Smith forms for adjacency matrices of circulant graphs

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Abstract

We calculate the Smith normal form of the adjacency matrix of each of the following graphs or their complements (or both): complete graph, cycle graph, square of the cycle, power graph of the cycle, distance matrix graph of cycle, Andrásfai graph, Doob graph, cocktail party graph, crown graph, prism graph, Möbius ladder. The proofs operate by finding the abelianisation of a cyclically presented group whose relation matrix is column equivalent to the required adjacency matrix.

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1. Introduction

The *circulant matrix* $\mathrm{circ}_n(a_0, \ldots, a_{n-1})$ is the $n \times n$ matrix whose first row is $(a_0, \ldots, a_{n-1})$ and where row $(i+1)$ $(0 \le i \le n-2)$ is a cyclic shift of row i by one column. A *circulant graph* is a graph that is isomorphic to a graph whose adjacency matrix is circulant. We shall write $A(\Gamma)$ for the adjacency matrix of a graph Γ. Given graphs Γ, Γ' if $\det(A(\Gamma)) \neq \det(A(\Gamma'))$ (in particular if precisely one of $A(\Gamma)$, $A(\Gamma')$ is singular) then Γ, Γ' are non-isomorphic. Similarly, if $\mathrm{rank}(A(\Gamma)) \neq \mathrm{rank}(A(\Gamma'))$ then Γ, Γ' are non-isomorphic. Singularity, rank, and determinants of various families of circulant graphs are considered (for example) in [3],[7],[16],[23].

For an $n \times n$ integer matrix M, the *Smith normal form* of M, written $\mathrm{SNF}(M)$, is the $n \times n$ diagonal integer matrix

$$S = \mathrm{diag}_n(d_0, \ldots, d_{n-1})$$

where $d_0, \ldots, d_{n-1} \in \mathbb{N} \cup \{0\}$ and $d_i | d_{i+1}$ $(0 \le i \le n-2)$ is such that there exist invertible integer matrices P, Q such that $PMQ = S$. The matrix S is unique and may be obtained from M using (integer) elementary row and column operations. Given graphs Γ, Γ' if $\mathrm{SNF}(A(\Gamma)) \neq \mathrm{SNF}(A(\Gamma'))$ then Γ, Γ' are non-isomorphic. Since also $\det(\mathrm{SNF}(A(\Gamma)) = |\det(A(\Gamma))|$ and $\mathrm{rank}(\mathrm{SNF}(A(\Gamma)) =$

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rank($A(\Gamma)$) the Smith normal form is a more refined invariant than (the absolute value of) the determinant and the rank.

The purpose of this article is to calculate the Smith normal form for the adjacency matrices of various families of circulant graphs. The calculations of determinant, rank and determination of singularity of adjacency matrices of circulant graphs cited above use a well known number theoretic expression for the eigenvalues of a circulant matrix and a graph theoretic formula for the determinant of the adjacency matrix of a graph, due to Harary [5, Proposition 7.2]. Our methods use techniques from combinatorial group theory, namely the application of Tietze transformations (the addition or removal of generators or relations to or from the presentation while leaving the group unchanged) on cyclic presentations. We now outline our method of proof; for further background we refer the reader to [14].

Let $P = \langle x_0, \ldots, x_{n-1} \mid r_0, \ldots, r_{m-1} \rangle$ be a group presentation defining a group G. For $0 \leq i \leq n-1$, $0 \leq j \leq m-1$ let a_{ij} denote the exponent sum of generator x_i in relator r_j (that is, the number of times x_i appears in r_j, counting multiplicities and sign). The matrix $M = \left(a_{ij}\right)_{n \times m}$ is called the *relation matrix* of P. The abelianization G^{ab} of G can be written in the form

$$\mathbb{Z}_{\delta_0} \oplus \cdots \oplus \mathbb{Z}_{\delta_{p-1}} \oplus \mathbb{Z}^q \tag{1}$$

where $1 < \delta_0 | \delta_1 | \cdots | \delta_{p-1}$ and $p + q \leq n$. These numbers determine the non-unity diagonal entries of the Smith normal form for the relation matrix of any presentation of G; in particular we have

$$\mathrm{SNF}(M) = \mathrm{diag}_n(1, \ldots, 1, \delta_0, \ldots, \delta_{p-1}, \underbrace{0, \ldots, 0}_{q}).$$

Let $w = w(x_0, \ldots, x_{n-1})$ be a word in generators $x_0, \ldots, x_{n-1}$. The *cyclically presented group* $G_n(w)$ is the group defined by the *cyclic presentation*

$$P_n(w) = \langle x_0, \ldots, x_{n-1} \mid w_0, \ldots, w_{n-1} \rangle$$

where $w_i = w(x_i, x_{i+1}, x_{i+2}, \ldots, x_{i+(n-1)})$ (subscripts mod n). The relation matrix of $P_n(w)$ is the circulant matrix $\mathrm{circ}_n(a_0, \ldots, a_{n-1})$ where for each $0 \leq i \leq n-1$ the entry a_i is the exponent sum of x_i in $w(x_0, \ldots, x_{n-1})$. Thus our method of proof is as follows. Given a circulant matrix $C = \mathrm{circ}_n(a_0, \ldots, a_{n-1})$ we define a suitable word $w(x_0, \ldots, x_{n-1})$ so that the exponent sum of x_i in $w(x_0, \ldots, x_{n-1})$ is a_i $(0 \leq i \leq n-1)$. We use Tietze transformations to simplify the presentation to obtain the abelianization G^{ab} in the form (1) and hence obtain $\mathrm{SNF}(C)$.

In principle, our proofs can be written purely in terms of elementary row and column operations, but cyclically presented groups provide a convenient framework and language within which to operate. We digress from this method of proof in Section 5 where we make use of multiplicativity results of Newman [17]. For matrices A, B we write $A \sim B$ to mean that B may be obtained from A by cyclically permuting columns. In Table 1 we summarize the graphs and adjacency matrices, and point to the result which gives the Smith form. (We write $\overline{\Gamma}$ to denote the complement of a graph Γ.)

2. Circulant graphs

In this section we review the definitions of various families of circulant graphs (the webpage [25] is a useful resource for such information) and, where necessary and possible, relabel vertices so that their adjacency matrices are column equivalent to a matrix of one of the following forms:

$$F_{n,s} = \mathrm{circ}_n(\underbrace{1,\dots,1}_{s},\underbrace{0,\dots,0}_{n-s}) \qquad (1 \le s \le n),$$
$$F_{n,s,r} = \mathrm{circ}_n(\underbrace{v,\dots,v}_{s},\underbrace{0,\dots,0}_{n-rs}) \qquad (rs-(r-1) \le n, 1 \le r),$$

where $v = (1,\underbrace{0,\dots,0}_{r-1})$,

$$G_{n,u} = \mathrm{circ}_n(1,\underbrace{0,\dots,0}_{u},1,\underbrace{0,\dots,0}_{n-u-2}) \qquad (0 \le u \le n-2),$$
$$H_{n,v} = \mathrm{circ}_n(\underbrace{1,\dots,1}_{v},0,\underbrace{1,\dots,1}_{v},\underbrace{0,\dots,0}_{n-2v-1}) \qquad (1 \le v \le \lfloor (n-1)/2 \rfloor).$$

Complete graph K_n

The complete graph K_n has adjacency matrix

$$\mathrm{circ}_n(0,1,\dots,1) \sim F_{n,n-1}. \tag{2}$$

Its complement consists of n isolated vertices, so its adjacency matrix is the $n \times n$ zero matrix.

Cycle graph C_n

The *cycle graph* C_n has adjacency matrix

$$\mathrm{circ}_n(0,1,\underbrace{0,\dots,0}_{n-3},1) \sim H_{n,1} = G_{n,1} \tag{3}$$

and its complement has adjacency matrix

$$\mathrm{circ}_n(0,0,\underbrace{1,\dots,1}_{n-3},0) \sim F_{n,n-3}. \tag{4}$$

Distance matrix graph of the cycle graph C_n^i

For a connected graph Γ of diameter d with vertices $v_0,\dots,v_{n-1}$, the *i'th distance matrix* $A_i(\Gamma)$ $(1 \le i \le d)$ is defined to be the $n\times n$ matrix whose (r,s)'th entry is 1 if $d(v_r,v_s) = i$ and 0 otherwise. The *i'th distance matrix graph* of Γ is the graph with adjacency matrix $A_i(\Gamma)$. The cycle C_n has diameter $\lfloor n/2 \rfloor$; for $1 \le i \le \lfloor n/2 \rfloor$ we denote its i'th distance matrix graph by C_n^i.

Consider first the case $i = \lfloor n/2 \rfloor$. If n is even then $C_n^{\lfloor n/2 \rfloor}$ has edges $(v_i, v_{i+n/2})$ $(0 \leq i \leq n-1)$ so it is the ladder rung graph, which we will consider separately (see equation (16), below). Suppose then that n is odd. Then $C_n^{\lfloor n/2 \rfloor}$ has edges $(v_i, v_{i+(n-1)/2})$, $(v_i, v_{i+(n+1)/2})$ $(0 \leq i \leq n-1)$. Since $(2, n) = 1$ the map $i \mapsto 2i$ is a bijection from the set $\{0, \ldots, n-1\}$ to itself. Therefore we may relabel the vertices according to the rule $v_i \mapsto v_{2i}$. Applying this, the edges become (v_j, v_{j-1}), (v_j, v_{j+1}) $(0 \leq j \leq n-1)$, $j = 2i$. That is, $C_n^{\lfloor n/2 \rfloor}$ is isomorphic to the cycle C_n which, again, we consider separately. We therefore consider the graph C_n^i for $1 \leq i < \lfloor n/2 \rfloor$, in which case it has the following adjacency matrix (see also [16]):

$$\mathrm{circ}_n(\underbrace{0, 0, \ldots, 0}_{i}, 1, \underbrace{0, \ldots, 0}_{n-2i-1}, 1, \underbrace{0, \ldots, 0}_{i-1}) \sim G_{n,2i-1} \tag{5}$$

and its complement has adjacency matrix

$$\mathrm{circ}_n(0, \underbrace{1, \ldots, 1}_{i-1}, 0, \underbrace{1, \ldots, 1}_{n-2i-1}, 0, \underbrace{1, \ldots, 1}_{i-1}). \tag{6}$$

