Exact Local Whittle Estimation of Fractionally Cointegrated Systems

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November 25, 2003

Abstract

Semiparametric estimation of a bivariate fractionally coitegrated system is considered. The new estimator employs the exact local Whittle approach developed by Shimotsu and Phillips (2003a) and estimates the two memory parameters jointly with the cointegrating vector. It permits both (asymptotically) stationary and nonstationary stochastic trends and/or equilibrium errors without relying on differencing or data tapering. Indeed, the asymptotic properties of the estimator depend only on the difference of the two memory parameters. The estimator of the memory parameters is shown to be consistent and asymptotically normally distributed in both stationary and nonstationary cases.

JEL Classification: C22

Key words and phrases: discrete Fourier transform, fractional cointegration, long memory, nonstationarity, semiparametric estimation, Whittle likelihood

1 Introduction

The analysis of the long-run equilibrium relationship between economic variables is now a common task in empirical econometric modeling. Cointegration methods have provided powerful tools for the analysis of these issues. Two random processes are said to be cointegrated if they have the same memory parameter but their linear combination has a smaller memory parameter. Cointegrated random processes form a long-run equilibrium relationship, in which the cointegrated processes are driven by a common stochastic trend and the equilibrium error has less persistence than the stochastic trend. The conventional cointegration modeling preassigns 1 as the value of the memory parameter of the stochastic trend and 0 as that of the equilibrium error. Therefore, a long-run equilibrium relationship is defined as the one between two $I(1)$ time series, where the equilibrium error is an $I(0)$ process.

*Currently at Department of Economics, Queen's University. The author thanks Marine Carrasco, Peter Phillips, and Werner Ploberger for helpful comments and the Cowles Foundation for hospitality during his stay from January 2002 to June 2003. Financial support from ESRC under Grant R000223986 is gratefully acknowledged.

The fractional cointegration analysis generalizes the $I(0)/I(1)$ cointegration analysis by allowing the memory parameter of the variables to be any real number. The system is driven by a common stochastic trend that has a memory parameter d_1 and is accompanied by an equilibrium error that has a memory parameter d_2. It provides a more flexible apparatus for analyzing long-run relationships between economic time series and enables more proper modelling of interdependence between them. For instance, consider the following two cases:

- Two time series have the same memory parameter $d_1 < 1$, and the equilibrium error has a memory parameter $d_2 < d_1$.
- Two time series are $I(1)$, but the equilibrium error has a memory parameter d that is between 0 and 1.

Clearly, the two time series form a long-run equilibrium in the above two cases, but the conventional $I(0)/I(1)$ cointegration modeling cannot accommodate them. When empirical researchers conduct the $I(0)/I(1)$ cointegration analysis with such data, it leads to either (i) a false rejection of the existence of an equilibrium relationship, or (ii) misspecification of the degree of persistence of the stochastic trend and/or the equilibrium error. Growing evidence shows that many economic time series have memory parameters between 0 and 1. Empirical analysis of fractional cointegration is also emerging, although it is still limited in number, mainly because of lack of general purpose inferential tools.

Given its attractiveness and relevance, theoretical studies of fractional cointegration have been emerging rapidly. One empirically appealing approach is to obtain estimates of d_1 and d_2 and conduct inference based on them. In the $I(0)/I(1)$ cointegration, the test of cointegration can be based on the unit root test applied to the OLS residuals from cointegrating regression, as proposed by Engle and Granger (1987) and analyzed by Phillips and Ouliaris (1990). This is because the OLS estimator converges at the rate n^{-1}, and the effect of estimating the cointegrating vector on the unit root testing vanishes in the limit.

Some studies seek to extend this residual-based approach to the estimation of the memory parameter of the equilibrium error, d_2, e.g., Hassler et al. (2000), Velasco (2003), and Nielsen (2002). They estimate the cointegrating vector first, either by the OLS or narrow-band least squares (NBLS) (Robinson and Marinucci, 2001), and then apply the semiparametric estimators to the residuals. They differ in details; Hassler et al.(2000) estimate d_2 by applying the log-periodogram regression of Robinson (1995a) to the regression residuals, whereas Velasco (2003) and Nielsen (2002) use a version of the two-step estimator of Lobato (1999) to jointly estimate d_1 and d_2 (and the cointegrating vector). In fact, these two-step estimators presuppose the existence of the first-stage estimator that is based on the regression residuals and converges at the same rate as the second-stage estimator.

Now the question is: to what extent does this residual-based approach work? In other words, how much of the effectiveness of the $I(0)/I(1)$ cointegrating regression carries itself over to the fractional cointegration? The answer is: not much, unfortunately. The $I(0)/I(1)$ cointegrating regression is n–consistent, because the $O_p(t^{1/2})$ signal from the regressor dominates the $O_p(1)$ noise of the regression error. In fractional cointegration, the stochastic order of the signal and noise becomes $O(t^{\max\{d_1-1/2,0\}})$ and $O(t^{\max\{d_2-1/2,0\}})$, respectively. As a result, when the difference between d_1 and d_2 is small, the cointegrating vector estimate converges at a too slow rate to validate the subsequent analysis based on the residuals. Indeed, the two-step procedure of Velasco (2003) requires $d_1 - d_2 > 1/2$, and Nielsen (2002) needs to assume that the long-run endogeneity between the stochastic trend and equilibrium error does not exist, by assuming that their long-run covariance matrix is diagonal.

These problems are reminiscent of the second-order bias in the $I(0)/I(1)$ cointegrating regression. In the $I(0)/I(1)$ cointegration, the OLS estimator has the second-order bias (Phillips and Durlauf, 1986), but the bias is not severe enough to cause problems for the residual-based unit root testing. In fractional cointegration, however, the slower rate of convergence of the cointegrating regression aggravates the second-order bias effect.

Given the difficulty in estimating d_2, some studies focus on testing the null hypothesis of no cointegration, e.g., by a Hausman-type test (Marinucci and Robinson, 2001) or estimating the rank of the (normalized) spectral density matrix at frequency zero (Robinson and Yajima, 2002, Chen and Hurvich, 2002). These procedures partly deprive the fractional cointegration of its flexibility and attractiveness, because they do not provide information about the persistence of the equilibrium error and the relative strength between the stochastic trend and equilibrium error. But they can accommodate multivariate models very easily, which are very relevant in applications, and compliment the approaches based on estimating the memory parameters in the system.

The above procedures, both the residual-based ones and the one based on the long-run covariance matrix, have an additional difficulty: prior to estimation, the researcher needs to know whether the value of the memory parameter d of each process in the system is larger or smaller than 1/2. This is because these procedures employ the semiparametric estimators of d that are proven to have a standard limiting distribution only for $-1/2 < d < 3/4$. Indeed, this poses serious problems for the following reasons:

1. Typically, whether $d \gtrless 1/2$ is unknown *a priori*; indeed, often empirical researchers want to *test* whether $d \gtrless 1/2$, because this determines whether the process is stationary (if $d > 1/2$) or nonstationary (if $d < 1/2$).

2. Because the value of d of most economic time series lies between 0 and 1, if two economic variables are cointegrated, then it is highly likely that the equilibrium error has d around 1/2.

3. We cannot construct a valid confidence interval for d that contains 1/2. In other words, the entire confidence interval must lie either below or above 1/2.

When $d > 1/2$, these semiparametric estimators exhibit nonstandard asymptotic behavior, such as nonnormal limit distribution and inconsistency, as shown e.g., by Phillips and Shimotsu (2003). As a result, the analysis and interpretation of the estimates are open to criticism and put empirical researchers in an awkward situation. Data tapering (Hurvich and Chen, 2000, Velasco, 2003) is one potential remedy to extend the range of the consistent estimation, but data tapering leads to a significant increase in the variance of the estimator.

The present paper develops a new estimation and inference method for bivariate fractionally cointegrated systems. It has two attractive features. First, it estimates the two memory parameters, d_1 and d_2, jointly with the cointegrating vector. This system-estimation approach is free from the second-order bias problem of the first-stage regression estimate, and, as a result, the new estimator does not need artificial restrictions such as $d_1 - d_2 > 1/2$ or the long-run exogeneity of the equilibrium error. Second, it requires no prior restrictions on the domain of d_1 and d_2. This is because it employs the exact local Whittle (ELW) approach developed by Shimotsu and Phillips (2003a). The ELW approach is based on the frequency domain Gaussian likelihood function (Whittle likelihood function) localized to the neighborhood of the origin and the discrete Fourier transform representation theory laid down by Phillips (1999). Shimotsu and Phillips (2003a) succeeded in showing the consistency and asymptotic normality of the ELW estimator for all values of d in the univariate case.

The developed estimator of (d_1, d_2) is consistent for any value of (d_1, d_2) in $[\Delta_1, \Delta_2]^2$ with $-\infty < \Delta_1 < \Delta_2 < \infty$, albeit we need to impose $\Delta_2 - \Delta_1 \leq 3/2$. The cointegrating vector is estimated $(n/m)^{\delta}$–consistently, where $\delta = d_1 - d_2$ and m is the number of frequencies included in the objective function. Regarding the asymptotic distribution, the estimator of (d_1, d_2) is asymptotically normally distributed when $\delta \in (0, \frac{3}{2}) \backslash \{\frac{1}{2}\}$. Therefore, the estimator imposes no restriction on the domain of δ for practical application and covers both stationary and nonstationary cases.

The remainder of the paper is organized as follows. Section 2 briefly reviews the model of fractional cointegration. Section 3 derives the asymptotic theory of exact local Whittle estimation of bivariate fractionally integrated processes. It serves as a precursor of the analysis of fractionally cointegrated systems. The asymptotic theory of the exact local Whittle estimation of fractionally cointegrated systems is developed in Section 4. Section 5 reports some simulation results. Some technical results are collected in Appendix A in Section 6. Proofs are given in Appendix B in Section 7.

2 Preliminaries

2.1 A model of fractional cointegration

We consider a model where the observed variables X_{1t} and X_{2t} are fractionally cointegrated. Specifically, X_{1t} and X_{2t} are generated by the model

$$\begin{cases} (1-L)^{d_1} X_{1t} = u_{1t} I\{t \geq 1\}, & t = 1, 2, \ldots, \\ (1-L)^{d_2} (X_{2t} - \beta X_{1t}) = u_{2t} I\{t \geq 1\}, & t = 1, 2, \ldots, \\ X_{1t} = X_{2t} = 0, & t \leq 0, \end{cases} \tag{1}$$

where $u_t = (u_{1t}, u_{2t})'$ is stationary with zero mean and spectral density matrix $f_u(\lambda)$. We assume $d_1 \geq d_2$. If $d_1 > d_2$, X_{1t} and X_{2t} are individually $I(d_1)$ because their d_1-th differences have spectral density that are bounded and bounded away from the origin. But their linear combination, $X_{2t} - \beta X_{1t}$, has a memory parameter d_2 that is smaller than d_1. Expanding the binomial in the first row of (1) gives the form

$$\sum_{k=0}^{t} \frac{(-d_1)_k}{k!} X_{1,t-k} = u_t I\{t \geq 1\}, \tag{2}$$

where

$$(d)_k = \frac{\Gamma(d+k)}{\Gamma(d)} = (d)(d+1)\ldots(d+k-1),$$

is Pochhammer's symbol for the forward factorial function and $\Gamma(\cdot)$ is the gamma function.

The model (1) provides a valid data-generating process for any value of (d_1, d_2). When $d_1 > 1/2$, X_{1t} is nonstationary, and when $d_1 < 1/2$, X_{1t} is asymptotically covariance stationary as shown by Robinson and Marinucci (2001). Therefore, it accommodates both nonstationary and asymptotically cases, and setting $d_1 = 1$ and $d_2 = 0$ gives the conventional $I(0)/I(1)$ cointegration.

For a vector time series a_t, define the discrete Fourier transform (dft) and the periodogram evaluated at the fundamental frequencies as

$$\begin{aligned} w_a(\lambda_j) &= \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} a_t e^{it\lambda_j}, \quad \lambda_j = \frac{2\pi j}{n}, \quad j = 1, \ldots, n, \\ I_a(\lambda_j) &= w_a(\lambda_j) w_a^*(\lambda_j), \end{aligned} \tag{3}$$

where x^* denotes the complex conjugate of x.

3 Multivariate exact local Whittle estimation

Before analyzing the estimation of the fractionally cointegrated system, it is useful to analyze the case where β is known in (1) so that we can concentrate on the estimation of d. Assume $\beta = 0$ without loss of generality, then the system reduces to a multivariate fractionally integrated process

$$\begin{cases} \begin{pmatrix} (1-L)^{d_1} & 0 \\ 0 & (1-L)^{d_2} \end{pmatrix} \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix} = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} I\{t \geq 1\}, & t = 1, 2, \ldots, \\ X_{1t} = X_{2t} = 0, & t \leq 0. \end{cases} \tag{4}$$

The (negative) Whittle likelihood of u_t based on frequencies up to λ_m and up to scale multiplication is

$$\sum_{j=1}^{m} \log(\det f_u(\lambda_j)) + \sum_{j=1}^{m} \operatorname{tr}\left[f_u(\lambda_j)^{-1} I_u(\lambda_j) \right], \tag{5}$$

where m is some integer less than n. Now we transform the likelihood function (5) to be data dependent. Define

$$I_{\Delta^d x}(\lambda_j) = w_{\Delta^d x}(\lambda_j) w_{\Delta^d x}^*(\lambda_j), \quad w_{\Delta^d x}(\lambda_j) = \begin{pmatrix} w_{\Delta^{d_1} x_1}(\lambda_j) \\ w_{\Delta^{d_2} x_2}(\lambda_j) \end{pmatrix}.$$

Lemma 6.1 in Appendix A provides an algebraic relationship that connects $w_u(\lambda_j)$ and $w_x(\lambda_j)$:

$$w_u(\lambda_j) = w_{\Delta^d x}(\lambda_j) = \Lambda_n(e^{i\lambda_j}; d) v_x(\lambda_j; d), \tag{6}$$

where

$$\begin{aligned} \Lambda_n(e^{i\lambda_j}; d) &= \begin{pmatrix} D_n(e^{i\lambda_j}; d_1) & 0 \\ 0 & D_n(e^{i\lambda_j}; d_2) \end{pmatrix}, \quad v_x(\lambda_j; d) = \begin{pmatrix} v_{x_1}(\lambda_j; d_1) \\ v_{x_2}(\lambda_j; d_2) \end{pmatrix}, \\ v_{x_a}(\lambda_j; d_a) &= w_{x_a}(\lambda_j) - D_n(e^{i\lambda_j}; d_a)^{-1} (2\pi n)^{-1/2} \widetilde{X}_{a,\lambda_j n}(d_a). \end{aligned}$$

Although $v_x(\lambda_j; d)$ is not a periodogram of X_t, we may view (6) as the frequency domain representation of X_t where $\Lambda_n(e^{i\lambda_j}; d)$ acts as a transfer function. Using (6) in conjunction with the local approximation $f_u(\lambda_j) \sim G$ and $|D_n(e^{i\lambda_j}; d_a)|^2 \sim \lambda_j^{2d_a}$, the objective function is simplified to

$$Q_m(G,d) = \frac{1}{m}\sum_{j=1}^{m}\left\{\log(\det G) - 2\log\left(\lambda_j^{d_1}+\lambda_j^{d_2}\right) + \mathrm{tr}\left[G^{-1}I_{\Delta^d x}(\lambda_j)\right]\right\}.$$

We propose to estimate (G, d) by minimising $Q_m(G, d)$, so that

$$\left(\widehat{G},\widehat{d}\right) = \underset{G\in(0,\infty)^2,\ d\in\Theta}{\arg\min}\ Q_m(G,d),$$

where $\Theta = \{[\Delta_1,\Delta_2]\ \times[\Delta_1,\Delta_2]\}$ is the space of the admissible values of d. In what follows, we distinguish the true values of the parameters by the notation $G^0 = f_u(0)$ and d^0. Concentrating $Q_m(G, d, \beta)$ with respect to G, the first order condition is

$$\widehat{G} = \frac{1}{m}\sum_{1}^{m}\mathrm{Re}\left[I_{\Delta^d x}(\lambda_j)\right].$$

Thus we find that $\widehat{d}$ satisfies

$$\widehat{d} = \underset{d\in\Theta}{\arg\min}\ R(d), \tag{7}$$

where

$$R(d) = \log\det\widehat{G}(d) - 2(d_1+d_2)\frac{1}{m}\sum_{1}^{m}\log\lambda_j,\quad \widehat{G}(d) = \frac{1}{m}\sum_{1}^{m}\mathrm{Re}[I_{\Delta^d x}(\lambda_j)]. \tag{8}$$

We call $\widehat{d}$ the exact local Whittle estimator of d.

3.1 Consistency

We introduce the following assumptions on m and the stationary component u_t in (4).

Assumption 1

$$f_u(\lambda) \sim G^0 \quad \text{as } \lambda \to 0+,$$

where G^0 is real, symmetric, finite, and positive definite.

Assumption 2

$$u_t - Eu_0 = A(L)\varepsilon_t = \sum_{j=0}^{\infty}A_j\varepsilon_{t-j},\quad \sum_{j=0}^{\infty}||A_j||^2 < \infty,$$

where $||\cdot||$ denotes the supremum norm and $E(\varepsilon_t|F_{t-1}) = 0$, $E(\varepsilon_t\varepsilon_t'|F_{t-1}) = I_q$ a.s., $t = 0, \pm1, \ldots$, in which F_t is the σ-field generated by ε_s, $s \le t$, and there exists a scaler random variable ε such that $E\varepsilon^2 < \infty$ and for all $\eta > 0$ and some $K > 0$, $\Pr(||\varepsilon_t|| > \eta) \le K\Pr(\varepsilon^2 > \eta)$.

Assumption 3 *In a neighborhood $(0,\delta)$ of the origin, $A(\lambda) = \sum_{j=0}^{\infty}A_je^{ij\lambda}$ is differentiable and*

$$\frac{\partial}{\partial\lambda}A_a(\lambda) = O(\lambda^{-1}) \quad \text{as } \lambda \to 0+,$$

where $A_a(\lambda)$ is the ath row of $A(\lambda)$.

Assumption 4

$$\frac{1}{m} + \frac{m(\log m)^{1/2}}{n} + \frac{\log n}{m^\gamma} \to 0 \quad \text{for any } \gamma > 0.$$

Assumption 5

$$\Delta_2 - \Delta_1 \leq 3/2.$$

Assumptions 1-3 are a version of a multivariate extension of Assumptions A1-A3 of Robinson (1995b), but we impose them in terms of u_t rather than X_t. They are analogous to the assumptions used in Lobato (1999). Assumption 4 is slightly stronger than Assumption A4 of Robinson (1995b). Assumption 5 restricts the length of the interval of the admissible estimates, although it imposes no restrictions on the value of d^0 itself. For economic data, we may safely assume $d_1^0, d_2^0 \geq 0$, then taking $[\Delta_1, \Delta_2] = [0, 1.5]$ makes $\widehat{d}$ consistent for any $d^0 \in [\Delta_1, \Delta_2]^2$.

Under these conditions we may now establish the consistency of $\widehat{d}$.

3.2 Theorem

Suppose X_t is generated by (4) and Assumptions 1-5 hold. Then, for $d^0 \in \Theta$, $\widehat{d} \to_p d^0$ as $n \to \infty$.

3.3 Asymptotic Normality

We introduce some further assumptions that are used to derive the limit distribution theory in this section.

Assumption 1′ *For $\beta \in (0, 2]$,*

$$\|A(\lambda) - A(0)\| = O(\lambda^\beta), \qquad \text{as } \lambda \to 0+.$$

Assumption 2′ *Assumption 2 holds and also for $a, b, c, d = 1, 2$,*

$$E(\varepsilon_{at}\varepsilon_{bt}\varepsilon_{ct}|F_{t-1}) = \mu_{abc} \quad a.s., \quad E(\varepsilon_{at}\varepsilon_{bt}\varepsilon_{ct}\varepsilon_{dt}|F_{t-1}) = \mu_{abcd}, \quad t = 0, \pm 1, \ldots,$$

where $|\mu_{abc}| < \infty$ and $|\mu_{abcd}| < \infty$.

Assumption 3′ *Assumption 3 holds.*

Assumption 4′ As $n \to \infty$,

$$\frac{1}{m} + \frac{m^{1+2\beta}(\log m)^2}{n^{2\beta}} + \frac{\log n}{m^\gamma} \to 0, \quad \text{for any } \gamma > 0.$$

Assumption 5′ *Assumption 5 holds.*

Assumption 6′ *Define* $\Gamma_j = E u_t u'_{t+j}$. *Uniformly in* $k = 0, 1, \ldots$

$$\left\|\sum_{j\geq k}\Gamma_j\right\| = O((\log(k+1))^{-4}), \quad \left\|\sum_{j\geq k}A_j\right\| = O((\log(k+1))^{-4}).$$

Assumptions 1′ implies and is stronger than

$$\left\|f_u(\lambda) - G^0\right\| = O(\lambda^{\beta}), \qquad \text{as } \lambda \to 0+, \tag{9}$$

which is analogous to Assumption A1 of Lobato (1999). It may be possible to relax Assumption 1′ to (9), but the proof would become more complicated. Assumptions 2′-4′ are comparable to Assumptions A2-A4 of Lobato (1999), where the only difference is an additional condition $\log n/m^{\gamma} \to 0$. Assumption 6′ controls the behavior of the tail sums of A_j and Γ_j and is a fairly mild condition. It allows for a pole and discontinuity in $f_u(\lambda)$ at $\lambda \neq 0$. For more details, see Phillips and Shimotsu (2003).

The following theorem establishes the asymptotic normality of the exact local Whittle estimator for $d_0 \in \mathrm{Int}(\Theta)$.

3.4 Theorem

Suppose X_t *is generated by (4) and Assumptions 1′-6′ hold. Then, for* $d^0 \in \mathrm{Int}(\Theta)$,

$$m^{1/2}\left(\widehat{d} - d_0\right) \to_d N(0, \Omega), \quad \Omega = 2(I_2 + G^0 \odot (G^0)^{-1}) + \frac{\pi^2 (G^0_{12})^2}{2\det G^0}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

where $\odot$ *denotes the Hadamard product.*

3.5 Remark

The limiting covariance matrix differs from that of Lobato (1999), where $\Omega = 2(I_2 + G^0 \odot (G^0)^{-1})$. This is because the approximation of the multivariate spectral density used in Lobato (1999) does not hold for a multivariate $I(d)$ process.

4 Exact local Whittle estimation of fractional cointegration

In the fractional cointegration case, the cointegrating vector β in (1) is unknown and hence needs to be estimated jointly with the memory parameter d. Following the same algebraic manipulation, we obtain the objective function

$$Q_m(G, d, \beta) = \frac{1}{m}\sum_{j=1}^{m}\left\{\log\det G - 2\log\left(\lambda_j^{d_1} + \lambda_j^{d_2}\right) + \mathrm{tr}\left[G^{-1} I_{\Delta^d x}(\lambda_j; \beta)\right]\right\},$$

where

$$I_{\Delta^d x}(\lambda_j; \beta) = w_{\Delta^d x}(\lambda_j; \beta)\, w^*_{\Delta^d x}(\lambda_j; \beta), \quad w_{\Delta^d x}(\lambda_j; \beta) = \begin{pmatrix} w_{\Delta^{d_1} x_1}(\lambda_j) \\ w_{\Delta^{d_2}(x_2 - \beta x_1)}(\lambda_j) \end{pmatrix}.$$

We propose to estimate (G, d, β) by minimising $Q_m(G, d, \beta)$, so that

$$\left(\widehat{G}, \widehat{d}, \widehat{\beta}\right) = \underset{G\in(0,\infty)^2,\ d\in\Theta,\ \beta\in B}{\arg\min} Q_m(G, d, \beta),$$

where Θ is defined above and B is a closed one-dimensional interval. We let β^0 denote the true value of β. Concentrating $Q_m(G, d, \beta)$ with respect to G, we find that $\widehat{d}$ and $\widehat{\beta}$ satisfies

$$(\widehat{d}, \widehat{\beta}) = \underset{d \in \Theta,\ \beta \in B}{\arg\min}\ R(d, \beta), \tag{10}$$

where

$$R(d, \beta) = \log\det \widehat{G}(d, \beta) - 2(d_1 + d_2)\frac{1}{m}\sum_{1}^{m} \log \lambda_j, \quad \widehat{G}(d, \beta) = \frac{1}{m}\sum_{1}^{m} \mathrm{Re}[I_{\Delta^d x}(\lambda_j; \beta)]. \tag{11}$$

We call $(\widehat{d}, \widehat{\beta})$ the exact local Whittle estimator of (d, β).

4.1 Consistency

Let $\delta = d_1^0 - d_2^0$. We need an additional assumption for the consistency of the ELW estimator of (d, β).

Assumption A Let $\theta_a = d - d_a$. Then $\Theta = \{[\Delta_1, \Delta_2] \times [\Delta_1, \Delta_2]\ \} \backslash (T_1 \cup T_2)$, where, for arbitrary small $\Delta > 0$,

$$\begin{aligned} T_1 &= \{|\theta_1 + 1/2| \le \Delta\} \cup \{|\theta_2 + 1/2| \le \Delta\} \cup \{|\theta_2 - \delta + 1/2| \le \Delta\} \cup \{|\theta_1 - \theta_2 + \delta| \le \Delta\}, \\ T_2 &= \{|\theta_1 - \theta_2| \le \Delta\} \cap \{\theta_1 \le -1/2 + \Delta\} \cap \{\theta_2 \le -1/2 + \Delta\} \cap \{\theta_2 - \delta \le -1/2 + \Delta\}. \end{aligned}$$

4.2 Theorem

Suppose X_t is generated by (1) and Assumptions 1-5 and Assumption A hold. Then, for $d^0 \in \Theta \backslash \{\delta = 1/2\}$, as $n \to \infty$,

$$\begin{aligned} &\begin{pmatrix} \widehat{d} - d^0 \\ (n/m)^{\delta}\,(\widehat{\beta} - \beta^0) \end{pmatrix} \to_p 0, \quad && \delta \in (0, \tfrac{3}{2}) \backslash \{\tfrac{1}{2}\}, \\ &(\widehat{d} - d^0) \to_p 0, && \delta = 0. \end{aligned}$$

4.3 Remark

When $\delta > 0$, $\widehat{d}$ is consistent and $\widehat{\beta}$ is $(n/m)^{\delta}$-consistent. When $\delta = 0$, $\widehat{\beta}$ is not consistent, but $\widehat{d}$ is still consistent. Inconsistency of $\widehat{\beta}$ follows because β is not identified when $\delta = 0$. But it still does not prevent $\widehat{d}$ from being consistent, because when $\delta = 0$, then any linear combination of X_{1t} and X_{2t} has a memory parameter $d_1^0 = d_2^0$.

