ON SETS WITH MORE RESTRICTED SUMS THAN DIFFERENCES

David Penman
Department of Mathematical Sciences, University of Essex, Wivenhoe Park, Colchester, CO4 3SQ, United Kingdom
dbpenman@essex.ac.uk

Matthew Wells
Department of Mathematical Sciences, University of Essex, Wivenhoe Park, Colchester, CO4 3SQ, United Kingdom
mwells@essex.ac.uk

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Abstract

Given a finite set A of integers, we define its restricted sumset $A\hat{+}A$ to be the set of sums of two *distinct* elements of A - a subset of the sumset $A+A$ - and its difference set $A-A$ to be the set of differences of two elements of A. We say A is a restricted-sum-dominant set if $|A\hat{+}A| > |A-A|$. Though intuition suggests that such sets should be rare, we present various constructions of such sets and prove that a positive proportion of subsets of $\{0, 1, \ldots n-1\}$ are restricted-sum-dominant sets. As a by-product, we improve on the previous record for the maximum value of $\ln(|A+A|)/\ln(|A-A|)$, and give some related discussion.

1. Introduction

Let A be a finite set of integers. We define its *sumset* $A+A$ to be $\{a+b : a, b \in A\}$, its *difference set* $A-A$ to be $\{a-b : a, b \in A\}$ and its *restricted sumset* $A\hat{+}A$ to be $\{a+b : a \neq b, a, b \in A\}$. It is a natural intuition that, since addition is commutative but subtraction is not, that 'often' we should have $|A+A| \leq |A-A|$. However it has been known for some time that this is not always the case: for example, the set $C = \{0, 2, 3, 4, 7, 11, 12, 14\}$, which is attributed to Conway, has $|C+C| = 26$, but $|C-C| = 25$. In this paper, sets with this property are called sum-dominant: in some other literature, they are described as MSTD (for 'more sums than differences') sets, see e.g. Nathanson [6]. It is now known by work of Martin and O'Bryant [5] that sum-dominant sets are less rare than they might initially appear: they prove that, for $n \geq 15$, the proportion of subsets of $\{0, 1, 2 \ldots n-1\}$

which are sum-dominant is at least 2×10^{-7}. The constant was sharpened, and the existence of a limit shown, by Zhao [11].

In this paper we investigate what might appear to be an even more demanding condition on a set, namely what we will call the restricted-sum-dominant property.

Definition 1. *A set A of integers is said to be* restricted-sum-dominant *if* $|A\hat{+}A| > |A - A|$.

There are examples of this. For example, we find the set from Hegarty [3]

$$A_{15} = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25, 29, 32, 33, 37, 40, 41, 42, 44, 45\}$$

has $|A_{15}\hat{+}A_{15}| = 86$ whilst $|A_{15} - A_{15}| = 83$.

Clearly any restricted-sum-dominant set is sum-dominant. The converse is false as Conway's set is sum-dominant but not restricted-sum-dominant ($|C\hat{+}C| = 21$).

Note that the property of being restricted-sum-dominant is preserved when we apply a bijection of the form $x \to ax + b$ with $a, b \in \mathbb{Z}$, $a \neq 0$. It therefore suffices to consider sets $A \subset \mathbb{Z}$ with $\min(A) = 0$ and $\gcd(A) = 1$. We shall refer to such sets as being *normalised*.

The organisation of this paper is as follows. In Section 2 we exhibit several sequences of restricted-sum-dominant sets, addressing some natural questions about the relative sizes of the restricted sumset and difference sets. In Section 3, we show that a strictly positive proportion of subsets of $\{0, 1, 2, \ldots n-1\}$ are restricted-sum-dominant sets. In Section 4 we obtain a new record high value of each of

$$f(A) = \frac{\ln(|A + A|)}{\ln(|A - A|)} \text{ and } g(A) = \frac{\ln(|A + A|/|A|)}{\ln(|A - A|/|A|)}$$

and give some related discussion. Finally, in Section 5 we improve somewhat the bounds on the order of the smallest restricted-sum-dominant set.

We shall, slightly unusually, use the notation $[a, b]$, when $a < b$ are integers, to denote $\{a, a + 1, \ldots b\}$.

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2. Explicit sequences of restricted-sum-dominant sets

Our first sequence of restricted-sum-dominant sets arose by considering the set $B = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 25, 28, 30, 32, 33\}$ which appears in [7] and [9] as a set of integers with $|B\hat{+}B| > |(B - B) \setminus \{0\}|$). We then noted that replacing 33 with 29 gives a 16 element restricted-sum-dominant set (which will be T_3' below). To get the subsequent terms of the sequence, we used (here and elsewhere in the paper) the idea from [9], Conjecture 6, that repetition of certain so-called interior blocks when the set is written in order as a sequence of differences can increase the size of the sumset more than the difference set: see [9] for details.

Theorem 2. *For every integer $j \geq 1$ we define*

$$\begin{aligned} T_j' = &\{0,2\} \cup \{1,9,\ldots,1+8j\} \cup \{4,12,\ldots,4+8j\} \\ &\cup \{5,13,\ldots,5+8j\} \cup \{6+8j, 8(j+1)\}. \end{aligned}$$

Then

$$\begin{aligned} T_j' \hat{+} T_j' &= [1, 6+8(2j+1)] \setminus \{8, 8(2j+1)\}, \\ T_j' + T_j' &= [0, 8(2j+2)] \setminus \{7+8(2j+1)\} \textit{ and} \\ T_j' - T_j' &= [-8(j+1), 8(j+1)] \setminus \{\pm 6, \ldots \pm (6+8(j-1))\}. \end{aligned}$$

Proof. We deal first with the restricted sumset. Since $0 \in T_j'$, $T_j' \setminus \{0\} \subseteq T_j' \hat{+} T_j'$, giving all elements congruent to 1,4 or 5 mod 8 less than $8(j+1)$. Also

$$\begin{aligned} 8(j+1) \hat{+} \{1,9,\ldots,1+8j\} &= \{1+8(j+1),\ldots,1+8(2j+1)\} \\ 8(j+1) \hat{+} \{4,12,\ldots,4+8j\} &= \{4+8(j+1),\ldots,4+8(2j+1)\} \\ 8(j+1) \hat{+} \{5,13,\ldots,5+8j\} &= \{5+8(j+1),\ldots,5+8(2j+1)\} \end{aligned}$$

so $T_j' \hat{+} T_j'$ contains all the elements congruent modulo 8 to 1,4 or 5 stated. For integers congruent to 2 modulo 8 the restricted sumset contains 0+2 and

$$\{1,9,\ldots,1+8j\} \hat{+} \{1,9,\ldots,1+8j\} = \{10,18,\ldots,2+8(2j-1)\}$$

gives most of the rest: the two missing elements are $(4+8j)+(6+8j) = 2+8(2j+1)$ and $4+8(j-1)+6+8j = 2+8(2j)$.

For integers congruent to 3 modulo 8, note that

$$\{1,9,\ldots,1+8j\} \hat{+} (2) = \{3,11,\ldots,3+8j\}$$

and

$$(6+8j) \hat{+} \{5,13,\ldots 5+8j\} = \{3+8(j+1),\ldots 3+8(2j+1)\}.$$

For integers congruent to 6 modulo 8,

$$\{1,9,\ldots,1+8j\} \hat{+} \{5,13,\ldots,5+8j\} = \{6,14,\ldots 6+8(2j)\}$$

and $(6+8j)+8(j+1) = 6+8(2j+1) \in T_j' \hat{+} T_j'$ also. The elements congruent to 7 modulo 8 are obtained from

$$(2) + \{5,13,\ldots,5+8j\} = \{7,15,\ldots,7+8j\}$$

and

$$(6+8j) + \{1,9,\ldots,1+8j\} = \{7+8j,\ldots,7+8(2j)\}$$

in $T'_j \hat{+} T'_j$. Finally, the required multiples of 8 are obtained from

$$\{4, 12, \ldots, 4 + 8j\} \hat{+} \{4, 12, \ldots, 4 + 8j\} = \{16, 24, \ldots, 8(2j)\}.$$

Finally we note that the alleged omitted elements $0, 8$ and $8(2j + 1)$ are not in $T'_j \hat{+} T'_j$. The claim for 0 is clear, the only way to get 8 is as $4 + 4$ which is not a restricted sum, for $8(2j + 1)$ the large elements of T'_j are $5 + 8j, 6 + 8j, 8(j + 1) \in T'_j$ but $3 + 8j, 2 + 8j, 8j \notin T'_j$ so it could only be obtained as $(4 + 8j) + (4 + 8j)$ which is not a restricted sum.

Next we address the sumset $T'_j + T'_j$. All we need do here is note that $0 = 0 + 0$, $8 = 4 + 4$, $7 + 8(2j + 1)$ is still not attained and that $8(2j + 2) = 8(j + 1) + 8(j + 1)$.

