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Dimension Theory in Iterated Local Skew Power Series Rings

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Abstract

Many well-known local rings, including soluble Iwasawa algebras and certain completed quantum algebras, arise naturally as iterated skew power series rings. We calculate their Krull and global dimensions, obtaining lower bounds to complement the upper bounds obtained by Wang. In fact, we show that many common such rings obey a stronger property, which we call triangularity, and which allows us also to calculate their classical Krull dimension (prime length). Finally, we correct an error in the literature regarding the associated graded rings of general iterated skew power series rings, but show that triangularity is enough to recover this result.

Keywords Iwasawa algebras · Skew power series rings · Local rings · Quantum algebras

Mathematics Subject Classification (2010) 16S34 · 16S36 · 16S99 · 16P60

Introduction

0.1 Skew Power Series Rings and Locality

The main objects of study of this paper are *iterated local skew power series rings* $k[[x_1; \sigma_1, \delta_1]] \dots [[x_n; \sigma_n, \delta_n]]$, as defined below.

Modelled on the notion of a skew polynomial extension $A[x; \sigma, \delta]$, *skew power series extensions* $B = A[[x; \sigma, \delta]]$ (Definition 1.2 below) were first introduced in [20] and [22], with a view to using them to study Iwasawa algebras: it was noted in [20] that certain Iwasawa algebras can be written as *iterated* skew power series extensions over a base field (usually $\mathbb{F}_p$) or complete discrete valuation ring (usually $\mathbb{Z}_p$).

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Unlike the case of skew polynomial extensions, it is *not* the case that an arbitrary skew derivation (σ, δ) on A gives rise to a well-defined ring $A[[x; \sigma, \delta]]$, due to possible convergence issues. For this reason, we work throughout the paper with a special class of skew power series extensions that are known to exist: we require A to have a unique maximal ideal $\mathfrak{m}$, with respect to which it is complete (and separated), and we stipulate that (σ, δ) should preserve the $\mathfrak{m}$-adic filtration in a certain way: see Definition 1.4. We will call these *local* skew power series extensions of A, or skew power series extensions of $(A, \mathfrak{m})$, to emphasise these stipulations. (This terminology is unambiguous: as we will show below, under our constraints, $A[[x; \sigma, \delta]]$ will also be a complete local ring, with maximal ideal generated by $\mathfrak{m}$ and x.)

There is a growing literature on (onefold) local skew power series extensions $S = R[[x; \sigma, \delta]]$: see, for instance, [21] and [14] for important milestones in this theory. We add some small results along these lines as part of Theorem A below.

However, very little work has been done to treat the *iterated* case so far. The most basic noncommutative case, that of *q-commutative power series rings* (completed quantum planes), has been studied in [15], and a few properties of general iterated local skew power series rings are established in [23], but the author is not aware of any other developments in the theory.

0.2 Iterated Skew Power Series Rings

Let $(R, \mathfrak{m})$ be a complete local ring. A ring S is an *(n-fold) iterated* skew power series extension of $(R, \mathfrak{m})$ if it can be written

$$S = R[[x_1; \sigma_1, \delta_1]] \dots [[x_n; \sigma_n, \delta_n]] \tag{0.1}$$

for suitable choices of (σ_i, δ_i): a more precise definition is given below. S will again turn out to be a complete local ring.

Our first results are as follows.

Theorem A *Let* $(R, \mathfrak{m})$ *be a complete local ring, and* $(S, \mathfrak{n})$ *an n-fold iterated local skew power series extension of* $(R, \mathfrak{m})$.

(i) *If* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is prime, then* $\mathrm{gr}_{\mathfrak{n}}(S)$ *and S are prime.*
(ii) *If R is noetherian, then S is faithfully flat over R.*
(iii) *If R is scalar local and* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is Auslander-regular, then* $\mathrm{gr}_{\mathfrak{n}}(S)$ *is Auslander-regular, S is Auslander-Gorenstein, and S is AS-Gorenstein.*
(iv) *If* $\mathrm{gr}_{\mathfrak{m}}(R)$ *is a noetherian integral domain which is a maximal order in its quotient ring, then both* $\mathrm{gr}_{\mathfrak{n}}(S)$ *and S have the same properties.*

Part (iii) of this theorem allows the use of dualising complexes in the sense of Yekutieli: see e.g. [26] and [27]. We take advantage of this in the proof of Theorem B.

Even under rather general conditions, we can understand much of the dimension theory of these rings: our second result, along these lines, complements the work of Wang [23].

Theorem B *Let k be a field, and let R be an n-fold iterated local skew power series extension of k. Then* $\mathrm{Kdim}(R) = \mathrm{gldim}(R) = \mathrm{injdim}(R) = \mathrm{Cdim}(R) = \mathrm{projdim}(k) = n$.

0.3 Triangularity

The process of *iteration* is often rather badly behaved. Suppose that $(R, \mathfrak{m})$ is a complete local ring and $S = R[[x_1; \sigma_1, \delta_1]] \dots [[x_n; \sigma_n, \delta_n]]$ is an n-fold iterated local skew power series extension. Write $R_0 = R$ and, for all $1 \le i \le n$, denote by R_i the i-fold subextension $R[[x_1; \sigma_1, \delta_1]] \dots [[x_i; \sigma_i, \delta_i]]$. Then each R_i is a local ring with some maximal ideal $\mathfrak{m}_i$. By definition, $(\sigma_{i+1}, \delta_{i+1})$ is always a skew derivation of $(R_i, \mathfrak{m}_i)$, but it does *not* necessarily restrict to a skew derivation of $(R_j, \mathfrak{m}_j)$ for any $0 \le j \le i-1$: see Non-examples 2.19 and 2.20

In the case when every $(\sigma_{i+1}, \delta_{i+1})$ restricts to a skew derivation of $(R_j, \mathfrak{m}_j)$ for *all* $0 \le j \le i-1$, we call $R[[x_1; \sigma_1, \delta_1]] \dots [[x_n; \sigma_n, \delta_n]]$ a *triangular* n-fold skew power series extension of $(R, \mathfrak{m})$. Note that triangularity is a property not just of the ring, but also of the choice of ordered generating set $(x_1, \dots, x_n)$ (cf. "upper triangular" matrices); rings that can be written in such a form by means of a change of variables could be called *triangularisable*. For the sake of simplicity, we will usually assume that triangularisable extensions have already been triangularised.

This appears to be a fairly restrictive condition: we expect that relatively few n-fold iterated local skew power series extensions of a ring $(R, \mathfrak{m})$ will be triangularisable unless R and n are small. But, in fact many well-known rings *are* triangularisable in this sense, including q-commutative skew power series rings (Example 2.14), Iwasawa algebras of supersoluble uniform groups (Examples 2.15–2.16), and various other completed quantum algebras (Examples 2.17–2.18).

Under reasonable conditions, we can often directly calculate the classical Krull dimension (prime length) $\mathrm{clKdim}(R)$, as defined in [17, 6.4.4]:

Theorem C *Let k be a division ring, and let R be an n-fold iterated local skew power series extension of k. If R is triangularisable, or R has pure automorphic type, then* $\mathrm{clKdim}(R) = n$.

0.4 Filtrations

So far we have not said much about the natural filtration on R, but this will be crucial for understanding its ideals in future work.

In this paper, a *filtration* on a ring R is a function $f : R \to \Gamma \cup \{\infty\}$, where Γ is an ordered group isomorphic to $\mathbb{Z}$, such that $f(0) = \infty$, $f(1) = 0$, $f(r+s) \ge \min\{f(r), f(s)\}$ and $f(rs) \ge f(r) + f(s)$ for all $r, s \in R$; and we will usually assume that such filtrations are *separated*, i.e. $f^{-1}(\infty) = \{0\}$. If I is an ideal of R satisfying $\bigcap_n I^n = \{0\}$, then the *I-adic filtration* f_I on R is given by $f_I(r) = \max\{n : r \in I^n\}$.

Given a filtration f on R, we can define additive subgroups $F_\gamma R := f^{-1}([\gamma, \infty])$ for all $\gamma \in \Gamma$. Conversely, given an appropriate decreasing family of additive subgroups $F_\gamma R$ for all $\gamma \in \Gamma$, we can define $f(r) = \max\{\gamma : r \in F_\gamma R\}$. We can also form the *associated graded ring* $\mathrm{gr}_f(R) := \bigoplus_{\gamma \in \Gamma} (F_\gamma R / F_{\gamma^+} R)$, where γ^+ is the immediate successor of γ in Γ. For the basics of filtrations and their associated graded rings, the reader may consult [18] or [12]. (Note that [12] uses a different notational convention, in which the family of additive subgroups is *increasing*.)

The family $\{F_\gamma R\}_{\gamma \in \Gamma}$ forms a fundamental system of neighbourhoods of 0 in R, under which R is a (Hausdorff) topological ring in the sense of [24, Definition 1.1]. We can form the *completion* $\widehat{R}_f$ of R with respect to f as a left R-module by setting $\widehat{R}_f :=$

$\varprojlim_{\gamma\in\Gamma}(R/F_\gamma R)$, which naturally inherits a filtration from f (see [12, Chapter 1, §§3.3–3.5] for more details of this construction): this in fact turns out to be a ring, and the canonical map $R \to \widehat{R}_f$ is a homomorphism of filtered rings [12, Chapter 1, §3.5, Proposition 3]. We will usually be interested in the case when R is *complete* with respect to f, i.e. the natural map $R \to \widehat{R}_f$ is an isomorphism.

Theorem D *Let $(R, \mathfrak{m})$ be a complete local ring, (σ, δ) a local skew derivation on R, and $S = R[[x; \sigma, \delta]]$ a local skew power series extension, with unique maximal ideal $\mathfrak{n}$. Then there exists a skew derivation $(\overline{\sigma}, \overline{\delta})$ of $\mathrm{gr}_{\mathfrak{m}}(R)$ such that the inclusion of graded rings $\mathrm{gr}_{\mathfrak{m}}(R) \to \mathrm{gr}_{\mathfrak{n}}(S)$ extends to an isomorphism $(\mathrm{gr}_{\mathfrak{m}}(R))[X; \overline{\sigma}, \overline{\delta}] \cong \mathrm{gr}_{\mathfrak{n}}(S)$ upon mapping X to $\mathrm{gr}(x)$. Moreover, $\overline{\delta} = 0$ if and only if $\delta(R) \subseteq \mathfrak{m}^2$ and $\delta(\mathfrak{m}) \subseteq \mathfrak{m}^3$.*

We give Example 1.15 to show that $\overline{\delta}$ can indeed be nonzero: this corrects a small error in [14] and [23]. However, in many nice cases of interest, we can get rid of $\overline{\delta}$ by modifying the filtration, as below.

Suppose A is a subring of R. We will say that a skew derivation (σ, δ) on R is *A-linear* if, for all $a \in A$, we have $\sigma(a) = a$ and $\delta(a) = 0$.

Theorem E *Let $(R, \mathfrak{m})$ be a complete local ring, and let*

$$S = R[[x_1; \sigma_1, \delta_1]] \ldots [[x_n; \sigma_n, \delta_n]]$$

be a triangular iterated local skew power series extension with maximal ideal $\mathfrak{n}$. Then there exists a filtration f of S, cofinal with the natural $\mathfrak{n}$-adic filtration and coinciding with the $\mathfrak{m}$-adic filtration on R, such that

$$\mathrm{gr}_f(S) \cong \mathrm{gr}_{\mathfrak{m}}(R)[X_1; \overline{\sigma_1}][X_2; \overline{\sigma_2}] \ldots [X_n; \overline{\sigma_n}]$$

is an iterated skew polynomial ring of pure automorphic type, where $X_i = \mathrm{gr}_f(x_i)$ for each $1 \le i \le n$.

Notations and Conventions

A ring R is *local* if it has a unique maximal (two-sided) ideal $\mathfrak{m}$ which is co-artinian, i.e. $R/\mathfrak{m}$ is a simple artinian ring. R is *scalar local* if additionally $\mathfrak{m}$ is maximal among *left* ideals (equivalently, right ideals: [9, Proposition and Definition 3.16]), i.e. $R/\mathfrak{m}$ is a division ring. We often write local rings as $(R, \mathfrak{m})$ for emphasis: unless specified, the filtration on R is $\mathfrak{m}$-adic. All local rings under consideration in this paper will be complete, and all filtrations are assumed to be separated.

We borrow standard notation from skew polynomial rings: when the derivation δ is zero, we will write the skew power series ring $R[[x; \sigma, \delta]]$ as $R[[x; \sigma]]$, and so on.

1 Preliminaries

1.1 Definitions and First Results

Definition 1.1 Let R be a ring. A *skew derivation* on R is a pair (σ, δ), where σ is a ring automorphism of R and δ is a *left σ-derivation* on R: that is, for all $a, b \in R$, $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$.