Power graph of the cycle graph $C_n^{(r)}$

Given a graph Γ and $r \geq 1$, the *r'th power graph* $\Gamma^{(r)}$ of Γ is the graph with the same set of vertices as Γ and edges (u, v) if $d(u, v) \leq r$ in Γ [20, page 333]. Therefore ([3]) the r'th power graph $C_n^{(r)}$ of the n-cycle C_n $(1 \leq r < \lfloor \frac{n}{2} \rfloor)$ has adjacency matrix

$$\mathrm{circ}_n(0, \underbrace{1, \ldots, 1}_{r}, \underbrace{0, \ldots, 0}_{n-2r-1}, \underbrace{1, \ldots, 1}_{r}) \sim H_{n,r} \tag{7}$$

and its complement has adjacency matrix

$$\mathrm{circ}_n(\underbrace{0, 0, \ldots, 0}_{r+1}, \underbrace{1, \ldots, 1}_{n-2r-1}, \underbrace{0, \ldots, 0}_{r}) \sim F_{n,n-2r-1}. \tag{8}$$

In particular, the square of the n-cycle, $C_n^{(2)}$ has adjacency matrix

$$\mathrm{circ}_n(0, 1, 1, \underbrace{0, \ldots, 0}_{n-5}, 1, 1) \sim H_{n,2} \tag{9}$$

and its complement has adjacency matrix

$$\mathrm{circ}_n(0, 0, 0, \underbrace{1, \ldots, 1}_{n-5}, 0, 0) \sim F_{n,n-5}. \tag{10}$$

(For $r \geq \lfloor \frac{n}{2} \rfloor$ the graph $C_n^{(r)}$ is the complete graph K_n.)

Andrásfai graph And(k)

For $k \geq 1$ the *Andrásfai graph And(k)* is the graph with vertices $v_0, \dots, v_{3k-2}$ and where (v_i, v_j) is an edge if and only if $(i-j) \bmod (3k-1) \in \{1, 4, 7, \dots, 3k-2\}$. (These were introduced in [1], and are considered in [12, Sections 6.10–6.12]; they also appear in [6].) In other words, for each $0 \leq i \leq 3k-1$ vertex v_i is connected by an edge to exactly the vertices v_{i+1},$v_{i+1+(1\cdot 3)}$,$v_{i+1+(2\cdot 3)}, \dots$, $v_{i+1+3(k-1)}$.

Since $(k, 3k-1) = 1$ we may relabel the vertices according to the rule $v_i \mapsto v_{ki}$. Since $3k \equiv 1 \bmod (3k-1)$, applying this means that each vertex v_j (where $j = 3i$, $0 \leq j \leq 3k-2$) is connected to exactly the vertices v_{j+k},v_{j+k+1},v_{j+k+2},..., v_{j+2k-1}, and therefore (in the new labelling) $And(k)$ has adjacency matrix

$$\mathrm{circ}_{3k-1}(\underbrace{0,\dots,0}_{k},\underbrace{1,\dots,1}_{k},\underbrace{0,\dots,0}_{k-1}) \sim F_{3k-1,k} \tag{11}$$

and its complement has adjacency matrix

$$\mathrm{circ}_{3k-1}(0,\underbrace{1,\dots,1}_{k-1},\underbrace{0,\dots,0}_{k},\underbrace{1,\dots,1}_{k-1}) \sim H_{3k-1,k-1}. \tag{12}$$

Doob graph D(r, t)

In constructing graphs whose adjacency matrices have the same determinant as that of a complete graph, Doob [9] introduced the graphs $D(r, t)$ $(r \geq 2, t \geq 1)$. (They are denoted $G(r, t)$ in [9] but we have changed notation to avoid clashes.) These are the graphs with $n = (r-1)t + 2$ vertices $v_0, \dots, v_{n-1}$ and where (v_i, v_j) is an edge if and only if $(i-j) \equiv 1 \bmod t$. They have adjacency matrices

$$\mathrm{circ}_{(r-1)t+2}(0,\underbrace{v,\dots,v}_{r},\underbrace{0,\dots,0}_{r-t+1}) \sim F_{(r-1)t+2,r,t} \tag{13}$$

where $v = (1,\underbrace{0,\dots,0}_{t-1})$, and their complements have adjacency matrices

$$\mathrm{circ}_{(r-1)t+2}(0,\underbrace{\bar{v},\dots,\bar{v}}_{r},\underbrace{1,\dots,1}_{r-t+1}) \tag{14}$$

where $\bar{v} = (0,\underbrace{1,\dots,1}_{t-1})$.

Note that $D(k, 3)$ is the Andrásfai graph $And(k)$. When t is odd, similar vertex relabelling arguments to those made above for $And(k)$ can be used to show that $D(r, t)$ has an adjacency matrix that is column equivalent to $F_{(r-1)t+2,r}$. Unlike in the $And(k)$ case, however, with such a relabelling the complement will not necessarily have an adjacency matrix that is column equivalent to some $H_{(r-1)t+2,v}$ and, for this reason, we will consider the graphs $And(k)$ separately from the graphs $D(r, t)$.

Cocktail party graph $Cock(n)$

The *Cocktail party graph*, or *hyperoctahedral graph*, $Cock(n)$ is constructed from the complete graph K_{2n} by removing n disjoint edges; it is a circulant graph with the following adjacency matrix ([5, page 17],[4, page 278]):

$$\mathrm{circ}_{2n}(0, \underbrace{1, \ldots, 1}_{n-1}, 0, \underbrace{1, \ldots, 1}_{n-1}) \sim H_{2n,n-1} \tag{15}$$

its complement is the *ladder rung graph* ([4, page 277]) with adjacency matrix

$$\mathrm{circ}_{2n}(\underbrace{0, \ldots, 0}_{n}, 1, \underbrace{0, \ldots, 0}_{n-1}) \sim I_{2n} \tag{16}$$

(where I_n denotes the $n \times n$ identity matrix).

Crown graph $Cr(n)$

The *crown graph* $Cr(n)$ is the graph with vertices $u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}$ and edges (u_i, v_j) $0 \le i \ne j \le n-1$. For odd n we may relabel the vertices as $x_0, \ldots, x_{2n-1}$ where $x_{2i} = u_i$, $x_{2i+n} = v_i$ $(0 \le i \le n-1)$, in which case the edges are (x_{2i}, x_{2j+n}) $0 \le i \ne j \le n-1$ so $Cr(n)$ is a circulant graph with adjacency matrix

$$\mathrm{circ}_{2n}(0, \underbrace{1, 0, 1, 0, \ldots, 1, 0}_{n-1}, 0, \underbrace{0, 1, 0, 1, \ldots, 0, 1}_{n-1}) \sim F_{2n,n-1,2} \tag{17}$$

and its complement has adjacency matrix

$$\mathrm{circ}_{2n}(0, \underbrace{0, 1, 0, 1, \ldots, 0, 1}_{n-1}, 1, \underbrace{1, 0, 1, 0, \ldots, 1, 0}_{n-1}). \tag{18}$$

Prism graph $P(n)$

The *prism graph* or *circular ladder graph* $P(n)$ is the skeleton of an n-prism. That is, it is the graph with $2n$ vertices $u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}$ where $u_0, \ldots, u_{n-1}$ form an n-cycle and $v_0, \ldots, v_{n-1}$ form an n-cycle and for each $0 \le i \le n-1$ there is an edge (u_i, v_i) (see [5, page 63] or [21, page 270]).

Suppose n is odd. If we relabel the vertices as $x_0, \ldots, x_{2n-1}$ where $x_{2i} = u_i, x_{2i+n} = v_i$ then the edges are (x_i, x_{i+2}), (x_i, x_{i-2}), (x_i, x_{i+n}) $(0 \le i \le 2n-1)$ so $P(n)$ is a (cubic) circulant graph. (In fact, this and the even Möbius ladder graph, defined below, are the only connected, cubic, circulant graphs – see [15, Theorem 2] or [7] – so, since for even n the graphs $P(n)$ are not circulant we will not consider them in this article.) We apply a further relabelling.

If $(n-1)/2$ is odd then $((n-1)/2, 2n) = 1$ so we may relabel the vertices according to the rule $x_i \mapsto x_{i(n-1)/2}$. Applying this, the edges become (x_j, x_{j+n-1}), (x_j, x_{j+n+1}), (x_j, x_{j+n}) $(0 \le j \le 2n-1)$, $j = i(n-1)/2$. If $(n-1)/2$ is even then $((n+1)/2, 2n) = 1$ so we may relabel the vertices according to the rule $x_i \mapsto x_{i(n+1)/2}$. Applying this, the edges again become

(x_j, x_{j+n+1}), (x_j, x_{j+n-1}), (x_j, x_{j+n}) $(0 \le j \le 2n-1)$, $j = i(n+1)/2$. Therefore in each case (in the new labelling) $P(n)$ has adjacency matrix

$$\mathrm{circ}_{2n}(\underbrace{0,\ldots,0}_{n-1},1,1,1,\underbrace{0,\ldots,0}_{n-2}) \sim F_{2n,3} \tag{19}$$

and its complement has adjacency matrix

$$\mathrm{circ}_{2n}(0,\underbrace{1,\ldots,1}_{n-2},0,0,0,\underbrace{1,\ldots,1}_{n-2}) \sim H_{2n,n-2}. \tag{20}$$

Möbius ladder graph L_n

The *Möbius ladder graph*, L_n, when n is even was introduced in [13] (see also [5, page 20],[2, page 171]) and consists of an n-cycle with each vertex connected by an edge to the opposite vertex. That is, it has vertices $v_0, \ldots, v_{n-1}$ and edges $(v_i, v_{i+1}), (v_i, v_{i-1}), (v_i, v_{i+n/2})$ $(0 \le i \le n-1)$. Thus the adjacency matrix is

$$\mathrm{circ}_n(0,1,\underbrace{0,\ldots,0}_{n/2-2},1,\underbrace{0,\ldots,0}_{n/2-2},1)$$

and the adjacency matrix for the complement is

$$\mathrm{circ}_n(0,0,\underbrace{1,\ldots,1}_{n/2-2},0,\underbrace{1,\ldots,1}_{n/2-2},0) \sim H_{n,n/2-2}. \tag{21}$$

