Sealed Bid Second Price Auctions with Discrete Bidding*

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Abstract

A single item is sold to two bidders by way of a sealed bid second price auction in which bids are restricted to a set of discrete values. Restricting attention to symmetric pure strategy behavior on the part of bidders, a unique equilibrium exists. When following these equilibrium strategies bidders may bid strictly above or below their valuation, implying that the item may be awarded to a bidder other than the high valuation bidder. In an auction with two acceptable bids, the expected revenue of the seller may be maximized by a high bid level not equal to the highest possible bidder valuation and may exceed the expected revenue from an analogous second price auction with continuous bidding (and no reserve price). With three acceptable bids, a revenue maximizing seller may choose unevenly spaced bids. With an arbitrary number of evenly spaced bids, as the number of acceptable bids is increased, the expected revenue of the seller and the probability of ex post inefficiency both may either increase or decrease.

Keywords: Auctions/Bidding; Auction Design; Discrete Bidding.

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1 Introduction

The existing literature on auctions focuses almost exclusively on situations in which any bid on a continuous interval can be submitted. In practice bidders are not able to choose their bid from a continuum of acceptable points. At the most basic level, the discrete nature of currency imposes a restriction on the acceptable bid levels. Additionally, the set of acceptable bids may be further restricted by the auctioneer. In this paper we analyze a sealed bid second price auction, in which acceptable bids are restricted to a set of discrete values.

In practice, the bid space is restricted in a discrete manner on most internet auction sites by the imposition of a positive bid increment. On eBay, this bid increment varies between five cents and one hundred dollars, depending upon the level of the current high bid. It should also be noted that most internet auctions are conducted in such a manner so as to make them strategically equivalent to second price auctions.[1]

Auctions in which the bid space is restricted in a discrete manner have been analyzed by Chwe (1989), Rothkopf and Harstad (1994), and Yu (1999). Chwe considers a sealed bid first price auction with discrete bidding. Specifically, he considers a situation in which there are n bidders, each with an independent private valuation drawn from the cumulative distribution function $F(v) = v$. The unique feature of his model is that bidders are not allowed to submit any bid on the interval $[0, \infty)$, but rather are restricted to choosing among M "evenly spaced" discrete bid levels $\{b_1, b_2, \ldots, b_M\}$ such that $b_i = \frac{i-1}{M}$. Since submitting a bid above ones valuation is a dominated strategy in a first price auction, Chwe is essentially allowing there to be an additional bid of $b_{M+1} = \frac{(M+1)-1}{M} = 1$. That is, he implicitly assumes that the highest acceptable bid is exactly equal to the highest possible bidder valuation. Chwe characterizes a unique symmetric pure strategy equilibrium for this auction. He subsequently argues that such an auction results in less revenue for the seller than the traditional continuous first price sealed bid auction. Further, as $M \to \infty$, the equilibrium bids converge to the equilibrium bids in the continuous case. Thus, the revenue of the seller in this discrete auction converges to the revenue of the continuous auction as $M \to \infty$. However, it is shown by way of example that the expected revenue of the seller is not monotonic in the number of acceptable bids.

Rothkopf and Harstad analyze an English auction in which bids are restricted to discrete values. It is important to note that (for bidders with independent, private valuations) an English auction with discrete bidding is *not* strategically equivalent to a sealed bid second price auction with discrete bidding, in contrast to the corresponding auctions with continuous bidding. To recognize this, first note that in an English auction submitting a bid above one's true valuation is dominated by submitting a bid at or below one's true valuation, be it an English auction with discrete or continuous bidding. Likewise, in a sealed bid second price auction with continuous bidding, submitting a bid above one's true valuation is dominated by submitting a bid equal to one's true valuation. However, as will be argued below, in a sealed bid second price auction with discrete bidding, submitting a bid above one's true valuation is not immediately dominated.

In the English auction analyzed by Rothkopf and Harstad n bidders (each with an independent, private valuation) are restricted to using $m + 1$ discrete bids, with the highest bid level exactly equal to the highest possible bidder valuation. Under numerous assumption on bidder

[1] For example, under the "proxy bidding" system on eBay (in which the highest bid competes at the minimum level necessary to be the leading bid), a bidder with an independent private valuation has a dominant strategy of submitting a bid equal to his true valuation, a point which has been recognized by Lucking-Reiley (2000a and 2000b).

behavior, expressions are determined for "expected seller revenue," "expected lost revenue" (the expected difference between the second highest bidder valuation and realized auction revenue), and "expected economic inefficiency" (the expected difference between the highest bidder valuation and the valuation of the winning bidder). When bidder valuations are uniformly distributed, expected seller revenue is maximized by bid levels which minimize expected inefficiency (this is not necessarily true if valuations are not uniformly distributed). Further, expected seller revenue may be maximized by having either an increasing, decreasing, or constant distance between subsequent bids (the result depending upon the number of bidders and distribution from which bidder valuations are drawn).

Yu examines each of the four common auction forms (sealed bid first price, sealed bid second price, English auction, and Dutch auction) under the assumption that bids are restricted to "evenly spaced" discrete values. The valuations of the n bidders are assumed to be independently drawn from a common distribution $F(v)$ such that $F(0) = 0$ and $F(1) = 1$. Yu assumes that bidders are restricted to choosing from $M + 1$ acceptable bid levels $\{b_1, b_2, \ldots, b_M, b_{M+1}\}$ such that $b_i = \frac{i-1}{M}$. Thus, she explicitly assumes that the highest acceptable bid level is exactly equal to the highest possible bidder valuation. It is shown that in each such auction a symmetric pure strategy equilibrium exists.[2] However, the sealed bid second price auction is no longer dominance solvable. Additionally, for the sealed bid second price auction, in equilibrium some bidders will bid above their valuation and some bidders will bid below their valuation. Finally, for each type of auction, as $M \to \infty$ equilibrium bids converge to the equilibrium bids in the corresponding continuous bid auction.

Before we look into the contributions made by this paper, it may be worthwhile to briefly mention of another vein in the literature. A few papers have looked into the phenomenon of "jump bidding" in ascending English auctions wherein bidders sometimes bid higher than what is necessary to be the current highest bidder. Isaac et.al.(2007), Avery (1998), Easley and Tenorio (2004) are few of the notable ones. As shall be subsequently seen, papers such as above are examining an auction format different from ours (open out-cry versus sealed bid) which do not always operate in the IPV domain. However we do feel obliged to mention them here, as the auctions described in them also demonstrate a discrete bid space similar to ours.

Turning to this paper, we consider a situation in which the discrete bid points need not be "evenly spaced," an assumption that was made by both Chwe and Yu.[3] Additionally, we do not assume that the highest acceptable bid point must be exactly equal to the highest possible bidder valuation, an assumption that was explicitly made by both Rothkopf and Harstad and Yu and implicitly made by Chwe.[4] A central point of our analysis is that these previous assumptions were quite restrictive. This point is made by illustrating that a revenue maximizing seller: may wish to set a high bid level which is not equal to the highest possible bidder valuation, and may wish to have discrete bids that are not evenly spaced. Throughout the present analysis, attention is restricted to a situation in which there are two bidders, each with an independent private valuation

[2] Yu does not show uniqueness of a symmetric pure strategy equilibrium.

[3] When reference is made to Chwe, it is simply done to note that the strategy space in the auction we are analyzing is different from the strategy space in the auction analyzed by Chwe. Since Chwe considered a first price auction whereas we are considering a second price auction, the payoff functions for bidders clearly differ. Thus, the equilibrium which we identify would not extend to a first price auction as examined by Chwe.

[4] Even though bidders in a sealed bid first price auction with discrete bidding (as analyzed by Chwe) would never submit a bid equal to the highest possible bidder valuation, making this assumption is restrictive when made in conjunction with the assumption of evenly spaced bids.

drawn from a common distribution. An examination of bidder behavior in such an environment is presented in Section 3. Restricting attention to symmetric pure strategy behavior on the part of bidders, a unique equilibrium is shown to exist. When bidders play according to this unique pair of strategies, the item may be awarded to the bidder with the lower valuation. That is, there is a strictly positive probability of allocative inefficiency.

Seller revenue is subsequently examined in Section 4. In order to demonstrate that the assumptions on the placement of the discrete bid points made by Chwe, Rothkopf and Harstad, and Yu were restrictive, we begin by considering situations in which there are only two acceptable bids (subsection 4.1) or three acceptable bids (subsection 4.2). When constrained to choosing two bid levels, a revenue maximizing seller may choose a level of the "highest acceptable bid" which is greater than or less than the highest possible valuation. In this case, such a revenue maximizing seller does not necessarily minimize the probability of allocative inefficiency. Additionally, a revenue maximizing seller constrained to choosing three discrete bid values may wish to set unevenly spaced bids. Finally, a numerical analysis of an auction with an arbitrary number of evenly spaced bids is conducted (subection 4.3), in order to examine how the equilibrium outcome changes as the distance between evenly spaced bids is reduced. As more discrete bid points below the highest possible bidder valuation become available (by way of the common distance between bids being reduced), both the expected revenue of the seller and the probability of ex post inefficiency may either increase or decrease. That is, reducing the spacing between discrete bid points will not always increase the expected revenue of the seller and will not always decrease the probability of ex post inefficiency. Section 5 concludes.

