Option Pricing : The Reduced-Form SDE Model

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Abstract

We use PDEs to describe the pricing process of options in an illiquid market. These equations are derived from stochastic differential equations built on the Ito process. With the help of Lie symmetry analysis, this paper focuses on the pricing of a model that incorporates the effect of large traders in an illiquid market. The non-linear partial differential equation representing this model incorporates a nonzero risk-neutral interest rate. This partial differential equation is singularly perturbed and quadratic in the highest derivative. Using the method of Lie symmetry analysis, we obtain symmetries in the mathematical package Program Lie, these symmetries are used to analyze the equation and to reduce the partial differential equation to ordinary differential equations. When the equations are solved, they yield group invariant solutions to the partial differential equation. We give a graphical representation of the obtained solutions. These invariant solutions are new to the field and can be used in place of simulations.

Keywords. Reduced-form SDE model, Lie algebras, symmetries, illiquid markets.

Key messages.

- Reduction of the impact on the stock price by large traders in an illiquid market
- Singularly perturbed PDE model with a quadratic gamma function
- Linearize a lower order derivative instead of the gamma function to obtain symmetries
- New invariant solutions for pricing options

1 Introduction

One of the most significant strides made in the study of Financial Mathematics was the Black-Scholes-Merton(BSM) model [11]. Up to this day, the Black-Scholes-Merton remains relevant in the pricing of portfolios and options. However, the Black-Scholes-Merton is based on assumptions that make it only applicable in a perfect world, with ideal markets. That said, markets are constantly bombarded with various frictions, thus making them far from ideal. For example, markets may be illiquid [12]. In these markets, the Black-Scholes-Merton model will not work. However, there can be numerous derivations from the Black-Scholes-Merton model employed in these illiquid markets.

In a bid to solve this problem, there have been many developments in various models. In this study, the focus is on one of the well known derivations which is the Reduced-Form Stochastic Differential Equation Model. Grull and Taschini derived first estimation methods which were used to calibrate reduced-form models. They went on to show how the prices varied in the article [13]. Graselli and Hurd [14], focused on the recoveries of par, treasury and market value of reduced-form models by using the example of a two-factor Gaussian model. Both articles, [14, 13], derived the reduced-form models from equilibrium models like the Black-Scholes-Merton model. Through calibrations to historical data, their studies looked at the stochastic behaviour of the reduced models in comparison to equilibrium models. In illiquid markets, one of the factors acting on the market is the ability to control the equilibrium of stock prices based on market share, especially for large traders. The reduced-form SDE model is applied to the pricing of portfolios, provided there exist large traders.

This is what led Bordag and Frey [3] to develop a model and consequently derive a PDE. However, not many methods can be employed to solve a PDE such as the reduced-form SDE model.

A method that has brought tremendous results and advancements in solving PDEs is that of Lie Symmetry Analysis [20]. Several fields of studies have made use of this method, for example, Caister, O'Hara and Govinder priced the Asian option in [7], Sinkala, Leach and O'Hara obtained group invariant solutions to a PDE model for zero-coupon bonds in [17], Sophocleous, O'Hara, and Leach obtained algebraic solutions for Stein-Stein model for stochastic volatility in [18], and also studied a model of stochastic volatility with time-dependent parameters in [19], just to mention a few. In this study, we solve a non-linear PDE through the method of Lie Symmetry Analysis; see [8] for an earlier example using Lie symmetries in finance, when the associated PDE is non-linear.

The remainder of the paper is as follows: In Section 2 we present the derivation to the nonlinear PDE and provide the first simple solution. In Section 3 we look at a more general approach to solving the PDE and the associated solution. We describe the method of Lie symmetry analysis in Section 4 and explain how to use it to find solutions to the PDE. In this section, we also look at the different invariant solutions arrived at, and provide their graphical representations. Section 5 contains a discussion of the holistic view of the paper and the significance of the solutions.

2 The non-linear differential equation

The reduced-form SDE model developed in [3] targets large traders in a market. Due to their considerable portion of market share, large traders can significantly influence markets. Bordag and Frey [3] established a model where $U(t, S)$ is the value of the option at time t, with the stock price at time t being S. One of the crucial aspects of the model is a liquidity parameter ρ, with σ being the volatility of the underlying asset price. Assuming that $\rho \geq 0$ and Φ is the semimartingale standing for the stock-trading strategy used by a trader, the stock price satisfies the SDE [4]

$$dS_t = \sigma S_t dW_t + \rho S_t d\Phi_t. \tag{1}$$

It is important to note that the strength of the impact on price is dependent on ρ. We see that when $\rho = 0$, the asset price follows a Black-Scholes-Merton model [4]. For their model, Bordag and Frey [4] followed the following reasoning:

1. Large traders use the trading strategy Φ with a corresponding stock price S^{Φ}. The corresponding Markovian strategy being $\Phi_t = \phi(t, S_t)$.