Assumption 3 restricts the possible range of $d - d^0$ in the domain of the optimization. It is necessary because the evaluation of the likelihood function becomes difficult on certain boundaries. Since Δ can be chosen arbitrary small and Θ contains a neighborhood of $\{(d_1, d_2) = (d_1^0, d_2^0)\}$, this assumption does not affect estimation in practice. We exclude the cases where $\delta = 1/2$ because of technical reasons. A longer proof would eliminate this restriction, but we chose not to do so to keep the proof simple.

4.4 Asymptotic Normality

The following theorem establishes the asymptotic normality of the exact local Whittle estimator.

4.5 Theorem

Suppose X_t is generated by (1) and Assumptions 1′- 6′ and Assumption A hold. Then, for $d_0 \in \mathrm{Int}(\Theta \backslash \{\delta = 1/2\})$, as $n \to \infty$,

(a) When $\delta \in (0, \frac{1}{2})$,

$$m^{1/2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (2\pi m/n)^{-\delta} \end{pmatrix} \begin{pmatrix} \widehat{d}_1 - d_1^0 \\ \widehat{d}_2 - d_2^0 \\ \beta - \beta^0 \end{pmatrix} \to_d N\left(0, \Xi_2^{-1} \Xi_1 \Xi_2^{-1}\right),$$

where Ξ_1 and Ξ_2 are symmetric, their upper-left (2,2) block are given by Ω, and

$$\begin{array}{lll} [\Xi_1]_{13} = -A - B, & [\Xi_1]_{23} = A + B, & [\Xi_1]_{33} = C, \\ [\Xi_2]_{13} = -A + B, & [\Xi_2]_{23} = A - B, & [\Xi_2]_{33} = C, \end{array}$$

where

$$\begin{aligned} A &= \frac{2G_{11}^0 G_{12}^0}{\det G^0} \cos(\frac{\pi\delta}{2}) \frac{2-\delta}{(1-\delta)^2}, \\ B &= \frac{\pi G_{11}^0 G_{12}^0}{\det G^0} \sin(\frac{\pi\delta}{2}) \frac{1}{1-\delta}, \\ C &= \frac{2(G_{11}^0)^2}{\det G^0} \cos^2\left(\frac{\pi\delta}{2}\right) \left[\frac{1}{1-2\delta} - \frac{1}{(1-\delta)^2}\right] + \frac{2(G_{11}^0)^2}{\det G^0} \sin^2\left(\frac{\pi\delta}{2}\right) \frac{1}{1-2\delta} \end{aligned}$$

(b) When $\delta \in (\frac{1}{2}, \frac{3}{2})$,

$$\begin{aligned} m^{1/2}\left(\widehat{d} - d_0\right) &\to {}_d N\left(0, \Omega^{-1}\right), \\ n^{\delta}(\widehat{\beta} - \beta^0) &= O_p(1). \end{aligned}$$

4.6 Remark

Velasco (2003) shows the asymptotic normality of the two-step estimator of a fractionally cointegrated system under restrictions on d_1, d_2 and m, including $\delta > 1/2$. Nielsen (2002) shows the asymptotic normality of the two-step estimator of a stationary fractionally cointegrated system under the restriction $G_{12}^0 = 0$. This theorem shows that those restrictions are unnecessary when d and β are jointly estimated.

5 Simulations

This section reports some simulations that were conducted to examine the finite sample performance of the developed estimator. We generate a fractionally cointegrated system according to (1) with $\beta = 3$. u_t is generated by $iidN(0, \Omega)$, where the diagonal elements of Ω were fixed to 1 and the off-diagonal elements of Ω were selected to be $\rho = (0.0, 0.3, 0.8)$. The bias, standard deviation, and root mean squared error (RMSE) were computed using 10,000 replications. Sample size and m were chosen to be $n = 200$ and $m = n^{0.6} = 24$. Values of d_1 were selected to be $(0.4, 0.6, 1.0)$, and the value of d_2 was fixed to 0.2. The joint estimation of (d_1, d_2, β), hereafter ELW estimation, is compared with the "naive" method, where d_1 is estimated from X_{1t} and d_2 is estimated from $X_{2t} - \widetilde{\beta} X_{1t}$, where $\widetilde{\beta}$ is the narrow-band least squares estimator with $m = 24$.

Table 1 shows the simulation results when the off-diagonal elements of Ω are 0, so there is no correlation between X_{1t} and $X_{2t} - \beta X_{1t}$. Although the "naive" estimator should have the same asymptotic variance as the ELW estimator in this case, the bias and RMSE of the ELW estimator are slightly smaller than those of the "naive" estimator. Not surprisingly, the ELW estimator of β is not very accurate when the difference between d_1 and d_2 is small. The narrow-band least squares estimate of β appears to be unbiased.

Table 1. Simulation Results
$n = 200$, $m = n^{0.6}$, $d_2 = 0.2$, $\rho = 0.0$

	"naive" estimation			ELW estimation		
	$\widehat{d_1}$	$\widehat{d_2}$	$\widehat{\beta}$	$\widehat{d_1}$	$\widehat{d_2}$	$\widehat{\beta}$
			$d_1 = 0.4$			
bias	-0.0678	-0.0467	0.0005	-0.0214	-0.0079	-17.357
s.d.	0.1472	0.1475	0.1221	0.1455	0.1342	332.53
RMSE	0.1620	0.1548	0.1221	0.1471	0.1344	332.99
			$d_1 = 0.6$			
bias	-0.0730	-0.0358	-0.0008	-0.0285	-0.0082	-4.6805
s.d.	0.1481	0.1496	0.0864	0.1466	0.1344	154.67
RMSE	0.1651	0.1538	0.0864	0.1494	0.1346	154.74
			$d_1 = 1.0$			
bias	-0.0712	-0.0110	0.0004	-0.0415	-0.0067	0.0006
s.d	0.1490	0.1362	0.0303	0.1516	0.1338	0.0382
RMSE	0.1651	0.1366	0.0303	0.1572	0.1340	0.0382

Table 2 shows the results for $\rho = 0.3$ and there is mild endogeneity. First, the narrow-band least square estimator of β exhibits a large bias when d_1 is 0.4 and 0.6. The results for the "naive" estimator of d_1 and d_2 are very similar to those in Table 1. The standard deviation and RMSE of the ELW estimator of d_1 and d_2 are similar to those in Table 1 when $d_1 = 0.4$ and 0.6, but they are smaller than those in Table 1 when $d_1 = 1.0$.

Table 2. Simulation Results
$n = 200$, $m = n^{0.6}$, $d_2 = 0.2$, $\rho = 0.3$

	"naive" estimation			ELW estimation		
	$\widehat{d_1}$	$\widehat{d_2}$	$\widehat{\beta}$	$\widehat{d_1}$	$\widehat{d_2}$	$\widehat{\beta}$
			$d_1 = 0.4$			
bias	-0.0675	-0.0474	0.2016	-0.0210	-0.0083	-4.1251
s.d	0.1473	0.1483	0.1175	0.1454	0.1347	330.79
RMSE	0.1620	0.1556	0.2333	0.1469	0.1349	330.82
			$d_1 = 0.6$			
bias	-0.0720	-0.0362	0.1048	-0.0276	-0.0097	0.1750
s.d	0.1477	0.1493	0.0837	0.1456	0.1337	115.08
RMSE	0.1643	0.1536	0.1341	0.1482	0.1341	115.08
			$d_1 = 1.0$			
bias	-0.0699	-0.0119	0.0104	-0.0358	-0.0109	-0.0019
s.d	0.1480	0.1371	0.0301	0.1416	0.1290	0.0394
RMSE	0.1637	0.1376	0.0318	0.1461	0.1294	0.0394

Table 3 shows the results for $\rho = 0.8$ and the endogeneity is strong. The bias in the narrow-band least square estimator of β is larger now. The standard deviation and RMSE of the ELW estimator is smaller than the case with $\rho = 0.3$, especially when d_1 is large. This corroborates the theoretical result, and the difference between the "naive" method and ELW estimation is even more significant. The ELW estimate of β is still not very accurate, however.

In sum, the simulation results demonstrate a good performance of the ELW estimator and the gain from estimating d and β jointly.

Table 3. Simulation Results
$n = 200$, $m = n^{0.6}$, $d_2 = 0.2$, $\rho = 0.8$

	"naive" estimation			ELW estimation		
	$\widehat{d}_1$	$\widehat{d}_2$	$\widehat{\beta}$	$\widehat{d}_1$	$\widehat{d}_2$	$\widehat{\beta}$
			$d_1 = 0.4$			
bias	-0.0593	-0.0476	0.5349	-0.0175	-0.0101	9.8749
s.d.	0.1418	0.1512	0.0812	0.1391	0.1307	184.68
RMSE	0.1538	0.1585	0.5410	0.1402	0.1311	184.94
			$d_1 = 0.6$			
bias	-0.0675	-0.0371	0.2800	-0.0201	-0.0139	1.3120
s.d.	0.1406	0.1494	0.0717	0.1259	0.1190	38.521
RMSE	0.1559	0.1540	0.2890	0.1275	0.1198	38.544
			$d_1 = 1.0$			
bias	-0.0608	-0.0139	0.0280	-0.0227	-0.0190	-0.0017
s.d.	0.1447	0.1389	0.0260	0.1117	0.1079	0.0296
RMSE	0.1569	0.1396	0.0382	0.1140	0.1096	0.0296

6 Appendix 1: Technical Lemmas

In this and the following section, C and ε denote generic constants such that $C \in (1,\infty)$ and $\varepsilon \in (0,1)$ unless specified otherwise, and they may take different values in different places. I_{yj} denotes $I_y(\lambda_j)$, w_{uj} denotes $w_u(\lambda_j)$, and similarly for other dft's and periodograms.

6.1 Lemma (Phillips, 1999, Theorem 2.2)

(a) If X_{at}, $a = 1, 2$, follows (1), then

$$w_{u_a}(\lambda) = D_n\left(e^{i\lambda}; d_a\right) w_{x_a}(\lambda) - (2\pi n)^{-1/2} e^{in\lambda} \widetilde{X}_{a,\lambda n}(d),$$

where $D_n(e^{i\lambda}; d_a) = \sum_{k=0}^{n} \frac{(-d_a)_k}{k!} e^{ik\lambda}$ and

$$\widetilde{X}_{a,\lambda n}(d) = \widetilde{D}_{n\lambda}\left(e^{-i\lambda}L; d_a\right) X_{an} = \sum_{p=0}^{n-1} \widetilde{d}_{a,\lambda p} e^{-ip\lambda} X_{a,n-p}, \quad \widetilde{d}_{a,\lambda p} = \sum_{k=p+1}^{n} \frac{(-d_a)_k}{k!} e^{ik\lambda}.$$

(b) If X_{at}, $a = 1, 2$, follows (1) with $d_a = 1$, then

$$w_{x_a}(\lambda)\left(1 - e^{i\lambda}\right) = w_{u_a}(\lambda) - (2\pi n)^{-1/2} e^{i\lambda} e^{in\lambda} X_{an}.$$

6.2 Lemma (Shimotsu and Phillips, 2003a, Lemma 5.5)

For $\kappa \in (0,1)$, *as* $m \to \infty$,

$$(a) \qquad \sup_{-C \le \gamma \le C} \left| \frac{1}{m} \sum_{j=[\kappa m]}^{m} \left(\frac{j}{m}\right)^{\gamma} - \int_{\kappa}^{1} x^{\gamma} dx \right| = O\left(m^{-1}\right),$$

$$(b) \qquad \begin{array}{c} \sup_{-C \le \gamma \le C} |m^{-1} \sum_{j=[\kappa m]}^{m} (j/m)^{\gamma}| = O\left(1\right), \\ \liminf_{m \to \infty} \inf_{-C \le \gamma \le C} |m^{-1} \sum_{j=[\kappa m]}^{m} (j/m)^{\gamma}| > \varepsilon > 0. \end{array}$$

6.3 Lemma (Shimotsu and Phillips, 2003a, Lemma 5.7)

For $p \sim m/e$ *as* $m \to \infty$, $\varepsilon \in (0, 0.1)$, *and* $\Delta \in (0, 1/(2e))$, *there exists* $\bar{\kappa} \in (0, 1/4)$ *such that, for sufficiently large* m *and all fixed* $\kappa \in (0, \bar{\kappa})$,

$$(a) \inf_{-C \le \gamma \le -1+2\Delta} \frac{1}{m} \sum_{j=[\kappa m]}^{m} \left(\frac{j}{p}\right)^{\gamma} \ge 1 + 2\varepsilon, \quad (b) \inf_{1 \le \gamma \le C} \frac{1}{m} \sum_{j=[\kappa m]}^{m} \left(\frac{j}{p}\right)^{\gamma} \ge 1 + 2\varepsilon.$$

6.4 Lemma

For $p \sim m/e$ *as* $m \to \infty$, $\varepsilon \in (0, 0.1)$, $\Delta \in (0, 1/(2e))$, *and* $\kappa \in (0, 1/4)$, *we have, for sufficiently large* m,

$$\inf_{-1+2\Delta \le \gamma \le 1} \frac{1}{m} \sum_{j=[\kappa m]}^{m} \left(\frac{j}{p}\right)^{\gamma} \ge 1 - \kappa^{2\Delta} + o(1).$$

6.5 Proof

It follows from Lemma 6.2 that

$$\frac{1}{m} \sum_{[\kappa m]}^{m} \left(\frac{j}{p}\right)^{\gamma} = \left(\frac{m}{p}\right)^{\gamma} \frac{1}{m} \sum_{[\kappa m]}^{m} \left(\frac{j}{m}\right)^{\gamma} = e^{\gamma} \int_{\kappa}^{1} x^{\gamma} dx + o(1) = \frac{e^{\gamma}(1 - \kappa^{\gamma+1})}{\gamma + 1} + o(1).$$

The stated result follows because $e^{\gamma}/(\gamma + 1) \ge 1$ for $\gamma \in [-1 + 2\Delta, 1]$. ■

6.6 Lemma

Suppose $(Y_{at}, Y_{bt}) = ((1-L)^{\theta_1} u_{1t} I\{t \ge 1\}, (1-L)^{\theta_2} u_{2t} I\{t \ge 1\})$. *Under the assumptions of Theorem 3.2, we have, for* $1 \le s \le r \le m$,

$$\max_{a,b=1,2} \sup_{|\theta_a|, |\theta_b| \le 1/2} \sum_{j=s}^{r} \left\{ \lambda_j^{-\theta_a - \theta_b} w_{y_a j} w_{y_b j}^{*} - e^{-\frac{\pi}{2}(\theta_a - \theta_b)i} G_{ab}^{0} \right\} = O_p(r^2 n^{-1} + r^{1/2} (\log n)^2) + o_p(r).$$

6.7 Proof

Rewrite the term inside the braces as

$$\begin{aligned} &\lambda_j^{-\theta_a - \theta_b} w_{y_a j} w_{y_b j}^{*} - \lambda_j^{-\theta_a - \theta_b} D_n(e^{i\lambda_j}; \theta_a) D_n^{*}(e^{i\lambda_j}; \theta_b) w_{u_a j} w_{u_b j}^{*} \\ &+ \left[\lambda_j^{-\theta_a - \theta_b} D_n(e^{i\lambda_j}; \theta_a) D_n^{*}(e^{i\lambda_j}; \theta_b) - e^{-\frac{\pi}{2}(\theta_a - \theta_b)i} \right] w_{u_a j} w_{u_b j}^{*} \end{aligned}$$

$$
\begin{aligned}
&+e^{-\frac{\pi}{2}(\theta_a-\theta_b)i}\left[w_{u_aj}w_{u_bj}^*-A_a(\lambda_j)I_{\varepsilon j}A_b^*(\lambda_j)\right]\\
&+e^{-\frac{\pi}{2}(\theta_a-\theta_b)i}\left[A_a(\lambda_j)I_{\varepsilon j}A_b^*(\lambda_j)-f_{u_{ab}}(\lambda_j)\right]\\
&+e^{-\frac{\pi}{2}(\theta_a-\theta_b)i}\left[f_{u_{ab}}(\lambda_j)-G_{ab}^0\right]\\
=\;& H_{1j}+H_{2j}+e^{-\frac{\pi}{2}(\theta_a-\theta_b)i}H_{3j}+e^{-\frac{\pi}{2}(\theta_a-\theta_b)i}H_{4j}+e^{-\frac{\pi}{2}(\theta_a-\theta_b)i}H_{5j},
\end{aligned}
$$

where $A_b^*(\lambda_j)$ denotes the bth column of $A_b^*(\lambda_j)$. From the proof of Theorem 2 of Robinson (1995a) (also see Robinson, 1995b, p. 1673), we have

$$
\begin{cases}
EI_{uj}=f_{uj}\{1+O(j^{-1}\log(j+1))\},\\
Ew_{u_aj}w_{\varepsilon j}^*=A_a(\lambda_j)/2\pi+O(j^{-1}\log(j+1)), \qquad j=1,\ldots,m.\\
EI_{\varepsilon j}=I_n/2\pi+O(j^{-1}\log(j+1)).
\end{cases}
\tag{12}
$$

From Lemma 8.4 of Phillips and Shimotsu (2003), for $j=1,\ldots,m$,

$$
\sup_{|\theta_a|,|\theta_b|\le 1/2}\left|\lambda_j^{-\theta_a-\theta_b}D_n(e^{i\lambda_j};\theta_a)D_n^*(e^{i\lambda_j};\theta_b)-e^{-\frac{\pi}{2}(\theta_a-\theta_b)i}\right|=O(\lambda_j)+O(j^{-1/2}).
$$

It follows that $E\sup_\theta\left|\sum_{j=s}^r H_{2j}\right|=O(r^2n^{-1}+r^{1/2})$. For the contributions from H_{1j}, we have

$$
\begin{aligned}
&\lambda_j^{-\theta_a-\theta_b}w_{y_aj}w_{y_bj}^*-\lambda_j^{-\theta_a-\theta_b}D_n(e^{i\lambda_j};\theta_a)D_n^*(e^{i\lambda_j};\theta_b)w_{u_aj}w_{u_bj}^*\\
=\;&\lambda_j^{-\theta_a}w_{y_aj}\left[\lambda_j^{-\theta_b}w_{y_bj}^*-\lambda_j^{-\theta_b}D_n^*(e^{i\lambda_j};\theta_b)w_{u_bj}^*\right]\\
&\left[\lambda_j^{-\theta_a}w_{y_aj}-\lambda_j^{-\theta_a}D_n(e^{i\lambda_j};\theta_a)w_{u_aj}\right]\lambda_j^{-\theta_b}D_n^*(e^{i\lambda_j};\theta_b)w_{u_bj}^*.
\end{aligned}
$$

From Lemma 6.1 and Lemma 5.3 of Shimotsu and Phillips (2003a), we obtain

$$
\lambda_j^{-\theta_b}w_{y_bj}=\lambda_j^{-\theta_b}D_n(e^{i\lambda_j};\theta_b)w_{u_bj}+R_{nj}(\theta_b), \tag{13}
$$

with $E\sup_{|\theta_b|\le 1/2}|R_{nj}(\theta_b)|^2=O(j^{-1}(\log n)^2)$. It follows from Cauchy-Schwartz inequality, (12), and (13) that

$$
E\sup_{|\theta_a|,|\theta_b|\le 1/2}\left|\lambda_j^{-\theta_a}w_{y_aj}\left[\lambda_j^{-\theta_b}w_{y_bj}^*-\lambda_j^{-\theta_b}D_n^*(e^{i\lambda_j};\theta_b)w_{u_bj}^*\right]\right|=O(j^{-1/2}(\log n)^2),
$$

and similarly for the second term on the right, giving $E\sup_\theta\left|\sum_{j=s}^r H_{1j}\right|=O(r^{1/2}(\log n)^2)$. For the contributions from H_{3j}, applying the decomposition

$$
H_{3j}=w_{u_aj}\left[w_{u_bj}^*-w_{\varepsilon j}^*A_b^*(\lambda_j)\right]+\left[w_{u_aj}-A_a(\lambda_j)w_{\varepsilon j}\right]w_{\varepsilon j}^*A_b^*(\lambda_j)
$$

and (12) gives $E\left|\sum_{j=s}^r H_{3j}\right|=O(r^{1/2}(\log n)^2)$. For H_{4j},we can apply the arguments in the proof of (C.3) in Lobato (1999, p.145) to show $\sum_{j=s}^r H_{4j}=o_p(r)+O_p(r^{1/2})$. Assumption 1 gives $\sum_{j=s}^r H_{5j}=o(r)$, and the stated result follows. ■

6.8 Lemma

For $\delta\in(-C,\frac{1}{2})$,

$$
\begin{aligned}
\xi\;&\equiv\;\frac{2\pi}{\sqrt{m}}\sum_1^m\operatorname{Im}\begin{bmatrix} w_{\varepsilon_1j}w_{\varepsilon_2j}^*\\ (j/m)^{-\delta}\,w_{\varepsilon_1j}w_{\varepsilon_2j}^*\end{bmatrix}\to_d N(0,\Sigma),\\
\Sigma\;&=\;\frac{1}{2}\begin{bmatrix}1 & (1-\delta)^{-1}\\ (1-\delta)^{-1} & (1-2\delta)^{-1}\end{bmatrix}.
\end{aligned}
$$

6.9 Proof

We show that $\eta\xi \to_d N(0, \eta'\Sigma\eta)$ for any $\eta = (\eta_1, \eta_2)$. First, observe that

$$\eta\xi = \eta_1 \frac{2\pi}{\sqrt{m}} \sum_1^m \mathrm{Im}\left[w_{\varepsilon_1 j} w^*_{\varepsilon_2 j}\right] + \eta_2 \frac{2\pi}{\sqrt{m}} \sum_1^m (j/m)^{-\delta}\, \mathrm{Im}\left[w_{\varepsilon_1 j} w^*_{\varepsilon_2 j}\right].$$

Proceeding in the same manner as in Robinson (1995b, pp.1644-47), we obtain

$$\begin{aligned} \eta\xi &= \frac{1}{n\sqrt{m}} \sum_t \sum_s \sum_j \sin(t-s)\lambda_j \left(\eta_1 + \eta_2 (j/m)^{-\delta}\right) \varepsilon_{1t}\varepsilon_{2s} \\ &= \sum_{t=1}^n \varepsilon_{1t} \sum_{s=1}^{t-1} \varepsilon_{2s} c_{t-s} - \sum_{t=1}^n \varepsilon_{2t} \sum_{s=1}^{t-1} \varepsilon_{1s} c_{t-s}; \\ c_s &= \frac{1}{n\sqrt{m}} \sum_{j=1}^m \left(\eta_1 + \eta_2 (j/m)^{-\delta}\right) \sin(s\lambda_j). \end{aligned}$$

Since the first and second terms in the second line are uncorrelated, the stated result follows if we show

$$\sum_{t=1}^n z_t \to_d N\left(0, \eta'\Sigma\eta/2\right); \quad z_t = \begin{cases} \varepsilon_{1t} \sum_{s=1}^{t-1} \varepsilon_{2s} c_{t-s}, & t \geq 2, \\ 0, & t = 1. \end{cases}$$

From Robinson (1995b, pp.1644-47), whose necessary conditions are

$$\textstyle\sum_{t=1}^{n-1} \sum_{s=1}^{n-t} c_s^2 \to \eta'\Sigma\eta/2, \quad n(\sum_{s=1}^n c_s^2)^2 \to 0, \quad n(\sum_{s=1}^n c_s^2)(\sum_{s=1}^n s c_s^2) \to 0. \qquad (14)$$

First,

$$\begin{aligned} \sum_{t=1}^{n-1} \sum_{s=1}^{n-t} c_s^2 &= \frac{1}{mn^2} \sum_{j=1}^m \sum_{t=1}^{n-1} \sum_{s=1}^{n-t} \left(\eta_1 + \eta_2 (j/m)^{-\delta}\right)^2 \sin^2(s\lambda_j) \\ &\quad + \frac{1}{2mn^2} \sum\sum_{j \neq k} \left(\eta_1 + \eta_2 (j/m)^{-\delta}\right)\left(\eta_1 + \eta_2 (k/m)^{-\delta}\right) \\ &\quad \times \sum_{t=1}^{n-1} \sum_{s=1}^{n-t} [\cos\{s(\lambda_j - \lambda_k)\} - \cos\{s(\lambda_j + \lambda_k)\}]. \end{aligned}$$

Using $\sum_{t=1}^{n-1} \sum_{s=1}^{n-t} \cos^2(s\lambda_j) = (n-1)^2/4$ (Robinson, 1995b, p. 1645) and $\sin^2(s\lambda_j) = 1 - \cos^2(s\lambda_j)$, the first term on the right is

$$\frac{1}{m} \sum_1^m \left(\eta_1 + \eta_2 (j/m)^{-\delta}\right)^2 \frac{1}{4} \to \frac{1}{4}\left(\eta_1^2 + \frac{2\eta_1\eta_2}{1-\delta} + \frac{\eta_2^2}{1-2\delta}\right).$$

The second term on the right is zero, because for $p \neq 0$

$$\sum_{t=1}^{n-1} \sum_{s=1}^{n-t} \cos(s\lambda_p) = \frac{\cos\lambda_p - 1}{4\sin^2 \lambda_p/2} - \frac{n-1}{2} = -\frac{1}{2} - \frac{n-1}{2} = -\frac{n}{2}.$$