Definition 1.2 Suppose we are given a skew derivation (σ, δ) on a ring R.

1. The *skew polynomial ring* $R[x; \sigma, \delta]$ is defined to be equal to $R[x]$ as a left R-module, with R-linear multiplication determined by the rule
$$xa = \sigma(a)x + \delta(a) \tag{1.1}$$
for all $a \in R$.
2. We may try to form the *skew power series ring* analogously. If the left R-module of formal power series $R[[x]]$ becomes a ring after we impose the (continuous, R-linear) multiplication rule (1.1), we call it a skew power series ring, and denote it $R[[x; \sigma, \delta]]$.

Unlike in the case of skew polynomials, we are not guaranteed that skew power series form a ring due to possible convergence issues. For instance, given some $a \in R$, we need to be able to write the product $(1 + x + x^2 + \dots)a$ as a (left) power series. This requires the element a to "move past" infinitely many terms in x, and as it does so, it generates infinite sums of elements of R: for instance, the constant term in the expression above must be $a + \delta(a) + \delta(\delta(a)) + \dots$, so we need this infinite sum to converge in R.

There are two obvious ways to ensure this converges. One, as in e.g. [2] and [10, §5] and others, is to insist that δ be *locally nilpotent*: that is, for each $a \in R$, there should be some $n \in \mathbb{N}$ such that $\delta^n(a) = 0$, ensuring that the sum above is finite. The other is to impose a complete topology on R and insist that δ respect it: for instance, we must have $\delta^n(a) \to 0$ as $n \to \infty$. We will focus entirely on the latter in this paper, as it is the primary case of interest in the Iwasawa case.

Definition 1.3 Let $(R, \mathfrak{m})$ be a complete local ring, and let (σ, δ) be a skew derivation on R. We say that (σ, δ) is a *local* skew derivation if the following three conditions are satisfied: (i) $\sigma(\mathfrak{m}) = \mathfrak{m}$, (ii) $\delta(R) \subseteq \mathfrak{m}$, and (iii) $\delta(\mathfrak{m}) \subseteq \mathfrak{m}^2$.

These conditions imply that $\sigma(\mathfrak{m}^i) = \mathfrak{m}^i$ and $\delta(\mathfrak{m}^i) \subseteq \mathfrak{m}^{i+1}$ for all i.

If these conditions are satisfied, and $\mathrm{gr}_{\mathfrak{m}}(R)$ is noetherian, then [14, §3.4] tells us that the ring $R[[x; \sigma, \delta]]$ defined above exists. This gives us a wealth of examples of such rings, many of which will be explored below. Note also that, while our conditions are more restrictive than those of [14], the same proof can be easily extended to show that skew power series rings over appropriately *filtered* rings exist.

Definition 1.4 The ring S is a *local skew power series extension of* R if R is a complete local ring and, for some $x \in S$, we have a local skew derivation (σ, δ) of R such that $S = R[[x; \sigma, \delta]]$.

We justify our use of the word "local":

Lemma 1.5 *Let $(R, \mathfrak{m})$ be a complete local ring, and let $S = R[[x; \sigma, \delta]]$ be a local skew power series extension. Then:*

(i) $\mathfrak{n} = \mathfrak{m} + xS = \mathfrak{m} + Sx$ *is the unique maximal ideal of S, and the natural inclusion $R \subseteq S$ induces an isomorphism $S/\mathfrak{n} \cong R/\mathfrak{m}$.*
(ii) $(S, \mathfrak{n})$ *is local. Moreover, $(S, \mathfrak{n})$ is scalar local if and only if $(R, \mathfrak{m})$ is scalar local.*
(iii) S *is $\mathfrak{n}$-adically complete.*

Proof (i) and (iii) follow from [22, §2]. For (ii), note that a local ring $(R, \mathfrak{m})$ is scalar local if and only if its residue ring $R/\mathfrak{m}$ is a division ring; the equivalence now follows from the isomorphism of part (i). □

We will also need a corollary of (i) later, which we record immediately:

Lemma 1.6 *Let $(R, \mathfrak{m})$ be a complete local ring, and let $S = R[[x; \sigma, \delta]]$ be a local skew power series extension. Then $\mathfrak{n}^j = \mathfrak{m}^j + \mathfrak{m}^{j-1}x + \cdots + \mathfrak{m}x^{j-1} + Sx^j$.*

Proof We prove this by induction on j. Lemma 1.5(i) shows that the claim is true for $j = 1$.

For $j > 1$: suppose that $\mathfrak{n}^{j-1} = \mathfrak{m}^{j-1} + \mathfrak{m}^{j-2}x + \cdots + \mathfrak{m}x^{j-2} + Sx^{j-1} = \sum_{k=1}^{j-1} \mathfrak{m}^k x^{j-1-k} + Sx^{j-1}$, and left-multiply by $\mathfrak{n} = \mathfrak{m} + Sx$: then

$$\begin{aligned}\mathfrak{n}^j &= (\mathfrak{m} + Sx)\left(\sum_{k=0}^{j-1} \mathfrak{m}^k x^{j-1-k} + Sx^{j-1}\right)\\ &= \sum_{k=0}^{j-1} \mathfrak{m}^{k+1}x^{j-1-k} + \sum_{k=0}^{j-1} Sx\mathfrak{m}^k x^{j-1-k} + SxSx^{j-1} + \mathfrak{m}Sx^{j-1}.\end{aligned}$$

The first sum is $\mathfrak{m}^j + \mathfrak{m}^{j-1}x + \cdots + \mathfrak{m}x^{j-1}$, which is part of our desired outcome. Using the facts that $\mathfrak{m}^l S = S\mathfrak{m}^l$ (see e.g. Lemma 2.3 below) and $S = R + Rx + \cdots + Rx^{l-1} + Sx^l$ (from the left R-module decomposition $S = R[[x]]$), we simplify the other terms in turn:

- $x\mathfrak{m}^k = \sigma(\mathfrak{m}^k)x + \delta(\mathfrak{m}^k)$, which is contained in $\mathfrak{m}^k x + \mathfrak{m}^{k+1}$ by assumption, and so
$$Sx\mathfrak{m}^k x^{j-1-k} \subseteq S\mathfrak{m}^k x^{j-k} + S\mathfrak{m}^{k+1}x^{j-1-k} = \mathfrak{m}^k Sx^{j-k} + \mathfrak{m}^{k+1}Sx^{j-1-k}.$$
Replacing S by $R + Rx + \cdots + Rx^{k-1} + Sx^k$ in the first term of the right-hand side shows that $\mathfrak{m}^k Sx^{j-k}$ is contained in $\mathfrak{m}^k x^{j-k} + Sx^j$, as required. Similarly, replacing S by $R + Rx + \cdots + Rx^k + Sx^{k+1}$ in the second term of the right-hand side shows that $\mathfrak{m}^{k+1}Sx^{j-1-k} \subseteq \mathfrak{m}^{k+1}x^{j-1-k} + Sx^j$.
- $SxSx^{j-1} = Sx(R + Sx)x^{j-1} \subseteq Sx^j + S\mathfrak{m}x^{j-1} + SxSx^j \subseteq Sx^j + \mathfrak{m}x^{j-1}$.
- $\mathfrak{m}Sx^{j-1} = \mathfrak{m}(R + Sx)x^{j-1} \subseteq \mathfrak{m}x^{j-1} + Sx^j$.

This concludes the proof. □

Notation 1.7 *The phrase "local skew power series extension" already borders on unwieldy, and will get worse when further adjectives are added below, so we make the following simplifications. As in the introduction, it will be convenient to explicitly mention local rings together with their maximal ideals – e.g. as $(R, \mathfrak{m})$ rather than simply R – and suppress the word "local". We will also usually abbreviate "skew power series" to "SPS-", and we will denote the class of SPS-extensions of $(R, \mathfrak{m})$ by* $\mathbf{SPS}(R, \mathfrak{m})$ *(or by* $\mathbf{SPS}(R)$ *when $\mathfrak{m}$ is understood from context).*

For instance, we may write "let R be a complete local ring with maximal ideal $\mathfrak{m}$, and let S be a local skew power series extension of R with maximal ideal $\mathfrak{n}$" simply as "let $(R, \mathfrak{m})$ be a complete local ring, and $(S, \mathfrak{n}) \in \mathbf{SPS}(R, \mathfrak{m})$".

Recall also the definition of (A-)linear from Section 0.4. *If (σ, δ) is an A-linear skew derivation on the complete local ring $(R, \mathfrak{m})$, and $S = R[[x; \sigma, \delta]]$, we may write $(S, \mathfrak{n}) \in$* $\mathbf{SPS}_A(R, \mathfrak{m})$ *to emphasise the A-linearity.*

Definition 1.8 The ring S is an *n-fold iterated SPS-extension of the complete local ring* $(R, \mathfrak{m})$ if there exists a nested sequence of rings

$$(R, \mathfrak{m}) = (R_0, \mathfrak{m}_0) < (R_1, \mathfrak{m}_1) < \cdots < (R_n, \mathfrak{m}_n) = (S, \mathfrak{n}) \tag{1.2}$$

such that, for each $1 \le i \le n$, $(R_i, \mathfrak{m}_i) \in \mathbf{SPS}(R_{i-1}, \mathfrak{m}_{i-1})$.

The class of n-fold iterated SPS-extensions of $(R, \mathfrak{m})$ will be denoted $\mathbf{SPS}^n(R, \mathfrak{m})$. Of course, we will write $S \in \mathbf{SPS}^n_A(R, \mathfrak{m})$ if all the skew derivations can be taken to be A-linear.

If $S \in \mathbf{SPS}^n(R, \mathfrak{m})$, the number n is called the *rank* of S (over R). (Later, we will see that this number can often be recovered from the global dimensions or Krull dimensions of R and S.)

Example 1.9 Let $(k, \mathfrak{m}_k) = (\mathbb{F}_p, 0)$ or $(\mathbb{Z}_p, p\mathbb{Z}_p)$ and let G be a soluble (solvable) *uniform* group of *rank n*, in the sense of [8]. Then the completed group ring kG satisfies $kG \in \mathbf{SPS}^n_k(k, \mathfrak{m}_k)$.

Several other examples are given later.

The following computational lemma will be useful.

Lemma 1.10 *Fix a positive integer n. Then, for $a \in R$, we have*

$$x^n a = \sum_{m \in M_n} m(\sigma, \delta)(a) x^{e(m)}$$

inside the skew polynomial ring $R[x; \sigma, \delta]$ or (if it exists) the skew power series ring $R[[x; \sigma, \delta]]$. Here, M_n is the set of formal (noncommutative) monomials $m = m(X, Y)$ of degree n in the variables X and Y, and $e(m)$ is the total degree of X in the monomial m.

Proof When $n = 1$, this is just the multiplication rule (1.1) in S. For $n > 1$, this follows by an easy induction: note that all elements of M_n are either of the form $Xm(X, Y)$ or of the form $Ym(X, Y)$ for some $m \in M_{n-1}$. □

1.2 Constructing Skew Derivations

In the rare case when we wish to *construct* a local skew derivation on an iterated local SPS-extension S, the following lemma may be of use.

Let $(R, \mathfrak{m})$ be a complete local ring, and $(S, \mathfrak{n}) \in \mathbf{SPS}^n_R(R, \mathfrak{m})$, say $S = R[[x_1; \sigma_1, \delta_1]] \dots [[x_n; \sigma_n, \delta_n]]$. Let τ be a fixed automorphism of S (necessarily preserving $\mathfrak{n}$).

Lemma 1.11 *Given any choice of $b_1, \dots, b_n \in \mathfrak{n}^2$, the assignment $d(x_i) = b_i$ for all $1 \le i \le n$ extends to a unique local R-linear τ-derivation d of S.*

Proof First, as the elements $x_1, \dots, x_n$ are R-linearly independent, the mapping $x_i \mapsto b_i$ (for all $1 \le i \le n$) extends uniquely to an R-linear map $d : Rx_1 \oplus \cdots \oplus Rx_n \to \mathfrak{n}^2$.

To extend d to an R-module homomorphism $S \to S$, it will suffice to define $d(m)$ for each ordered monomial m in the elements $x_1, \dots, x_n$. We do this inductively on the degree of the monomial as follows. If the monomial m is of degree 1, then $m \in \{x_1, \dots, x_n\}$, and the value of $d(m)$ is already known. Proceeding inductively, let $n > 1$: if m is a monomial of degree n, then it may be written uniquely as $x_j m'$, where m' is a monomial of degree $n-1$ in $x_j, x_{j+1}, \dots, x_n$. Then we will define $d(m) := d(x_j)m' + \tau(x_j)d(m')$.