2 Model

A single item is for sale by way of auction. There are two bidders, each with an independent private valuation, $v_i \in [v_L, v_H]$, drawn from a common continuous distribution $F(v)$, such that: $F(v_L) = 0$, $F(v_H) = 1$, and $f(v) = F'(v) > 0$ for all $v \in [v_L, v_H]$. Each bidder i simultaneously submits a sealed bid b_i. However, bids are constrained to a countable number of discrete points $B_0 < B_1 < \ldots < B_j < \ldots$. It is assumed throughout that $B_0 = v_L$.[5]

Once both bids have been submitted, the seller (weakly) orders the bids. If both bidders submit the same bid B_j, the bids are ordered randomly by the seller (with each ordering occurring with equal probability). The item is awarded to the bidder submitting the highest bid for an amount equal to the second highest bid submitted.[6]

It is well known that in a standard sealed bid second price auction with continuous bidding, bidders with independent private valuations have a dominant strategy of submitting a bid equal to their own valuation (Vickrey (1961)). However, if bids are constrained to a discrete set of points, a bidder may be unable to submit such a bid. This is precisely the case here, when valuations are drawn from a continuous distribution. It is not immediately clear what a bidder should do in such an environment. The first goal is to characterize optimal bidder behavior. Once this is done, the expected revenue of the seller is analyzed. An attempt is made to determine how the revenue of the seller changes as the discrete bid levels change and as additional bid levels are allowed.

[5] This is analogous to considering a traditional second price auction without a reserve price.

[6] For example, suppose the two bidders submit the following bids: $b_1 = B_2$ and $b_2 = B_2$. The bids will be order by the seller as either $b_1 \geq b_2$ or $b_2 \geq b_1$, each with equal probability. In either case, the second highest bid is equal to B_2. Thus, the item will be sold at a price of B_2 to either bidder 1 or bidder 2.

3 Bidder Behavior

Suppose there are an arbitrary number of discrete bid points, a finite number of which are less than v_H. If there are no acceptable bid levels greater than v_H, let B_k denote the largest acceptable bid level (clearly $B_k \leq v_H$ in this case). If there is at least one acceptable bid level (weakly) greater than v_H, let B_{k-1} denote the largest acceptable bid level (strictly) less than v_H and let B_k denote the smallest acceptable bid level (weakly) greater than v_H.[7]

3.1 Dominated Bidding Strategies

Consider a bidder with valuation v_i. Proposition 1 serves as an initial characterization of the behavior of such a bidder. The stated behavior is analogous to the weakly dominant strategy of "truthful bidding" in a traditional sealed bid second price auction. All results are proved in Appendix A.

Proposition 1 *Consider an arbitrary allowable bid level B_j. For a bidder with $v_i \leq B_j$, bidding B_j weakly dominates bidding above B_j; for a bidder with $v_i \geq B_j$, bidding B_j weakly dominates bidding below B_j.*

Considering a bidder with $v_i \in (B_{j-1}, B_j)$, Proposition 1 implies that all bids other than B_{j-1} and B_j are weakly dominated.[8] As in the proof of Proposition 1, define p_j to be the probability with which a rival bidder submits a bid of B_j. The following results follow immediately from Proposition 1 (and are thus stated without proof).

Corollary 1 *If $p_j > 0$, then a bidder with $v_i \geq B_j$ realizes a strictly higher payoff from bidding B_j than by bidding below B_j.*

Corollary 2 *If $p_j > 0$, then a bidder with $v_i \leq B_j$ realizes a strictly higher payoff from bidding B_j than by bidding above B_j.*

When characterizing bidder behavior, the search for an equilibrium will be restricted to symmetric, pure strategy bidding functions. It will be argued that within this restricted class of strategies, a unique equilibrium exists.

3.2 Symmetric Equilibrium Bidding Strategies

Focus on the payoff of a bidder with $v_i \in (B_{j-1}, B_j)$ from bidding either B_{j-1} or B_j (with the behavior of his rival fixed). The expected payoff of i from bidding B_j is

$$\pi_i(B_j, v_i) = \sum_{l=0}^{j-1}(v_i - B_l)p_l + (v_i - B_j)\frac{1}{2}p_j \tag{1}$$

[7] As will be seen, this seemingly inconsistent description of the bid levels will result in bidder i choosing from exactly $k+1$ bids in equilibrium in either case.

[8] Similarly, if the maximum acceptable bid is less than v_H, all bids other than B_k (the largest acceptable bid in this case) are weakly dominated for a bidder with $v_i \geq B_k$.

The difference between the expected payoff from bidding B_j versus B_{j-1} is

$$\pi_i(B_j, v_i) - \pi_i(B_{j-1}, v_i) = (v_i - B_j)\frac{1}{2}p_j + (v_i - B_{j-1})\frac{1}{2}p_{j-1}. \tag{2}$$

Given the results thus far, we establish the following:

Lemma 1 *For any pure strategy symmetric equilibrium, it must be that* $p_j > 0$ *for* $j = 0, 1, 2, \ldots, k$ *and* $\sum_{j=0}^{k} p_j = 1$.

Note that when describing a symmetric equilibrium, p_j refers to the probability with which each individual bidder chooses to submit the bid B_j. Lemma 1 implies that in any such equilibrium, all bids from B_0 up to B_k are submitted with positive probability (while bids of B_{k+1} and above are submitted with zero probability). Also observe that $\pi_i(B_j, v_i)$ is linear in v_i. Further, using equation (2) it can be easily seen that

$$\pi_i(B_j, v_L) < \pi_i(B_{j-1}, v_L). \tag{3}$$

Moreover,

$$\frac{\partial \pi_i(B_{j-1}, v_i)}{\partial v_i} < \frac{\partial \pi_i(B_j, v_i)}{\partial v_i}. \tag{4}$$

Consider $j = 1, 2, \ldots, k-1$. First substituting $v_i = B_j$ in (2) and then again $v_i = B_{j-1}$ in the same equation, we get the following respectively.

$$\pi_i(B_j, B_j) > \pi_i(B_{j-1}, B_j) \tag{5}$$

$$\pi_i(B_j, B_{j-1}) < \pi_i(B_{j-1}, B_{j-1}) \tag{6}$$

The above inequalities (3)–(6) imply that for any bidder i, $\pi_i(B_j, v_i)$ intersects $\pi_i(B_{j-1}, v_i)$ *from below* and that there exists an unique $c_j \in (B_{j-1}, B_j)$ at which both are equal.[9] Linearity of expected profit with respect to v_i ensures that the two functions intersect only once.

Finally, (5) and (6) also holds for $j = k$ implying that there exists an unique $c_k \in (B_{k-1}, B_k)$ at which expected profit functions are equal. In fact, it can be easily seen that $c_k \in (B_{k-1}, v_H)$. This is quite obvious if $B_k \leq v_H$. In case of $B_k > v_H$, this is ensured by Lemma 1. As $p_k > 0$, we must have $\pi_i(B_k, v_H) > \pi_i(B_{k-1}, v_H)$, otherwise p_k can never be positive in equilibrium.

Recall that for a bidder with $v_i \in (B_{j-1}, B_j)$, Proposition 1 implies that all bids other than B_{j-1} and B_j are weakly dominated. Focusing on B_j (the higher of these two remaining non-dominated bid points, which is greater than the valuation of the bidder) bidder i will consider submitting a bid of B_j in a second price environment precisely because doing so does not guarantee that he will have to pay an amount above his valuation. For this bidder, bidding B_j increases the likelihood of winning the auction, but at the risk of having to pay an amount above his valuation. A rational bidder will balance this benefit of bidding B_j (the greater likelihood of winning the auction) against this cost of bidding B_j (the possibility of having to pay an amount above his valuation).

For a bidder with $v_i \in (B_{j-1}, B_j)$, bidding B_{j-1} gives a strictly higher payoff *if and only if* his valuation is above a cutoff $c_j \in [B_{j-1}, B_j]$ while bidding B_j gives a strictly higher payoff *if*

[9] Note that these inequalities hold strictly (for all bids B_{j-1} and B_j) only if all bids are submitted with strictly positive probability. As stated by Lemma 1, this must be the case for any pure strategy symmetric equilibrium.

and only if his valuation is below c_j. Figure 1 in Appendix B illustrates this phenomenon for a bidder i facing a choice between bidding B_{j-1} and B_j.

Consider the bid points $B_0 < B_1 < \ldots < B_{k-1} < B_k$. For any such $k+1$ acceptable bid points, conjecture that there exists a unique set of $k+2$ values $c_0 < c_1 < \ldots < c_k < c_{k+1}$ such that $c_0 = v_L$, $c_j \in (B_{j-1}, B_j)$ for $j = 1, \ldots, k$, and $c_{k+1} = v_H$, with the interpretation that in equilibrium a bidder with valuation v_i will choose $b_i = B_j$ if and only if $v_i \in [c_j, c_{j+1})$ for $j = 0, 1, \ldots, k-1$ and choose $b_i = B_k$ if and only if $v_i \in [c_k, c_{k+1}]$.

If the rival of bidder i plays according to such a strategy characterized by $C = (c_0, c_1, \ldots, c_k, c_{k+1})$, bidder i with valuation v_i has an expected payoff of

$$\pi_i(B_0, v_i, C) = (v_i - B_0)\frac{1}{2}F(c_1)$$

from submitting a bid of $b_i = B_0$ and an expected payoff of

$$\pi_i(B_j, v_i, C) = \left\{\sum_{l=0}^{j-1}(v_i - B_l)\left[F(c_{l+1}) - F(c_l)\right]\right\} + (v_i - B_j)\frac{1}{2}\left[F(c_{j+1}) - F(c_j)\right]$$

from submitting a bid of $b_i = B_j$ for $j = 1, \ldots, k$.

In order for the conjectured values of $c_1 < \ldots < c_k$ to support an equilibrium, it must be that $\pi_i(B_{j-1}, c_j, C) = \pi_i(B_j, c_j, C)$, or equivalently $D_j(c_{j-1}, c_j, c_{j+1}) = \pi_i(B_{j-1}, c_j, C) - \pi_i(B_j, c_j, C) = 0$, for $j = 1, \ldots, k$. This leads to k conditions that must simultaneously be satisfied by the k values $c_1 < \ldots < c_k$. These k conditions can be expressed as

$$(B_j - c_j)\left[F(c_{j+1}) - F(c_j)\right] - (c_j - B_{j-1})\left[F(c_j) - F(c_{j-1})\right] = 0 \tag{7}$$

for $j = 1, \ldots, k$.[10] Theorem 1 characterizes equilibrium bidder behavior.