We finally deal with $T'_j - T'_j$. Given that $d \in T_j - T_j \iff -d \in T_j - T_j$ it suffices to consider the positive differences. Firstly we show that $\{6, \ldots, 6 + 8(j - 1)\} \notin T'_j - T'_j$. Given that T'_j has the form

$$T'_j = \{0, 1 + 8x, 2, 4 + 8y, 5 + 8z, 6 + 8j, 8(j + 1)\}$$

(where $0 \leq x, y, z, \leq j$), considering the difference set $T'_j - T'_j$ we see that the only difference of the form $6 + 8t$ (where t is a non-negative integer) is $6 + 8j$, as stated. To confirm $T'_j - T'_j$ does contain the other elements in the interval specified, note that, as $0 \in T'_j$, $T'_j \subseteq T'_j - T'_j$. The other elements are obtained as follows:

$$\begin{aligned}
\{1, 9, \ldots, 1 + 8j\} - (1) &= \{0, 8, \ldots, 8j\} \\
\{4, 12, \ldots, 4 + 8j\} - 1 &= \{3, 11, \ldots, 3 + 8j\} \\
\{4, 12, \ldots, 4 + 8j\} - 2 &= \{2, 10, \ldots, 2 + 8j\} \\
\{12, 20, \ldots, 4 + 8j\} - (5) &= \{7, 15, \ldots, 7 + 8(j - 1)\} \\
8(j + 1) - (1) &= 7 + 8j.
\end{aligned}$$

Thus all the elements of the right-hand side are in $T'_j - T'_j$ as required. □

Corollary 3. *For every integer $j \geq 1$ the set $T'_j \subset \mathbb{Z}$ has*

$$|T'_j| = 3j + 7,\ |T'_j \hat{+} T'_j| = 16j + 12, |T'_j + T'_j| = 16j + 16 \quad \textit{and} \quad |T'_j - T'_j| = 14j + 17.$$

Therefore

$$|T'_j \hat{+} T'_j| - |T'_j - T'_j| = 2j - 5, \quad |T'_j + T'_j| - |T'_j - T'_j| = 2j - 1$$

and T'_j is an restricted-sum-dominant set for every integer $j \geq 3$.

T'_3 of order 16 is one of the two smallest restricted-sum-dominant sets we have.

The set T'_j has a superset $T_j = T'_j \cup 1 + 8(j + 1)$, which is also restricted-sum-dominant for $j \geq 3$:

Theorem 4. *For every integer $j \geq 1$ define*

$$\begin{aligned} T_j = &\{0,2\} \cup \{1,9,\ldots,1+8(j+1)\} \cup \{4,12,\ldots,4+8j\} \\ &\cup \{5,13,\ldots,5+8j\} \cup \{6+8j, 8(j+1)\}. \end{aligned}$$

Then

$$\begin{aligned} T_j \hat{+} T_j &= [1, 1+8(2j+2)] \backslash \{8, 8(2j+1), 8(2j+2)\}, \\ T_j + T_j &= [0, 2+8(2j+2)] \text{ and} \\ T_j - T_j &= [-(1+8(j+1)), 1+8(j+1)] \backslash \{\pm 6, \ldots \pm (6+8(j-1))\}. \end{aligned}$$

Proof. Firstly since $T_j \supset T_j'$ we have $T_j \hat{+} T_j \supset [1, 6+8(2j+1)] \setminus \{8, 8(2j+1)\}$. With $1+8(j+1) \in T_j$ we now also have that

$$\begin{aligned} 8(j+1) + (1+8(j+1)) &= 1+8(2j+2) \quad \text{and} \\ (6+8j) + (1+8(j+1)) &= 7+8(2j+1) \end{aligned}$$

are in $T_j \hat{+} T_j$ as well. Furthermore

$$(1+8(j+1)) + (1+8(j+1)) = 2+8(2j+2) \in T_j + T_j.$$

This completes the claims for the sumset and restricted sumset, noting that clearly 8 and $8(2j+2)$ are not in $T_j \hat{+} T_j$ and checking that $8(2j+1) \notin T_j \hat{+} T_j$.

As regards the difference set, with $0 \leq x \leq j+1$ the positive differences resulting from the introduction of the new element have the form

$$\begin{aligned} &(1+8(j+1)) - \{0, 2, 1+8x, 4+8y, 5+8z, 6+8j, 8(j+1)\} \\ =&\{1+8(j+1), 8j+7, 8(j-x+1), 8(j-y)+5, 8(j-z)+4, 3, 1, 0\}. \end{aligned}$$

This shows that $T_j - T_j = T_j' - T_j' \cup \pm(1+8(j+1))$ and the result follows. $\square$

Corollary 5. *For every integer $j \geq 1$ the set $T_j \subset \mathbb{Z}$ has*

$$|T_j| = 3j+8,\ |T_j \hat{+} T_j| = 16j+14, |T_j + T_j| = 16j+19 \quad \textit{and} \quad |T_j - T_j| = 14j+19.$$

Therefore

$$|T_j \hat{+} T_j| - |T_j - T_j| = 2j-5, \quad |T_j + T_j| - |T_j - T_j| = 2j$$

and T_j is an restricted-sum-dominant set for every integer $j \geq 3$.

In [5], Martin and O'Bryant construct, for all integers x, subsets S of $[0, 17|x|]$ with $|S+S| - |S-S| = x$. Corollary 3 shows that for each positive odd integer x there is $T_j' \subset \mathbb{Z}$ with $|T_j' + T_j'| - |T_j' - T_j'| = x$, and Corollary 5 shows each positive even integer can be expressed as the difference of the cardinalities of the sumset and the difference set of some $T_j \subset \mathbb{Z}$.

Recall that the *diameter* of a finite set A of integers is $\max(A) - \min(A)$. There is some interest in finding sets of integers of small diameter with prescribed relationships between the order of the sumset (or restricted sumset) and the difference set: see e.g. [5] Theorem 4 where sets S_x of diameter at most $17|x|$ are constructed with $|S_x + S_x| - |S_x - S_x|$ equal to x. Our sets T'_j and T_j have respective diameters $8j + 8$ and $8j + 9$, which is smaller than the sets S_x in [5] for $j \geq 3$.

Further Corollary 5 makes it clear that the difference between the size of the restricted sumset and the difference set can be any odd positive integer. We will get any even difference for $|A \hat{+} A| - |A - A|$ in our next construction. This was motivated by the sum-dominant (but not restricted-sum-dominant) set called $A_{13} = \{0, 1, 2, 4, 7, 8, 12, 14, 15, 18, 19, 20\}$ in Hegarty [3]. We exhibit, addressing his remark about the desirability of generalising A_{13}, two infinite sequences of (eventually) restricted-sum dominant sets derived from A_{13} (which shall be our R_1).

Theorem 6. *For each integer $j \geq 1$ define $R_j \subset \mathbb{Z}$ to be the set*

$$\begin{aligned} R_j = &\{1, 4\} \cup \{0, 12, \ldots, 12j\} \cup \{2, 14, \ldots, 2 + 12j\} \\ &\cup \{7, 19, \ldots, 7 + 12j\} \cup \{8, 20, \ldots, 8 + 12j\} \cup \{3 + 12j, 6 + 12j\}. \end{aligned}$$

For each integer $j \geq 2$ we have

$$\begin{aligned} R_j \hat{+} R_j &= [1, 3 + 12(2j + 1)] \setminus \{\{17, \ldots, 5 + 12(j - 1)\} \cup \{12(2j), 12(2j + 1)\}\}, \\ R_j + R_j &= [0, 4 + 12(2j + 1)] \setminus \{17, \ldots, 5 + 12(j - 1)\} \quad \textit{and} \\ R_j - R_j &= [-(8 + 12j), 8 + 12j] \setminus \{\pm 9, \ldots, \pm(9 + 12(j - 1))\}. \end{aligned}$$

Proof. We first verify the claim for the restricted sumset. For multiples of 12,

$$\{0, 12, \ldots, 12j\} \hat{+} \{0, 12, \ldots, 12j\} = \{12, 24, \ldots, 12(2j - 1)\}.$$

The elements congruent to 1 modulo 12 are given by

$$(1) + \{0, 12, \ldots, 12j\} = \{1, 13, \ldots, 1 + 12j\}.$$

and

$$(6 + 12j) + \{7, 19, \ldots, 7 + 12j\} = \{1 + 12(j + 1), \ldots, 1 + 12(2j + 1)\}.$$

