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Collateral, liquidity and debt sustainability

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Abstract

We study the sustainability of public debt in a closed production economy where a benevolent government chooses fiscal policies, including haircuts on its outstanding debt, in a discretionary manner. Government bonds are held by domestic agents to smooth consumption over time and because they provide collateral and liquidity services. We characterize a recursive equilibrium where public debt amounts to a sizeable fraction of output in steady state and is nevertheless fully serviced by the government. In a calibrated economy, steady state debt amounts to around 84% of output, the government's default threshold is at around 94% of output, and the haircut on outstanding debt at this threshold is around 40%. Both reputational costs of default and contemporaneous costs due to lost collateral and liquidity are essential to generate these empirically plausible predictions.

JEL *classification*: E44, E62, H21, H63

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1 Introduction

The sustainability of sovereign debt has become a serious concern to investors and policy-makers during the recent financial crisis. In Europe fears of sovereign default and the associated rising borrowing costs have forced several countries to adopt severe fiscal austerity measures. Similarly, concerns about the sustainability of public debt have featured prominently in the debate on the *fiscal cliff* in the United States. They are also recurrent in Japan which faces the highest debt-to-GDP ratio among OECD countries.

The countries referred to above are developed economies where a substantial fraction of government debt is held domestically. Indeed, the sustainability of a country's sovereign debt when creditors are mostly domestic agents rather than international investors is not well understood. Empirical evidence on sovereign default incentives in such a situation is scarce. Theoretical academic work has mainly focused on the sustainability of *external* debt in developing economies – the empirically relevant case before the crisis. Against this background, the present paper seeks to contribute to the understanding of *internal* debt sustainability by studying the government's default incentives in a dynamic closed economy.

Our core framework is the standard model of optimal fiscal policy under discretion, pioneered by Lucas and Stokey (1983). We amend this core model by allowing the government to decide, in each period, on the fraction of outstanding debt it repays. Moreover, we introduce financial frictions. Firms must finance their wage bill in advance using collateralized loans; and the scale of profitable investment projects is limited by entrepreneurs' access to external finance. The existence of these frictions generates a role for government debt as *collateral* and, when it is tradable on a secondary market, as *private liquidity*.

Government revenue can only be generated by means of distortionary taxation, which gives rise to a time-consistency problem that manifests itself in two ways. First, since sovereign default effectively works as a lump-sum levy on households, there is an incentive for the government to default on its inherited liabilities unless such default is associated with costs. Second, the government's desire to minimize the interest payments on outstanding debt leads to an interest rate manipulation motive, which uniquely pins down the long-run level of debt. Key to this motive is the fact that public debt is priced not by risk-neutral international investors, but by domestic agents with finite intertemporal elasticity of substitution.

We analyze the implications of the government's lack of commitment in two steps. We first consider the case where the government decides sequentially about taxation, spending and its debt policy, but maintain the assumption that the government is committed to fully honor its outstanding debt. This allows us to uncover fundamental properties of optimal government policies in the face of the collateral and liquidity frictions constraining private agents, and to study the determination of steady state debt. In particular, we show that the steady state level of debt in our model is strictly positive, unlike in models which abstract from financial constraints and predict negative or zero long-run debt (Aiyagari, Marcet, Sargent, and Seppala, 2002; Debortoli and Nunes, 2013).

In a second step, we consider the case where the government's lack of intertemporal commitment also extends to the repayment of its debt. Two central features of our model, which differentiate our work from most previous research on sovereign default, are that government debt is held domestically and that we allow for fractional repayment rather than a binary default decision.[1] This changes the nature as well as the costs of government default. That debt is held by domestic agents implies that sovereign default does not involve a resource transfer between the domestic economy and the rest of the world but instead between the government and the private sector. That default can take the form of a fractional repayment implies that the government's optimal *haircut* decision will be determined endogenously such as to balance marginal benefits and costs of that policy.

We characterize optimal policies in a recursive equilibrium where the government defaults if its inherited debt exceeds an endogenously determined threshold level. Since public debt plays an essential role as collateral and as a source of liquidity, default leads to repercussions for financial intermediation. In detail, by reducing the amount of public debt available as collateral and liquidity, default induces costs that are proportional to the size of the haircut. Moreover, in line with the open-economy literature on sovereign debt (e.g. Arellano, 2008), we assume that default is also associated with fixed costs that result from the government's

[1] Both features have empirical support: On the basis of historical data for the period 1900-2010, Reinhart and Rogoff (2011) report that domestic debt accounts for about two thirds of overall public debt, and for even more in advanced economies. Cruces and Trebesch (2013) study sovereign debt restructurings between 1970 and 2010 and find an average sovereign haircut of 37%.

temporary exclusion from the bond market. During this period, outstanding bonds can no longer be traded on the secondary market and thus lose their liquidity value. Whereas the government is excluded from the primary bond market, it can still sell debt in the form of loans. Accordingly, in line with empirical evidence, the government is not forced to run a balanced budget during the exclusion spell. Loans to the government constitute pledgeable collateral, but – owing to their non-tradability – they do not provide liquidity. The government balances the costs of default against the additional tax distortions under full repayment. Since the latter are increasing in debt, there is a maximum sustainable level of debt, corresponding to more than 90% of output in our calibrated economy. For levels of debt in excess of this *fiscal limit*, the discretionary government optimally decides to exercise its default option. At the fiscal limt, the implied haircut is at about 40%. For higher levels of debt, the optimal haircut grows in line with the level of debt, resulting in a constant post-default level of debt.

Our model shares its focus on debt sustainability with the vast literature on sovereign debt and default. Following the seminal approach to international lending and sovereign default by Eaton and Gersovitz (1981), quantitative models have analyzed the dynamics of sovereign debt and default in small open economies (Aguiar and Gopinath, 2006; Arellano, 2008). There, debt is held externally, fiscal policy is largely absent, governments decide about default in a discretionary fashion, and costs of default are exogenous. Notable recent exceptions include the studies by Cuadra, Sanchez, and Sapriza (2010) who examine the role of fiscal policy, Mendoza and Yue (2012) who assess business cycle implications in an environment with endogenous default costs, and Adam and Grill (2012) who analyze optimal sovereign default as the solution to a Ramsey plan. D'Erasmo and Mendoza (2012) and Juessen and Schabert (2012) analyze the incentives for default on domestic debt. However, different from our paper, D'Erasmo and Mendoza (2012) focus on redistributive implications, while Juessen and Schabert (2012) consider a setup with risk neutral agents and exogenous default costs. Finally, Sosa-Padilla (2012) considers a framework with both external and domestic debt, where households face a static decision problem, domestic debt is held by risk neutral bankers, and the default decision is binary.

Our model emphasizes the role of endogenous default costs in the presence of financial frictions that can be mitigated by the issuance of public debt. This latter feature connects our paper to models with incomplete markets in the tradition of Aiyagari (1994). In this vein, Woodford (1990) and Holmstrom and Tirole (1998) show how public debt can help to relax financial constraints, while Aiyagari and McGrattan (1998) and Angeletos, Collard, Dellas, and Diba (2013) explore implications for optimal policy under commitment. Brutti (2011) and Gennaioli, Martin, and Rossi (2013) study sovereign default in three-period economies where sovereign default destroys firms' ability to insure against idiosyncratic shocks or the balance sheets of domestic banks, respectively. They find that financial frictions can render sizeable government debt levels sustainable even in the absence of reputational costs of default. Our paper examines the long-run implications of financial frictions on the government's default incentives in a fully dynamic environment, showing that reputational (fixed) costs of default are critical to generate sustainability of public debt in infinite horizon economies with fractional default.

Methodologically, our work is related to a number of recent papers invoking the optimal policy paradigm to study the determination of public debt under optimal discretionary fiscal policy. In a model without capital and with exogenous government expenditure, Krusell, Martin, and Rios-Rull (2006) uncover a multiplicity of steady states that are similar to those under full commitment. Considering endogenous government expenditure instead, Debortoli and Nunes (2013) establish convergence to zero long-run debt as a robust outcome driven by the government's interest rate manipulation motive. Our model nests their economy as a special case and inherits a generalized interest rate manipulation motive as an important force shaping the conduct of policy. Finally, in models with capital, public debt under discretionary fiscal policy has recently been studied by Klein, Krusell, and Rios-Rull (2008) and Ortigueira, Pereira, and Pichler (2012), among others.

