cyclic group or isomorphic to the direct product of the quaternion group Q_8 and a finite cyclic group.

2. Preliminaries

Our first lemma is immediate from the definition of $F^{k/l}(n)$.

Lemma 2.1.
(a) $F^{k/l}(2) \cong \mathbb{Z}_k * \mathbb{Z}_k$;
(b) $F^{k/0}(n)$ *is isomorphic to the free product of* n *copies of* $\mathbb{Z}_k$.

For $n \geq 1$ and $l \in \mathbb{Z}$ let

$$G(n, l) = \langle x_0, \ldots, x_{n-1} \mid x_i^l = x_{i+1}^l \ (0 \leq i < n) \rangle$$

(subscripts mod n). By relabelling the generators, we see that the group $F^{0/l}(n)$ is isomorphic to $G(n, l)$ if n is odd and is isomorphic to the free product of two copies of $G(n/2, l)$ if n is even. In this context we record the following:

Lemma 2.2. *Let $G = G(n,l)$ where $n \geq 2, l \geq 1$. Then there is a central extension $\mathbb{Z} \cong \langle x_0^l \rangle \hookrightarrow G(n,l) \twoheadrightarrow \underbrace{\mathbb{Z}_l * \cdots * \mathbb{Z}_l}_{n}$.*

Proof. Let $H = \langle x_0^l \rangle$, the subgroup of G, generated by x_0^l. Now, for each $0 \leq j < n$, $x_0^l x_j x_0^{-l} x_j^{-1} = x_j^l x_j x_j^{-l} x_j^{-1} = 1$, so $x_0^l \in Z(G)$, the centre of G. We have $G/H \cong \langle x_0, \ldots, x_{n-1} \mid x_0^l = \cdots = x_{n-1}^l = 1 \rangle$, which is isomorphic to the free product of n copies of $\mathbb{Z}_l$, and there is an epimorphism $G \to \mathbb{Z}$ given by sending each x_i to some fixed generator of $\mathbb{Z}$ so x_i (and in particular x_0) has infinite order, so $H \cong \mathbb{Z}$. □

Lemma 2.3. *For each $k, l \neq 0$ and $n \geq 2$ we have $F^{k/l}(n) \cong F^{(-k)/(-l)}(n) \cong F^{k/(-l)}(n) \cong F^{(-k)/l}(n)$.*

Proof. The isomorphism $F^{k/l}(n) \cong F^{(-k)/(-l)}(n)$ is obtained by replacing each generator by its inverse. We now show that $F^{k/(-l)}(n) \cong F^{(-k)/(-l)}(n)$; the final isomorphism $F^{(-k)/l}(n) \cong F^{k/l}(n)$ is then obtained from this by replacing each generator by its inverse.

The relations of $F^{k/(-l)}(n)$ are $x_i^{-l} x_{i+1}^k = x_{i+2}^{-l}$, which are equivalent to $x_{i+1}^k x_{i+2}^l = x_i^l$. Negating the subscripts and writing $j = -i$ these become $x_{j-1}^k x_{j-2}^l = x_j^l$; adding 2 to the subscripts gives $x_{j+1}^k x_j^l = x_{j+2}^l$. Inverting the relations gives $x_j^{-l} x_{j+1}^{-k} = x_{j+2}^{-l}$ which are the relations of $F^{(-k)/(-l)}(n)$, as required. □

Lemmas 2.1–2.3 allow us to assume $k, l \geq 1$.

For our next lemma, recall that a group is *large* if it has a finite index subgroup that maps onto the free group of rank 2, that a group mapping onto a large group is large, and that the free product of two non-trivial finite groups is large unless both groups have order 2 [34].

Lemma 2.4. *Let $n \geq 2$. For each $k, l \geq 1$ let $d = (k,l)$. If $d > 1$ then $F^{k/l}(n)$ is large unless $k = n = 2$, in which case $F^{k/l}(n) \cong D_\infty$, the infinite dihedral group.*

Proof. By killing x_i^d for each i we see that the group $F^{k/l}(n)$ maps onto the free product of n copies of $\mathbb{Z}_d$. Thus $F^{k/l}(n)$ is large if $d > 1$ except possibly if $d = 2$ and $n = 2$, in which case $F^{k/l}(n) \cong \mathbb{Z}_k * \mathbb{Z}_k$ by Lemma 2.1, which is large, unless $k = 1$ or 2. If $k = 1$ then $d = 1$, a contradiction; if $k = 2$ then $F^{k/l}(n) \cong \mathbb{Z}_2 * \mathbb{Z}_2 = D_\infty$. □

In the notation and terminology of [30, Chapter 5], writing $d = (k,l)$, we may express $F^{k/l}(n) = G_n(x_0^l x_1^k x_2^{-l})$ as a composite $G_n(v \circ u)$ where $u = x_0^d$, and $v = x_0^{l/d} x_1^{k/d} x_2^{-l/d}$. Since u is a positive word, by [30, Lemma 5.1.3.4] we then have $G_n(v) = F^{(k/d)/(l/d)}(n)$ embeds in $G_n(v \circ u) = F^{k/l}(n)$. We record this as:

Theorem 2.5 *([30, Chapter 5]). For each $k, l \geq 1$ let $d = (k,l)$. Then $F^{(k/d)/(l/d)}(n)$ embeds in $F^{k/l}(n)$.*

In the following corollary, and throughout this paper, by a *3-manifold group* we mean the fundamental group of a (not necessarily closed, compact, or orientable) 3-manifold.

Corollary 2.6. *Let $n \geq 2$, $k, l \geq 1$ and define $d = (k,l)$.*

(a) *Suppose $d > 1$. If $F^{(k/d)/(l/d)}(n)$ is not torsion-free then $F^{k/l}(n)$ is an infinite group that is not torsion-free; in particular, if $F^{(k/d)/(l/d)}(n)$ is a finite non-trivial group then $F^{k/l}(n)$ is an infinite group that is not torsion-free.*
(b) *Suppose $F^{(k/d)/(l/d)}(n)$ is not a 3-manifold group; then $F^{k/l}(n)$ is not a 3-manifold group.*
(c) *Suppose $F^{(k/d)/(l/d)}(n)$ is not the fundamental group of an orientable hyperbolic 3-orbifold of finite volume; then $F^{k/l}(n)$ is not the fundamental group of an orientable hyperbolic 3-orbifold of finite volume.*

Proof. (a) Since $F^{(k/d)/(l/d)}(n)$ is not torsion-free, it contains a non-trivial element of finite order, which is also an element of $F^{k/l}(n)$. (b) This holds since subgroups of 3-manifold groups are 3-manifold groups [17, Chapter 8]. (c) If $F^{(k/d)/(l/d)}(n)$ is not the fundamental group of an orientable hyperbolic 3-orbifold of finite volume, then there is no embedding of $F^{(k/d)/(l/d)}(n)$ into $PSL(2,\mathbb{C})$, the group of orientation preserving isometries of hyperbolic 3-space, and hence there is no embedding of $F^{k/l}(n)$ into $PSL(2,\mathbb{C})$, so $F^{k/l}(n)$ is not the fundamental group of an orientable hyperbolic 3-orbifold of finite volume. □

We say that a group G is a *q-generator group* ($q \geq 1$), or that G is *q-generated*, if it has a generating set with q generators. Starting with the case $l = 1$ we have:

Lemma 2.7. *Let $n \geq 2$, $k \geq 1$. Then $F^k(n)$ is 2-generated and can be generated by x_0 and x_1.*

Proof. The relations $x_{i+2} = x_i x_{i+1}^k$ allow each generator x_j ($2 \leq j < n$) to be written in terms of x_{j-1} and x_{j-2}, so only generators x_0, x_1 are needed. □

For the general case we have:

Lemma 2.8. *Let $n \geq 3$ be odd, $k, l \geq 1$, $(k,l) = 1$. Then $F^{k/l}(n)$ is $(n+1)/2$-generated. In particular, $F^{k/l}(3)$ is 2-generated and can be generated by x_0 and x_1.*