2. Applying the *Itô* formula on (1) shows S^{ϕ} as an *Itô* process with the dynamics

$$dS_t^{\phi} = v^{\phi}\left(t, S_t^{\phi}\right) S_t^{\phi} dW_t + b^{\phi}\left(t, S_t^{\phi}\right) S_t^{\phi} dt,$$

for adjusted volatility given by

$$v^{\phi}(t, S) = \frac{\sigma}{1 - \rho S \phi_S(t, S)}.$$

3. Using the adjusted volatility, the relation $\phi_S = u_{SS}$ and $V_t = u(t, S_t)$ yielded the below PDE.

Have you defined U? And its relationship to V?

$$U_t + \frac{\sigma^2 S^2 U_{SS}}{2(1-\rho S U_{SS})^2} = 0. \tag{2}$$

They assumed that the risk-free interest rate, r, equals zero. However, with the high volatility in financial markets caused by interest rates, there was a drive to determine the effect if $r \neq 0$. When r is incorporated the model becomes

$$U_t + rSU_S + \frac{\sigma^2 S^2 U_{SS}}{2(1-\rho S U_{SS})^2} - rU = 0. \tag{3}$$

This equation is the focus of our study. This is clearly a strong extension to Bordag and Frey in [4]. In this instance, it is important to note that there exists a payoff $w(S_T)$ for a function $w : [0, \infty) \to \mathbb{R}$ were the terminal condition at maturity date T for $S \geq 0$ is $U(T, S) = w(S)$. In the case of a European call option $w(S) = max\{S - K, 0\}$ where the strike price $K > 0$.

2.1 Hedging strategies

Definition 2.1. *For an option, delta Δ is the rate of change of the option price to the price of the underlying stock. It is the slope of the curve that relates the option price to the underlying stock price.*

$$\Delta = \frac{\partial U(t, S)}{\partial S}, \tag{4}$$

where $U(t, S)$ denotes the value of price[6].

.... the price of an option.

Definition 2.2. *The gamma Γ of a portfolio of options on an underlying stock is the rate of change of the portfolio's delta with respect to the price of the underlying stock. It is the second partial derivative of the portfolio with respect to stock price:*

$$\Gamma = \frac{\partial^2 U(t, S)}{\partial S^2}, \tag{5}$$

where $U(t, S)$ is defined above.

Taking a close look at the denominator for (3), it is important to note $\Gamma \neq \frac{1}{\rho S}$ since this will give a denominator that is equal to 0 and an undefined fraction.

2.2 Case when $\Gamma = 0$

In the case when \Gamma=0, and the corresponding portfolio is gamma neutral.

Equation (3) poses a unique challenge since Γ is quadratic. We assume that $\Gamma = 0$ and we obtain a simple non-trivial solution. This assumption implies that Δ is independent of the underlying stock S. The result of this is a straightforward solution to (3) of the form:

$$U(t, S) = a_0(t) + a_1(t)S, \tag{6}$$

where both $a_0(t)$ and $a_1(t)$ are arbitrary functions. Since we are studying a European Call option which is not deep in the money and also not deep out of the money, our terminal condition being when $\Gamma = 0$, we can write the trivial solution as:

call

$$U(t, S) = S - Ke^{-rt}. \tag{7}$$

where K is the option strike price. Mathematically, it is also important to note that by assuming that $U(t, S)$ is a linear combination of separable variables,

$$U(t, S) = X(S) + T(t), \tag{8}$$

the solution obtained is of a similar form as (7). However, this assumption does not yield any further results in the focus of this paper.

Figure 1 is an illustration of....

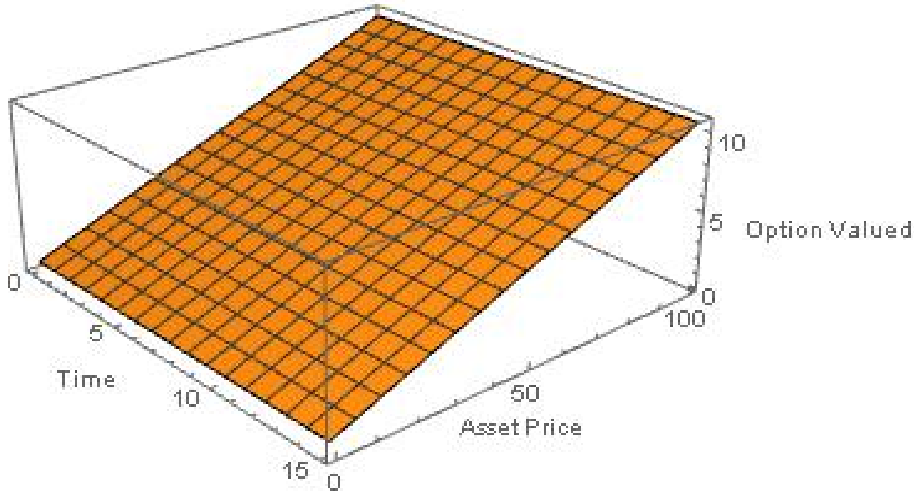

Figure 1: Case 2.1 when $\Gamma = 0$, $C_1 = 0.5$, $C_0 = 0.1$ and $r = 0.02$. Plot of the value of the option against the stock price and time. For the given range of stock price, S, the plot does shows that an increase in t will increase the value of the option, $U(t, S)$.