We may relabel the vertices to get a more convenient form of the adjacency matrix of L_n. If $n/2$ is even then $(n/2-1, n) = 1$ so we may relabel the vertices according to the rule $v_i \mapsto v_{(n/2-1)i}$. Applying this, the edges become $(v_j, v_{j+(n/2-1)})$, $(v_j, v_{j+(n/2+1)})$, $(v_j, v_{j+n/2})$ $(0 \le j \le n-1)$, $j = (n/2-1)i$, so L_n has adjacency matrix

$$\mathrm{circ}_n(\underbrace{0,\ldots,0}_{n/2-1},1,1,1,\underbrace{0,\ldots,0}_{n/2-2}) \sim F_{n,3}. \tag{22}$$

If $n/2$ is odd then $(n/2-2, n) = 1$ so we may relabel the vertices according to the rule $v_i \mapsto v_{(n/2-2)i}$. Applying this, the edges become $(v_j, v_{j+(n/2-2)})$, $(v_j, v_{j+(n/2+2)})$, $(v_j, v_{j+n/2})$ $(0 \le j \le n-1)$, $j = (n/2-2)i$, so L_n has adjacency matrix

$$\mathrm{circ}_n(\underbrace{0,\ldots,0}_{n/2-2},1,0,1,0,1,\underbrace{0,\ldots,0}_{n/2-3}) \sim F_{n,3,2}. \tag{23}$$

In [13] the graph L_n was also defined for odd n. In this case L_n consists of an n-cycle with each vertex connected by an edge to the two most opposite vertices. That is, it has vertices $v_0, \ldots, v_{n-1}$ and edges (v_i, v_{i+1}), (v_i, v_{i-1}), $(v_i, v_{i+(n-1)/2})$, $(v_i, v_{i+(n+1)/2})$ $(0 \le i \le n-1)$. Since $(2, n) = 1$ we may relabel the vertices according to the rule $v_i \mapsto v_{2i}$. Applying this, the edges become (v_j, v_{j+2}), (v_j, v_{j-2}), $(v_j, v_{j-1}), (v_j, v_{j+1})$ $(0 \le j \le n-1)$, $j = 2i$, so we see that L_n is isomorphic to the odd square cycle $C_n^{(2)}$.

Graph		**Matrix**	**SNF**
K_n		(2)$\sim F_{n,n-1}$	Cor. 3.4
C_n		(3)$\sim G_{n,1}$	Cor. 4.4
$\overline{C_n}$		(4)$\sim F_{n,n-3}$	Cor. 3.5
C_n^i	$1 \le i < \lfloor n/2 \rfloor$	(5)$\sim G_{n,2i-1}$	Cor. 4.3
$\overline{C_n^i}$	$1 \le i < \lfloor n/2 \rfloor$	(6)	?
$C_n^{(r)}$	$3 \le r < \lfloor n/2 \rfloor$	(7)$\sim H_{n,r}$	?
$C_n^{(2)}$		(9)$\sim H_{n,2}$	Cor. 7.2, Cor. 5.6
$\overline{C_n^{(r)}}$	$3 \le r < \lfloor n/2 \rfloor$	(8)$\sim F_{n,n-2r-1}$	Cor. 3.6
$\overline{C_n^{(2)}}$		(10)$\sim F_{n,n-5}$	Cor. 3.7
$And(k)$		(11)$\sim F_{3k-1,k}$	Cor. 3.9
$\overline{And(k)}$		(12)$\sim H_{3k-1,k-1}$	Cor. 9.2, Cor. 5.7
$D(r,t)$		(13)$\sim F_{(r-1)t+2,r,t}$	Cor. 3.10
$\overline{D(r,t)}$		(14)	?
$Cock(n)$		(15)$\sim H_{2n,n-1}$	Cor. 8.2
$\overline{Cock(n)}$		(16)$\sim I_{2n}$	I_{2n}
$Cr(n)$	(n odd)	(17)$\sim F_{2n,n-1,2}$	Cor. 3.11
$\overline{Cr(n)}$	(n odd)	(18)	Cor. 10.2
$P(n)$	(n odd)	(19)$\sim F_{2n,3}$	Cor. 3.8
$\overline{P(n)}$	(n odd)	(20)$\sim H_{2n,n-2}$	Cor. 5.4
L_n	(n even)	(22)$\sim F_{n,3}$, (23)$\sim F_{n,3,2}$	Cor. 3.12
$\overline{L_n}$	(n even)	(21)$\sim H_{n,n/2-2}$	Cor. 5.5, Cor. 6.3

Table 1: Summary of results.

3. Smith form of $F_{n,s}$ and applications

3.1. Cyclically presented group and Smith forms for $F_{n,s}$ and $F_{n,s,r}$

The matrix $F_{n,s}$ is the relation matrix of the presentation in the following theorem. This result was proved in [24], but as the proof is short and illustrates well the main technique of proof in this paper we include it again here.

Theorem 3.1 ([24]). $G_n(x_0x_1 \ldots x_{s-1}) \cong \mathbb{Z}_{s/(n,s)} * \mathbb{Z}^{(n,s)-1}$.

Proof. Let $\delta = (n,s)$. Then there exist $p, q \in \mathbb{Z}$ such that $\delta = ps + qn$ so $\delta \equiv ps \bmod n$. The relation $x_ix_{i+1} \ldots x_{i+s-1} = 1$ implies $x_i(x_{i+1} \ldots x_{i+s-1}x_{i+s}) = x_{i+s}$ so $x_i = x_{i+s}$ and hence $x_i = x_{i+s} = x_{i+2s} = \ldots = x_{i+ps}$. But $x_{i+ps} = x_{i+\delta}$ so we have $x_i = x_{i+\delta}$ for each $1 \le i \le n-1$. Eliminating generators $x_\delta, \ldots, x_{n-1}$ gives

$$G_n(x_0x_1 \ldots x_{s-1}) = \langle x_0, \ldots, x_{\delta-1} \mid (x_ix_{i+1} \ldots x_{i+\delta-1})^{s/\delta} = 1 \ (i = 0, \ldots, \delta-1) \rangle.$$

Now for $0 \le i \le \delta - 1$ we have that the relator $(x_ix_{i+1} \ldots x_{i+\delta-1})^{s/\delta}$ is a conjugate of $(x_0x_1\dots x_{\delta-1})^{s/\delta}$ so we can eliminate $\delta-1$ of the relators to get

$$\begin{aligned}G_n(x_0x_1\dots x_{s-1}) &= \langle x_0,\dots,x_{\delta-1} \mid (x_0x_1\dots x_{\delta-1})^{s/\delta}=1\rangle\\ &= \langle x_0,\dots,x_{\delta-1},z \mid (x_0x_1\dots x_{\delta-1})^{s/\delta}=1, z=x_0x_1\dots x_{\delta-1}\rangle\\ &= \langle x_0,\dots,x_{\delta-2},z \mid z^{s/\delta}=1\rangle\end{aligned}$$

and the result follows. □

Corollary 3.2.

$$G_n(x_0x_rx_{2r}\dots x_{(s-1)r}) \cong \mathbb{Z}_{s/(n/(n,r),s)}^{(n,r)} * \mathbb{Z}^{(n,r)((n/(n,r),s)-1)}.$$

Proof. A standard technique in the theory of cyclically presented groups is to partition generators and relators to express a cyclically presented group as a free product of others. Using this here we get that $G_n(x_0x_rx_{2r}\dots x_{(s-1)r})$ is isomorphic to the free product of (n,r) copies of $G_{n/(n,r)}(x_0x_1x_2\dots x_{s-1})$ so the result follows from Theorem 3.1. □

Applying Theorem 3.1 and Corollary 3.2 to the matrices $F_{n,s}$ and $F_{n,s,r}$ we get

Corollary 3.3. *(i)* $\mathrm{SNF}(F_{n,s}) = \mathrm{diag}_n(1,\dots,1,s/(n,s),\underbrace{0,\dots,0}_{(n,s)-1})$*;*

(ii) $\mathrm{SNF}(F_{n,s,r}) =$

$$\mathrm{diag}_n\left(1,\dots,1,\underbrace{\frac{s}{(n/(n,r),s)},\dots,\frac{s}{(n/(n,r),s)}}_{(n,r)},\ \underbrace{0,\dots\dots\dots,0}_{(n,r)((n/(n,r),s)-1)}\right).$$

When $(n,s)=1$ the determinant of $F_{n,s}$ was calculated in [18] and the Smith Form was obtained in [22, page 3]. The calculation of $\mathrm{SNF}(F_{n,s})$ and $\mathrm{SNF}(F_{n,s,r})$ (Corollary 3.3) have the following applications for circulant graphs

Corollary 3.4. $\mathrm{SNF}(A(K_n)) = \mathrm{diag}_n(1,\dots,1,n-1)$.

Corollary 3.4 generalizes [16, Lemma 1.6] which shows the well known result that $A(K_n)$ is non-singular for $n\geq 2$.

Corollary 3.5. $\mathrm{SNF}(A(\overline{C_n})) = \begin{cases}\mathrm{diag}_n(1,\dots,1,(n-3)/3,0,0) & \text{if } 3|n;\\ \mathrm{diag}_n(1,\dots,1,n-3) & \text{otherwise.}\end{cases}$

Corollary 3.5 generalizes a result of [8] which calculates $\mathrm{rank}(A(\overline{C_n}))$.

Corollary 3.6. $\mathrm{SNF}(A(\overline{C_n^{(r)}})) =$

$$\mathrm{diag}_n(1,\dots,1,(n-2r-1)/(n,2r+1),\ \underbrace{0,\dots,0}_{(n,2r+1)-1}\ )$$

(where $1\leq r<\lfloor n/2\rfloor$).

Corollary 3.6 generalizes [16, Corollary 2.18] which classifies when $A(\overline{C_n^{(r)}})$ is non-singular. A particular case is

Corollary 3.7.

$$\mathrm{SNF}(A(\overline{C_n^{(2)}})) = \begin{cases} \mathrm{diag}_n(1,\ldots,1,(n-5)/5,0,0,0,0) & \text{if } 5|n; \\ \mathrm{diag}_n(1,\ldots,1,n-5) & \text{otherwise.} \end{cases}$$

Corollary 3.3 also implies

Corollary 3.8. *Suppose n is odd. Then*

$$\mathrm{SNF}(A(P(n))) = \begin{cases} \mathrm{diag}_{2n}(1,\ldots,1,0,0) & \text{if } 3|n; \\ \mathrm{diag}_{2n}(1,\ldots,1,3) & \text{otherwise.} \end{cases}$$

Corollary 3.8 generalizes a result of [7] which calculates rank$(A(P(n))$ (for odd n).