The rest of this paper is organized as follows. In Section 2 we lay out our model economy. In Section 3 we examine optimal discretionary fiscal policy under the assumption of commitment to full debt repayment. In Section 4 we present our main results allowing for strategic default. We conclude in Section 5.

2 The Model

Our model builds upon the Lucas-Stokey real production economy with endogenous government spending studied by Debortoli and Nunes (2013). We extend their model by introducing financial frictions, which generate a role for public debt as a source of collateral and liquidity, and by allowing for outright default on government debt in the form of a fractional repayment decision. The economy is populated by households, firms and a government. There is a single non-storable output good, which is either consumed by households or transformed at a unitary rate into a public good by the government. Time is discrete.

2.1 Households

There is a continuum of measure one of identical, infinitely-lived households. The preferences of a representative household $j \in [0,1]$ are given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t^j, 1 - n_t^j, g_t), \tag{1}$$

where $\beta \in (0,1)$ is a time discount factor, c_t^j and n_t^j denote consumption and labor effort of household j, and g_t denotes the level of public good provision. The period utility function $u(\cdot)$ is assumed to be additively separable in its three arguments and twice continuously differentiable, with partial derivatives $u_c > 0$, $u_{cc} < 0$, $u_l > 0$, $u_{ll} \leq 0$, $u_g > 0$ and $u_{gg} \leq 0$.

Each household is composed of three types of members: workers, bankers and entrepreneurs.[2] Workers supply labor to competitive firms; the other agents either become bankers or get access to an entrepreneurial investment technology. The assignment to these two activities is stochastic; an individual agent becomes banker with probability $1 - \theta$ and entrepreneur with probability θ, respectively.

Household j enters period t with a stock of b_t^j government bonds. Initially, all bonds are held by bankers and entrepreneurs, with each of them holding the same amount b_t^j. Then, the household members separate, and individuals learn their type (banker or entrepreneur) before the government's policy decisions are announced.

[2]Each household comprises a continuum $[0,1]$ of workers and a continuum $[0,1]$ of agents who become either bankers or entrepreneurs.

Competitive firms and bankers

There is a continuum of measure one of perfectly competitive firms that take prices and wages as given. They have access to a production technology that transforms labor services into consumption goods at a unitary rate. Specifically, the technology allows the representative firm to produce

$$y_t^1 = \tilde{n}_t, \tag{2}$$

where $\tilde{n}_t$ denotes labor hired by the firm. Production is subject to a moral hazard problem which, in the absence of monitoring, makes it impossible for firms to pledge funds to workers and outside creditors. Firms must therefore finance their wage bill in advance, and they can do so using intra-period loans from financial intermediaries (bankers), who act as delegated monitors. In order to meet the firms' working capital requirement, bankers issue deposits contracts, d_t, to outside creditors (i.e., to workers from households other than their own; cf. Gertler and Karadi, 2011). However, although banks have a greater capacity to pledge funds to outside creditors, they are also subject to moral hazard. They can therefore only issue deposits if they are able to post collateral to cover at least a fraction $\xi^c \in (0, 1)$ of the amount issued. Government bonds are the sole source of collateral available to bankers, such that the collateral constraint facing a representative banker from household j is given by

$$d_t^j \leq \frac{\rho_t b_t^j}{\xi^c}, \tag{3}$$

where d_t^j denotes the deposits issued and ρ_t denotes the repayment rate on government bonds (see below). Note that the timing assumption underlying the collateral constraint (3) implies that the collateral can be seized by bank depositors at the end of the period when the bond price is equal to the repayment rate. Note also that the banking sector is competitive, and hence intra-period loans do not carry a positive interest rate unless the supply of loans is depressed by the bankers' availability of collateral. Aggregating across firms and bankers, equilibrium in the bank-intermediated market for intra-period loans implies that the economy's aggregate wage bill is constrained by

$$w_t \tilde{n}_t \leq \frac{(1-\theta)\rho_t b_t}{\xi^c}. \tag{4}$$

Entrepreneurs

Entrepreneurs have access to a profitable investment technology. Specifically, they can invest in projects that deliver a gross return $R > 1$ per unit of investment (both in consumption goods). Denoting by X_t^j the investment scale of the representative entrepreneur from household j, the investment technology is characterized by

$$y_t^{j,2} = R X_t^j. \tag{5}$$

Similar to the operation of banks, there is a moral hazard problem that limits entrepreneurs' access to external finance. As a consequence, internal investment, x_t^j, is necessary to attract external funds, e_t^j. External funds take the form of intra-period loans from workers and bankers that pay zero interest as there is no discounting within the period. To raise the consumption goods required for internal investment, entrepreneurs sell their liquid assets (government bonds) on the secondary market; hence, $x_t^j = z_t b_t^j$, where z_t denotes the bond's market price.[3] They then augment their internal funds by acquiring external funds subject to the constraint

$$e_t^j \leq \frac{x_t^j}{\xi^l}, \tag{6}$$

where $\xi^l \in (0,1)$. Constraint (6) is always binding when $R > 1$, resulting in an investment scale of $X_t^j = \frac{1+\xi^l}{\xi^l} z_t b_t^j$ per entrepreneur.

Aggregation

After production in the competitive and entrepreneurial sector has taken place, workers, bankers and entrepreneurs transfer their earnings back to the household. Consumption-savings decisions are then made at the household level; hence there is perfect consumption insurance within

[3] We assume that the secondary market for government debt is large enough to absorb the supply of bonds from entrepreneurs. Formally, $w_t n_t + (1 - w_t)\tilde{n}_t \geq \theta z_t b_t$, where variables without superscript denote economy-wide aggregates.

households.[4] Aggregating over household members, the total income of household j in period t is given by

$$I_t^j \;=\; w_t n_t^j + (1-w_t)\tilde{n}_t^j + \theta(R-1)\frac{1+\xi^l}{\xi^l} z_t b_t^j. \tag{7}$$

The first term on the right-hand side denotes the wage income earned by workers, the second term denotes overall profits between firms and bankers in the competitive sector, and the third term denotes entrepreneurs' net return from investment. Note that (7) does not include income from maturing government debt b_t^j.

2.2 The government

The government is benevolent and maximizes the utility (1) of the representative household. Its policy tools are a proportional income tax τ_t, the level of public good provision g_t, the issuance of new debt B_{t+1}, and the repayment rate on outstanding government debt, $\rho_t \in [0,1]$. The income tax is uniform across the different sources in (7); bond income is not taxed. The government's budget constraint reads

$$g_t + \rho_t B_t \leq (1-\tau_t)I_t + q_t B_{t+1}, \tag{8}$$

where I_t denotes aggregate income in the private sector and q_t denotes the price of a newly issued government bond that promises one unit of beginning-of-period wealth in $t+1$ but is subject to default risk.

The government cannot commit to following a fixed policy path over time. It can, however, make credible policy announcements within a given time period.[5] The period-t government announces its current policy choices $(\tau_t, g_t, B_{t+1}, \rho_t)$ before production takes place, but implements these policies only afterwards, concurrent with households' consumption-savings decisions. This timing structure implies that the government is a Stackelberg leader vis-à-vis the private sector. Table 1 summarizes the timing of events in any given period t.

[4]This property allows us to introduce financial frictions while keeping an otherwise standard representative agent framework (cf. Gertler and Karadi, 2011).

[5]Using the terminology of Ortigueira (2006), we assume intra-temporal commitment while abstracting from inter-temporal commitment.

Table 1: Timing of events in period t

1.	The household endows each of its bankers and entrepreneurs with b_t government bonds.
2.	The household members separate and individual types (banker or entrepreneur) are realized.
3.	The government announces its policies $(\tau_t, g_t, B_{t+1}, \rho_t)$.
4.	Bankers issue deposits, d_t, subject to collateral constraint (3) and make working capital loans to the competitive firms. The firms hire labor, $\tilde{n}_t$, subject to constraint (4). They produce $y_t^1 = \tilde{n}_t$ consumption goods.
5.	Entrepreneurs sell their government bonds to raise internal funds, x_t, and raise external funds, e_t, from workers and bankers subject to external finance constraint (6). They invest into projects of scale $X_t = x_t + e_t$, which return $y_t^2 = RX_t$ consumption goods.
6.	The government collects income taxes τ_t, transforms g_t units of the consumption good into a public good, repays a fraction ρ_t of the maturing debt B_t and issues new debt B_{t+1}. Households consume c_t and purchase newly issued government debt, b_{t+1}.

