Exact Local Whittle Estimation of Fractional Integration*

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Abstract

An exact form of the local Whittle likelihood is studied with the intent of developing a general purpose estimation procedure for the memory parameter (d) that applies throughout the stationary and nonstationary regions of d and which does not rely on tapering or differencing prefilters. The resulting exact local Whittle estimator is shown to be consistent and to have the same $N(0, \frac{1}{4})$ limit distribution for all values of d.

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Short Title: Exact Local Whittle Estimation

1 Introduction

Semiparametric estimation of the memory parameter (d) in fractionally integrated ($I(d)$) time series is appealing in empirical work because of the general treatment of the short memory component that it affords. Two common statistical procedures in this class are log periodogram (LP) regression and local Whittle (LW) estimation. LW estimation is known to be more efficient than LP regression in the stationary ($|d| < \frac{1}{2}$) case (Robinson, 1995), although numerical optimization methods are needed in calculation. Outside the stationary region, it is known that the asymptotic theory

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for the LW estimator is discontinuous at $d = \frac{3}{4}$ and again at $d = 1$, is awkward to use because of nonnormal limit theory and, worst of all, the estimator is inconsistent when $d > 1$ (Phillips and Shimotsu, 2001). Thus, the LW estimator is not a good general purpose estimator when the value of d may take on values in the nonstationary zone beyond $\frac{3}{4}$. Similar comments apply in the case of LP estimation (Kim and Phillips, 1999).

To extend the range of application of these semiparametric methods, data differencing and data tapering have been suggested (Velasco, 1999, Hurvich and Chen, 2000). These methods have the advantage that they are easy to implement and they make use of existing algorithms once the data filtering has been carried out. Differencing has the disadvantage that prior information is needed on the appropriate order of differencing. Tapering has the disadvantage that the filter distorts the trajectory of the data and inflates the asymptotic variance. In consequence, there is presently no general purpose efficient estimation procedure when the value of d may take on values in the nonstationary zone beyond $\frac{3}{4}$.

The present paper studies an exact form of the local Whittle estimator which does not rely on differencing or tapering and which seems to offer a good general purpose estimation procedure for the memory parameter that applies throughout the stationary and nonstationary regions of d. The estimator, which we call the exact LW estimator, is shown to be consistent and to have the same $N(0, \frac{1}{4})$ limit distribution for all values of d. The exact LW estimator therefore has the same limit theory outside the stationary region as the LW estimator has for stationary values of d. The approach therefore seems to offer a useful alternative for applied researchers who are looking for a general purpose estimator and want to allow for a substantial range of stationary and nonstationary possibilities for d. The method has the further advantage that it provides a basis for constructing valid asymptotic confidence intervals for d that are valid irrespective of the true value of the memory parameter. A minor disadvantage of the approach is that it involves a numerical optimization that is somewhat more demanding than LW estimation. Our experience from simulations indicates that the computation time required is about ten times that of the LW estimator and is well within the capabilities of a small notebook computer.

2 Preliminaries

2.1 A Model of Fractional Integration

We consider the fractional process X_t generated by the model

$$(1-L)^d X_t = u_t I\{t \geq 1\}, \quad t = 1, 2, \ldots \tag{1}$$

where u_t is stationary with zero mean and spectral density $f_u(\lambda)$. Expanding the binomial in (1) gives the form

$$\sum_{k=0}^{t} \frac{(-d)_k}{k!} X_{t-k} = u_t I\{t \geq 1\}, \tag{2}$$

where

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = (d)(d+1)\ldots(d+k-1),$$

is Pochhammer's symbol for the forward factorial function and $\Gamma(\cdot)$ is the gamma function. When d is a positive integer, the series in (2) terminates, giving the usual formulae for the model (1) in terms of the differences and higher order differences of $X_{t.}$ An alternate form for X_t is obtained by inversion of (1), giving a valid representation for all values of d

$$X_t = (1-L)^{-d} u_t I\{t \geq 1\} = \sum_{k=0}^{t-1} \frac{(d)_k}{k!} u_{t-k}. \tag{3}$$

Define the discrete Fourier transform (dft) and the periodogram of a time series a_t evaluated at the fundamental frequencies as

$$\begin{aligned} w_a(\lambda_s) &= \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} a_t e^{it\lambda_s}, \quad \lambda_s = \frac{2\pi s}{n}, s = 1, \ldots, n, \\ I_a(\lambda_s) &= w_a(\lambda_s) w_a(\lambda_s)^*. \end{aligned} \tag{4}$$

Our approach is to algebraically manipulate (2) so that it can be rewritten in a convenient form to accommodate dft's. The following lemma gives an exact expression that we use for the model in frequency domain form.

2.2 Lemma (Phillips, 1999, Theorem 2.2)

(a) If X_t follows (1), then

$$w_u(\lambda) = D_n\left(e^{i\lambda}; d\right) w_x(\lambda) - \frac{1}{\sqrt{2\pi n}} e^{in\lambda} \widetilde{X}_{\lambda n}(d), \tag{5}$$

where $D_n(e^{i\lambda}; d) = \sum_{k=0}^{n} \frac{(-d)_k}{k!} e^{ik\lambda}$ and

$$\widetilde{X}_{\lambda n}(d) = \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L; d\right) X_n = \sum_{p=0}^{n-1} \widetilde{d}_{\lambda p} e^{-ip\lambda} X_{n-p}, \quad \widetilde{d}_{\lambda p} = \sum_{k=p+1}^{n} \frac{(-d)_k}{k!} e^{ik\lambda}. \tag{6}$$

(b) If X_t follows (1) with $d = 1$, then

$$w_x(\lambda)\left(1 - e^{i\lambda}\right) = w_u(\lambda) - \frac{e^{i\lambda}}{\sqrt{2\pi n}} e^{in\lambda} X_n. \tag{7}$$

Note that $\widetilde{d}_{\lambda p} \equiv 0$ for $p \geq d$ when d is nonnegative integer, because $(-d)_k = 0$ for any integer $k > d$.

3 Exact Local Whittle Estimation

3.1 Exact Local Whittle Likelihood and Estimator

The (negative) Whittle likelihood based on frequencies up to λ_m and up to scale multiplication is

$$\sum_{j=1}^{m} \log f_u(\lambda_j) + \sum_{j=1}^{m} \frac{I_u(\lambda_j)}{f_u(\lambda_j)}, \tag{8}$$

where m is some integer less than n. As in Phillips (1999) define the new transform

$$\begin{aligned} v_x(\lambda_s;d) &= w_x(\lambda_s) - D_n\left(e^{i\lambda_s};d\right)^{-1}\frac{1}{\sqrt{2\pi n}}\widetilde{X}_{\lambda_s n}(d), \\ I_v(\lambda_s;d) &= v_x(\lambda_s;d)v_x(\lambda_s;d)^*, \end{aligned}$$

for which the relationship

$$v_x(\lambda_s;d) = D_n\left(e^{i\lambda_s};d\right)^{-1} w_u(\lambda_s), \quad I_v(\lambda_s;d) = \left|D_n\left(e^{i\lambda_s};d\right)\right|^{-2} I_u(\lambda_s),$$

holds exactly. Using this exact relationship in conjunction with the local approximation $f_u(\lambda_s) \sim G$, we can transform the likelihood function (8) to be data dependent, as suggested by Phillips (1999), to give the exact Whittle likelihood

$$L_m(G,d) = \frac{1}{m}\sum_{j=1}^{m}\left[\log\left(\left|D_n\left(e^{i\lambda_j};d\right)\right|^{-2}G\right) + \frac{\left|D_n\left(e^{i\lambda_j};d\right)\right|^2}{G}I_v(\lambda_j;d)\right].$$

Another exact relationship is

$$\Delta^d X_t = (1-L)^d X_t = u_t,$$

which leads to

$$w_{\Delta^d x}(\lambda_s) = w_u(\lambda_s) = D_n\left(e^{i\lambda_s};d\right)v_x(\lambda_s;d). \tag{9}$$

Use (9) to replace $|D_n(e^{i\lambda_j};d)|^2 I_v(\lambda_j;d)$ by $I_{\Delta^d x}(\lambda_j)$ and approximate $|D_n(e^{i\lambda_j};d)|^2$ by λ_j^{2d} (Phillips and Shimotsu, 2001). Then the objective function is simplified to

$$Q_m(G,d) = \frac{1}{m}\sum_{j=1}^{m}\left[\log\left(G\lambda_j^{-2d}\right) + \frac{1}{G}I_{\Delta^d x}(\lambda_j)\right].$$

We propose to estimate d and G by minimising $Q_m(G,d)$, so that

$$\left(\widehat{G},\widehat{d}\right) = \underset{G\in(0,\infty),\ d\in[\Delta_1,\Delta_2]}{\arg\min} Q_m(G,d),$$

where Δ_1 and Δ_2 are the lower and upper bound of the admissible values of d. We impose no restrictions on Δ_1 and Δ_2 except that $-\infty < \Delta_1 < \Delta_2 < \infty$. In what follows we distinguish the true values of the parameters by the notation $G_0 = f_u(0)$ and d_0. Concentrating $Q_m(G,d)$ with respect to G, we find that $\widehat{d}$ satisfies

$$\widehat{d} = \underset{d\in[\Delta_1,\Delta_2]}{\arg\min} R(d), \tag{10}$$

where

$$R(d) = \log\widehat{G}(d) - 2d\frac{1}{m}\sum_{1}^{m}\log\lambda_j, \quad \widehat{G}(d) = \frac{1}{m}\sum_{1}^{m}I_{\Delta^d x}(\lambda_j). \tag{11}$$

We call $\widehat{d}$ the exact LW estimator of d.

3.2 Consistency

We introduce the following assumptions on m and the stationary component u_t in (1).

Assumption 1

$$u_t = C\left(L\right)\varepsilon_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} j^{1/2}\left|c_j\right| < \infty, \quad C\left(1\right) \neq 0, \tag{12}$$

where $E(\varepsilon_t|F_{t-1}) = 0$, $E(\varepsilon_t^2|F_{t-1}) = 1$ *a.s.*, $t = 0, \pm 1, \ldots,$ *in which* F_t *is the* σ*-field generated by* ε_s, $s \leq t$, *and there exists a random variable* ε *such that* $E\varepsilon^2 < \infty$ *and for all* $\eta > 0$ *and some* $K > 0$, $\Pr(|\varepsilon_t| > \eta\ ) \leq K \Pr(|\varepsilon| > \eta)$.

Assumption 2 *As* $n \to \infty$,

$$\frac{1}{m} + \frac{m\left(\log n\right)^2\left(\log m\right)^3}{n} + \frac{\log n}{m^\gamma} \to 0 \quad \text{for any } \gamma > 0.$$

Under (12), the spectral density of u_t is $f_u(\lambda) = \frac{1}{2\pi}|C(e^{i\lambda})|^2$ and clearly satisfies

$$f_u\left(\lambda\right) \sim f_u\left(0\right) \in (0, \infty) \quad \text{as } \lambda \to 0+. \tag{13}$$

a local condition on $f_u\left(\lambda\right)$ analogous to Assumption A1 of Robinson (1995). Assumption 1 applies to u_t rather than X_t and does not assume differentiability of $f_u\left(\lambda\right)$ (c.f. A2 in Robinson). Assumption 2 is slightly stronger than Assumption A4 of Robinson (1995).

Under these conditions we may now establish the consistency of $\widehat{d}$.

3.3 Theorem

Suppose X_t *is generated by (1) and Assumptions 1 and 2 hold. Then, for* $d_0 \in \left[\Delta_1, \Delta_2\right]$, $\widehat{d} \to_p d_0$ *as* $n \to \infty$.

3.4 Asymptotic Normality

We introduce some further assumptions that are used to derive the limit distribution theory in this section.

Assumption 1′

(a) Assumption 1 holds and also

$$E(\varepsilon_t^3|F_{t-1}) = \mu_3 \quad a.s., \quad E(\varepsilon_t^4) = \mu_4, \quad t = 0, \pm 1, \ldots,$$

for finite constants μ_3 *and* μ_4.

(b) For some $\beta \in (0, 2]$,

$$f_u\left(\lambda\right) = f_u\left(0\right)\left(1 + O(\lambda^\beta)\right), \quad \text{as } \lambda \to 0+.$$

Assumption 2′ *As* $n \to \infty$,

$$\frac{1}{m} + \frac{m^{1+2\beta}(\log m)^2}{n^{2\beta}} + \frac{\log n}{m^\gamma} \to 0 \quad \text{for any } \gamma > 0.$$

Assumption 1′ is analogous to Assumptions A1′-A3′ of Robinson (1995). Robinson (1995) imposes the assumption on the spectral density of X_t and assumes differentiability. Assumption 2′ is comparable to Assumption A4′ of Robinson. The following theorem establishes the asymptotic normality of the exact local Whittle estimator for $d_0 \in \left(\Delta_1, \Delta_2\right)$.

3.5 Theorem

Suppose X_t is generated by (1) and Assumptions $1'$ and $2'$ hold. Then, for $d_0 \in (\Delta_1, \Delta_2)$,

$$m^{1/2}\left(\widehat{d} - d_0\right) \to_d N\left(0, \frac{1}{4}\right).$$

4 Simulations

This section reports some simulations that were conducted to examine the finite sample performance of the exact LW estimator (hereafter, exact estimator), the LW estimator (hereafter, untapered estimator) and the LW estimator with two types of tapering studied by Hurvich and Chen (2000) and Velasco (1999) with Bartlett's window (hereafter, tapered (HC) and tapered (V) estimator, respectively) . We generate $I(d)$ processes according to (3) with $u_t \sim iidN(0,1)$. Δ_1 and Δ_2 are set to -2 and 4. The bias, standard deviation, and mean squared error (MSE) were computed using 10,000 replications. Sample size and m were chosen to be $n = 500$ and $m = n^{0.65} = 56$. Values of d were selected in the interval $[-0.7, 2.3]$.

Table 1 shows the simulation results. The exact estimator has little bias for all values of d. The untapered estimator has a large negative bias for $d > 1$, corroborating the theoretical result that it converges to unity in probability (Phillips and Shimotsu, 2001). When $d < 1$, the exact and untapered estimators have similar variance and MSE. The variances of the tapered estimators are always larger than those of the exact and untapered estimator. Again this outcome corroborates the theoretical result that the asymptotic variance of the tapered estimators are larger ($3/(8m)$ and $1/(2m)$, respectively, for the HC and V tapered estimators). Tapered (HC) estimator has small bias and performs better than tapered (V) estimator. However, tapered (HC) estimator still has around 50% larger MSE than the exact estimator due to its larger variance.

Table 1. Simulation results: $n = 500$, $m = n^{0.65} = 56$

	Exact estimator			**Untapered estimator**		
d	bias	s.d.	MSE	bias	s.d.	MSE
-0.7	-0.0024	0.0787	0.0062	0.0363	0.0882	0.0091
-0.3	-0.0020	0.0774	0.0060	-0.0017	0.0776	0.0060
0.0	-0.0020	0.0776	0.0060	-0.0066	0.0773	0.0060
0.3	-0.0014	0.0770	0.0059	-0.0059	0.0771	0.0060
0.7	-0.0024	0.0787	0.0062	0.0088	0.0828	0.0069
1.3	-0.0033	0.0777	0.0060	-0.2102	0.0988	0.0539
1.7	-0.0029	0.0784	0.0061	-0.6279	0.1342	0.4122
2.3	-0.0020	0.0782	0.0061	-1.2632	0.1129	1.6084
	Tapered (HC) estimator			**Tapered (V) estimator**		
d	bias	s.d.	MSE	bias	s.d.	MSE
-0.7	0.0280	0.0972	0.0102	-0.0070	0.1230	0.0152
-0.3	0.0123	0.0978	0.0097	-0.0106	0.1210	0.0147
0.0	0.0043	0.0983	0.0097	-0.0115	0.1218	0.0150
0.3	-0.0007	0.0975	0.0095	-0.0110	0.1202	0.0146
0.7	-0.0076	0.0982	0.0097	-0.0068	0.1216	0.0148
1.3	-0.0084	0.0974	0.0096	0.0139	0.1243	0.0156
1.7	0.0005	0.0971	0.0094	0.0460	0.1286	0.0187
2.3	0.0525	0.0993	0.0126	-0.1776	0.1419	0.0517

Figures 1 and 2 plot kernel estimates of the densities of the four estimators for the values $d = -0.7, 0.3, 1.3$ and 2.3. The sample size and m were chosen as $n = 500$ and $m = n^{0.65}$, and 10,000 replications are used. When $d = -0.7$, the exact and tapered (V) estimators have symmetric distributions centred on -0.7, with the tapered estimator having a flatter distribution. The untapered and tapered (HC) estimators appear to be biased. When $d = 0.3$, the untapered and exact estimators have almost identical distributions, whereas the two tapered estimators have more dispersed distributions. When $d = 1.3$, the untapered estimator is centred on unity. In this case, the exact estimator seems to work well, having a symmetric distribution centred on 1.3. The tapered estimators have flatter distributions than the exact estimator but otherwise appear reasonable and they are certainly better than the inconsistent untapered estimator. When $d = 2.3$, the untapered and tapered (V) estimators appear centred on 1.0 and 2.0, respectively. In this case, the tapered (HC) estimator is upward biased. Again, the exact estimator has a symmetric distribution centred on the true value 2.3.