3 Lie Symmetry Analysis Methodology

3.1 Infinitesimal Transformations

Let

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{9}$$

be the infinitesimal operator or generator of G. The infinitesimal transformations can thus be re-written as

$$\overline{x}^i \approx (1 + aX)x^i, \overline{u}^\alpha \approx (1 + aX)u^\alpha \tag{10}$$

where a is an arbitrary constant.

Theorem 3.1. *Given the infinitesimal transformations* (10) or its symbol X, G is then obtained by solution of the Lie equations [2].

$$\frac{d\overline{x}^i}{da} = \xi^i(\overline{x}, \overline{u}), \frac{d\overline{u}^\alpha}{da} = \eta^\alpha(\overline{x}, \overline{u}),$$

subject to the conditions

$$\overline{x}^i|_{a=0} = x^i, \overline{u}^\alpha|_{a=0} = u^\alpha.$$

The theorem is employed to obtain one-parameter groups. The above transformations form a symmetry group G of the system if the function $\overline{u} = \overline{u}(\overline{x})$ satisfies

$$E^\sigma(\overline{x}, \overline{u}, \overline{u}_{(1)}, \dots, \overline{u}_{(k)}) = 0$$

whenever $u = u(x)$ satisfies the original differential equation

$$E^\sigma(x, u, u_{(1)}, \dots, u_{(k)}) = 0.$$

The one parameter group, $G^{[1]}$, is the first prolongation that acts in the space (x, u). Here $G^{[1]}$ is formed by transformations in $(x, u, u_{(1)})$ and the transformations $\overline{u}_{(1)} = \psi(x, u, u_{(1)}, a)$. To obtain the prolonged groups $G^{[2]}$ up to $G^{[k]}$ one uses the total derivative transforms.
Let the infinitesimal transformations of the prolonged groups be $G^{[1]}$ to $G^{[k]}$, then

$$\begin{aligned}
\overline{u}_i^\alpha &\approx u_i^\alpha + a\zeta_i^\alpha(x, u, u_{(1)}) \\
\overline{u}_{ij}^\alpha &\approx u_i^\alpha + a\zeta_{ij}^\alpha(x, u, u_{(1)}, u_{(2)}) \\
&\vdots \\
\overline{u}_{i_1 \dots i_k^\alpha} &\approx u_{i_j \dots i_k}^\alpha + a\zeta_{i_1 \dots i_k}^\alpha(x, u, u_{(1)}, \dots, u_{(k)}),
\end{aligned}$$

with prolongation formulas

$$\begin{aligned}
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j) \\
\zeta_{ij}^\alpha &= D_j(\zeta_i^\alpha) - u_{il}^\alpha D_j(\xi^l) \\
&\vdots \\
\zeta_{i_1 \dots i_k}^\alpha &= D_{i_j}(\zeta_{i_k \dots i_{k-1}}^\alpha) - u_{i_1 \dots i_{kl}}^\alpha D_j(\xi^l).
\end{aligned}$$

The generators of the prolonged groups are:

$$\begin{aligned} X^{[1]} = \xi^i(x,u)\frac{\partial}{\partial x^i} &+ \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x,u,u_{(1)}) \\ &\vdots \\ X^{[k]} = \xi^i(x,u)\frac{\partial}{\partial x^i} &+ \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x,u,u_{(1)}) \\ +\dots &+ \zeta^\alpha_{i_1\dots i_k}(x,u,\dots,u_{(k)})\frac{\partial}{\partial u^\alpha_{i_1\dots i_k}}, \end{aligned}$$

where X is the generator of the group G.

4 Solving the differential equation using Lie Symmetry Analysis

Olver describes how to obtain Lie symmetries from a PDE in [21], which are used to reduce the PDE. A computer package that is useful in obtaining the symmetries is Program Lie [1]. Essentially, the main idea is to reduce a PDE into a system of ordinary differential equations using symmetries, thus making it easier to solve the system.

Arbitrarily, the following holds:

$$E(t,S,U,U_t,U_S,U_{tS},U_{tt},U_{SS}) = U_t + rSU_S + \frac{\sigma^2 S^2 U_{SS}}{2(1-\rho S U_{SS})^2} - rU = 0. \tag{11}$$

However, (3) is a special type of PDE. It is known as a singularly perturbed PDE, so obtaining the symmetries by the usual method will not work. The fact that (3) is non-linear in the highest derivative of the function $U(t,S)$, but quadratic, poses a challenge in obtaining the symmetries. One possible approach to solve (3) requires both σ, the volatility of the underlying stock, and ρ, the liquidity parameter to equate to 0, i.e., $\rho = 0$ and $\sigma = 0$. However, this would defeat the purpose of this study as (3) is singularly perturbed. This type of differential equation results in complex functions that may be difficult to solve for symmetries. Frey dealt with an example of a singularly perturbed PDE in [9].