To show that this is indeed a τ-derivation, we must show that the value of $d(m)$ is well-defined regardless of how it is computed. More precisely: let m be an ordered monomial of total degree s,

$$m = x_{j_1} x_{j_2} \dots x_{j_s},$$

where $j_1 \le j_2 \le \cdots \le j_s$. Then, for each $1 \le r < s$, the monomial m can be written as the product $m = p_r q_r$, where $p_r = x_{j_1} x_{j_2} \dots x_{j_r}$ and $q_r = x_{j_{r+1}} x_{j_{r+2}} \dots x_{j_s}$. Then we may define:

$$d_r(m) = d(p_r)q_r + \tau(p_r)d(q_r).$$

Note that $d(m)$ was defined to be $d_1(m)$. In this notation, we must show that $d_1(m) = \cdots = d_{s-1}(m)$.

We will do this by induction on s. There is nothing to check when $s = 2$. Now suppose that, for some N, we have established $d_1(m) = \cdots = d_{t-1}(m)$ for all monomials m of total degree $2 \le t \le N$. Take a monomial m of total degree $N+1$: in the above notation, we will write it as $m = x_{j_1} x_{j_2} \dots x_{j_{N+1}} = p_r q_r$ for each $1 \le r \le N$.

Fix $1 \le r \le N-1$. Then $m = p_r q_r = p_{r+1} q_{r+1}$, where $q_r = x_{r+1} q_{r+1}$ and $p_{r+1} = p_r x_{r+1}$, i.e.

$$m = p_r \underbrace{x_{r+1} q_{r+1}}_{q_r} = \underbrace{p_r x_{r+1}}_{p_{r+1}} q_{r+1}.$$

So, exactly as above, we may calculate

$$\begin{aligned}
d_r(m) &= d(p_r)q_r + \tau(p_r)d(q_r) \\
&= d(p_r)x_{r+1}q_{r+1} + \tau(p_r)d(x_{r+1}q_{r+1}) \\
&= d(p_r)x_{r+1}q_{r+1} + \tau(p_r)[d(x_{r+1})q_{r+1} + \tau(x_{r+1})d(q_{r+1})] \\
&= [d(p_r)x_{r+1} + \tau(p_r)d(x_{r+1})]q_{r+1} + \tau(p_r x_{r+1})d(q_{r+1}) \\
&= d(p_r x_{r+1})q_{r+1} + \tau(p_r x_{r+1})d(q_{r+1}) \\
&= d(p_{r+1})q_{r+1} + \tau(p_{r+1})d(q_{r+1}) \\
&= d_{r+1}(m).
\end{aligned}$$

Hence these d_i are all equal, and in particular are equal to d. This shows that d is a τ-derivation as required. $\square$

1.3 The $\mathfrak{m}$-adic Filtration and Associated Graded Ring

Studying the natural filtrations on these rings and their associated graded rings is the key to understanding many of their basic properties. Throughout this paper, when $(R, \mathfrak{m})$ is a local ring, we will write $\mathrm{gr}(R)$ to mean $\mathrm{gr}_{\mathfrak{m}}(R)$ unless otherwise specified.

Properties 1.12 Let $(S, \mathfrak{n}) \in \mathbf{SPS}^n(R, \mathfrak{m})$.

(i) When R is a division ring (so that $\mathfrak{m} = 0$, it is easy to see that the rank n is uniquely defined, and can be recovered as $n = \dim_k(\mathfrak{n}/\mathfrak{n}^2)$.

(ii) The restriction of the $\mathfrak{n}$-adic filtration to R is the $\mathfrak{m}$-adic filtration, i.e. $R \cap \mathfrak{n}^i = \mathfrak{m}^i$. In particular, $\mathfrak{m}$ is generated in $\mathfrak{n}$-*adic degree* 1. This follows from Lemma 1.6.

(iii) Some results are already known due to [23, Corollary 2.9]. In particular:

  (a) If $\mathrm{gr}(R)$ is a domain (resp. right noetherian, resp. Auslander regular), then so is S.

(b) If $\mathrm{gr}(R)$ is right noetherian, then $\mathrm{r.Kdim}(S) \leq \mathrm{r.Kdim}(\mathrm{gr}(R))+n$ and $\mathrm{r.gldim}(S) \leq \mathrm{r.gldim}(\mathrm{gr}(R)) + n$.

We now calculate this associated graded ring inductively, and extend the list of results in (iii) above.

Let $(S, \mathfrak{n}) \in \mathbf{SPS}(R, \mathfrak{m})$, say $S = R[[x; \sigma, \delta]]$. Note that σ and δ induce maps on the graded ring $\mathrm{gr}(R)$ as follows: for all $r \in \mathfrak{m}^\lambda \setminus \mathfrak{m}^{\lambda+1}$, where $\lambda \in \mathbb{N}$, we have

- $\overline{\sigma}(r + \mathfrak{m}^{\lambda+1}) = \sigma(r) + \mathfrak{m}^{\lambda+1}$,
- $\overline{\delta}(r + \mathfrak{m}^{\lambda+1}) = \delta(r) + \mathfrak{m}^{\lambda+2}$.

These are graded endomorphisms of degrees 0 and ≥ 1 respectively. It is easy to check that $\overline{\sigma}$ is in fact a graded automorphism.

Lemma 1.13 *The unique homomorphism of graded rings* $(\mathrm{gr}(R))[X; \overline{\sigma}, \overline{\delta}] \to \mathrm{gr}(S)$, *extending the natural inclusion map* $\mathrm{gr}(R) \to \mathrm{gr}(S)$ *and sending* X *to* $\mathrm{gr}(x) = x + \mathfrak{n}^2$, *is an isomorphism of graded rings.*

Proof Noting that $S = \prod_{n\in\mathbb{N}} Rx^n$ as a left R-module, this follows from Property 1.12(ii). $\square$

Write $f_R : R \to \mathbb{N} \cup \{\infty\}$ for the natural $\mathfrak{m}$-adic filtration function on R, i.e. $f_R(r) = \ell$ if $r \in \mathfrak{m}^\ell \setminus \mathfrak{m}^{\ell+1}$, and similarly f_S for the $\mathfrak{n}$-adic filtration function on S.

Proof of Theorem D. The first claim follows from Lemma 1.13.

For the second, note that $f_R(\delta(r)) > f_R(r) + 1 \Leftrightarrow f_R(\delta(r)) \geq f_R(r) + 2$, as f_R has image in $\mathbb{N}$. This, in turn, is the same as the condition that $\delta(\mathfrak{m}^\ell) \subseteq \mathfrak{m}^{\ell+2}$ for all $\ell \in \mathbb{N}$, which implies that $\delta(R) \subseteq \mathfrak{m}^2$ and $\delta(\mathfrak{m}) \subseteq \mathfrak{m}^3$. The converse implication follows by induction: if $\delta(\mathfrak{m}) \subseteq \mathfrak{m}^3$ and $\delta(\mathfrak{m}^{\ell-1}) \subseteq \mathfrak{m}^{\ell+1}$, then

$$\delta(\mathfrak{m}^\ell) = \delta(\mathfrak{m} \cdot \mathfrak{m}^{\ell-1}) \subseteq \delta(\mathfrak{m})\mathfrak{m}^{\ell-1} + \sigma(\mathfrak{m})\delta(\mathfrak{m}^{\ell-1}) \subseteq \mathfrak{m}^3 \cdot \mathfrak{m}^{\ell-1} + \mathfrak{m} \cdot \mathfrak{m}^{\ell+1},$$

which is clearly contained in $\mathfrak{m}^{\ell+2}$ as required. $\square$

Remark 1.14 Property 1.12(ii) could be written as $f_S\left(\sum_{j\geq 0} r_j x^j\right) = \inf_{j\geq 0} \left\{f_R(r_j) + j\right\}$.

Now, it is easy to see that $\overline{\delta} = 0$ if and only if $f_R(\delta(r)) > f_R(r) + 1$ for all nonzero $r \in R$: indeed, if $f_R(r) = \ell$, then $f_S(xr) = f_S(\sigma(r)x) = \ell + 1$, and $xr - \sigma(r)x = \delta(r)$, but $f_S(\delta(r)) > \ell + 1$.

We give an example for which $\overline{\delta} \neq 0$, correcting a minor error in [23].

Example 1.15 Let k be a field. Take $R = k[[x]]$ with skew derivation (id, δ), where δ is the unique k-linear derivation on R satisfying $\delta(x) = x^2$. Form $S = R[[y; \delta]]$ (i.e. $R[[y; \mathrm{id}, \delta]]$). It is easy to see that the graded ring has nonzero $\overline{\delta}$. Set $X = x + \mathfrak{n}^2$ and $Y = y + \mathfrak{n}^2$ inside $\mathrm{gr}(S)$: then the multiplication in $\mathrm{gr}(S)$ is determined by the rule

$$YX = XY + X^2,$$

i.e. $\mathrm{gr}(S) \cong k[X][Y; \overline{\delta}]$, where $\overline{\delta}(X) = X^2$.

1.4 Lifting Properties from the Graded Ring

The results in Property 1.12(iii) were obtained by lifting properties from the graded ring. We obtain a few results along similar lines.

The proofs in [23] were given under the erroneous assumption that $\overline{\delta} = 0$ (in the notation of Lemma 1.13), but they remain true with essentially identical proofs even after removing this assumption.

In this subsection, we prove Theorem A, recording some further important properties that lift from the graded ring. Recall the statement and notation of Theorem A from the introduction: in particular, we are assuming $(S, \mathfrak{n}) \in \mathbf{SPS}^n(R, \mathfrak{m})$.

Proof of Theorem A

(i) If $\mathrm{gr}(R)$ is prime, then $\mathrm{gr}(S)$ is prime by Lemma 1.13 along with iterated application of [17, Theorem 1.2.9(iii)]. Hence S is prime by [12, II, Lemma 3.2.7].

(ii) As R is noetherian, it is in particular coherent, and so any direct product of flat R-modules is flat [7, Theorem 2.1(d) $\Longrightarrow$ (a)]. Writing $\mathfrak{m}$ and $\mathfrak{n}$ for the maximal ideals of R and S respectively, we have that $\mathfrak{m}S \subseteq \mathfrak{n} \neq S$, and so S is faithfully flat over R by [17, Proposition 7.2.3].

(iii) $\mathrm{gr}(S)$ is Auslander-regular by [12, III, Theorem 3.4.6(1)], and so S is Auslander-regular by Property 1.12(iii)(a), hence in particular S is Auslander-Gorenstein. As S is scalar local by Lemma 1.5(ii), we may apply [6, Lemma 4.3] to see that S is AS-Gorenstein.

(iv) It will suffice to show that $\mathrm{gr}(S)$ is a noetherian integral domain which is a maximal order, by [17, 5.1.6]. By invoking Lemma 1.13, we see that $\mathrm{gr}(S)$ is an iterated skew polynomial ring over $\mathrm{gr}(R)$, so it will suffice to argue inductively, and show that $\mathrm{gr}(R)[X; \overline{\sigma}, \overline{\delta}]$ has these properties for some skew derivation $(\overline{\sigma}, \overline{\delta})$. But $\mathrm{gr}(R)[X; \overline{\sigma}, \overline{\delta}]$ is a noetherian integral domain [17, Theorem 1.2.9(i, iv)]; and, as $\mathrm{gr}(R)$ is a noetherian maximal order, it is a Krull order in the sense of [16, §2.2], and so it follows from [16, Corollary 2.3.20] that $\mathrm{gr}(R)[X; \overline{\sigma}, \overline{\delta}]$ is also a maximal order.

□

2 Triangular Iterated Extensions

Let $(R, \mathfrak{m})$ be a complete local ring, and let $S \in \mathbf{SPS}^n(R, \mathfrak{m})$ for $n \geq 2$. Adopt the notation of Definition 1.8: suppose we are given a sequence of (onefold) local SPS-extensions

$$(R, \mathfrak{m}) = (R_0, \mathfrak{m}_0) < (R_1, \mathfrak{m}_1) < \cdots < (R_n, \mathfrak{m}_n) = (S, \mathfrak{n}),$$

where $R_i = R_{i-1}[[x_i; \sigma_i, \delta_i]]$ for each $1 \leq i \leq n$.

Here, each (σ_i, δ_i) is a skew derivation on R_{i-1}. However, iterating this procedure can lead to undesirable behaviour: (σ_i, δ_i) will usually *not* restrict to a skew derivation on R_j for $0 \leq j \leq i-2$, as the following example shows.

Example 2.1 Let k be a field. Let $R_0 = k[[Y]] \lneq k[[Y, Z]] = R_1$, so that R_0 has maximal ideal $\mathfrak{m}_0 = (Y)$. Take $S = R_1[[X; \sigma]]$, where σ is the unique k-linear automorphism of R_1 satisfying $\sigma(Y) = Z$, $\sigma(Z) = -Y$ (here $\delta = 0$). Then $S \in \mathbf{SPS}^2_k(R_0)$, but $\sigma(\mathfrak{m}_0) \not\subseteq \mathfrak{m}_0$.

