ON SETS WITH MORE RESTRICTED SUMS THAN DIFFERENCES

David Penman
Department of Mathematical Sciences, University of Essex, Wivenhoe Park, Colchester, CO4 3SQ, United Kingdom
dbenman@essex.ac.uk

Matthew Wells
Department of Mathematical Sciences, University of Essex, Wivenhoe Park, Colchester, CO4 3SQ, United Kingdom
mwells@essex.ac.uk

Received: , Revised: , Accepted: , Published:

Abstract

Given a finite set $A$ of integers, we define its restricted sumset $A + A$ to be the set of sums of two distinct elements of $A$ - a subset of the sumset $A + A$ - and its difference set $A - A$ to be the set of differences of two elements of $A$. We say $A$ is a restricted-sum-dominant set if $|A + A| > |A - A|$. Though intuition suggests that such sets should be rare, we present various constructions of such sets and prove that a positive proportion of subsets of $\{0, 1, \ldots, n - 1\}$ are restricted-sum-dominant sets. As a by-product, we improve on the previous record for the maximum value of $\ln(|A + A|)/\ln(|A - A|)$, and give some related discussion.

1. Introduction

Let $A$ be a finite set of integers. We define its sumset $A + A$ to be $\{a + b : a, b \in A\}$, its difference set $A - A$ to be $\{a - b : a, b \in A\}$ and its restricted sumset $A + A$ to be $\{a + b : a \neq b, a, b \in A\}$. It is a natural intuition that, since addition is commutative but subtraction is not, that ‘often’ we should have $|A + A| \leq |A - A|$. However it has been known for some time that this is not always the case: for example, the set $C = \{0, 2, 3, 4, 7, 11, 12, 14\}$, which is attributed to Conway, has $|C + C| = 26$, but $|C - C| = 25$. In this paper, sets with this property are called sum-dominant: in some other literature, they are described as MSTD (for ‘more sums than differences’) sets, see e.g. Nathanson [6]. It is now known by work of Martin and O’ Bryant [5] that sum-dominant sets are less rare than they might initially appear: they prove that, for $n \geq 15$, the proportion of subsets of $\{0, 1, 2 \ldots n - 1\}$
which are sum-dominant is at least $2 \times 10^{-7}$. The constant was sharpened, and the existence of a limit shown, by Zhao [11].

In this paper we investigate what might appear to be an even more demanding condition on a set, namely what we will call the restricted-sum-dominant property.

**Definition 1.** A set $A$ of integers is said to be restricted-sum-dominant if $|A+A| > |A-A|$.

There are examples of this. For example, we find the set from Hegarty [3]

$$A_{15} = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25, 29, 32, 33, 37, 40, 41, 42, 44, 45\}$$

has $|A_{15}+A_{15}| = 86$ whilst $|A_{15} - A_{15}| = 83$.

Clearly any restricted-sum-dominant set is sum-dominant. The converse is false as Conway’s set is sum-dominant but not restricted-sum-dominant ($|C+C| = 21$).

Note that the property of being restricted-sum-dominant is preserved when we apply a bijection of the form $x \rightarrow ax + b$ with $a, b \in \mathbb{Z}$, $a \neq 0$. It therefore suffices to consider sets $A \subset \mathbb{Z}$ with $\min(A) = 0$ and $\gcd(A) = 1$. We shall refer to such sets as being *normalised*.

The organisation of this paper is as follows. In Section 2 we exhibit several sequences of restricted-sum-dominant sets, addressing some natural questions about the relative sizes of the restricted sumset and difference sets. In Section 3, we show that a strictly positive proportion of subsets of $\{0, 1, 2, \ldots, n-1\}$ are restricted-sum-dominant sets. In Section 4 we obtain a new record high value of each of

$$f(A) = \frac{\ln(|A+A|)}{\ln(|A-A|)} \quad \text{and} \quad g(A) = \frac{\ln(|A+A|/|A|)}{\ln(|A-A|/|A|)}$$

and give some related discussion. Finally, in Section 5 we improve somewhat the bounds on the order of the smallest restricted-sum-dominant set.

We shall, slightly unusually, use the notation $[a, b]$, when $a < b$ are integers, to denote $\{a, a+1, \ldots, b\}$.

We are grateful to the referee for suggestions which have non-trivially improved the organisation and exposition of this paper, especially in Section 5.

2. Explicit sequences of restricted-sum-dominant sets

Our first sequence of restricted-sum-dominant sets arose by considering the set $B = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25, 28, 30, 32, 33\}$ which appears in [7] and [9] as a set of integers with $|B+B| > |(B-B) \setminus \{0\}|$). We then noted that replacing 33 with 29 gives a 16 element restricted-sum-dominant set (which will be $T_3$ below).

To get the subsequent terms of the sequence, we used (here and elsewhere in the paper) the idea from [9], Conjecture 6, that repetition of certain so-called interior
blocks when the set is written in order as a sequence of differences can increase the size of the sumset more than the difference set: see [9] for details.

**Theorem 2.** For every integer \( j \geq 1 \) we define

\[
T'_j = \{0, 2\} \cup \{1, 9, \ldots, 1 + 8j\} \cup \{4, 12, \ldots, 4 + 8j\} \\
\quad \cup \{5, 13, \ldots, 5 + 8j\} \cup \{6 + 8j, 8(j + 1)\}.
\]

Then

\[
T'_j + T'_j = [1, 6 + 8(2j + 1)] \setminus \{8, 8(2j + 1)\}, \\
T'_j + T'_j = [0, 8(2j + 2)] \setminus \{7 + 8(2j + 1)\} \text{ and} \\
T'_j - T'_j = [-8(j + 1), 8(j + 1)] \setminus \{-6, \ldots, \pm(6 + 8(j - 1))\}.
\]

**Proof.** We deal first with the restricted sumset. Since \( 0 \in T'_j \),

\[
T'_j \hat{\cup} T'_j = \{0\} \subseteq T'_j + T'_j,
\]
giving all elements congruent to 1, 4 or 5 mod 8 less than \( 8(j + 1) \).

For integers congruent to 2 mod 8, the restricted sumset contains 0+2 and

\[
8(j + 1) + \{1, 9, \ldots, 1 + 8j\} = \{1 + 8(j + 1), \ldots, 1 + 8(2j + 1)\}
\]

so \( T'_j \hat{\cup} T'_j \) contains all the elements congruent modulo 8 to 1, 4 or 5 stated. For integers congruent to 6 mod 8, note that

\[
8(j + 1) + \{5, 13, \ldots, 5 + 8j\} = \{5 + 8(j + 1), \ldots, 5 + 8(2j + 1)\}
\]

and (6 + 8j) + \{1, 9, \ldots, 1 + 8j\} = \{7 + 8j, \ldots, 7 + 8(2j)\}

For integers congruent to 3 mod 8, note that

\[
\{1, 9, \ldots, 1 + 8j\} \hat{+} (2) = \{3, 11, \ldots, 3 + 8j\}
\]

and

\[
(6 + 8j) \hat{+} \{5, 13, \ldots, 5 + 8j\} = \{3 + 8(j + 1), \ldots, 3 + 8(2j + 1)\}.
\]

For integers congruent to 6 mod 8,

\[
\{1, 9, \ldots, 1 + 8j\} \hat{+} \{5, 13, \ldots, 5 + 8j\} = \{6, 14, \ldots, 6 + 8(2j)\}
\]

and \((6 + 8j) + 8(j + 1) = 6 + 8(2j + 1) \in T'_j \hat{\cup} T'_j\) also. The elements congruent to 7 modulo 8 are obtained from

\[
(2) + \{5, 13, \ldots, 5 + 8j\} = \{7, 15, \ldots, 7 + 8j\}
\]

and

\[
(6 + 8j) + \{1, 9, \ldots, 1 + 8j\} = \{7 + 8j, \ldots, 7 + 8(2j)\}.
\]
Therefore, for every integer \( j \geq 1 \) the set \( T'_j \subset \mathbb{Z} \) has
\[
|T'_j| = 3j + 7, \quad |T'_j + T'_j| = 16j + 12, \quad |T'_j + T'_j| = 16j + 16 \quad \text{and} \quad |T'_j - T'_j| = 14j + 17.
\]

Therefore
\[
|T'_j + T'_j| - |T'_j - T'_j| = 2j - 5, \quad |T'_j + T'_j| - |T'_j - T'_j| = 2j - 1
\]
and \( T'_j \) is a restricted-sum-dominant set for every integer \( j \geq 3 \).