To help solve (3), instead of linearizing the highest derivative U_{SS} as stipulated in [1], we linearize U_t. This unconventional linearizing became one of the fundamentals of solving (3). Applying this method in Program Lie, we obtain the following Lie symmetries:

$$G_1 = \partial_t \tag{12}$$

$$G_2 = S\partial_U \tag{13}$$

$$G_3 = e^{rt}\partial_U \tag{14}$$

$$G_4 = U\partial_U + S\partial_S. \tag{15}$$

The given symmetries are vital in reducing (3) and ultimately solving it, meaning that equation (3) admits a non-trivial four-dimensional Lie algebra extended across the obtained generators. The algebra $\mathit{Diff}_{\triangle}(M)$ given as a linear combination of the symmetries:

$$G = \alpha_1 G_1 + \alpha_2 G_2 + \alpha_3 G_3 + \alpha_4 G_4. \tag{16}$$

with α_1, α_2, α_3 and α_4 being arbitrary constants. In as much as there are four arbitrary constants, α_4 will fall away due to the normalization of the associated symmetry, meaning that only α_1, α_2, and α_3 would be associated with the solutions. These constants eventually form part of the basis of the solution for $U(t,S)$, but have minimal effect in the movement of the value of the option with time t.

The expression G in (16) is known as an operator, and there exists $G^{[2]}$, which is the second prolongation of (16). This concept can be noted in [2, 10, 15].

In the process of solving for the value of the option $U(t,S)$, the linear combination (16) of symmetries amounts to the following system:

$$\frac{dt}{1} = \frac{dU}{\alpha_3 U + \alpha_2 e^{tr} + \alpha_1 S} = \frac{dS}{\alpha_3 S}. \tag{17}$$

The Lagrange system in (17) is solved to get an expression for the price of the underlying stock

	G_1	G_2	G_3	G_4
G_1	0	0	rG_3	0
G_2	0	0	0	0
G_3	$-rG_3$	0	0	G_3
G_4	0	0	$-G_3$	0

Table 1: The commutator table of symmetries $G_1, \ldots, G_4$. The table confirms the eligibility of the obtained symmetries to reduce the PDE in question.

at time ~~$S(t)$~~:

$$S(t) = pe^{\alpha_3 t}, \tag{18}$$

where p is a constant. Solving the following Lagrange system will give the value of the portfolio:

$$\frac{d(Ue^{-\alpha_3 t})}{dt} = \alpha_2 e^{t(r-\alpha_3)} + \alpha_1 S e^{-\alpha_3 t}. \tag{19}$$

4.1 Case when $\alpha_3 \neq r$

Reducing equation (19) will give the value of U:

$$U = \frac{\alpha_2}{r - \alpha_3} e^{t\alpha_3} - \alpha_1 \alpha_3 S + e^{\alpha_3 t} q(Se^{-\alpha_3 t}), \tag{20}$$

where $p = Se^{-\alpha_3 t}$ and $q(p)$ is the constant function. Now, applying (20) to the PDE (3) and simplifying will give a reduced ODE in p:

$$(\alpha_3 - r)q(p) - (\alpha_3 - r)pq_p(p) + \frac{\sigma^2 p^2 q_{pp}(p)}{2(1 - \rho p q_{pp})^2} = 0. \tag{21}$$

In the same way we are solving for (3), we now look for Lie symmetries for (21). By using the Wolfram Mathematica [22], we arrive at a set of symmetries. However, these symmetries come with their own sets of conditions in order for them to hold. For any given one of the symmetries obtained, either one of the following conditions has be true:

$$\rho = 0 \tag{22}$$

$$\sigma = 0 \tag{23}$$

$$r = \alpha_3. \tag{24}$$

The liquidity parameter ρ, as explained in Section 2, is the influence of the large trader in a market, and so a condition $\rho = 0$ would defeat the purpose of this research. In the same manner, the condition $\sigma = 0$ takes away the volatility of the market from the analysis, and would also lead to a different PDE given that fraction on (3) would fall away. Lastly, this subsection is looking at a specific case when $r \neq \alpha_3$, and so (24) directly opposes this. At this point, none of the symmetries obtained for (21) would add any value to the study of this section. The unique case when $\alpha_3 = r$ will then be looked at in the following sections.

One way of obtaining a solution for (3) in this subsection would be to exploit an assumption $q_{pp} = 0$. A simple solution exists for q_{pp} of the form,

$$q = q_0 + pq_1, \tag{25}$$

where q_0 and q_1 are arbitrary constants. Substituting (25) into (21) and solving gives $q_0 = 0$ and simplifies (25) to:

$$q = pq_1. \tag{26}$$

An invariant solution, therefore, exists when we substitute (26) into (20), to give:

$$U = \frac{\alpha_2}{r - \alpha_3} e^{t\alpha_3} - \alpha_1 \alpha_3 S + Sq_1. \tag{27}$$