Proof. Since $(k,l) = 1$ there exist $\alpha, \beta \in \mathbb{Z}$ such that $\alpha k + \beta l = 1$. The defining relations imply $x_{j+2}^l = x_j^l x_{j+1}^k$ and $x_{j+2}^k = x_{j+1}^{-l} x_{j+3}^l$. Thus

$$x_{j+2} = x_{j+2}^{\alpha k + \beta l} = (x_{j+2}^k)^\alpha (x_{j+2}^l)^\beta = (x_{j+1}^{-l} x_{j+3}^l)^\alpha (x_j^l x_{j+1}^k)^\beta$$

and so each generator x_{j+2} can be written in terms of x_j, x_{j+1}, x_{j+3}. We may therefore eliminate generators $x_{n-1}, x_{n-3}, \ldots, x_2$ in turn to leave a presentation with the $(n+1)/2$ generators $x_0, x_1, x_3 \ldots, x_{n-2}$. □

For the case $k = 1$ we can decrease the lower bound slightly:

Lemma 2.9. *Let $n \geq 3$, $l \geq 1$. If $n = 3$ then $F^{1/l}(n)$ is 2-generated, and if $n \geq 4$ then $F^{1/l}(n)$ is $\lfloor n/2 \rfloor$-generated.*

Proof. The relations $x_i^l x_{i+1} = x_{i+2}^l$ can be written $x_{i+1} = x_i^{-l} x_{i+2}^l$. If n is even, this allows all odd numbered generators to be eliminated, leaving an $n/2$-generator presentation. Suppose then that n is odd. Then we can eliminate $x_{2j+1} = x_{2j}^{-l} x_{2(j+1)}^l$ for each $0 \leq j \leq (n-3)/2$, leaving a presentation with $(n+1)/2$ generators $x_0, x_2, \ldots, x_{n-1}$. In doing so, the original relations $x_{n-2}^l x_{n-1} = x_0^l$ and $x_{n-1}^l x_0 = x_1^l$ become

$$(x_{n-3}^{-l} x_{n-1}^l)^l x_{n-1} = x_0^l, \tag{2}$$

$$x_{n-1}^l x_0 = (x_0^{-l} x_2^l)^l. \tag{3}$$

We may substitute the expression for x_0^l given by (2) into (3) which can then be used to eliminate x_0, leaving an $(n-1)/2$-generator presentation. □

The group $F^{k/l}(n)$ has an automorphism $\theta : x_i \mapsto x_{i+1}$ (subscripts mod n), called the *shift automorphism* and the corresponding split extension, called the *shift extension*,

$$E^{k/l}(n) = F^{k/l}(n) \rtimes_\theta \langle t \mid t^n \rangle$$

has a 2-generator, 2-relator presentation

$$E^{k/l}(n) = \langle x, t \mid t^n, x^l t x^k t x^{-l} t^{-2} \rangle$$

(which is obtained by rewriting $x_0 = x$ and $x_i = t^i x t^{-i}$ for $1 \leq i < n$).

We now turn to the groups $F^k(n)$. As remarked in [29, Remark 1] determining which groups $F^k(n)$ (where $n \geq 3$ and odd) are finite is a challenging problem and it is observed that $|F^2(3)| = 112$, $|F^3(3)| = 3528$ and that $F^2(5)$ is infinite. Using KBMAG [21] and the `NewmanInfinityCriterion` ([31]) command in GAP [14] we can prove certain groups $F^{k/l}(n)$ infinite. For example, we have the following result (further infinite groups will be exhibited in Example 5.14).

Lemma 2.10. *Let $n \in \{5, 7, 9\}$, $3 \leq k \leq 12$. Then $F^k(n)$ is infinite.*

Proof. If $(n, k) \neq (9, 3)$ the group $F^k(n)$ can be proved (automatic and) infinite using KBMAG. The group $F^3(9)$ maps onto $H = \langle x_0, \ldots, x_8 \mid x_i x_{i+1}^3 = x_{i+2}, x_i^{108}\ (0 \leq i < 9) \rangle$ which can be proved infinite using the `NewmanInfinityCriterion` command applied to the second derived subgroup H'' of H, with the prime $p = 7$. $\square$

3. Abelianisations

Knowledge of the order of the abelianisation $|F^{k/l}(n)^{\mathrm{ab}}|$ will be crucial to our later methods. In Theorem 3.1 we obtain a recurrence relation formula for this order. A version of this was asserted in [38, Lemma, page 238] but the formula there is not quite right (for instance, it incorrectly implies that $|F(n)^{\mathrm{ab}}|$ is even whenever n is odd). While this has no impact on the later arguments in [38], a correct formula is necessary for our arguments, so we include a proof. In Theorem 3.2 we express the order $|F^{k/l}(n)^{\mathrm{ab}}|$ as a polynomial in k and l, and in Corollaries 3.3–3.7 we derive consequences that will be used in later sections.

Define a sequence of natural numbers $V_j^{k/l}$ according to the following recurrence relation

$$V_1^{k/l} = k,\ V_2^{k/l} = k^2 + 2l^2,\ V_j^{k/l} = kV_{j-1}^{k/l} + l^2 V_{j-2}^{k/l} \quad (j \geq 3). \tag{4}$$

Theorem 3.1. *Let $k, l \geq 1$, $n \geq 2$. Then*

$$|F^{k/l}(n)^{\mathrm{ab}}| = \begin{cases} V_n^{k/l} & \text{if } n \text{ is odd;} \\ V_n^{k/l} - 2l^n & \text{if } n \text{ is even.} \end{cases}$$

Proof. The order $|F^{k/l}(n)^{\mathrm{ab}}|$ is equal to the resultant $|\mathrm{Res}(f(t), g(t))|$ where $f(t) = l + kt - lt^2$ is the representer polynomial of $F^{k/l}(n)$ and $g(t) = t^n - 1$ (see [25]). For each $j \geq 1$ define $u_j = V_j^{k/l}/l^j$ where $V_j^{k/l}$ is as defined at (4). Then $f(t)$ is the characteristic polynomial of the recurrence relation defining the sequence (u_j) and has distinct roots β_1, β_2, say. Then the sequence (u_j) has general solution $u_j = c_1\beta_1^j + c_2\beta_2^j$ (see for example Theorems 4.10, 4.11 of [32]). Putting $n = 1, 2$ into these solutions and solving for c_1, c_2 gives $c_1 = c_2 = 1$ and hence $u_j = \beta_1^j + \beta_2^j$. Then by [33, Lemma 2.1]

$$|\mathrm{Res}(f(t), g(t))| = |l^n(\beta_1^n - 1)(\beta_2^n - 1)| = |l^n\left((-1)^n + 1 - u_n\right)| = |l^n + (-l)^n - V_n^{k/l}|$$

as required. $\square$

This implies, for example, that for $k, l \geq 1$

$$|F^{k/l}(3)^{\mathrm{ab}}| = k^3 + 3kl^2, \tag{5}$$

and $F^{k/l}(n)$ is trivial if and only if $n \leq 2$ and $k = 1$.

Theorem 3.2. *Let $n, k, l \geq 1$ and let $N = \lfloor n/2 \rfloor$. Then*

$$V_n^{k/l} = \sum_{r=0}^{N} a_{n,r} k^{n-2r} l^{2r}$$

for integers $a_{n,r} \geq 1$ satisfying $a_{n,0} = 1$ for $n \geq 1$ and $a_{n,r} = a_{n-1,r} + a_{n-2,r-1}$ for $1 \leq r < N$, $n \geq 3$ and $a_{n,N} = n$ if n is odd and $a_{n,N} = 2$ if n is even.