\( T'_j \) of order 16 is one of the two smallest restricted-sum-dominant sets we have.

The set \( T'_j \) has a superset \( T_j = T'_j \cup 1 + 8(j + 1) \), which is also restricted-sum-dominant for \( j \geq 3 \):
Theorem 4. For every integer $j \geq 1$ define

\[ T_j = \{0, 2\} \cup \{1, 9, \ldots, 1 + 8(j + 1)\} \cup \{4, 12, \ldots, 4 + 8j\} \]
\[ \cup \{5, 13, \ldots, 5 + 8j\} \cup \{6 + 8j, 8(j + 1)\}. \]

Then

\[ T_j + T_j = [1, 1 + 8(2j + 2)] \setminus \{8, 8(2j + 1), 8(2j + 2)\}, \]
\[ T_j + T_j = [0, 2 + 8(2j + 2)] \text{ and} \]
\[ T_j - T_j = [-1 + 8(j + 1), 1 + 8(j + 1)] \setminus \{1 + 8(j + 1)\}. \]

Proof. Firstly since $T_j \supset T'_j$ we have $T_j + T_j \supset [1, 1 + 8(2j + 2)] \setminus \{8, 8(2j + 1)\}$. With $1 + 8(j + 1) \in T_j$ we now also have that

\[ 8(j + 1) + (1 + 8(j + 1)) = 1 + 8(2j + 2) \quad \text{and} \]
\[ (6 + 8j) + (1 + 8(j + 1)) = 7 + 8(2j + 1) \]

are in $T_j + T_j$ as well. Furthermore

\[ (1 + 8(j + 1)) + (1 + 8(j + 1)) = 2 + 8(2j + 2) \in T_j + T_j. \]

This completes the claims for the sumset and restricted sumset, noting that clearly $8$ and $8(2j + 2)$ are not in $T_j + T_j$ and checking that $8(2j + 1) \notin T_j + T_j$.

As regards the difference set, with $0 \leq x \leq j + 1$ the positive differences resulting from the introduction of the new element have the form

\[ (1 + 8(j + 1)) - \{0, 2, 1 + 8x, 4 + 8y, 5 + 8z, 6 + 8j, 8(j + 1)\} \]
\[ \quad = \{1 + 8(j + 1), 8j + 7, 8(j - x + 1), 8(j - y) + 5, 8(j - z) + 4, 3, 1, 0\}. \]

This shows that $T_j - T_j = T'_j - T'_j \cup \{1 + 8(j + 1)\}$ and the result follows. \(\square\)

Corollary 5. For every integer $j \geq 1$ the set $T_j \subset \mathbb{Z}$ has

\[ |T_j| = 3j + 8, \quad |T_j + T_j| = 16j + 14, \quad |T_j + T_j| = 16j + 19 \quad \text{and} \quad |T_j - T_j| = 14j + 19. \]

Therefore

\[ |T_j + T_j| - |T_j - T_j| = 2j - 5, \quad |T_j + T_j| - |T_j - T_j| = 2j \]

and $T_j$ is an restricted-sum-dominant set for every integer $j \geq 3$.

In [5], Martin and O’Byant construct, for all integers $x$, subsets $S$ of $[0, 17|x|]$ with $|S + S| - |S - S| = x$. Corollary 3 shows that for each positive odd integer $x$ there is $T_j' \subset \mathbb{Z}$ with $|T'_j + T'_j| - |T'_j - T'_j| = x$, and Corollary 5 shows each positive
even integer can be expressed as the difference of the cardinalities of the sumset and the difference set of some $T_j \subset \mathbb{Z}$.

Recall that the diameter of a finite set $A$ of integers is $\max(A) - \min(A)$. There is some interest in finding sets of integers of small diameter with prescribed relationships between the order of the sumset (or restricted sumset) and the difference set: see e.g. [5] Theorem 4 where sets $S_x$ of diameter at most $17|x|$ are constructed with $|S_x + S_x| - |S_x - S_x|$ equal to $x$. Our sets $T_j$ and $T_j'$ have respective diameters $8j + 8$ and $8j + 9$, which is smaller than the sets $S_x$ in [5] for $j \geq 3$.

Further Corollary 5 makes it clear that the difference between the size of the restricted sumset and the difference set can be any odd positive integer. We will get any even difference for $|A + A| - |A - A|$ in our next construction. This was motivated by the sum-dominant (but not restricted-sum-dominant) set called $A_{13} = \{0, 1, 2, 4, 7, 8, 12, 14, 15, 18, 19, 20\}$ in Hegarty [3]. We exhibit, addressing his remark about the desirability of generalising $A_{13}$, two infinite sequences of (eventually) restricted-sum dominant sets derived from $A_{13}$ (which shall be our $R_1$).

**Theorem 6.** For each integer $j \geq 1$ define $R_j \subset \mathbb{Z}$ to be the set

\[
R_j = \{1, 4\} \cup \{0, 12, \ldots, 12j\} \cup \{2, 14, \ldots, 2 \pm 12j\} \\
\cup \{7, 19, \ldots, 7 + 12j\} \cup \{8, 20, \ldots, 8 + 12j\} \cup \{3 + 12j, 6 + 12j\}.
\]

For each integer $j \geq 2$ we have

\[
R_j + R_j = [1, 3 + 12(2j + 1)] \setminus \{17, \ldots, 5 + 12(j - 1)\} \cup \{12(2j), 12(2j + 1)\}, \\
R_j + R_j = [0, 4 + 12(2j + 1)] \setminus \{17, \ldots, 5 + 12(j - 1)\} \text{ and} \\
R_j - R_j = [(8 + 12j), 8 + 12j] \setminus \{\pm 9, \ldots, \pm (9 + 12(j - 1))\}.
\]

**Proof.** We first verify the claim for the restricted sumset. For multiples of 12,

\[
\{0, 12, \ldots, 12j\} + \{0, 12, \ldots, 12j\} = \{12, 24, \ldots, 12(2j - 1)\}.
\]

The elements congruent to 1 modulo 12 are given by

\[
(1) + \{0, 12, \ldots, 12j\} = \{1, 13, \ldots, 1 + 12j\}.
\]

and

\[
(6 + 12j) + \{7, 19, \ldots, 7 + 12j\} = \{1 + 12(j + 1), \ldots, 1 + 12(2j + 1)\}.
\]

For those congruent to 2 modulo 12

\[
\{0, 12, \ldots, 12j\} + \{2, 14, \ldots, 2 + 12j\} = \{2, 14, \ldots, 2 + 12(2j)\}
\]

and also $(6 + 12j) + (8 + 12j) = 2 + 12(2j + 1) \in R_j + R_j$. For 3 modulo 12 clearly $3 = 1 + 2 \in R_j + R_j$ and the rest follow from

\[
\{7, 19, \ldots, 7 + 12j\} + \{8, 20, \ldots, 8 + 12j\} = \{15, 27, \ldots, 3 + 12(2j + 1)\}.
\]
For elements congruent to 4 modulo 12, we clearly have that 4 and 16 are in $R_j \hat{+} R_j$ as well as

$$\{8, 20, \ldots, 8 + 12j\} \hat{+} \{8, 20, \ldots, 8 + 12j\} = \{28, 40, \ldots, 4 + 12(2j)\}.$$ 

The elements congruent to 6 modulo 12 in $R_j \hat{+} R_j$ can be obtained as the union of

$$(4) \hat{+} \{2, 14, \ldots, 2 + 12j\} = \{6, 18, \ldots, 6 + 12j\}$$

and

$$(6 + 12j) + \{0, 12, \ldots, 12j\}.$$ 

The elements congruent to 7 (respectively 8) modulo 12 are obtained from

$$\{0, 12, \ldots, 12j\} \hat{+} \{7, 19, \ldots, 7 + 12j\} = \{7, 19, \ldots, 7 + 12(2j)\}.$$ 

and

$$\{0, 12, \ldots, 12j\} \hat{+} \{8, 20, \ldots, 8 + 12j\} = \{8, 20, \ldots, 8 + 12(2j)\}.$$ 