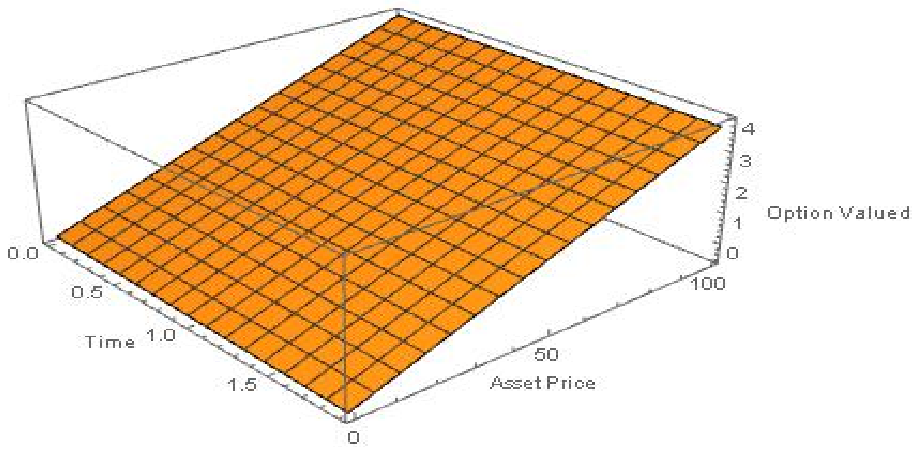

Figure 2: Case 4.1 when $r \neq \alpha_3$, $\alpha_2 = 0.063$, $\alpha_3 = 0.2$, $\alpha_1 = 0.03$, $q_1 = 0.05$ and $r = 0.04$. Plot of the value of the option against the stock price and time. For the given range of stock price S, the plot does shows that an increase in S will increase the value of the option $U(t, S)$.

4.2 Case when $r = \alpha_3$

In the event that $r = \alpha_3$, (19) reduces to the following differential,

$$\frac{d(Ue^{-rt})}{dt} = \alpha_2 + \alpha_1 p, \tag{28}$$

an immediate solution to (3) exists:

$$U = \alpha_2 t e^{rt} + \alpha_1 t S + q e^{rt}, \tag{29}$$

where $q(p)$ is a arbitrary function.

Substituting (29) into (3) will reduce (3) to an ODE:

$$\alpha_2 + \alpha_1 p + \frac{\sigma^2 p^2 q_{pp}(p)}{2(1 - \rho p q_{pp})^2} = 0. \tag{30}$$

4.2.1 $\alpha_1 = 0$

Equation (30) brings complexity, and one way of solving this would be to assume that $\alpha_1 = 0$. There exists a solution for the arbitrary function $q(p)$. Since equation (30) is quadratic, there would be two invariant solutions to (3), U_{a+} representing the value of the option for the specific solution to the quadratic equation and U_{a-} denoting the value of the option for the negative part of the quadratic solution following from (29):

$$\begin{aligned}U_{a+} =& e^{rt}(\alpha_2 t + C_1 + e^{-2rt}SC_2) - \frac{1}{4\alpha_2\rho^2} \\ &\Bigg[(4e^{-rt}S\alpha_2\rho + \frac{1}{2}e^{-2rt}S^2\sigma^2 - 4e^{-rt}S\alpha_2\rho \ln\left[e^{-rt}S\right] - 8e^{-rt}S\alpha_2\rho\ln\left[\sigma b_1 + b_2\right] \\ &+ \frac{32\alpha_2^2\rho^2\ln\left[\sigma b_1 + b_2\right]}{\sigma^2} - \frac{16\alpha_2^2\rho^2\ln\left[\sigma^2 b_1 + \sigma b_2\right]}{\sigma^2} + \frac{4\alpha_2\rho b_1 b2}{\sigma} - \frac{2\alpha_2\rho b_1 b_2}{\sigma} \\ &+ \frac{1}{2}\sigma b_2(e^{-rt}S)^{3/2}\Bigg],\end{aligned} \tag{31}$$

where

$$\begin{aligned}b_1 &= \sqrt{e^{-rt}S} \\ b_2 &= \sqrt{-8\alpha_2\rho + e^{-rt}S\sigma^2}.\end{aligned} \tag{32}$$

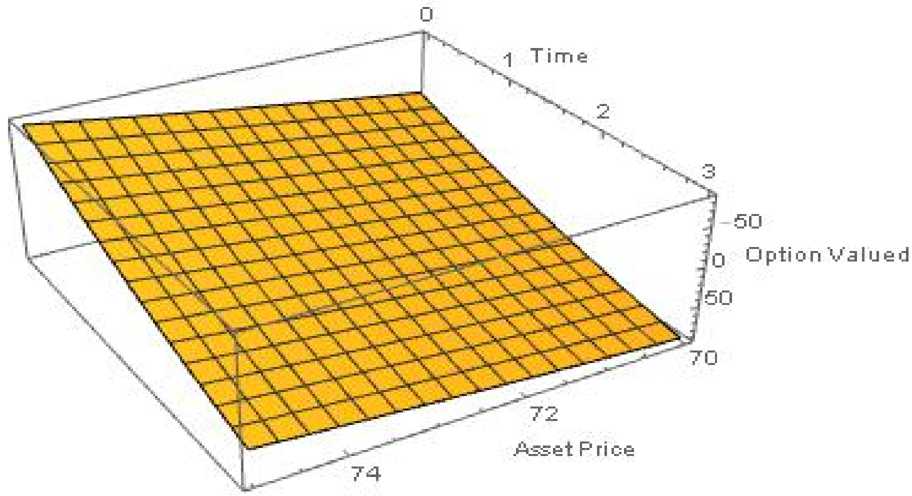

Figure 3: Case 4.2.1 when $r = \alpha_3$ and $\alpha_1 = 0$, $r = 0.05$, $\alpha_2 = 0.6$, $C_1 = 0.1$, $C_2 = 0.5$, $\sigma = 0.18$ and $\rho = 0.19$. First plot of the value of the option against the stock price and time. For the given range of stock price S, the plot does shows that an increase in S results in an increase in the value of the option $U(t, S)$.

