Employing zSlices Based General Type-2 Fuzzy Sets to Model Multi Level Agreement

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Abstract—**In this paper, we introduce the concept of Multi Level Agreement (MLA) based on (zSlices based) general type-2 fuzzy sets. We define the notion of MLA and describe how it can be computed based on a series of interval type-2 fuzzy sets. We provide examples, visualizing the nature of MLA sets and discuss their properties and interpretation. Moreover, we specifically address the reason for introducing MLA in order to allow the modeling of agreement in real world applications using fuzzy sets while still maintaining an uncertainty model and show that the use of general type-2 fuzzy sets is essential for MLA as classical sets, type-1 and interval type-2 fuzzy sets do not provide a degree of freedom which could be employed to model agreement.**

Keywords- general type-2 fuzzy sets, uncertainty modelling, fuzzy set parameters, agreement, multi level agreement, MLA, zSlices

I. INTRODUCTION

Type-2 Fuzzy Logic Systems have been a popular approach for a wide range of applications and while type-1 fuzzy logic has been at the forefront of the application domain, type-2 fuzzy logic first devised by Zadeh in [1] have gained popularity over the last 10 years. Within the type-2 research, interval type-2 fuzzy sets (a simplification of general type-2 fuzzy sets where the third dimension has been fixed at 1) have been strongly favored (until recently) over general type-2 fuzzy sets because of their reduced complexity and computational requirements.

As a series of developments such as [2]-[7] have addressed the most significant challenges in terms of complexity and computational requirements, the question of employing general type-2 fuzzy logic sets and systems can now be further explored. As a fundamental part of this process lies the question of how the additional degree of freedom in the form of the third dimension can be employed effectively. Further, how can it be interpreted and how do we address the interplay between the primary and secondary memberships as part of applications.

As part of this paper we are presenting the concept of Multi Level Agreement (MLA) for zSlices based general type-2 fuzzy sets [2],[8]. The concept of MLA was developed in order to facilitate the continuous operation of software agents in a pervasive computing context. Specifically, as part of our work we are aiming to develop adaptive systems that can support the user by adjusting for example the light or heating levels to her/his preference. As such preferences, as well as the sources of information (sensors) and outputs (e.g. lamps) change over time, such a system needs to be able to adapt to such varying conditions and be able to handle the large amount of uncertainties (from noise to changing user behavior) present as part of a real world application. The presence and requirements of a human user further increases the challenge as we are detailing below.

While the aim of this paper is foremost to develop the concepts of agreement and multi-leveled agreement for zSlices based general type-2 fuzzy sets, we refer to the example application of lighting adaptation which we address as part of our work in Ambient Intelligence (AmI) in order to illustrate the real world applicability of the proposed MLA approach and (zSlices based) general type-2 fuzzy sets in general. In other words, we are clarifying that MLA or general type-2 fuzzy sets and systems are not merely theoretical challenges but have a direct real world application. The concept of agreement for zSlices based general type-2 fuzzy sets has been previously introduced in a less formal approach in [9].

In order to provide an overview to the work presented here and in particular to explain the reasoning behind our approach, we will start by detailing the motivation for the proposed approach in Section I.A, followed by the reasoning for our choice of tools in Section I.B. Section I.C finally summarizes the specific aims and objectives for this paper.

A. *Motivation.*

The motivation for investigating agreement and MLA based on general type-2 fuzzy logic systems can be summarized in a series of points which directly relate to the real world application of general type-2 fuzzy systems in general and in our case, in an AmI context as part of an "Intelligent Home Environment". Specifically, by developing the approach presented in the paper we are aiming to address the following points:

- Develop a system which can autonomously adapt to changes in the environment over time.
- This system should be interpretable by the user, in other words, the user should be able to ask "Why is the system taking a specific decision?". This characteristic is an

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essential capability in a variety of applications but is particularly important in AmI applications where user understanding is the basis for creating user trust in the resulting system.

- The system should be location-independent in the sense that it – as a software agent – should be movable between locations with heterogeneous hardware setups and continue functioning. In our case we are specifically targeting the control of the ambient light level in different locations (home, at work, in the hotel room, etc. which may all have different models/types and different numbers of light sensors).

- The system should be able to deal with a variety of hardware devices (in our case light sensors) with heterogeneous device characteristics as well as other sources of noise and uncertainty. Specifically, the following factors in terms of light sensors were considered as part of the application presented here:

    - Different sensors return slightly different values, even if they are of the same model and manufacturer. They often return very different values if they are of different types, different manufacturers.

    - Different sensors return vastly different values depending on their position and orientation – in more general terms – their context. Examples include the different impact angle of the sun during the day, directly illuminating certain sensors at different times, sensor obstruction, both short term (user obstructs sensor) as well as long term changes happening in both abrupt and continuous fashions (e.g.: plant grows in front of sensor, sensor obstruction through furniture, etc.).

    - Medium to long term changes in the environment and devices, including weather changes, changes of season, sensor deterioration (e.g. dust obstruction), etc.