Corollary 3.9. $\mathrm{SNF}(A(And(k))) = \mathrm{diag}_{3k-1}(1,\ldots,1,k)$.

Corollary 3.10. *If t is odd then*

$$\mathrm{SNF}(A(D(r,t))) = \mathrm{diag}_{(r-1)t+2}\left(1,\ldots,1,\frac{r}{(r,t-2)},\underbrace{0,\ldots,0}_{(r,t-2)-1}\right);$$

if t is even then $\mathrm{SNF}(A(D(r,t))) =$

$$\mathrm{diag}_{(r-1)t+2}\left(1,\ldots,1,\frac{r}{(r,(t-2)/2)},\frac{r}{(r,(t-2)/2)},\underbrace{0,\ldots,0}_{(2r,t-2)-2}\right).$$

Theorem 9 (and Theorem 10) of [9] gives a formula for $\det(A(D(r,t)))$. Up to sign this follows directly from Corollary 3.10.

Conjecture 1 of [26] asserted that any graph with n vertices whose adjacency matrix has the same determinant as that of the complete graph is complete. The family of graphs $D(r,r+1)$ (where r is odd) were given in [19] as counterexamples. (Other counterexamples were given in [9].) That is, the graph $D(r,r+1)$ (with r odd) is not complete but its adjacency matrix has the same determinant as $K_n = K_{r^2+1}$. Corollaries 3.4 and 3.10 show that they do not have the same Smith form.

Corollary 3.11. *Let n be odd. Then*

$$\mathrm{SNF}(A(Cr(n))) = \mathrm{diag}_{2n}(1,\ldots,1,n-1,n-1).$$

Corollary 3.12. *If n is even then*

$$\mathrm{SNF}(A(L_n)) = \begin{cases} \mathrm{diag}_n(1,\ldots,1,0,0) & \text{if } n/2 \equiv 0 \bmod 6; \\ \mathrm{diag}_n(1,\ldots,1,0,0,0,0) & \text{if } n/2 \equiv 3 \bmod 6; \\ \mathrm{diag}_n(1,\ldots,1,3,3) & \text{if } n/2 \equiv \pm 1 \bmod 6; \\ \mathrm{diag}_n(1,\ldots,1,1,3) & \text{if } n/2 \equiv \pm 2 \bmod 6. \end{cases}$$

Corollary 3.12 generalizes a result of [7] which calculates $\mathrm{rank}(A(L(n))$ (for even n).

4. Smith form of $G_{n,u}$ and applications

4.1. Cyclically presented group and Smith form for $G_{n,u}$

The matrix $G_{n,u}$ is the relation matrix of the presentation in the following theorem.

Theorem 4.1. *Let $0 \le u \le n-2$. Then*

$$G_n(x_0x_{u+1}) \cong \begin{cases} \underbrace{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{(n,2(u+1))} & \text{if } (n,2(u+1)) = (n,u+1), \\ \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{(n,2(u+1))/2} & \text{otherwise.} \end{cases}$$

Proof. $G = G_n(x_0x_{u+1})$ has relators x_ix_{i+u+1} $(0 \le i \le n-1)$. Let $d = (n, 2(u+1))$, $b = (n, u+1)$. Now there exist $\alpha, \beta \in \mathbb{Z}$ such that $\alpha n + 2(u+1)\beta = d$ and so $2(u+1)\beta \equiv d \bmod n$. Then

$$x_i = x_{i+(u+1)}^{-1} = x_{i+2(u+1)} = x_{i+3(u+1)}^{-1} = \cdots = x_{i+2(u+1)\beta} = x_{i+d}$$

so

$$\begin{aligned} G &= \langle x_0, \ldots, x_{n-1} \mid x_i = x_{i+(u+1)}^{-1}, x_i = x_{i+d} \ (0 \le i \le n-1) \rangle \\ &= \langle x_0, \ldots, x_{d-1} \mid x_i = x_{i+(u+1)}^{-1} \ (0 \le i \le d-1) \rangle. \end{aligned}$$

If $b = d$ then $d | (u+1)$ so

$$G = \langle x_0, \ldots, x_{d-1} \mid x_i = x_i^{-1} \ (0 \le i \le d-1) \rangle \cong \underbrace{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{d}.$$

If $b \ne d$ then d is even so $(d/2)|b$ so $(d/2)|(u+1)$. But $d \not| (u+1)$ so $(u+1) \equiv d/2 \bmod d$. Therefore

$$\begin{aligned} G &= \langle x_0, \ldots, x_{d-1} \mid x_i = x_{i+d/2}^{-1} \ (0 \le i \le d-1) \rangle \\ &= \langle x_0, \ldots, x_{d/2-1} \mid \ \rangle \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{d/2}. \end{aligned}$$

□

Corollary 4.2. *Let* $0 \le u \le n-2$.

$$\mathrm{SNF}(G_{n,u}) = \begin{cases} \mathrm{diag}_n(1,\ldots,1,\underbrace{2,\ldots,2}_{(n,2(u+1))}) & \text{if } (n,2(u+1)) = (n,u+1); \\ \mathrm{diag}_n(1,\ldots,1,\underbrace{0,\ldots,0}_{(n,2(u+1))/2}) & \text{otherwise.} \end{cases}$$

We may apply Corollary 4.2 to the graph C_n^i $(1 \le i < \lfloor n/2 \rfloor)$ by calculating $\mathrm{SNF}(G_{n,u})$ where $u = 2i-1$. Write $n = 2^s m$, $i = 2^t j$ where $s,t \ge 0$ and m,j are odd. Then $(n,u+1) = 2^{\min(s,t+1)}(m,j)$ and $(n,2(u+1)) = 2^{\min(s,t+2)}(m,j)$ which are equal if and only if $s \le t+1$. Therefore we have the following.

Corollary 4.3. *Let* $n = 2^s m, i = 2^t j$ *where* $s,t \ge 0$ *and* m,j *are odd and* $1 \le i < \lfloor n/2 \rfloor$.

$$\mathrm{SNF}(A(C_n^i)) = \begin{cases} \mathrm{diag}_n(1,\ldots,1,\underbrace{2,\ldots,2}_{(n,2i)}) & \text{if } s \le t+1; \\ \mathrm{diag}_n(1,\ldots,1,\underbrace{0,\ldots,0}_{(n,4i)/2}) & \text{if } s > t+1. \end{cases}$$

Corollary 4.3 generalizes [16, Lemma 2.12], which states that $A(C_n^i)$ is singular if and only if $n \equiv 0 \bmod 4$ and $0 \le t \le s-2$. Putting $i=1$ gives

Corollary 4.4. $\mathrm{SNF}(A(C_n)) = \begin{cases} \mathrm{diag}_n(1,\ldots,1,0,0) & \text{if } n \equiv 0 \bmod 4; \\ \mathrm{diag}_n(1,\ldots,1,2,2) & \text{if } n \equiv 2 \bmod 4; \\ \mathrm{diag}_n(1,\ldots,1,1,2) & \text{if } n \equiv \pm 1 \bmod 4. \end{cases}$

Corollary 4.4 generalizes [11, Theorem 5] which calculates $\mathrm{rank}(A(C_n))$ and, up to sign, it generalizes [23, Theorem 2.3], which calculates $\det(A(C_n))$ (using graph theoretic methods).

5. Smith form of non-singular $H_{n,v}$

An easy calculation shows that

$$H_{n,v} = G_{n,v} F_{n,v}. \tag{24}$$

Corollaries 3.3 and 4.2 then imply

Lemma 5.1. *The matrix* $H_{n,v}$ *is non-singular if and only if* $(n,v) = 1$ *and* $(n,2(v+1)) = (n,v+1)$.

Theorem 2.2 of [3] (which is restated as Theorem 2.17 of [16]) which classifies when $A(C_n^{(r)})$ is non-singular follows immediately from this.

By [17, Theorem II. 15] if A,B are non-singular $n \times n$ matrices then the diagonal entry $\mathrm{SNF}(AB)_{i,i}$ is divisible by $\mathrm{SNF}(A)_{i,i}$ and by $\mathrm{SNF}(B)_{i,i}$ so equation (24) and Corollaries 3.3 and 4.2 (and the determinants of $F_{n,v}$, $G_{n,v}$, $H_{n,v}$ they imply) gives the following.

Theorem 5.2. *Let $d = (n, 2(v+1))$ and suppose $(n, v) = 1$ and $(n, 2(v+1)) = (n, v+1)$. Then* $\mathrm{SNF}(H_{n,v}) = \mathrm{diag}_n(1, \ldots, 1, \underbrace{2, \ldots, 2}_{d-1}, 2v)$.

In particular

Corollary 5.3. *Suppose n is odd. Then*

$$\mathrm{SNF}(H_{2n,n-2}) = \mathrm{diag}_{2n}(1, \ldots, 1, 2, 2(n-2)).$$

The calculation of $\mathrm{SNF}(H_{2n,n-2})$ for odd n (Corollary 5.3) has the following applications to circulant graphs.

Corollary 5.4. *Suppose n is odd. Then*

$$\mathrm{SNF}(A(\overline{P(n)})) = \mathrm{diag}_{2n}(1, \ldots, 1, 2, 2(n-2)).$$

Corollary 5.5. *Suppose $n \equiv 2 \bmod 4$. Then*

$$\mathrm{SNF}(A(\overline{L_n})) = \mathrm{diag}_n(1, \ldots, 1, 2, n-4).$$

Theorem 5.2 also implies the following.