This turns out to be a very natural stipulation to make when performing this iterative construction, and this motivates the definitions we make in this section.

2.1 Quotients by Stable Ideals

Recall the following basic definition (cf. [14, 3.13]).

Definition 2.2 Let R be a ring and I an ideal. If (σ, δ) is a skew derivation of R, then I is said to be a (σ, δ)-*ideal* if $\sigma(I) \subseteq I$ and $\delta(I) \subseteq I$.

This is a useful class of ideals because of results such as the following lemma:

Lemma 2.3 *Let* $(R, \mathfrak{m})$ *be a complete local ring, and let* $S = R[[x; \sigma, \delta]] \in \mathbf{SPS}(R, \mathfrak{m})$. *Suppose that* I *is a* (σ, δ)*-ideal of* R. *Then* $IS = SI$ *is a two-sided ideal, and* $S/IS \cong (R/I)[[x; \sigma, \delta]] \in \mathbf{SPS}(R/I, \mathfrak{m}/I)$.

Proof Note that [14, Setup 3.1(4)] is satisfied in our context, so we may apply [14, 3.13(ii) and Lemma 3.14(iv)]. □

We slightly extend this notion as follows.

Proposition 2.4 Let $(R, \mathfrak{m})$ be a complete local ring, and $S = R[[x_1; \sigma_1, \delta_1]] \ldots [[x_n; \sigma_n, \delta_n]] \in \mathbf{SPS}^n(R, \mathfrak{m})$. Suppose that I is a (σ_i, δ_i)-ideal of R for each $1 \leq i \leq n$. Then $IS = SI$ is a two-sided ideal, and

$$S/IS \cong (R/I)[[x_1; \sigma_1, \delta_1]] \ldots [[x_n; \sigma_n, \delta_n]] \in \mathbf{SPS}^n(R/I, \mathfrak{m}/I).$$

Proof Both claims follow from recursive application of Lemma 2.3. □

2.2 Iteratively Stable Presentations

Throughout, fix a complete local ring $(R, \mathfrak{m})$ and a ring $(S, \mathfrak{n}) \in \mathbf{SPS}^n(R, \mathfrak{m})$.

Definition 2.5 A *presentation* for S (over R) is a sequence of rings

$$(R, \mathfrak{m}) = (R_0, \mathfrak{m}_0) \lneq (R_1, \mathfrak{m}_1) \lneq \cdots \lneq (R_\ell, \mathfrak{m}_\ell) = (S, \mathfrak{n}) \tag{2.1}$$

where $(R_i, \mathfrak{m}_i) \in \mathbf{SPS}^{d_i}(R_{i-1}, \mathfrak{m}_{i-1})$ for each $1 \leq i \leq \ell$, so that each $d_i \geq 1$ and $d_1 + \cdots + d_\ell = n$. The number ℓ is called the *length* of the presentation.

Definition 2.6 Let (τ, ε) be an arbitrary local skew derivation of $(S, \mathfrak{n})$. We will say that the sequence (2.1) is *stabilised* by (τ, ε), or is (τ, ε)-*stable*, if $\tau(\mathfrak{m}_i) = \mathfrak{m}_i$, $\varepsilon(R_i) \subseteq \mathfrak{m}_i$ and $\varepsilon(\mathfrak{m}_i) \subseteq \mathfrak{m}_i^2$ for all $0 \leq i \leq \ell$.

We will say that the sequence (2.1) is *iteratively stable* if $\mathfrak{m}_i R_j$ is a two-sided ideal for each pair (i, j) satisfying $0 \leq i < j \leq \ell$. That is, writing $R_i = R_{i-1}[[x_{i,1}; \sigma_{i,1}, \delta_{i,1}]] \ldots [[x_{i,d_i}; \sigma_{i,d_i}, \delta_{i,d_i}]]$ for all $1 \leq i \leq \ell$: (2.1) is iteratively stable if and only if, for each $1 \leq i \leq \ell$, the subsequence

$$(R_0, \mathfrak{m}_0) \lneq \cdots \lneq (R_{i-1}, \mathfrak{m}_{i-1})$$

is $(\sigma_{i,h}, \delta_{i,h})$-stable for each $1 \leq h \leq d_i$.

Remark 2.7 Note that, as each $d_i \geq 1$, the length ℓ of any presentation (2.1) must be bounded above by the rank n of S over R.

Definition 2.8 We say that $R[[x_1; \sigma_1, \delta_1]] \dots [[x_n; \sigma_n, \delta_n]]$ is *triangular* (as an SPS-extension of R) if, writing $R_j = R[[x_1; \sigma_1, \delta_1]] \dots [[x_j; \sigma_j, \delta_j]]$ for the jth *subextension* and $\mathfrak{m}_j$ for its maximal ideal, the presentation

$$(R, \mathfrak{m}) = (R_0, \mathfrak{m}_0) \lneq (R_1, \mathfrak{m}_1) \lneq \cdots \lneq (R_n, \mathfrak{m}_n)$$

of length n is iteratively stable. Note that this depends not just on the ring, but on the choice of ordered topological generating set $(x_1, \dots, x_n)$ for the ring over R. An element of $\mathbf{SPS}^n(R, \mathfrak{m})$ is *triangularisable* if it can be written as a triangular SPS-extension of R, i.e. if it admits an iteratively stable presentation of length n beginning at R.

We will write $R[[x_1; \sigma_1, \delta_1]] \dots [[x_n; \sigma_n, \delta_n]] \in \mathbf{TSPS}^n(R, \mathfrak{m})$ to mean that $R[[x_1; \sigma_1, \delta_1]] \dots [[x_n; \sigma_n, \delta_n]]$ is triangular (as an SPS-extension of R).

Proposition 2.9 Let $(S, \mathfrak{n}) = R[[x_1; \sigma_1, \delta_1]] \dots [[x_n; \sigma_n, \delta_n]] \in \mathbf{TSPS}^n(R, \mathfrak{m})$, with jth subextension $(R_j, \mathfrak{m}_j)$ for each $0 \leq j \leq n$. Then $S/\mathfrak{m}_i S \in \mathbf{TSPS}^{n-i}(R/\mathfrak{m}, 0)$.

Proof It follows from the definition of triangularity that we may apply Proposition 2.4 to the extension $S \in \mathbf{TSPS}^{n-i}(R_i, \mathfrak{m}_i)$ and the ideal $\mathfrak{m}_i$, so that $S/\mathfrak{m}_i S \in \mathbf{SPS}^{n-i}(R_i/\mathfrak{m}_i, 0)$. But $R/\mathfrak{m} \cong R_i/\mathfrak{m}_i$ by repeated application of Lemma 1.5(i), and it is easy to check that the sequence

$$R_i/\mathfrak{m}_i \lneq R_{i+1}/\mathfrak{m}_i R_{i+1} \lneq \cdots \lneq R_n/\mathfrak{m}_i R_n = S/\mathfrak{m}_i S$$

is still triangular. $\square$

2.3 Rescaling Filtrations

Throughout this subsection we fix a triangular iterated local SPS-extension

$$(S, \mathfrak{n}) = R[[x_1; \sigma_1, \delta_1]] \dots [[x_n; \sigma_n, \delta_n]] \in \mathbf{TSPS}^n(R, \mathfrak{m})$$

with jth subextension $(R_j, \mathfrak{m}_j)$ for each $0 \leq j \leq n$.

Write f_R for the natural $\mathfrak{m}$-adic filtration function on R, and f_S for the $\mathfrak{n}$-adic filtration function on S (cf. Remark 1.14). Whenever $\alpha \in \mathbb{N}^n$, the multi-index notation $\boldsymbol{x}^\alpha$ means the (ordered) product $x_1^{\alpha_1} \dots x_n^{\alpha_n}$. Then we have

$$f_S\left(\sum_{\alpha \in \mathbb{N}^n} r_\alpha \boldsymbol{x}^\alpha\right) = \inf_{\alpha \in \mathbb{N}^n} \{f_R(r_\alpha) + \alpha_1 + \alpha_2 + \cdots + \alpha_n\}$$

for coefficients $r_\alpha \in R$. We will prove Theorem E by modifying this filtration. Given any set of coefficients $\{r_\alpha \in R \ : \ \alpha \in \mathbb{N}^n\}$, define the function

$$\tilde{f}_S\left(\sum_{\alpha \in \mathbb{N}^n} r_\alpha \boldsymbol{x}^\alpha\right) = \inf_{\alpha \in \mathbb{N}^n} \left\{f_R(r_\alpha) + \tfrac{1}{2}\alpha_1 + \tfrac{1}{2^2}\alpha_2 + \cdots + \tfrac{1}{2^n}\alpha_n\right\}.$$

Lemma 2.10 *The function $\tilde{f}_S$ satisfies the relation $\tilde{f}_S(a \pm b) \geq \min\{\tilde{f}_S(a), \tilde{f}_S(b)\}$ for all $a, b \in S$.*

Proof If $a = \sum a_\alpha \boldsymbol{x}^\alpha$ and $b = \sum b_\alpha \boldsymbol{x}^\alpha$ in the notation above, then $a \pm b = \sum (a_\alpha \pm b_\alpha)\boldsymbol{x}^\alpha$. But, given *any* $\beta \in \mathbb{N}^n$, by definition we have

$$\tilde{f}_S((a_\beta \pm b_\beta)\boldsymbol{x}^\beta) = f_R(a_\beta \pm b_\beta) + \tfrac{1}{2}\beta_1 + \tfrac{1}{2^2}\beta_2 + \cdots + \tfrac{1}{2^n}\beta_n.$$

Now, as f_R is known to be a filtration, it satisfies the relation $f_R(a_\beta + b_\beta) \geq \min\{f_R(a_\beta), f_R(b_\beta)\}$. Together with the above equality, this tells us that $\tilde{f}_S((a_\beta \pm b_\beta)\boldsymbol{x}^\beta) \geq \min\{\tilde{f}_S(a_\beta \boldsymbol{x}^\beta), \tilde{f}_S(b_\beta \boldsymbol{x}^\beta)\}$ for each $\beta \in \mathbb{N}^n$; now taking the infimum over all β gives the desired result. □

For the rest of this subsection, write $\tilde{f} = \tilde{f}_S$ for ease of notation, and set also $y = x_i$, $\sigma = \sigma_i$, $\delta = \delta_i$, so that we may view R_i as $R_{i-1}[[y; \sigma, \delta]]$.

Remark 2.11 For any choice of coefficients $s_j \in R_{i-1}$, we have $\tilde{f}(\sum_{j\geq 0} s_j y^j) = \inf_{j\geq 0}\{\tilde{f}(s_j) + \frac{j}{2^i}\}$.

Proposition 2.12 Suppose that $\tilde{f}|_{R_{i-1}}$ is a filtration, and that for any $b' \in R_{i-1}$ we have $\tilde{f}(\sigma(b')) = \tilde{f}(b')$ and $\tilde{f}(\delta(b')) \geq \tilde{f}(b') + \frac{1}{2^{i-1}}$. Then $\tilde{f}|_{R_i}$ is a filtration.

Proof In light of Lemma 2.10, we need to show that $\tilde{f}(ab) \geq \tilde{f}(a) + \tilde{f}(b)$ for all $a, b \in R_i$. We will write $a = \sum_{j\geq 0} a_j y^j$ and $b = \sum_{k\geq 0} b_k y^k$, where all the a_j and b_k are elements of R_{i-1}: then

$$\tilde{f}(ab) = \tilde{f}\left(\left(\sum_{j\geq 0} a_j y^j\right)\left(\sum_{k\geq 0} b_k y^k\right)\right) = \tilde{f}\left(\sum_{j,k\geq 0} a_j y^j b_k y^k\right) \geq \inf_{j,k\geq 0} \tilde{f}(a_j y^j b_k y^k).$$

As $\tilde{f}(a) + \tilde{f}(b) = \inf_{j\geq 0} \tilde{f}(a_j y^j) + \inf_{k\geq 0} \tilde{f}(b_k y^k)$, it will suffice to show that $\tilde{f}(a_j y^j b_k y^k) \geq \tilde{f}(a_j y^j) + \tilde{f}(b_k y^k)$ for all j, k.

By Lemma 1.10, we have

$$a_j y^j b_k y^k = \sum_{m\in M_j} a_j m(\sigma, \delta)(b_k) y^{e(m)+k},$$

where M_j is the set of all formal noncommutative monomials $m = m(X, Y)$ of total degree j, and $e(m)$ is the degree of X in the monomial m. By Lemma 2.10, it will suffice to show that the value under $\tilde{f}$ of each summand $a_j m(\sigma, \delta)(b_k) y^{e(m)+k}$ is at least $\tilde{f}(a_j y^j) + \tilde{f}(b_k y^k)$.