For those congruent to 2 modulo 12

$$\{0, 12, \ldots, 12j\} \hat{+} \{2, 14, \ldots, 2 + 12j\} = \{2, 14, \ldots, 2 + 12(2j)\}$$

and also $(6 + 12j) + (8 + 12j) = 2 + 12(2j + 1) \in R_j \hat{+} R_j$. For 3 modulo 12 clearly $3 = 1 + 2 \in R_j \hat{+} R_j$ and the rest follow from

$$\{7, 19, \ldots, 7 + 12j\} \hat{+} \{8, 20, \ldots, 8 + 12j\} = \{15, 27, \ldots, 3 + 12(2j + 1)\}.$$

For elements congruent to 4 modulo 12, we clearly have that 4 and 16 are in $R_j \hat{+} R_j$ as well as

$$\{8, 20, \ldots, 8 + 12j\} \hat{+} \{8, 20, \ldots, 8 + 12j\} = \{28, 40, \ldots, 4 + 12(2j)\}.$$

The elements congruent to 6 modulo 12 in $R_j \hat{+} R_j$ can be obtained as the union of

$$(4) \hat{+} \{2, 14, \ldots, 2 + 12j\} = \{6, 18, \ldots, 6 + 12j\}$$

and

$$(6 + 12j) + \{0, 12, \ldots, 12j\}.$$

The elements congruent to 7 (respectively 8) modulo 12 are obtained from

$$\{0, 12, \ldots, 12j\} \hat{+} \{7, 19, \ldots, 7 + 12j\} = \{7, 19, \ldots, 7 + 12(2j)\}.$$

and

$$\{0, 12, \ldots, 12j\} \hat{+} \{8, 20, \ldots, 8 + 12j\} = \{8, 20, \ldots, 8 + 12(2j)\}.$$

For 9 (respectively 10) modulo 12 use

$$\{2, 14, \ldots, 2 + 12j\} \hat{+} \{7, 19, \ldots, 7 + 12j\} = \{9, 21, \ldots, 9 + 12(2j)\}$$

respectively

$$\{2, 14, \ldots, 2 + 12j\} \hat{+} \{8, 20, \ldots, 8 + 12j\} = \{10, 22, \ldots, 10 + 12(2j)\}.$$

Finally the elements congruent to 11 modulo 12 are obtained from

$$(4) + \{7, 19, \ldots, 7 + 12j\} = \{11, 23, \ldots, 11 + 12j\}$$

and

$$(3 + 12j) + \{8, 20, \ldots, 8 + 12j\} = \{11 + 12j, \ldots, 11 + 12(2j)\}.$$

To see that the restricted sumset does not contain any of $\{17, \ldots, 5 + 12(j-1)\}$, note that none of the sumsets of the progressions with common difference 12 give elements which are congruent to 5 modulo 12 and neither can translates of the progressions by 1 or 4). The remaining elements congruent to 5 modulo 12 are obtained as clearly $5 \in R_j \hat{+} R_j$, and also

$$(3 + 12j) + \{2, 14, \ldots, 2 + 12j\} = \{5 + 12j, \ldots, 5 + 12(2j)\} \subseteq R_j \hat{+} R_j.$$

Finally, to see that $R_j \hat{+} R_j$ does not contain $12(2j)$ or $12(2j+1)$, note that it is impossible to obtain $12(2j)$ as a sum of distinct elements of R_j since the only elements of R_j greater than $12j$ are $S = \{2 + 12j, 3 + 12j, 6 + 12j, 7 + 12j, 8 + 12j\}$ but none of the numbers in $2(12j) - S$ (namely $10 + 12(j-1), 9 + 12(j-1),$

$6+12(j-1), 5+12(j-1), 4+12(j-1))$ are in R_j. Further as $12(j+1) \notin R_j$ $12(2j+1)$ is excluded from $R_j \hat{+} R_j$. This completes the argument for $R_j \hat{+} R_j$.

However, we do have that $12j+12j = 12(2j) \in R_j+R_j$ and $(6+12j)+(6+12j) = 12(2j+1) \in R_j + R_j$, so both these missing elements get into $R_j + R_j$. Since we readily see that none of the numbers congruent to 7 mod 12 ruled out of $R_j \hat{+} R_j$ are in $R_j + R_j$ either, the sumset is as stated.

To confirm the claim for the difference set as before we consider the positive differences. Writing R_j as

$$\{1, 4, 12w, 2+12x, 7+12y, 8+12z, 3+12j, 6+12j\}$$

the remainders which occur in $R_j - R_j$ are exactly the set $[0, 11] \setminus \{9\}$. On the other hand, to see that $R_j - R_j$ contains all the claimed differences, note that as $0 \in R_j$ we have $R_j \subset R_j - R_j$. Also the right hand sides of

$$\begin{aligned}
\{0, 12, \ldots, 12j\} - (1) &= \{-1, 11, \ldots, 11 + 12(j-1)\} \\
\{2, 14, \ldots, 2+12j\} - (1) &= \{1, 13, \ldots, 1+12j\} \\
\{7, 19, \ldots, 7+12j\} - (4) &= \{3, 15, \ldots, 3+12j\} \\
\{8, 20, \ldots, 8+12j\} - (4) &= \{4, 16, \ldots, 4+12j\} \\
\{7, 19, \ldots, 7+12j\} - (2) &= \{5, 17, \ldots, 5+12j\} \\
\{7, 19, \ldots, 7+12j\} - (1) &= \{6, 18, \ldots, 6+12j\} \\
\{2, 14, \ldots, 2+12j\} - (4) &= \{-2, 10, \ldots, 10+12(j-1)\}.
\end{aligned}$$

are in the difference set which completes the claim. □

Corollary 7. *For every integer $j \geq 2$ the set $R_j \subset \mathbb{Z}$ has*

$$|R_j| = 4j+8,\ |R_j \hat{+} R_j| = 23j+14, |R_j+R_j| = 23j+18 \quad \textit{and} \quad |R_j - R_j| = 22j+17.$$

Therefore

$$|R_j \hat{+} R_j| - |R_j - R_j| = j - 3, \quad |R_j + R_j| - |R_j - R_j| = j + 1$$

and R_j is an restricted-sum-dominant set for every integer $j \geq 4$.

This indeed confirms that any positive integer can be obtained as $|R_j \hat{+} R_j| - |R_j - R_j|$.

Our fourth sequence of sets, the M_js, also has R_1 (Hegarty's A_{13}) as its first member, but this time we focus not on prescribing $|M_j \hat{+} M_j| - |M_j - M_j|$ but instead on getting a reduced diameter $9 + 11j$ rather than the diameter $8 + 12j$ of R_j. (We were first led to this family by considering Marica's sum-dominant set [4] $M = \{1, 2, 3, 5, 8, 9, 13, 15, 16\}$, normalising it and trying to expand it to a restricted-sum-dominant set).

Theorem 8. *For $j \geq 1$ define*

$$\begin{aligned} M_j = &\{0,2\} \cup \{1,12,\ldots,1+11j\} \cup \{4,15,\ldots,4+11j\} \\ &\cup \{7,18,\ldots,7+11j\} \cup \{8,19,\ldots,8+11j\} \cup \{3+11j, 9+11j\} \end{aligned}$$

We then have that

$$\begin{aligned} M_j \hat{+} M_j &= [1, 6+11(2j+1)] \setminus \{3+11(2j+1)\}, \\ M_j + M_j &= [0, 7+11(2j+1)] \text{ and} \\ M_j - M_j &= [-(9+11j), 9+11j] \setminus \{\pm 9, \ldots, \pm(9+11(j-1))\}. \end{aligned}$$

Proof. Firstly we show that $M_j \hat{+} M_j$ consists of

$$\bigcup_{a=1,2,4,5,6} \{a, a+11, \ldots, a+11(2j+1)\}$$

and

$$\bigcup_{a=3,7,8,9,10,11} \{a, a+11, \ldots, a+11(2j)\}$$

and then show that the sumset contains the additional elements claimed. In the case where $a = 1$ we have

$$\{4,15,\ldots,4+11j\} \hat{+} \{8,19,\ldots,8+11j\} = \{12,23,\ldots,12+11(2j) = 1+11(2j+1)\}$$

and $0+1 \in M_j \hat{+} M_j$ also. For the case $a = 2$

$$\{1,12,\ldots,1+11j\} \hat{+} \{1,12,\ldots,1+11j\} = \{13,24,\ldots,2+11(2j-1)\}$$

and $0+2, (4+11(j-1))+(9+11j) = 2+11(2j)$, $(4+11j)+(9+11j) = 2+11(2j+1)$ are also in $M_j \hat{+} M_j$.

For the case $a = 4$,

$$\{7,18,\ldots,7+11j\} \hat{+} \{8,19,\ldots,8+11j\} = \{15,26,\ldots,15+11(2j) = 4+11(2j+1)\}$$

and $0+4 \in M_j \hat{+} M_j$.