Our analysis of the policy problem faced by a government under discretion proceeds in two steps. We first characterize optimal fiscal policies under the assumption that the government can and does commit to fully honor its outstanding debt. We then examine the optimal fiscal policy when the government has the option to default and study the sustainability of government debt.

3 Full debt repayment

In this section we study optimal tax, spending and debt policies under the assumption that the government commits to fully honor its outstanding debt, $\rho_t = 1$ for all t. This allows us to isolate the effects of financial frictions on the model's optimal policy prescriptions, particularly the determination of government debt. We start by examining the optimal choices of households for given fiscal policies.

3.1 Private-sector equilibrium

Households in our model are atomistic and take prices $(w_t, z_t, q_t)_{t=0}^{\infty}$ and policies $(\tau_t, g_t, B_{t+1})_{t=0}^{\infty}$ as given. They choose consumption, labor supply, labor demand, and savings to maximize their objective function (1). Adopting recursive notation and dropping the superscript j, the optimization problem faced by the representative household reads

$$\begin{aligned}
\tilde{V}(b;\tau,g,B) &= \max_{c,n,\tilde{n},b'} u(c,1-n,g)+\beta\tilde{V}(b';\tau',g',B') \\
&\quad -\lambda\Big(c+qb'-(1-\tau)\left[wn+(1-w)\tilde{n}_t+\theta(R-1)\frac{1+\xi^l}{\xi^l}zb\right]-b\Big) \\
&\quad -\mu\Big(w\tilde{n}-\frac{(1-\theta)b}{\xi^c}\Big).
\end{aligned}$$

The first-order conditions are

$$\frac{u_l}{u_c} = (1-\tau)w, \tag{9}$$

$$\mu = (1-\tau)\frac{u_c(1-w)}{w} \geq 0, \tag{10}$$

$$q = \beta\frac{u_c'}{u_c}\left\{1+(1-\tau')\left[z'\theta\frac{(R-1)(1+\xi^l)}{\xi^l}+(1-\theta)\frac{(1-w')}{\xi^c w'}\right]\right\}. \tag{11}$$

Condition (9) equates the marginal rate of substitution between leisure and consumption to the net wage. Condition (10) shows that the collateral constraint, if it is binding ($\mu > 0$), creates a wedge between the wage and the marginal product of labor; thus, whenever collateral is scarce, competitive bankers earn positive profits. The Euler equation (11) highlights the three roles played by government bonds in our model: (i) bonds allow households to shift consumption over time; (ii) bonds provide liquidity and hence allow households to increase entrepreneurial investment; and (iii) bonds are a source of collateral to bankers.

Note that in a private-sector equilibrium, since there is no discounting within the time period, the beginning-of-period price of a government bond must equal its repayment rate, that is, $z = \rho = 1$ in all periods. Introducing the liquidity premium π and the collateral premium ϕ,

$$\pi = \theta(1-\tau)(R-1)\frac{1+\xi^l}{\xi^l}, \tag{12}$$

$$\phi = (1-\theta)(1-\tau)\frac{(1-w)}{w\xi^c}, \tag{13}$$

we can thus write Euler equation (11) as

$$q \;=\; \beta\frac{u_c'}{u_c}\left(1+\pi'+\phi'\right). \tag{14}$$

Labor market clearing implies $\tilde{n}=n$ in a symmetric equilibrium, such that the household's budget constraint reads

$$c+qb'=(1-\tau)n+\left[1+(1-\tau)r\right]b,$$

where $r=\theta(R-1)\frac{1+\xi^l}{\xi^l}$. Eliminating the tax rate τ and the bond price q using (9) and (14), this expression can be rearranged as

$$u_c c+\beta u_c'(1+\pi'+\phi')b'=\frac{u_l}{w}n+u_c(1+\pi)b. \tag{15}$$

Finally, note that bond market clearing requires that $B=b$ in a private-sector equilibrium.

3.2 Optimal fiscal policy

Under our maintained assumption of lack of commitment, the government in a given time period can choose policy variables for that period but it cannot control policy variables for the future. To characterize the optimal policies we adopt a *primal approach.* Accordingly, the incumbent government directly chooses consumption c, labor n, and debt issuance b' for the current period, taking as given the policy rules $\{\hat{c},\hat{n},\hat{b}\}$ employed by future governments, and subject to the requirement that its choices are consistent with a private-sector equilibrium.

Inspection of implementability constraint (15) shows that, when $\rho=\rho'=1$, the aggregate state vector in our model consists of only one variable, b. The policy rules $\{\hat{c},\hat{n},\hat{b}\}$ are thus of the form $c=\hat{c}(b)$, $n=\hat{n}(b)$, and $b'=\hat{b}(b)$. Via equations (9), (12) and (13), these rules further imply decision rules for the tax rate, $\hat{\tau}(b)$, the liquidity premium, $\hat{\pi}(b)$, and the collateral premium, $\hat{\phi}(b)$, respectively. Plugging these functions into Euler equation (14), we can write the bond pricing function Q as

$$Q(u_c,b') \;=\; \beta\frac{u_c(\hat{c}(b'))}{u_c}\Big(1+\hat{\pi}(b')+\hat{\phi}(b')\Big). \tag{16}$$

Note that, as households have a finite intertemporal elasticity of substitution, the bond price depends on the current and future marginal utility of consumption.[6] Finally, note that the wage rate falls below labor productivity if firms' access to working capital loans is strictly constrained by the bankers' pledgeable collateral. This allows us to write the wage rate as a function

$$\omega(b,n) = \begin{cases} 1 & \text{if } (1-\theta)b > \xi^c n \\ \frac{(1-\theta)b}{\xi^c n} & \text{otherwise.} \end{cases} \tag{17}$$

Using the aggregate resource constraint to substitute for public consumption in the household utility function (1), the discretionary government's optimization problem under commitment to full debt repayment is then given by

$$\begin{aligned} V(b) = & \max_{c,n,b'} u(c, 1-n, n+rb-c) + \beta V(b') \\ & +\gamma \left(u_c c + u_c Q(u_c, b')b' - \frac{u_l}{\omega(b,n)} n - u_c(1+\pi)b \right), \end{aligned} \tag{18}$$

where γ is a non-negative Lagrangian multiplier and $V(b')$ is the continuation value function.

3.3 Recursive equilibrium

We study the government's optimal policy in a recursive equilibrium where agents choose their actions sequentially. A formal definition of the equilibrium is as follows.

Definition 1. *A recursive equilibrium under commitment to full debt repayment is a set of policy functions $\{\hat{c}, \hat{n}, \hat{b}\}$, a value function V and a bond pricing function Q such that:*

(i) given the value function V and the bond pricing function Q, the policy functions $\{\hat{c}, \hat{n}, \hat{b}\}$ solve the government's optimization problem (18);

(ii) given the policy functions $\{\hat{c}, \hat{n}, \hat{b}\}$, the bond pricing function Q satisfies (16);

[6] This property sets our model apart from related papers that determine bond prices on the basis of quasi-linear utility; see e.g., Juessen and Schabert (2012) and Sosa-Padilla (2012) who also consider environments with domestic debt. Similarly, in models of externally held debt, bonds are generally priced by risk neutral international investors.

(iii) given the policy functions $\{\hat{c}, \hat{n}, \hat{b}\}$, the value function satisfies the Bellman equation

$$V(b) = u(\hat{c}(b), 1 - \hat{n}(b), \hat{n}(b) + rb - \hat{c}(b)) + \beta V(\hat{b}(b)).$$

The first-order conditions characterizing the policy functions in a recursive equilibrium under full debt repayment are presented in the Appendix. Of particular interest is the *generalized Euler equation* (GEE) characterizing the optimal debt policy. It is given by

$$\gamma' \left\{ u_c'(1+\pi') - \frac{u_l' n'}{(w')^2}\omega_1' \right\} - u_g' r = \gamma u_c'(1 + \pi' + \phi') \left\{1 + \varepsilon_{b'}^q\right\}, \tag{19}$$

where $\varepsilon_{b'}^q$ denotes the elasticity of the bond price q with respect to changes in debt issuance b'.[7] The GEE equates the marginal cost of entering the next period with a higher stock of outstanding debt to the marginal benefit of relaxing implementability constraint (15) via issuing additional debt. For an economy without any role for government debt as collateral or liquidity, the case studied by Debortoli and Nunes (2013), the GEE simplifies to

$$\gamma' = \gamma\Big(1 + \varepsilon_{b'}^q\Big). \tag{20}$$

A steady state in their model is hence characterized by either $\gamma^* = 0$ or ${\varepsilon_{b'}^q}^* = 0$. The first case corresponds to an undistorted steady state where the government holds enough assets to implement the first-best allocation. The second case corresponds to a distorted steady state where either $Q_2^* = 0$, such that the bond price is locally invariant to changes in debt, or $b^* = 0$, such that changes of the bond price do not have budgetary effects.