Corollary 5.6. *If n is odd then*

$$\mathrm{SNF}(H_{n,2}) = \mathrm{SNF}(A(C_n^{(2)})) = \begin{cases} \mathrm{diag}_n(1, \ldots, 1, 2, 2, 4) & \text{if } 3|n; \\ \mathrm{diag}_n(1, \ldots, 1, 4) & \text{otherwise.} \end{cases}$$

Corollary 5.7. *Suppose k is even. Then*

$$\mathrm{SNF}(H_{3k-1,k-1}) = \mathrm{SNF}(A(\overline{And(k)})) = \mathrm{diag}_{3k-1}(1, \ldots, 1, 2(k-1)).$$

In Sections 6–9 we calculate the Smith form for $H_{n,v}$ in other cases that are relevant to the graphs we consider. While the Smith form for singular matrices $H_{n,v}$ in general eludes us, we can, however, obtain sharp bounds on the rank. Equation (24) and Corollaries 3.3,4.2 together with Sylvester's rank inequality (see, for example, [10, page 66]) imply the following.

Corollary 5.8.

$$\mathrm{rank}(H_{n,v}) \geq \begin{cases} n - (n, v) + 1 & \text{if } (n, 2(v+1)) = (n, v+1); \\ n - \frac{(n,2(v+1))}{2} - (n, v) + 1 & \text{otherwise.} \end{cases}$$

The bounds are the best possible as Lemma 5.1 shows that the bound of the first line can be attained, and the matrix $H_{4,1}$ shows that the bound of the second line can be attained.

6. Smith form of $H_{2n,n-2}$ for even n

When n is odd $\mathrm{SNF}(H_{2n,n-2})$ is calculated in Corollary 5.3, so we consider the case n even. The matrix $H_{2n,n-2}$ is the relation matrix of the presentation in Theorem 6.1.

Theorem 6.1. *Let* $G = G_{2n}((x_0x_1\ldots x_{n-3})(x_{n-1}x_n\ldots x_{2n-4}))$ *where* n *is even. Then*

$$G^{\mathrm{ab}} \cong \begin{cases} \mathbb{Z}_{n(n-2)} \oplus \mathbb{Z} & \text{if } n \equiv 0 \bmod 4; \\ \mathbb{Z}_{n(n-2)/4} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{if } n \equiv 2 \bmod 4. \end{cases}$$

Proof. We have

$$\begin{aligned} G^{\mathrm{ab}} &= \langle x_0, x_1, \ldots, x_{2n-1} \mid (x_ix_{i+1}\ldots x_{i+n-3})(x_{i+n-1}x_{i+n}\ldots x_{i+2n-4}) \\ &\qquad (0 \le i \le 2n-1)\rangle^{\mathrm{ab}} \\ &= \langle x_0, x_1, \ldots, x_{2n-1} \mid (x_ix_{i+n-1})(x_{i+1}x_{i+n})\ldots(x_{i+n-3}x_{i+2n-4}) \\ &\qquad (0 \le i \le 2n-1)\rangle^{\mathrm{ab}} \\ &= \langle x_0, x_1, \ldots, x_{2n-1}, y_0, y_1, \ldots, y_{2n-1} \mid y_iy_{i+1}\ldots y_{i+n-3} = 1, y_i = x_ix_{i+n-1} \\ &\qquad (0 \le i \le 2n-1)\rangle^{\mathrm{ab}} \end{aligned} \tag{25}$$

The relation $y_iy_{i+1}\ldots y_{i+n-3} = 1$ together with the relation $y_{i+1}\ldots y_{i+n-2} = 1$ imply $y_i = y_{i+n-2}$.

Suppose $n \equiv 0 \bmod 4$. Then $y_i = y_{i+(n-2)} = y_{i+2(n-2)} = \cdots = y_{i+\alpha(n-2)}$ ($\alpha \ge 1$). Now $n(n-2)/2 \equiv n \bmod 2n$, so putting $\alpha \equiv n/2$ gives $y_i = y_{i+n}$. But also $y_i = y_{i+n-2}$ so $y_{i+n-2} = y_{i+n}$ or equivalently $y_i = y_{i+2}$. Adding these relations to the presentation gives

$$\begin{aligned} G^{\mathrm{ab}} &= \langle x_0, x_1, \ldots, x_{2n-1}, y_0, y_1, \ldots, y_{2n-1} \mid y_iy_{i+1}\ldots y_{i+n-3} = 1, \\ &\qquad y_i = x_ix_{i+n-1}, y_i = y_{i+2}\ (0 \le i \le 2n-1)\rangle^{\mathrm{ab}} \\ &= \langle x_0, x_1, \ldots, x_{2n-1}, y_0, y_1, \ldots, y_{2n-1} \mid y_iy_{i+1}\ldots y_{i+n-3} = 1, y_i = x_ix_{i+n-1} \\ &\qquad (0 \le i \le 2n-1), y_{2j} = y_0, y_{2j+1} = y_1\ (0 \le j \le n-1)\rangle^{\mathrm{ab}} \\ &= \langle x_0, x_1, \ldots, x_{2n-1}, y_0, y_1 \mid (y_0y_1)^{(n-2)/2} = 1, x_{2j}x_{2j+n-1} = y_0, \\ &\qquad x_{2j+1}x_{2j+n} = y_1\ (0 \le j \le n-1)\rangle^{\mathrm{ab}} \\ &= \langle x_0, x_1, \ldots, x_{2n-1}, y_0, y_1, z \mid (y_0y_1)^{(n-2)/2} = 1, x_{2j}x_{2j+n-1} = y_0, \\ &\qquad x_{2j+1}x_{2j+n} = y_1\ (0 \le j \le n-1) z = y_0y_1\rangle^{\mathrm{ab}} \\ &= \langle x_0, x_1, \ldots, x_{2n-1}, y_0, z \mid z^{(n-2)/2} = 1, x_{2j}x_{2j+n-1} = y_0, \\ &\qquad x_{2j+1}x_{2j+n} = y_0^{-1}z\ (0 \le j \le n-1)\rangle^{\mathrm{ab}} \\ &= \langle x_0, x_1, \ldots, x_{2n-1}, y_0, z \mid z^{(n-2)/2} = 1, x_{2j+n+2}x_{2j+1} = y_0, \\ &\qquad x_{2j+1}x_{2j+n} = y_0^{-1}z\ (0 \le j \le n-1)\rangle^{\mathrm{ab}} \\ &= \langle x_0, x_2, \ldots, x_{2n-2}, y_0, z \mid z^{(n-2)/2} = 1, (x_{2j+n+2}^{-1}y_0)x_{2j+n} = y_0^{-1}z \\ &\qquad (0 \le j \le n-1)\rangle^{\mathrm{ab}} \end{aligned}$$

$$
\begin{aligned}
&= \langle x_0, x_2, \ldots, x_{2n-2}, y_0, z \mid z^{(n-2)/2} = 1, x_{2j} = x_{2j+2} y_0^{-2} z \ (0 \le j \le n-1) \rangle^{\mathrm{ab}} \\
&= \langle u_0, u_1, \ldots, u_{n-1}, y_0, z \mid z^{(n-2)/2} = 1, u_j = u_{j+1} y_0^{-2} z \ (0 \le j \le n-1) \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, z \mid z^{(n-2)/2} = 1, u_0 = u_0 (y_0^{-2} z)^n \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, z \mid z^{(n-2)/2} = 1, z^n = y_0^{2n} \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, z \mid z^{(n-2)/2} = 1, z^n = z^2, z^n = y_0^{2n} \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, z \mid z^{(n-2)/2} = 1, z^2 = y_0^{2n} \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, z, w \mid z^{(n-2)/2} = 1, w = y_0^n z^{-1}, w^2 = 1 \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, z, w \mid z^{(n-2)/2} = 1, z = y_0^n w^{-1}, w^2 = 1 \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, w \mid (y_0^n w^{-1})^{(n-2)/2} = 1, w^2 = 1 \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, w \mid y_0^{n(n-2)/2} = w^{(n-2)/2}, w^2 = 1 \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, w \mid y_0^{n(n-2)/2} = w, w^2 = 1 \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0 \mid y_0^{n(n-2)} \rangle^{\mathrm{ab}} \cong \mathbb{Z}_{n(n-2)} \oplus \mathbb{Z}.
\end{aligned}
$$

Suppose then that $n \equiv 2 \bmod 4$. By (25) $y_i = y_{i+n-2}$ so $y_i = y_{i+n-2} = y_{i+2n-4} = y_{i-4}$ so $y_i = y_{i+4}$. Adding these relations to (25) we get

$$
\begin{aligned}
G^{\mathrm{ab}} &= \langle x_0, x_1, \ldots, x_{2n-1}, y_0, y_1, \ldots, y_{2n-1} \mid y_i y_{i+1} \ldots y_{i+n-3} = 1, y_i = x_i x_{i+n-1}, \\
&\qquad y_i = y_{i+4} \ (0 \le i \le 2n-1) \rangle^{\mathrm{ab}} \\
&= \langle x_0, x_1, \ldots, x_{2n-1}, y_0, y_1, \ldots, y_{2n-1} \mid y_i y_{i+1} \ldots y_{i+n-3} = 1, \\
&\qquad y_i = x_i x_{i+n-1} \ (0 \le i \le 2n-1), y_{4j} = y_0, y_{4j+1} = y_1, y_{4j+2} = y_2, \\
&\qquad y_{4j+3} = y_3 \ (0 \le j \le n/2-1) \rangle^{\mathrm{ab}} \\
&= \langle x_0, x_1, \ldots, x_{2n-1}, y_0, y_1, y_2, y_3 \mid (y_0 y_1 y_2 y_3)^{(n-2)/4} = 1, x_{4j} x_{4j+n-1} = y_0, \\
&\qquad x_{4j+1} x_{4j+n} = y_1, x_{4j+2} x_{4j+n+1} = y_2, x_{4j+3} x_{4j+n+2} = y_3 \\
&\qquad (0 \le j \le n/2-1) \rangle^{\mathrm{ab}}.
\end{aligned}
$$