Proof. By the definition of $V_n^{k/l}$, the statement is true for $n = 1, 2$. Suppose that $n \geq 3$ and that the statement is true for all $3 \leq j < n$. If n is odd then

$$\begin{aligned}
V_n^{k/l} &= kV_{n-1}^{k/l} + l^2 V_{n-2}^{k/l} \\
&= k\left(\sum_{r=0}^{(n-1)/2} a_{n-1,r} k^{n-1-2r} l^{2r}\right) + l^2\left(\sum_{r=0}^{(n-3)/2} a_{n-2,r} k^{n-2-2r} l^{2r}\right) \\
&= \left(\sum_{r=0}^{(n-1)/2-1} a_{n-1,r} k^{n-2r} l^{2r} + \sum_{r=0}^{(n-3)/2-1} a_{n-2,r} k^{n-2-2r} l^{2+2r}\right) + nkl^{n-1} \\
&= \left(\sum_{r=0}^{(n-3)/2} a_{n-1,r} k^{n-2r} l^{2r} + \sum_{r=1}^{(n-3)/2} a_{n-2,r-1} k^{n-2r} l^{2r}\right) + nkl^{n-1} \\
&= a_{n-1,0} k^n + \left(\sum_{r=1}^{(n-3)/2} (a_{n-1,r} + a_{n-2,r-1}) k^{n-2r} l^{2r}\right) + nkl^{n-1} \\
&= \sum_{r=0}^{(n-1)/2} a_{n,r} k^{n-2r} l^{2r}
\end{aligned}$$

where $a_{n,0} = a_{n-1,0} = 1$, $a_{n,(n-1)/2} = n$ and $a_{n,r} = a_{n-1,r} + a_{n-2,r-1}$ for $1 \leq r \leq (n-3)/2$. A similar argument applies when n is even. $\square$

Corollary 3.3. *Let $k, l \geq 1$. Then for each $n \geq 2$ we have $|F^{k/l}(n+1)^{\mathrm{ab}}| > |F^{k/l}(n)^{\mathrm{ab}}|$. Hence if either $n \geq 3$ or ($n = 2$ and $k > 1$) then the shift automorphism θ of $F^{k/l}(n)$ has order n.*

Proof. If n is even then (since, by (4), (V_j) is increasing in j) we have $|F^{k/l}(n+1)^{\mathrm{ab}}| = V_{n+1}^{k/l} > V_n^{k/l} > V_n^{k/l} - 2l^n = |F^{k/l}(n)^{\mathrm{ab}}|$. If n is odd then

$$|F^{k/l}(n+1)^{\mathrm{ab}}| = V_{n+1}^{k/l} - 2l^{n+1} = \sum_{r=0}^{(n-1)/2} a_{n+1,r} k^{n+1-2r} l^{2r} > \sum_{r=0}^{(n-1)/2} a_{n,r} k^{n-2r} l^{2r} = |F^{k/l}(n)^{\mathrm{ab}}|$$

since $a_{n+1,0} = a_{n,0}$ and $a_{n+1,r} > a_{n,r}$ for any $r \geq 1$. Now let $n \geq 3$ and suppose that θ has order $m|n$. If $m = 1$ then $F^{k/l}(n) \cong \mathbb{Z}_k$, which contradicts $|F^{k/l}(n)^{\mathrm{ab}}| \geq k^2 + 2l^2$; if $2 \leq m < n$ then $F^{k/l}(n) \cong F^{k/l}(m)$ and so $|F^{k/l}(n)^{\mathrm{ab}}| = |F^{k/l}(m)^{\mathrm{ab}}|$, a contradiction. Finally, if $n = 2$, $k > 1$, and $m = 1$ then $F^{k/l}(n) \cong \mathbb{Z}_k$, which contradicts $|F^{k/l}(2)^{\mathrm{ab}}| = k^2$. $\square$

Corollary 3.4. *Suppose $n \geq 3$ is odd, $k, l \geq 1$, $(k, l) = 1$ where k is even. Then the 2-adic orders $v_2(k)$ and $v_2(V_n^{k/l})$ are equal.*

Proof. Let $k = 2^m q$ where q is odd and $m \geq 1$. We claim that for each $n \geq 1$ there exists some odd $\tilde{V}_n^{k/l}$ such that $V_n^{k/l} = 2^m \tilde{V}_n^{k/l}$ if n is odd and $V_n^{k/l} = 2\tilde{V}_n^{k/l}$ if n is even.

As at (4) we have $V_1^{k/l} = k = 2^m q = 2^m \tilde{V}_1^{k/l}$ where $\tilde{V}_1^{k/l} = q$ is odd, $V_2^{k/l} = k^2 + 2l^2 = 2\tilde{V}_2^{k/l}$ where $\tilde{V}_2^{k/l} = k^2/2 + l^2$, which is odd. Suppose that $n \geq 3$ and that the claim holds for all $j < n$. If n is odd, then

$$V_n^{k/l} = kV_{n-1}^{k/l} + l^2 V_{n-2}^{k/l} = k(2\tilde{V}_{n-1}^{k/l}) + l^2(2^m \tilde{V}_{n-2}^{k/l}) = 2^m \tilde{V}_n^{k/l}$$

where $\tilde{V}_n^{k/l} = 2q\tilde{V}_{n-1}^{k/l} + l^2 \tilde{V}_{n-2}^{k/l}$ is odd. Similarly if n is even, then

$$V_n^{k/l} = kV_{n-1}^{k/l} + l^2 V_{n-2}^{k/l} = k(2^m \tilde{V}_{n-1}^{k/l}) + l^2(2\tilde{V}_{n-2}^{k/l}) = 2\tilde{V}_n^{k/l}$$

where $\tilde{V}_n^{k/l} = 2^{m-1} k\tilde{V}_{n-1}^{k/l} + l^2 \tilde{V}_{n-2}^{k/l}$ is odd. $\square$

Corollary 3.5. *Let $n, k, l \geq 1$. Then $V_n^{k/l}$ is even if and only if either k is even or (l is odd and $n \equiv 0$ mod 3).*

Proof. Note that $F^{k/l}(n)$ maps onto $\mathbb{Z}_k$ (by sending each x_i to some fixed generator of $\mathbb{Z}_k$) so if k is even then $|F^{k/l}(n)^{\mathrm{ab}}|$ is even, so assume k is odd. If l is even then by (4), $V_n^{k/l} \equiv V_{n-1}^{k/l} \bmod 2$ for all $n \geq 2$ and $V_1^{k/l} = k$ is odd, and so $V_n^{k/l}$ is odd for all $n \geq 1$. If l is odd then $V_1^{k/l}$ and $V_2^{k/l}$ are odd, and $V_n^{k/l} \equiv V_{n-1}^{k/l} + V_{n-2}^{k/l} \bmod 2$ for all $n \geq 3$ and it follows that $V_n^{k/l}$ is even if and only if $n \equiv 0 \bmod 3$. $\square$

Corollary 3.6. *Let $n \geq 3$ be odd, $k, l \geq 1$, and suppose that $(k, l) = 1$.*

(a) *If $|F^{k/l}(n)^{\mathrm{ab}}|$ divides $(2l)^n$ then $k = l = 1$ and $n = 3$;*
(b) *if $|F^{k/l}(n)^{\mathrm{ab}}|$ divides $(2k)^n$ then $|F^{k/l}(n)^{\mathrm{ab}}|$ is even and k is odd.*

Proof. (a) First we claim that for each $n \geq 1$ we have $(V_n^{k/l}, l) = 1$ (which, by definition of $V_n^{k/l}$, is true for $n = 1, 2$). Suppose this statement is true for $j - 1$ where $j \geq 3$. Then

$$(V_j^{k/l}, l) = (kV_{j-1}^{k/l} + l^2 V_{j-2}^{k/l}, l) = (kV_{j-1}^{k/l}, l) = 1$$

so by induction, the statement is true for all $n \geq 1$. Therefore by Theorem 3.1 we have $(|F^{k/l}(n)^{\mathrm{ab}}|, l) = (V_n^{k/l}, l) = 1$ for all $n \geq 1$.

Suppose that $|F^{k/l}(n)^{\mathrm{ab}}|$ divides $(2l)^n$. Then $|F^{k/l}(n)^{\mathrm{ab}}|$ divides 2^n since $(|F^{k/l}(n)^{\mathrm{ab}}|, l) = 1$. By Theorem 3.1 we have $|F^{k/l}(n)^{\mathrm{ab}}| = V_n^{k/l}$ and by Theorem 3.2

$$V_n^{k/l} = k^n + \left(\sum_{r=1}^{N-1} a_{n,r} k^{n-2r} l^{2r} \right) + nkl^{n-1}$$

where $N = \left\lfloor \frac{n}{2} \right\rfloor$ and each $a_{n,r} \geq 1$ is an integer so in particular, $k^n < 2^n$ and $nkl^{n-1} < 2^n$ and hence $k = l = 1$. Then $V_j^{k/l}$ is a Lucas number, which therefore divides 2^n and since the only powers of 2 that appear in the Lucas sequence are 1,2,4 (see, for example, [6]) we have $n = 3$.