Hence, the first condition in (14) holds. For the other conditions in (14), first let

$$\begin{aligned} c_s &= \eta_1 \frac{1}{n\sqrt{m}} \sum_1^m \sin(s\lambda_j) + \eta_2 \frac{1}{n\sqrt{m}} \sum_1^m \left(\frac{j}{m}\right)^{-\delta} \sin(s\lambda_j) \\ &= \eta_1 c_{1s} + \eta_2 c_{2s}, \end{aligned}$$

and we derive the bound for c_s. Obviously $c_s = c_{n-s}$ and $|c_{1s}|, |c_{2s}| = O(m^{1/2}n^{-1})$. Furthermore, we have for $1 \le s \le n/2$

$$\begin{aligned} c_{1s} &= O(m^{-1/2}s^{-1}), \\ c_{2s} &= \frac{m^\delta}{n\sqrt{m}} \sum_1^{m-1} \left(j^{-\delta} - (j+1)^{-\delta}\right) \sum_1^j \sin(s\lambda_l) + \frac{m^{-1/2}}{n} \sum_1^m \sin(s\lambda_j) \\ &= O\left(m^{-1/2}s^{-1}\right), \quad \delta < 0, \\ c_{2s} &= \frac{1}{n\sqrt{m}} \sum_1^{n/s} \left(\frac{j}{m}\right)^{-\delta} \sin(s\lambda_j) + \frac{1}{n\sqrt{m}} \sum_{n/s+1}^{m} \left(\frac{j}{m}\right)^{-\delta} \sin(s\lambda_j) \\ &= O\left(\frac{m^\delta}{n\sqrt{m}} \sum_1^{n/s} j^{-\delta}\right) + O\left(\frac{m^\delta}{n\sqrt{m}} \left(\frac{n}{s}\right)^{-\delta} \max_M \left| \sum_{n/s+1}^{M} \sin(s\lambda_j) \right| \right) \\ &= O\left(n^{-\delta}m^{\delta-1/2}s^{\delta-1}\right), \quad \delta \in (0, 1/2), \end{aligned}$$

from the fact that $|\sum_1^m \sin(s\lambda_j)| = O(n/s)$ and Theorem 2.2 in Zygmund (1959, p.3). It follows that

$$\begin{aligned} \textstyle\sum_1^n |c_{1s}|^2 &= O\left(n^{-1} \textstyle\sum_1^n s^{-1}\right) = O(n^{-1}\log n), \\ \textstyle\sum_1^n |c_{2s}|^2 &= O\left(n^{-1} \textstyle\sum_1^n s^{-1}\right) = O(n^{-1}\log n), \quad \delta < 0, \\ \textstyle\sum_1^n |c_{2s}|^2 &= \textstyle\sum_1^n |c_{2s}|^{\frac{1}{1-\delta}} |c_{2s}|^{\frac{1-2\delta}{1-\delta}} = O\left(\textstyle\sum_1^n (n^{-\delta}m^{\delta-1/2}s^{\delta-1})^{\frac{1}{1-\delta}} (m^{1/2}n^{-1})^{\frac{1-2\delta}{1-\delta}}\right) \\ &= O\left(n^{-1} \textstyle\sum_1^n s^{-1}\right) = O(n^{-1}\log n), \quad \delta \in (0, 1/2), \end{aligned}$$

and

$$\begin{aligned} \textstyle\sum_1^n s|c_{1s}|^2 &= O\left(m^{-1} \textstyle\sum_1^n s^{-1}\right) = O(m^{-1}\log n), \\ \textstyle\sum_1^n s|c_{2s}|^2 &= \begin{cases} O\left(m^{-1} \sum_1^n s^{-1}\right) = O(m^{-1}\log n), & \delta < 0, \\ O\left(n^{-2\delta}m^{2\delta-1} \sum_1^n s^{2\delta-1}\right) = O(m^{2\delta-1}), & \delta \in (0, 1/2). \end{cases} \end{aligned}$$

Therefore, the second and third conditions in (14) hold, and the stated result follows. ■

7 Appendix 2: Proofs

7.1 Proof of Theorem 3.2

Define $\theta = (\theta_1, \theta_2)' = d - d^0$ and $S(d) = R(d) - R(d^0)$. Fix $1/2 > \rho > 0$, and for arbitrary small $\Delta > 0$, define $\Theta_1 = \{\theta : \theta \in [-1/2 + \Delta, 1/2]^2\}$. Without loss of generality, assume $\Delta < 1/8$. Then we have (c.f. Robinson, 1995b, p.1634)

$$\begin{aligned} \Pr\left(||d - d^0|| > \rho\right) &\le \Pr\left(\inf_{\overline{N}_\rho \cap \Theta} S(d) \le 0\right) \\ &\le \Pr\left(\inf_{\overline{N}_\rho \cap \Theta_1} S(d) \le 0\right) + \Pr\left(\inf_{\Theta \backslash \Theta_1} S(d) \le 0\right). \end{aligned}$$

Define

$$
\begin{aligned}
Y_t &\equiv \Delta^d X_t = \Delta^{d-d^0}\Delta^{d^0} X_t = \Delta^\theta u_t, \quad M_\infty(\theta) = \begin{pmatrix} \int_0^1 x^{2\theta_1} dx & \int_0^1 x^{\theta_1+\theta_2} dx \\ \int_0^1 x^{\theta_1+\theta_2} dx & \int_0^1 x^{2\theta_2} dx \end{pmatrix}, \\
\Lambda_j(\theta) &= \operatorname{diag}(\lambda_j^{-\theta_1}, \lambda_j^{-\theta_2}), \quad M_j(\theta) = \operatorname{diag}\{(j/m)^{\theta_1}, (j/m)^{\theta_2}\}.
\end{aligned}
$$

From the property of the determinant, we have

$$
\det \widehat{G}(d) = \left(\frac{2\pi m}{n}\right)^{2\theta_1+2\theta_2} \det\left(\frac{1}{m}\sum_1^m M_j(\theta)\Lambda_j(\theta)\operatorname{Re}[I_{yj}]\Lambda_j(\theta)M_j(\theta)\right),
$$

and

$$
S(d) = S_1(d) + S_2(d) + S_3(d) + S_4(d),
$$

where

$$
\begin{aligned}
S_1(d) &= \log\det\left(\frac{1}{m}\sum_1^m M_j(\theta)\Lambda_j(\theta)\operatorname{Re}[I_{yj}]\Lambda_j(\theta)M_j(\theta)\right) - \log\det\left(G^0 \odot M_\infty(\theta)\right), \\
S_2(d) &= \log\det\left(G^0 \odot M_\infty(\theta)\right) - \log\det G^0 + \log(2\theta_1+1) + \log(2\theta_2+1), \\
S_3(d) &= \log\det G^0 - \log\det \widehat{G}(d^0), \\
S_4(d) &= -2\left(\theta_1+\theta_2\right)\left(\frac{1}{m}\sum_1^m \log j - \log m\right) - \log(2\theta_1+1) - \log(2\theta_2+1).
\end{aligned}
$$

Since $m^{-1}\sum_1^m \log j - \log m + 1 = O(m^{-1}\log m)$ (see, e.g. Robinson, 1995b, Lemma 2),

$$
S_4(d) = 2\theta_1 - \log(2\theta_1+1) + 2\theta_2 - \log(2\theta_2+1) + O(m^{-1}\log m).
$$

Because $x - \log(1+x) \geq 0$ for $x \in (-1,\infty)$ and $x - \log(x+1) \geq x^2/6$ for $0 \leq |x| < 1$, for large n we have

$$
\inf_{\overline{N}_\rho \cap \Theta_1^a} S_4(d) \geq \rho^2/2.
$$

For $S_2(d)$, since $M_\infty(\theta)$ is positive semidefinite if $\theta_1, \theta_2 \geq -1/2 + \Delta$, it follows from Oppenheim's inequality (Lütkepohl, 1996, p.56) that

$$
\det\left(G^0 \odot M_\infty(\theta)\right) \geq \det G^0 \int_0^1 x^{2\theta_1} dx \int_0^1 x^{2\theta_2} dx = \frac{\det G^0}{(2\theta_1+1)(2\theta_2+1)},
$$

giving

$$
\inf_{\overline{N}_\rho \cap \Theta_1^a} S_2(d) \geq 0.
$$

Since $|\log(1+x)| \leq 2|x|$ for $|x| \leq 1/2$, we deduce that when $\varepsilon \leq 1$

$$
\Pr\left(\left|\log\det \widehat{G}(d^0) - \log\det G^0\right| \leq \varepsilon\right) \leq \Pr\left(\left|\frac{\det \widehat{G}(d^0) - \det G^0}{\det G^0}\right| \leq \frac{\varepsilon}{2}\right).
$$

Therefore, $\Pr(\inf_{\overline{N}_\rho \cap \Theta_1^a} S(d) \leq 0) \to 0$ follows if

$$
\det \widehat{G}(d^0) - \det G^0 = o_p(1), \tag{15}
$$

and

$$\Pr\left(\inf_{\overline{N}_\rho\cap\Theta_1^a}\left\{\det\left(\frac{1}{m}\sum_1^m M_j(\theta)\Lambda_j(\theta)\operatorname{Re}[I_{yj}]\Lambda_j(\theta)M_j(\theta)\right)\right.\right.$$
$$\left.\left. -\det\left(G^0\odot M_\infty(\theta)\right)\right\}\leq -\frac{\rho^2}{4}\right)\to 0, \tag{16}$$

as $n\to\infty$.

We proceed to show (15) and (16). From Robinson (1995b) Lemma 2, we have

$$\sup_{C\geq\gamma\geq\varepsilon}\left|\frac{\gamma}{m}\sum_1^m\left(\frac{j}{m}\right)^{\gamma-1}-1\right|=O\left(\frac{1}{m^\varepsilon}\right)\quad\text{as } m\to\infty,$$

for $\varepsilon\in(0,1]$ and $C\in(\varepsilon,\infty)$. It follows that

$$\det\left(\frac{1}{m}\sum_1^m M_j(\theta)G^0M_j(\theta)\right)-\det\left(G^0\odot M_\infty(\theta)\right)=O\left(m^{-2\Delta}\right),$$

uniformly in $\theta\in\Theta_1^a$. For any (2×2) Hermitian matrix A, we have

$$\det(\operatorname{Re}[A])=A_{11}A_{22}-\operatorname{Re}[A_{12}]^2\geq A_{11}A_{22}-\operatorname{Re}[A_{12}]^2-\operatorname{Im}[A_{12}]^2=\det A. \tag{17}$$

Let

$$\mathcal{E}(\theta)=\operatorname{diag}\{\exp(-i\pi\theta_1/2),\exp(-i\pi\theta_2/2)\},$$

and note that multiplying any matrix by $\mathcal{E}(\theta)$ and $\mathcal{E}(\theta)^*$ does not change its determinant. Therefore, the term inside the $\inf_{\overline{N}_\rho\cap\Theta_1^a}$ in (16) is no smaller than, apart from an $O(m^{-2\Delta})$ term,

$$\det\left(m^{-1}\sum_1^m M_j(\theta)\Lambda_j(\theta)I_{yj}\Lambda_j(\theta)M_j(\theta)\right)-\det\left(m^{-1}\sum_1^m M_j(\theta)\mathcal{E}(\theta)G^0\mathcal{E}(\theta)^*M_j(\theta)\right).$$

Thus, (16) follows if

$$\sup_{\Theta_1}\left\|\frac{1}{m}\sum_1^m M_j(\theta)\left\{\Lambda_j(\theta)I_{yj}\Lambda_j(\theta)-\mathcal{E}(\theta)G^0\mathcal{E}(\theta)^*\right\}M_j(\theta)\right\|\to 0. \tag{18}$$

From summation by parts (Robinson, 1995b, p. 1636), the left hand side of (18) is bounded by

$$\sum_{r=1}^{m-1}\left(\frac{j}{m}\right)^{2\Delta}\frac{1}{r^2}\sup_{\Theta_1}\left\|\sum_{j=1}^r\left\{\Lambda_j(\theta)I_{yj}\Lambda_j(\theta)-\mathcal{E}(\theta)G^0\mathcal{E}(\theta)^*\right\}\right\| \tag{19}$$
$$+\frac{1}{m}\sup_{\Theta_1}\left\|\sum_{j=1}^m\left\{\Lambda_j(\theta)I_{yj}\Lambda_j(\theta)-\mathcal{E}(\theta)G^0\mathcal{E}(\theta)^*\right\}\right\|. \tag{20}$$

Both (19) and (20) are $o_p(1)$ from Lemma 6.6, and (18) follows. (15) follows from the results derived above, because $\mathcal{E}(\theta)$ is an identity matrix when $\theta=0$.

We move to $\theta \in \Theta \backslash \Theta_1$. A little algebra gives

$$\begin{aligned} S(d) &= \log\det \frac{1}{m}\sum_1^m I_{\Delta^d xj} - \log\det\frac{1}{m}\sum_1^m I_{uj} \\ &\quad -2(\theta_1+\theta_2)\log\frac{2\pi}{n} - 2(\theta_1+\theta_2)\frac{1}{m}\sum_1^m \log j \\ &= \log\det \widehat{D}(d) - \log\det\widehat{D}(d^0), \end{aligned}$$

where

$$\widehat{D}(d) = \frac{1}{m}\sum_1^m P_j(\theta)\Lambda_j(\theta)\,\mathrm{Re}[I_{yj}]\Lambda_j(\theta)P_j(\theta),$$

$P_j(\theta)$ =diag$((j/p)^{\theta_1}, (j/p)^{\theta_2})$, and $p = \exp(m^{-1}\sum_1^m \log j) \sim m/e$ as $m \to \infty$. From the results for $d \in \Theta_1$, we have

$$\det \widehat{D}(d^0) \to_p \det G^0 \quad \text{as } n \to \infty.$$

Because $\log x$ is a strictly increasing function for $x > 0$, $\Pr(\inf_{\Theta\backslash\Theta_1} S(d) \leq 0)$ tends to 0 if, for arbitrary small $\eta > 0$,

$$\Pr(\inf_{\Theta\backslash\Theta_1} \det\widehat{D}(d) - \det G^0 \leq \eta) \to 0 \quad \text{as } n\to\infty. \tag{21}$$

From (17), we have

$$\det\widehat{D}(d) \geq \det\left[\frac{1}{m}\sum_1^m P_j(\theta)\Lambda_j(\theta)I_{yj}\Lambda_j(\theta)P_j(\theta)\right].$$

Furthermore, we have

$$\frac{1}{m}\sum_1^m P_j(\theta)\Lambda_j(\theta)I_{yj}\Lambda_j(\theta)P_j(\theta) = \frac{1}{m}\sum_1^m P_j(\theta)\Lambda_j(\theta)w_{yj}w_{yj}^*\Lambda_j^*(\theta)P_j^*(\theta),$$

which is a sum of m positive semidefinite matrices. For a fixed $\kappa \in (0, 1/4)$, define

$$\widehat{D}_\kappa(d) = \frac{1}{m}\sum_{j=[\kappa m]}^m P_j(\theta)\Lambda_j(\theta)I_{yj}\Lambda_j(\theta)P_j(\theta).$$

Then, it follows from Lütkepohl (1996, p.55) that

$$\det\left[\frac{1}{m}\sum_1^m P_j(\theta)\Lambda_j(\theta)I_{yj}\Lambda_j(\theta)P_j(\theta)\right] \geq \det\widehat{D}_\kappa(d),$$

giving $\det\widehat{D}(d) \geq \det\widehat{D}_\kappa(d)$.

We proceed to analyze the limit of $\widehat{D}_\kappa(d)$ for

$$\begin{aligned} &\Theta_2: \ \theta_1 \in [-1/2, 1/2], \quad \theta_2 \in [-1/2, -1/2+\Delta], \\ &\Theta_3: \ \theta_1 \in [-1/2, 1/2], \quad \theta_2 \in [1/2, 3/2], \\ &\Theta_4: \ \theta_1 \in [-1/2, 1/2], \quad \theta_2 \in [-3/2, -1/2], \\ &\Theta_5: \ \theta_1 \in [1/2, 3/2], \quad \theta_2 \in [1/2, 3/2], \\ &\Theta_6: \ \theta_1 \in [1/2, 3/2], \quad \theta_2 \in [-3/2, -1/2], \\ &\Theta_7: \ \theta_1 \in [-3/2, 1/2], \quad \theta_2 \in [-3/2, -1/2], \end{aligned}$$

and show that, for arbitrary small $\eta > 0$,

$$\Pr(\inf_\theta \det \widehat{D}_\kappa(d) - \det G^0 \leq \eta) \to 0 \quad \text{as } n \to \infty, \tag{22}$$

for $\Theta_2, \ldots, \Theta_7$. Then (21) follows because $\widehat{D}_\kappa(d)$ is symmetric in (θ_1, θ_2). Hereafter let $\sum'$ denote $\sum_{j=[\kappa m]}^m$. Observe that

$$\begin{aligned} \det \widehat{D}_\kappa(d) &= m^{-1} \textstyle\sum'(j/p)^{2\theta_1} \lambda_j^{-2\theta_1} I_{y_1 j} m^{-1} \sum'(j/p)^{2\theta_2} \lambda_j^{-2\theta_2} I_{y_2 j} \\ &\quad - \left| m^{-1} \textstyle\sum'(j/p)^{\theta_1} \lambda_j^{-\theta_1} w_{y_1 j}^{\theta_2} (j/p) \lambda_j^{-\theta_2} w_{y_2 j}^* \right|^2 . \end{aligned}$$

Define

$$M^\kappa(\theta) = \frac{1}{m} \begin{pmatrix} \sum'(j/p)^{2\theta_1} & \sum'(j/p)^{\theta_1+\theta_2} \\ \sum'(j/p)^{\theta_1+\theta_2} & \sum'(j/p)^{2\theta_2} \end{pmatrix}.$$

For $\theta \in \Theta_2$, from summation by parts and Lemmas 6.2 and 6.6, we obtain

$$\begin{aligned} \det \widehat{D}_\kappa(d) &= m^{-1} \textstyle\sum'(j/p)^{2\theta_1} G_{11}^0 m^{-1} \sum'(j/p)^{2\theta_2} G_{22}^0 \\ &\quad - \left(m^{-1} \textstyle\sum'(j/p)^{\theta_1} m^{-1} \sum'(j/p)^{\theta_2} \right)^2 (G_{12}^0)^2 + o_p(1) \\ &= \det \left[G^0 \odot M^\kappa(\theta) \right] + o_p(1), \end{aligned}$$

where $o_p(1)$ term is uniform in θ. Because $M^\kappa(\theta)$ is positive semidefinite, it follows from Oppenheim's inequality and Lemmas 6.3 (a) and 6.4 that, for sufficiently small (but fixed) κ and large n,

$$\begin{aligned} \inf_{\Theta_2} \det \left[G^0 \odot M^\kappa(\theta) \right] &\geq \det G^0 \inf_{\Theta_2} [m^{-1} \textstyle\sum'(j/p)^{2\theta_1} m^{-1} \sum'(j/p)^{2\theta_2}] \\ &\geq \det G^0 (1 + 3\eta / \det G^0) = \det G^0 + 3\eta, \end{aligned} \tag{23}$$

because $\theta_2 \leq -1/2 + \Delta$. Therefore, we have (22) for $\theta \in \Theta_2$.

Before proceeding to prove (22) for Θ_3, $\Theta_4, \ldots$, it is useful to collect some results from Shimotsu and Phillips (2003a) (hereafter simply SP). First, for $a = 1, 2$, we have

$$\lambda_j^{-\theta_a} w_{y_a j} = \begin{cases} W_j(\theta_a) + \lambda_j^{-\theta_a} (2\pi n)^{-1/2} e^{i\lambda_j} Z_{an}, & \theta_a \in [\frac{1}{2}, \frac{3}{2}], \\ W_j(\theta_a), & \theta_a \in [-\frac{1}{2}, \frac{1}{2}], \\ W_j(\theta_a) + \lambda_j^{-\theta_a} (2\pi n)^{-1/2} e^{i\lambda_j} (1 - e^{i\lambda_j})^{-1} Z_{an}, & \theta_a \in [-\frac{3}{2}, -\frac{1}{2}], \end{cases} \tag{24}$$

where $Z_{an} = \sum_{t=1}^n Y_{an}$ for $\theta_a \in [\frac{1}{2}, \frac{3}{2}]$, $Z_{an} = \Delta Y_{an}$ for $\theta_a \in [-\frac{3}{2}, -\frac{1}{2}]$, and

$$W_j(\theta_a) = D_{nj}(\theta_a) w_{u_a j} - \overline{U}_{a,nj}(\theta_a),$$

with $D_{nj}(\theta_a)$ and $\overline{U}_{a,nj}(\theta_a)$ satisfying

$$D_{nj}(\theta_a) = e^{-\frac{\pi}{2}\theta_a i} + O(\lambda_j) + O(j^{-1/2}), \quad \text{uniformly in } \theta_a, \tag{25}$$

$$E \sup_{\theta_a} |\overline{U}_{a,nj}(\theta_a)|^2 = O(j^{-1} (\log n)^2), \quad j = 1, \ldots, m. \tag{26}$$

The precise form of $D_{nj}(\theta_a)$ and $\overline{U}_{a,nj}(\theta_a)$ depends on the value of θ_a. For $\theta_a \in [\frac{1}{2}, \frac{3}{2}]$, (24) follows from SP equations (51) and (60) on pp. 27-28. The result for $\theta_a \in [-\frac{1}{2}, \frac{1}{2}]$ follows from Lemma 5.2 and 5.3 of SP. For $\theta_a \in [-\frac{3}{2}, -\frac{1}{2}]$, see SP equation (66) on page 29. In addition, using (25) and (26) with Lemma 6.2 and the arguments in the proof of Lemma 6.6 gives, for $a, b = 1, 2$, and uniformly in θ,

$$m^{-1} \textstyle\sum'(j/p)^{\theta_a+\theta_b} W_j(\theta_a) W_j^*(\theta_b) = e^{-\frac{\pi}{2}(\theta_a-\theta_b)i} m^{-1} \sum'(j/p)^{\theta_a+\theta_b} G_{ab}^0 + o_p(1), \tag{27}$$

In view of SP equation (62) and (26), we obtain, for $a, b = 1, 2$ and $\theta_a \in [\frac{1}{2}, \frac{3}{2}]$,

$$m^{-1}\sum{}'(j/p)^{\theta_a+\theta_b}W_j^*(\theta_b)\lambda_j^{-\theta_a}\,(2\pi n)^{-1/2}\,e^{i\lambda_j}Z_{an} = m^{-\theta_a}n^{\theta_a-1/2}Z_{an}O_p(k_n), \qquad (28)$$

where $k_n = m^{-1/2}\log n + mn^{-1} \to 0$, and $\theta_a \in [-\frac{3}{2}, -\frac{1}{2}]$,

$$\begin{aligned} & m^{-1}\sum{}'(j/p)^{\theta_a+\theta_b}W_j^*(\theta_b)\lambda_j^{-\theta_a}\,(2\pi n)^{-1/2}\,e^{i\lambda_j}(1-e^{i\lambda_j})^{-1}Z_{an} \\ = \; & m^{-\theta_a-1}n^{\theta_a+1/2}Z_{an}O_p(k_n). \end{aligned} \qquad (29)$$

We move to the proof of (22) for Θ_3. Hereafter all the $O_p(\cdot)$ and $o_p(\cdot)$ terms are uniform in θ. From (24)-(28), we obtain

$$m^{-1}\sum{}'(j/p)^{2\theta_1}\lambda_j^{-2\theta_1}I_{y_1j} = m^{-1}\sum{}'(j/p)^{2\theta_1}G_{11}^0 + o_p(1),$$

and

$$\begin{aligned} m^{-1}\sum{}'(j/p)^{2\theta_2}\lambda_j^{-2\theta_2}I_{y_2j} \;=\; & m^{-1}\sum{}'(j/p)^{2\theta_2}G_{22}^0 + (2\pi n)^{-1}m^{-1}\sum{}'(j/p)^{2\theta_2}\lambda_j^{-2\theta_2}Z_{2n}^2 \\ & + m^{-\theta_2}n^{\theta_2-1/2}Z_{2n}\cdot O_p(k_n) + o_p(1). \end{aligned}$$

From Lemma 6.3, we can express the second term on the right as

$$\xi(\theta_2)\,m^{-2\theta_2}n^{2\theta_2-1}Z_{2n}^2, \quad \inf_{\theta_2}\xi(\theta_2) > c > 0.$$

We also obtain

$$\begin{aligned} & m^{-1}\sum{}'(j/p)^{\theta_1}\lambda_j^{-\theta_1}w_{y_1j}(j/p)^{\theta_2}\lambda_j^{-\theta_2}w_{y_2j}^* \\ = \; & m^{-1}\sum{}'(j/p)^{\theta_1+\theta_2}e^{-\frac{\pi}{2}i(\theta_1-\theta_2)}G_{12}^0 + m^{-\theta_2}n^{\theta_2-1/2}Z_{2n}\cdot O_p(k_n) + o_p(1). \end{aligned}$$

Therefore, uniformly in θ,

$$\det\widehat{D}_\kappa(d) = D_1 + D_2 + D_3 + D_4 + o_p(1),$$

where

$$\begin{aligned} D_1 &= \det\left[G^0\odot\mathcal{E}(\theta)\odot M^\kappa(\theta)\odot\mathcal{E}^*(\theta)\right], \quad D_2 = \xi(\theta_2)\,m^{-2\theta_2}n^{2\theta_2-1}Z_{2n}^2, \\ D_3 &= \left[m^{-1}\sum{}'(j/p)^{2\theta_1}G_{11}^0 + o_p(1)\right]m^{-2\theta_2}n^{2\theta_2-1}Z_{2n}^2, \quad D_4 = m^{-\theta_2}n^{\theta_2-1/2}Z_{2n}\cdot O_p(k_n). \end{aligned}$$

For D_1 and D_3, Lemma 6.3 gives

$$\begin{aligned} \inf_{\Theta_2}D_1 &\geq \det G^0\inf_{\Theta_2}[m^{-1}\sum{}'(j/p)^{2\theta_1}m^{-1}\sum{}'(j/p)^{2\theta_2}] \geq \det G^0 + 3\eta. \\ \Pr(\inf_\theta D_3 &\geq 0) \to 1, \quad \text{as } n\to\infty. \end{aligned}$$

For D_2 and D_4, in view of the argument in SP pp. 28-29, we obtain

$$\Pr\left(\inf_\theta(D_2+D_4) \leq -\eta\right) \to 0, \quad \text{as } n\to\infty.$$

Therefore, (22) follows for $\theta\in\Theta_3$.

The proof of (22) for $\theta\in\Theta_4$ follows from essentially the same argument. Using

$$\lambda_j^{-\theta_2}w_{y_2j} = W_j(\theta_2) + \lambda_j^{-\theta_2}\,(2\pi n)^{-1/2}\,e^{i\lambda_j}(1-e^{i\lambda_j})^{-1}Z_{2n},$$

and (24)-(29) and proceeding in the same manner gives (22) for $\theta\in\Theta_4$.