For 9 (respectively 10) modulo 12 use

$$\{2, 14, \ldots, 2 + 12j\} \hat{+} \{7, 19, \ldots, 7 + 12j\} = \{9, 21, \ldots, 9 + 12(2j)\}$$

respectively

$$\{2, 14, \ldots, 2 + 12j\} \hat{+} \{8, 20, \ldots, 8 + 12j\} = \{10, 22, \ldots, 10 + 12(2j)\}.$$ 

Finally the elements congruent to 11 modulo 12 are obtained from

$$(4) + \{7, 19, \ldots, 7 + 12j\} = \{11, 23, \ldots, 11 + 12j\}$$

and

$$(3 + 12j) + \{8, 20, \ldots, 8 + 12j\} = \{11 + 12j, \ldots, 11 + 12(2j)\}.$$ 

To see that the restricted sumset does not contain any of $\{17, \ldots, 5 + 12(j - 1)\}$, note that none of the sumsets of the progressions with common difference 12 give elements which are congruent to 5 modulo 12 and neither can translates of the progressions by 1 or 4). The remaining elements congruent to 5 modulo 12 are obtained as clearly $5 \in R_j \hat{+} R_j$, and also

$$(3 + 12j) + \{2, 14, \ldots, 2 + 12j\} = \{5 + 12j, 7 + 12j, 9 + 12(2j)\} \subseteq R_j \hat{+} R_j.$$ 

Finally, to see that $R_j \hat{+} R_j$ does not contain $12(2j)$ or $12(2j + 1)$, note that it is impossible to obtain $12(2j)$ as a sum of distinct elements of $R_j$ since the only elements of $R_j$ greater than $12j$ are $S = \{2 + 12j, 3 + 12j, 6 + 12j, 7 + 12j, 8 + 12j\}$ but none of the numbers in $2(12j) - S$ (namely $10 + 12(j - 1), 9 + 12(j - 1)$,
6 + 12(j − 1), 5 + 12(j − 1), 4 + 12(j − 1)) are in $R_j$. Further as $12(j + 1) \notin R_j$ $12(2j + 1)$ is excluded from $R_j$. This completes the argument for $R_j < R_j$.

However, we do have that $12j + 12 = 12(2j) \in R_j + R_j$ and $(6 + 12j) + (6 + 12j) = 12j + 1 \in R_j + R_j$, so both these missing elements get into $R_j + R_j$. Since we readily see that none of the numbers congruent to 7 mod 12 ruled out of $R_j + R_j$ are in $R_j + R_j$ either, the sumset is as stated.

To confirm the claim for the difference set as before we consider the positive differences. Writing $R_j$ as

$$
\{1, 4, 12w, 2 + 12x, 7 + 12y, 8 + 12z, 3 + 12j, 4 + 12j\}
$$

the remainders which occur in $R_j − R_j$ are exactly the set $[0, 11] \setminus \{9\}$. On the other hand, to see that $R_j − R_j$ contains all the claimed differences, note that as $0 \in R_j$ we have $R_j \subset R_j − R_j$. Also the right hand sides of

$$
\{0, 12, \ldots, 12j\} − (1) = \{-1, 11, \ldots, 11 + 12(j − 1)\}
$$

$$
\{2, 14, \ldots, 2 + 12j\} − (1) = \{1, 13, \ldots, 1 + 12j\}
$$

$$
\{7, 19, \ldots, 7 + 12j\} − (4) = \{3, 15, \ldots, 3 + 12j\}
$$

$$
\{8, 20, \ldots, 8 + 12j\} − (4) = \{4, 16, \ldots, 4 + 12j\}
$$

$$
\{7, 19, \ldots, 7 + 12j\} − (2) = \{5, 17, \ldots, 5 + 12j\}
$$

$$
\{7, 19, \ldots, 7 + 12j\} − (1) = \{6, 18, \ldots, 6 + 12j\}
$$

$$
\{2, 14, \ldots, 2 + 12j\} − (4) = \{-2, 10, \ldots, 10 + 12(j − 1)\}.
$$

are in the difference set which completes the claim. □

**Corollary 7.** For every integer $j ≥ 2$ the set $R_j \subset \mathbb{Z}$ has

$$
|R_j| = 4j + 8, \ |R_j + R_j| = 23j + 14, \ |R_j + R_j| = 23j + 18 \quad \text{and} \quad |R_j − R_j| = 22j + 17.
$$

Therefore

$$
|R_j + R_j| − |R_j − R_j| = j − 3, \quad |R_j + R_j| − |R_j − R_j| = j + 1
$$

and $R_j$ is an restricted-sum-dominant set for every integer $j ≥ 4$.

This indeed confirms that any positive integer can be obtained as $|R_j + R_j| − |R_j − R_j|$. Our fourth sequence of sets, the $M_j$s, also has $R_1$ (Hegarty’s $A_{13}$) as its first member, but this time we focus not on prescribing $|M_j + M_j| − |M_j − M_j|$ but instead on getting a reduced diameter $9 + 11j$ rather than the diameter $8 + 12j$ of $R_j$. (We were first led to this family by considering Marica’s sum-dominant set $[4] M = \{1, 2, 3, 5, 8, 9, 13, 15, 16\}$, normalising it and trying to expand it to a restricted-sum-dominant set).
Theorem 8. For \( j \geq 1 \) define
\[
M_j = \{0, 2\} \cup \{1, 12, \ldots, 1 + 11j\} \cup \{4, 15, \ldots, 4 + 11j\} \\
\cup \{7, 18, \ldots, 7 + 11j\} \cup \{8, 19, \ldots, 8 + 11j\} \cup \{3 + 11j, 9 + 11j\}
\]
We then have that
\[
M_j + M_j = [1, 6 + 11(2j + 1)] \setminus \{3 + 11(2j + 1)\}, \\
M_j + M_j = [0, 7 + 11(2j + 1)] \text{ and} \\
M_j - M_j = [-9 + 11j, 9 + 11j] \setminus \{\pm 9, \ldots, \pm (9 + 11(j - 1))\}.
\]

Proof. Firstly we show that \( M_j + M_j \) consists of
\[
\bigcup_{a=1,2,4,5,6} \{a, a + 11, \ldots, a + 11(2j + 1)\}
\]
and
\[
\bigcup_{a=3,7,8,9,10,11} \{a, a + 11, \ldots, a + 11(2j)\}
\]
and then show that the sumset contains the additional elements claimed. In the case where \( a = 1 \) we have
\[
\{4, 15, \ldots, 4 + 11j\} + \{8, 19, \ldots, 8 + 11j\} = \{12, 23, \ldots, 12 + 11(2j) = 1 + 11(2j + 1)\}
\]
and \( 0 + 1 \in M_j + M_j \) also. For the case \( a = 2 \)
\[
\{1, 12, \ldots, 1 + 11j\} + \{1, 12, \ldots, 1 + 11j\} = \{13, 24, \ldots, 2 + 11(2j - 1)\}
\]
and \( 0 + 2, (4 + 11(j - 1)) + (9 + 11j) = 2 + 11(2j), (4 + 11j) + (9 + 11j) = 2 + 11(2j + 1) \)
are also in \( M_j + M_j \).

For the case \( a = 4 \),
\[
\{7, 18, \ldots, 7 + 11j\} + \{8, 19, \ldots, 8 + 11j\} = \{15, 26, \ldots, 15 + 11(2j) = 4 + 11(2j + 1)\}
\]
and \( 0 + 4 \in M_j + M_j \).

For the case \( a = 5 \),
\[
\{8, 19, \ldots, 8 + 11j\} + \{8, 19, \ldots, 8 + 11j\} = \{27, \ldots, 16 + 11(2j - 1) = 5 + 11(2j)\}
\]
and also \( 5 = 1 + 4, 16 = 12 + 4 \) and \( (7 + 11j) + (9 + 11j) = 5 + 11(2j + 1) \).