In sum, there seems to be little doubt from these results that the exact LW estimator is the best general purpose estimator over a wide range of d values.

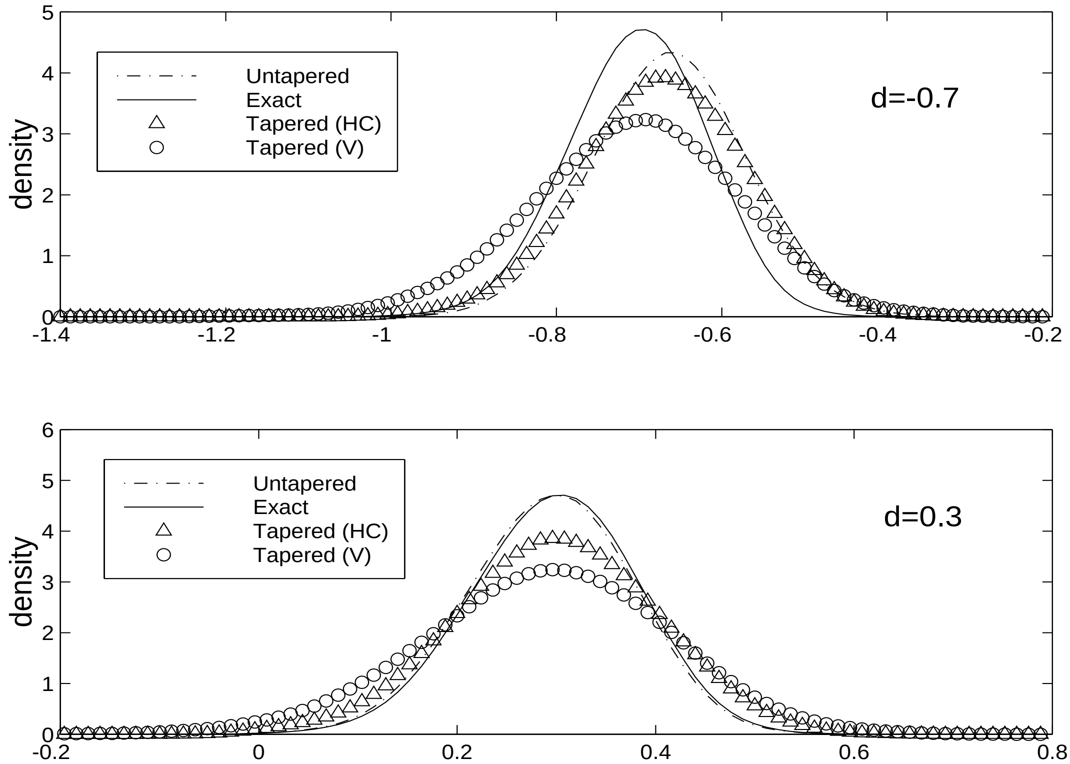

Figure 1: Densities of the four estimators: $n = 500$, $m = n^{0.65}$

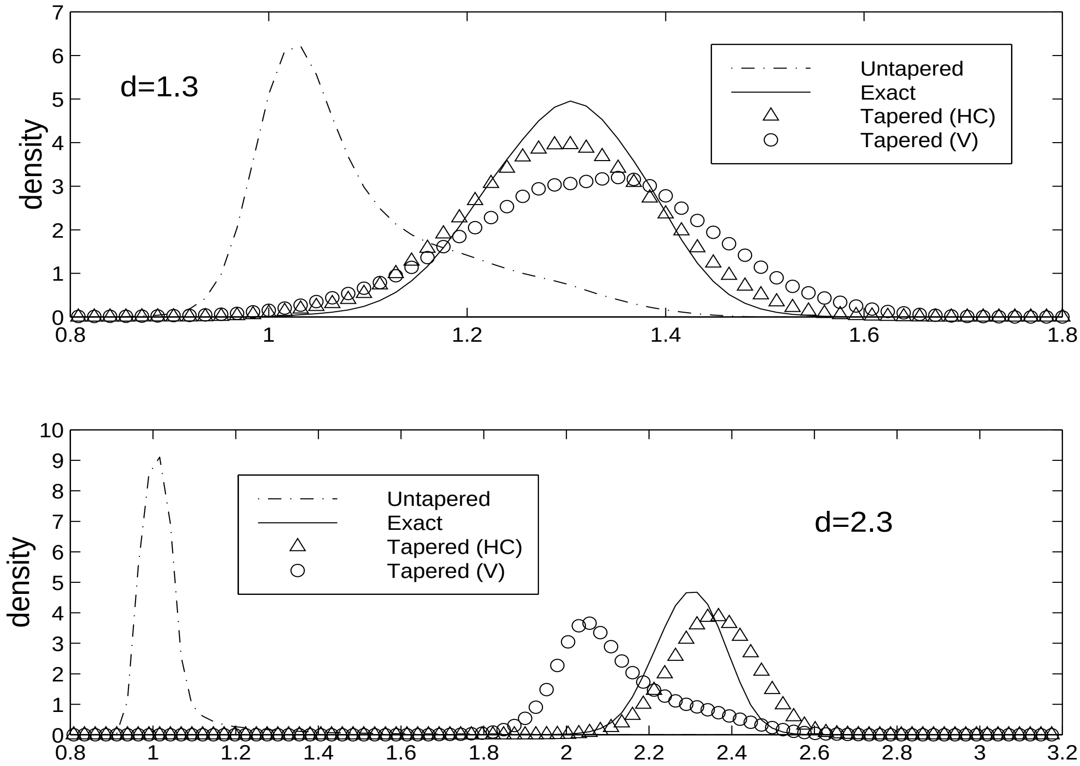

Figure 2: Densities of the four estimators: $n = 500$, $m = n^{0.65}$

5 Appendix A: Technical Lemmas

Lemma A0 is provided for reference, and the proof is omitted because it is trivial. Lemmas 5.1 - 5.3 extend Lemmas 8.1, 8.3, and 8.5 of Phillips and Shimotsu (2001) to hold uniformly in θ. Since their proofs are almost identical, we refer the reader to Phillips and Shimotsu for the proofs. In the following arguments, C and ε denote generic constants such that $C \in (1, \infty)$ and $\varepsilon \in (0, 1)$ unless specified otherwise.

Lemma A0 *As* $n \to \infty$,

(a) $\sup_{-1 \le \alpha \le C} \left| n^{-\alpha-1} (\log n)^{-1} \sum_{j=1}^{n} j^{\alpha} \right| = O(1)$.

(b) $\sup_{-C \le \alpha \le -1} \left| (\log n)^{-1} \sum_{j=1}^{n} j^{\alpha} \right| = O(1)$.

(c) For $1 \le p \le n$, $\sup_{-C \le \alpha \le -1} \left| p^{-\alpha-1} (\log n)^{-1} \sum_{j=p}^{n} j^{\alpha} \right| = O(1)$.

5.1 Lemma (Phillips and Shimotsu, Lemma 8.1)

Uniformly in $\theta \in [-1+\varepsilon, C]$ *and in* $s = 1, 2, \ldots, m$ *with* $m = o(n)$,

$$D_n\left(e^{i\lambda_s}; \theta\right) = \left(1 - e^{i\lambda_s}\right)^{\theta} + O\left(n^{-\theta} s^{-1}\right). \tag{14}$$

5.2 Lemma (Phillips and Shimotsu, Lemma 8.3)

Uniformly in $\theta \in [-1+\varepsilon, C]$ *and in* $s = 1, 2, \ldots, m$ *with* $m = o(n)$,

$$\begin{aligned}
\lambda_s^{-\theta}\left(1 - e^{i\lambda_s}\right)^{\theta} &= e^{-\frac{\pi}{2}\theta i} + O(\lambda_s), \\
\lambda_s^{-\theta} D_n\left(e^{i\lambda_s}; \theta\right) &= e^{-\frac{\pi}{2}\theta i} + O(\lambda_s) + O\left(s^{-1-\theta}\right), \\
\lambda_s^{-2\theta} D_n\left|\left(e^{i\lambda_s}; \theta\right)\right|^2 &= 1 + O\left(\lambda_s^2\right) + O\left(s^{-1-\theta}\right).
\end{aligned}$$

5.3 Lemma (Phillips and Shimotsu, Lemma 8.5)

For $p = 1, 2, \ldots, n$ *and* $s = 1, 2, \ldots, m$ *with* $m = o(n)$, *the following holds uniformly in* θ, p *and* s:

$$(a) \quad \widetilde{\theta}_{\lambda_s p} = \begin{cases} O(|p|_+^{-\theta} \log n), & \text{for } \theta \in [0, C], \\ O(n^{-\theta} \log n), & \text{for } \theta \in [-1+\varepsilon, 0], \end{cases} \tag{15}$$

$$(b) \quad \widetilde{\theta}_{\lambda_s p} = O(|p|_+^{-\theta-1} n s^{-1}), \quad \text{for } \theta \in [-1+\varepsilon, C], \tag{16}$$

where $|x|_+ = \max\{x, 1\}$.

5.4 Lemma

Let $\widetilde{U}_{\lambda n}(\theta) = \sum_{p=0}^{n-1} \widetilde{\theta}_{\lambda p} e^{-ip\lambda} u_{n-p}$. *Then, uniformly in* $s = 1, 2, \ldots, m$ *with* $m = o(n)$,

$$\begin{array}{llr}
(a) & E \sup_\theta n^{2\theta-1} s^{1-2\theta} (\log n)^{-6} |\widetilde{U}_{\lambda_s n}(\theta)|^2 = O(1), & \theta \in [0, \frac{1}{2}], \\
(b) & E \sup_\theta \left[n^{1-2\theta} s^{\theta-1} (\log n)^4 + n^{-2\theta} (\log n)^2\right]^{-1} |\widetilde{U}_{\lambda_s n}(\theta)|^2 = O(1), & \theta \in [-\frac{1}{2}, 0], \\
(c) & E \sup_\theta n^{2\theta-1} s (\log n)^{-4} |\widetilde{U}_{\lambda_s n}(\theta)|^2 = O(1), & \theta \in [-\frac{1}{2}, 0].
\end{array}$$

5.5 Proof

When $\theta = 0$, the result follows immediately because $\widetilde{U}_{\lambda_s n}(\theta) \equiv 0$. For $\theta \neq 0$, applying the BN decomposition

$$u_t = C(L)\varepsilon_t = C(1)\varepsilon_t - (1-L)\widetilde{\varepsilon}_t, \quad \widetilde{\varepsilon}_t = \sum_0^\infty \widetilde{c}_j \varepsilon_{t-j}, \quad \widetilde{c}_j = \sum_{j+1}^\infty c_s, \tag{17}$$

to $\widetilde{U}_{\lambda_s n}(\theta)$ in conjunction with Lemma 5.3 yields (see Phillips and Shimotsu, 2001)

$$\widetilde{U}_{\lambda_s n}(\theta) = C(1)\widetilde{\varepsilon}_{\lambda_s n}(\theta) + r_{ns}(\theta),$$

where

$$r_{ns}(\theta) = \begin{cases} O(\log n)\widetilde{\varepsilon}_n + \sum_1^{n-1} O(p^{-\theta-1})\widetilde{\varepsilon}_{n-p} + O(n^{-\theta}\log n)\widetilde{\varepsilon}_0, & \theta \geq 0, \\ O(n^{-\theta}\log n)\widetilde{\varepsilon}_n + \sum_1^{n-1} O(p^{-\theta-1})\widetilde{\varepsilon}_{n-p} + O(n^{-\theta}\log n)\widetilde{\varepsilon}_0, & \theta \leq 0, \end{cases}$$

uniformly in θ and $E|\widetilde{\varepsilon}_t|^2 < \infty$ for $t = 0, \ldots, n$. It follows that

$$E \sup_{\theta \in [0,1/2]} |r_{ns}(\theta)|^2 = O\left((\log n)^2\right), \quad E \sup_{\theta \in [-1/2,0]} n^{2\theta} |r_{ns}(\theta)|^2 = O\left((\log n)^2\right). \tag{18}$$

Since

$$\widetilde{\theta}_{\lambda_s p} = \Gamma\left(-\theta\right)^{-1} \sum_{p+1}^{n} k^{-\theta-1} e^{ik\lambda_s} + O\left(\sum_{p+1}^{n} k^{-\theta-2}\right),$$

we obtain

$$\widetilde{\varepsilon}_{\lambda_s n}\left(\theta\right) = \sum_{p=0}^{n-1} \widetilde{\theta}_{\lambda_s p} e^{-ip\lambda_s} \varepsilon_{n-p} = \widehat{\varepsilon}_{\lambda_s n}\left(\theta\right) + r'_{ns}\left(\theta\right),$$

where

$$\widehat{\varepsilon}_{\lambda_s n}\left(\theta\right) = \frac{1}{\Gamma\left(-\theta\right)} \sum_{p=0}^{n-1} \sum_{k=p+1}^{n} k^{-\theta-1} e^{i(k-p)\lambda_s} \varepsilon_{n-p}, \quad r'_{ns}\left(\theta\right) = \sum_{p=0}^{n-1} O\left(\sum_{p+1}^{n} k^{-\theta-2}\right) \varepsilon_{n-p}.$$

Since

$$\sup_{\theta \in [0,1/2]} \sum_{p+1}^{n} k^{-\theta-2} = O\left(|p|_{+}^{-1}\right), \qquad \sup_{\theta \in [-1/2,0]} n^{\theta} \sum_{p+1}^{n} k^{-\theta-2} = O\left(|p|_{+}^{-1}\right),$$

it follows that

$$E \sup_{\theta \in [0,1/2]} \left|r'_{ns}\left(\theta\right)\right|^2 = O\left(\left(\log n\right)^2\right), \quad E \sup_{\theta \in [-1/2,0]} n^{2\theta} \left|r'_{ns}\left(\theta\right)\right|^2 = O\left(\left(\log n\right)^2\right). \quad (19)$$

In view of (18), (19) and the fact $\sup_{\theta \in [-1/2,1/2]} \left|\Gamma\left(-\theta\right)^{-1}\right| < \infty$, the stated result follows if

$$E \sup_{\theta} \left| A_n\left(\theta\right) \sum_{p=0}^{n-1} \sum_{k=p+1}^{n} k^{-\theta-1} e^{i(k-p)\lambda_s} \varepsilon_{n-p} \right|^2 = O\left(1\right), \quad (20)$$

where

$$A_n\left(\theta\right) = \begin{cases} n^{\theta-1/2} s^{1/2-\theta} \left(\log n\right)^{-3}, & \theta \in [0,1/2], \\ n^{\theta-1/2} s^{1/2-\theta/2} \left(\log n\right)^{-2}, & \theta \in [-1/2,0], \end{cases}$$

and we suppress the subscript s from $A_n\left(\theta\right)$. Before showing (20), we slightly modify a result in Hansen (1996) and state it as a lemma.

Lemma H (Corollary of Hansen (1996) Theorem 2) *Let $\{X_{ni} : i \leq n; n = 1, 2, \ldots\}$ be a triangular array of random vectors defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\{\mathcal{F}_{ni}\}$ be an array of sub-σ-fields of $\mathcal{F}$, such that, for each n, $\{\mathcal{F}_{ni}\}$ is nondecreasing in i. Let F be a class of parametric functions $f_{ni}(x, \theta)$, where $\theta \in \Theta$, and Θ is a bounded subset of R^a. The elements $f_{ni} \in F$ satisfy the Lipschitz condition*

$$|f_{ni}(x, \theta) - f_{ni}(x, \theta')| \leq b_{ni}(x) |\theta - \theta'|^{\lambda}, \quad (21)$$

for some function $b_{ni}(\cdot)$ and some $\lambda > 0$. Define the empirical process operator ν_n by

$$\nu_n f\left(\theta\right) = n^{-1/2} \textstyle\sum_{i=1}^{n} (f_{ni}(X_{ni}, \theta) - E f_{ni}(X_{ni}, \theta)),$$

and let $||Z||_r$ denote the L^r norm for a random matrix Z, i.e. $||Z||_r = (E|Z|^r)^{1/r}$.