(b) Suppose that $|F^{k/l}(n)^{\mathrm{ab}}|$ divides $(2k)^n$. If $|F^{k/l}(n)^{\mathrm{ab}}|$ is odd, then it divides k^n which is impossible since $V_n^{k/l} > k^n$ by Theorem 3.2. Therefore $|F^{k/l}(n)^{\mathrm{ab}}|$ is even. Suppose for contradiction that k is even, say $k = 2^m q$ where q is odd and $m \geq 1$. Then by Theorem 3.1 and Corollary 3.4 we have $|F^{k/l}(n)^{\mathrm{ab}}|/2^m$ is odd, and so $|F^{k/l}(n)^{\mathrm{ab}}|/2^m$ divides q^n. But by Theorem 3.2

$$|F^{k/l}(n)^{\mathrm{ab}}|/2^m = a_{n,0} q^n 2^{m(n-1)} + \sum_{r=1}^{N} a_{n,r} k^{n-2r} l^{2r} 2^{-m} > q^n$$

since $N \geq 1$, a contradiction. Therefore k is odd. □

Corollary 3.7. *Suppose $n = pk > 7$ is odd, where $p \geq 1$, $k \geq 3$, $(p,k) = 1$, and $l \geq 1$. If $(k,l) = 1$ then $V_n^{k/l}$ does not divide $(2k)^n$.*

Proof. Suppose for contradiction that $V_n^{k/l}$ divides $(2k)^n$. By Theorem 3.2 we have

$$V_n^{k/l} = k^2\left(\left(k\sum_{r=0}^{(n-3)/2} a_{n,r}k^{n-2r-3}l^{2r}\right) + pl^{n-1}\right)$$

so $V_n^{k/l} \equiv 0 \bmod k^2$ and $(V_n^{k/l}/k^2, k) = 1$ and so $V_n^{k/l}/k^2$ divides 2^n. But

$$\begin{aligned}\frac{V_n^{k/l}}{k^2} &= \left(k\sum_{r=0}^{(n-3)/2} a_{n,r}k^{n-2r-3}l^{2r}\right) + pl^{n-1}\\ &> k^{n-2} + k^{n-4} \geq k^{n-4}(3^2+1) = 10k^{n-4} > 2^n\end{aligned}$$

since $n \geq 7$ and $k \geq 3$, a contradiction. □

4. Hyperbolic 3-orbifolds

In this section we prove the following.

Theorem 4.1. *Let $n, k, l \geq 1$, where n is odd. Then $F^{k/l}(n)$ is not the fundamental group of an orientable hyperbolic 3-orbifold (in particular, 3-manifold) of finite volume.*

Corollary 4.2. *Let $n, k, l \geq 1$, where n and k are odd. Then $F^{k/l}(n)$ is not the fundamental group of a hyperbolic 3-orbifold (in particular, 3-manifold) of finite volume.*

Note that we do not assume $(k,l) = 1$ in the hypotheses of Theorem 4.1 and Corollary 4.2. Our method of proof follows that introduced in [28] (for Fibonacci groups $F(n)$), and developed further in [2,7,37]. That is, supposing that $F^{k/l}(n)$ is the fundamental group of an orientable hyperbolic 3-orbifold of finite volume, then so is its shift extension $E^{k/l}(n) = \langle x, t \mid t^n, x^l t x^k t x^{-l} t^{-2}\rangle$, which is therefore isomorphic to a subgroup of $PSL(2,\mathbb{C})$. We show that a putative embedding in $PSL(2,\mathbb{C})$ would imply restrictions on the order of the abelianisation $F^{k/l}(n)^{\mathrm{ab}}$ and then use the results of Section 3 to show that these restrictions cannot occur.

Proof of Theorem 4.1. We prove the theorem in the case $(k,l) = 1$; the case $(k,l) > 1$ then follows from Corollary 2.6. If $n = 1$ then $F^{k/l}(n) \cong \mathbb{Z}_k$, so assume $n \geq 3$. Suppose for contradiction that $F^{k/l}(n)$ is the fundamental group of an orientable hyperbolic 3-orbifold of finite volume. By Corollary 3.3 the shift automorphism θ of $F^{k/l}(n)$ has order n so, as explained in the proof of [28, Theorem 3.1], it follows from the Mostow Rigidity Theorem that the shift extension $E = \langle x, t \mid t^n, x^l t x^k t x^{-l} t^{-2}\rangle$ of $F^{k/l}(n)$ is isomorphic to a subgroup of $PSL(2,\mathbb{C})$.

Therefore there exists a subgroup $\tilde{E}$ of $SL(2,\mathbb{C})$, which is the pre-image of E with respect to the canonical projection. Suppose that, for the generator x of E, the corresponding element in $\tilde{E}$ is the matrix $\tilde{x} = \begin{bmatrix} a & b \\ c & d\end{bmatrix} \in SL(2,\mathbb{C})$, where (as in the proof of [7, Theorem 3.1]) $bc \neq 0$, since E has finite covolume. For the generator $t \in E$, the corresponding element in $\tilde{E}$ is the matrix $\tilde{t} = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \in SL(2, \mathbb{C})$, where ζ is a primitive root of unity in $\mathbb{C}$ of order $2n$. Then the relation

$$tx^k t^{-1} = x^{-l} t^2 x^l t^{-2}$$

induces the relation

$$\begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^k \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}^l \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}^2 \begin{bmatrix} a & b \\ c & d \end{bmatrix}^l \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}^{-2} \tag{6}$$

where $\epsilon = \pm 1$. It was observed in [37, page 962] that in $SL(2, \mathbb{C})$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^j = \begin{bmatrix} S_j & bR_j \\ cR_j & T_j \end{bmatrix}$$

where $S_{j+1} = aS_j + bcR_j$, $T_{j+1} = dT_j + bcR_j$, and $R_{j+1} = S_j + dR_j$, with $S_1 = a$, $T_1 = d$ and $R_1 = 1$. Note that the determinant $S_j T_j - bcR_j^2 = 1$. Applying this formula to our case, the left hand side of (6) is

$$\begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \begin{bmatrix} S_k & bR_k \\ cR_k & T_k \end{bmatrix} \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} \epsilon S_k & \epsilon\zeta^2 bR_k \\ \epsilon\zeta^{-2} cR_k & \epsilon T_k \end{bmatrix}. \tag{7}$$

Similarly, the right hand side of (6) is

$$\begin{bmatrix} T_l & -bR_l \\ -cR_l & S_l \end{bmatrix} \begin{bmatrix} \zeta^2 & 0 \\ 0 & \zeta^{-2} \end{bmatrix} \begin{bmatrix} S_l & bR_l \\ cR_l & T_l \end{bmatrix} \begin{bmatrix} \zeta^{-2} & 0 \\ 0 & \zeta^2 \end{bmatrix} = \begin{bmatrix} T_l S_l - \zeta^{-4} bcR_l^2 & (\zeta^4 - 1) bT_l R_l \\ (\zeta^{-4} - 1) cS_l R_l & T_l S_l - \zeta^4 bcR_l^2 \end{bmatrix}. \tag{8}$$

Therefore, since $T_l S_l - bcR_l^2 = 1$, the equations (6), (7), (8) give

$$\begin{aligned} \begin{bmatrix} \epsilon S_k & \epsilon b\zeta^2 R_k \\ \epsilon\zeta^{-2} cR_k & \epsilon T_k \end{bmatrix} &= \begin{bmatrix} T_l S_l - \zeta^{-4} bcR_l^2 & (\zeta^4 - 1) bT_l R_l \\ (\zeta^{-4} - 1) cS_l R_l & T_l S_l - \zeta^4 bcR_l^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 + (1 - \zeta^{-4}) bcR_l^2 & (\zeta^4 - 1) bT_l R_l \\ (\zeta^{-4} - 1) cS_l R_l & 1 + (1 - \zeta^4) bcR_l^2 \end{bmatrix}. \end{aligned} \tag{9}$$