B. Choice of Computational Intelligence Tools.

While a large number of methods could be proposed to address the above factors, from supervised learning methods such as neural network based approaches to unsupervised learning methods (and of course combinations thereof), we have opted for establishing agreement for fuzzy sets, in particular MLA for zSlices based type-2 fuzzy sets for several reasons:

- Fuzzy Logic Systems (FLSs) have been shown to be an adequate methodology to deal with real world systems operating in real world conditions which entail the significant number of known sources but also particularly unknown sources of uncertainty which make traditional modeling extremely difficult. In the particular case discussed in this paper, this uncertainty is represented by the uncertainty about the sensor values which is due to imprecision – a "known unknown" (as the precise level of imprecision if rarely known for each specific sensor) as well as "unknown unknowns", i.e. other factors which influence the sensors, both internal (such as electrical interference, heat, etc.) and external (e.g. sensor obstruction, etc.)

- The notion of agreement and particularly MLA extends the ability of (general type-2) FLSs to handle uncertainty by allowing long term automatic adaptation to changes in the characteristics modeled by the fuzzy sets, i.e. the sensors in the case presented here. More information on the notion of agreement is provided in Section II.

- Fuzzy Logic rule based Systems are easily interpretable by humans at a high level (the linguistic rule based level). This characteristic is invaluable in the pervasive context where user understanding and thus trust is vital for the user acceptance of any automated system. Additionally, while some interval type-2 fuzzy systems and particularly type-1 fuzzy systems are subject to the potential creation of very large rule bases which significantly reduce their interpretability potential, the zSlices based fuzzy sets which incorporate "set agreement" provide the means of preserving a small, easily interpretable rule base.

- As the application of specifically general type-2 FLSs is still fairly new, we are aiming to show one example of how the additional degrees of freedom of general type-2 fuzzy logic sets can be employed as part of real world applications. As part of this effort we are introducing a clear description of the modelling "role" of each dimension of each type of fuzzy set as part of Section II.

C. The Aims and Objectives of this Paper.

As part of this paper, our aims and objectives are twofold: to demonstrate why general type-2 fuzzy sets are and can be useful for the modeling of specific concepts and most significantly, to present the notion of MLA, its background and areas of application.

While general type-2 FLSs have and are being used and discussed more and more, there is a feeling that the additional complexity of general type-2 fuzzy logic is not warranted by corresponding advances in performance or utility. In order to address this criticism which used to be a common criticism of type-1 fuzzy FLSs and respective publications (and is now mainly focused on type-2 FLSs), we are showing in this paper how the use of general type-2 FLSs has very specific advantages which are not available when employing other methods. As such, it is worth noting that "performance" is not a simple measure of a system or controller output but a measure of how satisfactory a specific system achieves its described goals – in a variety of aspects including precision, efficiency, interpretability, simplicity, maintainability, etc. At this point we would also like to clarify that this paper is not aiming to show that zSlices based general type-2 FLSs are the only or even the best solution to the given problem. We firmly believe that general type-2 fuzzy sets are one tool with a specific set of properties which can be used as part of certain applications and in general – there is not the perfect tool for any application of reasonable complexity: all available tools have their specific strengths and weaknesses.

Finally, and most importantly as part of this paper, we demonstrate the applicability of zSlices based general type-2 fuzzy sets and the novel introduced concepts of MLA for zSlices based general type-2 fuzzy sets. We explicitly show how MLA models can be computed and why only general type-2 fuzzy sets provide the required degrees of freedom.

Section II details the notion of agreement as employed in this paper and shows why classical as well as type-1 and interval type-2 sets cannot be employed to model MLA. Section III describes modeling MLA using zSlices based general type-2 fuzzy sets, followed by some examples in Section IV and conclusions in Section V.

II. THE NOTION OF AGREEMENT IN FUZZY SETS

A. The Meaning of "Agreement".

As part of this paper we are referring to "agreement" between sets as the notion that the specific sets overlap. In other words, the agreement of two sets A and B is the set constituted by the overlap of both sets. In set terms, this overlap is referred to as the intersection of A and B, denoted as $A \cap B$.

Further, consider a specific concept (such as size, weight, beauty, strength, light levels, temperature, etc.). The agreement (i.e. the intersection) between multiple sets describes the "common ground" expressed by the sets. Practically speaking, if for example several people provide an interval of medium temperature on a temperature scale, the intersection (an interval) of the provided intervals describes the least common denominator of the provided interpretations (in the form of intervals) by the individuals, in other words: their agreement on the meaning on the concept of "medium" temperature.

While so far "agreement" can be considered merely an interpretation of the logical intersection operation, this interpretation is essential for the development of the multi-leveled agreement approach in Section III.

It is important to note from the example above that agreement itself is expressed in the form of a set. As such, the notion of agreement is different from the notion of measures such as the similarity measures for fuzzy sets [10] which are expressed as a number $s \in [0,1]$ or other approaches such as consensus modeling [11]. This difference allows the computation with the output of an agreement operation using the same mechanisms as is used for the initial elements for which the agreement is computed (i.e. if the initial sets are type-1 fuzzy sets, the agreement is expressed as a type-1 fuzzy set) and is further investigated in the following sections.

B. The Meaning of "Multi-Leveled Agreement".

We refer to "Multi-Leveled Agreement" (MLA) to the notion of modeling agreement of multiple sets in a way that the resulting agreement set expresses the proportional level of agreement of its constituting sets, i.e. areas where multiple sets overlap are considered as more significant than areas where few sets overlap or even just one set exists.

For example, if three people define a fuzzy set for the linguistic label "comfortable indoor temperature", the three resulting sets will not be identical. The MLA of the three sets is itself a set which gives the most significance to the areas of the provided sets that are common to all three sets, less significance to areas which are common to only two of the three sets and finally low significance to the areas which belong only to one set.