Suppose for some $q \geq 2$ with $q > a/\lambda$ and each $\theta \in \Theta$, $\{f_{ni}(X_{ni}, \theta), \mathcal{F}_{ni}\}$ is a martingale difference array satisfying

$$\limsup_{n \to \infty} \left(n^{-1} \textstyle\sum_{i=1}^{n} ||f_{ni}(X_{ni}, \theta)||_q^2\right)^{1/2} < \infty, \quad (22)$$

and

$$\limsup_{n\to\infty} \left(n^{-1} \textstyle\sum_{i=1}^n ||b_{ni}(X_{ni})||_q^2\right)^{1/2} < \infty. \tag{23}$$

Then, for any $\theta_1 \in \Theta$ *there exists a constant* $C < \infty$ *such that*

$$\limsup_{n\to\infty} ||\sup_{\theta\in\Theta} |\nu_n f(\theta) - \nu_n f(\theta_1)|\, ||_q < C. \tag{24}$$

Proof The stated result follows from Theorem 2 of Hansen (1996) and its proof. First, Lemma 1 and Theorem 2 of Hansen (1996) still hold if we replace his $f(X_{ni},\theta)$ with $f_{ni}(X_{ni},\theta)$. Let C be a generic positive finite constant. Then, from equation (27) of Hansen (1996), for n sufficiently large and a sequence of integers $k(n)$ that satisfies $\sqrt{n}2^{-k(n)\lambda} \to 0$,

$$||\sup_{\theta\in\Theta} |\nu_n f(\theta) - \nu_n f(\theta_{k(n)})|\, ||_q < C, \tag{25}$$

where $\theta_{k(n)}$ is defined on page 356 in Hansen (1996). Furthermore, from equations (29) and (31) in Hansen (1996) and Minkowski's inequality it follows that, for a finite constant A,

$$\begin{aligned} ||\sup_{\theta\in\Theta} |\nu_n f(\theta_{k(n)}) - \nu_n f(\theta_1)|\, ||_q &\le ||\textstyle\sum_{k=2}^{k(n)} \sup_{\theta\in\Theta} |\nu_n f(\theta_k) - \nu_n f(\theta_{k-1})|\, ||_q \\ &\le \textstyle\sum_{k=2}^{k(n)} ||\sup_{\theta\in\Theta} |\nu_n f(\theta_k) - \nu_n f(\theta_{k-1})|\, ||_q \\ &\le \textstyle\sum_{k=2}^{\infty} A 2^{(a/q-\lambda)k} < C, \end{aligned} \tag{26}$$

because $q > a/\lambda$. The stated result follows from (25) and (26). ■

We proceed to show (20). Let

$$f_{np}(\theta) = \sqrt{n}A_n(\theta) \sum_{k=p+1}^{n} k^{-\theta-1} e^{i(k-p)\lambda_s},$$

so that

$$A_n(\theta) \sum_0^{n-1} \sum_{p+1}^{n} k^{-\theta-1} e^{i(k-p)\lambda_s} \varepsilon_{n-p} = \frac{1}{\sqrt{n}} \sum_{p=0}^{n-1} f_{np}(\theta)\, \varepsilon_{n-p} = \frac{1}{\sqrt{n}} \sum_{r=1}^{n} f_{n,n-r}(\theta)\, \varepsilon_r.$$

(20) holds if

$$\limsup_{n\to\infty} E\left|n^{-1/2} \textstyle\sum_0^{n-1} f_{np}(0)\, \varepsilon_{n-p}\right|^2 < \infty, \tag{27}$$

and we can apply Lemma H to $n^{-1/2}\sum_{r=1}^n (f_{n,n-r}(\theta)\,\varepsilon_r - E f_{n,n-r}(\theta)\,\varepsilon_r)$. Because $E f_{n,n-r}(\theta)\,\varepsilon_r = 0$ for any θ, in order to apply Lemma H with $q = 2$, $a = 1$ and $\lambda = 1$, it suffices to show

$$\begin{aligned} |f_{np}(\theta) - f_{np}(\theta')| &\le b_{np}|\theta - \theta'|, && (28) \\ \limsup_{n\to\infty} \left(n^{-1} \textstyle\sum_0^{n-1} |f_{np}(\theta)|^2\right)^{1/2} &< \infty, && (29) \\ \limsup_{n\to\infty} \left(n^{-1} \textstyle\sum_0^{n-1} |b_{np}|^2\right)^{1/2} &< \infty. && (30) \end{aligned}$$

First we show (27) and (29). Observe that

$$\sum_{p+1}^{n} k^{-\theta-1} e^{i(k-p)\lambda_s} \le (p+1)^{-\theta-1} \max_N \left|\sum_{p+1}^{N} e^{i(k-p)\lambda_s}\right| = O\left(|p|_+^{-\theta-1} n s^{-1}\right), \tag{31}$$

and

$$\sum_{p+1}^{n} k^{-\theta-1} e^{i(k-p)\lambda_s} = \begin{cases} O\left(|p|_+^{-\theta}\log n\right), & \theta\in[0,1/2], \\ O\left(n^{-\theta}\log n\right), & \theta\in[-1/2,0], \end{cases}$$

uniformly in θ. Then, for $\theta\in[0,1/2]$,

$$\begin{aligned} |f_{np}(\theta)|^2 &= O\left(n^{2\theta}s^{1-2\theta}|p|_+^{-2\theta}(\log n)^{-4}\right) \\ &= O\left(n|p|_+^{-1}(s|p|_+/n)^{1-2\theta}(\log n)^{-4}\right) = O\left(n|p|_+^{-1}(\log n)^{-4}\right),\ p\le n/s, \\ |f_{np}(\theta)|^2 &= O\left(n^{2\theta}s^{1-2\theta}|p|_+^{-2\theta-1}ns^{-1}(\log n)^{-5}\right) \\ &= O\left(n|p|_+^{-1}(s|p|_+/n)^{-2\theta}(\log n)^{-5}\right) = O\left(n|p|_+^{-1}(\log n)^{-5}\right),\ p\ge n/s, \end{aligned}$$

and for $\theta\in[-1/2,0]$,

$$\begin{aligned} |f_{np}(\theta)|^2 &= O\left(n^{2\theta}s^{1-\theta}|p|_+^{-\theta-1}ns^{-1}n^{-\theta}(\log n)^{-3}\right) \\ &= O\left(n|p|_+^{-1}(s|p|_+/n)^{-\theta}(\log n)^{-3}\right) = O\left(n|p|_+^{-1}(\log n)^{-3}\right),\ p\le n/s, \\ |f_{np}(\theta)|^2 &= O\left(n^{2\theta}s^{1-\theta}|p|_+^{-2\theta-2}n^2s^{-2}(\log n)^{-4}\right) \\ &= O\left(n|p|_+^{-1}(s|p|_+/n)^{-2\theta-1}s^{\theta}(\log n)^{-4}\right) = O\left(n|p|_+^{-1}(\log n)^{-4}\right),\ p\ge n/s. \end{aligned}$$

It follows that, for any $\theta\in[-1/2,1/2]$,

$$\limsup_{n\to\infty} E\left|n^{-1/2}\textstyle\sum_0^{n-1} f_{np}(\theta)\varepsilon_{n-p}\right|^2 = \limsup_{n\to\infty} n^{-1}\textstyle\sum_0^{n-1}|f_{np}(\theta)|^2<\infty,$$

giving (27) and (29).

We proceed to show (28) and (30). For $\theta\in[0,1/2]$, observe that

$$\frac{\partial}{\partial\theta}f_{np}(\theta) = \sqrt{n}A_n(\theta)\sum_{p+1}^{n}k^{-\theta-1}\left[\log(n/k)-\log s\right]e^{i(k-p)\lambda_s}.$$

Similarly as above, we have

$$\begin{aligned} \sum_{p+1}^{n}k^{-\theta-1}\log(n/k)e^{i(k-p)\lambda_s} &\le (p+1)^{-\theta-1}\log\left(\frac{n}{p+1}\right)\max_N\left|\sum_{p+1}^{N}e^{i(k-p)\lambda_s}\right| \\ &= O\left(|p|_+^{-\theta-1}ns^{-1}\log n\right), \end{aligned} \tag{32}$$

and

$$\sum_{p+1}^{n}k^{-\theta-1}\log(n/k)e^{i(k-p)\lambda_s} = \begin{cases} O\left(|p|_+^{-\theta}(\log n)^2\right), & \theta\in[0,1/2], \\ O\left(n^{-\theta}(\log n)^2\right), & \theta\in[-1/2,0], \end{cases} \tag{33}$$

uniformly in $\theta\in[-1/2,1/2]$. The same bounds hold for $\log s\sum_{p+1}^{n}k^{-\theta-1}e^{i(k-p)\lambda_s}$. Then, proceeding as for $|f_{np}(\theta)|^2$, we obtain uniformly in θ

$$\left|\frac{\partial}{\partial\theta}f_{np}(\theta)\right|^2 = O\left(n|p|_+^{-1}(\log n)^{-2}\right).$$

Now we consider the case $\theta \in [-1/2, 0]$. First observe

$$\frac{\partial}{\partial \theta} f_{np}(\theta) = \sqrt{n} A_{ns}(\theta) \sum_{p+1}^{n} k^{-\theta-1} \left[\log(n/k) - 0.5 \log s\right] e^{i(k-p)\lambda_s}.$$

Similarly as above, from (32) and (33) we obtain

$$\left| \frac{\partial}{\partial \theta} f_{np}(\theta) \right|^2 = O\left(n |p|_+^{-1} (\log n)^{-1} \right).$$

Since $f_{np}(\theta)$ is a differentiable function, the mean value theorem gives

$$|f_{np}(\theta) - f_{np}(\theta')| = \left| \frac{\partial}{\partial \theta} f_{np}(\theta_p) \right| |\theta - \theta'|,$$

where $\theta_p \in [\theta, \theta']$. Define $b_{np} = Bn^{1/2}|p|_+^{-1/2}(\log n)^{-1/2}$ for some large B, then b_{np} satisfies (28), and

$$\limsup_{n\to\infty} n^{-1} \textstyle\sum_0^{n-1} |b_{np}|^2 = B^2 \limsup_{n\to\infty} \sum_0^{n-1} |p|_+^{-1}(\log n)^{-1} < \infty,$$

giving (30) to complete the proof. ■

5.6 Lemma

$$\begin{array}{lll} (a) & E \sup_d n^{1-2d}(\log n)^{-4} X_n^2 = O(1), & d \in [1/2, C], \\ (b) & E \sup_d (\log n)^{-4} X_n^2 = O(1), & d \in [-C, 1/2]. \end{array}$$

5.7 Proof

¿From the proof of Lemma 8.11 of Phillips and Shimotsu (2001), we have

$$X_n = C(1) \sum_0^{n-1} \frac{(d)_k}{k!} \varepsilon_{n-k} + r_n(d) = \frac{C(1)}{\Gamma(d)} \sum_1^{n-1} k^{d-1} \varepsilon_{n-k} + r_n(d) + r_n'(d),$$

where

$$r_n(d) = \sum_0^{n-1} O\left(|k|_+^{d-2}\right) \widetilde{\varepsilon}_{n-k} + O\left(n^{d-1}\right) \widetilde{\varepsilon}_0, \quad r_n'(d) = \sum_0^{n-1} O\left(|k|_+^{d-2}\right) \varepsilon_{n-k}.$$

It follows that

$$\begin{array}{ll} E \sup_d n^{2-2d} |r_n(d) + r_n'(d)|^2 = O((\log n)^2), & d \in [1, C], \\ E \sup_d |r_n(d) + r_n'(d)|^2 = O((\log n)^2), & d \in [-C, 1]. \end{array}$$

We proceed to show

$$E \sup_d \left| \sum_1^{n-1} A_n(d) k^{d-1} \varepsilon_{n-k} \right|^2 = O(1); \quad A_n(d) = \begin{cases} n^{1/2-d}(\log n)^{-2}, & d \in [\frac{1}{2}, C], \\ (\log n)^{-2}, & d \in [-C, \frac{1}{2}], \end{cases} \tag{34}$$

then the required result follows. In view of the proof of Lemma 5.4 , (34) holds if

$$\begin{aligned} \sup_d \left| A_n(d) k^{d-1} \right| &= O\left(k^{-1/2} (\log n)^{-1/2}\right), \quad k = 1, \ldots n-1. \\ \sup_d \left| \frac{\partial}{\partial d} A_n(d) k^{d-1} \right| &= O\left(k^{-1/2} (\log n)^{-1/2}\right), \quad k = 1, \ldots n-1, \end{aligned}$$

Trivially the above two conditions are met, thereby giving the stated result. ■

5.8 Lemma

Define $J_n(L) = \sum_1^n \frac{1}{k} L^k$ *and* $D_n(L;d) = \sum_0^n \frac{(-d)_k}{k!} L^k$. *Then*

$$\begin{aligned} (a) \quad & J_n(L) = J_n\left(e^{i\lambda}\right) + \widetilde{J}_{n\lambda}\left(e^{-i\lambda}L\right)\left(e^{-i\lambda}L - 1\right), \\ (b) \quad & J_n(L) D_n(L;d) = J_n\left(e^{i\lambda}\right) D_n\left(e^{i\lambda};d\right) + D_n\left(e^{i\lambda};d\right) \widetilde{J}_{n\lambda}\left(e^{-i\lambda}L\right)\left(e^{-i\lambda}L - 1\right) \\ & \qquad + J_n(L) \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right)\left(e^{-i\lambda}L - 1\right), \end{aligned}$$

where

$$\begin{aligned} \widetilde{J}_{n\lambda}\left(e^{-i\lambda}L\right) &= \textstyle\sum_{p=0}^{n-1} \widetilde{j}_{\lambda p} e^{-ip\lambda} L^p, & \widetilde{j}_{\lambda p} &= \textstyle\sum_{p+1}^n \frac{1}{k} e^{ik\lambda}, \\ \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right) &= \textstyle\sum_{p=0}^{n-1} \widetilde{d}_{\lambda p} e^{-ip\lambda} L^p, & \widetilde{d}_{\lambda p} &= \textstyle\sum_{p+1}^n \frac{(-d)_k}{k!} e^{ik\lambda}. \end{aligned}$$

5.9 Proof

For part (a), see Phillips and Solo (1992, formula (32)). For part (b), from Lemma 2.1 of Phillips (1999) we have

$$D_n(L;d) = D_n\left(e^{i\lambda};d\right) + \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L;d\right)\left(e^{-i\lambda}L - 1\right).$$

and the stated result follows immediately. ■

5.10 Lemma

Uniformly in $p = 1, \ldots, n$ *and* $s = 1, \ldots, m$ *with* $m = o(n)$,

$$\begin{aligned} (a) \qquad & J_n\left(e^{i\lambda_s}\right) = -\log\lambda_s + \frac{i}{2}(\pi - \lambda_s) + O\left(\lambda_s^2\right) + O\left(s^{-1}\right), \\ (b) \qquad & \widetilde{j}_{\lambda_s p} = O\left(|p|_+^{-1} n s^{-1}\right), \\ (c) \qquad & \widetilde{j}_{\lambda_s p} = O(\log n). \end{aligned}$$

5.11 Proof

For (a), first we have

$$J_n\left(e^{i\lambda_s}\right) = \sum_1^n \frac{1}{k} e^{ik\lambda_s} = \sum_1^\infty \frac{1}{k} e^{ik\lambda_s} - \sum_{n+1}^\infty \frac{1}{k} e^{ik\lambda_s}. \tag{35}$$

The first term is (Zygmund, 1977, p.5)

$$\sum_1^\infty \frac{\cos k\lambda_s}{k} + i\sum_1^\infty \frac{\sin k\lambda_s}{k} = -\log\left|2\sin\frac{\lambda_s}{2}\right| + i\frac{1}{2}(\pi - \lambda_s).$$

Since $2\sin(\lambda_s/2) = \lambda_s + O(\lambda_s^3) = \lambda_s(1 + O(\lambda_s^2))$, it follows that

$$\sum_1^\infty \frac{1}{k} e^{ik\lambda_s} = -\log\lambda_s - \log\left(1 + O\left(\lambda_s^2\right)\right) + i\frac{1}{2}(\pi - \lambda_s) = -\log\lambda_s + O\left(\lambda_s^2\right) + \frac{i}{2}(\pi - \lambda_s).$$