Debortoli and Nunes (2013) show with a simple analytical example, as well as more general numerical examples, that steady states with $b^* = 0$ and $\hat{c}_b(b^*) > 0$, that is, a locally increasing consumption policy function, are generic in their economy.[8] This result is rooted in the *interest rate manipulation motive* faced by the government under lack of commitment. To understand the underlying intuition, note first that the bond pricing function in the absence of collateral

[7] Formally, $\varepsilon_{b'}^q = \frac{Q_2(u_c,b')b'}{Q(u_c,b')} = \frac{u_{cc}(b')}{u_c(b')}\hat{c}_b(b')b' + \frac{\left(\hat{\pi}_b(b')+\hat{\phi}_b(b')\right)b'}{\left(1+\hat{\pi}(b')+\hat{\phi}(b')\right)}$.

[8] Key to the emergence of an increasing consumption policy function is the fact that government expenditure is endogenous.

or liquidity premia is simply given by

$$Q(u_c, b') = \beta \frac{u_c(b')}{u_c}. \tag{21}$$

Accordingly, an increase in current consumption c raises q and thus reduces the interest rate on newly issued debt. At the same time, if the change was anticipated, it reduces the bond price one period in advance, q_{-1}. A government choosing an optimal policy path takes the effects in both time periods into account. A discretionary government, however, takes q_{-1} as given. Ignoring the (adverse) effect of a higher c on the past bond price q_{-1}, it chooses a higher level of consumption than prescribed by the optimal policy path.

Moreover, the incumbent government correctly foresees that its successor faces the same discretionary incentive to increase consumption. It therefore seeks to influence its successor's behavior via manipulation of the future state variable b'; it issues debt such as to induce the future government to *decrease* future consumption $\hat{c}(b')$. Given a consumption policy function that is increasing in debt, $\hat{c}_b(b') > 0$, this is achieved by decumulating debt. The incentive to decumulate debt is a recurrent phenomenon as long as there is a positive stock of debt outstanding, such that a steady state finally emerges at $b^* = 0$.

In our generalized model, where government bonds provide liquidity and collateral services, the zero long-run debt result no longer obtains. Instead, a positive steady state level of debt emerges generically.

Proposition 1. *If government bonds provide liquidity services, $r > 0$, or production in the competitive sector is subject to a collateral constraint, $\xi^c > 0$, the steady state features a strictly positive level of government debt, $b^* > 0$.*

Under a collateral role for government debt, positive steady state debt emerges by construction, for otherwise zero debt would imply zero production. However, a positive steady state level of debt emerges already if there is only a liquidity role but no collateral role for public debt. The intuition behind this finding is best understood as follows. If public debt has a role as private liquidity, the bond price includes a liquidity premium,

$$Q(u_c, b') = \beta \frac{u_c(b')}{u_c}(1 + \hat{\pi}(b')). \tag{22}$$

The current government again seeks to increase current bond prices q via manipulation of the future state b'. This is now achieved for changes in b' which induce an increase in $u_c'(1+\pi')$. While u_c' is decreasing in $\hat{c}(b')$, the opposite is true for the liquidity premium since $\pi_c = -\pi\frac{u_{cc}}{u_c} > 0$. Accordingly, there are conflicting motives for the manipulation of $\hat{c}(b')$ because, given $\hat{c}_b(b') > 0$, a decumulation of debt increases future marginal utility u_c' but decreases the future liquidity premium π'. These conflicting motives balance each other at a positive level of debt.

The finding of positive steady state debt might suggest that the accumulation of moderate levels of debt has positive welfare effects. The following Proposition examines this property.

Proposition 2. *The accumulation of moderate levels of debt has positive welfare effects if the return to investment in the entrepreneurial sector is sufficiently high,*

$$r > \frac{u_c}{u_l}\left(\frac{1-\frac{u_l}{u_g}}{\frac{u_l}{u_g}-\frac{u_{ll}}{u_l}n}\right), \tag{23}$$

or production in the competitive sector is subject to a collateral constraint, $\xi^c > 0$. Conversely, if $r = 0$ and $\xi^c = 0$, social welfare is monotonically decreasing in debt.

For the government's value function to be increasing in debt, the marginal benefit from a relaxation of the collateral constraint and/or from increased liquidity must exceed the marginal cost from increased taxation. In models of optimal fiscal policy under discretion, the marginal cost from taxation is generally increasing in the level of debt, suggesting that the value function in our model is of an inverted U-shape. Our numerical results presented in the following Section confirm this conjecture.

3.4 A calibrated economy

We now study optimal discretionary fiscal policy in a calibrated economy. The purpose of this exercise is to illustrate the key quantitative properties of optimal fiscal policy in a plausible economic environment.

Calibration

We consider an instantaneous utility function u that is additively separable and allows for curvature in all its arguments,

$$u(c, 1-n, g) = (1-\omega_g)\left[\omega_c \frac{c^{1-\sigma_c}-1}{1-\sigma_c} + (1-\omega_c)\frac{(1-n)^{1-\sigma_l}-1}{1-\sigma_l}\right] + \omega_g \frac{g^{1-\sigma_g}-1}{1-\sigma_g}, \tag{24}$$

where ω_c and ω_g denote preference weights on private and public consumption and σ_c, σ_l and σ_g are elasticities. The parameter values are selected as follows. The three elasticities σ_c, σ_l and σ_g are each set to the value 2, which is in the middle of the parameter range typically considered in the macroeconomic literature. The preference weights are chosen such that, in the model's steady state, $g^*/c^* = 0.25$ and $n^* = 0.3$; the resulting values are $\omega_c = 0.15$ and $\omega_g = 0.015$. The collateral parameter is set to $\xi^c = 0.4$, corresponding to a leverage ratio of 2.5. The parameter θ governs the relative importance of production in the competitive and entrepreneurial sectors. Beyond that, the parameters R, θ and ξ^l matter only jointly, as determinants of the return to entrepreneurial investment, r. The individual parameter values are selected in line with evidence from the Survey of Consumer Finances (SCF). As discussed in Moskowitz and Vissing-Jorgensen (2002), the SCF reports a median of the distribution of capital gains in private business investment of roughly 7%. This motivates our choice of $r = 0.07$. For simplicity, we set $\theta = 0.25$ and $\xi^l = \xi^c = 0.4$, implying $R = 1 + \frac{r\xi^l}{\theta(1+\xi^l)} = 1.08$. Finally, we choose the discount factor $\beta = 0.92$ to match an annual risk-free real interest rate of about 3% in the presence of a steady state liquidity premium. Our parameter choices are summarized in Table 2. For given parameters, we solve the model numerically, using a combination of standard projection and dynamic programming techniques.[9] In the following we present results for the model's steady state, policy functions and the social welfare function.

Steady state

The steady state values of key endogenous variables are presented in Table 3. Output in the competitive sector is roughly equal to $y^{1^*} = 0.3$, in line with our calibration target. Value added in the entrepreneurial sector is significantly smaller, $y^{2^*} = 0.019$, such that total output

[9]Details on our computational algorithm and the computer code are available upon request.