This notion of MLA, while very intuitive, cannot be modeled using classical sets or indeed type-1 or interval type-2 fuzzy sets as we are showing below. We are providing a detailed description of the MLA model for zSlices based general type-2 fuzzy sets at the heart of this paper in Section III.

C. Agreement of Classical Sets.

In terms of classical sets, the agreement of two sets reduces to their intersection as shown in Fig.1. However, as classical sets adopt a Boolean model of membership, i.e. an element is either a member of a set or it is not, classical sets cannot truly capture all the information of the MLA notion. As can be seen in Fig.1, the intersection $A \cap B$ captures the areas of A and B where both sets overlap, it however omits the remaining areas of A and B which would ideally be preserved and associated with a lesser degree of significance (agreement).

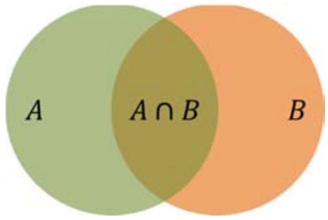


Figure 1. The agreement of two classical sets A and B defined by their intersection.

Fig.2 further clarifies the loss of information when extracting the agreement based on the intersection operation in classical sets. While the area of agreement of all three sets can be clearly defined by the intersection operation resulting in the set $A \cap B \cap C$, other relevant areas for the agreement are omitted.

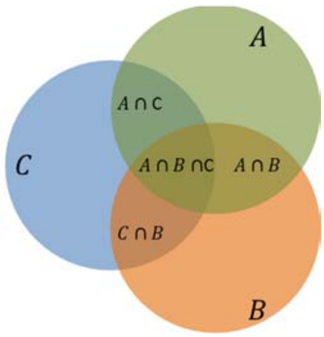


Figure 2. The agreement of three classical sets A, B and C defined by their intersection.

It should be noted that clearly, using classical sets it would be possible to capture more information on the agreement by defining several sets, for example in the case of Fig. 2, the 3-set agreement defined by $A \cap B \cap C$, the two-set agreement defined by $(A \cap B) \cup (A \cap C) \cup (B \cap C) \cup (A \cap B \cap C)$,as well as the single-set agreement defined by $A \cup B \cup C$. However, there is no mechanism to differentiate between the relevance of the three individual agreement-based groups (3-set, 2-set, single-set), i.e. the 3-set agreement reflecting the agreement of all three initial sets being the most relevant, followed by the 2-set agreement and the single-set agreement. As such, modeling MLA using classical sets is not possible.

D. *Agreement of Type-1 Fuzzy Sets.*

In type-1 fuzzy sets, the agreement of multiple sets is - similarly to classical sets - reduced to computing the intersection of the individual sets as shown in Fig. 3. While the individual intersections can express the agreement between the specific intersected sets, type-1 fuzzy sets do not offer a degree of freedom which would allow for the modeling of the different levels of significance in terms of agreement.

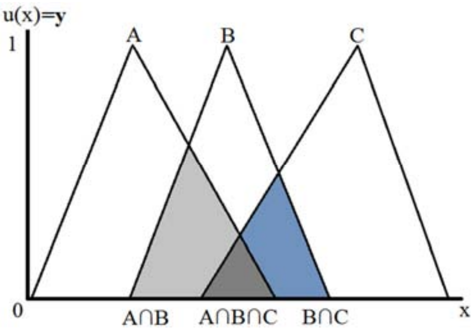


Figure 3. Intersection for three type-1 sets. (areas covered by intersections are shaded for clarity)

E. *Agreement of Interval Type-2 Fuzzy Sets.*

Interval type-2 fuzzy sets, while providing additional degrees of freedom through their associated Footprint Of Uncertainty (FOU) (which can be employed for uncertainty modeling) remain restricted in their ability to model MLA. Similarly to classical sets and type-1 fuzzy sets, while the intersection operation can be employed to extract the agreement between individual sets, interval type-2 fuzzy sets lack a degree of freedom to model the differing levels of agreement which result from computing the agreement for several initial interval type-2 fuzzy sets.

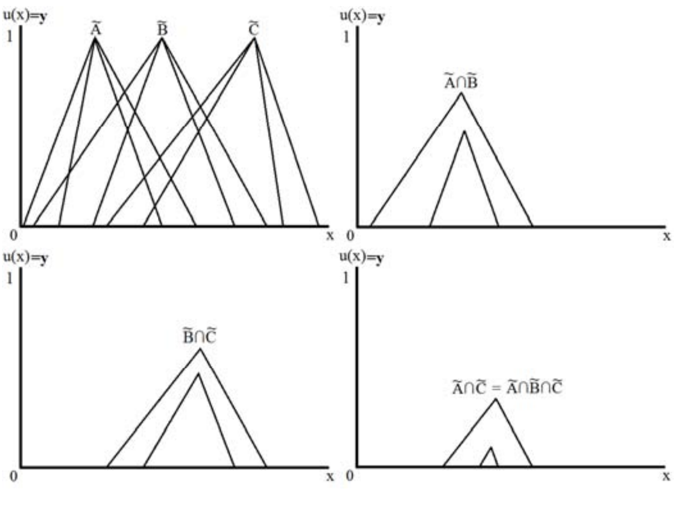


Figure 4. Example of the intersection of interval type-2 fuzzy sets. (expanded for clarity)

An example of the intersection operation applied for interval type-2 fuzzy sets is shown in Fig. 4. Note that it is a coincidence in our example that $A \cap C = A \cap B \cap C$, which however allows us to save space.