For the second term in (35), from Theorem 2.2 of Zygmund (1977, p.3) and the ordinary summation formula, we obtain

$$\sum_{n+1}^\infty \frac{1}{k} e^{ik\lambda_s} \leq n^{-1} \max_N \left|\sum_{n+1}^{n+N} e^{ik\lambda_s}\right| = O\left(s^{-1}\right),$$

giving (a). (b) and (c) follows from the fact that $|\widetilde{j}_{\lambda_s p}| \leq (p+1)^{-1} \max_{p+1\leq N\leq n} |\sum_{p+1}^{N} e^{ik\lambda_s}|$ and $\widetilde{j}_{\lambda_s p} = O\left(\sum_0^n |k|_+^{-1}\right)$. ■

5.12 Lemma

Suppose $Y_t = (1-L)^{\theta} u_t$. *Then, uniformly in* θ *and* $s = 1, \ldots, m$ *with* $m = o(n)$,

$$(a) \quad -w_{\log(1-L)y}(\lambda_s) = J_n\left(e^{i\lambda_s}\right) D_n\left(e^{i\lambda_s};\theta\right) w_u(\lambda_s) + n^{-1/2} V_{ns}(\theta),$$

where

$$\begin{array}{ll} E\sup_\theta n^{2\theta-1} s^{1-2\theta} (\log n)^{-8} |V_{ns}(\theta)|^2 = O(1), & \theta\in[0,1/2], \\ E\sup_\theta n^{2\theta-1} s (\log n)^{-6} |V_{ns}(\theta)|^2 = O(1), & \theta\in[-1/2,0]. \end{array}$$

$$(b) \quad -w_{\log(1-L)u}(\lambda_s) = J_n\left(e^{i\lambda_s}\right) w_u(\lambda_s) - C(1)(2\pi n)^{-1/2} \widetilde{J}_{n\lambda_s}\left(e^{-i\lambda_s}L\right)\varepsilon_n + r_{ns},$$

where $E|r_{ns}|^2 = O(n^{-1}(\log n)^2)$.

$$(c) \quad w_{(\log(1-L))^2 y}(\lambda_s) = J_n\left(e^{i\lambda_s}\right)^2 D_n\left(e^{i\lambda_s};\theta\right) w_u(\lambda_s) + n^{-1/2}\Psi_{ns}(\theta),$$

where

$$\begin{array}{ll} E\sup_\theta n^{2\theta-1} s^{1-2\theta} (\log n)^{-10} |\Psi_{ns}(\theta)|^2 = O(1), & \theta\in[0,1/2], \\ E\sup_\theta n^{2\theta-1} s (\log n)^{-8} |\Psi_{ns}(\theta)|^2 = O(1), & \theta\in[-1/2,0]. \end{array}$$

5.13 Proof

Recall that $Y_t = D_n(L;\theta) u_t$ and

$$\log(1-L) Y_t = \left(-L - L^2/2 - L^3/3 - \ldots\right) Y_t = -J_n(L) Y_t.$$

For part (a) and (b), from Lemma 5.8 (b) we have for all $t \leq n$

$$\begin{aligned} -\log(1-L) Y_t &= J_n(L) D_n(L;\theta) u_t \\ &= J_n\left(e^{i\lambda_s}\right) D_n\left(e^{i\lambda_s};\theta\right) u_t && (36) \\ &\quad + D_n\left(e^{i\lambda_s};\theta\right) \widetilde{J}_{n\lambda_s}\left(e^{-i\lambda_s}L\right)\left(e^{-i\lambda_s}L - 1\right) u_t && (37) \\ &\quad + J_n(L) \widetilde{D}_{n\lambda_s}\left(e^{-i\lambda_s}L;\theta\right)\left(e^{-i\lambda_s}L - 1\right) u_t. && (38) \end{aligned}$$

Since $\sum_{t=1}^n e^{it\lambda_s}(e^{-i\lambda_s}L - 1)u_t = -u_n$, taking the dft of (36) - (38) leaves us with

$$\begin{aligned} &J_n\left(e^{i\lambda_s}\right) D_n\left(e^{i\lambda_s};\theta\right) w_u(\lambda_s) - \frac{1}{\sqrt{2\pi n}} D_n\left(e^{i\lambda_s};\theta\right) \widetilde{J}_{n\lambda_s}\left(e^{-i\lambda_s}L\right) u_n \\ &- \frac{1}{\sqrt{2\pi n}} J_n(L) \widetilde{U}_{\lambda_s n}(\theta). && (39) \end{aligned}$$

Applying the BN decomposition to $\widetilde{J}_{n\lambda_s}(e^{-i\lambda_s}L)u_n$ yields

$$\begin{aligned} \widetilde{J}_{n\lambda_s}\left(e^{-i\lambda_s}L\right) u_n &= \sum_0^{n-1} \widetilde{j}_{\lambda_s p} e^{-ip\lambda_s} u_{n-p} = \sum_0^{n-1} \widetilde{j}_{\lambda_s p} e^{-ip\lambda_s} \left[C(1)\varepsilon_{n-p} - (1-L)\widetilde{\varepsilon}_{n-p}\right] \\ &= C(1) \widetilde{J}_{n\lambda_s}\left(e^{-i\lambda_s}L\right)\varepsilon_n - \sum_0^{n-1} \widetilde{j}_{\lambda_s p} e^{-ip\lambda_s} (1-L)\widetilde{\varepsilon}_{n-p}. && (40) \end{aligned}$$

For the first term in (40), from Lemma 5.10 we have

$$E\left|\widetilde{J}_{n\lambda_s}\left(e^{-i\lambda_s}L\right)\varepsilon_n\right|^2 = O\left((\log n)^2 + \sum_1^{n-1} p^{-1}ns^{-1}\log n\right) = O\left(ns^{-1}(\log n)^2\right).$$

The second term in (40) can be rewritten as follows:

$$\begin{aligned}
&\sum_0^{n-1}\widetilde{j}_{\lambda_s p}e^{-ip\lambda_s}(1-L)\widetilde{\varepsilon}_{n-p} \\
= {} & \widetilde{j}_{\lambda_s 0}\widetilde{\varepsilon}_n + \sum_1^{n-1}\widetilde{j}_{\lambda_s p}e^{-ip\lambda_s}\widetilde{\varepsilon}_{n-p} - \sum_1^{n-1}\widetilde{j}_{\lambda_s(p-1)}e^{-i(p-1)\lambda_s}\widetilde{\varepsilon}_{n-p} - \widetilde{j}_{\lambda_s(n-1)}e^{-i(n-1)\lambda_s}\widetilde{\varepsilon}_0 \\
= {} & \sum_1^{n-1}\left[\widetilde{j}_{\lambda_s p}e^{-ip\lambda_s} - \widetilde{j}_{\lambda_s(p-1)}e^{-i(p-1)\lambda_s}\right]\widetilde{\varepsilon}_{n-p} + \widetilde{j}_{\lambda_s 0}\widetilde{\varepsilon}_n - \widetilde{j}_{\lambda_s(n-1)}e^{-i(n-1)\lambda_s}\widetilde{\varepsilon}_0. \qquad (41)
\end{aligned}$$

$\widetilde{j}_{\lambda_s 0}$, $\widetilde{j}_{\lambda_s(n-1)} = O(\log n)$ from Lemma 5.10 (c), and from Lemma 5.10 (b) and the fact that $\widetilde{j}_{\lambda_s p} - \widetilde{j}_{\lambda_s(p-1)} = p^{-1}e^{ip\lambda_s}$ we obtain

$$\begin{aligned}
\widetilde{j}_{\lambda_s p}e^{-ip\lambda_s} - \widetilde{j}_{\lambda_s(p-1)}e^{-i(p-1)\lambda_s} &= \widetilde{j}_{\lambda_s p}\left[e^{-ip\lambda_s} - e^{-i(p-1)\lambda_s}\right] + e^{-i(p-1)\lambda_s}\left[\widetilde{j}_{\lambda_s p} - \widetilde{j}_{\lambda_s(p-1)}\right] \\
&= \widetilde{j}_{\lambda_s p}e^{-ip\lambda_s}\left(1-e^{i\lambda_s}\right) - e^{-i(p-1)\lambda_s}p^{-1}e^{ip\lambda_s} = O\left(p^{-1}\right).
\end{aligned}$$

Since $E|\widetilde{\varepsilon}_t|^2 < \infty$, it follows that $E\sup_s|(41)|^2 = O((\log n + \sum_1^n p^{-1})^2) = O((\log n)^2)$. Therefore, in view of the order of $D_n(e^{i\lambda_s};\theta)$ given by Lemma 5.1 we have

$$E\sup_\theta n^{2\theta-1}s^{1-2\theta}\left|D_n\left(e^{i\lambda_s};\theta\right)\widetilde{J}_{n\lambda_s}\left(e^{-i\lambda_s}L\right)u_n\right|^2 = O\left((\log n)^2\right), \qquad (42)$$

uniformly in s. Now we evaluate (39). Let

$$a_{ns}(\theta) = \begin{cases} n^{2\theta-1}s^{1-2\theta}(\log n)^{-8}, & \theta\in[0,1/2], \\ n^{2\theta-1}s(\log n)^{-6}, & \theta\in[-1/2,0]. \end{cases}$$

In view of the proof of Lemma 5.4, $L^p\widetilde{U}_{\lambda_s n}(\theta)$ has the same order as $\widetilde{U}_{\lambda_s n}(\theta)$. Thus

$$\begin{aligned}
&E\sup_\theta a_{ns}(\theta)\left|J_n(L)\widetilde{U}_{\lambda_s n}(\theta)\right|^2 \\
\le {} & E\sum_1^{n-1}\sum_1^{n-1}p^{-1}q^{-1}\sup_\theta(a_{ns}(\theta))^{1/2}\left|L^p\widetilde{U}_{\lambda_s n}(\theta)\right|\sup_\theta(a_{ns}(\theta))^{1/2}\left|L^q\widetilde{U}_{\lambda_s n}(\theta)\right| \\
\le {} & \sum_1^{n-1}\sum_1^{n-1}p^{-1}q^{-1}\left[E\sup_\theta a_{ns}(\theta)\left|L^p\widetilde{U}_{\lambda_s n}(\theta)\right|^2\right]^{1/2}\left[E\sup_\theta a_{ns}(\theta)\left|L^q\widetilde{U}_{\lambda_s n}(\theta)\right|^2\right]^{1/2} \\
= {} & O(1). \qquad (43)
\end{aligned}$$

Combining (42) and (43) gives part (a). If $d=0$, (39)$\equiv 0$ and $D_n(e^{i\lambda_s};0) = 1$, and part (a) follows immediately.

For part (b), observe that

$$\begin{aligned}
-w_{\log(1-L)u}(\lambda_s) &= J_n\left(e^{i\lambda_s}\right)w_u(\lambda_s) - C(1)(2\pi n)^{-1/2}\widetilde{J}_{n\lambda_s}\left(e^{-i\lambda_s}L\right)\varepsilon_n \\
&+ (2\pi n)^{-1/2}\textstyle\sum_0^{n-1}\widetilde{j}_{\lambda_s p}e^{-ip\lambda_s}(1-L)\widetilde{\varepsilon}_{n-p},
\end{aligned}$$

and finding the order of the last term gives the stated result.

For part (c), first from Lemma 5.8 and Lemma 2.1 of Phillips (1999) we have

$$\begin{aligned}
J_n(L)^2 &= J_n(L)\,[J_n(e^{i\lambda}) + \widetilde{J}_{n\lambda}(e^{-i\lambda}L)(e^{-i\lambda}L-1)] \\
&= J_n(L)\,J_n(e^{i\lambda}) + J_n(L)\,\widetilde{J}_{n\lambda}(e^{-i\lambda}L)(e^{-i\lambda}L-1) \\
&= J_n(e^{i\lambda})^2 + J_n(e^{i\lambda})\widetilde{J}_{n\lambda}(e^{-i\lambda}L)(e^{-i\lambda}L-1) \\
&\quad + J_n(L)\,\widetilde{J}_{n\lambda}(e^{-i\lambda}L)(e^{-i\lambda}L-1), \\
D_n(L;\theta) &= D_n(e^{i\lambda};\theta) + \widetilde{D}_{n\lambda}(e^{-i\lambda}L;\theta)(e^{-i\lambda}L-1).
\end{aligned}$$

It follows that

$$\begin{aligned}
(\log(1-L))^2\,Y_t &= J_n(L)^2\,D_n(L;\theta)\,u_t \\
&= J_n(e^{i\lambda})^2 D_n(e^{i\lambda};\theta)u_t \\
&\quad + D_n(e^{i\lambda};\theta)[J_n(e^{i\lambda}) + J_n(L)]\widetilde{J}_{n\lambda}(e^{-i\lambda}L)(e^{-i\lambda}L-1)u_t \\
&\quad + J_n(L)^2\,\widetilde{D}_{n\lambda}(e^{-i\lambda}L;\theta)(e^{-i\lambda}L-1)u_t.
\end{aligned}$$

Taking its dft gives

$$\begin{aligned}
& J_n(e^{i\lambda_s})^2 D_n(e^{i\lambda_s};\theta)w_u(\lambda_s) \\
& -\frac{1}{\sqrt{2\pi n}} D_n(e^{i\lambda_s};\theta)[J_n(e^{i\lambda_s}) + J_n(L)]\widetilde{J}_{n\lambda_s}(e^{-i\lambda_s}L)u_n \\
& -\frac{1}{\sqrt{2\pi n}} J_n(L)^2\,\widetilde{U}_{\lambda_s n}(\theta).
\end{aligned}$$

In view of (42) and (43), we obtain

$$\begin{aligned}
E\sup_{\theta} n^{2\theta-1}s^{1-2\theta}\left|D_n(e^{i\lambda_s};\theta)[J_n(e^{i\lambda_s}) + J_n(L)]\widetilde{J}_{n\lambda_s}(e^{-i\lambda_s}L)u_n\right|^2 &= O((\log n)^4), \\
E\sup_{\theta} a_{ns}(\theta)\left|J_n(L)^2\,\widetilde{U}_{\lambda_s n}(\theta)\right|^2 &= O((\log n)^2),
\end{aligned}$$

for $s = 1, \ldots, m$, and the stated result follows. ■

6 Appendix B: Proofs

6.1 Proof of consistency

Define $G(d) = G_0 \frac{1}{m}\sum_1^m \lambda_j^{2(d-d_0)}$ and $S(d) = R(d) - R(d_0)$. Rewrite $S(d)$ as follows:

$$\begin{aligned}
S(d) &= R(d) - R(d_0) \\
&= \log\frac{\widehat{G}(d)}{G(d)} - \log\frac{\widehat{G}(d_0)}{G_0} + \log\left(\frac{1}{m}\sum_{j=1}^{m} j^{2d-2d_0} \Big/ \frac{m^{2(d-d_0)}}{2(d-d_0)+1}\right) \\
&\quad -(2d-2d_0)\left[\frac{1}{m}\sum_{j=1}^{m}\log j - (\log m - 1)\right] \\
&\quad +(2d-2d_0) - \log(2(d-d_0)+1).
\end{aligned}$$

For arbitrary small $\Delta > 0$, define $\Theta_1^a = \{d : d_0 - \frac{1}{2} + \Delta \le d \le d_0 + \frac{1}{2}\}$, $\Theta_1^b = \{d : d_0 + \frac{1}{2} \le d \le \Delta_2\}$ and $\Theta_2 = \{d : \Delta_1 \le d \le d_0 - \frac{1}{2} + \Delta\}$, Θ_1^b and Θ_2 being possibly empty. Without loss of generality we assume $\Delta < \frac{1}{8}$ hereafter. In view of the arguments in Robinson (1995), $\widehat{d} \to_p d_0$ if

$$\sup_{\Theta_1^a} |T(d)| \to_p 0,$$

and

$$\Pr\left(\inf_{\Theta_1^b} S(d) \leq 0\right) \to 0, \quad \Pr\left(\inf_{\Theta_2} S(d) \leq 0\right) \to 0,$$

as $n \to \infty$, where

$$\begin{aligned} T(d) &= \log \frac{\widehat{G}(d_0)}{G_0} - \log \frac{\widehat{G}(d)}{G(d)} - \log\left(\frac{1}{m}\sum_{j=1}^{m} j^{2d-2d_0} / \frac{m^{2d-2d_0}}{2(d-d_0)+1}\right) \\ & \quad +(2d-2d_0)\left[\frac{1}{m}\sum_{1}^{m} \log j - (\log m - 1)\right]. \end{aligned}$$

Robinson (1995) shows that the fourth term on the right hand side is $O(\log m/m)$ uniformly in $d \in \Theta_1^a \cup \Theta_1^b$ and

$$\sup_{\Theta_1^a \cup \Theta_1^b} \left| \frac{2(d-d_0)+1}{m} \sum_{1}^{m} \left(\frac{j}{m}\right)^{2d-2d_0} - 1 \right| = O\left(\frac{1}{m^{2\Delta}}\right). \tag{44}$$