For the case \( a = 6 \)
\[
(2) + \{4, 15, \ldots, 4 + 11j\} = \{6, 17, \ldots, 6 + 11j\}
\]
\[
(9 + 11j) + \{8, 19, \ldots, 8 + 11j\} = \{6 + 11(j + 1), \ldots, 6 + 11(2j + 1)\}. 
\]
For the case $a = 3$
\[ \{7, 18, \ldots, 7 + 11j\} \hat{+} \{7, 18, \ldots, 7 + 11j\} = \{25, 36, \ldots, 3 + 11(2j)\} \]
and $3 = 1 + 2, 14 = 2 + 12$ are in $M_j \hat{+} M_j$.

For the case $a = 7$
\begin{align*}
(0) + \{7, 18, \ldots, 7 + 11j\} &= \{7, 18, \ldots, 7 + 11j\} \\
(3 + 11j) + \{4, 15, \ldots, 4 + 11j\} &= \{7 + 11j, \ldots, 7 + 11(2j)\}. \\
\end{align*}

For the case $a = 8$
\[ \{1, 12, \ldots, 1 + 11j\} \hat{+} \{7, 18, \ldots, 7 + 11j\} = \{8, 19, \ldots, 8 + 11(2j)\}. \]

For the case $a = 9$
\[ \{1, 12, \ldots, 1 + 11j\} \hat{+} \{8, 19, \ldots, 8 + 11j\} = \{9, 20, \ldots, 9 + 11(2j)\}. \]

For $a = 10$
\begin{align*}
(2) &\hat{+} \{8, 19, \ldots, 8 + 11j\} = \{10, 21, \ldots, 10 + 11j\} \\
(3 + 11j) &\hat{+} \{7, 18, \ldots, 7 + 11j\} = \{10 + 11j, \ldots, 10 + 11(2j)\}. \\
\end{align*}

For $a = 11$
\[ \{4, 15, \ldots, 4 + 11j\} \hat{+} \{7, 18, \ldots, 7 + 11j\} = \{11, 22, \ldots, 11 + 11(2j)\}. \]

To see that $3 + 11(2j + 1) \notin M_j \hat{+} M_j$, if it did not we would have a sum of the form $(a + 11j) + (c + 11j) = 14 + 22j$ from elements of $M_j$ with $a + c = 14$, however, since $a$ and $c$ are distinct elements of $\{1, 3, 4, 7, 8, 9\}$ this is impossible and hence $3 + 11(2j + 1) \notin M_j \hat{+} M_j$. This confirms the claim for the restricted sunset. Furthermore for each $m \in M_j$ the sumset contains $0, 2(7 + 11j) = 3 + 11(2j + 1)$ and $2(9 + 11j) = 7 + 11(2j + 1)$ which completes the claim for the sunset.

For the difference set to see that $\{-9, \ldots, -9 + 11(j - 1)\} \notin M_j - M_j$ let
\[ M_j = \{0, 2, 1 + 11w, 4 + 11x, 7 + 11y, 8 + 11z, 3 + 11j, 9 + 11j\}, \]
where $0 \leq w, x, y, z \leq j$. It suffices to consider just the positive differences. Calculation of $M_j - M_j$ reveals that the only positive difference congruent to 9 modulo 11 is $9 + 11j$ and 0, which is outside the range claimed.

To see that $M_j - M_j$ contains the remaining elements in the interval, firstly note that as $0 \in M_j$ we have $M_j - M_j \supseteq M_j$. Furthermore $M_j - M_j$ also contains the
right hand sides of the following:

\[
\begin{align*}
\{1, 12, \ldots, 1 + 11j\} - (1) &= \{0, 11, \ldots, 11j\} \\
\{4, 15, \ldots, 4 + 11j\} - (1) &= \{3, 14, \ldots, 3 + 11j\} \\
\{7, 18, \ldots, 7 + 11j\} - (1) &= \{6, 17, \ldots, 6 + 11j\} \\
\{1, 12, \ldots, 1 + 11j\} - (2) &= \{-1, 10, 21, \ldots, 10 + 11(j - 1)\} \\
\{4, 15, \ldots, 4 + 11j\} - (2) &= \{2, 13, \ldots, 2 + 11j\} \\
\{7, 18, \ldots, 7 + 11j\} - (2) &= \{5, 16, \ldots, 5 + 11j\} \\
9 + 11j - 0 &= 9 + 11j.
\end{align*}
\]

This completes the claim of the theorem.

\[\square\]

**Corollary 9.** For every integer \(j \geq 1\) the set \(M \subset \mathbb{Z}\) has

\[
|M_j| = 4j+8, \ |M_j + M_j| = 22j+16, \ |M_j + M_j| = 22j+19 \quad \text{and} \quad |M_j - M_j| = 20j+19.
\]

Hence

\[
|M_j + M_j| - |M_j - M_j| = 2j - 3, \quad |M_j + M_j| - |M_j - M_j| = 2j
\]

and \(M_j\) is an restricted-sum-dominant set for every \(j \geq 2\).

Note that the set \(M_2\) has slightly smaller diameter 31 than the other 16 element restricted-sum-dominant set \(T_3\).

Martin and O’Bryant refer to sets with \(|A + A| = |A - A|\) as sum-difference balanced. Similarly we can consider sets with \(|A + A| = |A - A|\) as restricted-sum-difference balanced. The results above show such sets exist (e.g. \(R_3\)). The smallest such set we have found has order 14: it is is

\[
M' = \{0, 1, 2, 4, 7, 8, 12, 14, 15, 19, 22, 25, 26, 27\},
\]

so \(|M' + M'| = ||1, 53\| \setminus \{43, 50\}| = 51\) and \(|M' - M'| = ||-27, 27\| \setminus \{\pm 9, \pm 16\}| = 51\). We show that by taking the union of translates of \(M'\) by non-negative integer multiples of its maximum element one can obtain arbitrarily large restricted-sum-difference balanced sets.

**Lemma 10.** Let \(k \geq 2\) and \(A_0 = A = \{0 = a_1 < a_2 < \cdots < a_k = m\} \subset \mathbb{Z}\) and \(A_i = A \cup (A + m) \cup \cdots \cup (A + im)\). Then

\[
\begin{align*}
|A_i + A_i| - |A_{i-1} + A_{i-1}| &= c_1 \quad \forall i \geq 2, \\
|A_i + A_i| - |A_{i-1} + A_{i-1}| &= c_1 \quad \forall i \geq 1
\end{align*}
\]

and

\[
|A_i - A_i| - |A_{i-1} - A_{i-1}| = c_2 \quad \forall i \geq 1.
\]

where \(c_1\) and \(c_2\) are positive constants.
Proof. We first note

\[ |A_i \hat{+} A_i| - |A_{i-1} \hat{+} A_{i-1}| = |(A_i \hat{+} A_i) \setminus (A_{i-1} \hat{+} A_{i-1})| \]

and show that the right-hand side is a constant by showing that the set of new elements introduced on each iteration is a translate of the set of new elements introduced on the previous iteration. We have

\[ A_i \hat{+} A_i = \bigcup_{r,s=0}^r ((A + rm) \hat{+} (A + sm)). \]

If \(|r-s| \geq 2\), it is clear that \(A + rm\) and \(A + sm\) are disjoint so their restricted sum is just their sum. If \(i-1 \geq r = s \geq 1\), then \((A + rm) \hat{+} (A + rm) = (A + (r-1)m) + (A + (r+1)m)\). The only case needing a little thought is \(|r-s| = 1\): without loss of generality, \(r = s + 1\). Then

\[(A + (s + 1)m) \hat{+} (A + sm) = \{a + b + (2s + 1)m : a + m \neq b\}\]

the only way we can have \(a + m = b\) is if \(a = 0, b = m\), but in this case

\[(0 + (s + 1)m) + (m + sm) = (m + (s + 1)m) \hat{+} (0 + sm)\]