III. MODELLING MULIT-LEVELED AGREEMENT BASED ON ZSLICES BASED GENERAL TYPE-2 FUZZY SETS.

A. *Overview*

As part of this Section we are presenting the details of the proposed approach for MLA modeling using zSlices based general type-2 fuzzy sets. We commence in Section III.B by giving a brief overview of zSlices based general type-2 fuzzy sets. Subsequently, in Section III.C we detail how the individual degrees of freedoms offered by zSlices based general type-2 fuzzy sets are employed to model both uncertainty within the specific membership function as well as MLA. Section III.D provides the details on how to compute the MLA set from a series of interval type-2 fuzzy sets while Section III.E concludes the Section by noting several significant observations.

B. *Overview of zSlices and zSlices Based General Type-2 Fuzzy Sets.*

zSlices are referred to as zSlices because they conceptually stem from the slicing of general type-2 fuzzy sets in the third dimension which is traditionally associated with the zAxis in mathematics. A zSlices based general type-2 fuzzy set has a number I of zLevels, where each zLevel is defined by a specific zSlice as described in this Section.

A zSlice $\tilde{Z}_i$ is equivalent to an interval type-2 fuzzy set with the exception that its membership grade $\mu_{\tilde{Z}_i(x,u)}$ in the third dimension is not fixed to 1 but is equal to z_i where $0 \leq z_i \leq 1$. Thus the zSlice $\tilde{Z}_i$ can be written as follows:

$$\tilde{Z}_i = \int_{x \in X} \int_{u_i \in J_{i_x}} z_i/(x, u_i) \tag{1}$$

Where at each x value (as shown in Fig. 5), zSlicing creates an interval set with height z_i and domain J_{i_x} which ranges from l_i to $r_i$as shown in Fig. 1b, $1 \leq i \leq I$, I is the number of zSlices (excluding $\tilde{Z}_0$) and $z_i = i/I$.

Thus (1) can be written as follows:

$$\tilde{Z}_i = \int_{x \in X} \int_{u_i \in [l_i, r_i]} z_i/(x, u_i) \tag{2}$$

$\tilde{Z}_0$ is considered as a special case with $z=0$, which does not contribute to the actual set and as such can be disregarded in normal computation as shown in [3].

A general type-2 fuzzy $\tilde{F}$ set can be seen equivalent to the collection of an infinite number of zSlices:

$$\tilde{F} = \int_{0 \leq i \leq I} \tilde{Z}_i \qquad I \to \infty \tag{3}$$

In a discrete universe of discourse Equation (4) can be rewritten as follows:

$$\tilde{F} = \sum_{i=1}^{I} \tilde{Z}_i \tag{4}$$

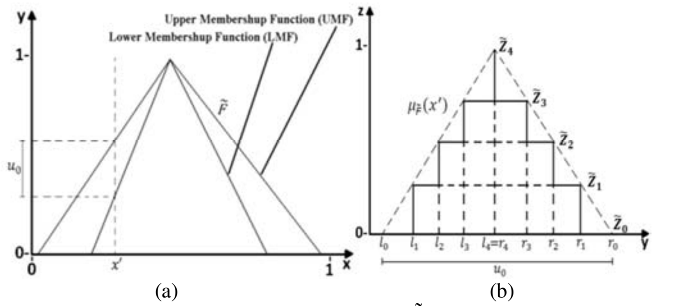


Figure 5. (a) Front view of a general type-2 set F̃. (b) Third dimension at x' of a zSlices based type-2 fuzzy set.

It should be noted that in Equation (4) the summation sign does not denote arithmetic addition but it denotes the union set theoretic operation [1]. The max operation to represent the union, hence the membership function $\mu_{\tilde{F}}(x')$ at x' of the zSlices based general type-2 fuzzy set $\tilde{F}$ shown in Fig.1b can be expressed as follows:

$$\mu_{\tilde{F}}(x') = \sum_{u \in J_{x'}} max(z_i/u) \ , J_{x'} \subseteq [0,1] \qquad (5)$$

where $0 \leq i \leq I$. It is worth noting that at x', $\mu_{\tilde{F}}(x')$ is a type-1 fuzzy set which is the vertical slice at x' of $\tilde{F}$. Fig. 5 shows a three dimensional diagram for an example of a general type-2 fuzzy set (shown in Fig. 5a and Fig. 5b) that is represented as a zSlices based general type-2 set (Fig. 5c and Fig. 5d) with I =3.

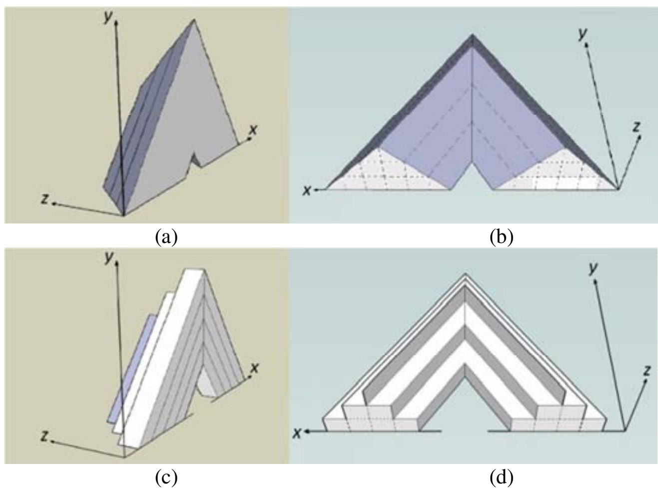


Figure 6. (a) Side view of a general type-2 fuzzy set, indicating three z levels on the third dimension. (b) Tilted rear/below view of the same set, indicating the position of the three zSlices (dashed lines). (c) Side view of the zSlice version of the set in (a)