For $\theta \in \Theta_5 = \{\theta_1, \theta_2 \in [1/2, 3/2]\}$, use (24) and rewrite $(j/p)^{\theta_a} \lambda_j^{-\theta_a} w_{y_a j}$, $a = 1, 2$, as

$$\begin{aligned} (j/p)^{\theta_a} \lambda_j^{-\theta_a} w_{y_a j} &= (j/p)^{\theta_a} W_j(\theta_a) + (j/p)^{\theta_a} \lambda_j^{-\theta_a} (2\pi n)^{-1/2} e^{i\lambda_j} Z_{an} \\ &= e^{i\lambda_j} \left[(j/p)^{\theta_a} W_j(\theta_a) e^{-i\lambda_j} + p^{-\theta_a} (2\pi)^{-\theta_a - 1/2} n^{\theta_a - 1/2} Z_{an} \right] \\ &= e^{i\lambda_j} (A_{aj} + B_a), \end{aligned}$$

and note that B_a does not depend on j, and B_a is a real number. It follows that

$$\begin{aligned} & \det \widehat{D}_\kappa(d) \\ = \; & m^{-1} \textstyle\sum' |A_{1j} + B_1|^2 \, m^{-1} \sum' |A_{2j} + B_2|^2 - \left| m^{-1} \sum' (A_{1j} + B_1) \left(A_{2j}^* + B_2 \right) \right|^2 \\ = \; & \left(m^{-1} \textstyle\sum' |A_{1j}|^2 + 2B_1 m^{-1} \sum' \mathrm{Re}[A_{1j}] + B_1^2 m^{-1} \sum' 1 \right) \\ & \times \left(m^{-1} \textstyle\sum' |A_{2j}|^2 + 2B_2 m^{-1} \sum' \mathrm{Re}[A_{2j}] + B_2^2 m^{-1} \sum' 1 \right) \\ & - \left| m^{-1} \textstyle\sum' A_{1j} A_{2j}^* + B_2 m^{-1} \sum' A_{1j} + B_1 m^{-1} \sum' A_{2j}^* + B_1 B_2 m^{-1} \sum' 1 \right|^2 . \end{aligned}$$

Using the fact

$$\begin{aligned} & \left(2B_1 m^{-1} \textstyle\sum' \mathrm{Re}[A_{1j}] + B_1^2 m^{-1} \sum' 1 \right) \left(2B_2 m^{-1} \sum' \mathrm{Re}[A_{2j}] + B_2^2 m^{-1} \sum' 1 \right) \\ & - \left| B_2 m^{-1} \textstyle\sum' A_{1j} + B_1 m^{-1} \sum' A_{2j}^* + B_1 B_2 m^{-1} \sum' 1 \right|^2 \\ = \; & 4 B_1 B_2 m^{-1} \textstyle\sum' \mathrm{Re}[A_{1j}] m^{-1} \sum' \mathrm{Re}[A_{2j}] - \left| B_2 m^{-1} \sum' A_{1j} + B_1 m^{-1} \sum' A_{2j}^* \right|^2 , \end{aligned}$$

and $m^{-1} \sum' A_{aj} = O_p(k_n)$ (follows from (28)) and a little algebra, we obtain

$$\begin{aligned} & \det \widehat{D}_\kappa(d) \\ = \; & m^{-1} \textstyle\sum' |A_{1j}|^2 m^{-1} \sum' |A_{2j}|^2 - \left| m^{-1} \sum' A_{1j} A_{2j}^* \right|^2 && (30) \\ & + m^{-1} \textstyle\sum' 1 \left[B_2^2 m^{-1} \sum' |A_{1j}|^2 + B_1^2 m^{-1} \sum' |A_{2j}|^2 - 2B_1 B_2 m^{-1} \sum' \mathrm{Re}[A_{1j} A_{2j}^*] \right] && (31) \\ & + B_1 B_2 O_p(k_n^2) + B_1^2 O_p(k_n^2) + B_2^2 O_p(k_n^2) + B_2 O_p(k_n) + B_1 O_p(k_n). \end{aligned}$$

From (27) and (23), (30) is bounded from below by

$$\inf_\theta \det(G^0 \odot M^\kappa(\theta)) - |o_p(1)| \geq \det G^0 + 3\eta - |o_p(1)|.$$

Because G^0 is positive definite, there exists $\zeta \in (0, 1)$ such that $(G_{12}^0)^2 = G_{11}^0 G_{12}^0 (1 - \zeta)^2$. For (31), observe that

$$\begin{aligned} & (1 - \zeta) \left(B_2^2 m^{-1} \textstyle\sum' |A_{1j}|^2 + B_1^2 m^{-1} \sum' |A_{2j}|^2 \right) - 2B_1 B_2 m^{-1} \sum' \mathrm{Re}[A_{1j} A_{2j}^*] \\ \geq \; & 2(1 - \zeta) \left(B_1^2 B_2^2 m^{-1} \textstyle\sum' |A_{1j}|^2 m^{-1} \sum' |A_{2j}|^2 \right)^{1/2} - 2B_1 B_2 m^{-1} \sum' \mathrm{Re}[A_{1j} A_{2j}^*] \\ \geq \; & 2|B_1 B_2| \left[(1 - \zeta)(G_{12}^0 G_{22}^0)^{1/2} - |G_{12}^0| \right] m^{-1} \textstyle\sum' (j/p)^{\theta_1 + \theta_2} - |B_1 B_2 o_p(1)| \\ = \; & -|B_1 B_2 o_p(1)|, \end{aligned}$$

where the third line follows from $m^{-1} \sum' A_{aj} A_{bj}^* = e^{-\frac{\pi}{2}(\theta_a - \theta_b) i} G_{ab}^0 m^{-1} \sum' (j/p)^{2\theta_a} + o_p(1)$ (from (27)) and Cauchy-Schwartz inequality. Therefore, (31) is bounded from below by, for sufficiently large n and a constant $c > 0$,

$$\begin{aligned} & \zeta m^{-1} \textstyle\sum' 1 \left[B_2^2 m^{-1} \sum' |A_{1j}|^2 + B_1^2 m^{-1} \sum' |A_{2j}|^2 \right] - |B_1 B_2 o_p(1)| \\ \geq \; & (1/2) \zeta 2c \left(B_2^2 G_{11}^0 + B_1^2 G_{22}^0 \right) - B_2^2 |o_p(1)| - B_1^2 |o_p(1)| - |B_1 B_2 o_p(1)|, \end{aligned}$$

because $m^{-1}\sum' 1 \sim 1-\kappa$ and $\inf_{\theta_a} m^{-1}\sum'(j/p)^{2\theta_a} \geq 2c$ from Lemma 6.3. Therefore, after collecting the terms and using $|xy| \leq x^2+y^2$, we obtain

$$\begin{aligned}\inf_\theta \det \widehat{D}_\kappa(d) \;\geq\; & \det G^0 + 2\eta \\ & +\zeta c/2\left(B_2^2 G_{11}^0 + B_1^2 G_{22}^0\right) - B_2|O_p(k_n)| - B_1|O_p(k_n)| \\ & +\eta - |o_p(1)| + B_1^2(\zeta c G_{11}^0/2 - |o_p(1)|) + B_2^2(\zeta c G_{22}^0/2 - |o_p(1)|).\end{aligned}$$

In view of the argument in SP pp. 28-29, the second line is larger than $-\eta$ with probability approaching to one (hereafter, wpa1). The third line is nonnegative wpa1, and (22) follows for $\theta \in \Theta_5$.

For $\theta \in \Theta_7$, a similar definition gives

$$\begin{aligned}(j/p)^{\theta_1}\lambda_j^{-\theta_1} w_{y_1 j} &= e^{i\lambda_j}(A_{1j}+B_1), \\ (j/p)^{\theta_2}\lambda_j^{-\theta_2} w_{y_2 j} &= e^{i\lambda_j}(A_{2j}+(1-e^{i\lambda_j})^{-1}B_2).\end{aligned}$$

It follows that

$$\begin{aligned}& \det \widehat{D}_\kappa(d) \\ =\; & m^{-1}\sum' |A_{1j}+B_1|^2\, m^{-1}\sum' \left|A_{2j}+(1-e^{i\lambda_j})^{-1}B_2\right|^2 \\ & -\left|m^{-1}\sum'(A_{1j}+B_1)\left(A_{2j}^*+(1-e^{-i\lambda_j})^{-1}B_2\right)\right|^2 \\ =\; & \left(m^{-1}\sum' |A_{1j}|^2 + 2B_1 m^{-1}\sum' \mathrm{Re}\,[A_{1j}] + (B_1)^2 m^{-1}\sum' 1\right) \\ & \times\left(m^{-1}\sum' |A_{2j}|^2 + 2B_2 m^{-1}\sum' \mathrm{Re}\left[(1-e^{-i\lambda_j})^{-1}A_{2j}\right] + (B_2)^2 m^{-1}\sum' \left|1-e^{i\lambda_j}\right|^{-2}\right) \\ & -\Big|m^{-1}\sum' A_{1j}A_{2j}^* + B_2 m^{-1}\sum'(1-e^{-i\lambda_j})^{-1}A_{1j} + B_1 m^{-1}\sum' A_{2j}^* \\ & +B_1B_2 m^{-1}\sum'(1-e^{-i\lambda_j})^{-1} + B_1 m^{-1}\sum' A_{2j}^* + B_1B_2 m^{-1}\sum'(1-e^{-i\lambda_j})^{-1}\Big|^2.\end{aligned}$$

A tedious algebra gives

$$\begin{aligned}& \det \widehat{D}_\kappa(d) \\ =\; & m^{-1}\sum' |A_{1j}|^2 m^{-1}\sum' |A_{2j}|^2 - \left|m^{-1}\sum' A_{1j}A_{2j}^*\right|^2 \\ & +m^{-1}\sum' |A_{1j}|^2 (B_2)^2 m^{-1}\sum' \left|1-e^{i\lambda_j}\right|^{-2} + m^{-1}\sum' |A_{2j}|^2 (B_1)^2 m^{-1}\sum' 1 \\ & -2B_1B_2\,\mathrm{Re}\left[m^{-1}\sum' A_{1j}A_{2j}^* m^{-1}\sum'(1-e^{i\lambda_j})^{-1}\right] \\ & +(B_1)^2(B_2)^2\left[m^{-1}\sum' 1 m^{-1}\sum' \left|1-e^{i\lambda_j}\right|^{-2} - \left|m^{-1}\sum'(1-e^{i\lambda_j})^{-1}\right|^2\right] \\ & +B_1 O_p(k_n) + B_2 n m^{-1} O_p(k_n) + (B_1)^2 B_2 n m^{-1} O_p(k_n) + B_1 (B_2)^2 n^2 m^{-2} O_p(k_n) \\ & +(B_2)^2 n^2 m^{-2} O_p(k_n^2) + (B_1)^2 O_p(k_n^2).\end{aligned}$$

The first two terms are bounded from below by $\det G^0 + 2\eta + o_p(1)$. The third and fourth terms are

$$\begin{aligned}& G_{11}^0 m^{-1}\sum'(j/p)^{2\theta_1}(B_2)^2 m^{-1}\sum' |1-e^{i\lambda_j}|^{-2} + G_{22}^0 m^{-1}\sum'(j/p)^{2\theta_2}(B_1)^2 m^{-1}\sum' 1 \\ & +(B_2)^2 o_p(n^2 m^{-2}) + (B_1)^2 o_p(1).\end{aligned}$$

A repeated use of Cauchy-Schwartz inequality gives

$$\begin{aligned}
& (1-\varepsilon)\left[\begin{array}{c} G^0_{11}m^{-1}\sum'(j/p)^{2\theta_1}(B_2)^2m^{-1}\sum'|1-e^{i\lambda_j}|^{-2} \\ +G^0_{22}m^{-1}\sum'(j/p)^{2\theta_2}(B_1)^2m^{-1}\sum' 1 \end{array}\right] \\
\geq\; & 2(1-\varepsilon)\left[\begin{array}{c} G^0_{11}m^{-1}\sum'(j/p)^{2\theta_1}(B_2)^2m^{-1}\sum'|1-e^{i\lambda_j}|^{-2} \\ \times G^0_{22}m^{-1}\sum'(j/p)^{2\theta_2}(B_1)^2m^{-1}\sum' 1 \end{array}\right]^{1/2} \\
\geq\; & 2|G^0_{12}|(m^{-1}\textstyle\sum'(j/p)^{\theta_1+\theta_2})(m^{-1}\sum'|1-e^{i\lambda_j}|^{-1}).
\end{aligned}$$

The fifth term is bounded by

$$\begin{aligned}
& |B_1||B_2|\left|m^{-1}\textstyle\sum' A_{1j}A^*_{2j}\right|m^{-1}\sum'|1-e^{i\lambda_j}|^{-1} \\
=\; & |G^0_{12}|(m^{-1}\textstyle\sum'(j/p)^{\theta_1+\theta_2})(m^{-1}\sum'|1-e^{i\lambda_j}|^{-1})+|B_1||B_2|o_p(nm^{-1}),
\end{aligned}$$

The sixth term is greater than, for $\eta>0$,

$$\eta(B_1)^2(B_2)^2n^2m^{-2},$$

by Cauchy-Schwartz inequality. Therefore, there exists $c>0$ such that

$$\begin{aligned}
\det\widehat{D}_\kappa(d) \;\geq\; & \det G^0+2\eta \\
& +cG^0_{11}(B_2)^2n^2m^{-2}+cG^0_{22}(B_1)^2+c(B_1)^2(B_2)^2n^2m^{-2} \\
& +o_p(1)+(B_2)^2o_p(n^2m^{-2})+(B_1)^2o_p(1)+|B_1||B_2|o_p(nm^{-1}) \\
& +B_1O_p(k_n)+B_2nm^{-1}O_p(k_n)+(B_1)^2B_2nm^{-1}O_p(k_n) \\
& +B_1(B_2)^2n^2m^{-2}O_p(k_n)+(B_2)^2n^2m^{-2}O_p(k_n^2)+(B_1)^2O_p(k_n^2).
\end{aligned}$$

On the right hand side, all the terms with $O_p(\cdot)$ and $o_p(\cdot)$ are dominated by the first four terms, and therefore we have(22) for $\theta\in\Theta_7$. The case with $\theta\in\Theta_6$ is analyzed in a similar way as $\theta\in\Theta_5$, and the proof is omitted. ■

7.2 Proof of Theorem 3.4

Theorem 3.2 holds under the current conditions and implies that with probability approaching to one, as $n\to\infty$, $\widehat{d}$ satisfies

$$0=\left.\frac{\mathrm{d}R(d)}{\mathrm{d}d}\right|_{\widehat{d}}=\left.\frac{\mathrm{d}R(d)}{\mathrm{d}d}\right|_{d^0}+\left(\left.\frac{\mathrm{d}^2R(d)}{\mathrm{d}d\mathrm{d}d'}\right|_{d^*}\right)(\widehat{d}-d^0),$$

where $||d^*-d^0||\leq||\widehat{d}-d^0||$. The stated result follows if the score vector at d^0 converges to $N(0,\Omega)$ in distribution and the Hessian at d^* converges uniformly to Ω in probability.

7.2.1 Score vector approximation

We show that for any 2×1 vector η

$$\eta'\sqrt{m}\left.\frac{\mathrm{d}R(d)}{\mathrm{d}d}\right|_{d^0}\to_d N(0,\eta'\Omega\eta),\quad \Omega=2(I_2+G^0\odot(G^0)^{-1})+\frac{\pi^2(G^0_{12})^2}{2\det G^0}\left[\begin{array}{cc}1&-1\\-1&1\end{array}\right].$$

Observe that

$$\sqrt{m}\frac{\partial R(d)}{\partial d_a}=-\frac{2}{\sqrt{m}}\sum_1^m\log\lambda_j+\mathrm{tr}\left(\widehat{G}\,(d)^{-1}\sqrt{m}\frac{\partial\widehat{G}\,(d)}{\partial d_a}\right).$$

Now

$$\sqrt{m}\left.\frac{\partial\widehat{G}\left(d\right)}{\partial d_1}\right|_{d^0}=\frac{1}{\sqrt{m}}\sum_1^m \operatorname{Re}\left.\left[\begin{array}{cc} 2\frac{\partial}{\partial d_1}w_{\Delta^{d_1}x_1j}w^*_{\Delta^{d_1}x_1j} & \frac{\partial}{\partial d_1}w_{\Delta^{d_1}x_1j}w^*_{\Delta^{d_2}x_2j} \\ \frac{\partial}{\partial d_1}w^*_{\Delta^{d_1}x_1j}w_{\Delta^{d_2}x_2j} & 0 \end{array}\right]\right|_{d^0}.$$

Using Lemmas 5.13 and 5.15 of SP, its $(1,1)$ element is

$$\frac{2}{\sqrt{m}}\sum_1^m \operatorname{Re}\left[w_{\log(1-L)u_1j}w^*_{u_1j}\right]=\frac{2}{\sqrt{m}}\sum_1^m \log\lambda_j I_{u_1j}+o_p\left(1\right).$$

Similarly, its $(2,2)$ element is $2m^{-1/2}\sum_1^m \log\lambda_j I_{u_2j}+o_p\left(1\right)$, and its off-diagonal elements are

$$\begin{aligned}
&\frac{1}{\sqrt{m}}\sum_1^m \operatorname{Re}\left[w_{\log(1-L)u_1j}w^*_{u_2j}\right]\\
=\;& -\frac{1}{\sqrt{m}}\sum_1^m \operatorname{Re}\left[J_n(e^{i\lambda_j})w_{u_1j}w^*_{u_2j}\right]+o_p\left(1\right)\\
=\;& \frac{1}{\sqrt{m}}\sum_1^m \log\lambda_j \operatorname{Re}\left[w_{u_1j}w^*_{u_2j}\right]-\frac{1}{2\sqrt{m}}\sum_1^m\left(\pi-\lambda_j\right)\operatorname{Im}\left[w_{u_1j}w^*_{u_2j}\right]+o_p\left(1\right).
\end{aligned}$$

The limit of $\sqrt{m}\,\partial\widehat{G}(d)/\partial d_2\big|_{d^0}$ is obtained in a similar manner.

Let i_a be a 2×2 matrix whose ath diagonal is one and all other elements are zero and define

$$\widehat{G}_1=m^{-1}\textstyle\sum_1^m \log\lambda_j \operatorname{Re}[I_{uj}].$$

Then it follows that, for $a=1,2$,

$$\begin{aligned}
\sqrt{m}\left.\frac{\partial\widehat{G}}{\partial d_a}\right|_{d^0} &= \sqrt{m}\left(i_a\widehat{G}_1+\widehat{G}_1 i_a\right)+(-1)^a H+o_p\left(1\right),\\
H &= \left[\begin{array}{cc} 0 & h\\ h & 0\end{array}\right],\quad h=\frac{1}{2\sqrt{m}}\sum_1^m\left(\pi-\lambda_j\right)\operatorname{Im}\left[w_{u_1j}w^*_{u_2j}\right].
\end{aligned}$$

Hereafter we use $\widehat{G}$ in place of $\widehat{G}(d)$ when its meaning is obvious. Let g^a and $A_a(0)'$ denote the ath row of $(G^0)^{-1}$ and ath column of $A(0)'$, respectively. Define

$$\begin{aligned}
S_{a1} &= -\frac{2}{\sqrt{m}}\sum_1^m \log\lambda_j+\sqrt{m}\operatorname{tr}\left(\widehat{G}^{-1}\left(i_a\widehat{G}_1+\widehat{G}_1 i_a\right)\right),\\
S_{a2} &= (-1)^a\operatorname{tr}(\widehat{G}^{-1}H),
\end{aligned}$$

so that

$$\sqrt{m}\,\partial R(d)/\partial d_a|_{d^0}=S_{a1}+S_{a2}+o_p(1).$$

Then we have

$$\begin{aligned}
S_{a1} &= 2\sqrt{m}\operatorname{tr}\left(\widehat{G}^{-1}\left[\widehat{G}_1-\frac{1}{m}\sum_1^m\log\lambda_j\widehat{G}\right]i_a\right)\\
&= 2\operatorname{tr}\left(\widehat{G}^{-1}\frac{1}{\sqrt{m}}\sum_1^m\nu_j\operatorname{Re}[I_{uj}]i_a\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2\text{tr}\left(\widehat{G}^{-1}\frac{1}{\sqrt{m}}\sum_1^m \nu_j A(0)\,\text{Re}[I_{\varepsilon j}]A(0)'\,i_a\right) + o_p(1) \\
&= \frac{2}{\sqrt{m}}\sum_1^m \nu_j\left\{g^a A(0)\,\text{Re}[I_{\varepsilon j}]A_a(0)' - 1\right\} + o_p(1),
\end{aligned}
$$

where $\nu_j = \log\lambda_j - m^{-1}\sum_1^m \log\lambda_j = \log j - m^{-1}\sum_1^m \log j$, and the third line follows from Lobato (1999, equation (C.2), p. 145) and $A(\lambda_j) = A(0) + O(j^\beta n^{-\beta})$. From the arguments in Lobato (1999, pp.141-43), it follows that

$$
\sum_{a=1}^{2}\eta_a S_{a1} \to_d N\left(0, \eta' E \eta\right), \quad E = 2(I_2 + G^0 \odot (G^0)^{-1}).
$$

It remains to derive the limit distribution of S_{a2}. Similarly as above, we obtain

$$
\begin{aligned}
S_{a2} &= -(-1)^a \frac{1}{\det\widehat{G}} 2[\widehat{G}(d)]_{12} h \\
&= -(-1)^a \frac{[\widehat{G}(d)]_{12}}{\det\widehat{G}} \frac{1}{\sqrt{m}}\sum_1^m (\pi - \lambda_j)\,\text{Im}\left[w_{u_1 j} w_{u_2 j}^*\right] \\
&= -(-1)^a \frac{[\widehat{G}(d)]_{12}}{\det\widehat{G}} \frac{1}{\sqrt{m}}\sum_1^m (\pi - \lambda_j)\left[A(0)\,\text{Im}[I_{\varepsilon j}]A(0)'\right]_{12} + o_p(1) \\
&= -(-1)^a \frac{G_{12}^0 \det A(0)}{\det G^0} \frac{\pi}{\sqrt{m}}\sum_1^m \text{Im}\left[w_{\varepsilon_1 j} w_{\varepsilon_2 j}^*\right] + o_p(1) \\
&= -(-1)^a \frac{\pi G_{12}^0}{\sqrt{\det G^0}} \frac{2\pi}{\sqrt{m}}\sum_1^m \text{Im}\left[w_{\varepsilon_1 j} w_{\varepsilon_2 j}^*\right] + o_p(1),
\end{aligned}
$$

since $\det G^0 = (2\pi)^{-2}(\det A(0))^2$. From Lemma 6.8, we have

$$
\frac{2\pi}{\sqrt{m}}\sum_1^m \text{Im}\left[w_{\varepsilon_a j} w_{\varepsilon_b j}^*\right] \to_d N(0, \frac{1}{2}), \quad a \neq b.
$$

Since $\text{Im}[w_{\varepsilon_1 j}^* w_{\varepsilon_2 j}]$ and $\text{Re}[I_{\varepsilon j}]$ are uncorrelated for all $1 \leq j, k \leq m$, it follows that

$$
\frac{\pi G_{12}^0}{\sqrt{\det G^0}} \frac{2\pi}{\sqrt{m}}\sum_1^m \text{Im}\left[w_{\varepsilon_1 j} w_{\varepsilon_2 j}^*\right] \to_d N\left(0, \frac{\pi^2 (G_{12}^0)^2}{2\det G^0}\right),
$$

and (S_{11}, S_{21}) and (S_{12}, S_{22}) are asymptotically independent. Therefore,

$$
\sqrt{m}\left.\frac{\mathrm{d}R(d)}{\mathrm{d}d}\right|_{d^0} \to_d N(0, \Omega). \tag{32}
$$

7.2.2 Hessian approximation

Fix $\varepsilon > 0$ and let $M = \{d : (\log n)^4 ||d - d^0|| < \varepsilon\}$. $\Pr(d^* \notin M)$ tends to zero from the proof of consistency, thus we assume $d^* \in M$ in the following. Observe that

$$\frac{\partial^2 R(d)}{\partial d_a \partial d_b} = \operatorname{tr}\left[-\widehat{G}^{-1}\frac{\partial \widehat{G}}{\partial d_a}\widehat{G}^{-1}\frac{\partial \widehat{G}}{\partial d_b} + \widehat{G}^{-1}\frac{\partial^2 \widehat{G}}{\partial d_a \partial d_b}\right]. \tag{33}$$

First we evaluate the first term on the right of (33). Let $J(\theta) = \operatorname{diag}(j^{\theta_a}, j^{\theta_b})$. Then

$$\begin{aligned} \widehat{G}^{-1}\frac{\partial \widehat{G}}{\partial d_a} &= \left[\frac{1}{m}\sum_1^m J(\theta)\,\Lambda_j(\theta)\operatorname{Re}[I_{\Delta^d xj}]\Lambda_j(\theta)J(\theta)\right]^{-1} \\ &\quad\times\left[\frac{1}{m}\sum_1^m J(\theta)\,\Lambda_j(\theta)\frac{\partial}{\partial d_a}\operatorname{Re}[I_{\Delta^d xj}]\Lambda_j(\theta)J(\theta)\right] \\ &= C(\theta)^{-1}\times B(\theta). \end{aligned}$$

First we derive the limit of

$$C(\theta) = \frac{1}{m}\sum_1^m \operatorname{Re}\left[\begin{array}{cc} j^{2\theta_1}\lambda_j^{-2\theta_1}I_{y1j} & j^{\theta_1+\theta_2}\lambda_j^{-\theta_1-\theta_2}w_{y_1j}w_{y_2j}^* \\ j^{\theta_1+\theta_2}\lambda_j^{-\theta_1-\theta_2}w_{y_1j}^*w_{y_2j} & j^{2\theta_2}\lambda_j^{-2\theta_2}I_{y_2j}\end{array}\right].$$

In view of the fact that

$$|j^{\theta_a}-1|/|\theta_a| \le (\log j)n^{|\theta_a|} \le (\log j)n^{1/\log n} = e\log j \quad \text{on } M, \tag{34}$$

and proof of consistency, we obtain

$$C(\theta) - G^0 = \frac{1}{m}\sum_1^m \operatorname{Re}[I_{uj}] - G^0 + o_p((\log n)^{-2}) = o_p((\log n)^{-2}),$$

uniformly in $\theta \in M$. Hereafter $o_p(\cdot)$ terms are all uniform in $\theta \in M$. For $B(\theta)$, from (34) and Lemmas 5.13 and 5.15 of SP, we have for $a = 1$

$$\begin{aligned} B(\theta) &= \frac{1}{m}\sum_1^m \operatorname{Re}\left[\begin{array}{cc} 2j^{2\theta_1}\lambda_j^{-2\theta_1}w_{\log(1-L)y_1j}w_{y_1j}^* & j^{\theta_1+\theta_2}\lambda_j^{-\theta_1-\theta_2}w_{\log(1-L)y_1j}w_{y_2j}^* \\ j^{\theta_1+\theta_2}\lambda_j^{-\theta_1-\theta_2}w_{\log(1-L)y_1j}^*w_{y_2j} & 0\end{array}\right] \\ &= \frac{1}{m}\sum_1^m(\log\lambda_j)\operatorname{Re}\left[\begin{array}{cc} 2I_{u_1j} & w_{u_1j}w_{u_2j}^* \\ w_{u_1j}^*w_{u_2j} & 0\end{array}\right] + o_p((\log n)^{-1}). \end{aligned}$$