We deduce that, for all \(i \geq 2\)

\[ A_i \hat{+} A_i = (A \hat{+} A) \cup (A + (A + m)) \cup \cdots \cup (A + A + (2i - 1)m) \cup (A \hat{+} A + 2im). \]

Similarly

\[ A_{i-1} \hat{+} A_{i-1} = (A \hat{+} A) \cup (A + A + m) \cup \cdots \cup (A \hat{+} A + (2i - 2)m). \]

Now some elements of \((A + A + (2i - 2)m) \setminus (A \hat{+} A + (2i - 2)m)\) may be in \(A + A + (2i - 3)m\) and thus in \(A_{i-1} \hat{+} A_{i-1}\). (Translates of \(A \hat{+} A\) by less than \((2i - 3)m\) need not be considered.) We have

\[(A_i \hat{+} A_i) \setminus (A_{i-1} \hat{+} A_{i-1}) = ((A + A + (2i - 2)m) \cup (A + A + (2i - 1)m) \cup (A \hat{+} A + 2im)) \setminus ((A + A + (2i - 3)m) \cup (A \hat{+} A + (2i - 2)m)). \quad (1)\]

Likewise

\[(A_{i+1} \hat{+} A_{i+1}) \setminus (A_i \hat{+} A_i) = ((A + A + 2im) \cup (A + A + (2i + 1)m) \cup (A \hat{+} A + (2i + 2)m)) \setminus ((A + A + (2i + 1)m) \cup (A \hat{+} A + (2i + 2)m)). \quad (2)\]

The right-hand side of (2) is a translation of the right-hand side of (1) by \(2m\). (To see this, note it is easy to check for sets of integers that if \(C_i + 2m = C_{i+1}\) and \(D_i + 2m = D_{i+1}\), then \((C_i \setminus D_i) + 2m = (C_{i+1} \setminus D_{i+1})\): apply this with the obvious choices of \(C_i\) and \(D_i\).) Thus

\[(A_{i+1} \hat{+} A_{i+1}) \setminus (A_i \hat{+} A_i) = ((A_i \hat{+} A_i) \setminus (A_{i-1} \hat{+} A_{i-1})) + 2m.\]
Since translation by a constant leaves the cardinality of the set difference unaltered it follows that
\[ |(A_{i+1} \hat{+} A_{i+1}) \setminus (A_i \hat{+} A_i)| = |(A_{i-1} \hat{+} A_{i-1}) \setminus (A_{i-1} \hat{+} A_{i-1})| \]
as required.
To see that
\[ |A_i + A_i| - |A_{i-1} + A_{i-1}| = |A_i \hat{+} A_i| - |A_{i-1} \hat{+} A_{i-1}| \] (3)
for all \( i \geq 1 \) we show that the number of additional elements \( A_i + A_i \) contains is constant. All the elements of
\((A + A) \setminus (A+ A)\)
extcept for \( 2m \), which is in \( A_i \hat{+} A_i \) for \( i \geq 1 \) due to \( 0 + 2m \), are excluded from \( A_i \hat{+} A_i \) for all \( i \geq 1 \). Similarly the elements of
\(((A + A) \setminus (A+ A)) + 2im\)
extcept for \( 2im \) are excluded from \( A_i \hat{+} A_i \). This means that for all \( i \geq 1 \)
\[ |A_i + A_i| - |A_{i-1} + A_{i-1}| = 2((A + A) \setminus (A+ A)| - 1). \]
In other words the difference between the cardinalities of the sumset and the restricted sumset is a constant for all \( i \geq 1 \) and (3) holds.
To verify the claim for the difference set, write
\[ A_i - A_i = \cup_{j=-i}^{i} (A - A + jm). \]
Thus we have
\[ (A_i - A_i) \setminus (A_{i-1} - A_{i-1}) = (A - A - im) \cup (A - A + im) \setminus \cup_{j=-i}^{i-1} (A - A - jm). \]
But the only sets in \( \cup_{j=-i}^{i-1} (A - A - jm) \) which could intersect \((A - A - im)\) or \((A - A + im)\) are for \( j = (i-1), j = (i-2) \) (which will intersect \( A - A - im \) in precisely the one element \((1 - i)m\)), \( j = -(i-2) \) (which will intersect it in precisely the one element \((i - 1)m\) and \( j = -(i-1) \). Thus for all \( i \geq 1 \)
\[ (A_i - A_i) \setminus (A_{i-1} - A_{i-1}) = ((A - (A + im)) \setminus (A - (A + (i-1)m))) \]
\[ \cup ((A - A + im) \setminus (A - A + (i-1)m)). \]
Similarly
\[ (A_{i+1} - A_{i+1}) \setminus (A_i - A_i) = ((A - (A + (i+1)m)) \setminus (A - (A + im))) \]
\[ \cup ((A - A + (i+1)m) \setminus (A - A + im)). \]
The sets \((A - (A + (i+1)m)) \setminus (A - (A + im))\) and \((A - A + (i+1)m) \setminus (A - A + im)\) are disjoint for all \(i \geq 1\). Also \((A - (A + (i+1)m)) \setminus (A - (A + im))\) is a translation of \((A - (A + im)) \setminus (A - (A + (i-1)m))\) by \(-m\) and \((A - A + (i+1)m) \setminus (A - A + im)\) is a translation of \((A - A + im) \setminus (A - A + (i-1)m)\) by \(m\). These translations leave the cardinalities of the sets unchanged, therefore

\[
| (A_{i+1} - A_{i+1}) \setminus (A_i - A_i) | = | (A_i - A_i) \setminus (A_{i-1} - A_{i-1}) |
\]

and the overall result follows.

Setting \(M'_i = M' \cup (M' + 27)\) we easily check

\[
|M'_1 + M'_1| = |[1, 107] \setminus \{97, 104\}| = |[-54, 54] \setminus \{\pm 36, \pm 43\}| = |M'_1 - M'_1|
\]

and \(M'_2 = M' \cup (M' + 27) \cup (M' + 54)\) gives

\[
|M'_2 + M'_2| = |[1, 161] \setminus \{151, 158\}| = |[-81, 81] \setminus \{\pm 63, \pm 70\}| = |M'_2 - M'_2|.
\]

It follows from Lemma 10 that

**Corollary 11.** There exist arbitrarily large restricted-sum-difference balanced subsets of \(Z\).

Our final sequence of restricted-sum-dominant sets is constructed with a view to obtaining high values of \(f(A)\) as defined in the introduction. Again, this set is a modification of one in [9], who describes \(Q_j \setminus \{1 + 4(4j + 7)\}\) for \(j = 1, 2, 3\) as sets giving large sumset relative to the difference set. Including \(1 + 4(4j + 7)\) increases the sumset but does not change the difference set.

**Theorem 12.** Let

\[
Q_j = \{0, 2, 4, 12\} \cup \{1, 5, \ldots, 1 + 4(4j + 8)\} \cup \{24, 40, \ldots, 8 + 16j\}
\]

if \(j \geq 1\). Then

\[
Q_j + Q_j = [1, 1 + 4(8j + 16)]
\]

\[
\setminus \{8, 20, 32, 4(8j + 4), 4(8j + 8), 4(8j + 11), 4(8j + 14), 4(8j + 16)\}
\]

for \(j \geq 2\), whilst

\[
Q_j + Q_j = [0, 2 + 4(8j + 16)] \setminus \{20, 32, 4(8j + 8), 4(8j + 11)\}
\]

for \(j \geq 1\) and

\[
Q_j - Q_j = [-1 + 4(4j + 8)], 1 + 4(4j + 8)] \setminus \pm \{6\}, \{14, \ldots, 14 + 16j\}, \{18, \ldots, 2 + 16j\}, \{26, \ldots, 10 + 16j\}, 6 + 16(j + 1)\}
\]

for \(j \geq 1\).
Proof. To verify these claims, consider elements of $Q_j$ in terms of the union of

$$Q \text{ odd} = \{1, 5, \ldots, 1 + 4(4j + 8)\}$$

and

$$Q \text{ even} = \{0, 2, 4, 12\} \cup \{24, \ldots, 8 + 16j\}$$

$$\cup \{4 + 16(j + 1), 12 + 16(j + 1), 14 + 16(j + 1), 16(j + 2)\}.$$

Firstly $Q_j \hat{+} Q_j$ contains all the odd numbers in the interval since we have

$$(0) \hat{+} \{1, 5, \ldots, 1 + 4(4j + 8)\} = \{1, 5, \ldots, 1 + 4(4j + 8)\}$$

$$(16j + 2) \hat{+} \{1, 5, \ldots, 1 + 4(4j + 8)\} = \{1 + 4(4j + 8), 5 + 4(4j + 8), \ldots, 1 + 4(8j + 16)\}$$

$$(2) \hat{+} \{1, 5, \ldots, 1 + 4(4j + 8)\} = \{3, 7, \ldots, 3 + 4(4j + 8)\}$$

$$14 + 16(j + 1) \hat{+} \{1, 5, \ldots, 1 + 4(4j + 8)\} = \{3 + 4(4j + 7), 7 + 4(4j + 7), \ldots, 3 + 4(8j + 15)\}.$$