Note that

$$\begin{aligned} \frac{\widehat{G}(d) - G(d)}{G(d)} &= \frac{m^{-1}\sum_1^m \lambda_j^{2(d-d_0)} \lambda_j^{2(d_0-d)} I_{\Delta^d x}(\lambda_j) - G_0 m^{-1} \sum_1^m \lambda_j^{2(d-d_0)}}{G_0 m^{-1} \sum_1^m \lambda_j^{2(d-d_0)}} \\ &= \frac{m^{-1}\sum_1^m (j/m)^{2(d-d_0)} \lambda_j^{2(d_0-d)} I_{\Delta^d x}(\lambda_j) - G_0 m^{-1} \sum_1^m (j/m)^{2(d-d_0)}}{G_0 m^{-1} \sum_1^m (j/m)^{2(d-d_0)}} \\ &= \frac{[2(d-d_0)+1]\, m^{-1}\sum_1^m (j/m)^{2(d-d_0)} \left[\lambda_j^{2(d_0-d)} I_{\Delta^d x}(\lambda_j) - G_0\right]}{[2(d-d_0)+1]\, G_0 m^{-1} \sum_1^m (j/m)^{2(d-d_0)}} \\ &= \frac{A(d)}{B(d)}. \end{aligned} \tag{45}$$

Therefore, by the fact that $\Pr(|\log Y| \geq \varepsilon) \leq 2\Pr(|Y-1| \geq \varepsilon/2)$ for any nonnegative random variable Y and $\varepsilon \leq 1$, $\sup_{\Theta_1^a} |T(d)| \to_p 0$ if

$$\sup_{\Theta_1^a} |A(d)/B(d)| \to_p 0. \tag{46}$$

Define $Y_t = (1-L)^d X_t$. Then

$$Y_t = (1-L)^{d-d_0} (1-L)^{d_0} X_t = (1-L)^{\theta} u_t I\{t \geq 1\},$$

where $\theta \equiv d - d_0$. Hereafter, we use the notation $Y_t \sim I(\alpha)$ when Y_t is generated by (1) with parameter α. So $Y_t \sim I(-\theta)$. Note that

$$d \in \Theta_1^a \Leftrightarrow -\tfrac{1}{2} + \Delta \leq \theta \leq \tfrac{1}{2}.$$

Applying Lemma 2.2 (a) to (Y_t, u_t) replacing the role of u_t, we obtain

$$w_y(\lambda_j) = w_u(\lambda_j) D_n\left(e^{i\lambda_j}; \theta\right) - \frac{1}{\sqrt{2\pi n}} \widetilde{U}_{\lambda_j n}(\theta), \tag{47}$$

and $A(d)$ can be written as, with $g = 2(d - d_0) + 1$,

$$A(d) = \frac{g}{m} \sum_{1}^{m} \left(\frac{j}{m}\right)^{2\theta} \left[\lambda_j^{-2\theta} I_y(\lambda_j) - G_0\right].$$

Hereafter let I_{yj} denote $I_y(\lambda_j)$, w_{uj} denote $w_u(\lambda_j)$, and similarly for other dft's and periodograms. From a similar argument as Robinson (1995, p. 1636), $\sup_{\Theta_1^a} |A(d)|$ is bounded by

$$\sum_{r=1}^{m-1} \left(\frac{r}{m}\right)^{2\Delta} \frac{1}{r^2} \sup_{\Theta_1^a} \left| \sum_{j=1}^{r} \left[\lambda_j^{-2\theta} I_{yj} - G_0\right] \right| + \frac{1}{m} \sup_{\Theta_1^a} \left| \sum_{j=1}^{m} \left[\lambda_j^{-2\theta} I_{yj} - G_0\right] \right|. \tag{48}$$

Now

$$\begin{aligned} & \lambda_j^{-2\theta} I_{yj} - G_0 \\ = \;& \lambda_j^{-2\theta} I_{yj} - \lambda_j^{-2\theta} \left|D_n(e^{i\lambda_j};\theta)\right|^2 I_{uj} + \left[\lambda_j^{-2\theta} \left|D_n(e^{i\lambda_j};\theta)\right|^2 - f_u(0)/f_u(\lambda_j)\right] I_{uj} \\ & + \left[I_{uj} - |C(e^{i\lambda_j})|^2 I_{\varepsilon j}\right] f_u(0)/f_u(\lambda_j) + f_u(0)(2\pi I_{\varepsilon j} - 1). \end{aligned} \tag{49}$$

¿From Lemma 5.2 and arguments in Phillips and Shimotsu (2001, pp. 18-19), for any $\eta > 0$

$$\begin{aligned} & \sum_{1}^{m} \left(\frac{r}{m}\right)^{2\Delta} \frac{1}{r^2} \sup_{\Theta_1^a} \sum_{1}^{r} \left| \left[\lambda_j^{-2\theta} \left|D_n(e^{i\lambda_j};\theta)\right|^2 - f_u(0)/f_u(\lambda_j)\right] I_{uj} \right. \\ & \left. + \left[I_{uj} - |C(e^{i\lambda_j})|^2 I_{\varepsilon j}\right] f_u(0)/f_u(\lambda_j) \right| = O_p\left(\eta + m^2 n^{-2} + m^{-2\Delta} + n^{-1/2}\right). \end{aligned}$$

Robinson (1995) shows that $\sum_1^m (r/m)^{2\Delta} r^{-2} |\sum_1^r (2\pi I_{\varepsilon j} - 1)| \to_p 0$. From (47), the fact that $||A|^2 - |B|^2| \le |A+B||A-B|$ and the Cauchy-Schwartz inequality we have

$$\begin{aligned} & E \sup_{\Theta_1^a} \left| \lambda_j^{-2\theta} I_{yj} - \lambda_j^{-2\theta} \left|D_n(e^{i\lambda_j};\theta)\right|^2 I_{uj} \right| \\ \le \;& \left(E \sup_{\Theta_1^a} \left| 2\lambda_j^{-\theta} D_n(e^{i\lambda_j};\theta) w_{uj} - \lambda_j^{-\theta} \frac{\widetilde{U}_{\lambda_j n}(\theta)}{\sqrt{2\pi n}} \right|^2 \right)^{1/2} \left(E \sup_{\Theta_1^a} \left| \lambda_j^{-\theta} \frac{\widetilde{U}_{\lambda_j n}(\theta)}{\sqrt{2\pi n}} \right|^2 \right)^{1/2} \end{aligned} \tag{50}$$

In view of Lemmas 5.2 and 5.4 (a) and (c), (50) is bounded by

$$j^{-1/2} (\log n)^3 + j^{-1} (\log n)^6 + j^{-\Delta} (\log n)^2 + j^{-2\Delta} (\log n)^4 = O\left(j^{-\Delta} (\log n)^6\right).$$

It follows that

$$\sum_{1}^{m-1} \left(\frac{r}{m}\right)^{2\Delta} \frac{1}{r^2} E \sup_{\Theta_1^a} \left| \sum_{1}^{r} \left[\lambda_j^{-2\theta} I_{yj} - \lambda_j^{-2\theta} \left|D_n(e^{i\lambda_j};\theta)\right|^2 I_{uj}\right] \right| = O_p\left(m^{-\Delta} (\log n)^6\right),$$

hence the first term in (48) is $o_p(1)$. Using the same technique, we can show that the second term in (48) is $o_p(1)$, and $\sup_{\Theta_1^a} |A(d)| \to_p 0$ follows. (44) gives $\sup_{\Theta_1} |B(d) - G_0| = O(m^{-2\Delta})$, and (46) follows.

Next we take care of $\Theta_1^b = \{d : d_0 + \frac{1}{2} \leq d \leq \Delta_2\} = \{\theta : \frac{1}{2} \leq \theta \leq \Delta_2 - d_0\}$. Note that

$$\begin{aligned}
S(d) &= \log \widehat{G}(d) - \log \widehat{G}(d_0) - 2(d-d_0)\frac{1}{m}\sum_1^m \log \lambda_j \\
&= \log \frac{1}{m}\sum_1^m I_{\Delta^d xj} - \log \frac{1}{m}\sum_1^m I_{\Delta^{d_0} xj} - 2(d-d_0)\log\frac{2\pi}{n} - 2(d-d_0)\frac{1}{m}\sum_1^m \log j \\
&= \log \frac{1}{m}\sum_1^m \lambda_j^{2(d-d_0)}\lambda_j^{2(d_0-d)} I_{\Delta^d xj} - \log \frac{1}{m}\sum_1^m I_{\Delta^{d_0} xj} \\
&\quad -2(d-d_0)\log\frac{2\pi}{n} - 2(d-d_0)\log p \\
&= \log \frac{1}{m}\sum_1^m (j/p)^{2\theta}\lambda_j^{-2\theta} I_{\Delta^d xj} - \log \frac{1}{m}\sum_1^m I_{\Delta^{d_0} xj} \\
&= \log \widehat{D}(d) - \log \widehat{D}(d_0).
\end{aligned}$$

where $p = \exp\left(m^{-1}\sum_1^m \log j\right) \sim m/e$ as $m \to \infty$. Since

$$\log \widehat{D}(d_0) - \log G_0 = \log\left(1 + G_0^{-1}\left(\frac{1}{m}\sum_1^m I_{uj} - G_0\right)\right) = o_p(1),$$

$\Pr(\inf_{\Theta_1^b} S(d) \leq 0)$ tends to 0 if, for $\delta \in (0, 0.01)$,

$$\Pr\left(\inf_{\Theta_1^b} \log \widehat{D}(d) - \log G_0 \leq \log(1+\delta)\right) = \Pr\left(\inf_{\Theta_1^b} \widehat{D}(d) - G_0 \leq \delta G_0\right) \to 0,$$

as $n \to 0$. Because

$$\begin{array}{ll}
\inf_{\Theta_1^b} (j/p)^{2\theta} \geq (j/p)^{2\Delta_2 - 2d_0}, & 1 \leq j \leq p, \\
\inf_{\Theta_1^b} (j/p)^{2\theta} \geq j/p, & p < j \leq m,
\end{array}$$

it follows that, for $d \in \Theta_1^b$,

$$\widehat{D}(d) \geq m^{-1}\textstyle\sum_1^m a_j \lambda_j^{-2\theta} I_{\Delta^d xj},$$

where, for a number $M \geq \max\{2\Delta_2 - 2\Delta_1, 2\}$

$$a_j = \begin{cases} (j/p)^M, & 1 \leq j \leq p, \\ j/p, & p < j \leq m. \end{cases} \tag{51}$$

Therefore, for $d \in \Theta_1^b$ we have

$$\widehat{D}(d) - G_0 \geq m^{-1}\textstyle\sum_1^m a_j(\lambda_j^{-2\theta} I_{yj} - G_0) + G_0 m^{-1}\sum_1^m (a_j - 1). \tag{52}$$

Before proceeding, collect the results concerning $\sum a_j$, $\sum a_j^2$, etc.

$$\begin{aligned}
&\text{(i)} & \sum_{1\leq j\leq m} a_j &= p^{-M}\sum_{1\leq j\leq p} j^M + p^{-1}\sum_{p<j\leq m} j = O(m), \\
&\text{(ii)} & \sum_{1\leq j\leq m} a_j^2 &= p^{-2M}\sum_{1\leq j\leq p} j^{2M} + p^{-2}\sum_{p<j\leq m} j^2 = O(m), \\
&\text{(iii)} & \sum_{p<j\leq m} a_j &\sim p^{-1}\int_p^m x\,dx = \frac{m^2-p^2}{2p} \sim \frac{p^2(e^2-1)}{2p} = \frac{e^2-1}{2e}m > 1.1m,
\end{aligned}$$

where the last inequality holds because $\left(e^2-1\right)/2e \doteq 1.17$. It follows that

$$G_0 m^{-1} \sum_1^m \left(a_j - 1\right) > G_0 \left(m^{-1} \sum_p^m a_j - 1\right) > 10\delta G_0,$$

as $n \to \infty$. In addition, for $-M - 1/2 \le \alpha \le C$,

$$\begin{aligned} \text{(iv)} \qquad & \sup_\alpha \left| m^{-\alpha-1} \sum_1^m a_j j^\alpha \right| \\ \le \; & \sup_\alpha \left| (p/m)^{\alpha+1} p^{-1} \sum_1^p (j/p)^{M+\alpha} \right| + \sup_\alpha \left| (p/m)^\alpha m^{-1} \sum_p^m (j/p)^{1+\alpha} \right| \\ = \; & O\left(p^{-1} \sum_1^p (j/p)^{-1/2} + m^{-1} \sum_p^m (j/p)^{1+C}\right) = O(1), \end{aligned}$$

and there exits $\kappa > c > 0$ such that

$$\text{(v)} \quad \min\{m^{-1} \sum_1^{m/4} a_j j^\alpha, m^{-1} \sum_{3m/4}^m a_j j^\alpha\} \ge \kappa m^\alpha,$$

uniformly in $\alpha \in [-C, C]$. This is because

$$\begin{aligned} m^{-\alpha} m^{-1} \sum_1^{m/4} a_j j^\alpha &\ge (m/p)^M m^{-1} \sum_1^{m/4} (j/m)^{M+\alpha} \ge m^{-1} \sum_1^{m/4} (j/m)^{M+C} \ge \kappa, \\ m^{-\alpha} m^{-1} \sum_{3m/4}^m a_j j^\alpha &\ge (m/p) m^{-1} \sum_{3m/4}^m (j/m)^{1+\alpha} \ge m^{-1} \sum_{3m/4}^m (j/m)^{1+C} \ge \kappa. \end{aligned}$$

We proceed to derive the limit of $\inf_\theta m^{-1} \sum_1^m a_j (\lambda_j^{-2\theta} I_{yj} - G_0)$ for subsets of Θ_1^b. For $\Theta_1^{b1} = \{\theta : \frac{1}{2} \le \theta \le \frac{3}{2}\}$, first define $D_{nj}(\theta) = \lambda_j^{-\theta}(1 - e^{i\lambda_j}) D_n(e^{i\lambda_j}; \theta - 1)$. Since $\theta - 1 \ge -1/2$, from Lemma 5.2 we have

$$D_{nj}(\theta) = e^{-\frac{\pi}{2}\theta i} + O(\lambda_j) + O(j^{-1/2}), \quad \text{uniformly in } \theta \in \Theta_1^{b1}. \tag{53}$$

Thus, we can easily show that $\sup_\theta \left| m^{-1} \sum_1^m a_j \left[|D_{nj}(\theta)|^2 I_{uj} - G_0 \right] \right| \to_p 0$. Now we evaluate $m^{-1} \sum_1^m a_j [\lambda_j^{-2\theta} I_{yj} - |D_{nj}(\theta)|^2 I_{uj}]$. From Lemma 2.2 (b) we have

$$w_{yj} = (1 - e^{i\lambda_j}) w_{zj} + (2\pi n)^{-1/2} e^{i\lambda_j} Z_n, \tag{54}$$

where $Z_n = \sum_{t=1}^n Y_t \sim I(1-\theta)$. ¿From this and (47) we obtain

$$\lambda_j^{-\theta} w_{yj} = D_{nj}(\theta) w_{uj} - \lambda_j^{-\theta}(1 - e^{i\lambda_j}) (2\pi n)^{-1/2} \widetilde{U}_{\lambda_j n}(\theta - 1) + \lambda_j^{-\theta} (2\pi n)^{-1/2} e^{i\lambda_j} Z_n.$$

It follows that $m^{-1} \sum_1^m a_j [\lambda_j^{-2\theta} I_{yj} - |D_{nj}(\theta)|^2 I_{uj}]$ consists of

$$m^{-1} \sum_1^m a_j \lambda_j^{-2\theta} |1 - e^{i\lambda_j}|^2 (2\pi n)^{-1} |\widetilde{U}_{\lambda_j n}(\theta - 1)|^2 \tag{55}$$

$$+ m^{-1} \sum_1^m a_j \lambda_j^{-2\theta} (2\pi n)^{-1} Z_n^2 \tag{56}$$

$$- m^{-1} \sum_1^m a_j D_{nj}(\theta)^* w_{uj}^* \lambda_j^{-\theta} (1 - e^{i\lambda_j}) (2\pi n)^{-1/2} \widetilde{U}_{\lambda_j n}(\theta - 1) \tag{57}$$

$$- m^{-1} \sum_1^m a_j \lambda_j^{-2\theta} (1 - e^{i\lambda_j}) (2\pi n)^{-1} \widetilde{U}_{\lambda_j n}(\theta - 1) e^{-i\lambda_j} Z_n \tag{58}$$

$$+ m^{-1} \sum_1^m a_j D_{nj}(\theta)^* w_{uj}^* \lambda_j^{-\theta} (2\pi n)^{-1/2} e^{i\lambda_j} Z_n, \tag{59}$$

and complex conjugates of (57)-(59). First, we state some results as lemmas, which will be used repeatedly. Lemma A is an immediate consequence of Lemma 5.4 (a) and (b) and its proof.