Table 2: Parameter values

Parameter	Value	Description
σ_c	2	elasticity of private consumption
σ_g	2	elasticity of public consumption
σ_l	2	elasticity of leisure
ω_c	0.15	weight of consumption (priv.+publ.) vs. leisure
ω_g	0.015	weight of public vs. private consumption
ξ^c	0.4	inverse of leverage ratio
r	0.07	private equity premium
θ	0.25	share of entrepreneurial investors
ξ^l	0.4	inverse of leverage ratio
R	1.08	gross return on investment projects
β	0.92	discount factor

is given by $y^* = 0.3220$. Private and public consumption amount to 80% and 20% of total output, respectively ($c^* = 0.2578$, $g^* = 0.0642$). The steady state level of debt is positive, in line with Proposition 1. In particular, our parameter choices imply a sizeable steady state debt level of $b^* = 0.2712$, which corresponds to a debt-to-GDP ratio equal to 84%. The steady state bond price $q^* = 0.97$ implies an annual interest rate close to our calibration target of 3%. The steady state tax rate is $\tau^* = 22.5\%$. Finally, the collateral constraint is not binding at the steady state, and the wage rate is thus equal to labor productivity, $w^* = 1$. Accordingly, the main driver behind the positive level of steady state debt is its liquidity role.

Table 3: Steady state values

Variable	Steady state
y^1	0.3030
y^2	0.0190
y	0.3220
c	0.2578
g	0.0642
b	0.2712
b/y	0.8422
q	0.9699
τ	0.2248
w	1.0000

Policy functions and welfare

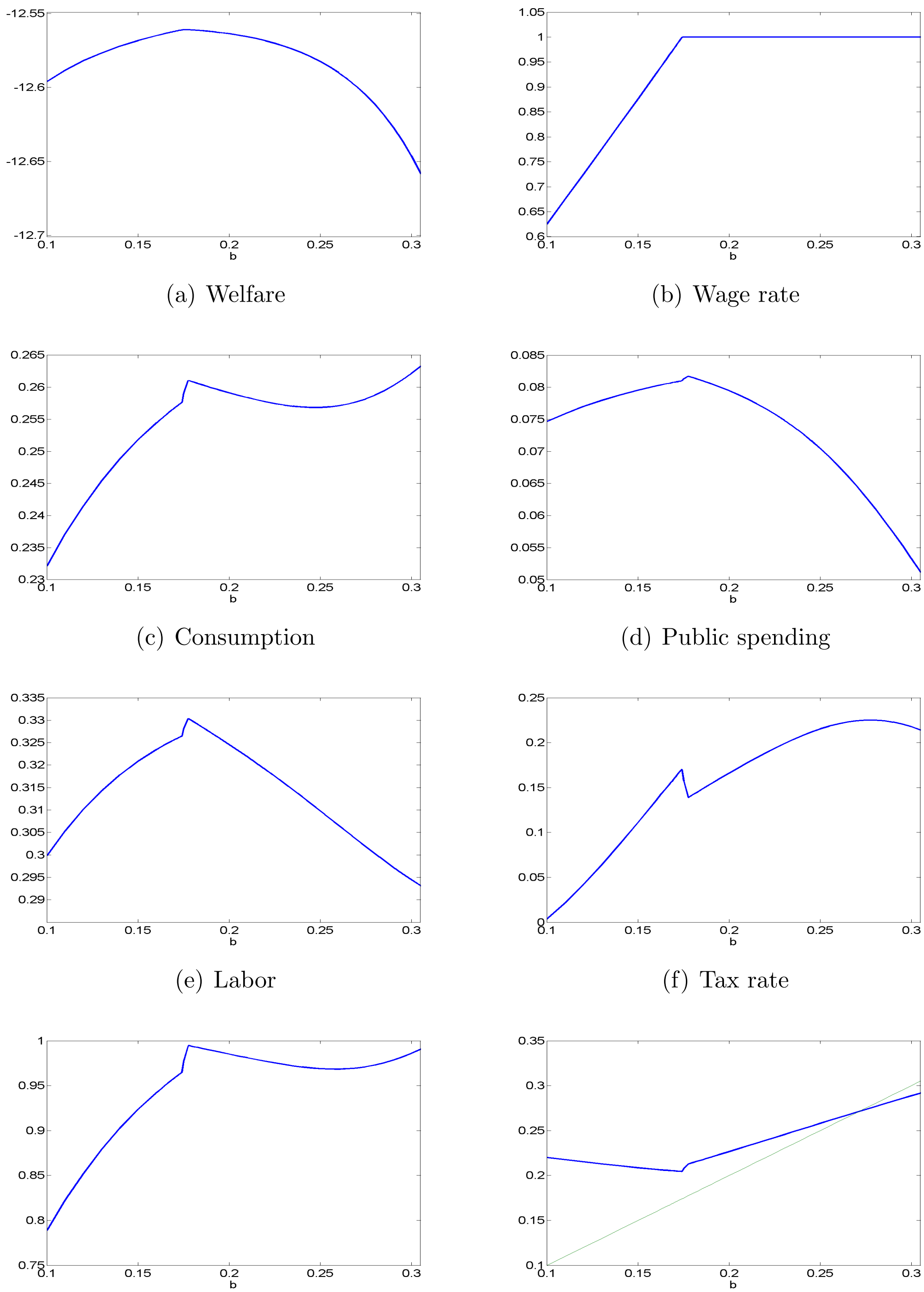
The optimal policy functions, displayed in Figure 1, are highly non-linear with kinks in the region of the state space where the collateral constraint kicks in. In fact, we can partition the state space $\mathbb{B} = [\underline{b}, \bar{b}]$ into three regions that differ in how optimal policies react to variations in the inherited debt level. In the first region, $\mathbb{B}^1 = [\underline{b}, b_1)$, debt is so low that the collateral constraint is strictly binding. In the second region, $\mathbb{B}^2 = [b_1, b_2)$, the collateral constraint is non-binding under optimal policies, but its existence nevertheless affects the government's optimal policy trade-offs. In the third region, $\mathbb{B}^3 = [b_2, \bar{b}]$, the collateral constraint has no distortionary effects on optimal policies.[10]

We now describe each region in detail, starting with $\mathbb{B}^3$. A first observation is that the steady state b^* is contained in this region. Public debt thus converges to a level that is sufficient to fully satiate the demand for collateral. Indeed, the optimal policy coincides with the one that would obtain in an otherwise identical model without collateral constraint ($\xi^c = 0$). The wage rate is constant and equal to labor productivity, $w = 1$. Labor supply and public consumption are monotonically decreasing in debt. By contrast, private consumption and the bond price are non-monotonic; they are decreasing in b for relatively low levels of debt in $\mathbb{B}^3$ and increasing in b for high levels of debt. In particular, we have $\hat{c}_b(b^*) > 0$. A reverse pattern is found for the tax rate, which reflects the government's effort to sustain liquidity premia $\pi = (1-\tau)r$ and thus bond prices. The debt policy function is increasing in b with a slope below one, indicating that the steady state b^* is stable. Finally, throughout $\mathbb{B}^3$, social welfare is monotonically decreasing in b. This illustrates that the adverse tax distortion effect resulting from a higher level of indebtedness dominates the positive liquidity effect.

In region $\mathbb{B}^2$ the collateral constraint is still non-binding under optimal policies, but its existence already distorts the optimal policy trade-off. In particular, the government relies more on taxation relative to debt issuance to finance public spending. In doing so, it depresses labor supply and prevents the collateral constraint from becoming binding. Hence, taxation is attractive relative to debt issuance because the income tax does not distort the equilibrium labor allocation. This is because, even under lower taxes, labor would still be depressed by the

[10] In our calibrated economy, $b_1 = 0.1745$ and $b_2 = 0.1800$, which corresponds to approximately 54% and 56% of steady state output, respectively.

Figure 1: Policy functions under commitment to full debt repayment



(a) Welfare

(b) Wage rate

(c) Consumption

(d) Public spending

(e) Labor

(f) Tax rate

(g) Bond price

(h) Debt issuance

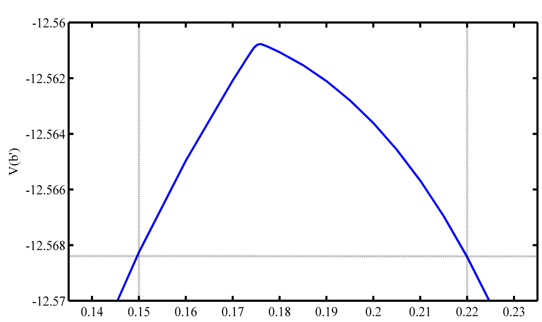
scarcity of pledgeable collateral.