C. Employing zSlices Based General Type-2 Fuzzy Sets for Simultaneous Modelling of Uncertainty and MLA.

One of the challenges of applying and employing general type-2 fuzzy sets is related to the complexity of their 3D nature. While the FOU of interval type-2 fuzzy sets has a clear interpretation in expressing uncertainty about the primary degree of membership of the fuzzy set, the secondary membership expressed in the third dimension (on the z-axis in terms of zSlices based general type-2 fuzzy sets) - while in concept useful for the more precise specification of the uncertainty distribution on the specific FOU - has been less well understood and employed.

As part of MLA, the modeling of the uncertainty encompassed in the fuzzy set is identical to that of interval type-2 fuzzy sets in the sense that it is expressed in the FOU of each zSlice. It should be noted that the uncertainty encompassed in the FOU relates to the uncertainty about the primary membership, i.e. the sensor value, the variable like temperature, tallness, etc.

However, the secondary membership i.e. the third dimension is employed to model the level of agreement. A higher secondary membership as such reflects a higher degree of agreement. As has been noted in Section III.B, a zSlices based general type-2 fuzzy set is based on a series of zSlices. As part of MLA modeling, the total number of zLevels I is equal to the number of constituting (or input) interval type-2 fuzzy sets and the agreement is modeled as follows (shown in more detail in Section IV):

- Areas which belong to only **one** interval type-2 fuzzy set are associated with a zLevel equal to $1/I$.
- Areas which belong to areas where at least **two** interval type-2 fuzzy sets intersect, i.e. "agree", are associated with the zLevel $2 * 1/I$.

⋮

- Areas where **all** interval type-2 fuzzy sets intersect, i.e. "agree", are associated with the zLevel $I * \frac{1}{I} = I$.

It should be noted that the number of zLevels can be reduced as for all zSlices based general type-2 fuzzy sets by relying on interpolation. However, the MLA agreement model will deteriorate in accuracy as a result.

D. Generating a MLA zSlices Based General Type-2 Fuzzy Set.

The generation of MLA zSlices based general type-2 is based on computing an MLA set based on a series of interval type-2 fuzzy sets. All interval type-2 fuzzy sets describe the same linguistic label but have (to be useful) heterogeneous sources. For example each interval type-2 set could model the interpretation of "warm" of a single person, incorporating the variation of this interpretation (for this one person) across the seasons (summer, winter, etc.) in its FOU. Another example could be the interval type-2 model of "low lightlevel" as perceived by one specific sensor and incorporating the measurement uncertainty of that sensor. While in the former example, computing the MLA set aims to retrieve the agreement between the individual people in terms of the meaning of "warm", in the latter example, computing the MLA for the sets "low lightlevel" of multiple sensors allows the extraction of a general notion of low lightlevel across all sensors.

The process of generating the zSlices based MLA set can be reduced to a recursive application of the fundamental set theoretic operations of intersection and union. As such, assume K source interval type-2 fuzzy sets $\widetilde{S_k}$, where $k \in \{1, \ldots, K\}$. Computing the MLA set $\tilde{Z}_{MLA}$ with a number of zLevels I

equal to K (and as such zSlices $\tilde{Z}_i$, where $i \in \{1, \dots, I\}$) involves the following series of steps:

- Establish the zSlice $\tilde{Z}_1$ as the union of all interval type-2 source sets associated with the zLevel $z_1 = 1/I$ as shown in Section III.B.:

$$\tilde{Z}_1 = z_1 / \bigcup_{k=1}^{K} \widetilde{S_k} \tag{6}$$

  $\tilde{Z}_1$ as the union of all source sets reflects the most basic level of agreement (where the minimum of sources that agree is 1) which we shall refer to as agreement level 1.

- Compute the intersections between all the combinations of source sets $\widetilde{S_k}$ to extract a series of intersections which together reflect agreement level 2, i.e. where at least 2 sources agree. We refer to these intersections as $\widetilde{S_2^m}$, where m designates an index for the intersection and $m \in \{1, \dots, M\}$, M being the number of intersections created (intersections that are equal to $\emptyset$ are omitted). For example, for three source sets $\widetilde{S_1}$, $\widetilde{S_2}$ and $\widetilde{S_3}$, compute $\widetilde{S_2^1} = \widetilde{S_1} \cap \widetilde{S_2}$, $\widetilde{S_2^2} = \widetilde{S_1} \cap \widetilde{S_3}$ and $\widetilde{S_2^3} = \widetilde{S_2} \cap \widetilde{S_3}$.