$$\begin{aligned} U_{a-} =& e^{rt}(\alpha_2 t + C_1 + e^{-2rt} S C_2) + \frac{1}{4\alpha_2\rho^2} \\ &\Bigg[-4e^{-rt}S\alpha_2\rho - \frac{1}{2}e^{-2rt}S^2\sigma^2 + 4e^{-rt}S\alpha_2\rho \ln\left[e^{-rt}S\right] - 8e^{-rt}S\alpha_2\rho\ln[\sigma d_1 + d_2] \\ &+ \frac{32\alpha_2^2\rho^2\ln[\sigma d_1 + d_2]}{\sigma^2} - \frac{16\alpha_2^2\rho^2\ln[\sigma^2 d_1 + \sigma d_2]}{\sigma^2} + \frac{4\alpha_2\rho d_1 d_2}{\sigma} - \frac{2\alpha_2\rho d_1 d_2}{\sigma} \\ &+ \frac{1}{2}\sigma d_2 (e^{-rt}S)^{3/2} \Bigg], \end{aligned} \tag{33}$$

where

$$\begin{aligned} d_1 &= \sqrt{e^{-rt}S} \\ d_2 &= \sqrt{-8\alpha_2\rho + e^{-rt}S\sigma^2}. \end{aligned} \tag{34}$$

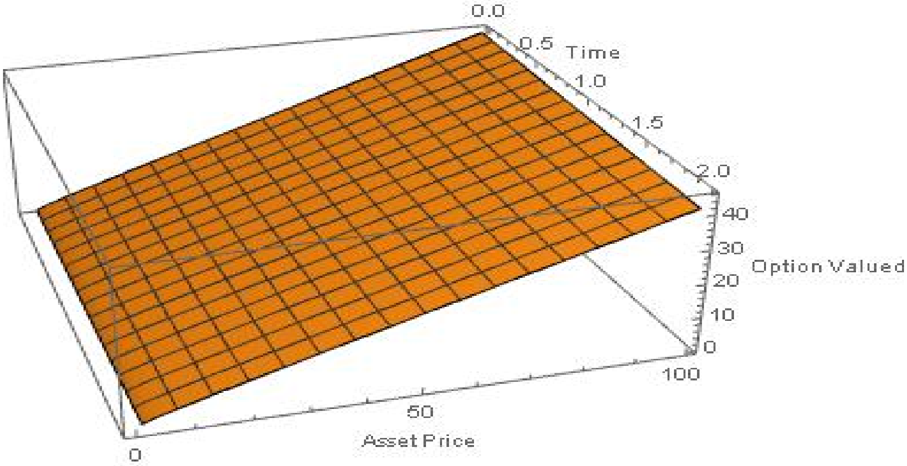

Figure 4: Case 4.2.1 when $r = \alpha_3$, $\alpha_1 = 0$, $r = 0.05$, $\alpha_2 = 0.01$, $C_1 = 0.001$, $C_2 = 0.005$, $\sigma = 0.25$ and $\rho = 0.9$. Second plot of the value of the option against the stock price and time. For the given range of stock price S, the plot does show that an increase in S results in an increase in the value of the option $U(t, S)$.

4.2.2 $\alpha_2 = 0$

Assuming $\alpha_2 = 0$ in solving for the arbitrary function $q(p)$ in (30) will give two invariant solutions as well, U_{b+} denoting the value of the option for the definite solution to the quadratic component of (29) and U_{b-} denoting the value of the option for the negative solution to the quadratic component of (29):

$$\begin{aligned} U_{b-} =& \alpha_1 tS + e^{rt}\Bigg[St\alpha_1 + e^{rt}\Bigg[C_3 + e^{-rt}SC_4 \\ &+ \frac{\left[-4\alpha_1\rho + \sigma^2 - \sqrt{-8\alpha_1\rho\sigma^2 + \sigma^4}\right]\left[e^{-rt}S + e^{-rt}S\ln\left[\frac{e^{rt}}{S}\right]\right]}{4\alpha_1\rho^2}\Bigg]\Bigg]. \end{aligned} \tag{35}$$