Write $e'(m) = j - e(m)$ for the degree of Y in m: then $m(\sigma, \delta)$ is the composite of $e(m)$ applications of σ (each of which preserves $\tilde{f}$ by assumption) and $e'(m)$ applications of δ (each of which increases $\tilde{f}$ by at least $\frac{1}{2^{i-1}}$ by assumption). That is,

$$\begin{aligned}
\tilde{f}(a_j m(\sigma, \delta)(b_k) y^{e(m)+k}) &= \tilde{f}(a_j m(\sigma, \delta)(b_k)) + \frac{e(m) + k}{2^i} \\
&= \tilde{f}(a_j)\tilde{f}(m(\sigma, \delta)(b_k)) + \frac{e(m) + k}{2^i} \\
&\geq \tilde{f}(a_j) + \tilde{f}(b_k) + \frac{e'(m)}{2^{i-1}} + \frac{e(m) + k}{2^i} \\
&\geq \tilde{f}(a_j) + \tilde{f}(b_k) + \frac{2j + k - e(m)}{2^i},
\end{aligned}$$

which attains its minimum value $\tilde{f}(a_j) + \tilde{f}(b_k) + \frac{j+k}{2^i}$ when $m(X, Y) = X^j$. But this minimum value is precisely $\tilde{f}(a_j y^j) + \tilde{f}(b_k y^k)$. □

Proposition 2.13 Suppose that $\tilde{f}|_{R_{i-1}}$ is a filtration.

(i) Suppose we are given an automorphism τ of R_i restricting to an automorphism of R_{i-1}, and that for any $b' \in R_{i-1}$, we have $\tilde{f}(\tau(b')) = \tilde{f}(b')$. Then, for any $b \in R_i$, we have $\tilde{f}(\tau(b)) = \tilde{f}(b)$.

(ii) Suppose we are given a local skew derivation (τ, ε) of R_i restricting to a local skew derivation of R_{i-1}, and that for any $b' \in R_{i-1}$, we have $\tilde{f}(\tau(b')) = \tilde{f}(b')$ and $\tilde{f}(\varepsilon(b')) \geq \tilde{f}(b') + \frac{1}{2^{i-1}}$. Then, for any $b \in R_i$, we have $\tilde{f}(\varepsilon(b)) \geq \tilde{f}(b) + \frac{1}{2^i}$.

Proof Write $b = \sum_{j\geq 0} b_j y^j$, for some choice of coefficients $b_j \in R_{i-1}$, so that we may evaluate $\tilde{f}(b)$ as $\inf_{j\geq 0}\{\tilde{f}(b_j y^j)\}$ by Remark 2.11.

(i) It follows from Lemma 2.10 that $\tilde{f}(\tau(b)) \geq \inf_{j\geq 0}\{\tilde{f}(\tau(b_j y^j))\}$. So we begin by showing that $\tilde{f}(\tau(b_j y^j)) \geq \tilde{f}(b_j y^j)$ for each j; then, taking infima over all j, we will have shown that $\tilde{f}(\tau(b)) \geq \tilde{f}(b)$. But since b and τ are arbitrary, we may then replace b by $\tau(b)$ and τ by τ^{-1} to get the reverse inequality. As τ restricts to an automorphism of R_{i-1}, we have $\tau(y^j) \in \mathfrak{n}^j$, which is equal to $\mathfrak{m}_{i-1}^j + \mathfrak{m}_{i-1}^{j-1} y + \cdots + \mathfrak{m}_{i-1} y^{j-1} + R_i y^j$ by Lemma 1.6. Write $\tau(y^j) = a_j + a_{j-1} y + \cdots + a_1 y^{j-1} + r y^j$ for some $a_k \in \mathfrak{m}_{i-1}^k$ and $r \in R_i$: then, left-multiplying by $\tau(b_j)$ and applying $\tilde{f}$, we get

$$\begin{aligned}\tilde{f}(\tau(b_j y^j)) &= \tilde{f}(\tau(b_j)a_j + \tau(b_j)a_{j-1}y + \cdots + \tau(b_j)a_1 y^{j-1} + \tau(b_j) r y^j)\\ &= \min\left\{\min_{1\leq k\leq j}\{\tilde{f}(\tau(b_j)a_k y^{j-k})\}, \tilde{f}(\tau(b_j) r y^j)\right\}.\end{aligned}$$

Each of the terms $\tilde{f}(\tau(b_j)a_k y^{j-k})$ can be directly evaluated as $\tilde{f}(\tau(b_j)a_k) + \frac{j-k}{2^i}$ by Remark 2.11, and since $\tilde{f}$ is a filtration on R_{i-1}, $\tilde{f}(\tau(b_j)a_k) \geq \tilde{f}(\tau(b_j)) + \tilde{f}(a_k)$. Putting this together,

$$\tilde{f}(\tau(b_j)a_k y^{j-k}) \geq \tilde{f}(\tau(b_j)) + \tilde{f}(a_k) + \frac{j-k}{2^i} \geq \tilde{f}(\tau(b_j)) + \frac{k}{2^{i-1}} + \frac{j-k}{2^i} > \tilde{f}(\tau(b_j)) + \frac{j}{2^i}.$$

For the remaining term: we may write $r = \sum_{l\geq 0} r_l y^l$ for coefficients $r_l \in R_{i-1}$, so that

$$\tilde{f}(\tau(b_j) r y^j) = \tilde{f}\left(\sum_{l\geq 0} \tau(b_j) r_l y^{j+l}\right) = \inf_{l\geq 0}\left\{\tilde{f}(\tau(b_j) r_l) + \frac{j+l}{2^i}\right\} \geq \tilde{f}(\tau(b_j)) + \frac{j}{2^i},$$

again by Remark 2.11. Now, again by Lemma 2.10, we can conclude that $\tilde{f}(\tau(b_j y^j)) \geq \tilde{f}(\tau(b_j)) + \frac{j}{2^i}$. But as $\tilde{f}$ is assumed τ-invariant on R_{i-1}, this right-hand side is equal to $\tilde{f}(b_j) + \frac{j}{2^i}$, which is just $\tilde{f}(b_j y^j)$, and so we are done.

(ii) Applying ε to both sides of the equality $b = \sum_{j\geq 0} b_j y^j$, and using the fact that ε is a τ-derivation, we get

$$\varepsilon(b) = \sum_{j\geq 0}[\varepsilon(b_j)y^j + \tau(b_j)\varepsilon(y^j)].$$

It will suffice to show that each $\tilde{f}(\varepsilon(b_j)y^j)$ and each $\tilde{f}(\tau(b_j)\varepsilon(y^j))$ is at least $\tilde{f}(b_j y^j) + \frac{1}{2^i} = \tilde{f}(b_j) + \frac{j+1}{2^i}$. For the first term, note that $\tilde{f}(\varepsilon(b_j)y^j) = \tilde{f}(\varepsilon(b_j)) + \frac{j}{2^i}$, and that $\tilde{f}(\varepsilon(b_j)) \geq \tilde{f}(b_j) + \frac{1}{2^{i-1}}$ by assumption. For the second, note that $\varepsilon(y^j) \in \mathfrak{n}^{j+1}$, which is equal to $\mathfrak{m}_{i-1}^{j+1} + \mathfrak{m}_{i-1}^{j} y + \cdots + \mathfrak{m}_{i-1} y^j + R_i y^{j+1}$ by Lemma 1.6, and repeat the argument of part (i) to show that $\tilde{f}(\tau(b_j)\varepsilon(y^j)) \geq \tilde{f}(b_j) + \frac{j+1}{2^i}$.

□

Proof of Theorem E Recall that $R_n = S$. We show by induction on i that

(a) given any local skew derivation (τ, ε) of R_i which stabilises the sequence $R = R_0 \lneq R_1 \lneq \cdots \lneq R_{i-1} \lneq R_i$, we have $\tilde{f}(\tau(b)) = \tilde{f}(b)$ and $\tilde{f}(\varepsilon(b)) \geq \tilde{f}(b) + \frac{1}{2^i}$ for any $b \in R_i$, and

(b) $\tilde{f}|_{R_i}$ is a filtration,

for all $0 \leq i \leq n$, and that

(c) $\mathrm{gr}_{\tilde{f}}(R_i) \cong \mathrm{gr}_{\tilde{f}}(R_{i-1})[X_i; \overline{\sigma_i}]$

for all $1 \leq i \leq n$. (Once we have proved that this is indeed a filtration, it will be clear from the definition that it is cofinal with the natural $\mathfrak{n}$-adic filtration.) □

Case $i = 0$. By definition, $R_0 = R$ and $\tilde{f}|_{R_0} = f_R$, which is known to be a filtration, so condition (b) is satisfied. In fact, as f_R is the $\mathfrak{m}$-adic filtration on R, condition (a) is just a restatement of the *locality* of (τ, ε).

Case $i > 0$. Suppose that R_{i-1} satisfies conditions (a) and (b).

Let (τ, ε) be a local skew derivation of R_i stabilising the sequence $R = R_0 \lneq R_1 \lneq \cdots \lneq R_{i-1} \lneq R_i$: then (τ, ε) restricts to a local skew derivation of R_{i-1} stabilising the sequence $R = R_0 \lneq R_1 \lneq \cdots \lneq R_{i-1}$, and so by induction we have that $\tilde{f}(\tau(b')) = \tilde{f}(b')$ and $\tilde{f}(\varepsilon(b')) \geq \tilde{f}(b') + \frac{1}{2^{i-1}}$ for any $b' \in R_{i-1}$. Condition (a) for R_i now follows from Proposition 2.13.

Similarly, as (σ_i, δ_i) is a local skew derivation of R_{i-1} stabilising the sequence $R = R_0 \lneq R_1 \lneq \cdots \lneq R_{i-1}$, by induction we have that $\tilde{f}(\sigma_i(b')) = \tilde{f}(b')$ and $\tilde{f}(\delta_i(b')) \geq \tilde{f}(b') + \frac{1}{2^{i-1}}$ for any $b' \in R_{i-1}$. Now Proposition 2.12 and the inductive hypothesis shows that R_i satisfies condition (b).

Finally, given $b' \in R_{i-1}$, we know already by induction that $\tilde{f}(\delta_i(b')) \geq \tilde{f}(b') + \frac{1}{2^{i-1}} > \tilde{f}(b') + \frac{1}{2^i} = \tilde{f}(\sigma_i(b')x_i)$. That is, applying $\tilde{f}$ to the equality $x_i b' = \sigma_i(b')x_i + \delta_i(b')$ gives $\tilde{f}(x_i b') = \tilde{f}(\sigma_i(b')x_i) < \tilde{f}(\delta_i(b'))$, showing that R_i satisfies condition (c). □

2.4 Examples

It turns out that many interesting iterated local SPS-extensions arising in nature are indeed triangularisable. Examples 2.14–2.16 below were important motivating examples for this paper.

Example 2.14 **q-commutative power series rings.** Let k be a field, and $q \in M_n(k^\times)$ a *multiplicatively antisymmetric* matrix (i.e. $q_{ij}q_{ji} = 1$ for all i and j) such that $q_{ii} = 1$ for all i. Then the *q-commutative power series ring* is the ring

$$R = k_q[[x_1, \ldots, x_n]],$$

defined as follows: $R \cong k[[x_1, \ldots, x_n]]$ as topological k-modules; and the multiplication is given by the n^2 relations $x_i x_j = q_{ij} x_j x_i$. It is easy to see that R is triangularisable.

For the following two examples, we assume familiarity with the notion of *uniform groups* [8] and *completed group rings* (also known as *Iwasawa algebras*) [13].

Example 2.15 **Nilpotent Iwasawa algebras.** Let G be a nilpotent uniform group. G admits a series

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

of closed subgroups of G such that G_{i-1} is normal in G and $G_i/G_{i-1} \cong \mathbb{Z}_p$ for all $1 \leq i \leq n$: an appropriate refinement of the isolated lower central series, defined in [25, Definition 3.7], will suffice. Hence G is supersoluble. Now it will follow from Example 2.16 below that its appropriate Iwasawa algebras are triangularisable.

Example 2.16 **Supersoluble Iwasawa algebras.** Let G be a supersoluble uniform group, so that we may fix a sequence

$$\{1\} = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

of closed normal subgroups of G such that $G_i/G_{i-1} \cong \mathbb{Z}_p$ for all $1 \leq i \leq n$. Let k be *either* the ring of integers of a finite extension of $\mathbb{Q}_p$ *or* a finite field extension of $\mathbb{F}_p$. Now, as in [22, Example 2.3], $kG \in \mathbf{SPS}_k^n(k)$ with presentation

$$k = kG_0 \leq kG_1 \leq \cdots \leq kG_n = kG,$$

and the condition that G_{i-1} should be normal in G ensures that this presentation is iteratively stable. Hence kG is triangular with respect to this presentation.