The union of the right hand sides of the above is indeed

$$\{1, 3, \ldots, 3 + 4(8j + 15), 1 + 4(8j + 16)\} = \{1, 3, \ldots, 1 + 2(4j + 8)\}.$$

To see that the sumset contains all the even elements claimed, note first that

$$Q \text{ odd} \hat{+} Q \text{ odd} \text{ gives the following elements congruent to 2 mod 4:}$$

$$Q \text{ odd} \hat{+} Q \text{ odd} = \{6, 10, \ldots, 2 + 4(8j + 15)\} \subseteq Q_j \hat{+} Q_j.$$  

Clearly 0 + 2 is also in $Q_j \hat{+} Q_j$, however whilst max$(Q_j + Q_j) = 2 + 4(8j + 16)$ this is not in the restricted sumset. As regards the multiples of four, clearly none of these can be obtained from $Q \text{ odd} \hat{+} Q \text{ odd}$ or $Q \text{ odd} \hat{+} Q \text{ even}$. To confirm the elements we claim to be excluded cannot be present note that $Q \text{ even}$ is symmetric w.r.t. 16(j+2): $Q \text{ even} = 16(j + 2) - Q \text{ even}$. Hence $Q \text{ even} \hat{+} Q \text{ even} = 16(2j + 4) - (Q \text{ even} \hat{+} Q \text{ even})$ and $Q \text{ even} + Q \text{ even} = 16(2j + 4) - (Q \text{ even} + Q \text{ even})$. The restricted sumset of the elements of $Q \text{ even}$ less than or equal to 32 is

$$\{0, 2, 4, 12, 24\} \hat{+} \{0, 2, 4, 12, 24\} = \{2, 4, 6, 12, 14, 16, 24, 26, 28, 36\}.$$  

Thus 0, 8, 20, 32 and 48 are excluded from $Q_j \hat{+} Q_j$. Whilst $Q_j + Q_j$ contains 0, 8 and 48 as the doubles of 0, 4 and 24 respectively, it is easy to check that neither 20 nor 0 are in $Q_j + Q_j$. By symmetry

$$16(2j + 4) - \{0, 8, 20, 32, 48\} = \{4(8j + 4), 4(8j + 8), 4(8j + 11), 4(8j + 14), 4(8j + 16)\}$$

which has empty intersection with $Q_j \hat{+} Q_j$.  

It remains to show that all other (relevant) multiples of 4 are in the (restricted) sumset; we consider the cases 0, 4, 8 and 12 modulo 16 separately. We have the following multiples of 16 in $Q_j^+$:

$$\{24, 40, \ldots, 16j + 8\}^+ \{24, 40, \ldots, 16j + 8\} = \{64, 80, \ldots, 16(2j)\}$$

$$(4 + 16(j + 1))^+ (12 + 16(j + 1)) = 4(8j + 12) = 16(2j + 3).$$

Furthermore $Q_j + Q_j$ contains 48 and 16($2j+1$) = 2($16j + 8$) and also 16($j + 2$) + 16($j + 2$) = $4(8j + 16)$ = 16($2j + 4$). We already saw 16($2j + 2$) = $4(8j + 8)$ is not in $Q_j + Q_j$.

We obtain those congruent to 4 modulo 16 from

$$(12)^+ \{24, 40, \ldots, 16j + 8\} = \{36, 52, \ldots, 4 + 16(j + 1)\}$$

$$(4)^+ (16(j + 2)) = 4 + 16(j + 2)$$

$$(12 + 16(j + 1))^+ \{24, \ldots, 8 + 16j\} = \{4 + 16(j + 3), \ldots, 4 + 16(2j + 2)\}$$

$$(4 + 16(j + 1))^+(16(j + 2)) = 4 + 16(2j + 3).$$

The elements congruent to 8 modulo 16 are given by

$$(0)^+ \{24, 40, \ldots, 8 + 16j\} = \{24, 40, \ldots, 8 + 16j\}$$

$$(4)^+ (4 + 16(j + 1)) = 8 + 16(j + 1)$$

$$(12)^+ (12 + 16(j + 1)) = 8 + 16(j + 2)$$

$$(16(j + 2))^+ \{24, 40, \ldots, 8 + 16j\} = \{8 + 16(j + 3), \ldots, 8 + 16(2j + 2)\}.$$
where $x \in \mathbb{Z}$ with $1 \leq x \leq j$. The positive differences of the elements of $Q_{\text{even}}$ are

\[
\{2, 4, 8, 10, 12, 12 + 16(x - 1), 4 + 16x, 6 + 16x, 8 + 16x, \\
12 + 16(j - x), 4 + 16(j - x + 1), 6 + 16(j - x + 1), 8 + 16(j - x + 1), \\
8 + 16j, 16(j + 1), 2 + 16(j + 1), 4 + 16(j + 1), 6 + 16(j + 1), \\
10 + 16(j + 1), 12 + 16(j + 1), 14 + 16(j + 1), 16(j + 2)\}.
\]

Thus none of the differences in $Q_j - Q_j$ have the form which we claim is excluded. To confirm the presence of the remaining differences we have that all the differences congruent to 1 modulo 4 are present since

\[
\{1, 5, \ldots, 1 + 4(4j + 8)\} - \{0\} = \{1, 5, \ldots, 1 + 4(4j + 8)\} \subseteq Q_j - Q_j.
\]

The elements congruent to 3 modulo 4 follow from

\[
\{1, 5, \ldots, 1 + 4(4j + 8)\} - \{2\} = \{-1, 3, \ldots, 3 + 4(4j + 7)\} \subseteq Q_j - Q_j.
\]

The multiples of 4 are obtained from

\[
\{1, 5, \ldots, 1 + 4(4j + 8)\} - \{1\} = \{0, 4, \ldots, 4(4j + 8)\}.
\]

For elements congruent to 2 mod 4, the only elements congruent to 2 mod 16 we are claiming to get are 2 and $2 + 16(j + 1)$; 2 is clearly in, and $2 + 16(j + 1) = 14 + 16(j + 1) - 12$.

The elements congruent to 6 modulo 16 arise can be obtained from

\[
\{24, 40, \ldots, 8 + 16j\} - \{2\} = \{22, 38, \ldots, 6 + 16j\}.
\]

The only elements congruent to 10 mod 16 we are claiming are $10 + 16(j + 1) = 12 + 16(j + 1) - 2$ and $10 = 12 - 2$. Finally the only element congruent to 14 mod 16 we claim is present is $14 + 16(j + 1) \in Q_j$.

**Corollary 13.** For the set $Q_j$ defined above we have

\[
|Q_j| = 5j + 17, |Q_j + Q_j| = 32j + 56 \text{ for } j \geq 2, |Q_j + Q_j| = 32j + 63 \text{ for } j \geq 1, \\
|Q_j - Q_j| = 26j + 61 \text{ for } j \geq 1
\]

(and $|Q_1 + Q_1| = 90$). Thus $Q_j$ is an restricted-sum-dominant set for all $j \geq 1$.