Comparing the terms on both sides of (9) gives

$$\begin{aligned} \epsilon b\zeta^2 R_k &= (\zeta^4 - 1) bT_l R_l, \\ \epsilon c\zeta^{-2} R_k &= (\zeta^{-4} - 1) cS_l R_l. \end{aligned}$$

Since $bc \neq 0$, we get

$$\begin{aligned} \epsilon R_k &= (\zeta^2 - \zeta^{-2}) R_l T_l, \\ \epsilon R_k &= -(\zeta^2 - \zeta^{-2}) R_l S_l, \end{aligned} \tag{10}$$

and, since ζ is a primitive root of unity of order $2n$ with n odd $\zeta^2 - \zeta^{-2} \neq 0$, so

$$R_l(T_l + S_l) = 0. \tag{11}$$

Suppose $R_l = 0$. Then $\epsilon T_k = \epsilon S_k = 1$ by (9) and $R_k = 0$ by (10). Hence $\tilde{x}^k = \begin{bmatrix} S_k & bR_k \\ cR_k & T_k \end{bmatrix} = \pm I$, so $x_i^{2k} = 1$ for all generators x_i of $F^{k/l}(n)$. Thus the order $|F^{k/l}(n)^{\mathrm{ab}}|$ divides $(2k)^n$ and then Corollary 3.6

implies that $|F^{k/l}(n)^{\mathrm{ab}}|$ is even and k is odd. Then by Corollary 3.5 l is odd and $n \equiv 0 \bmod 3$. The map from $F^{k/l}(n)$ to $F^{k/l}(3)$ (and hence from $F^{k/l}(n)^{\mathrm{ab}}$ to $F^{k/l}(3)^{\mathrm{ab}}$) sending x_i to $x_{i \bmod 3}$ is a surjective homomorphism. Hence x_0, x_1 and x_2 each has order dividing $2k$ in $(F^{k/l}(3))^{\mathrm{ab}}$ and so $|F^{k/l}(3)|^{\mathrm{ab}}$ divides $(2k)^3$. But $|F^{k/l}(3)|^{\mathrm{ab}} = k^3 + 3kl^2$ which divides $(2k)^3$, and so $k^2 + 3l^2$ divides $8k^2$. Therefore there exists a natural number m such that

$$l^2 = \frac{(8-m)k^2}{3m} \tag{12}$$

and so $m \in \{1, 2, \ldots, 7\}$. When $m = 1, 2, 3, 4, 5, 6, 7$, equation (12) implies $k = \sqrt{3}l/\sqrt{7}, l, 3l/\sqrt{5}, \sqrt{3}l, \sqrt{5}l, 3l, \sqrt{21}l$ respectively. Hence $m = 2$ or 6, and so either $k = l = 1$ or $k = 3$ and $l = 1$. If $k = l = 1$ then the result is given in [28, Theorem 3.1] so assume $k = 3, l = 1$, and therefore $F^{3/1}(n)$ divides $(2k)^n = 6^n$. If $9|n$ then $|F^{3/1}(9)^{\mathrm{ab}}| = 2^2 \cdot 3^3 \cdot 433$ divides $|F^{3/1}(n)^{\mathrm{ab}}|$, a contradiction. Thus $n = 3p$ for some p where $(p, 3) = 1$. If $n \geq 9$ then the result follows from Corollary 3.7 and if $n = 3$ then a computation in GAP shows that $F^{3/1}(3)$ is a finite group of order 3528 which cannot occur as subgroup of $PSL(2, \mathbb{C})$ (see, for example, [27, pages 152–154]).

Now suppose that $R_l \neq 0$. Then $T_l = -S_l$ by (11) so $\tilde{x}^l$ is traceless, and so $\tilde{x}^{2l} = -I$, where I is the identity element of $SL(2, \mathbb{C})$. Therefore $x^{2l} = I$ and so $x_i^{2l} = 1$ for all generators x_i of $F^{k/l}(n)$. Thus the order $|F^{k/l}(n)^{\mathrm{ab}}|$ divides $(2l)^n$. Hence by Corollary 3.6 $k = l = 1$, and $n = 3$, in which case $F^{k/l}(n) = F(3) \cong Q_8$, which is not the fundamental group of a hyperbolic 3-orbifold. $\square$

Proof of Corollary 4.2. Since k is odd, the defining relators of $F^{k/l}(n)$ imply that for each $0 \leq i < n$ generator $x_{i+1} = (x_{i+1}^{-(k-1)/2})^2 x_i^{-l} x_{i+2}^l$, which is a product of an even number of generators. Hence if G is the fundamental group of a hyperbolic 3-orbifold of finite volume, then that orbifold must be orientable, which is not possible by Theorem 4.1. $\square$

5. Torsion and asphericity

In this section we fix $w(n, k) = x_0^k x_1^k \ldots x_{n-1}^k \in F^{k/l}(n)$. Our starting point is the following result of Bardakov and Vesnin [2], who note that in the case $k = l = 1$ the words $w(n, 1) \in F(n)$ were first considered by Johnson [25].

Theorem 5.1 *([2, Proposition 3.1]). Suppose $n \geq 9$ is odd. Then $w(n, 1)$ is an element of order 2 in (the infinite group) $F(n)$.*

We have the following corollary concerning the asphericity of the relative presentation of the shift extension of $F^{k/l}(n)$, where the terms *relative presentation* and *aspherical* are as defined in [4].

Corollary 5.2 *(compare [4, Example 4.3(a)]). Suppose $n \geq 9$ is odd. Then the relative presentation $\langle G, x \mid xtxtx^{-1}t^{-2} \rangle$ (where $G = \langle t \mid t^n \rangle$) is not aspherical.*

(If $n \in \{3, 5, 7\}$ then $w(n, 1) = 1$ in the finite group $F(n)$.) Theorem 5.1 is significant because it gives examples of infinite cyclically presented groups with torsion. Indeed, in many studies (for example [15, 35, 2, 8, 12, 5, 36]) cyclically presented groups are proved infinite by showing that they are non-trivial and that the relative presentations of their shift extensions are aspherical, and deducing (by [15, Lemma 3.1], [3, Theorem 4.1(a)]) that the cyclic presentation is topologically aspherical, and hence that the cyclically presented group is torsion-free.

In this section we obtain similar results to Theorem 5.1 and Corollary 5.2 for groups $F^{k/l}(n)$ under certain conditions on k, l. Theorem 5.3(a) generalizes [25, Exercise 12, page 84] from $F(n)$ to $F^{k/l}(n)$; part

(b) generalizes the first part of the proof of [2, Proposition 3.1] (see also [25, Exercise 2, page 83]) from the groups $F(n)$ to the groups $F^{k/l}(n)$. We use the notation $[a,b]=a^{-1}b^{-1}ab$.

Theorem 5.3. *Let $n\geq 3$ be odd, $k,l\geq 1$ and let $w(n,k)=x_0^kx_1^k\dots x_{n-1}^k\in F^{k/l}(n)$. Then*

(a) $w(n,k)^2=1$;
(b) $w(n,k)=[x_0^l,x_{n-1}^l]$.

Proof. (a) We have

$$\begin{aligned}
w(n,k)^2 &= (x_0^kx_1^k)(x_2^kx_3^k)\dots(x_{n-1}^kx_0^k)\dots(x_{n-2}^kx_{n-1}^k)\\
&= (x_0^{-(l-k)}x_2^kx_2^{l-k})(x_2^{-(l-k)}x_4^kx_4^{l-k})\dots(x_{n-1}^{-(l-k)}x_1^kx_1^{l-k})\dots(x_{n-2}^{-(l-k)}x_0^kx_0^{l-k})\\
&= x_0^{-l}\left(x_0^kx_2^kx_4^k\dots x_{n-2}^k\right)x_0^l\\
&= x_0^{-l}\left((x_{n-1}^{-l}x_1^l)(x_1^{-l}x_3^l)(x_3^{-l}x_5^l)\dots(x_{n-3}^{-l}x_{n-1}^l)\right)x_0^l\\
&= 1.
\end{aligned}$$