Theorem 1 *A symmetric equilibrium exists in which each bidder i submits a bid of* $b_i = B_j$ *if and only if* $v_i \in [c_j, c_{j+1})$ *for* $j = 0, \ldots, k-1$ *and submits a bid of* $b_i = B_k$ *if and only if* $v_i \in [c_k, c_{k+1}]$*, where* $c_0 = v_L$*,* $c_{k+1} = v_H$*, and* $(c_1, \ldots, c_k)$ *are the unique values for which condition* (7) *is simultaneously satisfied for* $j = 1, \ldots, k$.

The unique values of $c_1 < \ldots < c_k$ for which condition (7) holds simultaneously for $j = 1, \ldots, k$ specify equilibrium bidder behavior in this auction. It is important to take note of two important characteristics of the equilibrium characterized in Theorem 1. First, it is unique in the class of symmetric pure strategy equilibria (follows from our discussion preceding the statement of the theorem above). And secondly, this equilibrium is dominance solvable to the extent of narrowing down to two bid points which are *nearest* to the valuation of any bidder.[11]

Figure 2 in Appendix B directly illustrates the equilibrium expected payoff for a bidder i from each allowable bid, in the case of four allowable bid points. As can be seen readily, for any $v_i \in (c_j, c_{j+1})$, bidding B_j fetches the highest payoff for bidder i, where $j = 0, 1, 2, 3$.

When following such a strategy, a bidder may choose to submit a bid exceeding his valuation. Recall that a bidder with $v_i \in (B_{j-1}, B_j)$ chooses to bid either B_{j-1} or B_j based upon a careful comparison of the benefit of bidding above his valuation (i.e., the increased likelihood of winning

[10] Recall that $c_0 = v_L$ and $c_{k+1} = v_H$. As a result, $F(c_0) = 0$ and $F(c_{k+1}) = 1$.

[11] Both characteristics have a nice resemblance to the case of standard second price auctions with a continuous bid space.

the auction when bidding B_j) to the cost of bidding above his valuation (i.e., the possibility of having to pay B_j, an amount greater than v_i). For a bidder with a valuation relatively close to B_j, the loss of having to pay a price above v_i is smaller than for a bidder with a valuation relatively close to B_{j-1}. Thus, for each $j = 1, \ldots, k$ the bidders with "relatively high valuations" within each range (specifically, $v_i \in [c_j, B_j)$) will choose to bid above their valuation, precisely because the gain from doing so exceeds the cost from doing so. Likewise, those bidders with "relatively low valuations" within each range (specifically, $v_i \in [B_{j-1}, c_j)$) will choose to not bid above their valuation, precisely because the cost from doing so exceeds the gain from doing so.

Further, note that the item is awarded to the bidder with the lower valuation with strictly positive probability. Specifically, whenever both bidders submit the same bid, the allocation of the item is inefficient with probability $\frac{1}{2}$. Thus, the total probability of such allocative inefficiency is

$$\Pr(I) = \sum_{j=0}^{k} \frac{1}{2} \left[F(c_{j+1}) - F(c_j)\right]^2 .$$

4 Seller Revenue

As with a traditional sealed bid second price auction, the revenue of the seller is equal to the second highest bid submitted. Therefore, the expected revenue of the seller depends critically upon the distribution of the second highest valuation of the two bidders, denoted $F_{(1)}(v) = 2F(v) - F(v)^2$. The expected revenue of the seller can be expressed as

$$\begin{aligned} \Pi_S &= B_0 F_{(1)}(c_1) + B_1[F_{(1)}(c_2) - F_{(1)}(c_1)] + \cdots \\ &\quad + B_{k-1}[F_{(1)}(c_k) - F_{(1)}(c_{k-1})] + B_k[1 - F_{(1)}(c_k)] \\ &= \sum_{j=0}^{k} B_j [F_{(1)}(c_{j+1}) - F_{(1)}(c_j)] \\ &= B_k - \sum_{j=1}^{k} (B_j - B_{j-1}) F_{(1)}(c_j). \end{aligned}$$

To demonstrate that the assumptions on the placement of the discrete bid points made in previous studies were in fact restrictive, we begin our examination of seller revenue by considering situations in which there are only two or only three acceptable bids. With only two bid levels, a revenue maximizing seller will not necessarily want to choose a bid exactly equal to the highest possible bidder valuation (a restriction on the highest acceptable bid made by Chwe, Rothkopf and Harstad, and Yu). When choosing the levels of three discrete bids, evenly spaced bids (a restriction on bid placement made by both Chwe and Yu) are not necessarily revenue maximizing. Finally, such an auction with an arbitrary number of evenly spaced bids (the highest of which need not be equal to the highest possible bidder valuation) is analyzed, in order to gain further insight into the outcome as the number of relevant bids and the distance between bids is changed.

4.1 Two Discrete Bids

Consider a situation in which there are only two acceptable bids, $B_0 = 0$ and $B_1 > 0$. When able to choose the value of B_1, an expected revenue maximizing seller will not always want to set $B_1 = v_H$ and will not always choose the value of B_1 which minimizes the probability of ex post inefficiency. To illustrate these points, consider a distribution function $F(\cdot)$ satisfying the additional assumptions that: $v_L = 0$; and $x\frac{f(x)}{1-F(x)}$ is strictly increasing in x. First note that for a general distribution function $F(\cdot)$, equilibrium bidder behavior defined by Condition (7) can be expressed as:

$$B_1\left[1 - F(c_1)\right] - c_1 = 0. \tag{8}$$

Letting $c_1(B_1)$ denote the value of c_1 for which this condition is satisfied, by the Implicit Function Theorem we have $c_1'(B_1) = \frac{1-F(c_1)}{1+B_1 f(c_1)} > 0$. That is, the value of c_1 is increasing in B_1. Therefore, there are two distinct effects on the expected revenue of the seller from a change in the value of B_1. An increase in B_1 will: increase expected revenue, since the seller will receive a greater amount when both bidders have valuations above c_1; but decrease expected revenue, since it becomes less likely that both bidders will have valuations above c_1. When choosing the value of B_1, an expected revenue maximizing seller will balance these two effects against one another. With only two discrete bids (and $B_0 = 0$), the expected revenue of the seller as a function of B_1 can be expressed as:

$$\begin{aligned}\Pi_S(B_1) &= B_1\left[1 - F_{(1)}\left(c_1(B_1)\right)\right] \\ &= B_1\left[1 - 2F\left(c_1(B_1)\right) + F\left(c_1(B_1)\right)^2\right].\end{aligned}$$

From here:

$$\Pi_S'(B_1) = \left[1 - F\left(c_1(B_1)\right)\right]^2 - 2B_1 f\left(c_1(B_1)\right) c_1'(B_1)\left[1 - F\left(c_1(B_1)\right)\right].$$

For an increase in B_1: the first term in the expression above captures the increase in expected revenue resulting from receiving a larger payment when both bidders bid B_1; the second term captures the decrease in expected revenue from having it be less likely that both bidders will submit a bid of B_1. Since $c_1'(B_1) = \frac{1-F(c_1)}{1+B_1 f(c_1)}$, this expression can be simplified to:

$$\Pi_S'(B_1) = \left[1 - F\left(c_1(B_1)\right)\right]^2\left\{1 - 2B_1\frac{f\left(c_1(B_1)\right)}{1 + B_1 f\left(c_1(B_1)\right)}\right\},$$

the sign of which is the same as the sign of $1 - B_1 f\left(c_1(B_1)\right)$. By Condition (8) the relation between B_1 and c_1 can be stated as $B_1 = \frac{c_1}{1-F(c_1)}$. Thus, the sign of $\Pi_S'(B_1)$ is the same as the sign of $1 - c_1\frac{f(c_1)}{1-F(c_1)}$. $1 - c_1\frac{f(c_1)}{1-F(c_1)}$ is equal to one for $c_1 = v_L = 0$ and tends to $-\infty$ as $c_1 \to v_H$. Further, under the assumption that $x\frac{f(x)}{1-F(x)}$ is increasing in x, $1 - c_1\frac{f(c_1)}{1-F(c_1)}$ is strictly decreasing in c_1 for $c_1 \in (0, v_H)$. Therefore, there exists a unique value $c_1^* \in (0, v_H)$ such that $1 - c_1\frac{f(c_1)}{1-F(c_1)}$ is equal to zero for $c_1 = c_1^*$, strictly positive for $c_1 < c_1^*$, and strictly negative for $c_1 > c_1^*$. It follows that there exists a unique B_1^* such that: $\Pi_S'(B_1^*) = 0$, $\Pi_S'(B_1) > 0$ for $B_1 < B_1^*$, and $\Pi_S'(B_1) < 0$ for $B_1 > B_1^*$. c_1^* is the unique value of c_1 and B_1^* is the unique value of B_1 for which the seller's expected revenue is maximized. Turning attention to the resulting probability of ex post inefficiency, when there are only two discrete bid points:

$$\Pr(I) = \frac{1}{2}\left(1 - 2F(c_1) + 2F(c_1)^2\right).$$

From this expression, $\frac{d\Pr(I)}{dc_1} = -f(c_1)\left[1 - 2F(c_1)\right]$, implying that the value of c_1 which minimizes the probability of ex post inefficiency is the unique value c_1^E for which $F\left(c_1^E\right) = \frac{1}{2}$. Let