For the case $a = 5$,

$$\{8,19,\ldots,8+11j\} \hat{+} \{8,19,\ldots,8+11j\} = \{27,\ldots,16+11(2j-1) = 5+11(2j)\}$$

and also $5 = 1+4$, $16 = 12+4$ and $(7+11j)+(9+11j) = 5+11(2j+1)$.

For the case $a = 6$

$$\begin{aligned} (2) + \{4,15,\ldots,4+11j\} &= \{6,17,\ldots,6+11j\} \\ (9+11j) + \{8,19,\ldots,8+11j\} &= \{6+11(j+1),\ldots,6+11(2j+1)\}. \end{aligned}$$

For the case $a = 3$

$$\{7, 18, \dots, 7 + 11j\}\hat{+}\{7, 18, \dots, 7 + 11j\} = \{25, 36, \dots, 3 + 11(2j)\}$$

and $3 = 1 + 2, 14 = 2 + 12$ are in $M_j\hat{+}M_j$.

For the case $a = 7$

$$(0) + \{7, 18, \dots, 7 + 11j\} = \{7, 18, \dots, 7 + 11j\}$$
$$(3 + 11j) + \{4, 15, \dots, 4 + 11j\} = \{7 + 11j, \dots, 7 + 11(2j)\}.$$

For the case $a = 8$

$$\{1, 12, \dots, 1 + 11j\}\hat{+}\{7, 18, \dots, 7 + 11j\} = \{8, 19, \dots, 8 + 11(2j)\}.$$

For the case $a = 9$

$$\{1, 12, \dots, 1 + 11j\}\hat{+}\{8, 19, \dots, 8 + 11j\} = \{9, 20, \dots, 9 + 11(2j)\}.$$

For $a = 10$

$$(2)\hat{+}\{8, 19, \dots, 8 + 11j\} = \{10, 21, \dots, 10 + 11j\}$$
$$(3 + 11j)\hat{+}\{7, 18, \dots, 7 + 11j\} = \{10 + 11j, \dots, 10 + 11(2j)\}.$$

For $a = 11$

$$\{4, 15, \dots, 4 + 11j\}\hat{+}\{7, 18, \dots, 7 + 11j\} = \{11, 22, \dots, 11 + 11(2j)\}.$$

To see that $3 + 11(2j + 1) \notin M\hat{+}M$, if it did not we would have a sum of the form $(a + 11j) + (c + 11j) = 14 + 22j$ from elements of M_j with $a + c = 14$, however, since a and c are distinct elements of $\{1, 3, 4, 7, 8, 9\}$ this is impossible and hence $3 + 11(2j + 1) \notin M_j\hat{+}M_j$. This confirms the claim for the restricted sumset. Furthermore for each $m \in M_j$ the sumset contains $0, 2(7 + 11j) = 3 + 11(2j + 1)$ and $2(9 + 11j) = 7 + 11(2j + 1)$ which completes the claim for the sumset.

For the difference set to see that $\{\pm 9, \dots, \pm(9 + 11(j - 1))\} \notin M_j - M_j$ let

$$M_j = \{0, 2, 1 + 11w, 4 + 11x, 7 + 11y, 8 + 11z, 3 + 11j, 9 + 11j\},$$

where $0 \le w, x, y, z \le j$. It suffices to consider just the positive differences. Calculation of $M_j - M_j$ reveals that the only positive difference congruent to 9 modulo 11 is $(9 + 11j) - 0$, which is outside the range claimed.

To see that $M_j - M_j$ contains the remaining elements in the interval, firstly note that as $0 \in M_j$ we have $M_j - M_j \supset M_j$. Furthermore $M_j - M_j$ also contains the right hand sides of the following:

$$\begin{aligned}
\{1,12,\ldots,1+11j\}-(1) &= \{0,11,\ldots,11j\}\\
\{4,15,\ldots,4+11j\}-(1) &= \{3,14,\ldots,3+11j\}\\
\{7,18,\ldots,7+11j\}-(1) &= \{6,17,\ldots,6+11j\}\\
\{1,12,\ldots,1+11j\}-(2) &= \{-1,10,21,\ldots,10+11(j-1)\}\\
\{4,15,\ldots,4+11j\}-(2) &= \{2,13,\ldots,2+11j\}\\
\{7,18,\ldots,7+11j\}-(2) &= \{5,16,\ldots,5+11j\}\\
9+11j-0 &= 9+11j.
\end{aligned}$$

This completes the claim of the theorem. □

Corollary 9. *For every integer $j \geq 1$ the set $M_j \subset \mathbb{Z}$ has*

$$|M_j| = 4j+8,\ |M_j\hat{+}M_j| = 22j+16, |M_j+M_j| = 22j+19 \quad \text{and} \quad |M_j-M_j| = 20j+19.$$

Hence

$$|M_j\hat{+}M_j| - |M_j - M_j| = 2j-3, \quad |M_j + M_j| - |M_j - M_j| = 2j$$

and M_j is an restricted-sum-dominant set for every $j \geq 2$.

Note that the set M_2 has slightly smaller diameter 31 than the other 16 element restricted-sum-dominant set T_3'.

Martin and O'Bryant refer to sets with $|A+A| = |A-A|$ as *sum-difference balanced*. Similarly we can consider sets with $|A\hat{+}A| = |A-A|$ as *restricted-sum-difference balanced*. The results above show such sets exist (e.g. R_3). The smallest such set we have found has order 14: it is is

$$M' = \{0,1,2,4,7,8,12,14,15,19,22,25,26,27\},$$

so $|M'\hat{+}M'| = |[1,53]\setminus\{43,50\}| = 51$ and $|M'-M'| = |[-27,27]\setminus\{\pm 9,\pm 16\}| = 51$. We show that by taking the union of translates of M' by non-negative integer multiples of its maximum element one can obtain arbitrarily large restricted-sum-difference balanced sets.

Lemma 10. *Let $k \geq 2$ and $A_0 = A = \{0 = a_1 < a_2 < \cdots < a_k = m\} \subset \mathbb{Z}$ and $A_i = A \cup (A+m) \cup \cdots \cup (A+im)$. Then*

$$\begin{aligned}
|A_i\hat{+}A_i| - |A_{i-1}\hat{+}A_{i-1}| &= c_1 \quad \forall\, i \geq 2,\\
|A_i + A_i| - |A_{i-1} + A_{i-1}| &= c_1 \quad \forall\, i \geq 1
\end{aligned}$$

and

$$|A_i - A_i| - |A_{i-1} - A_{i-1}| = c_2 \quad \forall\, i \geq 1.$$

where c_1 and c_2 are positive constants.

Proof. We first note

$$|A_i \hat{+} A_i| - |A_{i-1} \hat{+} A_{i-1}| = |(A_i \hat{+} A_i) \setminus (A_{i-1} \hat{+} A_{i-1})|$$

and show that the right-hand side is a constant by showing that the set of new elements introduced on each iteration is a translate of the set of new elements introduced on the previous iteration. We have

$$A_i \hat{+} A_i = \cup_{r,s=0}^{i} ((A + rm) \hat{+} (A + sm)).$$

If $|r - s| \geq 2$, it is clear that $A + rm$ and $A + sm$ are disjoint so their restricted sum is just their sum. If $i - 1 \geq r = s \geq 1$, then $(A + rm) \hat{+} (A + rm) = (A + (r-1)m) + (A + (r+1)m)$. The only case needing a little thought is $|r - s| = 1$: without loss of generality, $r = s + 1$. Then

$$(A + (s+1)m) \hat{+} (A + sm) = \{a + b + (2s+1)m : \, a + m \neq b\}$$

the only way we can have $a + m = b$ is if $a = 0, b = m$, but in this case

$$(0 + (s+1)m) + (m + sm) = (m + (s+1)m) \hat{+} (0 + sm)$$

We deduce that, for all $i \geq 2$

$$A_i \hat{+} A_i = (A \hat{+} A) \cup (A + (A + m)) \cup \cdots \cup (A + A + (2i-1)m) \cup (A \hat{+} A + 2im).$$

Similarly

$$A_{i-1} \hat{+} A_{i-1} = (A \hat{+} A) \cup (A + A + m) \cup \cdots \cup (A \hat{+} A + (2i-2)m).$$

Now some elements of $(A + A + (2i-2)m) \setminus (A \hat{+} A + (2i-2)m)$ may be in $A + A + (2i-3)m$ and thus in $A_{i-1} \hat{+} A_{i-1}$. (Translates of $A + A$ by less than $(2i-3)m$ need not be considered). We have

$$\begin{aligned}(A_i \hat{+} A_i) \setminus (A_{i-1} \hat{+} A_{i-1}) = ((A + A + (2i-2)m) \cup (A + A + (2i-1)m) \cup \\ (A \hat{+} A + 2im)) \setminus ((A + A + (2i-3)m) \cup (A \hat{+} A + (2i-2)m)).\end{aligned} \quad (1)$$