For each $0 \le j \le n/2-1$ let $a_j = x_{4j}$, $b_j = x_{4j+1}$, $c_j = x_{4j+2}$, $d_j = x_{4j+3}$. Then

$$
\begin{aligned}
G^{\mathrm{ab}} &= \langle a_j, b_j, c_j, d_j \ (0 \le j \le n/2-1), y_0, y_1, y_2, y_3 \mid (y_0 y_1 y_2 y_3)^{(n-2)/4} = 1, \\
&\qquad a_j b_{j+(n-2)/4} = y_0, b_j c_{j+(n-2)/4} = y_1, c_j d_{j+(n-2)/4} = y_2, \\
&\qquad d_j a_{j+(n+2)/4} = y_3 \rangle^{\mathrm{ab}} \\
&= \langle a_j, b_j, c_j, d_j \ (0 \le j \le n/2-1), y_0, y_1, y_2, y_3 \mid (y_0 y_1 y_2 y_3)^{(n-2)/4} = 1, \\
&\qquad a_j b_{j+(n-2)/4} = y_0, b_j c_{j+(n-2)/4} = y_1, c_j d_{j+(n-2)/4} = y_2, \\
&\qquad d_{j+(n-2)/4} = y_3 a_j^{-1} \rangle^{\mathrm{ab}} \\
&= \langle a_j, b_j, c_j \ (0 \le j \le n/2-1), y_0, y_1, y_2, y_3 \mid (y_0 y_1 y_2 y_3)^{(n-2)/4} = 1, \\
&\qquad a_j b_{j+(n-2)/4} = y_0, b_j c_{j+(n-2)/4} = y_1, c_j (y_3 a_j^{-1}) = y_2 \rangle^{\mathrm{ab}}
\end{aligned}
$$

$$
\begin{aligned}
&= \langle a_j, b_j, c_j \ (0 \le j \le n/2-1), y_0, y_1, y_2, y_3 \mid (y_0y_1y_2y_3)^{(n-2)/4} = 1, \\
&\qquad a_jb_{j+(n-2)/4} = y_0, b_jc_{j+(n-2)/4} = y_1, c_j = a_jy_2y_3^{-1}\rangle^{\mathrm{ab}} \\
&= \langle a_j, b_j, c_j \ (0 \le j \le n/2-1), y_0, y_1, y_2, y_3 \mid (y_0y_1y_2y_3)^{(n-2)/4} = 1, \\
&\qquad a_jb_{j+(n-2)/4} = y_0, b_jc_{j+(n-2)/4} = y_1, \\
&\qquad c_{j+(n-2)/4} = a_{j+(n-2)/4}y_2y_3^{-1}\rangle^{\mathrm{ab}} \\
&= \langle a_j, b_j \ (0 \le j \le n/2-1), y_0, y_1, y_2, y_3 \mid (y_0y_1y_2y_3)^{(n-2)/4} = 1, \\
&\qquad a_jb_{j+(n-2)/4} = y_0, b_j(a_{j+(n-2)/4}y_2y_3^{-1}) = y_1\rangle^{\mathrm{ab}} \\
&= \langle a_j, b_j \ (0 \le j \le n/2-1), y_0, y_1, y_2, y_3 \mid (y_0y_1y_2y_3)^{(n-2)/4} = 1, \\
&\qquad a_jb_{j+(n-2)/4} = y_0, b_j = y_1y_2^{-1}y_3a_{j+(n-2)/4}^{-1}\rangle^{\mathrm{ab}} \\
&= \langle a_j, b_j \ (0 \le j \le n/2-1), y_0, y_1, y_2, y_3 \mid (y_0y_1y_2y_3)^{(n-2)/4} = 1, \\
&\qquad a_jb_{j+(n-2)/4} = y_0, b_{j+(n-2)/4} = y_1y_2^{-1}y_3a_{j-1}^{-1}\rangle^{\mathrm{ab}} \\
&= \langle a_j \ (0 \le j \le n/2-1), y_0, y_1, y_2, y_3 \mid (y_0y_1y_2y_3)^{(n-2)/4} = 1, \\
&\qquad a_j(y_1y_2^{-1}y_3a_{j-1}^{-1}) = y_0\rangle^{\mathrm{ab}} \\
&= \langle a_j \ (0 \le j \le n/2-1), y_0, y_1, y_2, y_3 \mid (y_0y_1y_2y_3)^{(n-2)/4} = 1, \\
&\qquad a_j = y_0y_1^{-1}y_2y_3^{-1}a_{j-1} \ (0 \le j \le n/2-1)\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, y_2, y_3 \mid (y_0y_1y_2y_3)^{(n-2)/4} = 1, a_0 = (y_0y_1^{-1}y_2y_3^{-1})^{n/2}a_0\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, y_2, y_3, z, w \mid z^{(n-2)/4} = 1, z = y_0y_1y_2y_3, w^{n/2} = 1, \\
&\qquad w = y_0y_1^{-1}y_2y_3^{-1}\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, y_2, y_3, z, w \mid z^{(n-2)/4} = 1, z = y_0y_1y_2y_3, w^{n/2} = 1, \\
&\qquad y_3 = w^{-1}y_0y_1^{-1}y_2\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, y_2, z, w \mid z^{(n-2)/4} = 1, z = y_0y_1y_2(w^{-1}y_0y_1^{-1}y_2), w^{n/2} = 1\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, y_2, z, w \mid z^{(n-2)/4} = 1, w = z^{-1}y_0^2y_2^2, w^{n/2} = 1\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, y_2, z \mid z^{(n-2)/4} = 1, (z^{-1}y_0^2y_2^2)^{n/2} = 1\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, y_2, z \mid z^{(n-2)/4} = 1, (y_0y_2)^n = z^{n/2}\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, y_2, z \mid z^{(n-2)/4} = 1, z^{n/2} = z, (y_0y_2)^n = z^{n/2}\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, y_2, z \mid z^{(n-2)/4} = 1, (y_0y_2)^n = z\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, y_2 \mid (y_0y_2)^{n(n-2)/4} = 1\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, y_2, u \mid u^{n(n-2)/4} = 1, u = y_0y_2\rangle^{\mathrm{ab}} \\
&= \langle a_0, y_0, y_1, u \mid u^{n(n-2)/4} = 1\rangle^{\mathrm{ab}} \cong \mathbb{Z}_{n(n-2)/4} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.
\end{aligned}
$$

□

Corollary 6.2. *Suppose n is even. Then*

$$\mathrm{SNF}(H_{2n,n-2}) = \begin{cases} \mathrm{diag}_{2n}(1,\ldots,1,n(n-2),0) & \text{if } n\equiv 0 \bmod 4; \\ \mathrm{diag}_{2n}(1,\ldots,1,n(n-2)/4,0,0,0) & \text{if } n\equiv 2 \bmod 4. \end{cases}$$

This has the following consequence for circulant graphs.

Corollary 6.3. *Suppose $n\equiv 0$ mod 4. Then*

$$\mathrm{SNF}(A(\overline{L_n})) = \begin{cases} \mathrm{diag}_{n}(1,\ldots,1,n(n-4)/4,0) & \text{if } n\equiv 0 \bmod 8; \\ \mathrm{diag}_{n}(1,\ldots,1,n(n-4)/16,0,0,0) & \text{if } n\equiv 4 \bmod 8. \end{cases}$$

7. Smith form of $H_{n,2}$ for even n

When n is odd $\mathrm{SNF}(H_{n,2})$ is calculated in Corollary 5.6, so we consider the case n even. The matrix $H_{n,2}$ is the relation matrix of the presentation in the following theorem.

Theorem 7.1. *Suppose n is even. Then*

$$G_n(x_0x_1x_3x_4)^{\mathrm{ab}} \cong \begin{cases} \mathbb{Z}_{n/3}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & \text{if } 6|n; \\ \mathbb{Z}_n\oplus\mathbb{Z} & \text{otherwise.} \end{cases}$$

Proof. Let G be the group in the statement. Then

$$\begin{aligned} G^{\mathrm{ab}} &= \langle x_0,\ldots,x_{n-1} \mid x_ix_{i+3}x_{i+1}x_{i+4}=1\ (0\le i\le n-1)\rangle^{\mathrm{ab}} \\ &= \langle x_0,\ldots,x_{n-1},y_0,\ldots,y_{n-1} \mid y_iy_{i+1}=1, y_i=x_ix_{i+3}\ (0\le i\le n-1)\rangle^{\mathrm{ab}} \\ &= \langle x_0,\ldots,x_{n-1},y_0,\ldots,y_{n-1} \mid y_{i+1}=y_0^{(-1)^{i+1}}, \\ &\qquad\qquad y_i=x_ix_{i+3}\ (0\le i\le n-1)\rangle^{\mathrm{ab}} \\ &= \langle x_0,\ldots,x_{n-1},y_0 \mid y_0=y_0^{(-1)^n}, y_0^{(-1)^i}=x_ix_{i+3}\ (0\le i\le n-1)\rangle^{\mathrm{ab}} \\ &= \langle x_0,\ldots,x_{n-1},y_0 \mid y_0^{(-1)^i}=x_ix_{i+3}\ (0\le i\le n-1)\rangle^{\mathrm{ab}}. \end{aligned} \qquad (26)$$

Suppose $6\not|n$, that is, $(n,3)=1$. We have

$$x_0=y_0x_3^{-1}=y_0^2x_6=y_0^3x_9^{-1}=\cdots=y_0^{\alpha}x_{3\alpha}^{(-1)^\alpha}=\cdots=y_0^nx_{3n}^{(-1)^n}=y_0^nx_0.$$

(In particular, the derived relations $x_0=y_0^\alpha x_{3\alpha}^{(-1)^\alpha}$ imply the relations $x_{3\alpha}=x_0^{(-1)^\alpha}y_0^{(-1)^{\alpha+1}\alpha}$.) Now $\{x_i \mid 0\le i\le n-1\}=\{x_{3\alpha} \mid 1\le\alpha\le n\}$ so we may eliminate generators $x_1,\ldots,x_{n-1}$:

$$\begin{aligned} G^{\mathrm{ab}} &= \langle x_0,\ldots,x_{n-1},y_0 \mid x_{3\alpha}=x_0^{(-1)^\alpha}y_0^{(-1)^{\alpha+1}\alpha}\ (1\le\alpha\le n)\rangle^{\mathrm{ab}} \\ &= \langle x_0,y_0 \mid x_0=x_0^{(-1)^n}y_0^{(-1)^{n+1}n}\rangle^{\mathrm{ab}} \\ &= \langle x_0,y_0 \mid y_0^n=1\rangle^{\mathrm{ab}}\cong\mathbb{Z}_n\oplus\mathbb{Z}. \end{aligned}$$