Lemma A *For* $d \in [-1/2, 1/2]$,

$$\lambda_j^{-d} (2\pi n)^{-1/2} \widetilde{U}_{\lambda_j n} (d) = A_{nj}(d) + B_{nj}(d),$$

where

$$\begin{cases} E \sup_d |A_{nj}(d) + B_{nj}(d)|^2 = O(j^{-1}(\log n)^6), & d \geq 0, \\ E \sup_d |A_{nj}(d)|^2 = O(j^{-1/2}(\log n)^4), \ E \sup_d |B_{nj}(d)|^2 = O(jn^{-1}(\log n)^2), & d \leq 0. \end{cases}$$

Lemma B *For* $d \in [-1/2, 1/2]$, a_j *defined in (51), and a function* $C(\lambda_j; d)$ *such that* $|C(\lambda_j; d)| < \infty$ *uniformly in* λ_j *and* d, *we have*

$$E \sup_d \left| m^{-1} \textstyle\sum_1^m a_j C(\lambda_j; d) w_{uj} \lambda_j^{-d} (2\pi n)^{-1/2} \widetilde{U}_{\lambda_j n}(d) \right| \to 0.$$

Proof ¿From Lemma A, the above quantity is bounded by

$$\begin{aligned} &\begin{cases} O\left(m^{-1} \sum_1^m a_j j^{-1/2} (\log n)^3\right), & d \geq 0, \\ O\left(m^{-1} \sup_d \sum_1^m a_j \left[j^{-1/4} (\log n)^2 + j^{1/2} n^{-1/2} \log n\right]\right), & d \leq 0, \end{cases} \\ = \ & O\left(m^{-1/2} (\log n)^3 + m^{-1/4} (\log n)^2 + m^{1/2} n^{-1/2} \log n\right), \end{aligned}$$

giving the stated result. ■

Lemma C *For any* $a_j = \{(j/p)^\alpha$ *for* $1 \leq j \leq p$; $(j/p)^\beta$ *for* $p < j \leq m\}$, *where* $p \sim m/e$ *and* $\gamma \in [-C, C]$,

$$\left| m^{-1} \textstyle\sum_1^m a_j j^\gamma w_{uj} \right| \leq m^{-1} \textstyle\sum_1^m a_j j^{\gamma - 1/2} |\zeta_j|, \quad \textit{for large } n,$$

where $E|\zeta_j| < \infty$ *for* $j = 1, \ldots m$.

Proof As in Robinson (1995, p.1636), use summation by parts to obtain

$$\frac{1}{m} \sum_1^p a_j j^\gamma w_{uj} = \frac{1}{m} \sum_{r=1}^{p-1} [a_r r^\gamma - a_{r+1} (r+1)^\gamma] \sum_{j=1}^r w_{uj} + \frac{a_p p^\gamma}{m} \sum_1^p w_{uj}.$$

For $1 \leq r \leq p-1$, we have

$$a_r r^\gamma - a_{r+1} (r+1)^\gamma = a_r r^\gamma \left[1 - (1 + 1/r)^{\alpha + \gamma}\right] = O\left(a_r r^{\gamma - 1}\right),$$

because $|(1+x)^a - 1| \leq Cx$ uniformly in $0 \leq x \leq 1$. The same result holds for $p < r \leq m-1$. It follows that

$$\begin{aligned} \left| \frac{1}{m} \sum_1^p a_j j^\gamma w_{uj} \right| &\leq C \frac{1}{m} \sum_1^{p-1} a_r r^{\gamma - 1/2} \left| \frac{1}{\sqrt{r}} \sum_1^r w_{uj} \right| + \left| \frac{a_p p^\gamma}{m} \sum_1^p w_{uj} \right|, \\ \left| \frac{1}{m} \sum_{p+1}^m a_j j^\gamma w_{uj} \right| &\leq C \frac{1}{m} \sum_{p+1}^{m-1} a_r r^{\gamma - 1/2} \left| \frac{1}{\sqrt{r}} \sum_{p+1}^r w_{uj} \right| + \left| \frac{a_m m^\gamma}{m} \sum_{p+1}^m w_{uj} \right|, \end{aligned}$$

giving the required result. ■

(55) is almost surely nonnegative, and Lemma B gives (57)$\to_p 0$. We proceed to evaluate (56), (58) and (59). For (56), from (v) there exists $\eta > c > 0$ such that

$$(56) = (2\pi)^{-2\theta-1}\, n^{2\theta-1} Z_n^2 m^{-1} \textstyle\sum_1^m a_j j^{-2\theta} \geq_{a.s.} \eta |m^{-\theta} n^{\theta-1/2} Z_n|^2,$$

uniformly in θ. (58) is equal to

$$C_n(\theta)\, n^{\theta-1/2} Z_n m^{-1} \textstyle\sum_1^m a_j j^{-\theta} \lambda_j^{-\theta+1} n^{-1/2} |\widetilde{U}_{\lambda_j n}(\theta - 1)|, \tag{60}$$

where $C_n(\theta)$ is a generic function with $\sup_\theta |C_n(\theta)| < \infty$ for all n. First we consider $\theta \in [1, 3/2]$. Let Ω be the sample space with typical element ω and Θ be the domain of θ. From Lemma A, (60) can be written as

$$m^{-\theta} n^{\theta-1/2} |Z_n| R_n(\theta, \omega), \tag{61}$$

where

$$\sup_\theta |R_n(\theta, \omega)| = O_p(k_n); \quad k_n = m^{-1/2} (\log n)^3. \tag{62}$$

Define

$$\begin{aligned} \Omega_1 &= \left\{ (\omega, \theta) \in \Omega \times \Theta : m^{-\theta} n^{\theta-1/2} |Z_n| < k_n \log m \right\}, \\ \Omega_2 &= \left\{ (\omega, \theta) \in \Omega \times \Theta : m^{-\theta} n^{\theta-1/2} |Z_n| \geq k_n \log m \right\}, \end{aligned}$$

so that $\Omega_1 \cup \Omega_2 = \Omega \times \Theta$. Now for any $\varepsilon > 0$ we have

$$\begin{aligned} & \left\{ (\omega, \theta) : \eta \left| m^{-\theta} n^{\theta-1/2} Z_n \right|^2 + (61) \leq -\varepsilon \right\} \\ = & \left\{ (\omega, \theta) : \left(\eta \left| m^{-\theta} n^{\theta-1/2} Z_n \right|^2 + (61) \leq -\varepsilon \right) \cup \Omega_1 \right\} \\ & \cup \left\{ (\omega, \theta) : \left(\eta \left| m^{-\theta} n^{\theta-1/2} Z_n \right|^2 + (61) \leq -\varepsilon \right) \cup \Omega_2 \right\} \\ \subseteq & \left\{ (\omega, \theta) : \eta \left| m^{-\theta} n^{\theta-1/2} Z_n \right|^2 + k_n \log m \cdot R_n(\theta, \omega) \leq -\varepsilon \right\} \\ & \cup \left\{ (\omega, \theta) : m^{-\theta} n^{\theta-1/2} |Z_n| \left[\eta k_n \log m + R_n(\theta, \omega) \right] \leq -\varepsilon \right\} \\ \subseteq & \left\{ (\omega, \theta) : k_n \log m \cdot R_n(\theta, \omega) \leq -\varepsilon \right\} \cup \left\{ (\omega, \theta) : \eta k_n \log m + R_n(\theta, \omega) \leq 0 \right\}. \end{aligned}$$

It follows that

$$\begin{aligned} & \Pr\left(\inf_\theta \eta \left| m^{-\theta} n^{\theta-1/2} Z_n \right|^2 + (61) \leq -\varepsilon \right) \\ = & \Pr\left(\cup_\theta \left\{ (\omega, \theta) : \eta \left| m^{-\theta} n^{\theta-1/2} Z_n \right|^2 + (61) \leq -\varepsilon \right\} \right) \\ \leq & \Pr\left(\inf_\theta k_n \log m \cdot R_n(\theta, \omega) \leq -\varepsilon \right) + \Pr\left(\inf_\theta \eta k_n \log m + R_n(\theta, \omega) \leq 0 \right) \to 0, \end{aligned}$$

because $k_n^2 \log m \to 0$ and $R_n(\theta, \omega)$ is $O_p(k_n)$ uniformly in θ by virtue of (62). Hence, $(56) + (58) \geq -\varepsilon$ with probability approaching one. For $\theta \in [1/2, 1]$, from Lemma A, (60) is

$$m^{-\theta} n^{\theta-1/2} |Z_n| O_p(m^{-1/4} (\log n)^2) + m^{-\theta} n^{\theta-1/2} |Z_n| O_p(m^{1/2} n^{-1/2} \log n).$$

For $\theta \in [1/2, 3/2]$, (59) is equal to, from Lemma C,

$$\begin{aligned}
& C_n(\theta) n^{\theta-1/2} Z_n m^{-1} \sum_1^m a_j D_{nj}(\theta)^* j^{-\theta} w_{uj}^* e^{i\lambda_j} \\
= \; & C_n(\theta) n^{\theta-1/2} Z_n e^{-\frac{\pi}{2}\theta i} m^{-1} \sum_1^m a_j j^{-\theta} w_{uj}^* \\
& + C_n(\theta) n^{\theta-1/2} Z_n m^{-1} \sum_1^m a_j \left[O(\lambda_j) + O(j^{-1/2}) \right] j^{-\theta} w_{uj}^* \\
= \; & m^{-\theta} n^{\theta-1/2} |Z_n| \left[O_p(m^{-1/2}) + O_p(mn^{-1}) \right].
\end{aligned}$$

Therefore, provided that $mn^{-1}(\log n)^2 \log m \to 0$, (56) + 2 Re[(58) + (59)] $\geq -\delta G_0$ with probability approaching one. It follows that $\Pr(\inf_{\Theta_1^{b1}} \widehat{D}(d) - G_0 \leq \delta G_0) \to 0$ as $n \to \infty$.

For $\Theta_1^{b2} = \{\theta : \frac{3}{2} \leq \theta \leq \frac{5}{2}\}$, from applying (54) twice and (47), $\lambda_j^{-\theta} w_{yj}$ is equal to

$$\begin{aligned}
& \lambda_j^{-\theta}(1 - e^{i\lambda_j})^2 D_n(e^{i\lambda_j}; \theta - 2) w_{uj} - \lambda_j^{-\theta}(1 - e^{i\lambda_j})^2 (2\pi n)^{-1/2} \widetilde{U}_{\lambda_j n}(\theta - 2) \\
& + \lambda_j^{-\theta}(1 - e^{i\lambda_j}) (2\pi n)^{-1/2} e^{i\lambda_j} \sum_1^n Z_t + \lambda_j^{-\theta} (2\pi n)^{-1/2} e^{i\lambda_j} Z_n.
\end{aligned}$$

First we state a useful result as a lemma.

Lemma D *For large n, the following holds either for $j = 1, \ldots, m/4$ or $j = 3m/4, \ldots, m$.*

$$\left| (1 - e^{i\lambda_j}) \sum_1^n Z_t + Z_n \right|^2 \geq (1/4) \left[(\lambda_j \sum_1^n Z_t)^2 + Z_n^2 \right].$$

Proof Note that

$$|(1 - e^{i\lambda_j}) \sum_1^n Z_t + Z_n|^2 = ((1 - \cos\lambda_j) \sum_1^n Z_t + Z_n)^2 + (\sin\lambda_j \sum_1^n Z_t)^2.$$

Since $\sin\lambda \sim \lambda$ for $\lambda \sim 0$, for large n we have

$$(\sin\lambda_j \sum_1^n Z_t)^2 \geq (1/2)(\lambda_j \sum_1^n Z_t)^2. \tag{63}$$

Since $1 - \cos\lambda \geq 0$, if $sgn(\sum_1^n Z_t) = sgn(Z_n)$, then

$$|(1 - \cos\lambda_j) \sum_1^n Z_t + Z_n| = (1 - \cos\lambda_j) |\sum_1^n Z_t| + |Z_n| \geq |Z_n|, \tag{64}$$

and the required result follows immediately. When $sgn(\sum_1^n Z_t) \neq sgn(Z_n)$, without loss of generality assume $\sum_1^n Z_t > 0$ and $Z_n \leq 0$. Then $(1 - \cos\lambda_j) \sum_1^n Z_t$ is an increasing function of j. Now suppose $(1 - \cos\lambda_{m/2}) \sum_1^n Z_t + Z_n \geq 0$. Then, since $1 - \cos\lambda \sim \lambda^2$ for $\lambda \sim 0$,

$$(\lambda_{m/2})^2 \sum_1^n Z_t \geq (3/4)|Z_n|,$$

and it follows that

$$(1 - \cos\lambda_{3m/4}) \sum_1^n Z_t \geq (8/9)(\lambda_{3m/4})^2 \sum_1^n Z_t = 2(\lambda_{m/2})^2 \sum_1^n Z_t \geq (3/2)|Z_n|,$$

giving

$$(1 - \cos\lambda_j) \sum_1^n Z_t + Z_n \geq (1/2)|Z_n| \quad \text{for } j = 3m/4, \ldots, m. \tag{65}$$

Now suppose $(1 - \cos\lambda_{m/2}) \sum_1^n Z_t + Z_n < 0$. Then,

$$(\lambda_{m/2})^2 \sum_1^n Z_t \leq (3/2)|Z_n|,$$

and it follows that

$$(1-\cos\lambda_{m/4})\sum_1^n Z_t \le (4/3)(\lambda_{m/4})^2\sum_1^n Z_t = (1/3)(\lambda_{m/2})^2\sum_1^n Z_t \le (1/2)|Z_n|,$$

giving

$$(1-\cos\lambda_j)\sum_1^n Z_t + Z_n \le -(1/2)|Z_n| \quad \text{for } j=1,\ldots,m/4. \tag{66}$$

The stated result follows from (63)-(66). ■

¿From Lemma D and (v), there exists $\eta > c > 0$ such that

$$\begin{aligned} & (2\pi n)^{-1} m^{-1}\sum_1^m a_j\lambda_j^{-2\theta}|(1-e^{i\lambda_j})\sum_1^n Z_t + Z_n|^2 \\ \ge\ & (2\pi n)^{-1} m^{-1}(\textstyle\sum_1^{m/4} + \sum_{3m/4}^m)a_j\lambda_j^{-2\theta}|(1-e^{i\lambda_j})\sum_1^n Z_t + Z_n|^2 \\ \ge\ & \eta[m^{-2\theta+2}n^{2\theta-3}(\textstyle\sum_1^n Z_t)^2 + m^{-2\theta}n^{2\theta-1}Z_n^2], \end{aligned} \tag{67}$$

uniformly in θ. The terms involving the cross products of $w_{uj}, \widetilde{U}_{\lambda_j n}(\theta-2)$ and $\lambda_j^{-\theta}(1-e^{i\lambda_j})\sum_1^n Z_t + \lambda_j^{-\theta}Z_n$ are dominated by (67). For instance,

$$\begin{aligned} & m^{-1}\sum_1^m a_j\lambda_j^{-\theta}(1-e^{i\lambda_j})^2 D_n(e^{i\lambda_j};\theta-2)w_{uj}\lambda_j^{-\theta}(2\pi n)^{-1/2}e^{-i\lambda_j} \\ & \times[(1-e^{-i\lambda_j})\textstyle\sum_1^n Z_t + Z_n] \\ =\ & (m^{-\theta+1}n^{\theta-3/2}|\textstyle\sum_1^n Z_t| + m^{-\theta}n^{\theta-1/2}|Z_n|)[O_p(m^{-1/2}) + O_p(n^{-1}m)], \end{aligned}$$

and $\Pr(\inf_\theta \widehat{D}(d) - G_0 \le \delta G_0) \to 0$ follows. For larger values of θ, applying (54) repeatedly and the same argument establishes $\Pr(\inf_\theta \widehat{D}(d) - G_0 \le \delta G_0) \to 0$, albeit the expression of $\lambda_j^{-\theta}w_{yj}$ will contain $\sum_{k=1}^n\sum_{t=1}^k Z_t$, etc.