It follows that

$$B(\theta) = (i_1G_1 + G_1i_1) + o_p((\log n)^{-1}), \quad G_1 = G^0\frac{1}{m}\sum_1^m \log\lambda_j.$$

When $a = 2$, we obtain the same result with i_2 replacing i_1. Thus

$$\operatorname{tr}\left[-\widehat{G}^{-1}\frac{\partial \widehat{G}}{\partial d_a}\widehat{G}^{-1}\frac{\partial \widehat{G}}{\partial d_b}\right] = \operatorname{tr}\left[-(G^0)^{-1}(i_aG_1+G_1i_a)(G^0)^{-1}(i_bG_1+G_1i_b)\right] + o_p(1).$$

For the second term on the right of (33), first, when $a = b = 1$, we have

$$\begin{aligned} &\frac{1}{m}\sum_1^m J(\theta)\,\Lambda_j(\theta)\frac{\partial^2}{\partial d_1^2}\operatorname{Re}[I_{\Delta^d xj}]\Lambda_j(\theta)J(\theta) \\ =\ &\frac{1}{m}\sum_1^m \operatorname{Re}\left[\begin{array}{cc} j^{2\theta_1}\lambda_j^{-2\theta_1}\frac{\partial^2}{\partial d_1^2}I_{\Delta^{d_1}x_1j} & j^{\theta_1+\theta_2}\lambda_j^{-\theta_1-\theta_2}\frac{\partial^2}{\partial d_1^2}w_{\Delta^{d_1}x_1j}w_{\Delta^{d_2}x_2j}^* \\ j^{\theta_1+\theta_2}\lambda_j^{-\theta_1-\theta_2}\frac{\partial^2}{\partial d_1^2}w_{\Delta^{d_1}x_1j}^*w_{\Delta^{d_2}x_2j} & 0\end{array}\right]. \end{aligned}$$

Using Lemmas 5.13 and 5.15 of SP and the arguments above, its $(1,1)$ element is

$$4\frac{1}{m}\sum_{1}^{m}\{\mathrm{Re}[J_n(e^{i\lambda_j})]\}^2 I_{uj}+o_p(1)=4G_{11}^0\frac{1}{m}\sum_{1}^{m}(\log\lambda_j)^2+o_p(1).$$

Its off-diagonal elements are

$$\begin{aligned}\frac{1}{m}\sum_{1}^{m}\mathrm{Re}\left[J_n(e^{i\lambda_j})^2 w_{u_1j}w_{u_2j}^*\right]+o_p(1) &= \frac{1}{m}\sum_{1}^{m}\left[(\log\lambda_j)^2-\frac{\pi^2}{4}\right]\mathrm{Re}\left[w_{u_1j}w_{u_2j}^*\right]+o_p(1)\\ &= G_{12}^0\frac{1}{m}\sum_{1}^{m}(\log\lambda_j)^2-\frac{\pi^2}{4}G_{12}^0+o_p(1).\end{aligned}$$

When $a\neq b$, the diagonal elements of $m^{-1}\sum_1^m J(\theta)\,\Lambda_j(\theta)\frac{\partial^2}{\partial d_a\partial d_b}\,\mathrm{Re}[I_{\Delta^d xj}]\Lambda_j(\theta)J(\theta)$ are zero, and its off-diagonal elements are

$$\begin{aligned}\frac{1}{m}\sum_{1}^{m}\mathrm{Re}\left[|J_n(e^{i\lambda_j})|^2 w_{u_1j}w_{u_2j}^*\right]+o_p(1) &= \frac{1}{m}\sum_{1}^{m}\left[(\log\lambda_j)^2+\frac{\pi^2}{4}\right]\mathrm{Re}\left[w_{u_1j}w_{u_2j}^*\right]+o_p(1)\\ &= G_{12}^0\frac{1}{m}\sum_{1}^{m}(\log\lambda_j)^2+\frac{\pi^2}{4}G_{12}^0+o_p(1).\end{aligned}$$

Proceeding similarly for $a=b=2$, we obtain

$$\begin{aligned}\mathrm{tr}\left[\widehat{G}^{-1}\frac{\partial^2\widehat{G}}{\partial d_a\partial d_b}\right] &= \mathrm{tr}\left[(G^0)^{-1}\left(i_ai_bG_2+i_aG_2i_b+i_bG_2i_a+G_2i_ai_b\right)\right]\\ &\quad+\mathrm{tr}\left[(G^0)^{-1}(-1)^{I\{a=b\}}\frac{\pi^2}{4}\begin{bmatrix}0 & G_{12}^0\\ G_{12}^0 & 0\end{bmatrix}\right]+o_p(1),\\ G_2 &= G^0\frac{1}{m}\sum_{1}^{m}(\log\lambda_j)^2.\end{aligned}$$

Now

$$\begin{aligned}&\mathrm{tr}\left[-(G^0)^{-1}(i_aG_1+G_1i_a)(G^0)^{-1}(i_bG_1+G_1i_b)\right]\\ &+\mathrm{tr}\left[(G^0)^{-1}\left(i_ai_bG_2+i_aG_2i_b+i_bG_2i_a+G_2i_ai_b\right)\right]\\ =\;&\left(m^{-1}\textstyle\sum_1^m(\log\lambda_j)^2-(m^{-1}\sum_1^m\log\lambda_j)^2\right)\\ &\times\mathrm{tr}\left[(G^0)^{-1}\left(i_ai_bG^0+i_aG^0i_b+i_bG^0i_a+G^0i_ai_b\right)\right]\\ \to\;&(2(I_2+G^0\odot(G^0)^{-1})_{ab}.\end{aligned}$$

Finally, since

$$\mathrm{tr}\left[\frac{\pi^2}{4}G_0^{-1}\begin{bmatrix}0 & G_{12}^0\\ G_{12}^0 & 0\end{bmatrix}\right]=-\frac{\pi^2(G_{12}^0)^2}{2\det G^0},$$

it follows that

$$\mathrm{tr}\left[-\widehat{G}^{-1}\frac{\partial\widehat{G}}{\partial d_a}\widehat{G}^{-1}\frac{\partial\widehat{G}}{\partial d_b}+\widehat{G}^{-1}\frac{\partial^2\widehat{G}}{\partial d_a\partial d_b}\right]=\Omega_{ab}+o_p(1),$$

giving the limit of the Hessian. ■

7.3 Proof of Theorem 4.2

Define $\delta = d_1^0 - d_2^0 \geq 0$. Then we have

$$\begin{aligned}\Delta^{d_2}(X_{2t} - \beta X_{1t}) &= \Delta^{d_2}(X_{2t} - \beta^0 X_{1t}) + (\beta^0 - \beta)\Delta^{d_2} X_{1t} \\ &= \Delta^{\theta_2} u_{2t} + (\beta^0 - \beta)\Delta^{\theta_2 - \delta} u_{1t},\end{aligned}$$

because $\Delta^{d_2} X_{1t} = \Delta^{d_2 - d_1^0} u_{1t} = \Delta^{d_2 - d_2^0 + d_2^0 - d_1^0} u_{1t} = \Delta^{\theta_2 - \delta} u_{1t}$. Observe that

$$\begin{aligned}\lambda_j^{-\theta_2} w_{\Delta^{d_2}(x_2 - \beta x_1)j} &= \lambda_j^{-\theta_2} w_{\Delta^{\theta_2} u_2 j} + (\beta^0 - \beta)\lambda_j^{-\theta_2} w_{\Delta^{\theta_2 - \delta} u_1 j} \\ &= \lambda_j^{-\theta_2} w_{\Delta^{\theta_2} u_2 j} + (\beta^0 - \beta)\left(\frac{2\pi m}{n}\right)^{-\delta}\left(\frac{j}{m}\right)^{-\delta} \lambda_j^{\delta - \theta_2} w_{\Delta^{\theta_2 - \delta} u_1 j} \\ &= \lambda_j^{-\theta_2} w_{\Delta^{\theta_2} u_2 j} + \widetilde{\beta}\left(\frac{j}{m}\right)^{-\delta} \lambda_j^{\delta - \theta_2} w_{\Delta^{\theta_2 - \delta} u_1 j}, \qquad (35)\end{aligned}$$

where $\widetilde{\beta} = (\beta^0 - \beta)(2\pi m/n)^{-\delta}$.

Define $S(d, \beta) = R(d, \beta) - R(d^0, \beta^0)$ and $B_\rho = \{\beta : |\widetilde{\beta}| \leq \rho\}$. By a similar argument as the proof of Theorem 3.2, we obtain

$$\begin{aligned}&\Pr\left(\{\widehat{d} \in \overline{N}_\rho \cap \Theta\} \cup \{\widehat{\beta} \in \overline{B}_\rho \cap B\}\right) \\ \leq\ &\Pr\left(\inf_{\{d \in \overline{N}_\rho \cap \Theta\} \cup \{\beta \in \overline{B}_\rho \cap B\}} S(d, \beta) \leq 0\right) \\ \leq\ &\Pr\left(\inf_{\{d \in \overline{N}_\rho \cap \Theta\} \cap \{\beta \in B\}} S(d, \beta) \leq 0\right) + \Pr\left(\inf_{\{d \in N_\rho \cap \Theta\} \cap \{\beta \in \overline{B}_\rho \cap B\}} S(d, \beta) \leq 0\right).\end{aligned}$$

First we take care of the easiest case:

$$\begin{aligned} d &\in \Theta_1, \quad \theta_2 - \delta \in [-1/2 + \Delta, 1/2], \\ d &\in \Theta_1, \quad \theta_2 - \delta \in [-3/2, -1/2 - \Delta].\end{aligned}$$

In view of the proof of Theorem 3.2 for $d \in \Theta_1$ and the fact $N_\rho \subset \Theta_1$, it suffices to show, with probability approaching to one,

$$\begin{aligned}&\det\left(\frac{1}{m}\sum_1^m M_j(\theta)\Lambda_j(\theta) I_{\Delta^d x}(\lambda_j; \beta)\Lambda_j(\theta) M_j(\theta)\right) \qquad (36) \\ \geq\ &\begin{cases}(2\theta_1 + 1)^{-1}(2\theta_2 + 1)^{-1}\det G^0 + o_p(1), & \text{uniformly in } \{d \in \overline{N}_\rho \cap \Theta_1\} \cap \{\beta \in B\}, \\ (\theta_1 + \theta_2 + 1)^{-2}\det G^0 + \eta|\widetilde{\beta}|^2 \quad \text{for } \eta > 0, & \text{uniformly in } \{d \in N_\rho \cap \Theta_1\} \cap \{\beta \in \overline{B}_\rho \cap B\}.\end{cases}\end{aligned}$$

From (35) we have

$$\begin{aligned}&(j/m)^{2\theta_2}\lambda_j^{-2\theta_2} I_{\Delta^{d_2}(x_2 - \beta x_1)j} \\ =\ &(j/m)^{2\theta_2}\left|\lambda_j^{-\theta_2} w_{\Delta^{\theta_2} u_2 j}\right|^2 + 2\widetilde{\beta}(j/m)^{2\theta_2 - \delta}\,\mathrm{Re}\left[\lambda_j^{-\theta_2} w_{\Delta^{\theta_2} u_2 j}\lambda_j^{\delta - \theta_2} w^*_{\Delta^{\theta_2 - \delta} u_1 j}\right] \\ &+ |\widetilde{\beta}|^2 (j/m)^{2\theta_2 - 2\delta}\left|\lambda_j^{\delta - \theta_2} w_{\Delta^{\theta_2 - \delta} u_1 j}\right|^2, \\ &(j/m)^{\theta_1 + \theta_2}\lambda_j^{-\theta_1 - \theta_2} w_{\Delta^{d_1} x_1 j} w^*_{\Delta^{d_2}(x_2 - \beta x_1)j} \\ =\ &(j/m)^{\theta_1 + \theta_2}\lambda_j^{-\theta_1} w_{\Delta^{\theta_1} u_1 j}\lambda_j^{-\theta_2} w^*_{\Delta^{\theta_2} u_2 j} + \widetilde{\beta}(j/m)^{\theta_1 + \theta_2 - \delta}\lambda_j^{-\theta_1} w_{\Delta^{\theta_1} u_1 j}\lambda_j^{\delta - \theta_2} w^*_{\Delta^{\theta_2 - \delta} u_1 j}.\end{aligned}$$

For $\delta \geq 0$, applying Lemma 6.6 and proceeding as above, it follows that

$$\begin{aligned}(36) \quad = \quad & G^0_{11}G^0_{22}\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2} + o_p(1) \\ & +2\widetilde{\beta}\left[G^0_{11}G^0_{12}\cos(\pi\delta/2)\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2-\delta} + o_p(1)\right] \\ & +|\widetilde{\beta}|^2\left[(G^0_{11})^2\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2-2\delta} + o_p(1)\right] \\ & -(G^0_{12})^2(\textstyle\int_0^1 x^{\theta_1+\theta_2})^2 - 2\widetilde{\beta}\left[G^0_{11}G^0_{12}\cos(\pi\delta/2)\int_0^1 x^{\theta_1+\theta_2}\int_0^1 x^{\theta_1+\theta_2-\delta} + o_p(1)\right] \\ & -|\widetilde{\beta}|^2\left[(G^0_{11})^2(\textstyle\int_0^1 x^{\theta_1+\theta_2-\delta})^2 + o_p(1)\right]. \qquad (37)\end{aligned}$$

For $\{d \in \overline{N}_\rho \cap \Theta^a_1\} \cup \{\beta \in B\}$, (37) is

$$\begin{aligned}& \left[G^0_{11}G^0_{22} - (G^0_{12})^2\right]\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2} + o_p(1) \qquad (38) \\ & +(G^0_{12})^2\left[\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2} - (\textstyle\int_0^1 x^{\theta_1+\theta_2})^2\right] \qquad (39) \\ & +2\widetilde{\beta}\left[G^0_{11}G^0_{12}\cos(\pi\delta/2)\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2-\delta} - \int_0^1 x^{\theta_1+\theta_2}\int_0^1 x^{\theta_1+\theta_2-\delta} + o_p(1)\right] (40) \\ & +|\widetilde{\beta}|^2\left[(G^0_{11})^2\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2-2\delta} - (\textstyle\int_0^1 x^{\theta_1+\theta_2-\delta})^2 + o_p(1)\right]. \qquad (41)\end{aligned}$$

(38) is $(2\theta_1+1)^{-1}(2\theta_2+1)^{-1}\det G^0 + o_p(1)$.

The case where $\delta = 0$ is special, because (39)-(41) reduce to

$$\left[(G^0_{12})^2 + 2G^0_{11}G^0_{12}\widetilde{\beta}(1+o_p(1)) + |\widetilde{\beta}|^2(G^0_{11})^2(1+o_p(1))\right]\left[\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2} - (\textstyle\int_0^1 x^{\theta_1+\theta_2})^2\right]. \qquad (42)$$

Since $[\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2} - (\int_0^1 x^{\theta_1+\theta_2})^2] \in [0, C]$, $(G^0_{12})^2 + 2G^0_{11}G^0_{12}\widetilde{\beta} + |\widetilde{\beta}|^2(G^0_{11})^2 \geq 0$ and $|\widetilde{\beta}| < \infty$, we have $\Pr((42) \leq -\varepsilon) \to 0$ for any $\varepsilon > 0$.

For $\delta > 0$, first take care of the case when θ_1 and θ_2 are close to each other. Since $\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2-\delta} - \int_0^1 x^{\theta_1+\theta_2}\int_0^1 x^{\theta_1+\theta_2-\delta}$ is a continuous function of θ_1, θ_2, for any $\varepsilon > 0$, by taking Δ small we have uniformly in $|\theta_1 - \theta_2| \leq \Delta$

$$|(40)| \leq |\widetilde{\beta}|\left(\varepsilon + o_p(1)\right).$$

Since $\delta > 0$, (41) is bounded from below by

$$|\widetilde{\beta}|^2(C + o_p(1)),$$

where $C > 0$ does not depend on ε. Therefore, (40) + (41) is bounded from below by

$$|\widetilde{\beta}|\left[(C + o_p(1))|\widetilde{\beta}| - \varepsilon + o_p(1)\right]. \qquad (43)$$

When $|\widetilde{\beta}| > 2\varepsilon/C$, (43) is positive wpa1 and when $|\widetilde{\beta}| \leq 2\varepsilon/C$, (43) is no smaller than $-6\varepsilon^2/C$ wpa1. Make ε sufficiently small, then we have $(40) + (41) \geq -\varepsilon$ wpa1.

Now, for $|\theta_1 - \theta_2| > \Delta$, we have

$$\begin{aligned} & (39) + (41) \\ \geq \quad & 2|\widetilde{\beta}|G^0_{11}|G^0_{12}|\left[\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2} - (\textstyle\int_0^1 x^{\theta_1+\theta_2})^2\right]^{1/2} \\ & \times\left[\int_0^1 x^{2\theta_1}\int_0^1 x^{2\theta_2-2\delta} - (\textstyle\int_0^1 x^{\theta_1+\theta_2-\delta})^2\right]^{1/2}(1 + o_p(1)).\end{aligned}$$

Therefore, $\Pr((39) + (40) + (41) \leq -\varepsilon) \to 0$ if

$$\begin{aligned} & \left[\int_0^1 x^{2\theta_1} \int_0^1 x^{2\theta_2} - (\int_0^1 x^{\theta_1+\theta_2})^2\right] \left[\int_0^1 x^{2\theta_1} \int_0^1 x^{2\theta_2-2\delta} - (\int_0^1 x^{\theta_1+\theta_2-\delta})^2\right] \\ > & \left[\int_0^1 x^{2\theta_1} \int_0^1 x^{2\theta_2-\delta} - \int_0^1 x^{\theta_1+\theta_2} \int_0^1 x^{\theta_1+\theta_2-\delta}\right]^2. \end{aligned}$$

Expanding the above and factoring out $\int_0^1 x^{2\theta_1}$, then the above holds from Lemma S, and $\Pr((39)+(40)+(41) \leq -\varepsilon) \to 0$ follows. Since ε is arbitrary, the required result follows.

For $\{d \in \Theta_1^a\} \cup \{\beta \in \overline{B}_\rho \cap B\}$, (37) is

$$\left[G_{11}^0 G_{22}^0 - (G_{12}^0)^2\right] (\int_0^1 x^{\theta_1+\theta_2})^2 + o_p(1) \tag{44}$$
$$+G_{11}^0 G_{22}^0 \left[\int_0^1 x^{2\theta_1} \int_0^1 x^{2\theta_2} - (\int_0^1 x^{\theta_1+\theta_2})^2\right] \tag{45}$$
$$+2\widetilde{\beta} \left[G_{11}^0 G_{12}^0 \cos(\pi\delta/2) \int_0^1 x^{2\theta_1} \int_0^1 x^{2\theta_2-\delta} - \int_0^1 x^{\theta_1+\theta_2} \int_0^1 x^{\theta_1+\theta_2-\delta} + o_p(1)\right] \tag{46}$$
$$+|\widetilde{\beta}|^2 \left[(G_{11}^0)^2 \int_0^1 x^{2\theta_1} \int_0^1 x^{2\theta_2-2\delta} - (\int_0^1 x^{\theta_1+\theta_2-\delta})^2 + o_p(1)\right]. \tag{47}$$

Defining

$$\begin{aligned} S_1(d) &= \log\det\left(\frac{1}{m}\sum_1^m M_j(\theta)\Lambda_j(\theta) I_{\Delta^d x}(\lambda_j;\beta)\Lambda_j(\theta)M_j(\theta)\right) - \log\frac{\det G^0}{(\theta_1+\theta_2+1)^2}, \\ S_2(d) &= -2\log(\theta_1+\theta_2+1) - 2(\theta_1+\theta_2)\left(\frac{1}{m}\sum_1^m \log j - \log m\right), \end{aligned}$$

and $S_3(d)$ as in the proof of Theorem 3.2, we have $S(d) = S_1(d) + S_2(d) + S_3(d)$ and $S_2(d) \geq o_p(1)$. For $\delta = 0$, again (45)-(47) reduce to

$$\left[G_{11}^0 G_{22}^0 + 2\widetilde{\beta} G_{11}^0 G_{12}^0 (1+o_p(1)) + |\widetilde{\beta}|^2 (G_{11}^0)^2 (1+o_p(1))\right] \left[\int_0^1 x^{2\theta_1} \int_0^1 x^{2\theta_2} - (\int_0^1 x^{\theta_1+\theta_2})^2\right].$$

But $\int_0^1 x^{2\theta_1} \int_0^1 x^{2\theta_2} - (\int_0^1 x^{\theta_1+\theta_2})^2 = 0$ when $\theta_1 = \theta_2$, hence $\widehat{\beta}$ does not necessarily converge to β^0.

Move back to the case $\delta > 0$. It suffices to show that $(45) + (46) + (47) \geq \kappa|\widetilde{\beta}|$ wpa1. Again, for any $\varepsilon > 0$, there exists $\Delta > 0$ such that for $|\theta_1 - \theta_2| \leq \Delta$ we have

$$|(46)| \leq |\widetilde{\beta}| (\varepsilon + o_p(1)).$$

Since $|\theta_1 - \theta_2 + \delta| > \Delta > 0$, (47) is bounded from below by $C_1|\widetilde{\beta}|^2 + |\widetilde{\beta}|^2(C_2 + o_p(1))$, where $C_1, C_2 > 0$ does not depend on ε. Hence we obtain $(46) + (47) \geq -\varepsilon$ wpa1 for arbitrary small $\varepsilon > 0$.

For $|\theta_1 - \theta_2| > \Delta$, since

$$\begin{aligned} & (45) + (47) \\ \geq\ & 2|\widetilde{\beta}| G_{11}^0 (G_{11}^0 G_{22}^0)^{1/2} \left[\int_0^1 x^{2\theta_1} \int_0^1 x^{2\theta_2} - (\int_0^1 x^{\theta_1+\theta_2})^2\right]^{1/2} \\ & \times \left[\int_0^1 x^{2\theta_1} \int_0^1 x^{2\theta_2-2\delta} - (\int_0^1 x^{\theta_1+\theta_2-\delta})^2\right]^{1/2} (1+o_p(1)). \end{aligned}$$

and $G_{11}^0 G_{22}^0 > (G_{12}^0)^2$, the required result follows from Lemma S.

For $\theta_2 - \delta \in [-3/2, -1/2-\Delta]$, since

$$\lambda_j^{\delta-\theta_2} w^*_{\Delta^{\theta_2-\delta} u_1 j} = D_{nj}(\theta_2-\delta) w_{u_1 j} - C_{nj}(\theta_2) j^{\delta-\theta_2-1} n^{\theta_2-\delta+1/2} Y_{3n} + e_j,$$

where e_j is a generic reminder term that satisfies $m^{-1}\sum_1^m E(e_j)^2 = O(k_n^2)$, and

$$Y_{3n} = (1-L)^{\theta_2-\delta} u_{1n} I\{t \ge 1\}, \quad C_{nj}(\theta) = C(\theta) + O(\lambda_j),$$

where $C(\theta)$ is a generic function that does not depend on n and $0 < |C(\theta)| < \infty$. Then as in the proof for $\theta_2 \in [-3/2, -1/2-\Delta]$ we obtain

$$\begin{aligned}
&\frac{1}{m}\sum_{\kappa m}^{m}\left(\frac{j}{p}\right)^{2\theta_2} \lambda_j^{-2\theta_2} I_{\Delta^{d_2}(x_2-\beta x_1)j} \\
=\ & e^{2\theta_1} G^0_{22}\textstyle\int_\kappa^1 x^{2\theta_2} + o_p(1) + e^{2\theta_1} 2\widetilde{\beta}\left[G^0_{12}\cos(\pi\delta/2)\int_\kappa^1 x^{2\theta_2-\delta} + o_p(1)\right] \\
& + e^{2\theta_1}|\widetilde{\beta}|^2\left[G^0_{11}\textstyle\int_\kappa^1 x^{2\theta_2-2\delta} + o_p(1)\right] \\
& + \widetilde{\beta} m^{\delta-\theta_2-1} n^{\theta_2-\delta+1/2} Y_{3n} O_p(k_n) + |\widetilde{\beta}|^2 m^{\delta-\theta_2-1} n^{\theta_2-\delta+1/2} Y_{3n} O_p(k_n) \\
& + |\widetilde{\beta}|^2 m^{2\delta-2\theta_2-2} n^{2\theta_2-2\delta+1} |C(\theta)|^2 Y_{3n}^2, \\
&\left|\frac{1}{m}\sum_{\kappa m}^{m}\left(\frac{j}{p}\right)^{\theta_1+\theta_2} \lambda_j^{-\theta_1-\theta_2} w_{\Delta^{d_1} x_1 j} w^*_{\Delta^{d_2}(x_2-\beta x_1)j}\right|^2 \\
=\ & e^{\theta_1+\theta_2}(G^0_{12})^2(\textstyle\int_\kappa^1 x^{\theta_1+\theta_2})^2 + 2\widetilde{\beta}\left[G^0_{11}G^0_{12}\cos(\pi\delta/2)\int_\kappa^1 x^{\theta_1+\theta_2}\int_\kappa^1 x^{\theta_1+\theta_2-\delta} + o_p(1)\right] \\
& - |\widetilde{\beta}|^2\left[(G^0_{11})^2(\textstyle\int_\kappa^1 x^{\theta_1+\theta_2-\delta})^2 + o_p(1)\right] \\
& + \widetilde{\beta} m^{\delta-\theta_2-1} n^{\theta_2-\delta+1/2} Y_{3n} O_p(k_n) + |\widetilde{\beta}|^2 m^{\delta-\theta_2-1} n^{\theta_2-\delta+1/2} Y_{3n} O_p(k_n) \\
& + |\widetilde{\beta}|^2 m^{2\delta-2\theta_2-2} n^{2\theta_2-2\delta+1} Y_{3n}^2 o_p(1).
\end{aligned}$$

Therefore, all the additional terms are dominated by $|\widetilde{\beta}|^2 m^{2\delta-2\theta_2-2} n^{2\theta_2-2\delta+1} C(\theta)^2 Y_{3n}^2$ and their sum is larger than $-\varepsilon$ for any $\varepsilon > 0$ with probability approaching to 1. Thus the required result for $\{d \in \Theta_1\} \cap \{\beta \in \overline{B}_\rho \cap B\}$ also follows.