B_1^E denote the corresponding value of B_1 which minimizes $\Pr(I)$. The observation that c_1^E must satisfy $F\left(c_1^E\right) = \frac{1}{2}$ is rather intuitive. Inefficiency will result half of the time whenever both bidders have valuations on the same side of c_1. Therefore, (irrespective of the functional form of $F(\cdot)$) this probability of inefficiency is minimized by minimizing the likelihood that both bidder valuations will be on the same side of c_1. This is done only by setting a value of c_1 for which it is equally likely that any one bidder will have a valuation either above or below c_1: that is, a value such that $F\left(c_1^E\right) = \frac{1}{2}$. A final implication is that regardless of the functional form of $F(\cdot)$, the minimum probability of allocative inefficiency with only two discrete bid points is $\Pr(I)^E = \frac{1}{4}$. A comparison of B_1^* to v_H will allow us to see if an expected revenue maximizing seller would set B_1 equal to the highest possible bidder valuation, while a comparison of B_1^* to B_1^E (or of c_1^* to c_1^E) will allow us to determine if an expected revenue maximizing seller would choose B_1 to minimize the probability of ex post inefficiency. In general the seller will not necessarily want to set $B_1^* = v_H$ or $B_1^* = B_1^E$. To see this, consider $F(v) = v^\alpha$, with $\alpha \in (0, \infty)$. For this distribution function, $1 - c_1 \frac{f(c_1)}{1-F(c_1)} = 1 - \frac{\alpha c_1^\alpha}{1-c_1^\alpha}$. It follows that the values of c_1^* and B_1^* can be expressed as functions of α as:

$$c_1^*(\alpha) = \left(\frac{1}{1+\alpha}\right)^{\frac{1}{\alpha}} \qquad \text{and} \qquad B_1^*(\alpha) = \frac{(1+\alpha)^{\frac{\alpha-1}{\alpha}}}{\alpha}.$$

Proposition 2 characterizes $B_1^*(\alpha)$ in relation to $v_H = 1$, the highest possible bidder valuation.

Proposition 2 $B_1^*(1) = 1$; $B_1^*(\alpha) > 1$ *for* $\alpha < 1$; *and* $B_1^*(\alpha) < 1$ *for* $\alpha > 1$.

If bidder valuations are uniformly distributed between zero and one ($\alpha = 1$), then $B_1^*(\alpha) = 1 = v_H$. However, if bidder valuations are not uniformly distributed ($\alpha \neq 1$), then $B_1^*(\alpha) \neq 1 = v_H$. When interpreting the outcome for $\alpha \neq 1$, start by noting that when the seller sets a bid level B_1^* leading to c_1^*, the probability that any one bidder will choose to submit a bid of B_1^* is $1 - F\left(c_1^*\right) = \frac{\alpha}{1+\alpha}$. Since this probability is increasing in α, it follows that from the perspective of any one bidder it is: less likely that his rival will submit a bid of B_1^* when the value of α is smaller; and more likely that his rival will submit a bid of B_1^* when the value of α is larger. If $\alpha < 1$ (in which case bidder valuations are drawn from a distribution which is First Order Stochastically Dominated by the distribution function from which a uniformly distributed random variable is drawn), the expected revenue of the seller is maximized by a bid level strictly above the highest possible bidder valuation. In this case, any bidder is more willing to bid B_1^* precisely because it is less likely that he will actually have to pay B_1^* (since $1 - F\left(c_1^*\right)$ is smaller when α is smaller). Thus, the gain to the seller from choosing a relatively high value of B_1 outweighs the loss from doing so, implying that $B_1^* > 1$ is optimal. When $\alpha > 1$ (in which case bidder valuations are drawn from a distribution which First Order Stochastically Dominates the distribution function from which a uniformly distributed random variable is drawn), the expected revenue of the seller is maximized by a bid level strictly below the highest possible bidder valuation. In this case, any bidder is now less willing to bid B_1^* because he is more likely to have to pay B_1^* when doing so (since $1 - F\left(c_1^*\right)$ is larger when α is larger). As a result, the gain to the seller from choosing a higher value of B_1 is outweighed by the loss from choosing a higher value of B_1, so that $B_1^* < 1$ is best. In summary, the assumption in the existing literature that there must be a bid level exactly equal to the highest possible bidder valuation is restrictive, since an expected revenue maximizing seller (constrained to choosing a specific number of discrete bid points) will generally not want to have an acceptable bid level equal to the highest possible bidder valuation. Shifting focus to the probability of allocative inefficiency, note that when $F(v) = v^{\alpha}$ it follows that c_1^E and B_1^E can be expressed as functions of α as:

$$c_1^E(\alpha) = \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} \qquad \text{and} \qquad B_1^E(\alpha) = 2^{\frac{\alpha-1}{\alpha}}.$$

Proposition 3 characterizes $B_1^E(\alpha)$ in relation to $v_H = 1$, the highest possible bidder valuation.

Proposition 3 *$B_1^E(1) = 1$; $B_1^E(\alpha) < 1$ for $\alpha < 1$; and $B_1^E(\alpha) > 1$ for $\alpha > 1$.*

Together Propositions 2 and 3 allow for a straightforward comparison of the value of B_1 that maximizes the expected revenue of the seller (that is, $B_1^*(\alpha)$) to the value of B_1 that minimizes the probability of allocative inefficiency (that is, $B_1^E(\alpha)$).

When bidder valuations are uniformly distributed between zero and one ($\alpha = 1$) we have $B_1^*(1) = B_1^E(1) = 1$, implying that the probability of ex post inefficiency is minimized when the expected revenue of the seller is maximized. However, if bidder valuations are not uniformly distributed ($\alpha \neq 1$), then $B_1^*(\alpha) \neq B_1^E(\alpha)$. When $\alpha > 1$ (in which case bidder valuations are drawn from a distribution which First Order Stochastically Dominates the distribution function from which a uniformly distributed random variable is drawn), we have $B_1^*(\alpha) < 1 < B_1^E(\alpha)$. That is, the expected revenue of the seller is maximized by a value of B_1 strictly less than the value which minimizes the probability of allocative inefficiency. Finally, if $\alpha < 1$ (in which case bidder valuations are drawn from a distribution which is First Order Stochastically Dominated by the distribution function from which a uniformly distributed random variable is drawn), we observe $B_1^E(\alpha) < 1 < B_1^*(\alpha)$. In this case, the expected revenue of the seller is maximized by a value of B_1 strictly greater than the value which minimizes the probability of ex post inefficiency. In summary, for $F(v) = v^{\alpha}$, the expected revenue maximizing value of B_1 minimizes the probability of inefficiency only when bidder valuations are uniformly distributed.

Recall that when $\alpha = 1$, the expected revenue of the seller is maximized by $B_1^* = 1$. In this case, the resulting expected revenue of the seller is $\Pi_S(B_1^*) = \frac{1}{4}$, which is strictly less than the expected revenue of $\Pi_T^* = \frac{1}{3}$ from a traditional sealed bid second price auction in which any bid on the interval $[0, 1]$ could be submitted. However, for $\alpha \neq 1$, the seller may be able to realize a higher expected revenue from a sealed bid second price auction with only two discrete bid levels than from a traditional sealed bid second price auction in which any positive bid can be submitted.[12] To see this, begin by noting that with two discrete bid levels, the maximum expected revenue of the seller as a function of α is equal to

$$\Pi_S^*(\alpha) = B_1^*(\alpha)\left\{1 - F\left(c_1^*(\alpha)\right)\right\}^2 = \frac{\alpha}{(1+\alpha)^{\frac{\alpha+1}{\alpha}}}.$$

The expected revenue for the seller from a traditional second price auction with continuous bidding (and no reserve price) can be expressed as a function of α as

$$\Pi_T(\alpha) = \int_0^1 v\left(2\alpha v^{\alpha-1} - 2\alpha v^{2\alpha-1}\right) dv = \frac{2\alpha^2}{(1+\alpha)(2\alpha+1)}.$$

[12]Clearly the expected revenue from the discrete second price auction considered here cannot exceed that from the optimal auction characterized by Myerson (1981). However, a second price auction with a continuous bid space and no reserve price provides an appropriate benchmark for comparison, since in the discrete auction considered here (like those analyzed by Chwe (1989), Rothkopf and Harstad (1994), and Yu (1999)) the lowest acceptable bid is equal to the lowest possible bidder valuation.

The seller is able to realize a greater expected revenue from a discrete auction with only two bid levels so long as $\Pi_S^*(\alpha) > \Pi_T(\alpha)$, or equivalently

$$2\alpha\left[1-(1+\alpha)^{\frac{1}{\alpha}}\right]+1>0.$$

This condition is not satisfied for $\alpha = 1$, verifying the observation that when facing bidders with uniformly distributed valuations, the seller is able to realize a higher expected payoff from an analogous traditional auction with continuous bidding. For $\alpha \neq 1$ this inequality may or may not hold. It is straightforward to verify that this inequality is not satisfied for $\alpha = 2$ and $\alpha = \frac{1}{2}$, but is satisfied for $\alpha = \frac{1}{4}$ and $\alpha = \frac{1}{10}$.[13] Thus, it appears as if for α sufficiently low the seller realizes a higher expected revenue from a second price auction with only two discrete bids than from a traditional second price auction with continuous bidding.

4.2 Three Discrete Bids and Inter-Bid Spacing

Consider a discrete auction with three bids: $B_0 = 0$, B_1, and B_2. Suppose $F(v) = v$, so that bidder valuations are uniformly distributed. From subsection 4.1, we have that $B_1^* = 1$ maximizes the expected revenue of the seller when there are only two allowable bids. In the situation when there are three allowable bids, it can easily be seen that the seller has no incentive to have $B_1 > 1$.[14] Thus, when focusing on the expected revenue of the seller in an auction with three discrete bids, attention can be restricted to $B_1 \leq 1$.