**3. The proportion of restricted-sum-dominant sets is strictly positive**

Martin and O’Bryant prove that for $n \geq 15$ the number of sum-dominant subsets of $[0, n - 1]$ is at least $(2 \times 10^{-7})2^n$ (see Theorem 1 of [5]). Their result has been
improved by Zhao [11] who shows that the proportion of sum-dominant sets tends to a limit and that that limit is at least $4.28 \times 10^{-4}$. In this section we will show that the proportion of subsets of $\{0, 1, 2, \ldots, n-1\}$ which are restricted-sum-dominant is bounded below by a much weaker constant. It may well be that Zhao’s techniques, or others, can be modified to improve the result but at least a substantial piece of computation would appear to be required and our concern at present is simply to show that a positive proportion of sets are restricted-sum-dominant sets. Note that the fact that a positive proportion of sets have more differences than restricted sums is an immediate consequence of Theorem 14 in [5]. Many lemmas etc. in what follows are very slight modifications of corresponding results in [5] and we merely present these proofs without further comment. However the construction of the two ‘fringe sets’ $U$ and $L$ is notably more involved.

**Lemma 14.** Let $n, \ell$ and $u$ be integers such that $n \geq \ell + u$. Fix $L \subseteq [0, \ell - 1]$ and $U \subseteq [n-u, n-1]$. Suppose $R$ is a uniformly randomly selected subset of $[\ell, n-u-1]$ (where each element is chosen with probability 1/2) and set $A = L \cup R \cup U$. Then for every integer $k$ satisfying $2\ell - 1 \leq k \leq n - u - 1$, we have

$$P(k \notin A \hat{+} A) = \begin{cases} \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{(k+1)/2-\ell}, & \text{if } k \text{ is odd} \\ \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{k/2-\ell}, & \text{if } k \text{ is even} \end{cases}.$$ 

**Proof.** Define an indicator variable

$$X_j = \begin{cases} 1, & \text{if } j \in A, \\ 0, & \text{otherwise}. \end{cases}$$

Since $A = L \cup R \cup U$ the $X_j$ are independent random variables for $\ell \leq j \leq n-u-1$, each taking values 0 or 1 equiprobably. For $0 \leq j \leq \ell - 1$ and $n-u \leq j \leq n-1$ the values of $X_j$ are dictated by the choices of $L$ and $U$.

Now, $k \notin A \hat{+} A$ if and only if $X_jX_{k-j} = 0$ for all $0 \leq j \leq k/2 - 1$. ($j = k/2$ would not give a restricted sum). The random variables $X_jX_{k-j}$ for $0 \leq j \leq k/2$ are independent of each other. Hence

$$P(k \notin A \hat{+} A) = \prod_{0 \leq j \leq k/2 - 1} P(X_jX_{k-j} = 0).$$

When $k$ is odd we have

$$P(k \notin A \hat{+} A) = \prod_{j=0}^{\ell-1} P(X_jX_{k-j} = 0) \prod_{j=\ell}^{(k-1)/2} P(X_jX_{k-j} = 0)$$

$$= \prod_{j \in L} P(X_{k-j} = 0) \prod_{j=\ell}^{(k-1)/2} P(X_j = 0 \text{ or } X_{k-j} = 0) = \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{(k+1)/2-\ell}. $$
When \(k\) is even

\[
\mathbb{P}(k \notin A^{\hat{+}} A) = \prod_{j=0}^{\ell-1} \mathbb{P}(X_j, X_{k-j} = 0) \prod_{j=\ell}^{k/2-1} \mathbb{P}(X_j, X_{k-j} = 0) = \prod_{j \in L} \mathbb{P}(X_{k-j} = 0) \prod_{j=\ell}^{k/2-1} \mathbb{P}(X_j = 0 \text{ or } X_{k-j} = 0) = \left( \frac{1}{2} \right)^{|L|} \left( \frac{3}{4} \right)^{k/2-\ell}.
\]

**Lemma 15.** Let \(n, \ell, u, L, U, R\) and \(A\) be defined as in Lemma 14. Then for every integer \(k\) satisfying \(n + \ell - 1 \leq k \leq 2n - 2u - 1\), we have

\[
\mathbb{P}(k \notin A^{\hat{+}} A) = \begin{cases} \left( \frac{1}{2} \right)^{|U|} \left( \frac{3}{4} \right)^{n-(k+1)/2-u}, & \text{if } k \text{ is odd,} \\ \left( \frac{1}{2} \right)^{|U|} \left( \frac{3}{4} \right)^{n-1-k/2-u}, & \text{if } k \text{ is even.} \end{cases}
\]

**Proof.** This is similar to the previous lemma, but we consider different intervals for the summands. For \(k\) odd, we have

\[
\mathbb{P}(k \notin A^{\hat{+}} A) = \prod_{j=(k+1)/2}^{n-u-1} \mathbb{P}(X_j, X_{k-j} = 0) \prod_{j=n-u}^{n-1} \mathbb{P}(X_j, X_{k-j} = 0) = \prod_{j=(k+1)/2}^{n-u-1} \mathbb{P}(X_j = 0 \text{ or } X_{k-j} = 0) \prod_{j \in U} \mathbb{P}(X_{k-j} = 0) = \left( \frac{3}{4} \right)^{n-(k+1)/2-u} \left( \frac{1}{2} \right)^{|U|}.
\]

For \(k\) even, as \(k = k/2 + k/2\) is forbidden,

\[
\mathbb{P}(k \notin A^{\hat{+}} A) = \prod_{j=k/2+1}^{n-u-1} \mathbb{P}(X_j, X_{k-j} = 0) \prod_{j=n-u}^{n-1} \mathbb{P}(X_j, X_{k-j} = 0) = \prod_{j=k/2+1}^{n-u-1} \mathbb{P}(X_j = 0 \text{ or } X_{k-j} = 0) \prod_{j \in U} \mathbb{P}(X_{k-j} = 0) = \left( \frac{3}{4} \right)^{n-1-k/2-u} \left( \frac{1}{2} \right)^{|U|}.
\]

**Proposition 16.** Let \(n, \ell\) and \(u\) be integers such that \(n \geq \ell + u\). Fix \(L \subseteq [0, \ell - 1]\) and \(U \subseteq [n - u, n - 1]\). Suppose \(R\) is a uniformly randomly selected subset of \([\ell, n-u-1]\) (where each element is chosen, independently of all other elements,
with probability 1/2) and set \( A = L \cup R \cup U \). Then for every integer \( k \) satisfying \( 2\ell - 1 \leq n - u - 1 \),
\[
\mathbb{P}(2\ell - 1, n - u - 1] \cup [n + \ell - 1, 2n - 2u - 1] \subseteq A^+A) > 1 - 8(2^{-|L|} + 2^{-|U|}).
\]

Proof. We crudely estimate
\[
\mathbb{P}(2\ell - 1, n - u - 1] \cup [n + \ell - 1, 2n - 2u - 1] \not\subseteq A^+A)
\leq \sum_{k=2\ell-1}^{n-u-1} \mathbb{P}(k \notin A^+A) + \sum_{k=n+\ell-1}^{2n-2u-1} \mathbb{P}(k \notin A^+A).
\]
The left summation of the line above can be bounded using Lemma 14:
\[
\sum_{k=2\ell-1}^{n-u-1} \mathbb{P}(k \notin A^+A) < \sum_{k>2\ell-1}^{\text{k even}} \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{(k+1)/2-\ell} + \sum_{k>2\ell-1}^{\text{k odd}} \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{k/2-\ell}
= \left(\frac{1}{2}\right)^{|L|} \sum_{m=0}^{\infty} \left(\frac{3}{4}\right)^{m} \left(\frac{1}{2}\right)^{|L|} \sum_{m=0}^{\infty} \left(\frac{3}{4}\right)^{m} = 8 \left(\frac{1}{2}\right)^{|L|}.
\]
The summation on the right can be bounded similarly, using Lemma 15, to give
\[
\sum_{k=n+\ell-1}^{2n-2u-1} \mathbb{P}(k \notin A^+A) < 8 \left(\frac{1}{2}\right)^{|U|}.
\]
Thus \( \mathbb{P}(2\ell, n - u - 1] \cup [n + \ell - 1, 2n - 2u - 1] \subseteq A^+A) \) is bounded above by
\[
8((1/2)^{|L|} + (1/2)^{|U|}),
\]
which is equivalent to the claim of Proposition 16.