(b) We have

$$\begin{aligned}
w(n,k) &= x_0^kx_1^kx_2^kx_3^kx_4^kx_5^k\dots x_{n-1}^k\\
&= x_0^{-l}x_0^k(x_0^lx_1^k)x_2^kx_3^kx_4^kx_5^k\dots x_{n-1}^k\\
&= x_0^{-l}x_0^k(x_2^l)x_2^kx_3^kx_4^kx_5^k\dots x_{n-1}^k\\
&= x_0^{-l}x_0^kx_2^k(x_2^lx_3^k)x_4^kx_5^k\dots x_{n-1}^k\\
&= \cdots\\
&= x_0^{-l}x_0^kx_2^kx_4^kx_6^k\dots x_{n-1}^kx_{n-1}^l\\
&= x_0^{-l}(x_{n-1}^{-l}x_1^l)(x_1^{-l}x_3^l)(x_3^{-l}x_5^l)(x_5^{-l}x_7^l)\dots(x_{n-2}^{-l}x_0^l)x_{n-1}^l\\
&= x_0^{-l}x_{n-1}^{-l}x_0^lx_{n-1}^l. \quad \square
\end{aligned}$$

As we now show, in many cases (for odd n) we have $w(n,k)\neq 1$, and so $w(n,k)$ is an element of order 2. Corollary 5.4 generalizes the second part of the proof of [2, Proposition 3.1] from the groups $F(n)$ to the groups $F^k(n)$, showing that for odd n the group $F^k(n)$ is not torsion-free. This is in contrast to the case where n is even where, if either $k=1$ and $n\geq 8$ or $k\geq 2$ and $n\geq 6$ the group $F^k(n)$ (being the fundamental group of a hyperbolic manifold [16, Theorem C], [29, Theorem 3]) is torsion-free.

Corollary 5.4. *Let $n\geq 3$ be odd, $k\geq 1$, $G=F^k(n)$ and let $w(n,k)=x_0^kx_1^k\dots x_{n-1}^k\in G$. Then the normal closure of $w(n,k)$ in G is equal to the derived subgroup of G. Thus $w(n,k)=1$ if and only if G is abelian. In particular, G is not torsion-free.*

Proof. By Theorem 5.3 $w(n,k)=[x_0,x_1]$, and by Lemma 2.7 G is generated by x_0,x_1 so the derived subgroup $G'=\langle\langle w\rangle\rangle^G$. For the 'in particular', note that if G is infinite, then since G^{ab} is finite, w is of order 2, so G is not torsion-free, and if G is finite then it is not torsion-free, since it is non-trivial. $\square$

For the case $n=3$ we have the following:

Corollary 5.5. *Let $k,l\geq 1$, $(k,l)=1$, $G=F^{k/l}(3)$ and let $w(3,k)=x_0^kx_1^kx_2^k\in G$. Then the normal closure of $w(3,k)$ in G is equal to the derived subgroup of G. Thus $w(3,k)=1$ if and only if G is abelian. In particular, G is not torsion-free.*

Proof. Let $w = w(3,k)$, $N = \langle\langle w \rangle\rangle^G$. By Theorem 5.3 $w = [x_0^l, x_1^l] \in G'$, so N is a subgroup of G'. We shall show that G/N is abelian, and so G' is a subgroup of N, and hence $N = G'$. The 'in particular' will follow as in the proof of Corollary 5.4.

In G/N we have $x_0^k x_2^k x_1^k = (x_2^{-l} x_1^l)(x_1^{-l} x_0^l)(x_0^{-l} x_2^l) = 1$ and $x_0^k x_1^k x_2^k = w(3,k) = 1$ and hence $x_1^k x_0^k = x_2^{-k} = x_0^k x_1^k$, $x_2^k x_1^k = x_0^{-k} = x_1^k x_2^k$, $x_0^k x_2^k = x_1^{-k} = x_2^k x_0^k$. That is, $[x_j^k, x_{j+1}^k] = 1$ for each $0 \le j \le 2$. Moreover, by Theorem 5.3(b), for each $0 \le j \le 2$ we have $1 = \theta^{j+1}(w) = \theta^{j+1}([x_0^l, x_2^l]) = [x_{j+1}^l, x_j^l]$.

The relations $x_j^l x_{j+1}^k = x_{j+2}^l$ and $[x_j^k, x_{j+2}^k] = 1$ in G/N imply

$$x_j^k x_{j+2}^l = x_j^k x_j^l x_{j+1}^k = x_j^l x_j^k x_{j+1}^k = x_j^l x_{j+1}^k x_j^k = x_{j+2}^l x_j^k.$$

Hence

$$x_{j+2}^k x_j^l = x_{j+2}^k x_{j+1}^l x_{j+2}^k = x_{j+1}^l x_{j+2}^k x_{j+2}^k = x_j^l x_{j+2}^k.$$

Since $(k,l) = 1$ there exist $\alpha, \beta \in \mathbb{Z}$ such that $\alpha k + \beta l = 1$. Then for each j we have

$$\begin{aligned} x_j x_{j+2} &= x_j^{\alpha k+\beta l} x_{j+2}^{\alpha k+\beta l} = x_j^{\alpha k} x_j^{\beta l} x_{j+2}^{\beta l} x_{j+2}^{\alpha k} = x_j^{\alpha k} x_{j+2}^{\beta l} x_j^{\beta l} x_{j+2}^{\alpha k} = x_{j+2}^{\beta l} x_j^{\alpha k} x_j^{\beta l} x_{j+2}^{\alpha k} \\ &= x_{j+2}^{\beta l} x_j^{\beta l} x_j^{\alpha k} x_{j+2}^{\alpha k} = x_{j+2}^{\beta l} x_j^{\beta l} x_{j+2}^{\alpha k} x_j^{\alpha k} = x_{j+2}^{\beta l} x_{j+2}^{\alpha k} x_j^{\beta l} x_j^{\alpha k} = x_{j+2}^{\alpha k+\beta l} x_j^{\alpha k+\beta l} = x_{j+2} x_j. \end{aligned}$$

Hence G/N is abelian. □

Example 5.6. Let $G = F^{1/2}(5)$. Using GAP we see that $G/\langle\langle w(5,1)\rangle\rangle^G \cong \mathbb{Z}_{101} = G^{\mathrm{ab}}$. Moreover, a computation using KBMAG shows that G is infinite, so non-abelian, and thus $w(5,1) \neq 1$ in $F^{1/2}(5)$, which is therefore not torsion-free.

Thus Corollaries 5.4, 5.5 and Example 5.6 give cases where $w(n,k) = 1$ is equivalent to $F^{k/l}(n)$ being abelian. We expect that in most cases $F^{k/l}(n)$ (n odd) is not abelian, and thus $w(n,k) \neq 1$. However, in some cases $F^{k/l}(n)$ is abelian. The cases we know of are $F(5) \cong \mathbb{Z}_{11}$, $F(7) \cong \mathbb{Z}_{29}$, $F^{1/2}(3) \cong \mathbb{Z}_{13}$ and $F^{2/3}(3) \cong \mathbb{Z}_{62}$. It would be interesting to know if there are any further cases. We know of the following finite non-abelian groups $F^{k/l}(n)$: $F(3) \cong Q_8$; $F^{1/3}(3)$ of order 3584; $F^{1/4}(3)$ of order 392; $F^2(3)$ of order 112; $F^3(3)$ of order 3528; $F^{3/2}(3)$ of order 504. In each of these cases $n = 3$ so $w(3,k) \neq 1$ by Corollary 5.5.

We now turn to the question of asphericity.