Finally, we consider the case

$$\begin{aligned}
I \ &: \ \theta_1 \in [-3/2, -1/2-\Delta], \quad \theta_2 \in [-1/2+\Delta, 1/2], \quad \theta_2 - \delta \in [-1/2+\Delta, 1/2], \\
II \ &: \ \theta_1 \in [-3/2, -1/2-\Delta], \quad \theta_2 \in [-1/2+\Delta, 1/2], \quad \theta_2 - \delta \in [-3/2, -1/2-\Delta].
\end{aligned}$$

From the results in the proof of Theorem 3.2 we obtain

$$\begin{aligned}
&\frac{1}{m}\sum_{\kappa m}^{m}\left(\frac{j}{p}\right)^{2\theta_1} \lambda_j^{-2\theta_1} I_{\Delta^{d_1} x_1 j} \\
=\ & e^{2\theta_1} G^0_{11}\textstyle\int_\kappa^1 x^{2\theta_1} + m^{-2\theta_1-2} n^{2\theta_1+1} |C(\theta)|^2 Y_{1n}^2 + m^{-\theta_1-1} n^{\theta_1+1/2} Y_{1n} O_p(k_n) + o_p(1),
\end{aligned}$$

For case I, we have

$$\begin{aligned}
&\frac{1}{m}\sum_{\kappa m}^{m}\left(\frac{j}{p}\right)^{2\theta_2} \lambda_j^{-2\theta_2} I_{\Delta^{d_2}(x_2-\beta x_1)j} \\
=\ & e^{2\theta_1} G^0_{22}\textstyle\int_\kappa^1 x^{2\theta_2} + o_p(1) + e^{2\theta_1} 2\widetilde{\beta}\left[G^0_{12}\cos(\pi\delta/2)\int_\kappa^1 x^{2\theta_2-\delta} + o_p(1)\right] \\
& + e^{2\theta_1}|\widetilde{\beta}|^2\left[G^0_{11}\textstyle\int_\kappa^1 x^{2\theta_2-2\delta} + o_p(1)\right],
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{1}{m}\sum_{\kappa m}^{m}\left(\frac{j}{p}\right)^{\theta_1+\theta_2}\lambda_j^{-\theta_1-\theta_2}w_{\Delta^{d_1}x_1j}w^*_{\Delta^{d_2}(x_2-\beta x_1)j}\right|^2\\
=\;& e^{\theta_1+\theta_2}(G^0_{12})^2(\textstyle\int_\kappa^1 x^{\theta_1+\theta_2})^2+2\widetilde{\beta}\left[G^0_{11}G^0_{12}\cos(\pi\delta/2)\int_\kappa^1 x^{\theta_1+\theta_2}\int_\kappa^1 x^{\theta_1+\theta_2-\delta}+o_p(1)\right]\\
&-|\widetilde{\beta}|^2\left[(G^0_{11})^2(\textstyle\int_\kappa^1 x^{\theta_1+\theta_2-\delta})^2+o_p(1)\right]\\
&+m^{-2\theta_1-2}n^{2\theta_1+1}Y^2_{1n}O_p(k_n)+m^{-\theta_1-1}n^{\theta_1+1/2}Y_{1n}O_p(k_n).
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{1}{m}\sum_{\kappa m}^{m}\left(\frac{j}{p}\right)^{2\theta_2}\lambda_j^{-2\theta_2}I_{\Delta^{d_2}(x_2-\beta x_1)j}\\
\geq\;& [1-\cos(\pi\delta/2)]e^{2\theta_1}\left(G^0_{22}\textstyle\int_\kappa^1 x^{2\theta_2}+|\widetilde{\beta}|^2\left[G^0_{11}\int_\kappa^1 x^{2\theta_2-2\delta}+o_p(1)\right]\right)\\
&+e^{2\theta_1}\cos(\pi\delta/2)\left[G^0_{22}\textstyle\int_\kappa^1 x^{2\theta_2}+2\widetilde{\beta}G^0_{12}\left(\int_\kappa^1 x^{2\theta_2-\delta}+o_p(1)\right)\right.\\
&\left.+|\widetilde{\beta}|^2\left(G^0_{11}\textstyle\int_\kappa^1 x^{2\theta_2-2\delta}+o_p(1)\right)\right]\\
\geq\;& \eta,
\end{aligned}$$

we have

$$\det D_\kappa(d)\geq \det G^0+\eta,\quad \text{wpa1.}$$

For case II, let

$$\begin{aligned}
A_{1j} &= \left(\frac{j}{p}\right)^{\theta_1}\left[D_{nj}(\theta_1)w_{u_1j}+e_j\right],\quad A_{2j}=p^{-\theta_1}(2\pi)^{-\theta_1-1/2}n^{\theta_1-1/2}Z_{1n},\\
B_{1j} &= \left(\frac{j}{p}\right)^{\theta_2}\left[D_{nj}(\theta_2)w_{u_2j}+\widetilde{\beta}D_{nj}(\theta_2-\delta)w_{u_1j}+e_j\right]\\
B_{2j} &= p^{-\theta_2}j^{\delta-1}n^{\theta_2-\delta+1/2}Y_{3n},
\end{aligned}$$

so that

$$(j/p)^{\theta_1}\lambda_j^{-\theta_1}w_{\Delta^{d_1}x_1j}=A_{1j}+B_{1j},\quad (j/p)^{\theta_1}\lambda_j^{-\theta_1}w_{\Delta^{d_2}(x_2-\beta x_1)j}=A_{2j}+B_{2j},$$

then the required result follows from the proof of Theorem 3.2. ■

Lemma S *For $\delta>0$ and $\Delta>0$, define*

$$\begin{aligned}
S(\theta;\delta) &= \textstyle\int x^{2\theta_1}\int x^{2\theta_2}\int x^{2\theta_2-2\delta}-\int x^{2\theta_2}(\int x^{\theta_1+\theta_2-\delta})^2-\int x^{2\theta_2-2\delta}(\int x^{\theta_1+\theta_2})^2\\
&\quad -\textstyle\int x^{2\theta_1}(\int x^{2\theta_2-\delta})^2+2\int x^{2\theta_2-\delta}\int x^{\theta_1+\theta_2}\int x^{\theta_1+\theta_2-\delta},
\end{aligned}$$

where the domain of integration is $[0,1]$ when $\theta_1,\theta_2,\theta_2-\delta\geq -1/2+\Delta$ and $[\kappa,1]$ otherwise. Then, by choosing $\kappa>0$ small, we have $\inf_{\theta\in\Theta\backslash T}S(\theta;\delta)>0$, where

$$\begin{aligned}
T &= \{|\theta_1-\theta_2+\delta|\leq\Delta\}\cup\{|\theta_1+1/2|\leq\Delta\}\cup\{|\theta_2+1/2|\leq\Delta\}\\
&\quad\cup\{|\theta_2-\delta+1/2|\leq\Delta\}.
\end{aligned}$$

and $T=\{|\theta_1-\theta_2|\leq\Delta\}\cup T$ when $\theta_1,\theta_2,\theta_2-\delta\geq -1/2+\Delta$ *or* $\theta_1,\theta_2,\theta_2-\delta\leq -1/2+\Delta$.

Proof There are six possible cases

$$
\begin{array}{rlll}
\text{I}: & \theta_1 \geq -1/2+\Delta, & \theta_2 \geq -1/2+\Delta, & \theta_2-\delta \geq -1/2+\Delta, \\
\text{II}: & \theta_1 \geq -1/2+\Delta, & \theta_2 \leq -1/2-\Delta, & \theta_2-\delta \leq -1/2-\Delta, \\
\text{III}: & \theta_1 \geq -1/2+\Delta, & \theta_2 \geq -1/2-\Delta, & \theta_2-\delta \leq -1/2-\Delta, \\
\text{IV}: & \theta_1 \leq -1/2-\Delta, & \theta_2 \geq -1/2-\Delta, & \theta_2-\delta \geq -1/2-\Delta, \\
\text{V}: & \theta_1 \leq -1/2-\Delta, & \theta_2 \leq -1/2-\Delta, & \theta_2-\delta \leq -1/2-\Delta, \\
\text{VI}: & \theta_1 \leq -1/2-\Delta, & \theta_2 \geq -1/2-\Delta, & \theta_2-\delta \leq -1/2-\Delta.
\end{array}
$$

Define

$$
g(\theta;\delta) = \frac{\frac{1}{2\theta_1+1}\frac{1}{2\theta_2+1}\frac{1}{2\theta_2-2\delta+1} - \frac{1}{2\theta_2+1}\frac{1}{(\theta_1+\theta_2-\delta+1)^2} - \frac{1}{2\theta_2-2\delta+1}\frac{1}{(\theta_1+\theta_2+1)^2}}{-\frac{1}{2\theta_1+1}\frac{1}{(2\theta_2-\delta+1)^2} + 2\frac{1}{2\theta_2-\delta+1}\frac{1}{\theta_1+\theta_2+1}\frac{1}{\theta_1+\theta_2-\delta+1}}.
$$

For case I, an elementary algebra shows that

$$
S(\theta;\delta) = g(\theta;\delta) = A(\theta)^{-1}(\theta_1-\theta_2)^2(\theta_1-\theta_2+\delta)^2\delta^2 \geq (\sup_\theta A(\theta))^{-1}\delta^2\Delta^4 > 0,
$$

where $A(\theta)$ is the least common denominator of the terms in $g(\theta;\delta)$, and $A(\theta) \in (0,\infty)$ uniformly in θ. Hereafter all the inequalities hold uniformly in θ. For case V, we obtain $S(\theta;\delta) = \kappa^{2\theta_1+4\theta_2-2\delta+3}[-g(\theta;\delta) + C(\theta;\delta)\kappa^{2\Delta}]$, where $|C(\theta;\delta)| < \infty$, and $A(\theta) \in (-\infty, 0)$. Taking κ small makes the terms inside the brackets positive.

For case II, first there exists $\eta > 0$ such that

$$
\textstyle\int x^{2\theta_1}[\int x^{2\theta_2}\int x^{2\theta_2-2\delta} - (\int x^{2\theta_2-\delta})^2] \geq \kappa^{4\theta_2-2\delta+2}(4\eta + C(\theta;\delta)\kappa^{2\Delta}) \geq 3\eta\kappa^{4\theta_2-2\delta+2},
$$

by taking κ small. Also for small κ we obtain

$$
\textstyle\int x^{2\theta_2}(\int x^{\theta_1+\theta_2-\delta})^2 \leq \left\{ \begin{array}{ll} C\kappa^{2\theta_2+1}\log\kappa, & \text{for } \theta_1+\theta_2-\delta \geq -1 \\ C\kappa^{2\theta_1+4\theta_2-2\delta+3}\log\kappa, & \text{for } \theta_1+\theta_2-\delta \leq -1 \end{array} \right\} \leq \eta\kappa^{4\theta_2-2\delta+2},
$$

where C is a finite constant, because $2\theta_1 \geq -1+2\Delta$. Similarly we have $\int x^{2\theta_2-2\delta}(\int x^{\theta_1+\theta_2})^2 \leq \eta\kappa^{4\theta_2-2\delta+2}$. It follows that $S(\theta;\delta) \geq \eta\kappa^{4\theta_2-2\delta+2}$, giving the required result. For case III, in an analogous manner we obtain

$$
\begin{array}{rcl}
\int x^{2\theta_1}\int x^{2\theta_2}\int x^{2\theta_2-2\delta} & \geq & 4\eta\kappa^{2\theta_2-2\delta+1}, \\
\int x^{2\theta_2}(\int x^{\theta_1+\theta_2-\delta})^2,\ \int x^{2\theta_2-2\delta}(\int x^{\theta_1+\theta_2})^2,\ \int x^{2\theta_1}(\int x^{2\theta_2-\delta})^2 & \leq & \eta\kappa^{2\theta_2-2\delta+1},
\end{array}
$$

and $S(\theta;\delta) \geq \eta\kappa^{4\theta_2-2\delta+2}$ follows. The remaining cases are proven in a similar way. ■

7.4 Proof of Theorem 4.5

Let $\tau = (d_1, d_2, \beta)$. By the same reasoning as the proof of Theorem 3.4, the stated result follows if we derive the limit of the score vector at τ^0 and the Hessian at τ^* where $||\tau^* - \tau^0|| \leq ||\hat{\tau} - \tau^0||$. For $\delta \in (0, 1/2)$, let B_n =diag$(1, 1, (2\pi m/n)^\delta)$. We show

$$
B_n\sqrt{m}\frac{\mathrm{d}R(d^0,\beta^0)}{\mathrm{d}\tau} \to_d N(0,\Xi_1), \quad B_n\left(\frac{\mathrm{d}^2R(d^*,\beta^*)}{\mathrm{d}\tau\mathrm{d}\tau'}\right)B_n \to_p \Xi_1,
$$

then $B_n^{-1}(\hat{\tau} - \tau^0) \to_d N(0, (\Xi_1)^{-1})$ follows because

$$
B_n\left(\frac{\mathrm{d}^2R(d^*,\beta^*)}{\mathrm{d}\tau\mathrm{d}\tau'}\right)B_nB_n^{-1}(\hat{\tau}-\tau^0) = B_n\sqrt{m}\frac{\mathrm{d}R(d^0,\beta^0)}{\mathrm{d}\tau}.
$$

7.4.1 Part I: $\delta \in (0, \frac{1}{2})$

Score vector approximation The partial derivative of $R(d, \beta)$ with respect to d at τ^0 is the same as those of $R(d)$ at d^0. For the partial derivative with respect to β, we have

$$[B_n]_3 \sqrt{m} \frac{\partial R(d,\beta)}{\partial \beta} = \operatorname{tr}\left(\widehat{G}(d,\beta)^{-1} [B_n]_3 \sqrt{m} \frac{\partial \widehat{G}(d,\beta)}{\partial \beta} \right),$$

where $[B_n]_3$ denotes the $(3,3)$ element of B_n. Now

$$[B_n]_3 \sqrt{m} \frac{\partial \widehat{G}(d^0,\beta^0)}{\partial \beta} = -\left(\frac{2\pi m}{n}\right)^{\delta} \frac{1}{\sqrt{m}} \sum_1^m \operatorname{Re} \begin{bmatrix} 0 & w_{u_1 j} w^*_{\Delta^{-\delta} u_1 j} \\ w^*_{u_1 j} w_{\Delta^{-\delta} u_1 j} & 2 w_{\Delta^{-\delta} u_1 j} w^*_{u_2 j} \end{bmatrix}.$$

Its off-diagonal elements are

$$\begin{aligned} & -\left(\frac{2\pi m}{n}\right)^{\delta} \frac{1}{\sqrt{m}} \sum_1^m \operatorname{Re}\left[w^*_{u_1 j} \lambda_j^{-\delta} \left(D_{nj}(-\delta) w_{u_1 j} - \lambda_j^{\delta} \frac{\widetilde{U}_{1,\lambda_j n}(-\delta)}{\sqrt{2\pi n}} \right) \right] \\ = & -\frac{1}{\sqrt{m}} \sum_1^m \left(\frac{j}{m}\right)^{-\delta} \operatorname{Re}\left[D_{nj}(-\delta) \right] I_{u_1 j} + o_p(1), \end{aligned}$$

in view of the arguments on pp. 22-24 of Shimotsu and Phillips (2003b). Similarly, its $(2,2)$ element is

$$\begin{aligned} & -\frac{2}{\sqrt{m}} \sum_1^m \left(\frac{j}{m}\right)^{-\delta} \operatorname{Re}\left[D_{nj}(-\delta) \right] \operatorname{Re}\left[w_{u_1 j} w^*_{u_2 j} \right] \\ & -\frac{2}{\sqrt{m}} \sum_1^m \left(\frac{j}{m}\right)^{-\delta} \operatorname{Im}\left[D_{nj}(-\delta) \right] \operatorname{Im}\left[w_{u_1 j} w^*_{u_2 j} \right] + o_p(1). \end{aligned}$$

It follows that

$$[B_n]_3 \sqrt{m} \left. \frac{\partial R(d,\beta)}{\partial \beta} \right|_{\tau^0} = \frac{\det G^0}{\det \widehat{G}} (S_{31} + S_{32}) + o_p(1),$$

where

$$\begin{aligned} S_{31} &= \frac{1}{\det G^0} \frac{2}{\sqrt{m}} \sum_1^m \left(\frac{j}{m}\right)^{-\delta} \operatorname{Re}\left[D_{nj}(-\delta) \right] \left(\widehat{G}_{12} I_{u_1 j} - \widehat{G}_{11} \operatorname{Re}\left[w_{u_1 j} w^*_{u_2 j} \right] \right), \\ S_{32} &= -\frac{\widehat{G}_{11}}{\det G^0} \frac{2}{\sqrt{m}} \sum_1^m \left(\frac{j}{m}\right)^{-\delta} \operatorname{Im}\left[D_{nj}(-\delta) \right] \operatorname{Im}\left[w_{u_1 j} w^*_{u_2 j} \right]. \end{aligned}$$

First we derive the limit distribution of S_{31}. Later we show that S_{31} and S_{32} are asymptotically independent. Rewrite S_{31} as

$$\begin{aligned} & \frac{1}{\det G^0} \frac{2}{\sqrt{m}} \sum_1^m \left(\frac{j}{m}\right)^{-\delta} \operatorname{Re}\left[D_{nj}(-\delta) \right] \left(G^0_{12} I_{u_1 j} - G^0_{11} \operatorname{Re}\left[w_{u_1 j} w^*_{u_2 j} \right] \right) \\ & + \frac{1}{\det G^0} \frac{2}{\sqrt{m}} \sum_1^m \left(\frac{j}{m}\right)^{-\delta} \\ & \times \left\{ \operatorname{Re}\left[D_{nj}(-\delta) \right] (\widehat{G}_{12} - G^0_{12}) I_{u_1 j} - (\widehat{G}_{11} - G^0_{11}) \operatorname{Re}\left[w_{u_1 j} w^*_{u_2 j} \right] \right\} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{\sqrt{m}}\sum_1^m\left(\frac{j}{m}\right)^{-\delta}\mathrm{Re}\left[D_{nj}(-\delta)\right]g^2\,\mathrm{Re}\left[\begin{array}{c} I_{u_1j} \\ w_{u_1j}w^*_{u_2j}\end{array}\right] && (48)\\
&\quad -\left[\frac{1}{1-\delta}\cos\left(\frac{\pi\delta}{2}\right)g^2+o_p(1)\right]\frac{2}{\sqrt{m}}\sum_1^m\mathrm{Re}\left[\begin{array}{c} I_{u_1j}-G^0_{11} \\ w_{u_1j}w^*_{u_2j}-G^0_{12}\end{array}\right]. && (49)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
(48) &= -\frac{2}{\sqrt{m}}\sum_1^m\left(\frac{j}{m}\right)^{-\delta}\mathrm{Re}\left[D_{nj}(-\delta)\right]g^2A(0)\,\mathrm{Re}[I_{\varepsilon j}]A_1(0)'+o_p(1)\\
&= -\frac{2}{\sqrt{m}}\sum_1^m\left(\frac{j}{m}\right)^{-\delta}\mathrm{Re}\left[D_{nj}(-\delta)\right]g^2A(0)\left(\frac{1}{2\pi n}\sum_{t=1}^n\varepsilon_t\varepsilon_t'-\frac{I_2}{2\pi}\right)A_1(0)'+o_p(1)\\
&\quad -\frac{2}{\sqrt{m}}\sum_1^m\left(\frac{j}{m}\right)^{-\delta}\mathrm{Re}\left[D_{nj}(-\delta)\right]g^2A(0)\frac{1}{2\pi n}\sum\sum_{t\neq s}^n\varepsilon_t\varepsilon_s'\cos(t-s)\lambda_jA_1(0)',
\end{aligned}
$$

because $g^2C(1)C_1(1)'=0$. The first term on the right is $o_p(1)$ because $\left\|n^{-1}\sum_1^n\varepsilon_t\varepsilon_t'-I_2\right\|=O_p(n^{-1/2})$ from Lobato (1999) p. 149. Similarly,

$$
\begin{aligned}
(49) &= -\left[\frac{\cos(\pi\delta/2)}{1-\delta}g^2+o_p(1)\right]\frac{2}{\sqrt{m}}\sum_1^m A(0)\left(\mathrm{Re}[I_{\varepsilon j}]-\frac{I_2}{2\pi}\right)A_1(0)'+o_p(1)\\
&= -\frac{\cos(\pi\delta/2)}{1-\delta}\frac{2}{\sqrt{m}}\sum_1^m g^2A(0)\frac{1}{2\pi n}\sum\sum_{t\neq s}^n\varepsilon_t\varepsilon_s'\cos(t-s)\lambda_jA_1(0)'+o_p(1).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
S_{31} &= \frac{2}{\sqrt{m}}\sum_1^m a_{nj}g^2A(0)\frac{1}{2\pi n}\sum\sum_{t\neq s}^n\varepsilon_t\varepsilon_s'\cos(t-s)\lambda_jA_1(0)'+o_p(1),\\
\alpha_{nj} &= -\left(\frac{j}{m}\right)^{-\delta}\mathrm{Re}\left[D_{nj}(-\delta)\right]-\frac{\cos(\pi\delta/2)}{1-\delta}.
\end{aligned}
$$

In the same manner as Lobato (1999), we can rewrite this as $\sum_{t=1}^n\varepsilon_t'\sum_{s=1}^{t-1}\widetilde{\Theta}_{t-s}\varepsilon_s$, where

$$
\begin{aligned}
\widetilde{\Theta}_t &= (\pi\sqrt{m}n)^{-1}\sum_{j=1}^m\widetilde{\Omega}_j\cos(t\lambda_j),\\
\widetilde{\Omega}_j &= \alpha_{nj}\left[A(0)'(g^2)'(A_1(0))'+A_1(0)'g^2A(0)\right].
\end{aligned}
$$

Combining the above with the results from the proof of Theorem 3.4, with S_{11} and S_{21} defined there, we obtain

$$
\eta_1S_{11}+\eta_2S_{21}+\eta_3S_{31}=\textstyle\sum_{t=1}^n\varepsilon_t'\sum_{s=1}^{t-1}\Theta_{t-s}\varepsilon_s,
$$

where

$$
\begin{aligned}
\Theta_t &= (\pi\sqrt{m}n)^{-1}\sum_{j=1}^m\Omega_j\cos(t\lambda_j),\\
\Omega_j &= \sum_{a=1}^2\eta_a\nu_j\left[A(0)'(g^a)'(A_a(0))'+A_a(0)'g^aA(0)\right]\\
&\quad +\eta_3\alpha_{nj}\left[A(0)'(g^2)'(A_1(0))'+A_1(0)'g^2A(0)\right].
\end{aligned}
$$

The bounds of $\|\Theta_t\|^2$ are given by $(\log m)^2$ times the bound of c_s in Lemma 6.8. Therefore, we need only to check the limit of

$$\frac{1}{m}\sum_1^m \operatorname{tr}\left[\frac{1}{4\pi^2}\Omega_j'\Omega_j\right].$$

From Lobato (1999), the terms in $m^{-1}\sum_1^m \operatorname{tr}[\Omega_j'\Omega_j/(4\pi^2)]$ that involve only η_1 and η_2 tend to $\sum_{a=1}^2\sum_{b=1}^2 \eta_a\eta_b E_{ab}$. The terms with η_3^2 in $\operatorname{tr}[\Omega_j'\Omega_j]$ are

$$\begin{aligned}
&\operatorname{tr}\left\{\begin{array}{c} \eta_3^2\alpha_{nj}^2\left[A(0)'(g^2)'(A_1(0))' + A_1(0)'g^2A(0)\right]' \\ \times\left[A(0)'(g^2)'(A_1(0))' + A_1(0)'g^2A(0)\right]\end{array}\right\} \\
=\ & 2\operatorname{tr}\left[\eta_3^2\alpha_{nj}^2 A_1(0)'g^2A(0)A(0)'(g^2)'(A_1(0))'\right] \\
& +2\operatorname{tr}\left[\eta_3^2\alpha_{nj}^2 A_1(0)'g^2A(0)A_1(0)'g^2A(0)\right] \\
=\ & (4\pi^2)2\eta_3^2\alpha_{nj}^2 G_{11}^0\left[\begin{array}{cc} 0 & 1\end{array}\right](g^2)' = (4\pi^2)2\eta_3^2\alpha_{nj}^2(G_{11}^0)^2/\det G^0.
\end{aligned}$$

The terms with $\eta_a\eta_3$ in $\operatorname{tr}[\Omega_j'\Omega_j]$ are

$$\begin{aligned}
&2\operatorname{tr}\left[\begin{array}{c} \eta_a\eta_3\nu_j\alpha_{nj}\left[A(0)'(g^a)'(A_a(0))' + A_a(1)'g^aA(0)\right]' \\ \times\left[A(0)'(g^2)'(A_1(1))' + A_1(0)'g^2A(0)\right]\end{array}\right] \\
=\ & 4\operatorname{tr}\left[\eta_a\eta_3\nu_j\alpha_{nj}A_a(0)'g^aA(0)A(0)'(g^2)'(A_1(0))'\right] \\
& +4\operatorname{tr}\left[\eta_a\eta_3\nu_j\alpha_{nj}A_a(0)'g^aA(0)A_1(0)'g^2A(0)\right] \\
=\ & (4\pi^2)4\eta_a\eta_3\nu_j\alpha_{nj}\left\{\begin{array}{ll} G_{11}^0[\begin{array}{cc}1 & 0\end{array}](g^2)', & \text{for } a=1, \\ G_{12}^0[\begin{array}{cc}0 & 1\end{array}](g^2)', & \text{for } a=2,\end{array}\right. \\
=\ & (4\pi^2)4\eta_a\eta_3\nu_j\alpha_{nj}(-1)^a G_{11}^0G_{12}^0/\det G^0.
\end{aligned}$$

Therefore, the terms in $m^{-1}\sum_1^m \operatorname{tr}[\Omega_j'\Omega_j]/(4\pi^2)$ that involve η_3 are, from Lemma NB,

$$\begin{aligned}
&\frac{1}{\det G^0}\left(\eta_3^2(G_{11}^0)^2\frac{2}{m}\sum_1^m\alpha_{nj}^2 + \eta_a\eta_3(-1)^aG_{11}^0G_{12}^0\frac{4}{m}\sum_1^m\nu_j\alpha_{nj}\right) \\
\to\ & \frac{2}{\det G^0}\eta_3^2\cos^2\left(\frac{\pi\delta}{2}\right)(G_{11}^0)^2\left(\frac{1}{1-2\delta} - \frac{1}{(1-\delta)^2}\right) \\
& +\frac{4}{\det G^0}\eta_a\eta_3(-1)^a\cos\left(\frac{\pi\delta}{2}\right)G_{11}^0G_{12}^0\frac{2-\delta}{(1-\delta)^2}.
\end{aligned}$$