Suppose then that $6|n$ and set $n=3k$ for some even k. By (26) we have

$$G^{\mathrm{ab}}=\langle x_0,\ldots,x_{3k-1},y_0 \mid y_0^{(-1)^i}=x_ix_{i+3}\ (0\le i\le 3k-1)\rangle^{\mathrm{ab}}$$

$$
\begin{aligned}
&= \langle x_0, \ldots, x_{3k-1}, y_0 \mid y_0^{(-1)^j} = x_{3j}x_{3(j+1)}, y_0^{(-1)^{j+1}} = x_{3j+1}x_{3(j+1)+1}, \\
&\qquad y_0^{(-1)^j} = x_{3j+2}x_{3(j+1)+2}\ (0 \le j \le k-1)\rangle^{\mathrm{ab}} \\
&= \langle x_0, \ldots, x_{3k-1}, y_0 \mid x_{3(j+1)} = x_{3j}^{-1}y_0^{(-1)^j}, x_{3(j+1)+1} = x_{3j+1}^{-1}y_0^{(-1)^{j+1}}, \\
&\qquad x_{3(j+1)+2} = x_{3j+2}^{-1}y_0^{(-1)^j}\ (0 \le j \le k-1)\rangle^{\mathrm{ab}} \\
&= \langle x_0, \ldots, x_{3k-1}, y_0 \mid x_{3j} = x_{3(j-1)}^{-1}y_0^{(-1)^{j-1}}, x_{3j+1} = x_{3(j-1)+1}^{-1}y_0^{(-1)^j}, \\
&\qquad x_{3j+2} = x_{3(j-1)+2}^{-1}y_0^{(-1)^{j-1}}\ (0 \le j \le k-1)\rangle^{\mathrm{ab}}.
\end{aligned}
$$

The relations $x_{3j} = x_{3(j-1)}^{-1}y_0^{(-1)^{j-1}}$ are equivalent to the relations $x_{3j} = x_0^{(-1)^j}y_0^{j(-1)^{j-1}}$; the relations $x_{3j+1} = x_{3(j-1)+1}^{-1}y_0^{(-1)^j}$ are equivalent to the relations $x_{3j+1} = x_1^{(-1)^j}y_0^{j(-1)^j}$; the relations $x_{3j+2} = x_{3(j-1)+2}^{-1}y_0^{(-1)^{j-1}}$ are equivalent to the relations $x_{3j+2} = x_2^{(-1)^j}y_0^{j(-1)^{j-1}}$. This gives

$$
\begin{aligned}
G^{\mathrm{ab}} &= \langle x_0, \ldots, x_{3k-1}, y_0 \mid x_{3j} = x_0^{(-1)^j}y_0^{j(-1)^{j-1}}, x_{3j+1} = x_1^{(-1)^j}y_0^{j(-1)^j}, \\
&\qquad x_{3j+2} = x_2^{(-1)^j}y_0^{j(-1)^{j-1}}\rangle^{\mathrm{ab}} \\
&= \langle x_0, x_1, x_2, y_0 \mid x_0 = x_0^{(-1)^k}y_0^{k(-1)^{k-1}}, x_1 = x_1^{(-1)^k}y_0^{k(-1)^k}, \\
&\qquad x_2 = x_2^{(-1)^k}y_0^{k(-1)^{k-1}}\rangle^{\mathrm{ab}} \\
&= \langle x_0, x_1, x_2, y_0 \mid y_0^k = 1\rangle^{\mathrm{ab}} \cong \mathbb{Z}_k \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}.
\end{aligned}
$$

□

Corollary 7.2. *Suppose n is even. Then*

$$
\mathrm{SNF}(H_{n,2}) = \mathrm{SNF}(A(C_n^{(2)})) = \begin{cases} \mathrm{diag}_n(1, \ldots, 1, n/3, 0, 0, 0) & \text{if } 6|n; \\ \mathrm{diag}_n(1, \ldots, 1, n, 0) & \text{otherwise.} \end{cases}
$$

8. Smith form of $H_{2n,n-1}$

The matrix $H_{2n,n-1}$ is the relation matrix of the presentation in the following theorem.

Theorem 8.1.

$$
G_{2n}((x_0x_1 \ldots x_{n-2})(x_nx_{n+1} \ldots x_{2n-2}))^{\mathrm{ab}} \cong \mathbb{Z}_{n-1} \oplus \mathbb{Z}^n.
$$

Proof. Let G be the group in the statement. Then

$$
\begin{aligned}
G^{\mathrm{ab}} &= \langle x_0, \ldots, x_{2n-1} \mid (x_ix_{i+1} \ldots x_{i+n-2})(x_{i+n}x_{i+n+1} \ldots x_{i+2n-2}) = 1 \\
&\qquad (0 \le i \le 2n-1)\rangle^{\mathrm{ab}} && (27) \\
&= \langle x_0, \ldots, x_{2n-1} \mid x_ix_{i+1} \ldots x_{i+2n-2} = x_{i+n-1}\ (0 \le i \le 2n-1)\rangle^{\mathrm{ab}} && (28) \\
&= \langle x_0, \ldots, x_{2n-1} \mid (x_ix_{i+n})(x_{i+1}x_{i+n+1}) \ldots (x_{i+n-2}x_{i+2n-2}) = 1 \\
&\qquad (0 \le i \le 2n-1)\rangle^{\mathrm{ab}}. && (29)
\end{aligned}
$$

By the i'th relation of (28) $x_{i+1}\dots x_{i+2n-2}=x_i^{-1}x_{i+(n-1)}$; by the $(i+1)$'th relation $x_{i+1}\dots x_{i+2n-2}=x_{i+2n-1}^{-1}x_{i+n}$. Therefore $x_ix_{i+n}=x_{i+2n-1}x_{i+n-1}$, so $x_ix_{i+n}=x_{i-1}x_{i+n-1}$ or equivalently $x_ix_{i+n}=x_{i+1}x_{i+n+1}$. Adding these to presentation (29) we get

$$\begin{aligned} G^{\mathrm{ab}} &= \langle x_0,\dots,x_{2n-1} \mid (x_ix_{i+n})(x_{i+1}x_{i+n+1})\dots(x_{i+n-2}x_{i+2n-2})=1,\\ &\qquad x_ix_{i+n}=x_{i+1}x_{i+n+1}\ (0\le i\le 2n-1)\rangle^{\mathrm{ab}}\\ &= \langle x_0,\dots,x_{2n-1} \mid (x_ix_{i+n})^{n-1}=1, x_ix_{i+n}=x_{i+1}x_{i+n+1}\\ &\qquad (0\le i\le 2n-1)\rangle^{\mathrm{ab}}\\ &= \langle x_0,\dots,x_{2n-1},y_0,\dots,y_{2n-1} \mid (x_ix_{i+n})^{n-1}=1, x_ix_{i+n}=x_{i+1}x_{i+n+1},\\ &\qquad y_i=x_ix_{i+n}\ (0\le i\le 2n-1)\rangle^{\mathrm{ab}}\\ &= \langle x_0,\dots,x_{2n-1},y_0,\dots,y_{2n-1} \mid y_i^{n-1}=1, y_i=y_{i+1}, y_i=x_ix_{i+n}\\ &\qquad (0\le i\le 2n-1)\rangle^{\mathrm{ab}}\\ &= \langle x_0,\dots,x_{2n-1},y_0 \mid y_0^{n-1}=1, y_0=x_ix_{i+n}\ (0\le i\le 2n-1)\rangle^{\mathrm{ab}}\\ &= \langle x_0,\dots,x_{n-1},y_0 \mid y_0^{n-1}=1\rangle^{\mathrm{ab}}\cong \mathbb{Z}_{n-1}\oplus\mathbb{Z}^n. \end{aligned}$$

□

Corollary 8.2.

$$\mathrm{SNF}(H_{2n,n-1})=\mathrm{SNF}(A(Cock(n)))=\mathrm{diag}_{2n}(1,\dots,1,n-1,\underbrace{0,\dots,0}_{n}).$$

9. Smith form of $H_{3k-1,k-1}$

For odd k the matrix $H_{3k-1,k-1}$ is the relation matrix of the presentation in Theorem 9.1.

Theorem 9.1. *Suppose k is odd. Then*

$$G_{3k-1}((x_0x_1\dots x_{k-2})(x_kx_{k+1}\dots x_{2k-2}))^{\mathrm{ab}}\cong \mathbb{Z}_{(3k-1)(k-1)/2}\oplus\mathbb{Z}.$$

Proof. Let G be the group in the statement. Then

$$\begin{aligned} G^{\mathrm{ab}} &= \langle x_0,\dots,x_{3k-2} \mid (x_ix_{i+1}\dots x_{i+k-2})(x_{i+k}x_{i+k+1}\dots x_{i+2k-2})\\ &\qquad (0\le i\le 3k-2)\rangle^{\mathrm{ab}}\\ &= \langle x_0,\dots,x_{3k-2},y_0,\dots,y_{3k-2} \mid y_iy_{i+1}\dots y_{i+k-2}=1,\ y_i=x_ix_{i+k}\\ &\qquad (0\le i\le 3k-2)\rangle^{\mathrm{ab}}. \end{aligned}$$

By the i'th relator $y_i=(y_{i+1}\dots y_{i+k-2})^{-1}$ so $y_iy_{i+k-1}^{-1}=(y_{i+1}\dots y_{i+k-1})^{-1}$, which is the inverse of the $(i+1)$'th relator, so $y_i=y_{i+k-1}$. Therefore $y_i=$