In an auction with three discrete bids, the equilibrium behavior of bidders with uniformly distributed valuations is dictated by the unique values of c_1 and c_2 satisfying the following conditions:

$$(B_1-c_1)(c_2-c_1)-c_1^2=0$$

and

$$(B_2-c_2)(1-c_2)-(c_2-B_1)(c_2-c_1)=0.$$

Let $c_1(B_1,B_2)$ and $c_2(B_1,B_2)$ denote these values. The expected revenue of the seller in this case can be expressed as:

$$\begin{aligned}\Pi_S(B_1,B_2) &= B_2-B_1\left[2c_1(B_1,B_2)-c_1(B_1,B_2)^2\right]\\ &\quad -(B_2-B_1)\left[2c_2(B_1,B_2)-c_2(B_1,B_2)^2\right].\end{aligned}$$

Constraining $B_2 = 2B_1$, we have $\widehat{c}_1(B_1) = c_1(B_1, 2B_1)$ and $\widehat{c}_2(B_1) = c_2(B_1, 2B_1)$. Such evenly spaced bids lead to an expected revenue

$$\begin{aligned}\widehat{\Pi}_S(B_1) &= \Pi_S(B_1,2B_1)\\ &= B_1\left\{2-\left[2\widehat{c}_1(B_1)-\widehat{c}_1(B_1)^2\right]-\left[2\widehat{c}_2(B_1)-\widehat{c}_2(B_1)^2\right]\right\}.\end{aligned}$$

Based upon a numerical analysis, $\widehat{\Pi}_S(B_1)$ is maximized by $\widehat{B}_1^* \approx .45528513$. Since the two bid points are constrained to be evenly spaced, this leads to $\widehat{B}_2^* \approx .91057027$. For these bid levels, $\widehat{c}_1 \approx .27018243$ and $\widehat{c}_2 \approx .66455019$, resulting in $\widehat{\Pi}_S^* \approx .29373188$.

[13] Numerical calculations suggest that this inequality holds if and only if $\alpha < \bar{\alpha}$, with $\bar{\alpha} \approx .37341058$.

[14] If $B_1 \geq 1$, then bidding B_1 dominates B_2 for each bidder. Since bidders will never bid B_2, the value of B_2 becomes irrelevant. From the results in subsection 4.1, it follows that $B_1 = 1$ (along with any $B_2 \geq B_1$) results in a greater expected revenue than any $B_1 > 1$ (along with any $B_2 \geq B_1$).

In order to illustrate that the expected revenue of the seller is not maximized by evenly spaced bids, it is sufficient to identify a pair of unevenly spaced bids $\left(\widetilde{B}_1, \widetilde{B}_2\right)$ which result in $\widetilde{\Pi}_S > \widehat{\Pi}^*_S$. For specified values of $\widetilde{B}_1$ and $\widetilde{B}_2$, the resulting equilibrium values of $\widetilde{c}_1$ and $\widetilde{c}_2$ can be approximated numerically. Doing so, $\widetilde{B}_1 = .65$ and $\widetilde{B}_2 = .7$ lead to $\widetilde{c}_1 \approx .33097279$ and $\widetilde{c}_2 \approx .67433844$, so that $\widetilde{\Pi}_S \approx .29624109 > .29373188 \approx \widehat{\Pi}^*_S$.

Although the expected revenue maximizing values of B_1 and B_2 have not been determined, it has been shown that the revenue maximizing bid points are not evenly spaced. Thus, the assumption in previous studies that discrete bid points must be evenly spaced is restrictive in the sense that an expected revenue maximizing seller (constrained to choosing a specific number of discrete bid points) may wish to set unevenly spaced bid points.

4.3 Some Numerical Results with Evenly Spaced Bids

Theorem 1 specifies equilibrium behavior for bidders, characterized by the unique values $c_1 < \ldots < c_k$ which simultaneously satisfy the k conditions specified by Condition (7). Because of the complexity of the conditions specifying equilibrium bidder behavior, it is not possible to solve for these values for the general case of $k + 1 > 2$ relevant discrete bid points. In order to further analyze this situation, equilibrium values of $c_1 < \ldots < c_k$ are approximated numerically, assuming bidder valuations are independently drawn from $F(v) = v^\alpha$ (with $\alpha \in (0, \infty)$).

Instead of attempting to determine the optimal (expected revenue maximizing) discrete bid points, it is simply assumed that the bids are "evenly spaced." The common distance between any two bids is denoted by t, implying $B_j = jt$.[15] As this common distance between bids is decreased, the number of acceptable bid levels below the highest possible bidder valuation may become greater. It is of particular interest to determine the behavior of both the expected revenue of the seller and the probability of ex post inefficiency as this common distance between bid points is reduced.

Approximate equilibrium values of $c_1 < \ldots < c_k$ have been determined numerically for different combinations of α and t. Of primary interest are the corresponding values of Π_S and $\Pr(I)$. When examining the value of Π_S, a comparison is made to the appropriate value of Π_T, the expected revenue from a traditional sealed bid second price auction with continuous bidding (with no reserve price). The numerical results are summarized in Tables 1 through 3 in Appendix C. It will be noticed that some of the rows in each table are "starred", i.e. they have an asterisk marking them. These rows correspond to those values of t for which $B_k = v_H = 1$.

Table 1 specifies the approximated values of Π_S, the expected revenue from the sealed bid second price auction with discrete bidding. In general Π_S is non-monotonic in t, which can be seen by considering $\alpha = \frac{1}{2}$ (or $\alpha = \frac{1}{4}$). Further, for the reported values of $\alpha \geq \frac{2}{3}$, Π_S increases as t is decreased, whereas for $\alpha = \frac{1}{10}$, Π_S decreases as t is decreased. As would be expected, Π_S appears to be monotonically increasing in α for each fixed value of t.

The results in Table 1 also allow for an examination of the change in expected revenue as a result of allowing additional bids. With $t = .5$ there are three relevant bid points: 0, $.5$, and 1. If $t = .25$, then in addition to these three bid points, bids of $.25$ and $.75$ are also acceptable. For the reported values of $\alpha \geq \frac{1}{2}$, allowing additional bids of $.25$ and $.75$ leads to an increase in the expected revenue of the seller. However, for $\alpha = \frac{1}{10}$ and $\alpha = \frac{1}{4}$, the expected revenue of the seller

[15] However, attention is not restricted to situations in which $B_k = v_H$.

decreases as a result of this change. From here it is clear that as a result of allowing additional bids, the revenue of the seller will not always increase, but may actually decrease.

The expected revenue in a traditional continuous second price auction is reported in the second to last row of Table 1. The sole purpose of calculating Π_T is to have a frame of reference to which we can compare Π_S.[16]

Table 2 reports the value of $\Pi_T - \Pi_S$ in each case considered. A negative value indicates that the seller realizes a higher expected revenue from the discrete auction under consideration, while a positive value indicates that the seller realizes a higher expected revenue from a traditional auction with continuous bidding. Since Π_T does not depend upon t, the behavior of this difference as t is decreased must mirror the behavior of Π_S as t is decreased. Additionally, while this difference typically increases as α is increased for a fixed value of t, this is not always the case (see for example the change as α is increased from $\frac{1}{10}$ to $\frac{1}{4}$ for the reported values of $t \leq \frac{1}{2}$). Finally, the negative values reported for $t \geq 1$ for $\alpha = \frac{1}{10}$ and $\alpha = \frac{1}{4}$ immediately verify the earlier observations that $\Pi_S^*(\alpha) > \Pi_T(\alpha)$ for these values of α, where Π_S^* denotes the maximum revenue of the seller when choosing the level of a single bid point $B_1 > 0$ (with $B_0 = 0$).

Finally, Table 3 reports the corresponding probability of ex post inefficiency, denoted $\Pr(I)$. This is simply the probability with which the item is awarded to the bidder with the lower valuation. $\Pr(I)$ is clearly not monotonic in α. Further, for any fixed value of α, this probability typically decreases as t becomes smaller. In fact, for the reported values this is always the case if we restrict attention to the values of t for which $B_k = v_H$ (that is, the "starred rows"). However, upon examination of Table 3 in its entirety, we see that this is not always the case. Rather, in general the probability of ex post inefficiency may actually increase as the common distance between the evenly spaced discrete bid points is decreased. For example, an increase in the value of $\Pr(I)$ is observed for both $\alpha = 4$ and $\alpha = 10$ as t is decreased from $t = 2$ to $t = 1$, from $t = 0.75$ to $t = 0.5$, from $t = 0.4$ to $t = 1/3$, and finally from $t = 0.225$ to $t = 0.2$. This provides an additional reason for concerning ourselves with situations in which $B_k \neq v_H$, since if we simply focused on cases with $B_k = v_H$ we may incorrectly infer that a decrease in the distance between evenly spaced bid points must decrease $\Pr(I)$.

5 Conclusion

A situation in which a single item is sold to two bidders by way of a sealed bid second price auction in which bids are restricted to a set of discrete values was studied. In contrast to previous studies, bids need not be "evenly spaced" and the highest acceptable bid need not be equal to the highest possible bidder valuation.

Bidder behavior in such an environment is analyzed. (Within the class of symmetric pure strategy equilibria) a unique equilibrium is shown to exist, under which bidders may choose to bid either strictly above or strictly below their independent private valuation. As a result,

[16] Recall that, as noted in footnote 12, the expected revenue from the discrete auction analyzed here must be lower than the expected revenue from the optimal auction. From Myerson (1981), a seller facing two bidders with independent private valuations drawn from $F(v) = v^\alpha$ maximizes expected revenue by imposing a reserve price of $r^* = \left(\frac{1}{1+\alpha}\right)^{1/\alpha}$ in a sealed bid second price auction. The resulting expected revenue of $\Pi_O = \frac{2\alpha}{(1+\alpha)(1+2\alpha)}\left(\alpha + \left(\frac{1}{1+\alpha}\right)^{(1+\alpha)/\alpha}\right)$ is reported in the last row of Table 1 for each α under consideration.

allocative efficiency may be sacrificed in that the item will be awarded to the bidder with the lower valuation with strictly positive probability.