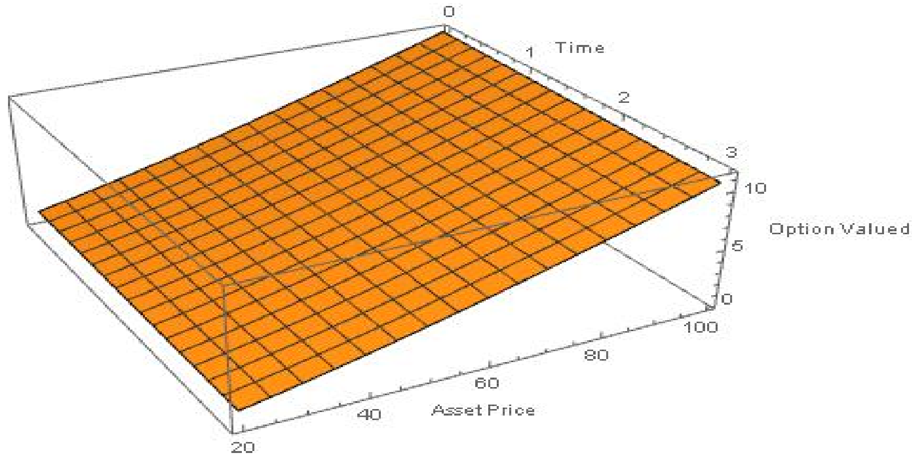

Figure 5: Case 4.2.2 when $r = \alpha_3$, $\alpha_2 = 0$, $r = 0.05$, $\alpha_1 = -0.001$, $C_1 = 0.001$, $C_2 = 0.005$, $\sigma = 0.25$ and $\rho = 0.9$. First plot of the value of the option against the stock price and time. For the given range of stock price S, the plot does shows that an increase in S results in an increase in the value of the option $U(t, S)$.

$$\begin{aligned} U_{b+} =& \alpha_1 t S + e^{rt}\Bigg[S t \alpha_1 + e^{rt}\Bigg[C_3 + e^{-rt} S C_4 \\ &+ \frac{\left[-4\alpha_1\rho + \sigma^2 + \sqrt{-8\alpha_1\rho\sigma^2 + \sigma^4}\right]\left[e^{-rt}S + e^{-rt}S\ln\left[\frac{e^{rt}}{S}\right]\right]}{4\alpha_1\rho^2}\Bigg]\Bigg]. \end{aligned} \tag{36}$$

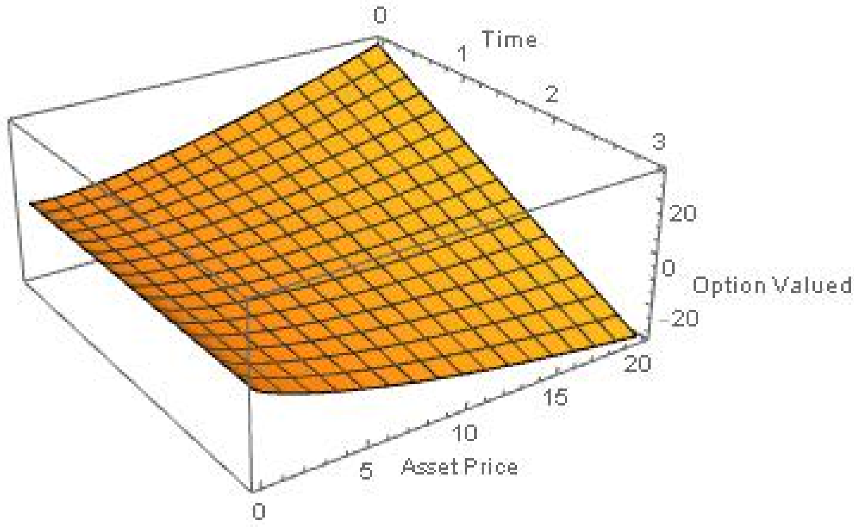

Figure 6: Case 4.2.2 when $r = \alpha_3$, $\alpha_2 = 0$, $r = 0.05$, $\alpha_1 = -1$, $C_1 = 0.01$, $C_2 = 0.05$, $\sigma = 0.18$ and $\rho = 1.5$. Second plot of the value of the option against the stock price and time. For the given range of stock price S, the plot does show that an increase in S results in an increase in the value of the option $U(t, S)$.

4.2.3 $\alpha_2 = 0$ and $\alpha_1 = \frac{-\sigma^2}{8\rho}$

There exists another pair of invariant solutions to (3) assuming $\alpha_2 = 0$ and $\alpha_1 = \frac{-\sigma^2}{8\rho}$ following solving (30):

$$\begin{aligned} U_{c-} =& \frac{-\sigma^2}{8\rho} tS + e^{rt}\Bigg[-\frac{St\sigma^2}{8\rho} + e^{rt}\Bigg[C_5 + e^{-rt}SC_6 \\ &+ \frac{\left(3 - 2\sqrt{2}\right)\left(-e^{-rt}S + e^{-rt}S\ln\left[e^{-rt}S\right]\right)}{\rho}\Bigg]\Bigg]. \end{aligned} \tag{37}$$

$$\begin{aligned} U_{c+} =& \frac{-\sigma^2}{8\rho} tS + e^{rt}\Bigg[-\frac{St\sigma^2}{8\rho} + e^{rt}\Bigg[C_5 + e^{-rt}SC_6 \\ &+ \frac{\left(3 + 2\sqrt{2}\right)\left(-e^{-rt}S + e^{-rt}S\ln\left[e^{-rt}S\right]\right)}{\rho}\Bigg]\Bigg]. \end{aligned} \tag{38}$$