Now we consider $\Theta_2 = \{\theta : \Delta_1 - d_0 \le \theta \le -\frac{1}{2} + \Delta\}$. Since

$$\begin{aligned} \inf_{\Theta_2}(j/p)^{2\theta} &\ge (j/p)^{2\Delta-1}, & 1 \le j \le p, \\ \inf_{\Theta_2}(j/p)^{2\theta} &\ge (j/p)^{2\Delta_1-2d_0}, & p < j \le m, \end{aligned}$$

it follows that $\widehat{D}(d) \ge m^{-1}\sum_1^m a_j\lambda_j^{-2\theta}I_{\Delta^d xj}$ for $d \in \Theta_2$, where

$$a_j = \begin{cases} (j/p)^{2\Delta-1}, & 1 \le j \le p, \\ (j/p)^{2\Delta_1-2d_0}, & p < j \le m. \end{cases} \tag{68}$$

Therefore, for $d \in \Theta_2$ we have

$$\widehat{D}(d) - G_0 \ge m^{-1}\sum_1^m a_j(\lambda_j^{-2\theta}I_{yj} - G_0) + G_0 m^{-1}\sum_1^m (a_j - 1).$$

As $m \to \infty$, $p \sim m/e$, and we can also show that $m^{-1}\sum_1^m (a_j-1) > 2\delta$ when $\Delta < 1/(2e)$, $\sum_1^m a_j = O(m)$ and $\sum_1^m a_j^2 = O(m^{2-4\Delta})$. Furthermore, $m^{-1}\sum_1^m a_j j^\alpha = O(m^\alpha \log m + m^{-2\Delta}\log m)$ uniformly in $\alpha \in [-C, C]$ because

$$\begin{aligned} & \sup_\alpha\left|m^{-\max\{\alpha,-2\Delta\}}m^{-1}\textstyle\sum_1^p a_j j^\alpha\right| \\ =\ & O\left(\sup_{\alpha\ge-2\Delta} m^{-1}\textstyle\sum_1^p (j/p)^{2\Delta-1+\alpha} + \sup_{\alpha\le-2\Delta}\sum_1^p j^{2\Delta-1+\alpha}\right) = O(\log m), \\ & \sup_{\alpha\le C}\left|m^{-\alpha}m^{-1}\textstyle\sum_{p+1}^m a_j j^\alpha\right| = O\left(\sup_{\alpha\le C} m^{-1}\textstyle\sum_{p+1}^m (j/p)^{2\Delta_1-2d_0+\alpha}\right) = O(1). \end{aligned}$$

We proceed to derive the limit of $m^{-1}\sum_1^m a_j(\lambda_j^{-2\theta}I_{yj} - G_0)$ for subsets of Θ_2. First, for $\Theta_2^a = \{\theta : -\frac{1}{2} \le \theta \le -\frac{1}{2} + \Delta\}$, $\sup_\theta|m^{-1}\sum_1^m a_j[\lambda_j^{-2\theta}|D_n(e^{i\lambda_j};\theta)|^2 I_{uj} - G_0]| \to_p 0$ can be shown as above, and from (47) $\lambda_j^{-2\theta}I_{yj} - \lambda_j^{-2\theta}|D_n(e^{i\lambda_j};\theta)|^2 I_{uj}$ is equal to

$$\lambda_j^{-2\theta}(2\pi n)^{-1}|\widetilde{U}_{\lambda_j n}(\theta)|^2 - 2\,\mathrm{Re}[\lambda_j^{-2\theta}D_n(e^{i\lambda_j};\theta)w_{uj}(2\pi n)^{-1/2}\widetilde{U}_{\lambda_j n}(\theta)^*]. \tag{69}$$

The contribution to $m^{-1}\sum_1^m a_j[\lambda_j^{-2\theta}I_{yj}-\lambda_j^{-2\theta}|D_n(e^{i\lambda_j};\theta)|^2 I_{uj}]$ from the first term is almost surely nonnegative. Before proceeding to evaluate the contribution from the second term, we state a general result analogous to Lemma B.

Lemma E *For $d\in[-1/2,1/2]$, a_j defined in (68), and a function $C(\lambda_j;d)$ such that $|C(\lambda_j;d)|<\infty$ uniformly in λ_j and d, we have*

$$E\sup_d\left|m^{-1}\textstyle\sum_1^m a_j C(\lambda_j;d)\,w_{uj}\lambda_j^{-d}(2\pi n)^{-1/2}\widetilde{U}_{\lambda_j n}(d)\right|\to 0.$$

Proof ¿From Lemma A, the above function is bounded by

$$\begin{aligned}&\left\{\begin{array}{ll} m^{-1}\sum_1^m a_j j^{-1/2}(\log n)^3, & d\geq 0,\\ m^{-1}\sum_1^m a_j\left[j^{-1/4}(\log n)^2+j^{1/2}n^{-1/2}\log n\right], & d\leq 0,\end{array}\right.\\ =\;& O(m^{-2\Delta}(\log n)^4+m^{1/2}n^{-1/2}\log n),\end{aligned}$$

giving the stated result. ■

The contribution from the second term in (69) is $o_p(1)$ by Lemma E. Therefore, $\Pr(\inf_{\Theta_2^a}\widehat{D}(d)-G_0\leq\delta G_0)\to 0$ as $n\to\infty$.

We move to $\Theta_2^b=\{\theta:-3/2\leq\theta\leq-1/2\}$. Note that $Y_t\sim I(-\theta)$ and $\Delta Y_t\sim I(-\theta-1)$. Since

$$w_{yj}=(1-e^{i\lambda_j})^{-1}w_{\Delta yj}-(1-e^{i\lambda_j})^{-1}(2\pi n)^{-1/2}e^{i\lambda_j}Y_n, \tag{70}$$

$\lambda_j^{-\theta}w_{yj}$ is equal to

$$\begin{aligned}&\lambda_j^{-\theta}(1-e^{i\lambda_j})^{-1}D_n(e^{i\lambda_j};\theta+1)w_{uj}\\ &-\lambda_j^{-\theta}(1-e^{i\lambda_j})^{-1}(2\pi n)^{-1/2}\widetilde{U}_{\lambda_j n}(\theta+1)-\lambda_j^{-\theta}(1-e^{i\lambda_j})^{-1}(2\pi n)^{-1/2}e^{i\lambda_j}Y_n.\end{aligned}$$

Since $\theta+1\geq-1/2$, with a slight abuse of notation we have

$$\lambda_j^{-\theta}(1-e^{i\lambda_j})^{-1}D_n(e^{i\lambda_j};\theta+1)\equiv D_{nj}(\theta)=e^{-\frac{\pi}{2}\theta i}+O(\lambda_j)+O(j^{-1/2}).$$

Therefore, apart from $o_p(1)$ terms, $m^{-1}\sum_1^m a_j[\lambda_j^{-2\theta}I_{yj}-G_0]$ consists of

$$m^{-1}\textstyle\sum_1^m a_j\lambda_j^{-2\theta}|1-e^{i\lambda_j}|^{-2}(2\pi n)^{-1}Y_n^2 \tag{71}$$
$$+m^{-1}\textstyle\sum_1^m a_j\lambda_j^{-2\theta}|1-e^{i\lambda_j}|^{-2}(2\pi n)^{-1}|\widetilde{U}_{\lambda_j n}(\theta+1)|^2 \tag{72}$$
$$+m^{-1}\textstyle\sum_1^m a_j\lambda_j^{-2\theta}|1-e^{i\lambda_j}|^{-2}(2\pi n)^{-1}\widetilde{U}_{\lambda_j n}(\theta+1)^* e^{i\lambda_j}Y_n \tag{73}$$
$$-m^{-1}\textstyle\sum_1^m a_j D_{nj}(\theta)^* w_{uj}^*\lambda_j^{-\theta}(1-e^{i\lambda_j})^{-1}(2\pi n)^{-1/2}\widetilde{U}_{\lambda_j n}(\theta+1) \tag{74}$$
$$-m^{-1}\textstyle\sum_1^m a_j D_{nj}(\theta)^* w_{uj}^*\lambda_j^{-\theta}(1-e^{i\lambda_j})^{-1}(2\pi n)^{-1/2}e^{i\lambda_j}Y_n, \tag{75}$$

and complex conjugates of (73)-(75). For $\theta\in[-1,-1/2]$, first note that

$$\begin{aligned}&(71)+(72)+2\,\mathrm{Re}[(73)]\\ =\;& m^{-1}\textstyle\sum_1^m a_j\lambda_j^{-2\theta}|1-e^{i\lambda_j}|^{-2}(2\pi n)^{-1}|e^{i\lambda_j}Y_n+\widetilde{U}_{\lambda_j n}(\theta+1)|^2\geq_{a.s.}0.\end{aligned}$$

(74) is $o_p(1)$ by Lemma E. Observe that uniformly in $-1\leq\alpha<C$

$$\begin{aligned}&m^{-1}\textstyle\sum_1^m a_j D_{nj}(\theta)\,j^\alpha w_{uj}[1+O(\lambda_j)]\\ =\;& e^{-\frac{\pi}{2}\theta i}m^{-1}\textstyle\sum_1^m a_j j^\alpha w_{uj}+m^{-1}\sum_1^m a_j[O(\lambda_j)+O(j^{-1/2})]j^\alpha w_{uj}\\ =\;& O_p\left(m^{\alpha-1/2}\log m+m^{-2\Delta}\log m\right)+O_p\left(n^{-1}m^{\alpha+1}\right).\end{aligned} \tag{76}$$

Since $n^{\theta+1/2}Y_n = O_p((\log n)^2)$ from Lemma 5.6 (a), it follows that

$$\begin{aligned}
|(75)| &\leq C|n^{\theta+1/2}Y_n||m^{-1}\textstyle\sum_1^m a_j D_{nj}(\theta)^* j^{-\theta-1}w_{uj}^*[1+O(\lambda_j)]| \\
&= O_p\left((m^{-2\Delta}+n^{-1}m)(\log n)^2\log m\right),
\end{aligned}$$

giving $\Pr(\inf_\theta \widehat{D}(d) - G_0 \leq \delta G_0) \to 0$. For $\theta \in [-3/2,-1]$, from an analogous argument as before there exists $\kappa > 0$ such that

$$\min\{m^{-1}\textstyle\sum_1^{m/4} a_j j^\alpha, m^{-1}\sum_{3m/4}^m a_j j^\alpha\} \geq \kappa m^\alpha, \tag{77}$$

uniformly in $\alpha \in [-C,C]$. Therefore, (71) is almost surely larger than $\eta|n^{\theta+1/2}m^{-\theta-1}Y_n|^2$ for some $\eta > c > 0$. (72) is almost surely nonnegative, and (74) is $o_p(1)$ by Lemma E. We proceed to show that (73) and (75) are dominated by (71). From Lemma A, (73) is equal to

$$\begin{aligned}
& C_n(\theta)\,n^{\theta+1/2}Y_n m^{-1}\textstyle\sum_1^m a_j j^{-\theta-1}\lambda_j^{-\theta-1}n^{-1/2}|\widetilde{U}_{\lambda_j n}(\theta+1)| \\
&= n^{\theta+1/2}|Y_n|O_p((m^{-\theta-5/4}+m^{-2\Delta})(\log n)^3) \\
&\quad n^{\theta+1/2}|Y_n|O_p(m^{-\theta-1/2}n^{-1/2}\log n\log m) \\
&= m^{-\theta-1}n^{\theta+1/2}|Y_n|O_p(m^{-2\Delta}(\log n)^3) \\
&\quad +m^{-\theta-1}n^{\theta+1/2}|Y_n|O_p(m^{1/2}n^{-1/2}\log n\log m).
\end{aligned}$$

¿From (76), (75) is equal to

$$\begin{aligned}
& C_n(\theta)\,n^{\theta+1/2}Y_n m^{-1}\textstyle\sum_1^m a_j D_{nj}(\theta)^* j^{-\theta-1}w_{uj}^* \\
&= m^{-\theta-1}n^{\theta+1/2}|Y_n|[O_p(m^{-2\Delta}\log m)+O_p(mn^{-1})],
\end{aligned}$$

Therefore, by the same argument as the one used for Θ_1^b, we have $(71)+2\,\mathrm{Re}[(73)+(75)] \geq -\delta G_0$ with probability approaching one provided $n^{-1}m(\log n)^2(\log m)^3 \to 0$. Hence $\Pr(\inf_\theta \widehat{D}(d) - G_0 \leq \delta G_0) \to 0$. Finally, we consider $\Theta_2^c = \{\theta : -5/2 \leq \theta \leq -3/2\}$. Applying (70) twice and (47), $\lambda_j^{-\theta}w_{yj}$ is equal to

$$\begin{aligned}
& \lambda_j^{-\theta}(1-e^{i\lambda_j})^{-2}D_n(e^{i\lambda_j};\theta+2)w_{uj} - \lambda_j^{-\theta}(1-e^{i\lambda_j})^{-2}(2\pi n)^{-1/2}\widetilde{U}_{\lambda_j n}(\theta+2) \\
& -\lambda_j^{-\theta}(1-e^{i\lambda_j})^{-1}(2\pi n)^{-1/2}e^{i\lambda_j}Y_n - \lambda_j^{-\theta}(1-e^{i\lambda_j})^{-2}(2\pi n)^{-1/2}e^{i\lambda_j}\Delta Y_n.
\end{aligned}$$

Neglecting the $o_p(1)$ and a.s. nonnegative terms, $m^{-1}\sum_1^m a_j[\lambda_j^{-2\theta}I_{yj} - G_0]$ consists of

$$m^{-1}\textstyle\sum_1^m a_j\lambda_j^{-2\theta}|1-e^{i\lambda_j}|^{-4}(2\pi n)^{-1}|(1-e^{i\lambda_j})Y_n+\Delta Y_n|^2 \tag{78}$$

$$+m^{-1}\textstyle\sum_1^m a_j\lambda_j^{-\theta-2}(2\pi n)^{-1}\widetilde{U}_{\lambda_j n}(\theta+2)^*(\lambda_j^{-\theta-1}Y_n+\lambda_j^{-\theta-2}\Delta Y_n)(1+O(\lambda_j)) \tag{79}$$

$$+m^{-1}\textstyle\sum_1^m a_j[1+O(\lambda_j)+O(j^{-1/2})]w_{uj}^*(2\pi n)^{-1/2}(\lambda_j^{-\theta-1}Y_n+\lambda_j^{-\theta-2}\Delta Y_n), \tag{80}$$

and complex conjugates of (79) and (80). In view of Lemma D and (77) we have

$$(78) \geq_{a.s.} \eta|m^{-\theta-1}n^{\theta+1/2}Y_n|^2 + \eta|m^{-\theta-2}n^{\theta+3/2}\Delta Y_n|^2.$$

¿From Lemma A, (79) is, for $\theta \in [-2,-3/2]$,

$$\begin{aligned}
& n^{\theta+1/2}|Y_n|O_p(m^{-\theta-3/2}(\log n)^4) + n^{\theta+3/2}|\Delta Y_n|O_p((m^{-\theta-5/2}+m^{-2\Delta})(\log n)^4) \\
&= m^{-\theta-1}n^{\theta+1/2}|Y_n|O_p(m^{-1/2}(\log n)^4) + O_p(m^{-2\Delta}(\log n)^6),
\end{aligned}$$

because $n^{\theta+3/2}\Delta Y_n = O_p((\log n)^2)$. For $\theta \in [-5/2, -2]$, (79) is

$$n^{\theta+1/2}|Y_n|O_p(m^{-\theta-5/4}(\log n)^3) + n^{\theta+3/2}|\Delta Y_n|O_p((m^{-\theta-9/4} + m^{-2\Delta})(\log n)^3) \quad (81)$$

$$+n^{\theta+1/2}|Y_n|O_p(m^{-\theta}n^{-1}\log n) + n^{\theta+3/2}|\Delta Y_n|O_p(m^{-\theta-1}n^{-1}\log n). \quad (82)$$

The first term in (81) is $m^{-\theta-1}n^{\theta+1/2}|Y_n|O_p(m^{-1/4}(\log n)^4)$, and the second term in (81) is

$$\begin{cases} O_p(m^{-2\Delta}(\log n)^5), & \theta \in [-9/4 + 2\Delta, -2], \\ m^{-\theta-2}n^{\theta+3/2}|\Delta Y_n|O_p(m^{-1/4}(\log n)^3), & \theta \in [-5/2, -9/4 + 2\Delta]. \end{cases}$$