Subsequently, expected seller revenue in such an auction is examined. The assumptions on the placement of the discrete bid points made in previous studies are shown to be restrictive by way of analyzing seller revenue in such an auction with only two bids and in such an auction with only three bids. With only two bid levels, an expected revenue maximizing seller may wish to set the higher acceptable bid either strictly above or strictly below the highest possible bidder valuation. Further, with only two acceptable bid levels it is possible for the expected revenue of the seller to be greater than the expected revenue in an analogous second price auction with continuous bidding (and no reserve price). Considering a situation with three discrete bid levels, when bidder valuations are uniformly distributed it is shown that having evenly spaced bids is not optimum.

Finally, a numerical analysis was conducted for an auction with an arbitrary number of evenly spaced bids. It is often the case that the seller may realize a higher expected revenue in such an auction than in a traditional auction with continuous bidding (and no reserve price). Further, the numerical analysis illustrates that the expected revenue of the seller may either increase or decrease as the common distance between bids is decreased. While the probability of ex post inefficiency often decreases as the distance between bids is decreased, this is not always the case. That is, the probability of ex post inefficiency may increase as the distance between evenly spaced discrete bids is reduced.

Appendix

A Proof of the Main Results

A.1 Proof of Proposition 1

Arbitrarily fix the behavior of the rival of bidder i. Let $\pi_i(B_m, v_i)$ denote the expected payoff for a bidder with valuation v_i from bidding B_m. It must be shown that: for a bidder with $v_i \leq B_j$, $\pi_i(B_j, v_i) \geq \pi_i(B_m, v_i)$ for every $B_m > B_j$; for a bidder with $v_i \geq B_j$, $\pi_i(B_j, v_i) \geq \pi_i(B_m, v_i)$ for every $B_m < B_j$.

Let p_j denote the probability with which the rival of bidder i submits a bid of B_j. The expected payoff of i from bidding B_m is

$$\pi_i(B_m, v_i) = \sum_{l=0}^{m-1}(v_i - B_l)p_l + (v_i - B_m)\frac{1}{2}p_m$$

Note that $\pi_i(B_m, v_i) - \pi_i(B_{m-h}, v_i)$ can be expressed as

$$\frac{1}{2}\left\{(v_i - B_m)\,p_m + (v_i - B_{m-h})\,p_{m-h}\right\} + \sum_{l=m-h+1}^{m-1}(v_i - B_l)p_l$$

for $h = 2, \ldots, m$ and as

$$\frac{1}{2}\left\{(v_i - B_m)\,p_m + (v_i - B_{m-1})\,p_{m-1}\right\}$$

for $h = 1$.

For a bidder with $v_i \leq B_{m-h}$, $\pi_i(B_m, v_i) - \pi_i(B_{m-h}, v_i) \leq 0$. Thus for a bidder with $v_i \leq B_j$, $\pi_i(B_j, v_i) \geq \pi_i(B_m, v_i)$ for all $B_m > B_j$. Similarly, for a bidder with $v_i \geq B_m$, $\pi_i(B_m, v_i) - \pi_i(B_{m-h}, v_i) \geq 0$. Thus for a bidder with $v_i \geq B_j$, $\pi_i(B_j, v_i) \geq \pi_i(B_m, v_i)$ for all $B_m < B_j$. ***Q.E.D.***

A.2 Proof of Lemma 1

It must be shown that in any symmetric, pure strategy equilibrium, $0 < p_j < 1$ for all $j = 0, 1, \ldots, k$ and $\sum_{j=0}^{k} p_j = 1$. We prove this stepwise.

Step 1. To begin with consider p_0. It can easily be seen that in any symmetric equilibrium, $0 < p_0 < 1$. Suppose, to the contrary, $p_0 = 1$. Then every bidder has an incentive to deviate unilaterally and bid B_1 and win the object for sure while paying B_0. Now suppose $p_0 = 0$. A bidder with $v_i \in [v_L, B_1)$ has a negative expected payoff in this case (since he will either obtain the item and pay more than his valuation or not obtain the item and realize a payoff of zero). By deviating to a bid of B_0 he is guaranteed a certain payoff of zero (since he will now never obtain the item). Hence it must be true that, in equilibrium, $0 < p_0 < 1$.

Step 2. Next we show that for any $j = 1, 2, \ldots, (k-1)$, if $p_0, \ldots, p_{j-1}$ are each strictly positive and $\sum_{l=0}^{j-1} p_l < 1$, then it must be true that $p_j > 0$ and $\sum_{l=0}^{j} p_l < 1$. We show this inductively.

Conjecture that there exists an equilibrium with $p_j = 0$. Focus on a bidder with $v_i \in (B_{j-1}, B_j)$. By Corollary 1 such a bidder realizes a strictly higher payoff from bidding B_{j-1} than from bidding less than B_{j-1}. Further, if $p_j = 0$ then $\pi_i(B_j, v_i) - \pi_i(B_{j-1}, v_i) = \frac{1}{2} p_{j-1}(v_i - B_{j-1})$, which is strictly positive for such a bidder. Thus, bidding B_j would result in a strictly higher payoff than bidding B_{j-1} or below.

Since $\sum_{l=0}^{j-1} p_l < 1$ it follows that positive probability must be placed on some bid above B_j. Let B_{j+m} be the lowest bid above B_j on which positive probability is placed. That is, $p_j = \ldots = p_{j+m-1} = 0$ while $p_{j+m} > 0$. It follows that $\pi_i(B_j, v_i) = \ldots = \pi_i(B_{j+m-1}, v_i) = \sum_{l=0}^{j-1} (v_i - B_l) p_l$, but $\pi_i(B_{j+m}, v_i) = \sum_{l=0}^{j-1} (v_i - B_l) p_l + (v_i - B_{j+m}) \frac{1}{2} p_{j+m}$. For a bidder with $v_i \in (B_{j-1}, B_j)$ it is clear that $\pi_i(B_j, v_i) > \pi_i(B_{j+m}, v_i)$. Further, by Proposition 1 bidding B_{j+m} weakly dominates bidding B_{j+m+1} or above for such a bidder. Thus, a bidder with $v_i \in (B_{j-1}, B_j)$ would strictly increase his expected payoff by deviating from the conjectured equilibrium and instead submitting a bid of B_j. Hence, in equilibrium we cannot have $p_j = 0$, but must rather have $p_j > 0$.

Now conjecture that $\sum_{l=0}^{j} p_l = 1$, which would imply that all bids above B_j are submitted with zero probability. From here, bidding B_j leads to a payoff of $\pi_i(B_j, v_i) = \sum_{l=0}^{j-1} (v_i - B_l) p_l + (v_i - B_j) \frac{1}{2} p_j$ while bidding B_{j+1} or above leads to $\pi_i(B_{j+1}, v_i) = \ldots = \pi_i(B_k, v_i) =$

$\sum_{l=0}^{j}(v_i - B_l)p_l$. The latter of these is clearly larger for any bidder with $v_i \geq B_j$. Further, by Proposition 1 bidding B_j weakly dominates bidding B_{j-1} or below for such a bidder. Thus, a bidder with $v_i \geq B_j$ could strictly increase his expected payoff by deviating from the conjectured equilibrium and instead submitting a bid above B_j. Hence, it must be true that $\sum_{l=0}^{j} p_l < 1$ in equilibrium.

Given that $p_0 > 0$, the above has to hold for $j = 1$. Similarly, it has to hold for all $j = 1, 2, \ldots, (k-1)$.

Step 3. Finally, if $\sum_{l=0}^{k-1} p_l < 1$, then it must be true that $p_k > 0$ and $\sum_{l=0}^{k} p_l = 1$. To see why $p_k > 0$, note that if $p_k = 0$ then a bidder with valuation $v_i = v_H$ would have a higher payoff from bidding B_k as opposed to anything less (since $\pi_i(B_k, v_H) = \sum_{l=0}^{k-1}(v_H - B_l)p_l$ is greater than $\pi_{B_{k-1},v_H} = \sum_{l=0}^{k-2}(v_H - B_l)p_l + (v_i - B_{k-1})\frac{1}{2}p_{k-1}$, along with the fact that bidding B_{k-1} weakly dominates bidding below B_{k-1} for such a bidder). To see why $\sum_{l=0}^{k} p_l = 1$, first note that either B_k is the highest acceptable bid or there are other acceptable bids above B_k. In the former case all weight must be on bids B_0 through B_k, since there are no other acceptable bids. In contrast, if there are other acceptable bids above B_k, then (by the way in which B_k was defined) it must be that $B_k \geq v_H$. By Corollary 2 it follows that in this case bidding B_{k+1} or above results in a strictly lower payoff than bidding B_k.

This proves our claim that in any symmetric, pure strategy equilibrium, $0 < p_j < 1$ for all $j = 0, 1, \ldots, k$ and $\sum_{j=0}^{k} p_j = 1$. *Q.E.D.*

A.3 Proof of Theorem 1

It must be shown that, for arbitrary acceptable bid levels, there exist unique values C such that condition (7) holds for $j = 1, \ldots, k$; and for this unique C, neither bidder has an incentive to unilaterally deviate from the conjectured strategy pair.