We demonstrate the existence of some supersoluble, non-nilpotent uniform groups of small rank below. In both cases, uniformity can be easily checked using [8, Theorem 4.5].

(i) Let Γ_1 and Γ_2 be copies of $\mathbb{Z}_p$, and take the continuous group homomorphism $\rho : \Gamma_2 \to \mathrm{Aut}(\Gamma_1) \cong \mathbb{Z}_p^\times$ sending a generator of Γ_2 to $1 + p \in \mathbb{Z}_p^\times$. Form the semidirect product $G = \Gamma_1 \rtimes \Gamma_2$. Then G is a supersoluble but non-nilpotent uniform group of dimension 2.

(ii) Take p to be a prime congruent to 1 mod 4, so that $i := \sqrt{-1} \in k$. Let $A = \overline{\langle y, z\rangle} \cong \mathbb{Z}_p^2$ and $B = \overline{\langle x\rangle} \cong \mathbb{Z}_p$, and fix the left action of B on A, say $\rho : B \to \mathrm{Aut}(A)$, defined by $\rho(x)(y) = yz^p$ and $\rho(x)(z) = zy^{-p}$. Then $G = B \ltimes A$ is easily checked to be a soluble, non-nilpotent uniform group of dimension 3, and the chain of normal subgroups

$$1 \leq \overline{\langle yz^i\rangle} \leq B \leq A$$

shows that G is in fact supersoluble. (Compare Non-example 2.20.)

The next two examples are not crucial for the current paper, but we include them to illustrate the wide applicability of Theorems B, C and E.

Example 2.17 **Completed quantised k-algebras.** Let k be a field, $\Gamma = (\gamma_{ij}) \in M_n(k^\times)$ a multiplicatively antisymmetric matrix, and $P = (p_1, \ldots, p_n) \in (k^\times)^n$, $Q = (q_1, \ldots, q_n) \in (k^\times)^n$ two vectors with $p_i \neq q_i$ for all $1 \leq i \leq n$. Horton's algebra $R = K_{n,\Gamma}^{P,Q}(k)$ (defined in [11, Definition 1.1]), a simultaneous generalisation of quantum symplectic space and quantum Euclidean $2n$-space, can be written as an iterated skew polynomial ring,

$$k[x_1][y_1; \tau_1][x_2; \sigma_2][y_2; \tau_2, \delta_2] \ldots [x_n; \sigma_n][y_n; \tau_n, \delta_n],$$

where the σ_i (for $2 \leq i \leq n$) and τ_i (for $1 \leq i \leq n$) are k-linear automorphisms, and each δ_i (for $2 \lneq i \lneq n$) is a k-linear τ_i-derivation. In [23, §3.2], it is proved that the I-adic completion $\hat{R}$ of R, where $I = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$, is an iterated skew power series extension of k:

$$\hat{R} = k[[x_1]][[y_1; \tau_1]][[x_2; \sigma_2]][[y_2; \tau_2, \delta_2]] \ldots [[x_n; \sigma_n]][[y_n; \tau_n, \delta_n]] \in \mathbf{SPS}_k^{2n}(k). \quad (2.2)$$

We do not spell out the relations in full: see [11] or [23, §3.2] for details. It is only necessary, for our purposes, to know the following:

- $\sigma_i(x_j)$, $\tau_i(x_j)$, $\delta_i(x_j)$ are scalar multiples of x_j for all $j < i$;
- $\sigma_i(y_j)$, $\tau_i(y_j)$, $\delta_i(y_j)$ are scalar multiples of y_j for all $j < i$;
- $\tau_i(x_i)$ is a scalar multiple of x_i for all i;
- $\delta_i(x_i)$ is a k-linear combination of the elements $y_l x_l$ for all $l < i$.

It now follows by an easy calculation that the saturated presentation associated to (2.2) is iteratively stable, and hence that $\hat{R}$ is triangularisable.

Example 2.18 **Completed quantum matrix algebras.** Let k be a field, $\lambda \in k^\times$ a scalar, and $\mathbf{p} = (p_{ij}) \in M_n(k^\times)$ a multiplicatively antisymmetric matrix (i.e. $p_{ij}p_{ji} = 1$ for all i, j). Then the *multiparameter quantum $n \times n$ matrix algebra $R = \mathcal{O}_{\lambda,\mathbf{p}}(M_n(k))$* can be defined (see e.g. [3, Definition I.2.2]) as a skew polynomial ring in n^2 variables labelled $X_{i,j}$ for each $1 \leq i, j \leq n$, in which k is central. Again, we do not spell out the relations in full, but we note:

- $X_{l,m}X_{i,j}$ is a linear combination of $X_{i,j}X_{l,m}$ and $X_{i,m}X_{l,j}$ when $l > i$ and $m > j$;
- $X_{l,m}X_{i,j}$ is a scalar multiple of $X_{i,j}X_{l,m}$ whenever either $l \leq i$ or $j \leq m$.

In [23, §3.2] it is proved that the I-adic completion $\hat{R}$ of R, where I is the ideal generated by the n^2 variables $X_{i,j}$ for $1 \leq i, j \leq n$, is an iterated skew power series extension of k satisfying the same relations: $\hat{R} \in \mathbf{SPS}_k^{n^2}(k)$. But the "obvious" saturated presentations, e.g. those associated to

$$k[[X_{1,1}]][[X_{1,2}]][[X_{1,3}]] \ldots [[X_{n,n}]]$$

(where we have omitted the skew derivations for readability) are usually *not* iteratively stable. (For a counterexample, take R to be the usual quantum 2×2 matrix algebra $\mathcal{O}_q(M_2(k))$, given by $n = 2$, $\mathbf{p} = \left(\begin{smallmatrix} 1 & q \\ q^{-1} & 1 \end{smallmatrix}\right)$, and $\lambda = q^{-2}$ for any $q \in k^\times$. See [3, Definition I.1.7]) for the relations in this case.)

We fix this by adjoining the variables in the following order:

- at the *0th stage*, adjoin the "antidiagonal" elements $X_{i,j}$ satisfying $|i + j - (n + 1)| = 0$, in any order;
- at the *1st stage*, adjoin those $X_{i,j}$ satisfying $|i + j - (n + 1)| = 1$;
- at the *2nd stage*, adjoin those $X_{i,j}$ satisfying $|i + j - (n + 1)| = 2$;

and so on, until finally all variables have been adjoined at the end of the $(n - 1)$th stage.

Diagrammatically:

$$\underset{\text{0th stage}}{\begin{pmatrix} & & & & \iddots \\ & & & \circledast & \\ & & \circledast & & \\ & \circledast & & & \\ \circledast & & & & \end{pmatrix}} \rightsquigarrow \underset{\text{1st stage}}{\begin{pmatrix} & & & \iddots & \iddots \\ & & \circledast & \ast & \iddots \\ & \circledast & \ast & \circledast & \\ \circledast & \ast & \circledast & & \\ \ast & \circledast & & & \end{pmatrix}} \rightsquigarrow \underset{\text{2nd stage}}{\begin{pmatrix} & & \iddots & \iddots & \iddots \\ & \circledast & \ast & \ast & \iddots \\ \circledast & \ast & \ast & \ast & \iddots \\ \ast & \ast & \ast & \circledast & \\ \ast & \ast & \circledast & & \end{pmatrix}} \rightsquigarrow \cdots$$

It is easy to verify that such a presentation of $\hat{R}$ is iteratively stable. The only case in which there is anything to prove is when $l > i$ and $m > j$: multiplying $X_{l,m}$ by $X_{i,j}$ results in a term involving the variables $X_{i,m}$ and $X_{l,j}$, and we must check that each of the variables $X_{i,m}$ and $X_{l,j}$ has been adjoined *before* we adjoin both of $X_{l,m}$ and $X_{i,j}$.

More precisely, let $M = |\max\{i + j, l + m\} - (n + 1)|$ be the first stage after which both $X_{l,m}$ and $X_{i,j}$ have been adjoined, and likewise let $N = |\max\{i + m, l + j\} - (n + 1)|$ be the first stage after which both $X_{i,m}$ and $X_{l,j}$ have been adjoined. Then it is easy to see that the inequalities $l > i$ and $m > j$ imply

$$i + j < i + m < l + m,$$
$$i + j < l + j < l + m,$$

and hence $N < M$.

The existence of such an iteratively stable presentation shows that $\hat{R}$ is triangularisable.

2.5 Non-Examples

Showing that an extension is *not* triangularisable appears to involve lots of tedious calculation, but we give two examples which seem of interest to the theory.

Non-example 2.19 **An extension of pure automorphic type.** Let k be a field such that $\sqrt{-1} \notin k$, and let $S = k[[Y, Z]][[X; \sigma]]$, where σ is a k-linear automorphism of the commutative power series ring $k[[Y, Z]]$ defined by $\sigma(Y) = Z, \sigma(Z) = -Y$: that is, $ZX = XY$ and $YX = -XZ$ inside S.

By construction, $S \in \mathbf{SPS}_k^3(k)$.

Suppose that S is triangularisable, and denote its maximal ideal by $\mathfrak{n}$. Then there must exist an iteratively stable presentation

$$k \lneq (R_1, \mathfrak{m}_1) \lneq (R_2, \mathfrak{m}_2) \lneq (S, \mathfrak{n})$$

for S. We will show that no such presentation can exist. Write $I = \mathfrak{m}_1 S$ and $J = \mathfrak{m}_2 S$: then, by Lemma 1.5(i), $I = sS$ and $J = sS + tS$ with $s, t \in \mathfrak{n}$, and $s, t \notin \mathfrak{n}^2$ by Property 1.12(ii).

For computation purposes, we will write

- $s = aX + bY + cZ + \varepsilon$,
- $t = \alpha X + \beta Y + \gamma Z + \varepsilon'$,

where $\varepsilon, \varepsilon' \in \mathfrak{n}^2$, and $a, b, c, \alpha, \beta, \gamma \in k$ are constants, which must satisfy $(a, b, c) \neq (0, 0, 0) \neq (\alpha, \beta, \gamma)$ by Property 1.12(ii).

As S is triangular, $I = \mathfrak{m}_1 S$ must in fact be a two-sided ideal. This places some restrictions on possible choices for s, which we now compute.

Henceforth, we work in $S/\mathfrak{n}^3$ for ease of computation. We have

$$Xs \equiv aX^2 + bXY + cXZ, \qquad sX \equiv aX^2 + cXY - bXZ \mod \mathfrak{n}^3,$$

and so

$$Xs - sX = (b - c)XY + (b + c)XZ \mod \mathfrak{n}^3.$$

But $Xs - sX \in I = sS$, so we must have $(b - c)XY + (c + b)XZ \equiv s\alpha \mod \mathfrak{n}^3$ for some $\alpha \in S$. As the left-hand side belongs to $\mathfrak{n}^2$, we must have $\alpha \in \mathfrak{n}$, and so $\varepsilon\alpha \in \mathfrak{n}^3$. Writing therefore $\alpha \equiv d_X X + d_Y Y + d_Z Z \mod \mathfrak{n}^2$ for some $d_X, d_Y, d_Z \in k$, we see that

$$(b - c)XY + (b + c)XZ \equiv (aX + bY + cZ)(d_X X + d_Y Y + d_Z Z) \mod \mathfrak{n}^3.$$

Multiplying out the right-hand side:

$$\begin{aligned}(b - c)XY + (b + c)XZ &\equiv ad_X X^2 + (ad_Y + cd_X)XY + (ad_Z - bd_X)XZ \\ &\quad + bd_Y Y^2 + (bd_Z + cd_Y)YZ + cd_Z Z^2 \qquad \mod \mathfrak{n}^3.\end{aligned}$$

Now, equating the coefficients of each monomial on both sides, some tedious case-checking shows that the only solution to this congruence is $b = c = d_X = d_Y = d_Z = 0$. Hence we have $s = aX + \varepsilon$, so that $a \neq 0$.

Now, as S is triangular, $J = \mathfrak{m}_2 S$ must also be a two-sided ideal, and so we calculate the restrictions that this places on t. It is easy to see that $R_2 = k[[s]][[t; \tau, \delta]]$ for some local skew derivation (τ, δ) of $k[[s]]$, and that this means there must be a unit $\eta \in k[[s]]^\times$ and an element $\theta \in k[[s]]$ such that $\tau(s) = \eta s$ and $\delta(s) = \theta s^2$. (Note that $\eta \not\equiv 0 \mod \mathfrak{n}^2$.) Hence we have $\tau(s) \equiv \eta s \equiv \eta a X \mod \mathfrak{n}^2$ and $\delta(s) \equiv \theta s^2 \equiv a^2\theta X^2 \mod \mathfrak{n}^3$, and so

$$(\alpha X + \beta Y + \gamma Z)\eta a X \equiv \eta a X(\alpha X + \beta Y + \gamma Z) + a^2\theta X^2 \mod \mathfrak{n}^3.$$

Multiplying out again:

$$a^2\theta X^2 + \eta a(\beta - \gamma)XY + \eta a(\beta + \gamma)XZ \equiv 0 \mod \mathfrak{n}^3,$$

from which we see immediately that $\beta = \gamma = \theta = 0$, and hence $t = \alpha X + \varepsilon'$.