It follows that

$$\sum_{a=1}^3 \eta_a S_{a1} \to_d N(0, \eta'\mathcal{I}_1\eta),$$

where the upper-left (2×2) block of $\mathcal{I}_1$ is given by the first term of Ω, and $(3,1)$, $(3,2)$, $(3,3)$ elements of $\mathcal{I}_1$ are given by

$$-\frac{2G_{11}^0G_{12}^0}{\det G^0}\cos(\tfrac{\pi\delta}{2})\frac{2-\delta}{(1-\delta)^2},\quad \frac{2G_{11}^0G_{12}^0}{\det G^0}\cos(\tfrac{\pi\delta}{2})\frac{2-\delta}{(1-\delta)^2},\quad \frac{2(G_{11}^0)^2}{\det G^0}\cos^2\left(\tfrac{\pi\delta}{2}\right)\left[\frac{1}{1-2\delta} - \frac{1}{(1-\delta)^2}\right].$$

It remains to derive the limit distribution of S_{32}. We have

$$\begin{aligned}
S_{32} &= -\frac{\widehat{G}_{11}\det A(0)}{\det G^0}\frac{2}{\sqrt{m}}\sum_1^m\left(\frac{j}{m}\right)^{-\delta}\operatorname{Im}[D_{nj}(-\delta)]\operatorname{Im}\left[w_{\varepsilon 1j}w_{\varepsilon 2j}^*\right] + o_p(1) \\
&= -\frac{2G_{11}^0}{\sqrt{\det G^0}}\sin\left(\frac{\pi}{2}\delta\right)\frac{2\pi}{\sqrt{m}}\sum_1^m\left(\frac{j}{m}\right)^{-\delta}\operatorname{Im}\left[w_{\varepsilon 1j}w_{\varepsilon 2j}^*\right] + o_p(1).
\end{aligned}$$

In view of the proof of Theorem 3.4, we obtain

$$\begin{aligned}\sum_{a=1}^{3}\eta_a S_{a2} &= (\eta_1-\eta_2)\frac{\pi G_{12}^0}{\sqrt{\det G^0}}\frac{2\pi}{\sqrt{m}}\sum_1^m \operatorname{Im}\left[w_{\varepsilon_1 j}w_{\varepsilon_2 j}^*\right] \\ &\quad -\eta_3\frac{2G_{11}^0}{\sqrt{\det G^0}}\sin\left(\frac{\pi}{2}\delta\right)\frac{2\pi}{\sqrt{m}}\sum_1^m\left(\frac{j}{m}\right)^{-\delta}\operatorname{Im}\left[w_{\varepsilon_1 j}w_{\varepsilon_2 j}^*\right]+o_p(1) \\ &\to {}_dN(0,\eta'\mathcal{I}_2\eta),\end{aligned}$$

where the upper-left (2×2) block of $\mathcal{I}_2$ is given by the second term of Ω, and $(3,1)$, $(3,2)$, $(3,3)$ elements of $\mathcal{I}_2$ are given by

$$-\tfrac{\pi G_{11}^0 G_{12}^0}{\det G^0}\sin(\tfrac{\pi\delta}{2})\tfrac{1}{1-\delta},\quad \tfrac{\pi G_{11}^0 G_{12}^0}{\det G^0}\sin(\tfrac{\pi\delta}{2})\tfrac{1}{1-\delta},\quad \tfrac{2(G_{11}^0)^2}{\det G^0}\sin^2\left(\tfrac{\pi\delta}{2}\right)\tfrac{1}{1-2\delta}.$$

Therefore, $\sum_1^3\eta_a S_{a2}$ and $\sum_1^3\eta_a S_{a1}$ are asymptotically independent to each other, and the required result follows from $\widehat{G}\to_p G$ and $\Xi=\mathcal{I}_1+\mathcal{I}_2$.

Lemma NB

$$\begin{aligned}\frac{1}{m}\sum_1^m\alpha_{nj}^2 &\to \cos^2\left(\frac{\pi\delta}{2}\right)\left(\frac{1}{1-2\delta}-\frac{1}{(1-\delta)^2}\right), \\ \frac{1}{m}\sum_1^m\nu_j\alpha_{nj} &\to \cos\left(\frac{\pi\delta}{2}\right)\frac{2-\delta}{(1-\delta)^2}.\end{aligned}$$

Proof Since for $\alpha>-1$ we have

$$\begin{aligned}m^{-1}\textstyle\sum_1^m(j/m)^\alpha\log j &= (\alpha+1)^{-1}\log m-(\alpha+1)^{-2}+o(1), \\ m^{-1}\textstyle\sum_1^m(j/m)^\alpha m^{-1}\sum_1^m\log j &= (\alpha+1)^{-1}\log m+(\alpha+1)^{-1}+o(1),\end{aligned}$$

it follows that

$$\begin{aligned}\frac{1}{m}\sum_1^m\alpha_{nj}^2 &= \frac{1}{m}\sum_1^m\left(\left(\frac{j}{m}\right)^{-\delta}\operatorname{Re}\left[D_{nj}(-\delta)\right]+\frac{\cos(\pi\delta/2)}{1-\delta}\right)^2 \\ &\to \cos^2\left(\frac{\pi\delta}{2}\right)\left(\frac{1}{1-2\delta}-\frac{1}{(1-\delta)^2}\right), \\ \frac{1}{m}\sum_1^m\nu_j\alpha_{nj} &= -\frac{1}{m}\sum_1^m\left(\left(\frac{j}{m}\right)^{-\delta}\operatorname{Re}\left[D_{nj}(-\delta)\right]+\frac{\cos(\pi\delta/2)}{1-\delta}\right)\left(\log j-\frac{1}{m}\sum_1^m\log j\right) \\ &= -\frac{1}{m}\sum_1^m\left(\frac{j}{m}\right)^{-\delta}\operatorname{Re}\left[D_{nj}(-\delta)\right]\left(\log j-\frac{1}{m}\sum_1^m\log j\right) \\ &\to \cos\left(\frac{\pi\delta}{2}\right)\left(\frac{1}{(1-\delta)^2}+\frac{1}{1-\delta}\right)=\cos\left(\frac{\pi\delta}{2}\right)\frac{2-\delta}{(1-\delta)^2},\end{aligned}$$

giving the stated result. ■

Hessian Let $M = \{\tau : B_n^{-1}(\log n)^4 ||\tau - \tau^0|| < \varepsilon\}$. Again $\Pr(\tau^* \notin M)$ tends to zero from the proof of consistency, hence hereafter we assume $\tau \in M$. First we show that the terms involving β^* are negligible, i.e.,

$$B_n \left(\left. \frac{\mathrm{d}^2 R(d,\beta)}{\mathrm{d}\tau \mathrm{d}\tau'} \right|_{\tau^*} \right) B_n = B_n \left(\left. \frac{\mathrm{d}^2 R(d,\beta)}{\mathrm{d}\tau \mathrm{d}\tau'} \right|_{d^*,\beta^0} \right) B_n + o_p(1). \tag{50}$$

$\tau^* \in M$ implies that $\widetilde{\beta}^* = (2\pi m/n)^{-\delta} (\beta^0 - \beta^*) = O((\log n)^{-4})$. From the proof of Theorem 4.2, we have

$$\begin{aligned} \lambda_j^{-\theta_2} w_{\Delta^{d_2}(x_2 - \beta x_1) j} &= \lambda_j^{-\theta_2} w_{\Delta^{\theta_2} u_2 j} + \lambda_j^{-\theta_2} (\beta^0 - \beta) w_{\Delta^{\theta_2 - \delta} u_1 j} \\ &= \lambda_j^{-\theta_2} w_{y_2 j} + (j/m)^{-\delta} \left(\frac{2\pi m}{n} \right)^{-\delta} (\beta^0 - \beta) \lambda_j^{\delta - \theta_2} w_{\Delta^{\theta_2 - \delta} u_1 j} \\ &= \lambda_j^{-\theta_2} w_{y_2 j} + O((\log n)^{-4}) (j/m)^{-\delta} \lambda_j^{\delta - \theta_2} w_{\Delta^{\theta_2 - \delta} u_1 j}. \end{aligned}$$

Recall that $Y_{2t} = \Delta^{\theta_2} u_{2t} I\{t \geq 1\}$. Thus, the $(1,2)$ element of $J(\theta) \Lambda_j(\theta) \frac{\partial \widehat{G}(d,\beta)}{\partial d_1} \Lambda_j(\theta) J(\theta)$ is

$$\begin{aligned} & \frac{1}{m} \sum_1^m j^{\theta_1 + \theta_2} \lambda_j^{-\theta_1 - \theta_2} \operatorname{Re} \left[w_{\log(1-L) \Delta^{d_1} x_1 j} w^*_{\Delta^{d_2}(x_2 - \beta x_1) j} \right] \\ = & \frac{1}{m} \sum_1^m j^{\theta_1 + \theta_2} \lambda_j^{-\theta_1 - \theta_2} \operatorname{Re} \left[w_{\log(1-L) y_1 j} w^*_{y_2 j} \right] + O_p((\log n)^{-3}), \end{aligned}$$

uniformly in $\tau \in M$. Hereafter $O_p(\cdot)$ and $o_p(\cdot)$ terms are all uniform in $\tau \in M$. Applying a similar argument to the other elements give

$$J(\theta) \Lambda_j(\theta) \frac{\partial \widehat{G}}{\partial d_1} \Lambda_j(\theta) J(\theta) \bigg|_{\tau^*} = J(\theta) \Lambda_j(\theta) \frac{\partial \widehat{G}}{\partial d_1} \Lambda_j(\theta) J(\theta) \bigg|_{d^*,\beta^0} + O_p((\log n)^{-3}),$$

and similarly we can show (50). Therefore, hereafter all the derivatives are evaluated at (d^*, β^0).

For the derivatives with respect to d, we have from (50) and the proof of Theorem 3.4

$$\frac{\partial^2 R(d,\beta)}{\partial d_a \partial d_b} \to_p \Omega_{ab}.$$

For the derivatives with respect to β, first we evaluate

$$\begin{aligned} & [B_n]_3 J(\theta) \Lambda_j(\theta) \frac{\partial \widehat{G}}{\partial \beta} \Lambda_j(\theta) J(\theta) \\ = & -\left(\frac{2\pi m}{n} \right)^{\delta} \frac{1}{m} \sum_1^m \operatorname{Re} \begin{bmatrix} 0 & j^{\theta_1 + \theta_2} \lambda_j^{-\theta_1 - \theta_2} w_{y_1 j} w^*_{\Delta^{d_2} x_1 j} \\ j^{\theta_1 + \theta_2} \lambda_j^{-\theta_1 - \theta_2} w_{y_1 j} w^*_{\Delta^{d_2} x_1 j} & 2 j^{2\theta_2} \lambda_j^{-2\theta_2} w_{\Delta^{d_2} x_1 j} w^*_{\Delta^{d_2}(x_2 - \beta x_1) j} \end{bmatrix}. \end{aligned} \tag{51}$$

By the same argument as the one in the proof of Theorem 3.4, its diagonal elements are

$$-\frac{1}{m} \sum_1^m \left(\frac{j}{m} \right)^{-\delta} \operatorname{Re} \left[\lambda_j^{-\theta_1} w_{y_1 j} \lambda_j^{-\theta_2 + \delta} w^*_{\Delta^{\theta_2 - \delta} u_1 j} \right] + o_p((\log n)^{-1})$$

$$
\begin{aligned}
&= -\frac{1}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-\delta}\mathrm{Re}\left[D_{nj}(-\delta)^{*}\right]I_{u_1j}+o_p((\log n)^{-1})\\
&= -G_{11}^{0}\cos\left(\frac{\pi}{2}\delta\right)\frac{1}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-\delta}+o_p((\log n)^{-1}).
\end{aligned}
$$

Its $(2,2)$ element is

$$
\begin{aligned}
&-\frac{2}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-\delta}\mathrm{Re}\left[\lambda_j^{-\theta_2+\delta}w_{\Delta^{\theta_2-\delta}u_1j}\lambda_j^{-\theta_2}w_{y_2j}^{*}\right]+o_p((\log n)^{-1})\\
&= -\frac{2}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-\delta}\mathrm{Re}\left[D_{nj}(-\delta)\right]\mathrm{Re}\left[w_{u_1j}w_{u_2j}^{*}\right]+o_p((\log n)^{-1})\\
&= -2G_{12}^{0}\cos\left(\frac{\pi}{2}\delta\right)\frac{1}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-\delta}+o_p((\log n)^{-1}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
[B_n]_3 J(\theta)\,\Lambda_j(\theta)\frac{\partial\widehat{G}}{\partial\beta}\Lambda_j(\theta)J(\theta) &= -\cos\left(\frac{\pi}{2}\delta\right)\frac{1}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-\delta}G_3+o_p((\log n)^{-1}),\\
G_3 &= \begin{bmatrix} 0 & G_{11}^{0}\\ G_{11}^{0} & 2G_{12}^{0}\end{bmatrix}.
\end{aligned}
$$

Next, for $([B_n]_3)^2 J(\theta)\,\Lambda_j(\theta)(\partial^2\widehat{G}/\partial\beta^2)\Lambda_j(\theta)J(\theta)$, its (2,2) element is

$$
\frac{2}{m}\sum_{1}^{m}j^{2\theta_2}\left(\frac{j}{m}\right)^{-2\delta}\mathrm{Re}\left[\lambda_j^{-2\theta_2+2\delta}I_{\Delta^{\theta_2-\delta}u_1j}\right]=2G_{11}^{0}\frac{1}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-2\delta}+o_p(1),
$$

and the other elements are zero. Therefore,

$$
\begin{aligned}
\left[B_n\frac{\mathrm{d}^2R(d,\beta)}{\mathrm{d}\tau\mathrm{d}\tau'}B_n\right]_{33} &= ([B_n]_3)^2\mathrm{tr}\left[-\widehat{G}^{-1}\frac{\partial\widehat{G}}{\partial\beta}\widehat{G}^{-1}\frac{\partial\widehat{G}}{\partial\beta}+\widehat{G}^{-1}\frac{\partial^2\widehat{G}}{\partial\beta^2}\right]\\
&= \frac{\cos^2(\pi\delta/2)}{(1-\delta)^2}\mathrm{tr}\left[-(G^0)^{-1}G_3(G^0)^{-1}G_3\right]+\frac{2(G_{11}^{0})^2}{\det G^0}\frac{1}{1-2\delta}+o_p(1)\\
&= \frac{2(G_{11}^{0})^2}{\det G^0}\left[\frac{1}{1-2\delta}-\cos^2\left(\frac{\pi\delta}{2}\right)\frac{1}{(1-\delta)^2}\right]+o_p(1).
\end{aligned}
$$

We proceed to evaluate

$$
[B_n]_3 J(\theta)\,\Lambda_j(\theta)\frac{\partial^2\widehat{G}}{\partial d_a\partial\beta}\Lambda_j(\theta)J(\theta)\,. \tag{52}
$$

When $a=1$, its diagonal elements are zero, and its off-diagonal elements are

$$
\begin{aligned}
&-\frac{1}{m}\sum_{1}^{m}j^{\theta_1+\theta_2}\left(\frac{j}{m}\right)^{-\delta}\left[\log\lambda_j\cos\left(\frac{\pi}{2}\delta\right)+\frac{\pi}{2}\sin\left(\frac{\pi}{2}\delta\right)\right]I_{u_1j}+o_p((\log n)^{-1})\\
&= -G_{11}^{0}\cos\left(\frac{\pi}{2}\delta\right)\frac{1}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-\delta}\log\lambda_j-G_{11}^{0}\frac{\pi}{2}\sin\left(\frac{\pi}{2}\delta\right)\frac{1}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-\delta}\\
&\quad+o_p((\log n)^{-1}).
\end{aligned}
$$

When $a = 2$, its (1,1) element is zero, and its off-diagonal elements are

$$-G_{11}^{0}\cos\left(\frac{\pi}{2}\delta\right)\frac{1}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-\delta}\log\lambda_j$$
$$+G_{11}^{0}\frac{\pi}{2}\sin\left(\frac{\pi}{2}\delta\right)\frac{1}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-\delta}+o_p((\log n)^{-1}).$$

Since

$$\begin{aligned}&\frac{\partial^2}{\partial\beta\partial d_2}I_{\Delta^{d_2}(x_2-\beta x_1)j}\\
=\ &\frac{\partial}{\partial\beta}2\operatorname{Re}\left[w_{\log(1-L)\Delta^{d_2}(x_2-\beta x_1)j}w^*_{\Delta^{d_2}(x_2-\beta x_1)j}\right]\\
=\ &-2\operatorname{Re}\left[w_{\log(1-L)\Delta^{d_2}x_1j}w^*_{\Delta^{d_2}(x_2-\beta x_1)j}+w_{\log(1-L)\Delta^{d_2}(x_2-\beta x_1)j}w^*_{\Delta^{d_2}x_1j}\right],\end{aligned}$$

its (2,2) element is

$$\begin{aligned}&-\frac{2}{m}\sum_{1}^{m}j^{2\theta_2}\lambda_j^{-2\theta_2}\operatorname{Re}\left[w_{\log(1-L)\Delta^{d_2}x_1j}w^*_{\Delta^{d_2}(x_2-\beta x_1)j}+w_{\log(1-L)\Delta^{d_2}(x_2-\beta x_1)j}w^*_{\Delta^{d_2}x_1j}\right]\\
=\ &-\frac{2}{m}\sum_{1}^{m}\operatorname{Re}\left[J_n(e^{i\lambda_j})D_{nj}(-\delta)w_{u_1j}w^*_{u_2j}+J_n(e^{i\lambda_j})D_{nj}(-\delta)^*w^*_{u_1j}w_{u_2j}\right]+o_p(1)\\
=\ &-\frac{4}{m}\sum_{1}^{m}(j/m)^{-\delta}\log\lambda_j\cos\left(\frac{\pi}{2}\delta\right)\operatorname{Re}\left[w_{u_1j}w^*_{u_2j}\right]+o_p(1)\\
=\ &4G_{12}^{0}\cos\left(\frac{\pi}{2}\delta\right)\frac{1}{m}\sum_{1}^{m}(j/m)^{-\delta}\log\lambda_j+o_p(1).\end{aligned}$$

It follows that

$$\begin{aligned}(52)\ =\ &-\cos\left(\frac{\pi}{2}\delta\right)(i_aG_3+G_3i_a)\frac{1}{m}\sum_{1}^{m}(j/m)^{-\delta}\log\lambda_j\\
&+(-1)^a\frac{\pi}{2}\sin\left(\frac{\pi}{2}\delta\right)(i_1G_3+G_3i_1)\frac{1}{m}\sum_{1}^{m}(j/m)^{-\delta}+o_p(1).\end{aligned}$$

Therefore, since $\widehat{G}_1 = G^0m^{-1}\sum_1^m\log\lambda_j + o_p((\log n)^{-1})$,

$$\begin{aligned}&\left(\frac{2\pi m}{n}\right)^{\delta}\operatorname{tr}\left[-\widehat{G}^{-1}\frac{\partial\widehat{G}}{\partial\beta}\widehat{G}^{-1}\frac{\partial\widehat{G}}{\partial d_a}+\widehat{G}^{-1}\frac{\partial^2\widehat{G}}{\partial\beta\partial d_a}\right]\\
=\ &\cos\left(\frac{\pi}{2}\delta\right)\operatorname{tr}\left[(G_0)^{-1}\left(i_a\widehat{G}_1+\widehat{G}_1i_a\right)(G_0)^{-1}G_3\right]\frac{1}{m}\sum_{1}^{m}(j/m)^{-\delta}\\
&-\cos\left(\frac{\pi}{2}\delta\right)\operatorname{tr}\left[(G_0)^{-1}(i_aG_3+G_3i_a)\right]\frac{1}{m}\sum_{1}^{m}(j/m)^{-\delta}\log\lambda_j\\
&+(-1)^a\frac{\pi}{2}\sin\left(\frac{\pi}{2}\delta\right)\operatorname{tr}\left[(G_0)^{-1}(i_1G_3+G_3i_1)\right]\frac{1}{m}\sum_{1}^{m}(j/m)^{-\delta}+o_p(1)\end{aligned}$$

$$
\begin{aligned}
&= -\cos\left(\frac{\pi}{2}\delta\right)\operatorname{tr}\left[(G_0)^{-1}(i_aG_3+G_3i_a)\right] \\
&\quad \left(\frac{1}{m}\sum_1^m (j/m)^{-\delta}\log\lambda_j - \frac{1}{m}\sum_1^m (j/m)^{-\delta}\frac{1}{m}\sum_1^m \log\lambda_j\right) \\
&\quad -(-1)^a\frac{\pi}{2}\sin\left(\frac{\pi}{2}\delta\right)\frac{2G_{11}^0G_{12}^0}{\det G^0}\frac{1}{m}\sum_1^m (j/m)^{-\delta} + o_p(1) \\
&= \cos\left(\frac{\pi}{2}\delta\right)\operatorname{tr}\left[(G_0)^{-1}(i_aG_3+G_3i_a)\right]\frac{2-\delta}{(1-\delta)^2} \\
&\quad -(-1)^a\pi\sin\left(\frac{\pi}{2}\delta\right)\frac{G_{11}^0G_{12}^0}{\det G^0}\frac{1}{1-\delta} + o_p(1) \\
&= (-1)^a\cos\left(\frac{\pi}{2}\delta\right)\frac{2G_{11}^0G_{12}^0}{\det G^0}\frac{2-\delta}{(1-\delta)^2} \\
&\quad -(-1)^a\pi\sin\left(\frac{\pi}{2}\delta\right)\frac{G_{11}^0G_{12}^0}{\det G^0}\frac{1}{1-\delta} + o_p(1),
\end{aligned}
$$

7.4.2 Part II: $\delta \in (\frac{1}{2}, \frac{3}{2})$

Define $B_n = \text{diag}(1, 1, n^{-\delta}m^{1/2})$. We show

$$
\begin{aligned}
B_n^{-1}\sqrt{m}\frac{\mathrm{d}R(d^0,\beta^0)}{\mathrm{d}\tau} &= \begin{bmatrix} x \\ y \end{bmatrix}, \quad x \to_d N(0,\Omega), \quad y = O_p(1), \\
B_n\left(\frac{\mathrm{d}^2R(d^*,\beta^*)}{\mathrm{d}\tau\mathrm{d}\tau'}\right)B_n &\to_d \begin{bmatrix} \Omega & 0 \\ 0 & \xi \end{bmatrix},
\end{aligned}
$$

where

$$
\xi = O_p(1), \quad \Pr(|\xi| < \varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.
$$

then $\sqrt{m}(\widehat{d} - d^0) \to_d N(0, \Omega^{-1})$ follows.