For $D_k(c_{k-1}, c_k, c_{k+1}) = (B_k - c_k)[1 - F(c_k)] - (c_k - B_{k-1})[F(c_k) - F(c_{k-1})]$ observe that $D_k(c_{k-1}, min\{v_H, B_k\}, c_{k+1}) < 0$ and $D_k(c_{k-1}, B_{k-1}, c_{k+1}) > 0$. Further, $\frac{\partial D_k}{\partial c_k} = -[1 - F(c_{k-1})] - f(c_k)(B_k - B_{k-1}) < 0$, implying that for any $c_{k-1} < B_{k-1}$ there exists a unique $c_k \in (B_{k-1}, min\{v_H, B_k\})$ such that $D_k(c_{k-1}, c_k, c_{k+1}) = 0$. Letting $c_k(c_{k-1})$ denote this value, by the Implicit Function Theorem

$$c_k'(c_{k-1}) = \frac{(c_k - B_{k-1})f(c_{k-1})}{[1 - F(c_{k-1})] + (B_k - B_{k-1})f(c_k)}.$$

Clearly $c_k'(c_{k-1}) \in \left(0, \frac{f(c_{k-1})}{f(c_k(c_{k-1}))}\right)$.

Now consider

$$D_j(c_{j-1}, c_j, c_{j+1}(c_j)) = (B_j - c_j)[F(c_{j+1}(c_j)) - F(c_j)] - (c_j - B_{j-1})[F(c_j) - F(c_{j-1})],$$

presuming that an examination of $D_{j+1}(c_j, c_{j+1}, c_{j+2}(c_{j+1}))$ has lead to the defining of $c_{j+1}(c_j) \in (B_j, B_{j+1})$ such that $c'_{j+1}(c_j) \in \left(0, \frac{f(c_j)}{f(c_{j+1}(c_j))}\right)$. Note that: $D_j(c_{j-1}, B_{j-1}, c_{j+1}(B_{j-1})) > 0$ and $D_j(c_{j-1}, B_j, c_{j+1}(B_j)) < 0$. Further, (since $c'_{j+1}(c_j) < \frac{f(c_j)}{f(c_{j+1}(c_j))}$):

$$\begin{aligned}\frac{\partial D_j}{\partial c_j} &= -[F(c_{j+1}(c_j)) - F(c_{j-1})] - f(c_j)(B_j - B_{j-1}) + (B_j - c_j) f(c_{j+1}(c_j))\, c'_{j+1}(c_j) \\ &< -[F(c_{j+1}(c_j)) - F(c_{j-1})] - f(c_j)(c_j - B_{j-1}) < 0.\end{aligned}$$

Thus, for any $c_{j-1} < B_{j-1}$ there exists a unique $c_j \in (B_{j-1}, B_j)$ for which $D_j(c_{j-1}, c_j, c_{j+1}(c_j)) = 0$. Letting $c_j(c_{j-1})$ denote this value,

$$c'_j(c_{j-1}) = \frac{(c_j - B_{j-1}) f(c_{j-1})}{[F(c_{j+1}(c_j)) - F(c_{j-1})] + f(c_j)(B_j - B_{j-1}) - (B_j - c_j) f(c_{j+1}(c_j))\, c'_{j+1}(c_j)}.$$

Since $c'_{j+1}(c_j) \in \left(0, \frac{f(c_j)}{f(c_{j+1}(c_j))}\right)$, it follows that $c'_j(c_{j-1}) \in \left(0, \frac{f(c_{j-1})}{f(c_j(c_{j-1}))}\right)$.

Finally, consider $D_1(c_0, c_1, c_2(c_1))$ (supposing that an examination of $D_2(c_1, c_2, c_3(c_2))$ has led to the defining of $c_2(c_1)$ such that $c'_2(c_1) \in \left(0, \frac{f(c_1)}{f(c_2(c_1))}\right)$). First note that $D_1(c_0, c_0, c_2(c_0)) > 0$ and $D_1(c_0, B_1, c_2(B_1)) < 0$. Further,

$$\begin{aligned}\frac{\partial D_1}{\partial c_1} &= -F(c_2(c_1)) - f(c_1)(B_1 - B_0) + (B_1 - c_1) f(c_2(c_1))\, c'_2(c_1) \\ &< -F(c_2(c_1)) - f(c_1)(c_1 - B_0) < 0.\end{aligned}$$

Thus, there exists a unique $c_1 \in (c_0, B_1)$ for which $D_1(c_0, c_1, c_2(c_1)) = 0$.

From here it follows that, for any arbitrary bid levels, there exists a unique vector $C = (c_0, c_1, \ldots, c_k, c_{k+1})$, with $c_0 = v_L$, $c_j \in (B_{j-1}, B_j)$ for $j = 1, \ldots, k-1$, $c_k \in (B_{k-1}, min\{v_H, B_k\})$, and $c_{k+1} = v_H$, such that condition (7) holds for $j = 1, \ldots, k$ simultaneously.

It is clear that for this unique C, neither bidder wishes to unilaterally deviate from the conjectured strategy pair. *Q.E.D.*

A.4 Proof of Proposition 2

Defining $g(\alpha) = (1+\alpha)^{\frac{\alpha-1}{\alpha}}$, we have $B_1^*(\alpha) = \frac{g(\alpha)}{\alpha}$. It follows that $B_1^*(1) = 1$.

Since

$$g'(\alpha) = (1+\alpha)^{-\frac{1}{\alpha}} \left\{ \frac{(1+\alpha)\ln(1+\alpha) + \alpha(\alpha-1)}{\alpha^2} \right\},$$

we have that

$$\frac{dB_1^*(\alpha)}{d\alpha} = \frac{\alpha g'(\alpha) - g(\alpha)}{\alpha^2} = \frac{(1+\alpha)\ln(1+\alpha) - 2\alpha}{(1+\alpha)^{\frac{1}{\alpha}}\alpha^3}.$$

The sign of this expression is determined by the sign of

$$\eta(\alpha) = (1+\alpha)\ln(1+\alpha) - 2\alpha.$$

Note that $\eta(0) = 0$. Further, $\eta'(\alpha) = \ln(1+\alpha) - 1$, which implies $\eta'(0) = -1 < 0$. Additionally, $\eta''(\alpha) = \frac{1}{1+\alpha} > 0$. From here we have that there exists a unique $\tilde{\alpha} > 0$ such that $\eta(\alpha) < 0$ for

$\alpha \in (0, \tilde{\alpha})$ and $\eta(\alpha) > 0$ for $\alpha \in (\tilde{\alpha}, \infty)$. As a result, $B_1^*(\alpha)$ is decreasing in α for $\alpha \in (0, \tilde{\alpha})$ and increasing in α for $\alpha \in (\tilde{\alpha}, \infty)$. That is, $B_1^*(\alpha)$ is a quasi-convex function.

Since $B_1^*(\alpha)$ is less than $\frac{1+\alpha}{\alpha}$, and $\lim\limits_{\alpha \to \infty} \frac{1+\alpha}{\alpha} = 1$, it follows that $\lim\limits_{\alpha \to \infty} B_1^*(\alpha) \leq 1$. From here, by the quasi-convexity of $B_1^*(\alpha)$, $B_1^*(\alpha) > 1 = v_H$ for $\alpha < 1$ and $B_1^*(\alpha) < 1 = v_H$ for $\alpha > 1$. *Q.E.D.*

A.5 Proof of Proposition 3

Consider the function, $B_1^E(\alpha) = 2^{\frac{\alpha-1}{\alpha}}$. Begin by noting that $B_1^E(1) = 1$. Further,

$$\frac{dB_1^E(\alpha)}{d\alpha} = \frac{2^{\frac{\alpha-1}{\alpha}} \ln(2)}{\alpha^2},$$

which is strictly positive for all $\alpha \in (0, \infty)$. From here: $B_1^E(\alpha) < 1$ for $\alpha < 1$, and $B_1^E(\alpha) > 1$ for $\alpha > 1$. *Q.E.D.*

B Figures

Figure 1: Optimal choice of B_j vs. B_{j-1}.

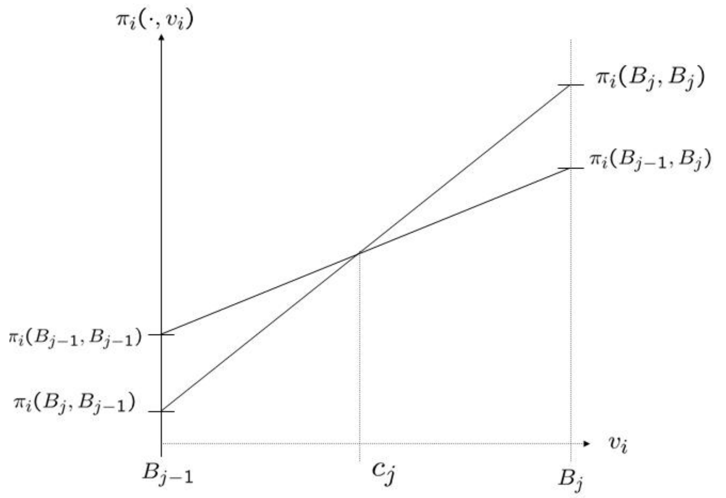
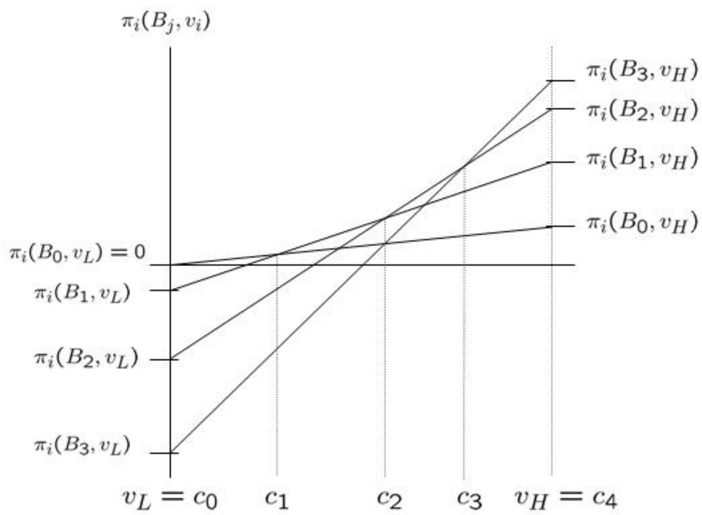


Figure 2: Bidder Equilibrium with four bid points.