This implies that the images of s and t in $\mathfrak{n}/\mathfrak{n}^2$, and hence also in $\mathfrak{m}_2/\mathfrak{m}_2^2$, are linearly dependent, and so $\dim_k(\mathfrak{m}_2/\mathfrak{m}_2^2) \leq 1$. But $R_2 \in \mathbf{SPS}_k^2(k)$ by construction, and Property 1.12(i) tells us that we should have $\dim_k(\mathfrak{m}_2/\mathfrak{m}_2^2) = 2$. This is a contradiction.

Non-example 2.20 **A soluble Iwasawa algebra.**

This example is similar to the previous in many ways. Compare also Example 2.16(ii).

Fix a prime p congruent to 3 mod 4, so that $\sqrt{-1} \notin \mathbb{F}_p$. Let $A = \overline{\langle y, z\rangle} \cong \mathbb{Z}_p^2$ and $B = \overline{\langle x\rangle} \cong \mathbb{Z}_p$, and form the semidirect product $G = B \ltimes A$ as in Example 2.16(ii). Construct its $\mathbb{F}_p$-Iwasawa algebra $S = \mathbb{F}_p G$.

Writing $X = x - 1$, $Y = y - 1$, $Z = z - 1$, we may easily calculate (see e.g. [22, Example 2.3]) that

$$S = \mathbb{F}_p[[Y, Z]][[X; \sigma, \delta]],$$

where $\sigma(Y) = Y + (1 + Y)Z^p$, $\sigma(Z) = Z + (-Y + Y^2 - Y^3 + \dots)^p(1 + Z)$, and $\delta = \sigma - \mathrm{id}$. Clearly, $S \in \mathbf{SPS}^3_{\mathbb{F}_p}(\mathbb{F}_p)$.

Suppose that S is triangularisable, and denote its maximal ideal by $\mathfrak{n}$. Then there must exist an iteratively stable presentation

$$\mathbb{F}_p \lneq (R_1, \mathfrak{m}_1) \lneq (R_2, \mathfrak{m}_2) \lneq S$$

for S: in particular, by Proposition 2.9, S must admit a quotient $S/\mathfrak{m}_1 S \in \mathbf{TSPS}^2_{\mathbb{F}_p}(\mathbb{F}_p)$. We will show that this leads to a contradiction.

As in Non-example 2.19, take $I = \mathfrak{m}_1 S = sS$, and write $s = aX + bY + cZ + \varepsilon$, where $\varepsilon \in \mathfrak{n}^2$, and $a, b, c \in \mathbb{F}_p$. By triangularity of S, I must be a two-sided ideal; this time, we calculate restrictions on s by working in $S/\mathfrak{n}^{p+1}$. We have

$$\sigma(Y) \equiv Y + Z^p, \qquad \sigma(Z) \equiv Z - Y^p, \quad \mod \mathfrak{n}^{p+1},$$

and so we must have

$$Xs - sX \equiv bZ^p - cY^p \mod \mathfrak{n}^{p+1}.$$

But $Xs - sX \in I = sS$, so we must have $bZ^p - cY^p \equiv s\alpha \mod \mathfrak{n}^{p+1}$ for some $\alpha \in S$. As the left-hand side belongs to $\mathfrak{n}^p$, we must have $\alpha \in \mathfrak{n}^{p-1}$, and so $\varepsilon\alpha \in \mathfrak{n}^{p+1}$. So this equation becomes

$$bZ^p - cY^p \equiv (aX + bY + cZ)\left(\sum_\gamma d_\gamma X^{\gamma_1} Y^{\gamma_2} Z^{\gamma_3}\right) \mod \mathfrak{n}^{p+1},$$

for some choices of $d_\gamma \in \mathbb{F}_p$, where this sum ranges over all $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ with $p - 1 \leq \gamma_1 + \gamma_2 + \gamma_3 \leq p$.

Suppose that, for our given a, b, c, we have a solution $\{d_\gamma\}$ to this congruence. Then, multiplying out the right-hand side, and writing $\mathbf{e}_1 = (1, 0, 0), \mathbf{e}_2 = (0, 1, 0), \mathbf{e}_3 = (0, 0, 1)$:

$$bZ^p - cY^p \equiv \sum_{\gamma'} \left(ad_{\gamma'-\mathbf{e}_1} + bd_{\gamma'-\mathbf{e}_2} + cd_{\gamma'-\mathbf{e}_3}\right) X^{\gamma'_1} Y^{\gamma'_2} Z^{\gamma'_3} \mod \mathfrak{n}^{p+1},$$

where for convenience we set $d_\gamma = 0$ if any of the γ_i is equal to -1. We may eliminate any term in the sum of total degree not equal to p, as all nonzero monomials appearing on the left hand side have degree p, so this sum may be taken to range over all $\gamma' = (\gamma'_1, \gamma'_2, \gamma'_3)$ with $\gamma'_1 + \gamma'_2 + \gamma'_3 = p$; furthermore, as there are no terms in $X^{\gamma'_1} Y^{\gamma'_2} Z^{\gamma'_3}$ on the left hand side except those with $\gamma_1 = 0$, we may also eliminate all monomials of nonzero degree in X on the right. We can now rewrite this congruence as

$$bZ^p - cY^p \equiv \sum_{i=0}^{p} \left(bd_{(0,i-1,p-i)} + cd_{(0,i,p-i-1)}\right) Y^i Z^{p-i} \mod \mathfrak{n}^{p+1}.$$

It is easy to see that, if $b \neq 0$, then $c \neq 0$, and vice-versa. We assume for contradiction that $b \neq 0 \neq c$: we will show that, in this case, the above congruence cannot hold for any choice of $\{d_\gamma\}$.

Indeed, equating monomial coefficients on the left and right hand sides:

$$\begin{aligned} b &= cd_{(0,0,p-1)} && (i = 0) \\ 0 &= bd_{(0,m-1,p-m)} + cd_{(0,m,p-m-1)} && (i = m : 1 \leq m \leq p-1) \\ -c &= bd_{(0,p-1,0)} && (i = p). \end{aligned}$$

On the one hand, multiplying the equations labelled $(i = 0)$ and $(i = p)$ together, we get

$$-bc = bcd_{(0,0,p-1)}d_{(0,p-1,0)},$$

i.e. $d_{(0,0,p-1)}d_{(0,p-1,0)} = -1$. On the other hand, multiplying the equation labelled $(i = m)$ by $b^{p-m}c^m$ and rearranging for each $1 \leq m \leq p-1$, we get

$$b^{p-m+1}c^m d_{(0,m-1,p-m)} = -b^{p-m}c^{m+1}d_{(0,m,p-m-1)};$$

substituting each one into the next, we eventually get

$$b^p c d_{(0,0,p-1)} = bc^p d_{(0,p-1,0)}$$

as p is odd; and since $b, c \in \mathbb{F}_p$, we have $b^p = b$ and $c^p = c$. This tells us that $d_{(0,0,p-1)} = d_{(0,p-1,0)}$, and denoting this common value by d, we have shown that we must have $d^2 = -1 \in \mathbb{F}_p$, which is a contradiction.

Hence we have shown that $b = c = 0$, and so $s = aX + \varepsilon$. But now

$$sY - Ys \equiv aZ^p \mod \mathfrak{n}^{p+1},$$

and a very similar (but easier) calculation shows that we must have $a = 0$. Hence $s = \varepsilon \in \mathfrak{n}^2$: that is, $\mathfrak{m}_1$ is generated in $\mathfrak{n}$-adic degree ≥ 2. This contradicts Property 1.12(ii).

3 Dimension Theory

Many of the results in this section can be slightly extended; but we will not always strive for full generality, and often work over a field or a division ring for simplicity.

3.1 Krull Dimension

Let k be a division ring, and $(R, \mathfrak{m}) \in \mathbf{SPS}^n(k)$ a complete local ring. Property 1.12(v)(c) implies that $\mathrm{Kdim}(R) \leq n$: in this subsection, we show that this is always an equality.

For any ring R, write $\mathcal{I}_r(R)$ for the lattice of right ideals of R. The following is the result corresponding to [14, 3.14(ii)] in the case when I is *not* a (σ, δ)-ideal.

Proposition 3.1 Let R be a complete local ring, $S = R[[x; \sigma, \delta]] \in \mathbf{SPS}(R)$, and let I be a right ideal of R. Set $I[[x; \sigma, \delta]] := \{\sum a_i x^i : a_i \in I\}$. Then $I[[x; \sigma, \delta]]$ is a right ideal of S. Moreover, the map $\theta : \mathcal{I}_r(R) \to \mathcal{I}_r(S)$ sending I to $I[[x; \sigma, \delta]]$ is a strictly increasing poset map.

Proof $I[[x; \sigma, \delta]]$ is clearly an additive subgroup of S.

Take $r \in R$ and $a = \sum_{i=0}^{\infty} a_i x^i \in I[[x; \sigma, \delta]]$. Then we may evaluate ar inside S:

$$\begin{aligned} ar &= \sum_{i=0}^{\infty} a_i x^i r \\ &= \sum_{i=0}^{\infty} \sum_{m \in M_i} a_i m(\sigma, \delta)(r) x^{e(m)} \end{aligned}$$

in the notation of Lemma 1.10. Now, $m(\sigma, \delta)(r) \in R$, so $a_i m(\sigma, \delta)(r) \in I$, and hence $ar \in I[[x; \sigma, \delta]]$.

We need to check that this gives a well-defined right S-action on $I[[x; \sigma, \delta]]$ – in other words, that the right actions of the elements xr and $\sigma(r)x + \delta(r)$ agree for all $r \in R$. But this is already true *a fortiori*, as the right action of S on $I[[x; \sigma, \delta]]$ is just induced by multiplication inside the ring S.

Finally, it is clear that, if $I_1 \leq I_2$, then $\theta(I_1) \leq \theta(I_2)$; and, if $J = \theta(I)$, then we may recover I as J/Jx. This shows that θ is a strict map of posets. $\square$

Lemma 3.2 *Let $(R, \mathfrak{m})$ be a complete local ring with $R/\mathfrak{m} = k$. Let $Y \subsetneq X$ be adjacent right ideals of R: then, as right S-modules, $\theta(X)/\theta(Y)$ is isomorphic to $S/\mathfrak{m}S \cong k[[\overline{x}; \overline{\sigma}]]$, a skew power series ring of automorphic type.*

Proof Let θ continue to denote the map $\mathcal{I}_r(R) \to \mathcal{I}_r(S)$ defined in Proposition 3.1.

$Y \subset X$ are adjacent if and only if X/Y is a simple right module. In particular, the annihilator $\mathrm{ann}(X/Y)_R$ must be the unique maximal right ideal $\mathfrak{m}$ of R, and so we must have $X/Y \cong R/\mathfrak{m} \cong k$.

The proposed isomorphism is obvious on the level of abelian groups, and it is easy to see that right multiplication by $x \in S$ is the same as right multiplication by $\overline{x} \in k[[\overline{x}; \overline{\sigma}]]$. It remains to check the R-action. Recall, from Lemma 1.10, that

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) r = \sum_{i=0}^{\infty} \sum_{m \in M_i} a_i m(\sigma, \delta)(r) x^{e(m)}$$

for all $a_i \in X, r \in R$. But, for a given $m \in M_i$, if $m(\sigma, \delta)$ contains an instance of δ (i.e. if $e(m) < i$), then $m(\sigma, \delta)(r) \in \mathfrak{m}$, and hence $a_i m(\sigma, \delta)(r) \in X\mathfrak{m} \subseteq Y$. This implies that

$$\left(\sum_{i=0}^{\infty} \overline{a_i x^i}\right) \overline{r} = \sum_{i=0}^{\infty} \overline{a_i \sigma^i(r) x^i}$$

inside X/Y, as required. $\square$

Theorem 3.3 *Let k be a division ring, and $S \in$ $\mathbf{SPS}^n(k)$. Then* $\mathrm{Kdim}(S) = \mathrm{Kdim}(\mathrm{gr}(S)) = n$.

Proof The proof of this theorem closely follows some of the methods of [1]. We will calculate the *right* Krull dimension of S, but the calculation of the left Krull dimension is identical. Let θ continue to denote the map $\mathcal{I}_r(R) \to \mathcal{I}_r(S)$ of Proposition 3.1.