Finally, in region $\mathbb{B}^1$ the collateral constraint is strictly binding. The scarcity of collateral constrains labor demand, such that the market clearing wage rate falls short of labor productivity ($w < 1$) and output, private consumption and public consumption are depressed. Since the collateral constraint is less stringent the higher the initial debt stock, the policy functions $\hat{n}$, $\hat{w}$, $\hat{c}$ and $\hat{g}$ are increasing in b. The same is true for the equilibrium bond price under the optimal debt policy.

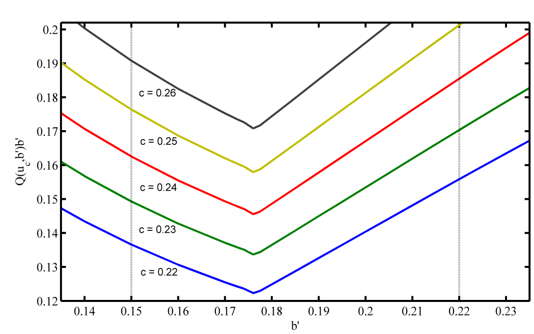
The debt Laffer curve

Inspection of the debt policy function $\hat{b}$ allows for the further observation that, independent of the initial debt stock, the government always issues an amount of bonds that is sufficient to ensure a non-binding collateral constraint in the future. To understand the intuition behind this finding, first note that the social welfare function has an inverted U-shape as prescribed by Proposition 2. Specifically, the welfare function is initially upward-sloping in region $\mathbb{B}^1$, where the collateral constraint is strictly binding, and later downward-sloping. Given this inverted U-shape, for each possible choice $b' \in \mathbb{B}^1$ there hence exists an alternative choice $\tilde{b}' > b'$ such that $V(\tilde{b}') = V(b')$. Since b' and $\tilde{b}'$ deliver the same continuation payoff to the government, a necessary condition for b' to be optimal is to generate a higher current revenue from debt creation compared to $\tilde{b}'$. Formally, given the optimal choice of current consumption, $c = \hat{c}(b)$, the debt issuance b' can be an optimal choice only if $Q(u_c, b')b' > Q(u_c, \tilde{b}')\tilde{b}'$. Figure 2 shows that

Figure 2: Welfare and the debt Laffer curve



(a) Value function



(b) Revenue from debt issuance

this is generically not the case in our calibrated economy. In particular, the figure shows that debt choices $b' \in \mathbb{B}^1$ generate a *lower* current revenue than the corresponding choices $\tilde{b}' \in \mathbb{B}^3$.[11] This pattern reflects an underlying debt Laffer curve – a situation where a marginal increase in the quantity of debt issued is associated with a reduction in the revenue for the government from that operation. Facing declining bond prices $Q(u_c, b')$ associated with suboptimal, low choices of $b' \in \mathbb{B}^1$, the government thus responds by an aggressive debt policy in order to escape the Laffer curve region.[12]

4 Fractional default

We now examine the properties of optimal fiscal policy when the government does no longer commit to full debt repayment. Instead, it decides in a discretionary manner on the fraction $\rho \in [0, 1]$ of outstanding debt it repays. Our focus is on the optimal fractional default decision and the maximum level of debt that can be sustained without default in equilibrium.

Understanding the optimal discretionary repayment policy requires consideration of the costs of default. There are two dimensions to these costs in our model. First, in line with much of the sovereign debt literature, we assume that there are reputational costs. Following a default, the government is excluded from the primary bond market, and outstanding bonds can no longer be traded on the secondary market.[13] The duration of the market exclusion is stochastic; with a constant probability α an excluded government can re-access the bond market in the next period. Unlike most of the literature, however, we assume that during the bond market exclusion the government can still sell debt in the form of loans.[14] The difference between

[11] As an example, consider an inherited level of debt $b = 0.1$ and the two alternative debt choices $b' = 0.15$ and $\tilde{b}' = 0.22$. These two choices deliver the exact same continuation welfare level $V(0.15) = V(0.22) = -12.5685$. Yet, the current revenue from issuing $\tilde{b}' = 0.22$ exceeds the current revenue from issuing $b' = 0.15$ for all possible values of c, including the optimal one at $c \approx 0.23$.

[12] Formally, the marginal revenue from issuing additional debt b' is given by $\frac{dQ(u_c,b')b'}{db'} = Q(u_c, b')\left\{1 + \varepsilon^q_{b'}\right\}$. Accordingly, there is a debt Laffer curve whenever $\varepsilon^q_{b'} < -1$. From the GEE (19), $-V_b(b') = \gamma u'_c(1 + \pi' + \phi')\left\{1 + \varepsilon^q_{b'}\right\}$. Since $\gamma u'_c(1 + \pi' + \phi') > 0$, it follows that $V_b(b') > 0$ if and only if $\varepsilon^q_{b'} < -1$. Hence, an aggressive debt policy that escapes the Laffer curve region implies a level of future debt such that $V_b(b') < 0$.

[13] Consistent with this assumption is the empirical evidence presented in Bai, Julliard, and Yuan (2012). These authors analyze Eurozone sovereign bond markets in the period 2006-2012 and find that secondary market liquidity has been significantly reduced during the recent crisis, with markets basically drying up in countries that received a bailout (Greece and Portugal).

[14] Note that the complete exclusion also from the primary market for debt considered in the literature (cf. Arellano, 2008) has the counterfactual implication of zero outstanding debt following a default.

bonds and loans lies in their tradability on the secondary market. While bonds are readily marketable, loans must be held to maturity. Loans can thus be used as collateral by bankers in the same way as government bonds, but they are not a source of liquidity for entrepreneurs.

Second, there are further contemporaneous costs associated with a default, since a haircut on bonds reduces the amount of pledgeable collateral available to private agents. If the haircut is large enough such as to make the bankers' collateral constraint binding, this induces output losses in the competitive sector. Note that the costs via reduced collateral depend on the size of the implemented haircut, whereas the repercussions of market exclusion are of a fixed cost nature.

4.1 Optimal fiscal policy

It is convenient to cast the incumbent government's optimal policy problem under the option to default as a two stage decision problem. The government first decides whether or not to repay the entirety of its outstanding debt. Conditional on this decision, the government then chooses its relevant policy instruments.

Optimal policies under the option to default depend not only on the level of the government's outstanding debt. Instead, it is also payoff-relevant whether maturing debt is in the form of bonds or loans and whether the government can issue bonds or not. We capture this by an indicator variable $s \in \{f, a, e\}$, where f indicates that bond markets are fully operational (i.e., both maturing and new government debt is in the form of bonds), a indicates that the government has only loans outstanding but can issue bonds on the primary market, and e indicates that the government is and remains excluded from the bond market.[15]

Define $V_f^o(b)$ as the value function for a government that has the option to default and starts the current period with b outstanding bonds. This value function satisfies

$$V_f^o(b) = \max\{V_f^{nd}(b), V_f^d(b)\}, \tag{25}$$

where $V_f^{nd}(b)$ is the value conditional on full repayment ($\rho = 1$) and $V_f^d(b)$ is the value condi-

[15]There is no need to consider mixed portfolios consisting of bonds and loans: Since bonds carry a liquidity premium, the government prefers to issue debt in the form of bonds whenever this is possible. The underlying pricing functions for bonds and loans are formally presented in the Appendix.