Likewise

$$\begin{aligned}(A_{i+1} \hat{+} A_{i+1}) \setminus (A_i \hat{+} A_i) = ((A + A + 2im) \cup (A + A + (2i+1)m) \cup \\ (A \hat{+} A + (2i+2)m)) \setminus ((A + A + (2i-1)m) \cup (A \hat{+} A + (2i)m)).\end{aligned} \quad (2)$$

The right-hand side of (2) is a translation of the right-hand side of (1) by $2m$. (To see this, note it is easy to check for sets of integers that if $C_i + 2m = C_{i+1}$ and $D_i + 2m = D_{i+1}$, then $(C_i \setminus D_i) + 2m = (C_{i+1} \setminus D_{i+1})$: apply this with the obvious choices of C_i and D_i). Thus

$$(A_{i+1} \hat{+} A_{i+1}) \setminus (A_i \hat{+} A_i) = ((A_i \hat{+} A_i) \setminus (A_{i-1} \hat{+} A_{i-1})) + 2m.$$

Since translation by a constant leaves the cardinality of the set difference unaltered it follows that

$$|(A_{i+1}\hat{+}A_{i+1}) \setminus (A_i\hat{+}A_i)| = |(A_i\hat{+}A_i) \setminus (A_{i-1}\hat{+}A_{i-1})|$$

as required.

To see that

$$|A_i + A_i| - |A_{i-1} + A_{i-1}| = |A_i\hat{+}A_i| - |A_{i-1}\hat{+}A_{i-1}| \tag{3}$$

for all $i \geq 1$ we show that the number of additional elements $A_i + A_i$ contains is constant. All the elements of

$$(A + A) \setminus (A\hat{+}A)$$

except for $2m$, which is in $A_i\hat{+}A_i$ for $i \geq 1$ due to $0+2m$, are excluded from $A_i\hat{+}A_i$ for all $i \geq 1$. Similarly the elements of

$$((A + A) \setminus (A\hat{+}A)) + 2im$$

except for $2im$ are excluded from $A_i\hat{+}A_i$. This means that for all $i \geq 1$

$$|A_i + A_i| - |A_i\hat{+}A_i| = 2(|(A + A) \setminus (A\hat{+}A)| - 1).$$

In other words the difference between the cardinalities of the sumset and the restricted sumset is a constant for all $i \geq 1$ and (3) holds.

To verify the claim for the difference set, write

$$A_i - A_i = \cup_{j=-i}^{i}(A - A + jm).$$

Thus we have

$$\begin{aligned} &(A_i - A_i) \setminus (A_{i-1} - A_{i-1}) \\ = (A - A - im) \cup (A - A + im) &\setminus \cup_{j=-(i-1)}^{i-1}(A - A - jm). \end{aligned}$$

But the only sets in $\cup_{j=-(i-1)}^{i-1}(A - A - jm)$ which could intersect $(A - A - im)$ or $(A - A + im)$ are for $j = (i - 1)$, $j = (i - 2)$ (which will intersect $A - A - im$ in precisely the one element $(1-i)m$), $j = -(i-2)$ (which will intersect it in precisely the one element $(i - 1)m$) and $j = -(i - 1)$. Thus for all $i \geq 1$

$$\begin{aligned} (A_i - A_i) \setminus (A_{i-1} - A_{i-1}) =& ((A - (A + im)) \setminus (A - (A + (i - 1)m))) \\ & \cup ((A - A + im) \setminus (A - A + (i - 1)m)). \end{aligned}$$

Similarly

$$\begin{aligned} (A_{i+1} - A_{i+1}) \setminus (A_i - A_i) =& ((A - (A + (i + 1)m)) \setminus (A - (A + im))) \\ & \cup ((A - A + (i + 1)m) \setminus (A - A + im)). \end{aligned}$$

The sets $(A-(A+(i+1)m))\setminus(A-(A+im))$ and $(A-A+(i+1)m)\setminus(A-A+im)$ are disjoint for all $i \geq 1$. Also $(A-(A+(i+1)m))\setminus(A-(A+im))$ is a translation of $(A-(A+im))\setminus(A-(A+(i-1)m))$ by $-m$ and $(A-A+(i+1)m)\setminus(A-A+im)$ is a translation of $(A-A+im)\setminus(A-A+(i-1)m)$ by m. These translations leave the cardinalities of the sets unchanged, therefore

$$|(A_{i+1}-A_{i+1})\setminus(A_i-A_i)| = |(A_i-A_i)\setminus(A_{i-1}-A_{i-1})|$$

and the overall result follows. □

Setting $M_1' = M' \cup (M'+27)$ we easily check

$$|M_1' \hat{+} M_1'| = |[1,107]\setminus\{97,104\}| = |[-54,54]\setminus\{\pm 36, \pm 43\}| = |M_1' - M_1'|$$

and $M_2' = M' \cup (M'+27) \cup (M'+54)$ gives

$$|M_2' \hat{+} M_2'| = |[1,161]\setminus\{151,158\}| = |[-81,81]\setminus\{\pm 63, \pm 70\}| = |M_2' - M_2'|.$$

It follows from Lemma 10 that

Corollary 11. *There exist arbitrarily large restricted-sum-difference balanced subsets of* $\mathbb{Z}$.

Our final sequence of restricted-sum-dominant sets is constructed with a view to obtaining high values of $f(A)$ as defined in the introduction. Again, this set is a modification of one in [9], who describes $Q_j \setminus \{1+4(4j+7)\}$ for $j = 1, 2, 3$ as sets giving large sumset relative to the difference set. Including $1+4(4j+7)$ increases the sumset but does not change the difference set.

Theorem 12. *Let*

$$\begin{aligned} Q_j =& \{0,2,4,12\} \cup \{1,5,\ldots,1+4(4j+8)\} \cup \{24,40,\ldots,8+16j\} \\ & \cup \{4+16(j+1), 12+16(j+1), 14+16(j+1), 16(j+2)\} \end{aligned}$$

for an integer $j \geq 1$. *Then*

$$\begin{aligned} Q_j \hat{+} Q_j =& [1, 1+4(8j+16)] \\ & \setminus \{8,20,32,48,4(8j+4),4(8j+8),4(8j+11),4(8j+14),4(8j+16)\} \end{aligned}$$

for $j \geq 2$, *whilst*

$$Q_j + Q_j = [0, 2+4(8j+16)] \setminus \{20, 32, 4(8j+8), 4(8j+11)\}$$

for $j \geq 1$ *and*

$$\begin{aligned} Q_j - Q_j =& [-(1+4(4j+8)), 1+4(4j+8)] \setminus \pm\{\{6\},\{14,\ldots,14+16j\}, \\ & \{18,\ldots,2+16j\},\{26,\ldots,10+16j\}, 6+16(j+1)\} \end{aligned}$$

for $j \geq 1$.

Proof. To verify these claims, consider elements of Q_j in terms of the union of

$$Q_{\text{odd}} = \{1, 5, \ldots, 1 + 4(4j + 8)\}$$

and

$$\begin{aligned} Q_{\text{even}} =& \{0, 2, 4, 12\} \cup \{24, \ldots, 8 + 16j\} \\ & \cup \{4 + 16(j + 1), 12 + 16(j + 1), 14 + 16(j + 1), 16(j + 2)\}. \end{aligned}$$

Firstly $Q_j \hat{+} Q_j$ contains all the odd numbers in the interval since we have

$$\begin{aligned} (0)\hat{+}\{1, 5, \ldots, 1 + 4(4j + 8)\} =& \{1, 5, \ldots, 1 + 4(4j + 8)\} \\ 16(j + 2)\hat{+}\{1, 5, \ldots, 1 + 4(4j + 8)\} =& \{1 + 4(4j + 8), 5 + 4(4j + 8), \\ & \ldots, 1 + 4(8j + 16)\} \\ (2)\hat{+}\{1, 5, \ldots, 1 + 4(4j + 8)\} =& \{3, 7, \ldots, 3 + 4(4j + 8)\} \\ 14 + 16(j + 1)\hat{+}\{1, 5, \ldots, 1 + 4(4j + 8)\} =& \{3 + 4(4j + 7), 7 + 4(4j + 7), \\ & \ldots, 3 + 4(8j + 15)\}. \end{aligned}$$

The union of the right hand sides of the above is indeed

$$\{1, 3, \ldots, 3 + 4(8j + 15), 1 + 4(8j + 16)\} = \{1, 3, \ldots, 1 + 2(4(4j + 8))\}.$$

To see that the sumset contains all the even elements claimed, note first that $Q_{\text{odd}} \hat{+} Q_{\text{odd}}$ gives the following elements congruent to 2 mod 4:

$$Q_{\text{odd}} \hat{+} Q_{\text{odd}} = \{6, 10, \ldots, 2 + 4(8j + 15)\} \subseteq Q_j \hat{+} Q_j.$$