Score vector We show

$$
\sqrt{m}\frac{\partial R(d^0,\beta^0)}{\partial\beta} = O_p(n^\delta m^{-1/2}).
$$

First, we have

$$
\begin{aligned}
&\sqrt{m}\frac{\partial R(d^0,\beta^0)}{\partial\beta} \\
&= \operatorname{tr}\left(\widehat{G}(d,\beta)^{-1}\sqrt{m}\left.\frac{\partial R(d,\beta)}{\partial\beta}\right|_{\tau^0}\right) \\
&= \frac{2}{\det\widehat{G}}\left[\widehat{G}_{12}\frac{1}{\sqrt{m}}\sum_1^m \operatorname{Re}\left[w_{u_1j}w^*_{\Delta^{-\delta}u_1j}\right] - \widehat{G}_{11}\frac{1}{\sqrt{m}}\sum_1^m \operatorname{Re}\left[w_{\Delta^{-\delta}u_1j}w^*_{u_2j}\right]\right].
\end{aligned}
$$

Since

$$
\lambda_j^\delta w^*_{\Delta^{-\delta}u_1j} = D_{nj}(-\delta)w^*_{u_1j} - \frac{\lambda_j^\delta}{1-e^{i\lambda_j}}\frac{\widetilde{U}_{1,\lambda_jn}(1-\delta)}{\sqrt{2\pi n}} - \frac{\lambda_j^\delta}{1-e^{i\lambda_j}}\frac{e^{i\lambda_j}Y_{3n}}{\sqrt{2\pi n}},
$$

where $Y_{3n} = \Delta^{-\delta} u_{1n} I\{t \geq 1\}$, we obtain $m^{-1/2} \sum_1^m \operatorname{Re}\left[w_{u_1 j} w^*_{\Delta^{-\delta} u_1 j}\right] = I + II + III$, where

$$\begin{aligned} I &= \frac{1}{\sqrt{m}} \sum_1^m \lambda_j^{-\delta} \operatorname{Re}\left[D_{nj}(-\delta)\right] I_{u_1 j}, \\ II &= -\frac{1}{\sqrt{m}} \sum_1^m \operatorname{Re}\left[w_{u_1 j} \frac{1}{1 - e^{i\lambda_j}} \frac{\widetilde{U}_{1,\lambda_j n}(1-\delta)}{\sqrt{2\pi n}}\right], \\ III &= -\frac{1}{\sqrt{m}} \sum_1^m \operatorname{Re}\left[w_{u_1 j} \frac{1}{1 - e^{i\lambda_j}} \frac{e^{i\lambda_j} Y_{3n}}{\sqrt{2\pi n}}\right]. \end{aligned}$$

We show II and III are $O_p(n^\delta m^{-1/2})$. First we take care of II. Since $1 - \delta \in (0, \frac{1}{2})$, Lemmas 8.7 and 8.9 of Phillips and Shimotsu (2003b) give

$$\begin{aligned} II &= O_p\left(\frac{1}{\sqrt{m}} \sum_1^m j^{-1} n^{1/2} n^{1/2-(1-\delta)} j^{1-\delta-1/2}\right) \\ &= O_p\left(n^\delta m^{-1/2} \sum_1^m j^{-\delta-1/2}\right) = O_p(n^\delta m^{-1/2}). \end{aligned}$$

For III, since $n^{1/2-\delta} Y_{3n} \to_d N(0, \sigma^2)$ by the standard MDS-CLT,

$$\begin{aligned} III &= -\frac{Y_{3n}}{\sqrt{2\pi n}} \frac{1}{\sqrt{m}} \sum_1^m \operatorname{Re}\left[w_{u_1 j} \frac{e^{i\lambda_j}}{1 - e^{i\lambda_j}}\right] \\ &= m^{-1/2} n^{1/2} Y_{3n} \sum_1^m w_{u_1 j} O(j^{-1}) \\ &= n^\delta m^{-1/2} (n^{1/2-\delta} Y_{3n}) \left[\sum_1^m A^1(\lambda_j) w_{\varepsilon j} O(j^{-1}) + O_p(n^{-1/2} \log m)\right] \\ &= O_p(n^\delta m^{-1/2}), \end{aligned}$$

where $A^1(\lambda_j)$ denotes the first row of $A(\lambda_j)$. A similar calculation for $m^{-1/2} \sum_1^m \operatorname{Re}\left[w_{\Delta^{-\delta} u_1 j} w^*_{u_2 j}\right]$ gives

$$\begin{aligned} & \frac{\det \widehat{G}}{2} \operatorname{tr}\left(\widehat{G}(d,\beta)^{-1} \sqrt{m} \frac{\partial \widehat{G}(d,\beta)}{\partial \beta}\right) \\ = \; & \widehat{G}_{12} \frac{1}{\sqrt{m}} \sum_1^m \lambda_j^{-\delta} \operatorname{Re}\left[D_{nj}(-\delta)\right] I_{u_1 j} \\ & - \widehat{G}_{11} \frac{1}{\sqrt{m}} \sum_1^m \lambda_j^{-\delta} \operatorname{Re}\left[D_{nj}(-\delta)\right] \operatorname{Re}\left[w_{u_1 j} w^*_{u_2 j}\right] + O_p(n^\delta m^{-1/2}). \end{aligned}$$

Rewrite the first two terms on the right as

$$\begin{aligned} & \left(\widehat{G}_{12} - G^0_{12}\right) \frac{1}{\sqrt{m}} \sum_1^m \lambda_j^{-\delta} \operatorname{Re}\left[D_{nj}(-\delta)\right] I_{u_1 j} \\ & - \left(\widehat{G}_{11} - G^0_{11}\right) \frac{1}{\sqrt{m}} \sum_1^m \lambda_j^{-\delta} \operatorname{Re}\left[D_{nj}(-\delta)\right] \operatorname{Re}\left[w_{u_1 j} w^*_{u_2 j}\right] \end{aligned}$$

$$+\frac{1}{\sqrt{m}}\sum_{1}^{m}\lambda_j^{-\delta}\,\mathrm{Re}\left[D_{nj}(-\delta)\right]\mathrm{Re}\left[G_{12}^{0}I_{u_1j}-G_{11}^{0}w_{u_1j}w_{u_2j}^{*}\right].$$

The first two terms are

$$O_p(m^{-1/2})O_p(n^{\delta}m^{-1/2}\sum_{1}^{m}j^{-\delta})=O_p(n^{\delta}m^{-\delta})=o_p(n^{\delta}m^{-1/2}).$$

The third term is

$$\begin{aligned}
&\frac{1}{\sqrt{m}}\sum_{1}^{m}\lambda_j^{-\delta}\,\mathrm{Re}\left[D_{nj}(-\delta)\right]\left[\begin{array}{ccc} G_{12}^{0} & \vdots & -G_{11}^{0}\end{array}\right]\mathrm{Re}[I_{uj}]\\
=\;&\frac{\det G}{\sqrt{m}}\sum_{1}^{m}\lambda_j^{-\delta}\,\mathrm{Re}\left[D_{nj}(-\delta)\right]g^{1}\,\mathrm{Re}\left[A(\lambda_j)I_{\varepsilon j}A_1(\lambda_j)'\right]\\
&+O_p(n^{\delta}m^{-1/2}\sum_{1}^{m}j^{-\delta}n^{-1/2})\\
=\;&\frac{\det G}{\sqrt{m}}\sum_{1}^{m}\lambda_j^{-\delta}\,\mathrm{Re}\left[D_{nj}(-\delta)\right]g^{1}\,\mathrm{Re}\left[A(0)I_{\varepsilon j}A_1(0)'\right]\\
&+O_p(n^{\delta-1/2}m^{1/2-\delta})+O_p(n^{\delta}m^{-1/2}\sum_{1}^{m}j^{-\delta}j^{\beta}n^{-\beta})\\
=\;&\frac{\det G}{\sqrt{m}}\sum_{1}^{m}\lambda_j^{-\delta}\,\mathrm{Re}\left[D_{nj}(-\delta)\right]g^{1}\,\mathrm{Re}\left[A(0)I_{\varepsilon j}A_1(0)'\right]+O_p(n^{\delta}m^{-1/2}),
\end{aligned}$$

because

$$\begin{aligned}
n^{\delta-1/2}m^{1/2-\delta} &= n^{-1/2}m^{1-\delta}n^{\delta}m^{-1/2},\\
n^{\delta}m^{-1/2}\sum_{1}^{m}j^{-\delta}j^{\beta}n^{-\beta} &= n^{\delta}m^{-1/2}O(m^{\beta-\delta+1}n^{-\beta}).
\end{aligned}$$

The first term on the last line has mean zero and variance

$$O\left(n^{2\delta}m^{-1}\sum_{1}^{m}j^{-2\delta}\right)=O\left(n^{2\delta}m^{-1}\right),$$

giving

$$\mathrm{tr}\left(\widehat{G}\,(d,\beta)^{-1}\sqrt{m}\frac{\partial\widehat{G}(d,\beta)}{\partial\beta}\right)=O_p(n^{\delta}m^{-1/2}).$$

When $\delta\in(1,\frac{3}{2})$, we have

$$\begin{aligned}
I &= O_p\left(\frac{1}{\sqrt{m}}\sum_{1}^{m}\lambda_j^{-\delta}\right)=O_p(n^{\delta}m^{-1/2}),\\
II &= O_p\left(\frac{1}{\sqrt{m}}\sum_{1}^{m}j^{-1}\sqrt{n}n^{1/2-(1-\delta)}j^{-1/2}\right)=O_p\left(n^{\delta}m^{-1/2}\sum_{1}^{m}j^{-3/2}\right)=O_p\left(n^{\delta}m^{-1/2}\right),\\
III &= O_p\left(n^{\delta}m^{-1/2}\right).
\end{aligned}$$

When $\delta = 1$, from Robinson and Marinucci (2001) Theorems 4.3 and 5.1,

$$\frac{1}{\sqrt{m}}\sum_1^m \mathrm{Re}\left[w_{u_1 j} w^*_{\Delta^{-1} u_1 j}\right] = m^{-1/2} n \left(\frac{1}{n}\sum_1^m \mathrm{Re}\left[w_{u_1 j} w^*_{\Delta^{-1} u_1 j}\right]\right) = O_p(m^{-1/2} n),$$

and similarly for $m^{-1/2}\sum_1^m \mathrm{Re}\left[w_{\Delta^{-\delta} u_1 j} w^*_{u_2 j}\right]$.

Hessian

Showing $\Pr(d^* \notin M)$ First, for $\delta \in (\frac{1}{2}, 1)$, let $M = \left\{\tau : (\log n)^{10}||\tau - \tau^0|| < \varepsilon\right\}$. In the proof of consistency, let $\rho = \varepsilon(\log n)^{-10}$. Since $\theta_2 \in N_\rho$, for large n we have

$$\begin{aligned}\inf_{\{d\in N_\rho\}\cap\{\beta\in\overline{B}_\rho\cap B\}} S(d,\beta) &= \inf_{\{d\in N_\rho\}\cap\{\theta_2-\delta\in[-3/2,-1/2]\}\cap\{\beta\in\overline{B}_\rho\cap B\}} S(d,\beta) \\ &\geq \eta|\widetilde{\beta}| + O_p((\log n)^{-11}).\end{aligned}$$

It follows that $\Pr(d^* \notin M) \to 0$.

Next, for $\delta \in [1, \frac{3}{2})$, let $M = \{\tau : \{(\log n)^{10}||\tau - \tau^0|| < \varepsilon\} \cap \{(\log n)^6 m^{\delta-1}|\widetilde{\beta}| < \varepsilon\}\}$. We proceed to show $\Pr(d^* \notin M) \to 0$. We have $\theta_2 - \delta \in [-3/2, -1/2]$ for large n if $d \in N_\rho$ and $k_n^2(\log n)^{12} \to 0$. Furthermore, $\det X$ has a term

$$\eta|\widetilde{\beta}| m^{2\delta-2\theta_2-2} n^{2\theta_2-2\delta+1} Y_{3n}^2.$$

Therefore, it suffices to show

$$n^{\theta_2-\delta+1/2} Y_{3n} \to_d N(0,\sigma^2), \quad \sigma^2 > 0.$$

Recall $Y_{3n} = (1-L)^{\theta_2-\delta} u_{1n} I\{t \geq 1\}$ with $\theta_2 - \delta \in [-3/2, -1/2]$. Then rewrite

$$u_{1n} = \sum_{j=0}^{\infty} A_{j,11}\varepsilon_{1,n-j} + \sum_{j=0}^{\infty} A_{j,12}\varepsilon_{2,n-j},$$

and let

$$u_{1n}^a = \sum_{j=0}^{\infty} A_{j,11}\varepsilon_{1,n-j} = \overline{A}(L)\varepsilon_{1n}, \quad \overline{A}(L) = \sum_{j=0}^{\infty} A_{j,11} L^j.$$

We show that $n^{\theta_2-\delta+1/2}(1-L)^{\theta_2-\delta} u_{1n}^a I\{t \geq 1\}$ converges to a normal random variable.

From the proof of Lemma 5.6 of SP, we have

$$\begin{aligned}n^{\theta_2-\delta+1/2}(1-L)^{\theta_2-\delta} u_{1n}^a I\{t \geq 1\} &= n^{\theta_2-\delta+1/2}\sum_{k=0}^{n-1}\frac{(\delta-\theta_2)_k}{k!}\overline{A}(L)\varepsilon_{1,n-k} \\ &= n^{\theta_2-\delta+1/2}\frac{\overline{C}(1)}{\Gamma(\delta-\theta_2)}\sum_{k=1}^{n-1} k^{\delta-\theta_2-1}\varepsilon_{1,n-k} + o_p(1).\end{aligned}$$

And

$$\begin{aligned}n^{\theta_2-\delta+1/2}\sum_{k=1}^{n-1} k^{\delta-\theta_2-1}\varepsilon_{1,n-k} &= n^{\theta_2-\delta+1/2}\sum_{k=1}^{n-1} k^{\delta-1}\varepsilon_{1,n-k} + n^{\theta_2-\delta+1/2}\sum_{k=1}^{n-1} k^{\delta-1} O((\log n)^{-3})\varepsilon_{1,n-k} \\ &\to {}_d N(0,\sigma_a^2), \quad \sigma_a^2 > 0,\end{aligned}$$

by the standard martingale CLT.

The limit of the Hessian For $J(\theta)\,\Lambda_j(\theta)\ \frac{\partial \widehat{G}(d,\beta)}{\partial d_a}\Lambda_j(\theta)\ J(\theta)$, we have as before

$$\begin{aligned}
& \lambda_j^{-\theta_2} w_{\Delta^{d_2}(x_2-\beta x_1)j} \\
= \;& \lambda_j^{-\theta_2} w_{y_2 j} + (j/m)^{-\delta}\widetilde{\beta}\lambda_j^{\delta-\theta_2} w_{\Delta^{\theta_2-\delta}u_1 j} \\
= \;& \lambda_j^{-\theta_2} w_{y_2 j} + O((\log n)^{-4})(j/m)^{-\delta}a_j + \widetilde{\beta}O(m^\delta j^{-\theta_2-1})n^{\theta_2-\delta+1/2}Y_{3n}.
\end{aligned}$$

Since $n^{\theta_2-\delta+1/2}Y_{3n}=O_p(1)$, for instance, the $(1,2)$ element of $J(\theta)\,\Lambda_j(\theta)\ \frac{\partial \widehat{G}(d,\beta)}{\partial d_1}\Lambda_j(\theta)$ $J(\theta)$ is

$$\frac{1}{m}\sum_1^m j^{\theta_1+\theta_2}\lambda_j^{-\theta_1-\theta_2}\,\mathrm{Re}\left[w_{\log(1-L)y_1 j}w_{y_2 j}^*\right]+O_p(1)\widetilde{\beta}m^{\delta-1}O_p((\log n)^3)+O_p((\log n)^{-3}).$$

Hence the second term is $o_p(1)$ both when $\delta\in(\frac{1}{2},1)$ and $\delta\in[1,\frac{3}{2})$.

For $(2\pi m/n)^\delta\, J(\theta)\,\Lambda_j(\theta)\frac{\partial\widehat{G}}{\partial\beta}\Lambda_j(\theta)J(\theta)$, since

$$\begin{aligned}
\lambda_j^{-\theta_2+\delta}w_{\Delta^{\theta_2-\delta}u_1 j} &= D_{nj}(\theta_2-\delta)w_{u_1 j}+e_j+\frac{\lambda_j^{-\theta_2+\delta}}{1-e^{i\lambda_j}}\frac{e^{i\lambda_j}Y_{1n}}{\sqrt{2\pi n}} \\
&= D_{nj}(\theta_2-\delta)w_{u_1 j}+e_j+Cj^{-\theta_2+\delta-1}n^{1/2-\delta+\theta_2}Y_{1n}(1+O(\lambda_j)),
\end{aligned}$$

its diagonal elements and $(2,2)$ element have an extra term

$$\begin{aligned}
m^{\delta-1}n^{1/2-\delta+\theta_2}Y_{3n}\sum_1^m \mathrm{Re}\left[\lambda_j^{-\theta_1}w_{y_1 j}\right]j^{-\theta_2-1}(1+O(\lambda_j)) &= o_p(m^{\delta-1/2}), \\
m^{\delta-1}n^{1/2-\delta+\theta_2}Y_{3n}\sum_1^m \mathrm{Re}\left[\lambda_j^{-\theta_2}w_{y_2 j}^*\right]j^{-\theta_2-1}(1+O(\lambda_j)) &= o_p(m^{\delta-1/2}),
\end{aligned}$$

respectively. Therefore, since $\delta>1/2$, we have

$$J(\theta)\,\Lambda_j(\theta)\frac{\partial\widehat{G}}{\partial\beta}\Lambda_j(\theta)J(\theta)=O_p(n^\delta m^{-\delta})+o_p(n^\delta m^{-1/2})=o_p(n^\delta m^{-1/2}).$$

For $(2\pi m/n)^{2\delta}\, J(\theta)\,\Lambda_j(\theta)\frac{\partial^2\widehat{G}}{\partial\beta^2}\Lambda_j(\theta)J(\theta)$, its $(2,2)$ element is

$$\frac{2}{m}\sum_1^m j^{2\theta_2}\left(\frac{j}{m}\right)^{-2\delta}\lambda_j^{-2\theta_2+2\delta}I_{\Delta^{\theta_2-\delta}u_1 j}.$$

From Lemma H,

$$\begin{aligned}
\frac{2}{m}\sum_1^m j^{2\theta_2}\left(\frac{j}{m}\right)^{-2\delta}\lambda_j^{-2\theta_2+2\delta}I_{\Delta^{\theta_2-\delta}u_1 j} &= \frac{2}{m}\sum_1^m\left(\frac{j}{m}\right)^{-2\delta}\lambda_j^{2\delta}I_{\Delta^{-\delta}u_1 j}+o_p(m^{2\delta-1}) \\
&= 2(2\pi)^{2\delta-1}m^{2\delta-1}n^{-2\delta}\sum_1^m I_{\Delta^{-\delta}u_1 j}+o_p(m^{2\delta-1}).
\end{aligned}$$

From Theorem 4.5 and 5.1 of Robinson and Marinucci (2001), we have

$$\lim_{n\to\infty}E\left[n^{-2\delta}\sum_1^m I_{\Delta^{-\delta}u_1 j}\right]>C>0,\quad \lim_{n\to\infty}Var\left[n^{-2\delta}\sum_1^m I_{\Delta^{-\delta}u_1 j}\right]<\infty,$$

giving

$$\frac{2}{m}\sum_{1}^{m} j^{2\theta_2}\left(\frac{j}{m}\right)^{-2\delta} \operatorname{Re}\left[\lambda_j^{-2\theta_2+2\delta} I_{\Delta^{\theta_2-\delta}u_{1}j}\right] = \frac{1}{\pi} m^{2\delta-1}\xi,$$

where

$$\xi = O_p(1), \quad \Pr(|\xi| < \varepsilon) \to 0 \quad \text{as } \varepsilon \to 0.$$

It follows that

$$\begin{aligned} n^{-2\delta} m \left[B_n \frac{\mathrm{d}^2 R(d,\beta)}{\mathrm{d}\tau \mathrm{d}\tau'} B_n\right]_{33} &= m^{1-2\delta}(m/n)^{2\delta} \operatorname{tr}\left[-\widehat{G}^{-1}\frac{\partial \widehat{G}}{\partial \beta}\widehat{G}^{-1}\frac{\partial \widehat{G}}{\partial \beta} + \widehat{G}^{-1}\frac{\partial^2 \widehat{G}}{\partial \beta^2}\right] \\ &= \frac{G_{11}^0}{\det G^0}\frac{1}{\pi} n^{1-2\delta}\frac{1}{n}\sum_{t=1}^{n}(\Delta^{-\delta}u_{1t})^2 + o_p(1). \end{aligned}$$

Similarly, the elements of $(2\pi m/n)^{\delta}\, J(\theta)\, \Lambda_j(\theta) \frac{\partial^2 \widehat{G}}{\partial d_a \partial \beta} \Lambda_j(\theta) J(\theta)$ are all $o_p(m^{\delta-1/2})$. Therefore,

$$n^{-\delta} m^{1/2}\left[B_n \frac{\mathrm{d}^2 R(d,\beta)}{\mathrm{d}\tau\ \mathrm{d}\tau'} B_n\right]_{a3} = m^{1/2-\delta}(m/n)^{\delta}\ \left[B_n \frac{\mathrm{d}^2 R(d,\beta)}{\mathrm{d}\tau\ \mathrm{d}\tau'} B_n\right]_{a3} = o_p(1).$$

Lemma H For $\delta > 1/2$ and $|\theta_2| < (\log n)^{-5}$, uniformly in θ_2 we have

$$\frac{1}{m}\sum_{1}^{m} j^{2\theta_2}\left(\frac{j}{m}\right)^{-2\delta} \lambda_j^{-2\theta_2+2\delta} I_{\Delta^{\theta_2-\delta}u_{1}j} = \frac{1}{m}\sum_{1}^{m}\left(\frac{j}{m}\right)^{-2\delta} \lambda_j^{2\delta} I_{\Delta^{-\delta}u_{1}j} + o_p(m^{2\delta-1}).$$

Proof Observe that

$$\frac{1}{m}\sum_{1}^{m} j^{2\theta_2}\left(\frac{j}{m}\right)^{-2\delta} \lambda_j^{-2\theta_2+2\delta} I_{\Delta^{\theta_2-\delta}u_{1}j} = m^{2\delta-1}\sum_{1}^{m} j^{2\theta_2} j^{-2\delta}\left[\lambda_j^{-2\theta_2+2\delta} I_{\Delta^{\theta_2-\delta}u_{1}j}\right],$$

and

$$\begin{aligned} \lambda_j^{-2\theta_2+2\delta} I_{\Delta^{\theta_2-\delta}u_{1}j} = &\Bigg| \frac{\lambda_j^{\delta-\theta_2}}{1-e^{i\lambda_j}} D_n(e^{i\lambda_j};\theta_2-\delta+1) w_{u_1 j} \\ &- \frac{\lambda_j^{\delta-\theta_2}}{1-e^{i\lambda_j}} \frac{\widetilde{U}_{1,\lambda_j n}(\theta_2-\delta+1)}{\sqrt{2\pi n}} - \frac{\lambda_j^{\delta-\theta_2}}{1-e^{i\lambda_j}} \frac{e^{i\lambda_j} Y_{3n}(\delta-\theta_2)}{\sqrt{2\pi n}} \Bigg|^2 . \end{aligned}$$

where $Y_{3n}(\delta-\theta_2) = \Delta^{\theta_2-\delta}u_{1t} I\{t \geq 1\}$.

We show

$$\frac{\lambda_j^{\delta-\theta_2}}{1-e^{i\lambda_j}} D_n(e^{i\lambda_j};\theta_2-\delta+1) = \frac{\lambda_j^{\delta}}{1-e^{i\lambda_j}} D_n(e^{i\lambda_j};-\delta+1) + O\left((\log n)^{-4}\right), \tag{53}$$

$$\frac{\lambda_j^{\delta-\theta_2}}{1-e^{i\lambda_j}} \frac{\widetilde{U}_{1,\lambda_j n}(\theta_2-\delta+1)}{\sqrt{2\pi n}} = \frac{\lambda_j^{\delta}}{1-e^{i\lambda_j}} \frac{\widetilde{U}_{1,\lambda_j n}(-\delta+1)}{\sqrt{2\pi n}} + O_p((\log n)^{-4}), \tag{54}$$

$$\frac{\lambda_j^{\delta-\theta_2}}{1-e^{i\lambda_j}} \frac{e^{i\lambda_j} Y_{3n}(\delta-\theta_2)}{\sqrt{2\pi n}} = \frac{\lambda_j^{\delta}}{1-e^{i\lambda_j}} \frac{e^{i\lambda_j} Y_{3n}(\delta)}{\sqrt{2\pi n}} + O_p(j^{\delta}(\log n)^{-4}), \tag{55}$$

then the required result follows in view of the order of the terms. (53) follows if, for $\alpha, \alpha' \in [-1/2, 1/2]$ such that $|\alpha - \alpha'| \leq (\log n)^{-5}$,

$$D_n(e^{i\lambda_j};\alpha) - D_n(e^{i\lambda_j};\alpha') = O(\lambda_j^{\alpha}(\log n)^{-4}).$$

where

$$D_n(e^{i\lambda_j};\alpha) = \sum_{k=0}^{n} \frac{(-\alpha)_k}{k!} e^{ik\lambda_j}.$$

Now

$$\begin{aligned} & D_n(e^{i\lambda_j};\alpha) - D_n(e^{i\lambda_j};\alpha') \\ = & \sum_{k=0}^{n} \frac{(-\alpha)_k}{k!} e^{ik\lambda_j} - \sum_{k=0}^{n} \frac{(-\alpha')_k}{k!} e^{ik\lambda_j} \\ = & \sum_{k=0}^{\infty} \left[\frac{(-\alpha)_k}{k!} e^{ik\lambda_j} - \frac{(-\alpha')_k}{k!} e^{ik\lambda_j}\right] - \sum_{k=n+1}^{\infty} \left[\frac{(-\alpha)_k}{k!} - \frac{(-\alpha')_k}{k!}\right] e^{ik\lambda_j}. \end{aligned}$$

The first term is (see PS p.12)

$$(1 - e^{i\lambda_j})^{\alpha} - (1 - e^{i\lambda_j})^{\alpha'} = O\left(\lambda_j^{\alpha}(\log n)^{-4}\right).$$

For the second term,

$$\begin{aligned} & \sum_{k=n+1}^{\infty} \left[\frac{(-\alpha)_k}{k!} - \frac{(-\alpha')_k}{k!}\right] e^{ik\lambda_j} \\ = & \sum_{k=n+1}^{\infty} \left[\frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)} - \frac{\Gamma(k-\alpha')}{\Gamma(k+1)\Gamma(-\alpha')}\right] e^{ik\lambda_j} \\ = & \frac{1}{\Gamma(-\alpha)} \sum_{k=n+1}^{\infty} \frac{\Gamma(k-\alpha) - \Gamma(k-\alpha')}{\Gamma(k+1)} e^{ik\lambda_j} \\ & + \left[\frac{1}{\Gamma(-\alpha)} - \frac{1}{\Gamma(-\alpha')}\right] \sum_{k=n+1}^{\infty} \frac{\Gamma(k-\alpha')}{\Gamma(k+1)} e^{ik\lambda_j}. \end{aligned}$$

The second terms is $O(n^{-\alpha} j^{-1} (\log n)^{-4})$, because $\frac{1}{\Gamma(-\alpha)} - \frac{1}{\Gamma(-\alpha')} = \frac{\alpha}{\Gamma(1-\alpha)} - \frac{\alpha'}{\Gamma(1-\alpha')} = O\left(|\alpha - \alpha'|\right)$. For the first term,

$$\frac{\Gamma(k-\alpha) - \Gamma(k-\alpha')}{\Gamma(k+1)} = k^{-\alpha-1} - k^{-\alpha'-1} + O(k^{-\alpha-2}) + O(k^{-\alpha'-2}).$$

And we have

$$\sum_{n+1}^{\infty} O(k^{-\alpha-2}) e^{ik\lambda_j} = O(n^{-\alpha-1}), \quad \sum_{n+1}^{\infty} O(k^{-\alpha'-2}) e^{ik\lambda_j} = O(n^{-\alpha'-1}) = O(n^{-\alpha-1}).$$

So

$$n^{-\alpha-1} = \lambda_j^{\alpha}(\log n)^{-4} O(m^{-\alpha} n^{-1} (\log n)^4) = o(\lambda_j^{\alpha}(\log n)^{-4}).$$

Next,

$$\begin{aligned}
\sum_{k=n+1}^{\infty}\left(k^{-\alpha-1}-k^{-\alpha'-1}\right)e^{ik\lambda_j} &= \sum_{k=n+1}^{\infty}k^{-\alpha-1}\left(1-k^{\alpha-\alpha'}\right)e^{ik\lambda_j} \\
&= \left(\alpha-\alpha'\right)\sum_{k=n+1}^{\infty}k^{-\alpha-1}k^{\overline{\alpha}_k}(\log k)e^{ik\lambda_j} \\
&\leq \left(\alpha-\alpha'\right)n^{-\alpha-1+\overline{\alpha}_k}(\log n)\max_N\left|\sum_{k=n+1}^{n+N}e^{ik\lambda_j}\right| \\
&= O(n^{-\alpha}j^{-1}(\log n)^{-4}) = o(\lambda_j^{\alpha}(\log n)^{-4}),
\end{aligned}$$

where the second equality follows because $k^{\alpha-\alpha'} = 1 + k^{\overline{\alpha}_k}(\log k)\left(\alpha-\alpha'\right)$ where $\overline{\alpha}_k \in [0, \alpha-\alpha']$, and the third inequality follows from Theorem 2.2 of Zygmund (1959) p.3 and the fact that $k^{-\alpha-1}k^{\overline{\alpha}_k}(\log k)$ is a nonnegative and nonincreasing function for large n. Hence we have shown (53).

(54) and (55) follow from the proof of Lemma 5.3 of SP. ■

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