$y_{i+(k-1)} = y_{i+2(k-1)} = y_{i+3(k-1)} = y_{i+(3k-1)-2} = y_{i-2}$ so $y_i = y_{i+2}$. Adding these relations to the presentation we get

$$
\begin{aligned}
G^{\mathrm{ab}} &= \langle x_0, \ldots, x_{3k-2}, y_0, \ldots, y_{3k-2} \mid y_i y_{i+1} \ldots y_{i+k-2} = 1, \\
&\qquad y_i = x_i x_{i+k}, y_i = y_{i+2} \ (0 \le i \le 3k-2) \rangle^{\mathrm{ab}} \\
&= \langle x_0, \ldots, x_{3k-2}, y_0, \ldots, y_{3k-2} \mid y_i y_{i+1} \ldots y_{i+k-2} = 1,\ y_i = x_i x_{i+k} \\
&\qquad (0 \le i \le 3k-2),\ y_{2j} = y_0, y_{2j+1} = y_1 \ (0 \le j \le (3k-3)/2) \rangle^{\mathrm{ab}} \\
&= \langle x_0, \ldots, x_{3k-2}, y_0, y_1 \mid (y_0 y_1)^{(k-1)/2} = 1,\ x_{2j} x_{2j+k} = y_0, \\
&\qquad x_{2j+1} x_{2j+k+1} = y_1 (0 \le j \le (3k-3)/2 \rangle^{\mathrm{ab}} \\
&= \langle x_0, \ldots, x_{3k-2}, y_0, y_1, z \mid z^{(k-1)/2} = 1,\ z = y_0 y_1,\ x_{2j} x_{2j+k} = y_0, \\
&\qquad x_{2j+1} x_{2j+k+1} = z y_0^{-1} \ (0 \le j \le (3k-3)/2 \rangle^{\mathrm{ab}} \\
&= \langle x_0, \ldots, x_{3k-2}, y_0, z \mid z^{(k-1)/2} = 1,\ x_{2j-(k-1)} x_{2j+1} = y_0, \\
&\qquad x_{2j+1} x_{2j+k+1} = z y_0^{-1} \ (0 \le j \le (3k-3)/2 \rangle^{\mathrm{ab}} \\
&= \langle x_0, \ldots, x_{3k-2}, y_0, z \mid z^{(k-1)/2} = 1,\ x_{2j-(k-1)} x_{2j+1} = y_0, \\
&\qquad x_{2j+1} = z y_0^{-1} x_{2j+k+1}^{-1} \ (0 \le j \le (3k-3)/2 \rangle^{\mathrm{ab}} \\
&= \langle x_0, x_2, x_4 \ldots, x_{3k-3}, y_0, z \mid z^{(k-1)/2} = 1, \\
&\qquad x_{2j-(k-1)} (z y_0^{-1} x_{2j+k+1}^{-1}) = y_0 \ (0 \le j \le (3k-3)/2 \rangle^{\mathrm{ab}} \\
&= \langle x_0, x_2, x_4 \ldots, x_{3k-3}, y_0, z \mid z^{(k-1)/2} = 1,\ x_{2j-(k-1)} x_{2j+k+1}^{-1} = y_0^2 z^{-1} \\
&\qquad (0 \le j \le (3k-3)/2 \rangle^{\mathrm{ab}} \\
&= \langle x_0, x_2, x_4 \ldots, x_{3k-3}, y_0, z \mid z^{(k-1)/2} = 1,\ x_{2j} x_{2j+2k}^{-1} = y_0^2 z^{-1} \\
&\qquad (0 \le j \le (3k-3)/2 \rangle^{\mathrm{ab}} \\
&= \langle u_0, u_1, \ldots, u_{(3k-3)/2}, y_0, z \mid z^{(k-1)/2} = 1,\ u_j u_{j+k}^{-1} = y_0^2 z^{-1} \\
&\qquad (0 \le j \le (3k-3)/2 \rangle^{\mathrm{ab}} \\
&= \langle u_0, u_1, \ldots, u_{(3k-3)/2}, y_0, z, w \mid z^{(k-1)/2} = 1,\ w = y_0^2 z^{-1},\ u_j = w u_{j+k} \\
&\qquad (0 \le j \le (3k-3)/2 \rangle^{\mathrm{ab}}.
\end{aligned}
$$

Now $u_j = w u_{j+k} = w^2 u_{j+2k} = \cdots = w^{\alpha} u_{j+\alpha k}$ $(\alpha \ge 1)$. Putting $\alpha = (3k-1)/2$ gives $u_j = w^{(3k-1)/2} u_{j+(3k-1)k/2} = w^{(3k-1)/2} u_j$. Eliminating $u_1, \ldots, u_{(3k-3)/2}$ gives

$$
\begin{aligned}
G^{\mathrm{ab}} &= \langle u_0, y_0, z, w \mid z^{(k-1)/2} = 1,\ w = y_0^2 z^{-1},\ u_0 = w^{(3k-1)/2} u_0 \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, z, w \mid z^{(k-1)/2} = 1,\ w = y_0^2 z^{-1},\ w^{(3k-1)/2} = 1 \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, z \mid z^{(k-1)/2} = 1,\ (y_0^2 z^{-1})^{(3k-1)/2} = 1 \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, z \mid z^{(k-1)/2} = 1,\ y_0^{3k-1} = z^{(3k-1)/2} \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, z \mid z^{(k-1)/2} = 1,\ y_0^{3k-1} = z^{3(k-1)/2} z \rangle^{\mathrm{ab}} \\
&= \langle u_0, y_0, z \mid z^{(k-1)/2} = 1,\ y_0^{3k-1} = z \rangle^{\mathrm{ab}}
\end{aligned}
$$

$$
\begin{aligned}
&= \langle u_0, y_0 \mid y_0^{(3k-1)(k-1)/2} \rangle^{\mathrm{ab}} \\
&\cong \mathbb{Z}_{(3k-1)(k-1)/2} \oplus \mathbb{Z}.
\end{aligned}
$$

□

Corollary 9.2. *Suppose k is odd. Then*

$$\mathrm{SNF}(H_{3k,k-1}) = \mathrm{SNF}(A(\overline{And(k)})) = \mathrm{diag}_{3k-1}(1, \ldots, 1, (3k-1)(k-1)/2, 0).$$

10. Smith form of adjacency matrix for $\overline{Cr(n)}$

The adjacency matrix for $\overline{Cr(n)}$ (18) does not fit into any of our families $F_{n,s}$, $F_{n,s,r}, G_{n,u}, H_{n,v}$, so we deal with it separately. It is the adjacency matrix of the group in the following theorem.

Theorem 10.1. *Suppose n is odd. Then $G_{2n}((x_2x_4 \ldots x_{2(n-1)})x_n)^{\mathrm{ab}} \cong \mathbb{Z}_{n-2} \oplus \mathbb{Z}^{n-1}$.*

Proof. Let G be the group in the statement. Then

$$
\begin{aligned}
G^{\mathrm{ab}} &= \langle x_0, \ldots, x_{2n-1} \mid x_{i+2}x_{i+4} \ldots x_{i+2(n-1)} = x_{i+n}^{-1}\ (0 \le i \le 2n-1) \rangle^{\mathrm{ab}} \\
&= \langle x_0, \ldots, x_{2n-1}, y_0, \ldots, y_{n-1}, z_0, \ldots, z_{n-1} \mid x_{i+2}x_{i+4} \ldots x_{i+2(n-1)} = x_{i+n}^{-1} \\
&\qquad (0 \le i \le 2n-1), y_j = x_{2j}, z_j = x_{2j+1}\ (0 \le j \le n-1) \rangle^{\mathrm{ab}} \\
&= \langle x_0, \ldots, x_{2n-1}, y_0, \ldots, y_{n-1}, z_0, \ldots, z_{n-1} \mid x_{i+2}x_{i+4} \ldots x_{i+2(n-1)} = x_{i+n}^{-1} \\
&\qquad (0 \le i \le 2n-1), y_j = x_{2j}, z_j = x_{2j+1}, y_{j+(n+1)/2} = x_{2j+n+1}, \\
&\qquad z_{j+(n-1)/2} = x_{2j+n}\ (0 \le j \le n-1) \rangle^{\mathrm{ab}} \\
&= \langle y_0, \ldots, y_{n-1}, z_0, \ldots, z_{n-1} \mid y_{i+1}y_{i+2} \ldots y_{i+(n-1)} = z_{i+(n-1)/2}^{-1}, \\
&\qquad z_{i+1}z_{i+2} \ldots z_{i+(n-1)} = y_{i+(n+1)/2}^{-1}\ (0 \le i \le n-1) \rangle^{\mathrm{ab}} \\
&= \langle y_0, \ldots, y_{n-1}, z_0, \ldots, z_{n-1} \mid y_iy_{i+1}y_{i+2} \ldots y_{i+(n-1)} = y_iz_{i+(n-1)/2}^{-1}, \\
&\qquad z_iz_{i+1}z_{i+2} \ldots z_{i+(n-1)} = z_iy_{i+(n+1)/2}^{-1}\ (0 \le i \le n-1) \rangle^{\mathrm{ab}} \\
&= \langle y_0, \ldots, y_{n-1}, z_0, \ldots, z_{n-1}, Y, Z \mid Y = y_0y_1 \ldots y_{n-1}, Z = z_0z_1 \ldots z_{n-1}, \\
&\qquad Y = y_iz_{i+(n-1)/2}^{-1}, Z = z_iy_{i+(n+1)/2}^{-1}\ (0 \le i \le n-1) \rangle^{\mathrm{ab}} \\
&= \langle y_0, \ldots, y_{n-1}, Y, Z \mid Y = y_0y_1y_2 \ldots y_{n-1}, Z = Z^nY, \\
&\qquad Y = y_i(Zy_i)^{-1}\ (0 \le i \le n-1) \rangle^{\mathrm{ab}} \\
&= \langle y_0, \ldots, y_{n-1}, Y, Z \mid Y = y_0y_1y_2 \ldots y_{n-1}, Z = Z^nY, Z = Y^{-1} \rangle^{\mathrm{ab}} \\
&= \langle y_0, \ldots, y_{n-1}, Y \mid Y = y_0y_1y_2 \ldots y_{n-1}, Y^{n-2} \rangle^{\mathrm{ab}} \\
&= \langle y_0, \ldots, y_{n-2}, y_{n-1}, Y \mid Y(y_0y_1y_2 \ldots y_{n-2})^{-1} = y_{n-1}, Y^{n-2} \rangle^{\mathrm{ab}} \\
&= \langle y_0, \ldots, y_{n-2}, Y \mid Y^{n-2} \rangle^{\mathrm{ab}} \\
&\cong \mathbb{Z}_{n-2} \oplus \mathbb{Z}^{n-1}.
\end{aligned}
$$

□

Corollary 10.2. *Let n be odd. Then*

$$\mathrm{SNF}(A(\overline{Cr(n)})) = \mathrm{diag}_{2n}(1, \ldots, 1, n-2, \underbrace{0, \ldots, 0}_{n-1}).$$

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