tional on partial default ($\rho < 1$). The no-default value function is the solution to

$$\begin{aligned} V_f^{nd}(b) &= \max_{c,n,b'} u(c, 1-n, n+rb-c) + \beta V_f^o(b') \\ &\quad + \gamma \left(u_c c + u_c Q^b(u_c, b')b' - \frac{u_l}{\omega(b,n)} n - u_c(1+\pi)b \right), \end{aligned} \tag{26}$$

where $\omega(b, n)$ and $Q^b(u_c, b')$ are the pricing functions for labor and newly issued bonds, respectively. Similar to bonds, loans promise one unit of consumption in the next period; they also serve as collateral but owing to their non-tradability offer no liquidity services. The government's value function under default is hence given by

$$V_f^d(b) = \max_{\rho \in [0,1]} \tilde{V}^d(\rho b), \tag{27}$$

where

$$\begin{aligned} \tilde{V}^d(\rho b) &= \max_{c,n,\ell'} u(c, 1-n, n-c) + \beta W^o(\ell') \\ &\quad + \gamma \left(u_c c + u_c Q^\ell(u_c, \ell')\ell' - \frac{u_l}{\omega(\rho b, n)} n - u_c \rho b \right) \end{aligned} \tag{28}$$

is the value function conditional on a given repayment rate $\rho < 1$, and ℓ' and $Q^\ell(u_c, \ell')$ denote newly issued loans and the underlying pricing function, respectively. This formulation makes clear that what ultimately matters for allocations and welfare is the *effective state* ρb. Since this state can be regulated via the repayment policy ρ subject to $\rho b \leq b$, the value function $V_f^d(b)$ is necessarily *non-decreasing* over the entire state space. Specifically, $V_f^d(b)$ is increasing whenever the optimal default policy prescribes full debt repayment, and constant whenever the optimal default policy prescribes partial default. Finally, $W^o(\ell)$ is the value function of a government that starts the period with ℓ outstanding loans,

$$W^o(\ell) = \alpha \max\{W_a^{nd}(\ell), W^d(\ell)\} + (1-\alpha)\max\{W_e^{nd}(\ell), W^d(\ell)\}, \tag{29}$$

where $W_a^{nd}(\ell)$ is the value function conditional on full repayment of a government that regains access to the bond market in the beginning of the period, $W_e^{nd}(\ell)$ is the no-default value function of a government that remains excluded from the bond market, and $W^d(\ell)$ is the value function conditional on default. These functions satisfy

$$\begin{aligned} W_a^{nd}(\ell) &= \max_{c,n,b'} u(c, 1-n, n-c) + \beta V_f^o(b') \\ &\quad + \gamma \left(u_c c + u_c Q^b(u_c, b') b' - \frac{u_l}{\omega(\ell, n)} n - u_c \ell \right), \end{aligned} \tag{30}$$

$$\begin{aligned} W_e^{nd}(\ell) &= \max_{c,n,\ell'} u(c, 1-n, n-c) + \beta W^o(\ell') \\ &\quad + \gamma \left(u_c c + u_c Q^\ell(u_c, \ell') \ell' - \frac{u_l}{\omega(\ell, n)} n - u_c \ell \right), \end{aligned} \tag{31}$$

$$W^d(\ell) = \max_{\rho \in [0,1]} \tilde{W}^d(\rho\ell), \tag{32}$$

where $\tilde{W}^d(\rho\ell)$ denotes the value function conditional on a given repayment rate ρ on loans,

$$\begin{aligned} \tilde{W}^d(\rho\ell) &= \max_{c,n,\ell'} u(c, 1-n, n-c) + \beta W^o(\ell') \\ &\quad + \gamma \left(u_c c + u_c Q^\ell(u_c, \ell') \ell' - \frac{u_l}{\omega(\rho\ell, n)} n - u_c \rho\ell \right). \end{aligned} \tag{33}$$

It is not necessary to index $W^d(\ell)$ by a or e, since default precludes the current government's option of immediate bond market access. Moreover, because default hampers the liquidity of maturing bonds, the value of defaulting is independent of whether outstanding liabilities are in the form of bonds or loans, that is, $V_f^d(x) = W^d(x)$ and $\tilde{V}^d(x) = \tilde{W}^d(x)$, where x denotes the (effective) amount of outstanding liabilities. Finally, note also that $W_e^{nd}(\ell) = \tilde{W}^d(\rho\ell)$ for $\rho = 1$. Accordingly, $W_e^{nd}(\ell) = W^d(\ell)$ whenever the optimal default policy prescribes full debt repayment.

4.2 Recursive equilibrium

A recursive equilibrium under the option to default can be defined analogous to Definition 1. This is conceptually straightforward but requires cumbersome notation, and hence we relegate the formal equilibrium definition to the Appendix.

In a recursive equilibrium under the option to default, the government fully repays its outstanding bonds if $V_f^{nd}(b) \geq V^d(b)$. Similarly, it fully repays its outstanding loans if $W_s^{nd}(\ell) \geq W^n(\ell)$, where the indicator $s \in \{a, e\}$ makes clear that this decision may depend on whether the government regains market access. When the government defaults, its optimal repayment decision is given by $\hat{\rho}_f(b) = \arg\max_\rho \tilde{V}^d(\rho b)$ and $\hat{\rho}_s(\ell) = \arg\max_\rho \tilde{W}^d(\rho\ell)$, $s \in \{a, e\}$, respectively.

Inspection of the government's policy problems in Section 4.1 allows us to derive further characteristics of the equilibrium default decision. Under the premise that the welfare functions conditional on no default are either monotonically decreasing or of an inverse U-shape,[16] the following result obtains.

Proposition 3. *In a recursive equilibrium under the option to default, the government defaults if and only if its inherited debt exceeds a threshold level that depends on the form of outstanding liabilities and the government's access to the primary bond market. Specifically, there exist default thresholds $\bar{\ell}_e^d < \bar{\ell}_a^d < \bar{b}_f^d$ such that the optimal repayment policy is characterized by*

$$\hat{\rho}_f(b) = \begin{cases} 1 & \text{if } b \leq \bar{b}_f^d \\ \underline{x}/b & \text{if } b > \bar{b}_f^d \end{cases}, \qquad \text{and} \qquad \hat{\rho}_s(\ell) = \begin{cases} 1 & \text{if } \ell \leq \bar{\ell}_s^d \\ \underline{x}/\ell & \text{if } \ell > \bar{\ell}_s^d \end{cases},$$

where $s \in \{a, e\}$ and $\underline{x}$ is the lowest level of effective debt that maximizes post-default welfare, $\underline{x} = \arg\max_x \tilde{V}^d(x)$. At $\underline{x}$ the collateral constraint is strictly binding.

The intuition behind this result is readily seen. Recall that the value functions conditional on default are non-decreasing over the entire state space, and constant whenever the optimal policy prescribes $\rho < 1$. Denote this constant level of welfare by $\bar{V}^d$. Given the premise underlying Proposition 3, the value functions conditional on full repayment are monotonically decreasing or inverse U-shaped. Hence there exist unique default thresholds $\bar{\ell}_e^d < \bar{\ell}_a^d < \bar{b}_f^d$, implicitly defined by $W_e^{nd}(\bar{\ell}_e^d) = \bar{V}^d$, $W_a^{nd}(\bar{\ell}_a^d) = \bar{V}^d$ and $V_f^{nd}(\bar{b}^d) = \bar{V}^d$, respectively. Moreover, due to the benefits of having access to an operational bond market, $V_f^{nd}(x) > W_a^{nd}(x) > W_e^{nd}(x)$ globally.[17] Hence, $\bar{\ell}_e^d < \bar{\ell}_a^d < \bar{b}_f^d$.

More generally, Proposition 3 has two important implications. First, it establishes $\bar{b}_f^d$ as the maximum sustainable level of public debt; we denote this threshold by *fiscal limit.* Sec-

[16] Given the economic structure of our model, it is natural to expect this property to hold. Our numerical results of Sections 3 and 4 confirm this; however, a formal proof is not available.

[17] The first inequality follows because the liquidity services of maturing bonds are valuable, $u_g r - \gamma u_c \pi = r[u_g - \gamma u_l] > 0$. The second inequality follows because, relative to loans, there is a liquidity premium on newly issued bonds, $Q^b(u_c, x') > Q^\ell(u_c, x')$ for all $x' \in \{b', \ell'\}$.

ond, the proposition makes clear that the optimal haircut reduces effective debt to the unique level that maximizes post-default welfare. This post-default level of effective debt is given by $\underline{x} = \bar{\ell}_e^d$ regardless of the type of maturing debt and current bond market access. Moreover, as a result of balancing the marginal benefits and costs of default, this level necessarily induces a strictly binding collateral constraint. The role of public debt in providing collateral and liquidity services is thus an important force in disciplining the discretionary government's default incentives. But a government that exercises its default option will always find it optimal to make the post-default level of debt so scarce that financial intermediation is hampered.

4.3 A calibrated economy

We now explore the quantitative implications of the recursive equilibrium under the option to default within the calibrated economy introduced in Section 3.4. The model parameters are kept unchanged, as summarized in Table 2. In addition, we need to pin down the reentry probability α. For our benchmark scenario we choose $\alpha = 0.5$, which implies that, on average, the bond market is impaired during the default period and the two following periods. The implied duration of two years is consistent with the empirical evidence reported by Bai, Julliard, and Yuan (2012), and it is also broadly in line with estimates reported in the sovereign debt literature (cf. Cruces and Trebesch, 2013).