Clearly $0 + 2$ is also in $Q_j \hat{+} Q_j$, however whilst $\max(Q_j + Q_j) = 2 + 4(8j + 16)$ this is not in the restricted sumset. As regards the multiples of four, clearly none of these can be obtained from $Q_{\text{odd}} \hat{+} Q_{\text{odd}}$ or $Q_{\text{odd}} \hat{+} Q_{\text{even}}$. To confirm the elements we claim to be excluded cannot be present note that Q_{even} is symmetric w.r.t. 16(j+2): $Q_{even} = 16(j + 2) - Q_{even}$. Hence $Q_{even} \hat{+} Q_{even} = 16(2j + 4) - (Q_{even} \hat{+} Q_{even})$ and $Q_{even} + Q_{even} = 16(2j+4) - (Q_{even} + Q_{even})$. The restricted sumset of the elements of Q_{even} less than or equal to 32 is

$$\{0, 2, 4, 12, 24\} \hat{+} \{0, 2, 4, 12, 24\} = \{2, 4, 6, 12, 14, 16, 24, 26, 28, 36\}.$$

Thus $0, 8, 20, 32$ and 48 are excluded from $Q_j \hat{+} Q_j$. Whilst $Q_j + Q_j$ contains $0, 8$ and 48 as the doubles of $0, 4$ and 24 respectively, it is easy to check that neither 20 nor 32 are in $Q_j + Q_j$. By symmetry

$$16(2j+4) - \{0, 8, 20, 32, 48\} = \{4(8j+4), 4(8j+8), 4(8j+11), 4(8j+14), 4(8j+16)\}$$

which has empty intersection with $Q_j \hat{+} Q_j$.

It remains to show that all other (relevant) multiples of 4 are in the (restricted) sumset; we consider the cases 0,4,8 and 12 modulo 16 separately. We have the following multiples of 16 in $Q_j \hat{+} Q_j$:

$$\{24, 40, \ldots, 16j+8\} \hat{+} \{24, 40, \ldots, 16j+8\} = \{64, 80, \ldots, 16(2j)\}$$
$$(4+16(j+1)) \hat{+} (12+16(j+1)) = 4(8j+12) = 16(2j+3).$$

Furthermore $Q_j + Q_j$ contains 48 and $16(2j+1) = 2(16j+8)$ and also $16(j+2) + 16(j+2) = 4(8j+16) = 16(2j+4)$. We already saw $16(2j+2) = 4(8j+8)$ is not in $Q_j + Q_j$.

We obtain those congruent to 4 modulo 16 from

$$(12) \hat{+} \{24, 40, \ldots, 16j+8\} = \{36, 52, \ldots, 4+16(j+1)\}$$
$$(4) \hat{+} (16(j+2)) = 4+16(j+2)$$
$$(12+16(j+1)) \hat{+} \{24, \ldots, 8+16j\} = \{4+16(j+3), \ldots, 4+16(2j+2)\}$$
$$(4+16(j+1)) \hat{+} (16(j+2)) = 4+16(2j+3).$$

The elements congruent to 8 modulo 16 are given by

$$(0) \hat{+} \{24, 40, \ldots, 8+16j\} = \{24, 40, \ldots, 8+16j\}$$
$$(4) \hat{+} (4+16(j+1)) = 8+16(j+1)$$
$$(12) \hat{+} (12+16(j+1)) = 8+16(j+2)$$
$$(16(j+2)) \hat{+} \{24, 40, \ldots, 8+16j\} = \{8+16(j+3), \ldots, 8+16(2j+2)\}.$$

Also $(12+16(j+1)) + (12+16(j+1)) = 8+16(2j+3) \in Q_j + Q_j$. Finally the elements congruent to 12 modulo 16 follow from

$$(4) \hat{+} \{24, \ldots, 8+16j\} = \{28, \ldots, 12+16j\}$$
$$(0) \hat{+} (12+16(j+1)) = 12+16(j+1)$$
$$(4+16(j+1)) \hat{+} \{24, \ldots, 8+16j\} = \{12+16(j+2), \ldots, 12+16(2j+1)\}$$
$$(12+16(j+1)) \hat{+} (16(j+2)) = 12+16(2j+3).$$

We now deal with the difference set. Again, it suffices to consider the non-negative differences. Since all the differences which we claim are excluded are even we need only consider differences of pairs of elements of Q_j of the same parity and therefore divide into cases accordingly. The non-negative elements of $Q_{\text{odd}} - Q_{\text{odd}}$ are

$$\{0, 4, \ldots, 4(4j+8)\}.$$

The even elements of Q_j have the form

$$Q_{\text{even}} = \{0, 2, 4, 12, 8+16x, 4+16(j+1), 12+16(j+1), 14+16(j+1), 16(j+2)\}$$

where $x \in \mathbb{Z}$ with $1 \le x \le j$. The positive differences of the elements of Q_{even} are

$$\{2, 4, 8, 10, 12, 12 + 16(x-1), 4 + 16x, 6 + 16x, 8 + 16x,$$
$$12 + 16(j-x), 4 + 16(j-x+1), 6 + 16(j-x+1), 8 + 16(j-x+1),$$
$$8 + 16j, 16(j+1), 2 + 16(j+1), 4 + 16(j+1), 8 + 16(j+1),$$
$$10 + 16(j+1), 12 + 16(j+1), 14 + 16(j+1), 16(j+2)\}.$$

Thus none of the differences in $Q_j - Q_j$ have the form which we claim is excluded. To confirm the presence of the remaining differences we have that all the differences congruent to 1 modulo 4 are present since

$$\{1, 5, \ldots, 1 + 4(4j+8)\} - \{0\} = \{1, 5, \ldots, 1 + 4(4j+8)\} \subseteq Q_j - Q_j.$$

The elements congruent to 3 modulo 4 follow from

$$\{1, 5, \ldots, 1 + 4(4j+8)\} - \{2\} = \{-1, 3, \ldots, 3 + 4(4j+7)\} \subseteq Q_j - Q_j.$$

The multiples of 4 are obtained from

$$\{1, 5, \ldots, 1 + 4(4j+8)\} - \{1\} = \{0, 4, \ldots, 4(4j+8)\}.$$

For elements congruent to 2 mod 4, the only elements congruent to 2 mod 16 we are claiming to get are 2 and $2 + 16(j+1)$; 2 is clearly in, and $2 + 16(j+1) = 14 + 16(j+1) - 12$.

The elements congruent to 6 modulo 16 arise can be obtained from

$$\{24, 40, \ldots, 8 + 16j\} - \{2\} = \{22, 38, \ldots, 6 + 16j\}.$$

The only elements congruent to 10 mod 16 we are claiming are $10 + 16(j+1) = 12 + 16(j+1) - 2$ and $10 = 12 - 2$. Finally the only element congruent to 14 mod 16 we claim is present is $14 + 16(j+1) \in Q_j$. □

Corollary 13. *For the set Q_j defined above we have*

$$|Q_j| = 5j + 17, |Q_j \hat{+} Q_j| = 32j + 56 \text{ for } j \ge 2, |Q_j + Q_j| = 32j + 63 \text{ for } j \ge 1,$$
$$|Q_j - Q_j| = 26j + 61 \text{ for } j \ge 1$$

(and $|Q_1 \hat{+} Q_1| = 90$). Thus Q_j is an restricted-sum-dominant set for all $j \ge 1$.

3. The proportion of restricted-sum-dominant sets is strictly positive

Martin and O'Bryant prove that for $n \ge 15$ the number of sum-dominant subsets of $[0, n-1]$ is at least $(2 \times 10^{-7})2^n$ (see Theorem 1 of [5]). Their result has been improved by Zhao [11] who shows that the proportion of sum-dominant sets tends to a limit and that that limit is at least 4.28×10^{-4}. In this section we will show that the proportion of subsets of $\{0,1,2,\dots n-1\}$ which are restricted-sum-dominant is bounded below by a much weaker constant. It may well be that Zhao's techniques, or others, can be modified to improve the result but at least a substantial piece of computation would appear to be required and our concern at present is simply to show that a positive proportion of sets are restricted-sum-dominant sets. Note that the fact that a positive proportion of sets have more differences than restricted sums is an immediate consequence of Theorem 14 in [5]. Many lemmas etc. in what follows are very slight modifications of corresponding results in [5] and we merely present these proofs without further comment. However the construction of the two 'fringe sets' U and L is notably more involved.