When $n = 0$, we have $S = \mathrm{gr}(S) = k$, and there is nothing to prove. We proceed by induction on the rank of S.

Let $R \in \mathbf{SPS}^{n-1}(k)$, with $S = R[[x; \sigma, \delta]] \in \mathbf{SPS}(R)$. By the inductive hypothesis, we know that

$$\mathrm{Kdim}(R) = \mathrm{Kdim}(\mathrm{gr}(R)) = n - 1.$$

We also know, by [23, Corollary 2.9(ii)], that $\mathrm{gr}(S) \cong \mathrm{gr}(R)[\overline{x}; \overline{\sigma}]$, a skew polynomial ring of automorphic type, and so $\mathrm{Kdim}(\mathrm{gr}(S)) = \mathrm{Kdim}(\mathrm{gr}(R)) + 1 = n$ by [17, Proposition 6.5.4(i)].

Now, given arbitrary adjacent right ideals $Y \subsetneq X$ of R, we know already from Lemma 3.2 that $\theta(X)/\theta(Y) \cong k[[\overline{x}; \overline{\sigma}]]$ as right S-modules: in particular, $\theta(X)/\theta(Y)$ is not artinian as a right S-module, so we must have $\mathrm{Kdim}_S(\theta(X)/\theta(Y)) \geq 1$. Now applying [1, Lemma 2.3], we see that $n = \mathrm{Kdim}(R) + 1 \leq \mathrm{Kdim}(S)$.

Together with the inequality of [23, Corollary 2.9(iv)], we see that

$$n \leq \mathrm{Kdim}(S) \leq \mathrm{Kdim}(\mathrm{gr}(R)) + 1 = n,$$

and the result follows. $\square$

3.2 Classical Krull Dimension

Theorem 3.4 *Let k be a division ring, and $R \in \mathbf{SPS}^n(k)$ be triangularisable. Then* $\mathrm{clKdim}(R) = n$.

Proof Triangularise R as $R = k[[x_1;\sigma_1,\delta_1]]\dots[[x_n;\sigma_n,\delta_n]] \in \mathbf{TSPS}^n(k)$, with jth subextension $(R_j, \mathfrak{m}_j)$ for $0 \le j \le n$.

To obtain a lower bound on the classical Krull dimension, consider the chain of ideals $\{\mathfrak{m}_i R\}_{i=0}^n$ (these are two-sided by Proposition 2.4). This chain has length n, and again by Proposition 2.9, the quotient rings are iterated local skew power series rings over k, and are hence prime by Theorem A(i). Hence $\mathrm{clKdim}(R) \ge n$. The reverse inequality is given by noting that $\mathrm{clKdim}(R) \le \mathrm{Kdim}(R)$ (e.g. [17, Lemma 6.4.5]), and that here $\mathrm{Kdim}(R) = n$ by Theorem 3.3. □

Remark 3.5 Triangularisability is a sufficient, but not a necessary, condition to have $\mathrm{clKdim}(R) = \mathrm{Kdim}(R)$, as shown by the following proposition together with Non-example 2.19.

Proposition 3.6 Let k be a division ring, and let $R \in \mathbf{SPS}^n(k)$ have *pure automorphic* type. Then $\mathrm{clKdim}(R) = n$.

Proof Suppose $R = k[[x_1;\sigma_1]][[x_2;\sigma_2]]\dots[[x_n;\sigma_n]]$. Note that (x_n) is a two-sided ideal of R, and

$$R/(x_n) \cong k[[x_1;\sigma_1]][[x_2;\sigma_2]]\dots[[x_{n-1};\sigma_{n-1}]].$$

The result follows by induction on n. □

Proof of Theorem C This is contained within Theorem 3.4 and Proposition 3.6. □

3.3 Global Dimension

Theorem 3.7 *Let $(R, \mathfrak{m})$ be a complete local ring and $S \in \mathbf{SPS}(R, \mathfrak{m})$. Then $\mathrm{Ext}^i_R(M, R) \cong \mathrm{Ext}^{i+1}_S(M, S)$ as abelian groups for all i and all left S-modules M that are finitely generated as left R-modules.*

This follows from a similar argument to [19, Proposition 1.4], after minor adjustments. Below we outline a brief sketch proof, broadly adopting the notation of [19], with the relevant adjustments made. The reader may also consult the similar result [20, Theorem 3.1] for more details.

Sketch proof. Write $S = R[[x;\sigma,\delta]]$.

If M' is any left S-module, we can define endomorphisms of abelian groups

- $a_l(M') : S \otimes_R M' \to S \otimes_R M'$, given by $a_l(s \otimes m) = sx \otimes m - s \otimes xm$ for all $s \in S$ and $m \in M'$,
- $b_l(M') : S \otimes_R M' \to M'$, given by $b_l(s \otimes m) = sm$,

and similarly if M' is a right S-module we can define

- $a_r(M') : M' \otimes_R S \to M' \otimes_R S$, given by $a_r(m \otimes s) = m \otimes xs - mx \otimes s$.

Moreover, if M' is any left S-module, there is a continuous right action of S on $\mathrm{Hom}_R(M', R)$ as follows: for all $f \in \mathrm{Hom}_R(M', R)$ and $m \in M'$, set $(fx)(m) = \sigma^{-1}(f(xm)) - \sigma^{-1}\delta(f(m))$ and $(fr)(m) = f(m)\sigma^{-1}(r)$. Now we may define further endomorphisms (of abelian groups)

- $F(M') : \mathrm{Hom}_R(M', S) \to \mathrm{Hom}_R(M', S)$, given by $F(f)(m) = xf(m) - f(xm)$,
- $\varphi : \mathrm{Hom}_R(M', R) \otimes_R S \to \mathrm{Hom}_R(M', S)$, given by $\varphi(f \otimes s)(m) = f(m)s$.

Then the following diagram of right S-modules commutes:

$$\begin{tikzcd}
\mathrm{Hom}_R(M', R) \otimes_R S \arrow[r, "(\sigma\otimes\mathrm{id})\circ a_r"] \arrow[d, "\varphi"'] & \mathrm{Hom}_R(M', R) \otimes_R S \arrow[d, "\varphi"] \\
\mathrm{Hom}_R(M', S) \arrow[r, "F(M')"'] & \mathrm{Hom}_R(M', S).
\end{tikzcd}$$

(For readability, we have suppressed the argument of $a_r = a_r(\mathrm{Hom}_R(M', S))$, and will continue to do so throughout the rest of the proof.)

Given a left S-module M that is finitely generated as a left R-module, we note first that

$$0 \longrightarrow S \otimes_R M \xrightarrow{a_l} S \otimes_R M \xrightarrow{b_l} M \longrightarrow 0 \tag{3.1}$$

is a short exact sequence of left S-modules, and we may deduce the associated long exact sequence

$$\begin{aligned}
\cdots &\xrightarrow{\partial} \mathrm{Ext}^i_S(M, S) \xrightarrow{\mathrm{Ext}^i_S(b_l,S)} \mathrm{Ext}^i_S(S \otimes_R M, S) \xrightarrow{\mathrm{Ext}^i_S(a_l,S)} \mathrm{Ext}^i_S(S \otimes_R M, S) \\
&\xrightarrow{\partial} \mathrm{Ext}^{i+1}_S(M, S) \xrightarrow{\mathrm{Ext}^{i+1}_S(b_l,S)} \mathrm{Ext}^{i+1}_S(S \otimes_R M, S) \xrightarrow{\mathrm{Ext}^{i+1}_S(a_l,S)} \mathrm{Ext}^{i+1}_S(S \otimes_R M, S) \\
&\longrightarrow \quad \cdots
\end{aligned} \tag{3.2}$$

Now, taking $X^\bullet \to M$ to be an appropriate projective resolution of M as a left S-module, the commutative diagram above gives a commutative diagram of homology

$$\begin{tikzcd}
H(\mathrm{Hom}_R(X^\bullet, R) \otimes_R S) \arrow[r, "H((\sigma\otimes\mathrm{id})\circ a_r)"] \arrow[d, "H(\varphi)"'] & H(\mathrm{Hom}_R(X^\bullet, R) \otimes_R S) \arrow[d, "H(\varphi)"] \\
H(\mathrm{Hom}_R(X^\bullet, S)) \arrow[r, "H(F)"'] & H(\mathrm{Hom}_R(X^\bullet, S)).
\end{tikzcd}$$

As in [19], standard arguments in homological algebra allow us to identify the respective complexes and morphisms in the above diagram with

$$\begin{tikzcd}
\mathrm{Ext}_R(M, R) \otimes_R S \arrow[r, "(\sigma\otimes\mathrm{id})\circ a_r"] \arrow[d, "\wr"'] & \mathrm{Ext}_R(M, R) \otimes_R S \arrow[d, "\wr"] \\
\mathrm{Ext}_S(S \otimes_R M, S) \arrow[r, "\mathrm{Ext}_S(a_l,S)"'] & \mathrm{Ext}_S(S \otimes_R M, S).
\end{tikzcd}$$

The map $(\sigma \otimes \mathrm{id}) \circ a_r$ is injective: indeed, the top row of this commutative square fits into a twisted right-hand version of the exact sequence (3.1). In particular, this means that the maps $\mathrm{Ext}^i_S(a_l, S)$ of the long exact sequence (3.2) are injective, and so we may complete the above diagram to the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^i_R(M,R)\otimes_R S \arrow[r, "(\sigma\otimes\mathrm{id})\circ a_r"] \arrow[d, "\wr"'] & \mathrm{Ext}^i_R(M,R)\otimes_R S \arrow[r, "b_r\circ(\sigma^{-1}\otimes\mathrm{id})"] \arrow[d, "\wr"] & \mathrm{Ext}^i_R(M,R) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ext}^i_S(S\otimes_R M,S) \arrow[r, "\mathrm{Ext}^i_S(a_l,S)"'] & \mathrm{Ext}^i_S(S\otimes_R M,S) \arrow[r, "\partial"'] & \mathrm{Ext}^{i+1}_S(M,S) \arrow[r] & 0,
\end{tikzcd}$$

from which we may deduce that the final vertical map is also an isomorphism. □

Corollary 3.8 *Let k be a field, $(R,\mathfrak{m}) \in \mathbf{SPS}^n(k,0)$, and $(S,\mathfrak{n}) = R[[x;\sigma,\delta]] \in \mathbf{SPS}(R,\mathfrak{m})$. Then we have* $\mathrm{projdim}_S(k) = \mathrm{projdim}_R(k) + 1$ *and* $\mathrm{gldim}(S) = \mathrm{gldim}(R) + 1$.

Proof We have augmentation maps $R \to R/\mathfrak{m} \cong k$ and $S \to S/\mathfrak{n} \cong k$, so by [4, Corollary 3.7], we have $\mathrm{gldim}(R) = \mathrm{projdim}_R(k)$ and $\mathrm{gldim}(S) = \mathrm{projdim}_S(k)$.

Suppose that $\mathrm{projdim}_R(k) = n < \infty$. Then by [5, VI, Ex. 9], we have $\mathrm{Ext}^n_R(k,R) \neq 0$, and hence by Theorem 3.7, $\mathrm{Ext}^{n+1}_S(k,S) \neq 0$, so that $\mathrm{projdim}_S(k) \geq n+1$.

Now assume (for contradiction) that $\mathrm{projdim}_S(k) \geq n+2$. By the same argument, this gives us $\mathrm{Ext}^{n+2}_S(k,S) \neq 0$, and hence $\mathrm{Ext}^{n+1}_R(k,R) \neq 0$. But this implies that $\mathrm{projdim}_R(k) \geq n+1$, a contradiction. So we must have $\mathrm{projdim}_S(k) = n+1 = \mathrm{projdim}_R(k)+1$.

Finally, as $\mathrm{projdim}_k(k) = 0$, it is easy to see by induction on the rank of R (as an iterated local skew power series ring over k) that we always have $\mathrm{projdim}_R(k) < \infty$. □

Corollary 3.9 *Let k be a field and $R \in \mathbf{SPS}^n(k,0)$. Then* $\mathrm{injdim}_R(R) = \mathrm{projdim}_R(k) = \mathrm{gldim}(R) = n$.

Proof The claim $\mathrm{projdim}_R(k) = \mathrm{gldim}(R) = n$ follows from the above corollary. Now, if A is any R-module, then $\mathrm{Ext}^{n+1}_R(A,R) = 0$ by [5, VI, Proposition 2.1], so $\mathrm{injdim}_R(R) \leq n$; but again by [5, VI, Ex. 9] we have $\mathrm{Ext}^n_R(k,R) \neq 0$, so $\mathrm{injdim}_R(R) \geq n$. □

Proof of Theorem B This follows from Theorem 3.3 and Corollary 3.9, except for the claim about the canonical dimension $\mathrm{Cdim}(R)$, which follows from [6, Proposition 4.2(1)]. □

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