C Numerical Results

Table 1: Seller's Revenue for Discrete Auction

	Alpha Values								
Increment	1/10	1/4	1/2	2/3	1	3/2	2	4	10
2.00	0.032580	0.081913	0.143594	0.174756	0.222222	0.271088	0.304806	0.376253	0.440493
1.00*	0.027199	0.075905	0.145898	0.185037	0.250000	0.324718	0.381966	0.524889	0.697357
0.75	0.026213	0.077005	0.153974	0.198259	0.272940	0.359697	0.426100	0.587733	0.758068
0.50*	0.024416	0.076355	0.159726	0.208988	0.292893	0.390059	0.463021	0.629304	0.776064
0.4	0.023502	0.076079	0.163088	0.215176	0.304222	0.406902	0.483094	0.652247	0.803395
1/3*	0.022719	0.075460	0.164526	0.218219	0.310000	0.415138	0.492292	0.660467	0.808426
0.30	0.022293	0.075155	0.165485	0.220181	0.313749	0.420712	0.498903	0.668410	0.817459
0.25*	0.021560	0.074468	0.166496	0.222539	0.318386	0.427401	0.506495	0.676162	0.823800
0.225	0.021158	0.074076	0.167026	0.223812	0.320955	0.431266	0.511122	0.682126	0.830921
0.2*	0.020717	0.073589	0.167361	0.224805	0.323026	0.434252	0.514487	0.685489	0.833130
0.15	0.019728	0.072432	0.167890	0.226697	0.327186	0.440521	0.521947	0.695119	0.844307
0.125*	0.019164	0.071718	0.167960	0.227391	0.328865	0.443033	0.524885	0.698661	0.847488
0.1*	0.018546	0.070913	0.167927	0.227966	0.330379	0.445357	0.527681	0.702526	0.852292
0.0825	0.018074	0.070287	0.167827	0.228269	0.331279	0.446754	0.529376	0.704981	0.855666
1/15*	0.017616	0.069674	0.167680	0.228467	0.331964	0.447822	0.530673	0.706899	0.858417
0.06	0.017412	0.069402	0.167604	0.228531	0.332220	0.448227	0.531172	0.707693	0.859798
0.05*	0.017094	0.068977	0.167470	0.228597	0.332547	0.448741	0.531795	0.708624	0.861163
0.04*	0.016759	0.068534	0.167319	0.228637	0.332824	0.449181	0.532332	0.709473	0.862624
0.025*	0.016217	0.067836	0.167064	0.228646	0.333131	0.449671	0.532933	0.710444	0.864424
Traditional	0.015152	0.066667	0.166667	0.228571	0.333333	0.450000	0.533333	0.711111	0.865801
Optimal	0.068257	0.154048	0.265432	0.324179	0.416667	0.515146	0.584653	0.734889	0.871994

Table 2: Revenue Loss from Discrete Auction

	Alpha Values								
Increment	1/10	1/4	1/2	2/3	1	3/2	2	4	10
2	-0.017429	-0.015246	0.023073	0.053815	0.111111	0.178912	0.228527	0.334858	0.425308
1.00*	-0.012047	-0.009238	0.020769	0.043534	0.083333	0.125282	0.151367	0.186223	0.168444
0.75	-0.011061	-0.010339	0.012692	0.030312	0.060394	0.090303	0.107233	0.123379	0.107733
0.50*	-0.009265	-0.009688	0.006940	0.019584	0.040440	0.059941	0.070313	0.081807	0.089737
0.4	-0.008350	-0.009413	0.003578	0.013395	0.029112	0.043098	0.050239	0.058864	0.062406
1/3*	-0.007567	-0.008794	0.002140	0.010353	0.023333	0.034862	0.041042	0.050644	0.057375
0.3	-0.007141	-0.008488	0.001182	0.008390	0.019585	0.029288	0.034431	0.042701	0.048342
0.25*	-0.006409	-0.007801	0.000171	0.006032	0.014947	0.022599	0.026838	0.034949	0.042001
0.225	-0.006006	-0.007409	-0.000359	0.004759	0.012378	0.018734	0.022211	0.028985	0.034880
0.20*	-0.005566	-0.006922	-0.000694	0.003767	0.010307	0.015748	0.018846	0.025622	0.032671
0.15	-0.004577	-0.005765	-0.001224	0.001874	0.006148	0.009479	0.011386	0.015992	0.021494
0.125*	-0.004013	-0.005051	-0.001293	0.001180	0.004469	0.006967	0.008448	0.012450	0.018313
0.1*	-0.003394	-0.004246	-0.001260	0.000605	0.002955	0.004643	0.005652	0.008585	0.013509
0.0825	-0.002923	-0.003621	-0.001160	0.000302	0.002054	0.003246	0.003957	0.006130	0.010135
1/15*	-0.002464	-0.003007	-0.001013	0.000105	0.001369	0.002178	0.002660	0.004212	0.007384
0.06	-0.002261	-0.002735	-0.000937	0.000040	0.001113	0.001773	0.002162	0.003419	0.006003
0.05*	-0.001943	-0.002311	-0.000803	-0.000026	0.000786	0.001259	0.001539	0.002487	0.004638
0.04*	-0.001608	-0.001868	-0.000652	-0.000066	0.000509	0.000819	0.001001	0.001638	0.003177
0.025*	-0.001066	-0.001169	-0.000397	-0.000075	0.000202	0.000329	0.000401	0.000667	0.001377

Table 3: Probability of ex-post Inefficiency

	Alpha Values								
Increment	1/10	1/4	1/2	2/3	1	3/2	2	4	10
2	0.388657	0.338580	0.303848	0.291780	0.277778	0.267381	0.262015	0.254391	0.250942
1.00*	0.362278	0.300397	0.263932	0.254878	0.250000	0.254878	0.263932	0.300397	0.362278
0.75	0.345442	0.271880	0.226490	0.214226	0.205377	0.206527	0.212638	0.238715	0.274686
0.50*	0.322774	0.236742	0.187015	0.175738	0.171573	0.180910	0.194430	0.241304	0.309451
0.4	0.309726	0.216003	0.161817	0.149174	0.142928	0.148877	0.158548	0.190508	0.231779
1/3*	0.299417	0.200690	0.145579	0.133785	0.130000	0.139889	0.153454	0.200522	0.275591
0.3	0.293415	0.191584	0.134985	0.122703	0.117867	0.125637	0.136732	0.173879	0.227501
0.25*	0.283315	0.177007	0.119625	0.107819	0.104135	0.113494	0.126241	0.171559	0.248760
0.225	0.277550	0.168723	0.110524	0.098472	0.093990	0.101466	0.111972	0.147954	0.202218
0.20*	0.271265	0.160085	0.101899	0.090284	0.086612	0.095177	0.106935	0.149843	0.226908
0.15	0.256371	0.140215	0.082025	0.070757	0.066710	0.073190	0.082406	0.116084	0.174288
0.125*	0.247336	0.128858	0.071535	0.060859	0.057267	0.063687	0.072831	0.108325	0.180142
0.1*	0.236672	0.116042	0.060139	0.050113	0.046651	0.052062	0.059927	0.091306	0.158541
0.0825	0.227830	0.105944	0.051650	0.042239	0.038906	0.043448	0.050172	0.077401	0.137272
1/15*	0.218406	0.095728	0.043575	0.034916	0.031845	0.035679	0.041446	0.065434	0.122034
0.06	0.213885	0.091018	0.039994	0.031687	0.028674	0.032020	0.037120	0.058177	0.106225
0.05*	0.206278	0.083402	0.034517	0.026882	0.024163	0.027115	0.031648	0.050935	0.099178
0.04*	0.197325	0.074899	0.028747	0.021896	0.019464	0.021860	0.025589	0.041680	0.083516
0.025*	0.179694	0.059634	0.019452	0.014135	0.012291	0.013817	0.016246	0.026959	0.056635

References

[1] Avery, C., 1998, "Strategic Jump Bidding in English Auctions" Review of Economic Studies 65, 185-210.

[2] Chwe, M. S-Y., 1989, "The Discrete Bid First Auction," Economics Letters 31, 303-6.

[3] Easley, R.F. and Rafael Tenorio, 2004, "Jump Bidding Strategies in Internet Auctions," Management Science 50, 1407-1419.

[4] Isaac, R.M., T.C. Salmon and A. Zillante, 2007, "A Theory of Jump Bidding in Ascending Auctions," Journal of Economic Behavior and Organization 62, 144-164.

[5] Lucking-Reiley, D., 2000a, "Auctions on the Internet: What's Being Auctioned, and How?," Journal of Industrial Economics 48, 227-252.

[6] Lucking-Reiley, D., 2000b, "Vickrey Auctions in Practice: From Nineteenth Century Philately to Twenty-first Century E-Commerce," Journal of Economic Perspectives 14, 183-92.

[7] Myerson, R. B., 1981, "Optimal Auction Design," Mathematics of Operations Research 6, 58-73.

[8] Rothkopf, M. H. and R. M. Harstad, 1994, "On the Role of Discrete Bid Levels in Oral Auctions," European Journal of Operational Research 74, 572-581.

[9] Vickrey, W., 1961, "Counterspeculation, Auctions, and Competitive Sealed Tenders," Journal of Finance 16, 8-37.

[10] Yu, J., 1999, "Discrete Approximation of Continuous Allocation Mechanisms," PhD Thesis, California Institute of Technology, Division of Humanities and Social Sciences.