  Compute the union of all the resulting intersections and associate it with zLevel $z_2 = 2/I$ to extract zSlice $\tilde{Z}_2$. In terms of our example: $\tilde{Z}_2 = z_2/\left(\widetilde{S_2^1} \cup \widetilde{S_2^2} \cup \widetilde{S_2^3}\right)$. To summarize:

$$\tilde{Z}_2 = z_2 / \left\{ \begin{matrix} \left(\widetilde{S_1} \cap \widetilde{S_2}\right) \cup \left(\widetilde{S_1} \cap \widetilde{S_3}\right) \cup \dots \cup \left(\widetilde{S_1} \cap \widetilde{S_K}\right) \cup \\ \left(\widetilde{S_2} \cap \widetilde{S_3}\right) \cup \left(\widetilde{S_2} \cap \widetilde{S_4}\right) \cup \dots \cup \left(\widetilde{S_2} \cap \widetilde{S_K}\right) \cup \\ \dots \cup \\ \left(\widetilde{S_{K-1}} \cap \widetilde{S_K}\right) \end{matrix} \right\} \tag{7}$$

  As such, we have extracted a zSlice reflection the agreement of at least 2 of the initial source sets.

- Proceed by extracting the intersections of larger combinations to compute the MLA zSlice for the subsequent agreement levels until no intersections are found. For each agreement level, the union of all extracted intersections defines the zSlices for this specific level. For example, for zSlice $\tilde{Z}_3$, we compute all 3-way intersections $\widetilde{S_3^m}$ possible and subsequently compute their union:

$$\tilde{Z}_3 = z_3 / \left\{ \begin{matrix} \left(\widetilde{S_1} \cap \widetilde{S_2} \cap \widetilde{S_3}\right) \cup \left(\widetilde{S_1} \cap \widetilde{S_2} \cap \widetilde{S_4}\right) \cup \dots \cup \left(\widetilde{S_1} \cap \widetilde{S_2} \cap \widetilde{S_K}\right) \cup \\ \left(\widetilde{S_1} \cap \widetilde{S_3} \cap \widetilde{S_4}\right) \cup \left(\widetilde{S_1} \cap \widetilde{S_3} \cap \widetilde{S_5}\right) \cup \dots \cup \left(\widetilde{S_1} \cap \widetilde{S_3} \cap \widetilde{S_K}\right) \cup \\ \dots \cup \\ \left(\widetilde{S_1} \cap \widetilde{S_{K-1}} \cap \widetilde{S_K}\right) \cup \\ \left(\widetilde{S_2} \cap \widetilde{S_3} \cap \widetilde{S_4}\right) \cup \left(\widetilde{S_2} \cap \widetilde{S_3} \cap \widetilde{S_5}\right) \cup \dots \cup \left(\widetilde{S_2} \cap \widetilde{S_3} \cap \widetilde{S_K}\right) \cup \\ \left(\widetilde{S_2} \cap \widetilde{S_4} \cap \widetilde{S_5}\right) \cup \left(\widetilde{S_2} \cap \widetilde{S_4} \cap \widetilde{S_6}\right) \cup \dots \cup \left(\widetilde{S_2} \cap \widetilde{S_4} \cap \widetilde{S_K}\right) \cup \\ \dots \cup \\ \left(\widetilde{S_2} \cap \widetilde{S_{K-1}} \cap \widetilde{S_K}\right) \cup \\ \\ \dots \\ \left(\widetilde{S_{K-2}} \cap \widetilde{S_{K-1}} \cap \widetilde{S_K}\right) \end{matrix} \right\} \tag{8}$$

  As part of our example, the only 3-way combination and thus resulting zSlices is: $\tilde{Z}_3 = z_3/\left(\widetilde{S_1} \cap \widetilde{S_2} \cap \widetilde{S_3}\right)$.

A visual example of this process is provided in Section IV.

By extracting the MLA zSlices based general type-2 fuzzy set zSlice by zSlice, we successively extract the models for higher levels of agreement. Naturally, the "size" of the intersections (and as such the zSlices) becomes "smaller" as the number of intersecting sets increases with increasing zLevel. This is intuitive - similar to the decreasing amount of agreement on a topic by people as the number of people who compare their opinion increases. Importantly however, the MLA zSlices based general type-2 set, by relying on the third dimension to model the level of agreement, can capture where most, or two, three, four,… people agree while preserving the "opinions" of each individual. This is impossible using classical, type-1 or interval type-2 fuzzy sets as they lack an additional degree of freedom which could be employed to capture this information.

In the following Section we highlight a series of important observations on computing the MLA model, while Section III.F provides examples of MLA sets computed using the process described in this Section.

E. Significant observations on computing MLA.

Several aspects of computing MLA are non-intuitive in the first instance and/or require further explanation, we address what we feel are the most significant three aspects:

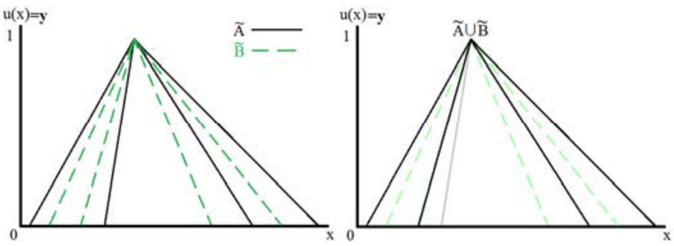
1) Potentially non-intuitve results of computing the intersection and union.