Lemma 14. *Let n,ℓ and u be integers such that $n\geq \ell+u$. Fix $L\subseteq[0,\ell-1]$ and $U\subseteq[n-u,n-1]$. Suppose R is a uniformly randomly selected subset of $[\ell,n-u-1]$ (where each element is chosen with probability 1/2) and set $A=L\cup R\cup U$. Then for every integer k satisfying $2\ell-1\leq k\leq n-u-1$, we have*

$$\mathbb{P}(k\notin A\hat{+}A)=\begin{cases}\left(\frac{1}{2}\right)^{|L|}\left(\frac{3}{4}\right)^{(k+1)/2-\ell}, & \text{if } k \text{ is odd},\\ \left(\frac{1}{2}\right)^{|L|}\left(\frac{3}{4}\right)^{k/2-\ell}, & \text{if } k \text{ is even}.\end{cases}$$

Proof. Define an indicator variable

$$X_j=\begin{cases}1, \text{ if } j\in A,\\ 0, \text{ otherwise}.\end{cases}$$

Since $A=L\cup R\cup U$ the X_j are independent random variables for $\ell\leq j\leq n-u-1$, each taking values 0 or 1 equiprobably. For $0\leq j\leq \ell-1$ and $n-u\leq j\leq n-1$ the values of X_j are dictated by the choices of L and U.

Now, $k\notin A\hat{+}A$ if and only if $X_jX_{k-j}=0$ for all $0\leq j\leq k/2-1$. ($j=k/2$ would not give a *restricted* sum). The random variables X_jX_{k-j} for $0\leq j\leq k/2$ are independent of each other. Hence

$$\mathbb{P}(k\notin A\hat{+}A)=\Pi_{0\leq j\leq k/2-1}\mathbb{P}(X_jX_{k-j}=0).$$

When k is odd we have

$$\begin{aligned}\mathbb{P}(k\notin A\hat{+}A)&=\prod_{j=0}^{\ell-1}\mathbb{P}(X_jX_{k-j}=0)\prod_{j=\ell}^{(k-1)/2}\mathbb{P}(X_jX_{k-j}=0)\\ &=\prod_{j\in L}\mathbb{P}(X_{k-j}=0)\prod_{j=\ell}^{(k-1)/2}\mathbb{P}(X_j=0\text{ or }X_{k-j}=0)=\left(\frac{1}{2}\right)^{|L|}\left(\frac{3}{4}\right)^{(k+1)/2-\ell}.\end{aligned}$$

When k is even

$$\begin{aligned}\mathbb{P}(k \notin A\hat{+}A) &= \prod_{j=0}^{\ell-1} \mathbb{P}(X_j X_{k-j} = 0) \prod_{j=\ell}^{k/2-1} \mathbb{P}(X_j X_{k-j} = 0) \\ &= \prod_{j\in L} \mathbb{P}(X_{k-j} = 0) \prod_{j=\ell}^{k/2-1} \mathbb{P}(X_j = 0 \text{ or } X_{k-j} = 0) = \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{k/2-\ell}.\end{aligned}$$

□

Lemma 15. *Let n, ℓ, u, L, U, R and A be defined as in Lemma 14. Then for every integer k satisfying $n + \ell - 1 \le k \le 2n - 2u - 1$, we have*

$$\mathbb{P}(k \notin A\hat{+}A) = \begin{cases} \left(\frac{1}{2}\right)^{|U|} \left(\frac{3}{4}\right)^{n-(k+1)/2-u}, & \text{if } k \text{ is odd,} \\ \left(\frac{1}{2}\right)^{|U|} \left(\frac{3}{4}\right)^{n-1-k/2-u}, & \text{if } k \text{ is even.} \end{cases}$$

Proof. This is similar to the previous lemma, but we consider different intervals for the summands. For k odd, we have

$$\begin{aligned}\mathbb{P}(k \notin A\hat{+}A) &= \prod_{j=(k+1)/2}^{n-u-1} \mathbb{P}(X_j X_{k-j} = 0) \prod_{j=n-u}^{n-1} \mathbb{P}(X_j X_{k-j} = 0) \\ &= \prod_{j=(k+1)/2}^{n-u-1} \mathbb{P}(X_j = 0 \text{ or } X_{k-j} = 0) \prod_{j\in U} \mathbb{P}(X_{k-j} = 0) \\ &= \left(\frac{3}{4}\right)^{n-(k+1)/2-u} \left(\frac{1}{2}\right)^{|U|}.\end{aligned}$$

For k even, as $k = k/2 + k/2$ is forbidden,

$$\begin{aligned}\mathbb{P}(k \notin A\hat{+}A) &= \prod_{j=k/2+1}^{n-u-1} \mathbb{P}(X_j X_{k-j} = 0) \prod_{j=n-u}^{n-1} \mathbb{P}(X_j X_{k-j} = 0) \\ &= \prod_{j=k/2+1}^{n-u-1} \mathbb{P}(X_j = 0 \text{ or } X_{k-j} = 0) \prod_{j\in U} \mathbb{P}(X_{k-j} = 0) \\ &= \left(\frac{3}{4}\right)^{n-1-k/2-u} \left(\frac{1}{2}\right)^{|U|}.\end{aligned}$$

□

Proposition 16. *Let n, ℓ and u be integers such that $n \ge \ell + u$. Fix $L \subseteq [0, \ell - 1]$ and $U \subseteq [n - u, n - 1]$. Suppose R is a uniformly randomly selected subset of $[\ell, n - u - 1]$ (where each element is chosen, independently of all other elements,* *with probability 1/2) and set $A = L \cup R \cup U$. Then for every integer k satisfying $2\ell - 1 \le n - u - 1$,*

$$\mathbb{P}([2\ell-1, n-u-1] \cup [n+\ell-1, 2n-2u-1] \subseteq A\hat{+}A) > 1 - 8(2^{-|L|} + 2^{-|U|}).$$

Proof. We crudely estimate

$$\begin{aligned}
&\mathbb{P}([2\ell-1, n-u-1] \cup [n+\ell-1, 2n-2u-1] \not\subseteq A\hat{+}A) \\
&\le \sum_{k=2\ell-1}^{n-u-1} \mathbb{P}(k \notin A\hat{+}A) + \sum_{k=n+\ell-1}^{2n-2u-1} \mathbb{P}(k \notin A\hat{+}A).
\end{aligned}$$

The left summation of the line above can be bounded using Lemma 14:

$$\begin{aligned}
\sum_{k=2\ell-1}^{n-u-1} \mathbb{P}(k \notin A\hat{+}A) &< \sum_{\substack{k \ge 2\ell-1 \\ k \text{ odd}}} \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{(k+1)/2-\ell} + \sum_{\substack{k \ge 2\ell-1 \\ k \text{ even}}} \left(\frac{1}{2}\right)^{|L|} \left(\frac{3}{4}\right)^{k/2-\ell} \\
&= \left(\frac{1}{2}\right)^{|L|} \sum_{m=0}^{\infty} \left(\frac{3}{4}\right)^{m} + \left(\frac{1}{2}\right)^{|L|} \sum_{m=0}^{\infty} \left(\frac{3}{4}\right)^{m} = 8\left(\frac{1}{2}\right)^{|L|}.
\end{aligned}$$

The summation on the right can be bounded similarly, using Lemma 15, to give

$$\sum_{k=n+\ell-1}^{2n-2u-1} \mathbb{P}(k \notin A\hat{+}A) < 8\left(\frac{1}{2}\right)^{|U|}.$$

Thus $\mathbb{P}([2\ell, n-u-1] \cup [n+\ell-1, 2n-2u-1] \subseteq A\hat{+}A)$ is bounded above by $8((1/2)^{|L|} + (1/2)^{|U|})$, which is equivalent to the claim of Proposition 16. ☐

We now come to the main result. Whilst the respective lower and upper fringes $U = \{0, 2, 3, 7, 8, 9, 10\}$ and $L = \{n-11, n-10, n-9, n-8, n-6, n-3, n-2, n-1\}$ used by Martin and O'Bryant are sufficient for the sum-dominant case these fall some way short of what is required for a restricted-sum-dominant result. However we can again use Spohn's idea of repeating interior blocks. After a few iterations we get the new fringes, which we shall henceforth refer to as L and U, to fit with the earlier lemmas. Thus from now on

$$\begin{aligned}
L = \{&0, 2, 3, 7, 9, 10, 14, 16, 17, 21, 23, 24, 28, 30, 31, 35, \\
&37, 38, 42, 44, 45, 49, 51, 52, 56, 57, 58, 59, 60\}, \\
U = n - \{&59, 58, 57, 55, 52, 51, 50, 48, 45, 44, 43, 41, 38, 37, 36, 34, 31, \\
&30, 29, 27, 24, 23, 22, 20, 17, 16, 15, 13, 10, 9, 8, 6, 3, 2, 1\}.
\end{aligned}$$

Theorem 17. *For $n \ge 120$, the number of restricted-sum-dominant subsets of $[0, n-1]$ is at least $(7.52 \times 10^{-37})2^n$.*