(82) is $m^{-\theta-1}n^{\theta+1/2}|Y_n|O_p\ (mn^{-1}\log n) + m^{-\theta-2}n^{\theta+3/2}|\Delta Y_n|O_p(mn^{-1}\log m)$. Finally, (80) is

$$n^{\theta+1/2}|Y_n|O_p(m^{-\theta-3/2}\log m) + n^{\theta+3/2}|\Delta Y_n|O_p((m^{-\theta-5/2} + m^{-2\Delta})\log m)$$
$$+n^{\theta+1/2}|Y_n|O_p\ (m^{-\theta}n^{-1}\log n) + n^{\theta+3/2}|\Delta Y_n|O_p(m^{-\theta-1}n^{-1}\log m).$$

Therefore, (79) and (80) are either $o_p(1)$ or dominated by (78), and from the same argument as above we have $\Pr(\inf_\theta \widehat{D}(d) - G_0 \leq \delta G_0) \to 0$. For smaller d, we use (47) repeatedly and the expression of $\lambda_j^{-\theta}w_{yj}$ will contain $\Delta^2 Y_n$, $\Delta^3 Y_n$, ..., but the same reasoning gives the required result and completes the proof. ■

6.2 Proof of asymptotic normality

Theorem 3.3 holds under the current conditions and implies that with probability approaching 1, as $n \to \infty$, $\widehat{d}$ satisfies

$$0 = R'(\widehat{d}) = R'(d_0) + R''(d^*)(\widehat{d} - d_0), \quad (83)$$

where $|d^* - d_0| \leq |\widehat{d} - d_0|$. From the fact

$$\begin{aligned} \frac{\partial}{\partial d} w_{\Delta^d xs} &= \frac{\partial}{\partial d}\frac{1}{\sqrt{2\pi n}}\sum_1^n e^{i\lambda_s t}(1-L)^d X_t = \frac{1}{\sqrt{2\pi n}}\sum_1^n e^{i\lambda_s t}\log(1-L)(1-L)^d X_t, \\ \frac{\partial^2}{\partial d^2} w_{\Delta^d xs} &= \frac{1}{\sqrt{2\pi n}}\sum_1^n e^{i\lambda_s t}(\log(1-L))^2(1-L)^d X_t, \end{aligned}$$

we obtain

$$R''(d) = \frac{\widehat{G}_2(d)\widehat{G}(d) - \widehat{G}_1(d)^2}{\widehat{G}(d)^2} = \frac{\widetilde{G}_2(d)\widetilde{G}_0(d) - \widetilde{G}_1(d)^2}{\widetilde{G}_0(d)^2},$$

where

$$\begin{aligned} \widehat{G}_1(d) &= \frac{1}{m}\sum_1^m \frac{\partial}{\partial d}\left[w_{\Delta^d xj}w^*_{\Delta^d xj}\right] = \frac{1}{m}\sum_1^m 2\,\mathrm{Re}\left[w_{\log(1-L)\Delta^d xj}w^*_{\Delta^d xj}\right], \\ \widehat{G}_2(d) &= \frac{1}{m}\sum_1^m \frac{\partial^2}{\partial d^2}\left[w_{\Delta^d xj}w^*_{\Delta^d xj}\right] = \frac{1}{m}\sum_1^m W_x(L,d,j), \\ W_x(L,d,j) &= 2\,\mathrm{Re}\left[w_{(\log(1-L))^2\Delta^d xj}w^*_{\Delta^d xj}\right] + 2I_{\log(1-L)\Delta^d xj}, \\ \widetilde{G}_0(d) &= \frac{1}{m}\sum_1^m j^{2\theta}\lambda_j^{-2\theta}I_{yj}, \quad \widetilde{G}_1(d) = \frac{1}{m}\sum_1^m j^{2\theta}\lambda_j^{-2\theta}2\,\mathrm{Re}\left[w_{\log(1-L)yj}w^*_{yj}\right], \\ \widetilde{G}_2(d) &= \frac{1}{m}\sum_1^m j^{2\theta}\lambda_j^{-2\theta}W_y(L,0,j). \end{aligned}$$

Fix $\varepsilon > 0$ and let $M = \{d : (\log n)^4 |d - d_0| < \varepsilon\}$. $\Pr(d^* \notin M)$ tends to zero because $\sup_{\Theta_1^a} |A(d)/B(d)| = o_p((\log n)^{-8})$. Thus we assume $d \in M$ in the following discussion on $\widetilde{G}_k(d)$. Now we derive the approximation of $\widetilde{G}_k(d)$ for $k = 0, 1, 2$. For $\widetilde{G}_0(d)$, from a similar decomposition as (49) along with (50), we obtain

$$\sup_{\theta \in M} \left| \lambda_j^{-2\theta} I_{yj} - I_{uj} \right| = O_p\left(j^{-\Delta} (\log n)^6 + jn^{-1} \right).$$

Thus, in view of the fact that

$$|j^{2\theta} - 1| / |2\theta| \leq (\log j)\, n^{2|\theta|} \leq (\log j)\, n^{1/\log n} = e \log j \quad \text{on } M,$$

we have

$$\sup_M \left| \widetilde{G}_0(d) - G_0 \right| = \sup_M \left| \frac{1}{m} \sum_1^m j^{2\theta} I_{uj} - G_0 \right| + o_p((\log n)^{-2}) = o_p((\log n)^{-2}).$$

For $\widetilde{G}_1(d)$, from Lemma 5.12 we have

$$\begin{aligned}
& \lambda_j^{-2\theta} w_{\log(1-L)yj} w_{yj}^* + J_n\left(e^{i\lambda_j}\right) I_{uj} \\
= \; & J_n\left(e^{i\lambda_j}\right) \left[1 - \lambda_j^{-2\theta} \left| D_n(e^{i\lambda_j}; \theta) \right|^2 \right] I_{uj} \\
& - J_n\left(e^{i\lambda_j}\right) \lambda_j^{-\theta} D_n(e^{i\lambda_j}; \theta) w_{uj} \cdot \lambda_j^{-\theta} (2\pi n)^{-1/2} \widetilde{U}_{\lambda_j n}(\theta)^* \\
& - \lambda_j^{-\theta} D_n(e^{i\lambda_j}; \theta)^* w_{uj}^* \cdot \lambda_j^{-\theta} (2\pi n)^{-1/2} V_{nj}(\theta) - \lambda_j^{-2\theta} (2\pi n)^{-1} \widetilde{U}_{\lambda_j n}(\theta)^* V_{nj}(\theta).
\end{aligned}$$

Then, using a similar line of argument as above, we can easily show

$$\frac{1}{m} \sum_1^m \sup_M j^{2\theta} \left| \lambda_j^{-2\theta} w_{\log(1-L)yj} w_{yj}^* + J_n\left(e^{i\lambda_j}\right) I_{uj} \right| = o_p\left((\log n)^{-1} \right).$$

It follows that

$$\begin{aligned}
& \sup_M \left| \widetilde{G}_1(d) + \frac{1}{m} \sum_1^m 2 \operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right) \right] I_{uj} \right| \\
= \; & \sup_M \left| \frac{1}{m} \sum_1^m \left(1 - j^{2\theta}\right) 2 \operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right) \right] I_{uj} \right| + o_p\left((\log n)^{-1} \right) = o_p\left((\log n)^{-1} \right).
\end{aligned}$$

For $\widetilde{G}_2(d)$, the same line of argument as above with Lemma 5.12 (c) gives

$$\begin{aligned}
& \sup_M \left| \widetilde{G}_2(d) - \frac{1}{m} \sum_1^m \left[2 \operatorname{Re} J_n\left(e^{i\lambda_j}\right)^2 + 2 J_n\left(e^{i\lambda_j}\right) J_n\left(e^{i\lambda_j}\right)^* \right] I_{uj} \right| \\
= \; & \sup_M \left| \widetilde{G}_2(d) - \frac{1}{m} \sum_1^m 4 \left\{ \operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right) \right] \right\}^2 I_{uj} \right| = o_p(1).
\end{aligned}$$

Therefore, with probability approaching one,

$$\begin{aligned}
R''(d^*) &= \left[\widetilde{G}_2(d^*) \widetilde{G}_0(d^*) - \widetilde{G}_1(d^*)^2 \right] \left[\widetilde{G}_0(d^*) \right]^{-2} \\
&= \left[\left\{ \frac{1}{m} \sum_1^m 4 \left\{ \operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right) \right] \right\}^2 I_{uj} + o_p(1) \right\} \left\{ G_0 + o_p\left((\log n)^{-2} \right) \right\} \right.
\end{aligned}$$

$$-\left\{-\frac{1}{m}\sum_1^m 2\operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right)\right]I_{uj}+o_p\left((\log n)^{-1}\right)\right\}^2\Bigg]\left[\widetilde{G}_0\left(d^*\right)\right]^{-2}$$

$$=\frac{G_0\frac{1}{m}\sum_1^m 4\left\{\operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right)\right]\right\}^2 I_{uj}-\left\{\frac{1}{m}\sum_1^m 2\operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right)\right]I_{uj}\right\}^2+o_p(1)}{G_0^2+o_p((\log n)^{-2})}$$

$$=4m^{-1}\sum_1^m\left\{\operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right)\right]\right\}^2-4\left\{m^{-1}\sum_1^m\operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right)\right]\right\}^2+o_p(1). \quad (84)$$

¿From Lemma 5.10 (a) we have

$$\begin{aligned}\operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right)\right] &= -\log\lambda_j+O(1/j)+O\left(j^2/n^2\right),\\ \left\{\operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right)\right]\right\}^2 &= (\log\lambda_j)^2+O\left(j^{-1}\log n\right)+O\left(j^2n^{-2}\log n\right).\end{aligned}$$

It follows that

$$\begin{aligned}m^{-1}\sum_1^m\left\{\operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right)\right]\right\}^2 &= m^{-1}\sum_1^m(\log\lambda_j)^2+o(1),\\ \left\{m^{-1}\sum_1^m\operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right)\right]\right\}^2 &= \left(m^{-1}\sum_1^m\log\lambda_j\right)^2+o(1).\end{aligned}$$

Therefore, (84)/4 is, apart from $o_p(1)$ terms,

$$\begin{aligned}& m^{-1}\sum_1^m(\log\lambda_j)^2-\left(m^{-1}\sum_1^m\log\lambda_j\right)^2\\ =\;& m^{-1}\sum_1^m\left(\log\frac{2\pi}{n}+\log j\right)^2-\left(m^{-1}\sum_1^m\left(\log\frac{2\pi}{n}+\log j\right)\right)^2\\ =\;& m^{-1}\sum_1^m(\log j)^2-\left(m^{-1}\sum_1^m\log j\right)^2\to 1,\end{aligned}$$

and $R''(d^*)=4+o_p(1)$ follows.

Now we find the limit distribution of $m^{1/2}R'(d_0)$. Since $w_{us}=C(e^{i\lambda_s})w_{\varepsilon s}+r_{ns}$ where $E|r_{ns}|^2=O(n^{-1})$ uniformly in s (Hannan, 1970, p.248), in view of Lemma 5.12 (b) and its proof we obtain

$$-w_{\log(1-L)us}w_{us}^*=J_n(e^{i\lambda_s})I_{us}-\frac{C(1)}{\sqrt{2\pi n}}\widetilde{J}_{n\lambda_s}\left(e^{i\lambda_s}L\right)\varepsilon_n C(e^{i\lambda_s})^*w_{\varepsilon s}^*+R_{ns},$$

where $E|R_{ns}|=O(n^{-1/2}\log n)$ uniformly in s. It follows that $m^{1/2}\widehat{G}_1(d_0)$ is equal to

$$-\frac{1}{\sqrt{m}}\sum_1^m 2\operatorname{Re}\left[J_n\left(e^{i\lambda_j}\right)\right]I_{uj} \quad (85)$$

$$+\frac{C(1)}{\sqrt{m}}\sum_1^m 2\operatorname{Re}\left[\widetilde{J}_{n\lambda_j}\left(e^{i\lambda_j}L\right)\frac{\varepsilon_n}{\sqrt{2\pi n}}C(e^{i\lambda_j})^*w_{\varepsilon j}^*\right]+O_p\left(m^{1/2}n^{-1/2}\log m\right) \quad (86)$$

¿From Lemma 5.10 (a),

$$(85)=\frac{2}{\sqrt{m}}\sum_1^m(\log\lambda_j)I_{uj}+O_p\left(m^{5/2}n^{-2}\right)+O_p\left(m^{-1/2}\log m\right).$$

For the fist term in (86), in view of the fact that

$$w_\varepsilon(\lambda_j)^*=\frac{1}{\sqrt{2\pi n}}\sum_{p=1}^n e^{-ip\lambda_j}\varepsilon_p=\frac{1}{\sqrt{2\pi n}}\sum_{q=0}^{n-1}e^{iq\lambda_j}\varepsilon_{n-q},$$

we obtain the decomposition

$$
\begin{aligned}
& \frac{1}{\sqrt{m}} \sum_1^m \widetilde{J}_{n\lambda_j}\left(e^{i\lambda_j}L\right) \frac{\varepsilon_n}{\sqrt{2\pi n}} C(e^{i\lambda_j})^* w_{\varepsilon j}^* \\
= \quad & \frac{1}{\sqrt{m}} \sum_1^m \frac{C(e^{i\lambda_j})^*}{2\pi n} \left(\sum_0^{n-1} \widetilde{j}_{\lambda_j p} e^{-ip\lambda_j} \varepsilon_{n-p} \right) \left(\sum_0^{n-1} e^{iq\lambda_j} \varepsilon_{n-q} \right). \qquad (87)
\end{aligned}
$$

Because ε_t are iid, the second moment of (87) is bounded by

$$
\begin{aligned}
& \frac{1}{mn^2} \sum_{j=1}^m \sum_{k=1}^m \sum_0^{n-1} \left|\widetilde{j}_{\lambda_j p}\right| \left|\widetilde{j}_{\lambda_k p}\right| + \frac{2}{mn^2} \sum_{j=1}^m \sum_{k=1}^m \sum_0^{n-1} \left|\widetilde{j}_{\lambda_j p}\right| \sum_0^{n-1} \left|\widetilde{j}_{-\lambda_k r}\right| \qquad (88) \\
& + \frac{1}{mn^2} \sum_{j=1}^m \sum_{k=1}^m \sum_0^{n-1} \left|\widetilde{j}_{\lambda_j p}\right| \left|\widetilde{j}_{-\lambda_k p}\right| \sum_0^{n-1} e^{iq(\lambda_j - \lambda_k)}. \qquad (89)
\end{aligned}
$$

(88) is bounded by

$$
\frac{1}{mn^2} \sum_1^m \sum_1^m \left[\sum_0^{n-1} (\log n)^2 + \sum_0^{n-1} \frac{n}{j|p|_+} \sum_0^{n-1} \frac{n}{k|r|_+} \right] = O\left(mn^{-1} (\log n)^2 + m^{-1} (\log n)^4 \right),
$$

and in view of the fact that $\sum_{q=0}^{n-1} e^{iq(\lambda_j - \lambda_k)} = nI\{j = k\}$, (89) is bounded by

$$
\frac{1}{mn} \sum_1^m \sum_0^{n-1} \left|\widetilde{j}_{\lambda_j p}\right|^2 = O\left(\frac{1}{mn} \sum_1^m \sum_0^{n-1} j^{-1} |p|_+^{-1} n \log n \right) = O\left(m^{-1} (\log n)^3 \right),
$$

giving (86)$=o_p(1)$. Therefore, we obtain $m^{1/2}\widehat{G}_1(d_0) = 2m^{-1/2}\sum_1^m (\log \lambda_j) I_{uj} + o_p(1)$, and it follows that

$$
\begin{aligned}
& m^{1/2} R'(d_0) = m^{1/2} \left[\frac{\widehat{G}_1(d_0)}{\widehat{G}(d_0)} - 2\frac{1}{m} \sum_1^m \log \lambda_j \right] \\
= \quad & \frac{2m^{-1/2} \sum_1^m (\log \lambda_j) I_{uj} + o_p(1) - \left(\frac{1}{m} \sum_1^m \log \lambda_j \right) 2m^{-1/2} \sum_1^m I_{uj}}{m^{-1} \sum_1^m I_{uj}} \\
= \quad & \frac{2m^{-1/2} \sum_1^m (\log \lambda_j - \frac{1}{m} \sum_1^m \log \lambda_j) I_{uj} + o_p(1)}{G_0 + o_p(1)} \\
= \quad & \frac{2m^{-1/2} (\sum_1^m \log j - \frac{1}{m} \sum_1^m \log j)(2\pi I_{\varepsilon j} - 1) + O_p(m^{\beta + 1/2} n^{-\beta} \log m) + o_p(1)}{1 + o_p(1)} \\
\to_d \quad & N(0, 4),
\end{aligned}
$$

where the last line follows from Robinson (1995), to complete the proof. ■

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