Corollary 5.7. *Suppose $n \ge 3$ is odd and let $k, l \ge 1$. If $F^{k/l}(n)$ is finite or $w(n,k) \neq 1$ in $F^{k/l}(n)$ then the relative presentation $\mathcal{P} = \langle G, x \mid x^l t x^k t x^{-l} t^{-2} \rangle$ (where $G = \langle t \mid t^n \rangle$) is not aspherical.*

Proof. The group $G(\mathcal{P})$ defined by $\mathcal{P}$ is isomorphic to the shift extension of $F^{k/l}(n)$ so $F^{k/l}(n)$ is isomorphic to a subgroup of $G(\mathcal{P})$. By [24, Section 3] (due to Serre), if the relative presentation $\mathcal{P}$ is aspherical then every finite subgroup of $G(\mathcal{P})$ is conjugate to a subgroup of G (see also [4, Theorem 2.4(c)]). If $F^{k/l}(n)$ is finite and conjugate to a subgroup of G then $F^{k/l}(n)$ is abelian of order at most n; but by Theorem 3.1 $|F^{k/l}(n)^{\mathrm{ab}}| \ge |F(n)^{\mathrm{ab}}| > n$ for all $n \ge 3$, a contradiction. If $w(n,k) \neq 1$ then it generates a cyclic subgroup of $F^{k/l}(n)$ of order 2 which, since n is odd, is not conjugate to a subgroup of G. Thus $\mathcal{P}$ is not aspherical. □

Corollary 5.8. *Suppose $n \ge 3$ is odd and $k \ge 1$. Then the relative presentation $\mathcal{P} = \langle G, x \mid x t x^k t x^{-1} t^{-2} \rangle$ (where $G = \langle t \mid t^n \rangle$) is not aspherical.*

Proof. By Corollary 5.7 we may assume $w(n,k) = 1$, so $F^k(n)$ is abelian, by Corollary 5.4. But then $F^{k/l}(n)$ is finite, so the result follows from Corollary 5.7. □

Note that Corollary 5.8 generalizes Corollary 5.2.

Corollary 5.9. *Suppose $k, l \geq 1$, $(k, l) = 1$. Then the relative presentation $\mathcal{P} = \langle G, x \mid x^l t x^k t x^{-l} t^{-2} \rangle$ (where $G = \langle t \mid t^3 \rangle$) is not aspherical.*

Proof. By Corollary 5.7 we may assume $w(3, k) = 1$, so $F^{k/l}(3)$ is abelian, by Corollary 5.5. But then $F^{k/l}(3)$ is finite, so the result follows from Corollary 5.7. $\square$

We now introduce the following quotients of groups $F^{k/l}(n)$. For each $n \geq 2$, $k, l \geq 1$ and each $\Omega \geq 0$ define

$$F^{k/l}(n; \Omega) = \langle x_0, \ldots, x_{n-1} \mid x_i^l x_{i+1}^k = x_{i+2}^l, x_i^\Omega = 1 \ (0 \leq i < n) \rangle.$$

Lemma 5.10. *Let $m \geq 3$, $K, L \geq 1$, $(K, L) = 1$, $\Omega \geq 0$. Suppose $F^{K/L}(m; \Omega)$ is infinite (resp. is non-cyclic, resp. is non-abelian, resp. is non-solvable). Then for all n, k, l where $k \equiv \pm K$ mod Ω, $l \equiv \pm L$ mod Ω, $n \equiv 0$ mod m, the group $F^{k/l}(n)$ is infinite (resp. is non-cyclic, resp. is non-abelian, resp. is non-solvable). Further, if n/m is odd and $w(m, K) \neq 1$ in $F^{K/L}(m; \Omega)$ then $w(n, k) \neq 1$ in $F^{k/l}(n)$.*

Proof. Let $\epsilon = \pm 1, \delta = \pm 1$, $n \equiv 0 \bmod m$, $k \equiv \epsilon K \bmod \Omega$, $l \equiv \delta L \bmod \Omega$. Let $\phi : F^{k/l}(n) \to F^{k/l}(m)$ be the natural epimorphism given by $\phi(x_i) = x_{i \bmod m}$. We have $F^{k/l}(m) \cong F^{\epsilon k/\delta l}(m)$ by Lemma 2.3, so it maps onto $F^{K/L}(m; \Omega)$. If this latter group is infinite (or is non-cyclic or is non-abelian, or is non-solvable) then the same therefore holds for $F^{k/l}(n)$. It remains to show that if n/m is odd and $w(m, K) \neq 1$ in $F^{K/L}(m; \Omega)$ then $w(n, k) \neq 1$ in $F^{k/l}(n)$.

Now if n/m is odd then $\phi(w(n, k)) = w(m, k)^{n/m} = (w(m, k)^2)^{(n/m-1)/2} w(m, k) = w(m, k) \in F^{k/l}(m)$. By adjoining the relators x_i^Ω $(0 \leq i < m)$ the group $F^{k/l}(m)$ maps onto $F^{\epsilon K/\delta L}(m; \Omega) \cong F^{K/L}(m; \Omega)$. Thus if $w(m, K) \neq 1$ in $F^{K/L}(m; \Omega)$ then $w(m, K) \neq 1$ in $F^{k/l}(m)$ and hence $w(n, k) \neq 1$ in $F^{k/l}(n)$. $\square$

In Corollaries 5.11, 5.12, 5.13 we give applications of Lemma 5.10 and in Example 5.14 we give further examples of groups $F^{k/l}(m; \Omega)$ to which Lemma 5.10 can usefully be applied.

Corollary 5.11. *Suppose $n \equiv 3$ mod 6, $k, l \geq 1$, $(k, l) = 1$. If $F^{k/l}(3)$ is non-abelian then $w(n, k) \neq 1$ in $F^{k/l}(n)$.*

Proof. By Corollary 5.5 if $F^{k/l}(3)$ is non-abelian then $w(3, k) \neq 1$ in $F^{k/l}(3) = F^{k/l}(3; 0)$ so the result follows from Lemma 5.10. $\square$

In cases where the order of the generators x_i of $F^{K/L}(m)$ is known and finite we can set Ω equal to that order. However, it can be fruitful to set Ω to be a proper divisor of that order. Both instances are exhibited in the proof of the following corollary, where the order of generators x_i of $F^{1/1}(3)$ is equal to 4 (and we set $\Omega = 4$); whereas the order of the generators x_i of the groups $F^{1/3}(3)$, $F^{1/4}(3)$, $F^2(3)$, $F^{3/2}(3)$ is 28, 49, 14, 63, respectively (and we set $\Omega = 7$).

Corollary 5.12.

(a) *If k and l are odd and $n \equiv 3$ mod 6 then $w(n, k) \neq 1$ in $F^{k/l}(n)$.*
(b) *If $(\pm k \bmod 7, \pm l \bmod 7) \in \{(1, 3), (2, 1), (3, 2)\}$ and $n \equiv 3$ mod 6 then $w(n, k) \neq 1$ in $F^{k/l}(n)$.*

Proof. (a) This follows from Lemma 5.10 by observing that $F^{1/1}(3; 4) \cong Q_8$ and so $w(3, 1) \neq 1$ in this group by Corollary 5.5. (b) This follows from Lemma 5.10 by observing that $F^{1/3}(3; 7) \cong F^{2/1}(3; 7) \cong F^{3/2}(3; 7)$ is a non-abelian group (of order 56) and so $w(3, 1) \neq 1$ in this group by Corollary 5.5. $\square$

Corollary 5.13. *If $(k,l)=1$, k is even, $l \equiv 3 \bmod 6$, and $n \equiv 0 \bmod m$, where $m \in \{5,7\}$ then $F^{k/l}(n)$ is infinite.*

Proof. The hypotheses imply $k \equiv \pm 2 \bmod 6$ and $l \equiv 3 \bmod 6$. For $m \in \{5,7\}$ computations in GAP show that $F^{2/3}(m;6)$ has an index 5 subgroup with infinite abelianisation. Therefore $F^{2/3}(m;6)$ is infinite, and the result follows from Lemma 5.10. □

Example 5.14.

(a) $F^{1/3}(5;6) \cong PSL(2,11)$; $F^{3/1}(3;36)$ is a non-abelian, solvable group of order 3528; $F^{3/2}(3;63)$ is a non-abelian, solvable group of order 504; $F^{3/1}(3;6) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_6$.

(b) Computations with the `NewmanInfinityCriterion` command [31] in GAP (applied to the derived subgroup or second derived subgroup) show that the following groups $F^{k/l}(n;\Omega)$ are infinite, and therefore $w(n,k) \neq 1$ in $F^{k/l}(n)$ by Corollaries 5.4 and 5.5: $F^{1/1}(9;76)$, $F^{3/1}(9;108)$, $F^{2/1}(17;206)$, $F^{2/1}(21;98)$, $F^{2/1}(23;94)$, $F^{3/5}(3;126)$, $F^{1/11}(3;182)$. The Ω values are selected as divisors of the order of $F^{k/l}(n)^{\mathrm{ab}}$ that are large enough for the quotient $F^{k/l}(n,\Omega)$ to be infinite yet small enough for the `NewmanInfinityCriterion` command to complete.