While the computation of the intersection and union of interval type-2 fuzzy sets are extremely common and well known, it has been our experience that their effect visible through the visualization of the resulting sets can appear non-intuitive. We are highlighting this point as it is significant when considering visualized MLA sets (as in Section IV) which are based directly on the union and intersection operations. In order to address this concern we briefly recapitulate the formulae for both operations based on the vertical slices $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ of the interval type-2 fuzzy sets $\tilde{A}$ and $\tilde{B}$:

$$\tilde{A} \sqcap \tilde{B} \Leftrightarrow \mu_{\tilde{A} \sqcap \tilde{B}} = \mu_{\tilde{A}}(x) \sqcap \mu_{\tilde{B}}(x) = \sum_{k \in [min(l_A, l_B), min(r_A, r_B)]} 1/k, \forall x \in X \tag{9}$$

$$\tilde{A} \sqcup \tilde{B} \Leftrightarrow \mu_{\tilde{A} \sqcup \tilde{B}} = \mu_{\tilde{A}}(x) \sqcup \mu_{\tilde{B}}(x) = \sum_{k \in [max(l_A, l_B), max(r_A, r_B)]} 1/k, \forall x \in X \tag{10}$$

, where l and r represent the left and right endpoints of the intervals formed by the respective memberships $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$.



(a) (b)

Figure 7. The union of interval type-2 fuzzy sets. (a) Two interval type-2 sets $\tilde{A}$ (black) and $\tilde{B}$(dashed green). (b) Their union (in bold black).

(9) and (10) result in the perhaps at first non-intuitive nature of the resulting intersections and unions, for example the union of two interval type-2 sets , as can be seen from Fig. 7, does not "contain" the FOU of both sets (as might seem intuitive at first) - it is described by the "higher" of both the lower and upper membership functions.

2) A deviation from the established zSlices model in terms of the overlapping FOUs.

Traditionally, the third dimension of general type-2 fuzzy sets has been considered as a dimension to further describe the nature of the uncertainty distribution of the second dimension (i.e. a function of the second dimension as shown in Fig. Figure 5b).

It is important to note that this relationship does not exist in MLA zSlices based general type-2 fuzzy sets. Here, the third dimension describes the level of agreement which is not directly related to the uncertainty distribution modeled in the second dimension. This allows the creation of MLA zSlices based general type-2 fuzzy sets with a structure as depicted in Fig. 8. (Please note that Fig. 8 depicts the x-z plane and not the y-z plane as shown in Fig. 5b.)

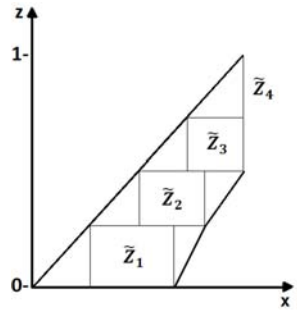


Figure 8. Structure of MLA zSlices based general type-2 fuzzy sets. (x-z plane view where y=0)

This special type of zSlices based general type-2 fuzzy sets where the third dimension is – to some extent – disconnected in meaning from the second dimension requires a review of some of the aspects of the zSlices based theory established in [3], in particular that the FOU of "higher" zSlices is not necessarily "contained" within the lower zSlices' FOUs. We will address these modifications in a future publication.

Nevertheless, as during computation all zLevels (and as such zSlices) are computed individually (as shown in [3]), MLA zSlices based general type-2 fuzzy sets can be employed without complications as part of applications.

Finally, it should be noted that as part of computing the MLA model, the information encoded in the individual source sets' FOUs is employed to create the new FOUs of the individual agreement level zSlices. In other words, while the uncertainty model expressed through the primary membership is not directly related to the agreement model expressed in the secondary membership, both memberships do interact as is expected: for example if several people provide agree on a specific concept, we should be able to define this concept at the different levels of agreement with more, respectively less precision. The process is further described and demonstrated in Section IV.

3) The potential creation of non-convex and/or non-continuous zSlices.

As the extraction of the zSlices based MLA agreement sets relies on the computation of the union of multiple discrete intersections (as shown in the previous section), the resulting sets are potentially non-continuous and/or non-convex.

As standard fuzzy logic theory requires fuzzy sets to be both continuous and convex, we apply interpolation and our convexity algorithm introduced in [12] to create theory-compliant fuzzy sets. It is important to note that the accurate MLA model may be non-continuous/non-convex but a continuous/convex approximation can be employed for computation. This is the reason why the proposed extraction method for MLA has consciously not been further modified to (directly) provide continuous/convex sets.

IV. Multi Level Agreement Examples

While it was our aim to only include real-world data based examples, the resulting sets generally do not lend themselves for a clear visualization as part of a paper. Hence, we are providing examples based on a series of manually designed interval type-2 fuzzy sets depicted in Fig. 9a, Fig. 9b and Fig. 9c.

We are providing step-by-step visualizations of the steps for the computation of MLA as set out in Section III.D for the MLA of the first two sets. As such, Fig. 10a depicts the union of both sets as shown in (6) which will form the basis for zSlice 1. Fig. 10b subsequently shows the intersection of both sets which, as shown in (7), provides the basis for zSlice 2.

Fig. 11a shows the resulting MLA zSlices based general type-2 fuzzy set based on the interval type-2 sets shown in Fig. 9a and Fig. 9b.