We now come to the main result. Whilst the respective lower and upper fringes
\( U = \{0, 2, 3, 7, 8, 9, 10\} \) and \( L = \{n - 11, n - 10, n - 9, n - 8, n - 6, n - 3, n - 2, n - 1\} \)
used by Martin and O’Bryant are sufficient for the sum-dominant case these fall
some way short of what is required for a restricted-sum-dominant result. However
we can again use Spohn’s idea of repeating interior blocks. After a few iterations
we get the new fringes, which we shall henceforth refer to as \( L \) and \( U \), to fit with
the earlier lemmas. Thus from now on
\[
L = \{0, 2, 3, 7, 9, 10, 14, 16, 17, 21, 23, 24, 28, 30, 31, 35, 37, 38, 42, 44, 45, 49, 51, 52, 56, 57, 58, 59, 60\},
\]
\[
U = n - \{59, 58, 57, 55, 52, 51, 50, 48, 45, 44, 43, 41, 38, 37, 36, 34, 31, 30, 29, 27, 24, 23, 22, 20, 17, 16, 15, 13, 10, 9, 8, 6, 3, 2, 1\}.
\]

**Theorem 17.** For \( n \geq 120 \), the number of restricted-sum-dominant subsets of
\( [0, n - 1] \) is at least \((7.52 \times 10^{-37})2^n\).
Theorem 18. Given a subset \( S \) of an arithmetic progression \( P \) of length \( n \) for every positive integer \( n \), we have

\[
\sum_{S \subseteq P} |S+\hat{S}| = 2^n(2n-15) + \begin{cases} 26 \cdot 3^{(n-1)/2}, & \text{if } n \text{ is odd}, \\ 15 \cdot 3^{n/2}, & \text{if } n \text{ is even}. \end{cases} \tag{4}
\]

Thus \( \frac{1}{2^n} \sum_{S \subseteq P} |S+\hat{S}| \sim 2n - 15 \). This combined with Martin and O’Bryant’s Theorem 3, that \( \frac{1}{2^n} \sum_{S \subseteq P} |S-S| \sim 2n - 7 \) gives that on average the difference set has eight elements more than the restricted sumset. Details will appear in [10].
4. How much larger can the sumset be?

As in section 4 of [3] we consider this question in terms of $f(A) = \ln|A + A|/\ln|A - A|$ (and the analogous quantity $\hat{f}(A) = \ln|\hat{A} + \hat{A}|/\ln|\hat{A} - \hat{A}|$). It is known - see e.g. [1] - that $\frac{3}{4} \leq f(A) \leq 4\frac{3}{4}$. The reason for considering the ratio of logarithms rather than (say) the ratio is explained in [3] in terms of the base expansion method. Some authors, e.g. Granville in [2], prefer to use $g(A) = \ln(|A + A|/|A|)/\ln(|A - A|/|A|)$ for which the analogous bounds are $1/2 \leq g(A) \leq 2$.

Hegarty’s set $A_{15}$ is easily checked to have $f(A_{15}) = 1.0208\ldots$, which is often quoted as the largest known value of $f(A)$. In fact, the set $X$ (our $T_2$) which Hegarty uses to write $A_{15} = X \cup (X + 20)$ already does fractionally better:

**Lemma 19.** Let $X = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25\}$. Then $X + X = [0, 50]$ but $X - X = [-25, 25]\{-6, \pm 14\}$. Thus $f(X) = \ln(51)/\ln(47) \simeq 1.0212$.

**Proof.** This is just a short calculation. We do better than either of these using the sets $Q_j$ at the end of Section 2.

**Theorem 20.** There is a set $A$ of integers for which

$$f(A) = \frac{\ln(|A + A|)}{\ln(|A - A|)} \simeq 1.030597781\ldots$$

and another set $B$ of integers for which

$$\hat{f}(B) = \frac{\ln(|\hat{B} + \hat{B}|)}{\ln(|\hat{B} - \hat{B}|)} \simeq 1.028377107\ldots$$

**Proof.** Take $A = Q_{10}$ for the first claim and $A = Q_{19}$ for the second claim.

It is easy to check that neither any other $Q_j$, nor any of the $T_j$, $T'_j$, $M_j$ or $R_j$ give better results than the two $Q_j$s listed above.

The function $g$ has a slightly different behaviour, as it is monotone increasing as $j$ increases in our sequences. The result here is

**Theorem 21.** Given $\epsilon > 0$, there is a set $C$ of integers for which

$$g(C) = \frac{\ln(|C + C|/|C|)}{\ln(|C - C|/|C|)} > \frac{\ln(32/5)}{\ln(26/5)} - \epsilon \simeq 1.125944426$$

**Proof.** Take $Q_j$ for $j$ sufficiently large. (For comparison, $g(A_{15}) \simeq 1.0717$).

The corresponding suprema are $\ln(16/3)/\ln(14/3) \simeq 1.0867$ for both $(g(T_j))$ and $(g(T'_j))$, $\ln(23/4)/\ln(11/2) \simeq 1.0261$ for $(g(R_j))$ and $\ln(11/2)/\ln(5) \simeq 1.0592$ for $(g(M_j))$. None of these do as well as the supremum for the $(Q_j)$. 

Note also that because the sumsets and restricted sumsets in each of our families $T'_j, T_j, M_j, R_j$ and $Q_j$ only differ in order by a constant, the function
\[
\hat{g}(A) = \frac{\ln(|A+A|/|A|)}{\ln(|A-A|/|A|)}
\]
will give similar insights to $g$.

5. The smallest order of a restricted-sum-dominant set

We noted above that we have two restricted-sum-dominant sets of order 16, namely $T'_3$ and $M_2$: we know of no smaller examples. In this section we reduce the range in which the smallest restricted-sum-dominant set can be.

Hegarty ([3], Theorem 1) proves that no seven element subset of the integers is sum-dominant, and that up to linear transformations Conway’s set is the unique eight element sum-dominant subset of $\mathbb{Z}$. As Conway’s set is not a restricted-sum-dominant set there is no eight element restricted-sum-dominant set of integers.

Further Hegarty finds all nine-element sum-dominant sets $A$ of integers with the additional property that for some $x \in A + A$ there are at least four ordered pairs $(a, a') \in A \times A$ with $a + a' = x$. There are, up to linear transformations, nine such sets, listed in [3] as $A_2$ and $A_4$ through to $A_{11}$. It is easy to check that none of these nine sets is restricted-sum-dominant.

Thus, the only possible nine element restricted-sum-dominant sets of integers have the property that for every $x \in A + A$ there are fewer than four ordered pairs $(a, a')$ such that $x = a + a'$. This condition implies that there is no solution of $x + y = u + v$ with $x, y, u, v$ all distinct, so such a set is a weak Sidon set in the sense of Ruzsa [8].

Defining $\delta(n)$ for $n \in A - A$ to be the number of ordered pairs $(x, y)$ such that $x - y = n$, it is shown in the proof of Theorem 4.7 in [8] that for a weak Sidon set, $\delta(n) \leq 2$ whenever $n \neq 0$ and at most $2|A|$ elements $n$ have $\delta(n) = 2$.

Thus, noting 0 has $|A| = 9$ representations and putting $m = |A - A|$, \[81 \leq 9 + (2 \times 9) \times 2 + (m - 19) \Rightarrow m \geq 55\]
so if such a set were to be sum-dominant its sumset would have to have order at least 56. But of course $|A + A| \leq 9 \times 10 / 2 = 45$, and we have proven

**Theorem 22.** All sum-dominant sets of integers of order 9 are linear transformations of one of Hegarty’s nine sets $A_2$ and $A_4$ to $A_{11}$. None of these is restricted-sum-dominant, so there is no restricted-sum-dominant set of order 9.

We thus know that the smallest restricted-sum-dominant set of integers has order between 10 and 16. It appears a non-trivial computational challenge to find the order of the smallest restricted-sum-dominant set.
References