6. 3-manifold groups

Theorem 5.1 was used in [23] to obtain the following result.

Theorem 6.1 *([23, Theorem 3]). If $n \geq 3$ is odd then $F(n)$ is a 3-manifold group if and only if $n = 3,5,7$, in which case $F(n) \cong Q_8, \mathbb{Z}_{11}, \mathbb{Z}_{29}$, respectively.*

In this section we use the results of Section 5 to prove the following corresponding result to Theorem 6.1 for the groups $F^k(n)$ and $F^{k/l}(3)$. As reported earlier, the groups $F(5), F(7), F^{1/2}(3), F^{2/3}(3)$ are cyclic and $F(3) \cong Q_8$, and we expect this to be the only non-cyclic 3-manifold group among the groups $F^k(n)$ and $F^{k/l}(3)$. Part (a) of Theorem 6.2 is in contrast to the case when n is even, where $F^k(n)$ is the fundamental group of a hyperbolic 3-manifold if either $k=1$ and $n \geq 8$ [16, Theorem C] or $k \geq 2$ and $n \geq 6$ [29, Theorem 3].

Theorem 6.2.

(a) *Let $n \geq 3$ be odd, $k \geq 1$. If $F^k(n)$ is a non-cyclic 3-manifold group then k is odd, $n \equiv 3 \bmod 6$ and $F^k(n) \cong Q_8 \times \mathbb{Z}_{V_n^{k/1}/4}$.*

(b) *Let $k,l \geq 1$, $(k,l)=1$. If $F^{k/l}(3)$ is a non-cyclic 3-manifold group then k and l are odd and $F^{k/l}(3) \cong Q_8 \times \mathbb{Z}_{V_3^{k/l}/4}$.*

We first extract an argument from the proof of Theorem 6.1 and apply it to groups $F^{k/l}(n)$:

Lemma 6.3. *Let $n \geq 3$ be odd, $k,l \geq 1$, let $G = F^{k/l}(n)$ and let $w(n,k) = x_0^k x_1^k \ldots x_{n-1}^k \in G$. If $w(n,k) \neq 1$ then G is not a 2-generator, infinite, 3-manifold group. In particular:*

(a) *if $n \geq 3$ is odd and $k \geq 1$ then $F^k(n)$ is not an infinite 3-manifold group;*

(b) *if $k,l \geq 1$, where $(k,l)=1$, then $F^{k/l}(3)$ is not an infinite 3-manifold group.*

Proof. Suppose G is a 2-generator, infinite, 3-manifold group. By Theorem 5.3 we have $w(n,k) \in G'$. Therefore the subgroup $< w(n,k) > \cong \mathbb{Z}_2$ is an orientation preserving subgroup of $G = \pi_1(M)$ of finite order. Then by [13, Theorem 8.2] (see also [17, Theorem 9.8]) we have $M = R \# M_1$ where R is closed and orientable, $\pi_1(R)$ is finite, and $< w(n,k) >$ is conjugate to a subgroup of $\pi_1(R)$. Since G is infinite we have $\pi_1(M_1) \neq 1$ and since it can be generated by two elements $\pi_1(R)$ and $\pi_1(M_1)$ are each cyclic. But the derived subgroup of a free product of cyclic groups is free, contradicting the fact that $w(n,k) \in G'$ is an element of order two.

Part (a) (resp. Part (b)) follows since $F^k(n)$ (resp. $F^{k/l}(3)$) is 2-generated by Lemma 2.7 (resp. Lemma 2.8) and if it is infinite then $w(n,k) \neq 1$ by Theorem 3.1 and Corollary 5.4 (resp. Corollary 5.5). □

To consider when $F^k(n)$ and $F^{k/l}(3)$ can be finite 3-manifold groups we need the following classification of finite 3-manifold groups (see [22, Section 2] or [1, Section 1.5]) and their derived subgroups.

Theorem 6.4. *Suppose G is a finite 3-manifold group. Then either G is cyclic or $G \cong H \times \mathbb{Z}_p$ where $p \geq 1$ is coprime to $|H|$ and H is as in one of the following cases:*

(i) $H = P_{48} = \langle x, y \mid x^2 = (xy)^3 = y^4, x^4 = 1\rangle$, *with* $H/H' \cong \mathbb{Z}_2$, $H' \cong SL(2,3)$ *and* $H'/H'' \cong \mathbb{Z}_3$;
(ii) $H = P_{120} = \langle x, y \mid x^2 = (xy)^3 = y^5, x^4 = 1\rangle$, *a perfect group;*
(iii) $H = Q_{4m} = \langle x, y \mid x^2 = (xy)^2 = y^m\rangle$, $m \geq 2$, *with* $H/H' \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ *(m even),* $H/H' \cong \mathbb{Z}_4$ *(m odd), and* $H' \cong \mathbb{Z}_m$;
(iv) $H = D_{2^m(2n+1)} = \langle x, y \mid x^{2^m} = 1, y^{2n+1} = 1, xyx^{-1} = y^{-1}\rangle$, $m, n \geq 1$, *with* $H/H' \cong \mathbb{Z}_{2^m}$ *and* $H' \cong \mathbb{Z}_{2n+1}$;
(v) $H = P'_{8\cdot 3^m} = \langle x, y, z \mid x^2 = (xy)^2 = y^2, zxz^{-1} = y, zyz^{-1} = xy, z^{3^m} = 1\rangle$, $m \geq 1$, *with* $H^{\mathrm{ab}} \cong \mathbb{Z}_{3^m}$ *and* $H' \cong Q_8$.

We now prove Theorem 6.2.

Proof of Theorem 6.2. Let $G = F^{k/l}(n)$ where $k, l \geq 1$, $(k,l) = 1$ and either $l = 1$ or $n = 3$. By Lemma 6.3 we may assume that G is a finite, non-cyclic, 3-manifold group.

Observe that in each case $w = w(n,k) \neq 1$ by Corollaries 5.4 and 5.5, and that G is generated by x_0 and x_1 by Lemmas 2.7 and 2.8. Suppose that $G \cong H \times \mathbb{Z}_p$ where H is one of the groups in (i)–(v) of Theorem 6.4 and $(p, |H|) = 1$. Then the derived subgroup D of G is isomorphic to the derived subgroup of H. By Theorem 5.3 the element $w(n,k) = [x_0^l, x_1^l]$ has order 2 in G. Corollaries 5.4 and 5.5 imply that D is the normal closure of w in G and so $D^{\mathrm{ab}} \cong \mathbb{Z}_2^d$ for some $d \geq 0$, which gives a contradiction if H is the group in part (i) or (iv).

If $H = P_{120}$ or $H = P'_{8\cdot 3^m} \cong Q_8 \rtimes \mathbb{Z}_{3^m}$ (as in parts (ii) or (v)) then H has a unique element h of order 2, and the normal closure $\langle\langle h\rangle\rangle^H$ is not isomorphic to the derived subgroup of H. Therefore $H \times \mathbb{Z}_p$ has a unique element of order 2, namely $(h, 0)$, and the normal closure $\langle\langle (h,0)\rangle\rangle^{H\times\mathbb{Z}_p}$ is not the derived subgroup of $H \times \mathbb{Z}_p \cong G$, a contradiction (since the normal closure $\langle\langle w\rangle\rangle^G = D$).

If $H = Q_{4m}$ for some $m \geq 2$ (as in part (iii)), then $D \cong \mathbb{Z}_m$, so $m = 2$ and hence $G \cong Q_8 \times \mathbb{Z}_p$, where $p = V_n^{k/l}/4$. Therefore there is an epimorphism $G \twoheadrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and since the images of generators x_0, x_1 have order 2 there is also an epimorphism $F^{k/l}(n;2) \twoheadrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$. If k is even then $(k,l) = 1$ implies l is odd so $F^{k/l}(n;2) \cong \mathbb{Z}_2$, a contradiction. Therefore k is odd. Since $|G^{\mathrm{ab}}| = V_n^{k/l}$ is even the remaining conditions on k, l, n follow from Corollary 3.5. □

Acknowledgements

The authors thank Jim Howie for helpful comments on a draft and the referee for the careful reading of the article.

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