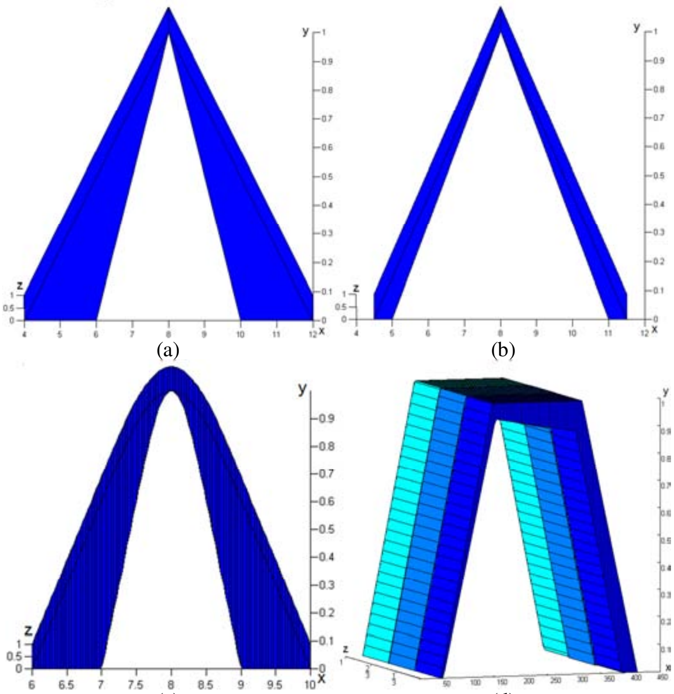
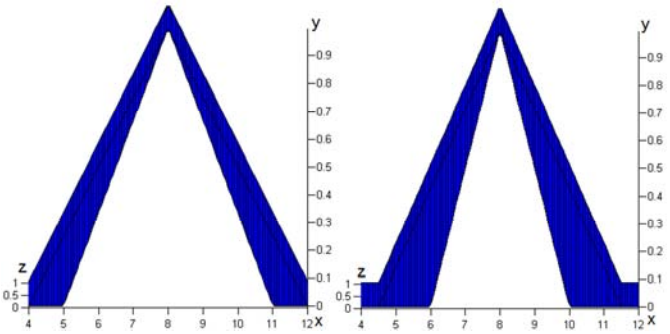


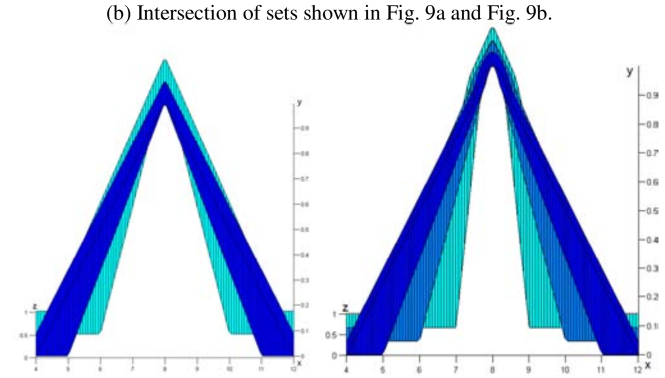
Figure 9. Three example interval type-2 fuzzy sets: (a), (b) and (c). (d) is a real-world data based example of a ambient light level created by computing the MLA for the models of three individual days.

Fig. 11b subsequently shows the MLA zSlices based general type-2 fuzzy set computed based on the three interval type-2 sets shown in Fig. 9a, Fig. 9b and Fig. 9c. It can be noted how different types of interval type-2 fuzzy sets (triangular in Fig. 9a, Fig. 9b and Gaussian (modified) in Fig. 9c) are combined as part of the MLA model. In other words, the MLA set can be computed based on any types and number of interval type-2 source sets. As the number of source sets increased, the number of zLevels increases – in the case of the examples here, from two in Fig. 11a to three in Fig. 11b.



(a) (b)

Figure 10. Interval type-2 sets: (a) Union of sets shown in Fig. 9a and Fig. 9b. (b) Intersection of sets shown in Fig. 9a and Fig. 9b.



(a) (b)

Figure 11. MLA zSlices based general type-2 sets. (a) MLA computed on sets shown in Fig. 9a and Fig. 9b. (b) MLA computed on sets shown in Fig. 9a, Fig. 9b and Fig. 9c.

Fig. 9d shows an example of an MLA set modeling "low" ambient light level which has been created based on three interval type-2 fuzzy sets which in turn were created based on real-world data from a series of light sensors in our Ambient Intelligence lab (the iSpace at the University of Essex). As the light levels vary from day to day, the MLA model allows us to maintain a consistent fuzzy set model over time by incorporating each daily model.

V. Conclusions & Future Work

As part of this paper we have presented the concept of Multi Level Agreement (MLA) for zSlices based general type-2 fuzzy sets. We have defined the notion of MLA sets and provided the details to compute MLA sets based on interval type-2 fuzzy sets and have provided examples visualizing the nature of MLA sets.

Moreover, while the core of the paper has been devoted to the presentation of MLA, we have provided an in-depth overview of the reasons for and potential benefits of employing MLA based type-2 fuzzy sets and systems. As such, we have clarified why the presented approach was devised while also highlighting the requirement of general type-2 fuzzy sets in order to process complex concepts such as MLA which is not possible using classical, type-1 or interval type-2 fuzzy sets.

General type-2 fuzzy sets and systems are still in a very early stage and while recent years have provided significant advances in addressing the technical challenges of employing general type-2 fuzzy sets, the nature, interpretation and usefulness of the secondary membership in general type-2 fuzzy sets has barely been researched and offers a very large number of open questions (and unasked questions).

In the future we are aiming to show the use of MLA for zSlices based general type-2 fuzzy sets as part of real world applications, in particular in the context of modeling agreement for concepts based on information gathered from people in an AmI context while continuing to investigate the relationship between the individual modeling dimensions in general type-2 fuzzy sets.

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