Proof. With L and U as just defined, one can check that

$$U - L = [n-119, n-1] \setminus \{n-7, n-14, n-21, n-28, \\ n-35, n-42, n-49, n-56\}.$$

Now since $n-7, n-14, n-21, n-28, n-35, n-42, n-49, n-56 \notin U-L$ it follows that $\pm(n-7), \pm(n-14), \pm(n-21), \pm(n-28), \pm(n-35), \pm(n-42), \pm(n-49), \pm(n-56) \notin A - A \subseteq [-(n-1), n-1]$. With eight pairs of differences excluded from $A - A$ we have $|A - A| \le 2n - 17$. On the other hand one can check

$$\begin{aligned} L\hat{+}L =& [0, 120] \setminus \{0, 1, 4, 6, 8, 15, 22, 29, 36, 43, 50, 120\} \\ U\hat{+}L = U + L =& [n-59, n+59] \\ U\hat{+}U =& [2n-118, 2n-2] \setminus \{2n-118, 2n-6, 2n-2\}. \end{aligned}$$

Hence for $120 \le n \le 178$ we have that $A\hat{+}A$ contains

$$[0, 2n-2] \setminus \{0, 1, 4, 6, 8, 15, 22, 29, 36, 43, 50, 120, 2n-118, 2n-6, 2n-2\}$$

so that $|A\hat{+}A| \ge 2n - 16$. There are $n - 120$ numbers between 61 and $n - 60$ inclusive. Therefore the number of such A is 2^{n-120}.

For $n \ge 178$ applying Proposition 16 with $\ell = 61$ and $u = 59$ implies that when A is chosen uniformly randomly from all such sets, the probability that $A\hat{+}A$ contains $[61, n-60] \cup [n+60, 2n-119]$ is at least

$$1 - 8(2^{-|L|} + 2^{-|U|}) = 1 - 8(2^{-29} + 2^{-35}) = \frac{4294967231}{4294967296}.$$

That is, there are at least $2^{n-120}\frac{4294967231}{4294967296} > (7.52 \times 10^{-37})2^n$ such sets A with

$$A\hat{+}A = [0, 2n-2] \setminus \{0, 1, 4, 6, 8, 15, 22, 29, 36, 43, 50, 120, 2n-118, 2n-6, 2n-2\},$$

whilst at the same time eight pairs of differences are excluded from $A - A$. Thus all such sets A are restricted-sum-dominant sets. □

Martin and O'Bryant's Lemma 7 and Theorem 16 for a subset S of an arithmetic progression of length n can also be adapted to give the following result.

Theorem 18. *Given a subset S of an arithmetic progression P of length n for every positive integer n, we have*

$$\sum_{S \subseteq P} |S\hat{+}S| = 2^n(2n-15) + \begin{cases} 26 \cdot 3^{(n-1)/2}, & \text{if } n \text{ is odd,} \\ 15 \cdot 3^{n/2}, & \text{if } n \text{ is even.} \end{cases} \tag{4}$$

Thus $\frac{1}{2^n}\sum_{S \subseteq P}|S\hat{+}S| \sim 2n - 15$. This combined with Martin and O'Bryant's Theorem 3, that $\frac{1}{2^n}\sum_{S \subseteq P}|S - S| \sim 2n - 7$ gives that on average the difference set has eight elements more than the restricted sumset. Details will appear in [10].

4. How much larger can the sumset be?

As in section 4 of [3] we consider this question in terms of $f(A) = \ln|A+A|/\ln|A-A|$ (and the analogous quantity $\hat{f}(A) = \ln|A\hat{+}A|/\ln|A-A|$). It is known - see e.g. [1] - that $\frac{3}{4} \leq f(A) \leq \frac{4}{3}$. The reason for considering the ratio of logarithms rather than (say) the ratio is explained in [3] in terms of the *base expansion method*. Some authors, e.g. Granville in [2], prefer to use $g(A) = \ln(|A+A|/|A|)/\ln(|A-A|/|A|)$ for which the analogous bounds are $1/2 \leq g(A) \leq 2$.

Hegarty's set A_{15} is easily checked to have $f(A_{15}) = 1.0208\ldots$, which is often quoted as the largest known value of $f(A)$. In fact, the set X (our T_2) which Hegarty uses to write $A_{15} = X \cup (X+20)$ already does fractionally better:

Lemma 19. *Let $X = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25\}$. Then $X + X = [0, 50]$ but $X - X = [-25, 25]\backslash\{\pm 6, \pm 14\}$. Thus $f(X) = \ln(51)/\ln(47) \simeq 1.0212$.*

Proof. This is just a short calculation. □

We do better than either of these using the sets Q_j at the end of Section 2.

Theorem 20. *There is a set A of integers for which*

$$f(A) = \frac{\ln(|A+A|)}{\ln(|A-A|)} \simeq 1.030597781\ldots$$

and another set B of integers for which

$$\hat{f}(B) = \frac{\ln(|B\hat{+}B|)}{\ln(|B-B|)} \simeq 1.028377107\ldots$$

Proof. Take $A = Q_{10}$ for the first claim and $A = Q_{19}$ for the second claim. □

It is easy to check that neither any other Q_j, nor any of the T_j, T'_j, M_j or R_j give better results than the two Q_js listed above.

The function g has a slightly different behaviour, as it is monotone increasing as j increases in our sequences. The result here is

Theorem 21. *Given $\epsilon > 0$, there is a set C of integers for which*

$$g(C) = \frac{\ln(|C+C|/|C|)}{\ln(|C-C|/|C|)} > \frac{\ln(32/5)}{\ln(26/5)} - \epsilon \simeq 1.125944426$$

Proof. Take Q_j for j sufficiently large. □

(For comparison, $g(A_{15}) \simeq 1.0717$).

The corresponding suprema are $\ln(16/3)/\ln(14/3) \simeq 1.0867$ for both $(g(T'_j))$ and $(g(T_j))$, $\ln(23/4)/\ln(11/2) \simeq 1.0261$ for $(g(R_j))$ and $\ln(11/2)/\ln(5) \simeq 1.0592$ for $(g(M_j))$. None of these do as well as the supremum for the (Q_j).

Note also that because the sumsets and restricted sumsets in each of our families T_j', T_j, M_j, R_j and Q_j only differ in order by a constant, the function

$$\hat{g}(A) = \frac{\ln(|A\hat{+}A|/|A|)}{\ln(|A-A|/|A|)}$$

will give similar insights to g.

5. The smallest order of a restricted-sum-dominant set

We noted above that we have two restricted-sum-dominant sets of order 16, namely T_3' and M_2: we know of no smaller examples. In this section we reduce the range in which the smallest restricted-sum-dominant set can be.

Hegarty ([3], Theorem 1) proves that no seven element subset of the integers is sum-dominant, and that up to linear transformations Conway's set is the unique eight element sum-dominant subset of $\mathbb{Z}$. As Conway's set is not a restricted-sum-dominant set there is no eight element restricted-sum-dominant set of integers.

Further Hegarty finds all nine-element sum-dominant sets A of integers with the additional property that for some $x \in A + A$ there are at least four ordered pairs $(a, a') \in A \times A$ with $a + a' = x$. There are, up to linear transformations, nine such sets, listed in [3] as A_2 and A_4 through to A_{11}. It is easy to check that none of these nine sets is restricted-sum-dominant.

Thus, the only possible nine element restricted-sum-dominant sets of integers have the property that for every $x \in A + A$ there are fewer than four ordered pairs (a, a') such that $x = a + a'$. This condition implies that there is no solution of $x + y = u + v$ with x, y, u, v all distinct, so such a set is a weak Sidon set in the sense of Ruzsa [8].

Defining $\delta(n)$ for $n \in A - A$ to be the number of ordered pairs (x, y) such that $x - y = n$, it is shown in the proof of Theorem 4.7 in [8] that for a weak Sidon set, $\delta(n) \leq 2$ whenever $n \neq 0$ and at most $2|A|$ elements n have $\delta(n) = 2$.

Thus, noting 0 has $|A| = 9$ representations and putting $m = |A - A|$,

$$81 \leq 9 + (2 \times 9) \times 2 + (m - 19) \Rightarrow m \geq 55$$

so if such a set were to be sum-dominant its sumset would have to have order at least 56. But of course $|A + A| \leq 9 \times 10/2 = 45$, and we have proven

Theorem 22. *All sum-dominant sets of integers of order 9 are linear transformations of one of Hegarty's nine sets A_2 and A_4 to A_{11}. None of these is restricted-sum-dominant, so there is no restricted-sum-dominant set of order 9.*

We thus know that the smallest restricted-sum-dominant set of integers has order between 10 and 16. It appears a non-trivial computational challenge to find the order of the smallest restricted-sum-dominant set.

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