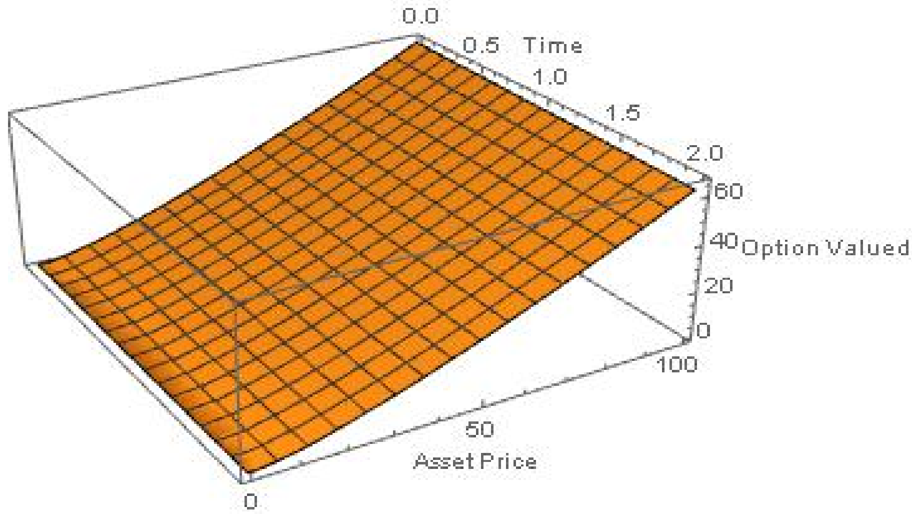

Figure 7: Case 4.2.3 when $r = \alpha_3$, $\alpha_2 = 0$, $\alpha_1 = \frac{-\sigma^2}{8\rho}$, $r = 0.05$, $C_1 = 0.1$, $C_2 = 0.01$, $\sigma = 0.25$, $\rho = 0.95$. First plot of the value of the option against the stock price and time. For the given range of stock price S, the plot does show that an increase in S has a negligible impact on the value of the option $U(t, S)$.

5 Discussion

This study focuses on the model (3), which is suitable for pricing in an illiquid market, taking into consideration the effect of large traders. To solve this model, the method of Lie symmetry analysis was mainly used. The symmetries admitted by (3) were obtained. The solutions obtained are exact and have not yet been obtained or explored. This represents the value of a portfolio $U(t, S)$.

In Section 2, we obtained solution (7), with Figure 1 as the graphical representation. The method used in Section 3 produces a solution in the same format as (7). At this point, it is important to note that this solution (7) does not have either ρ or σ, which brings an interesting notion to the effect of large traders.

In Section 2, we discussed the fact that ρ represents the strength of the impact on the price caused by large traders. The assumption is that $U_{SS} = 0$ will give a PDE free of ρ. As the volatility (ρ) approaches zero, the PDE turns to assume the Black-Scholes-Merton model and this is another study that can be pursued. The solution (7) suggests that there exist conditions that can give a value of the option that would not be impacted by the entry of large traders into the market. One can even argue that (7) can be likened to the Black-Scholes-Merton solutions since the Black-Scholes-Merton model exists when the liquidity parameter $\rho = 0$.

Further research would be warranted to prove this argument. In the same manner, one can also argue that (7) shows that the volatility, σ, of the underlying stock price would not affect the value of the option.

Further to this study would be an investigation on the replacement of the large trader by high speed, small value, but high volume transactions. In this case, speed would most likely be a variable. The reason that this notion would add value in the market is the notion that comes with these numerous high-speed trades that would speak to the volume of traders with a shared belief in a particular underlying stock, as opposed to one large trader whom the market might not have confidence in. Within this reasoning, one large trader may be contrary to the rest of the market, and intuitively their entry into the market would have minimal to no influence on the price associated.

Second to this build would be an investigation that quantifies a large trade. This research looks at a large trader in umbrella form, but it would be essential to identify at which point a trade can be classified as significant.

Section 4 employs the method of Lie Symmetry. This gave a whole host of solutions (27), (31), (32), (33), (34), (35), and (36) all with an existing interest rate r. Apart from these solutions suggesting that a large trader has an impact on the price of the underlying stock, they can also be utilized in an illiquid market with high volatility where the interest rate is not equal to zero. In as much as these solutions have not yet been explored, their implication fits the qualitative analysis associated with an illiquid market and is also in line with Bordag and Frey's research on a similar market [4]. A further study on the extent to which the value of the option can be impacted by a large trader would be a worthy follow-up to this study. This further study could then be useful in determining at what point a trader can be classified as a large trader.

In conclusion, the study of this paper validates the traditional qualitative analysis that suggests that a large trader will have an impact on the underlying stock of a given portfolio, and has also uncovered the possibility of a large trader-free portfolio. The obtained solutions can be used to validate solutions obtained using numerical methods.

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