Figure 3 shows the value functions of the government under the option to default. The top panel contrasts the government's value function conditional on no default (V^{nd}) and on default (V^d) when bond markets are fully operational. Under full repayment, the government's value function is of an inverse U-shape. Under partial default, it is monotonically increasing for low levels of debt and constant from the threshold level $\underline{x} = 0.1705$ onwards. The two value functions intersect at the fiscal limit $\bar{b}_f^d = 0.2975$, which corresponds to roughly 94% of steady state output. Importantly, $\bar{b}^d > b^*$, and hence the government fully repays its debt at the steady state.

The bottom panel of Figure 3 contrasts the government's value functions W_a^{nd}, W_e^{nd} and W^d. The two functions W_a^{nd} and W_e^{nd} are again of an inverted U-shape, in line with our previous discussion. Conditional on regaining market access, the government fully honors its debt up to the point where W_a^{nd} and W^d intersect, which corresponds to $\bar{\ell}_a^d = 0.2630$ in our calibrated

Figure 3: Value functions

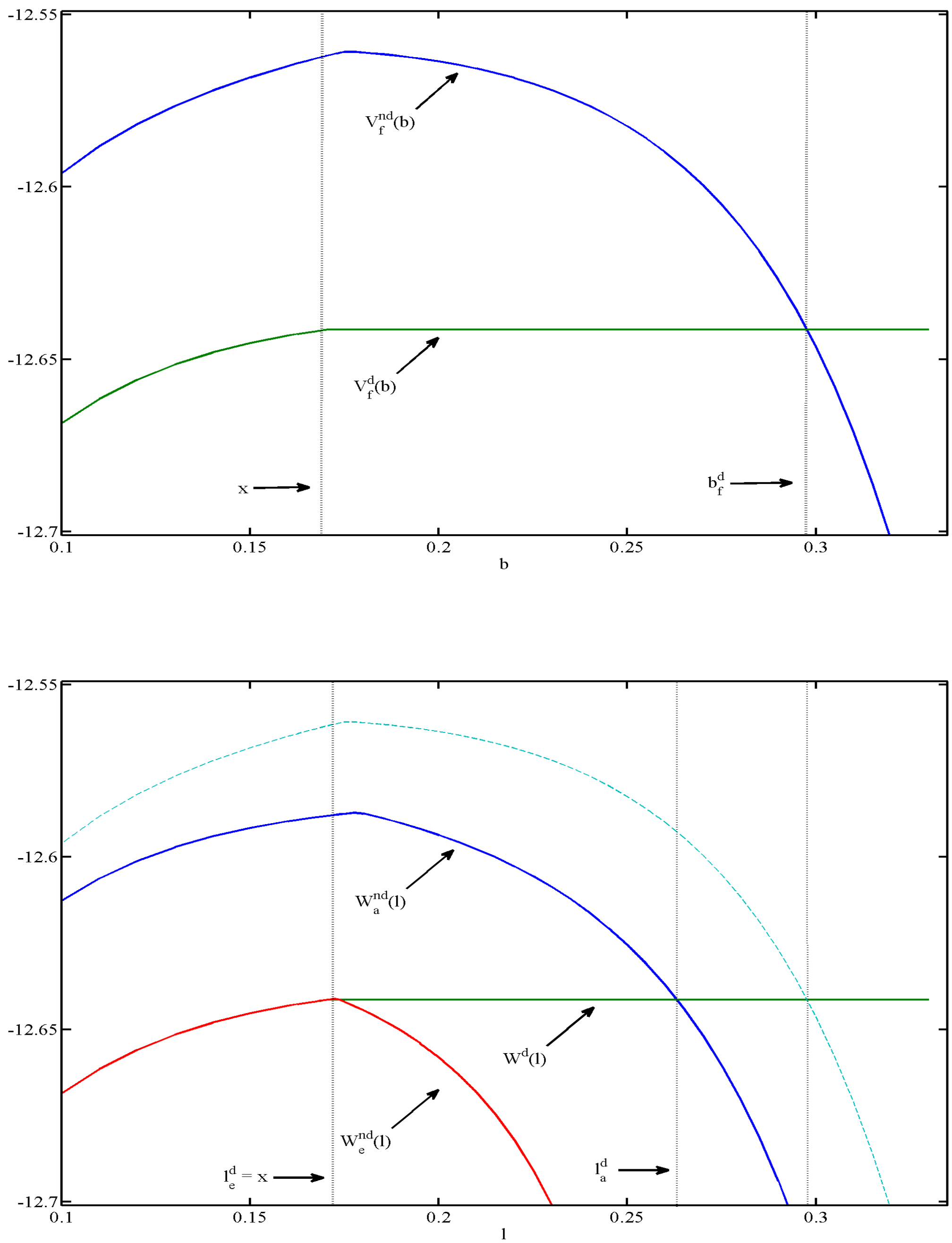
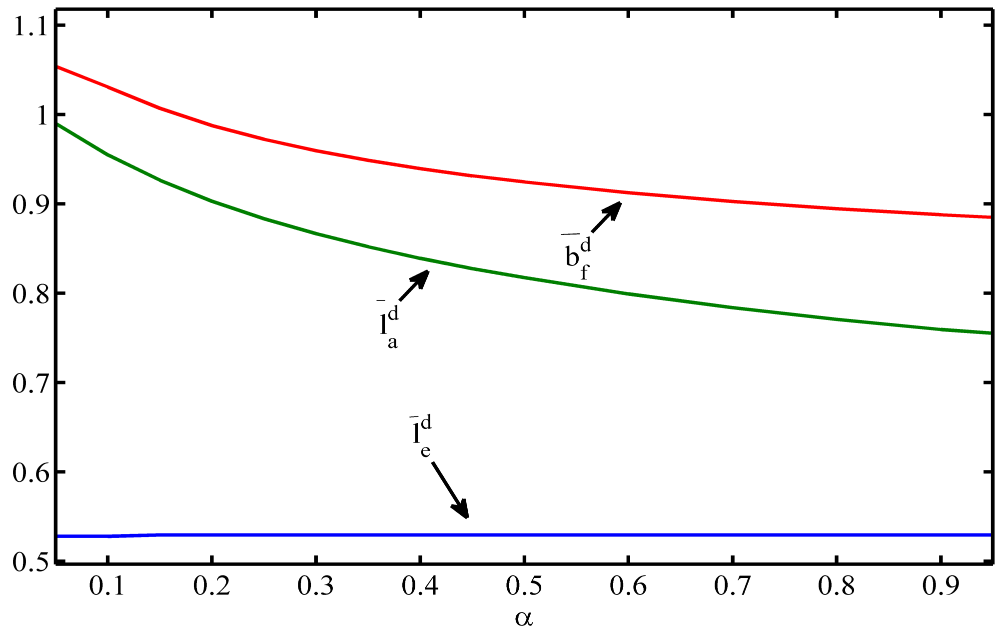


Figure 4: Default thresholds as a fraction of steady state output



economy. Finally, note that $W_e^{nd}(\ell) = W^d(\ell)$ if and only if debt is below $\underline{x}$, and hence the government under market exclusion fully honors its debt up to this threshold, $\bar{\ell}_e^d = \underline{x}$.

Figure 4 further illustrates how the three default thresholds $\{\bar{b}_f^d, \bar{\ell}_a^d, \bar{\ell}_e^d\}$ depend on the market re-access probability α. We observe that $\bar{b}_f^d$ and $\bar{\ell}_a^d$ are both monotonically decreasing in α. This reflects that a higher probability of market re-access lowers the cost of the bond market exclusion triggered by default; hence, the maximum sustainable level of debt is reduced. Quantitatively, however, an increase in α above our benchmark of $\alpha = 0.5$ has only relatively minor consequences for the fiscal limit: expressed as a fraction of steady state output, it changes from 94% for $\alpha = 0.5$ to 89% for $\alpha = 0.9$. Finally, the default threshold $\bar{\ell}_e^d$ is independent of α because the government is already exluded from the bond market and thus incurs only the contemporaneous costs due to the reduction in pledgeable collateral. At the threshold $\bar{\ell}_e^d$ these costs exactly balance the benefits of default due to reduced tax distortions. As also the benefits are independent of the re-access probability, so is the default threshold $\bar{\ell}_e^d$.

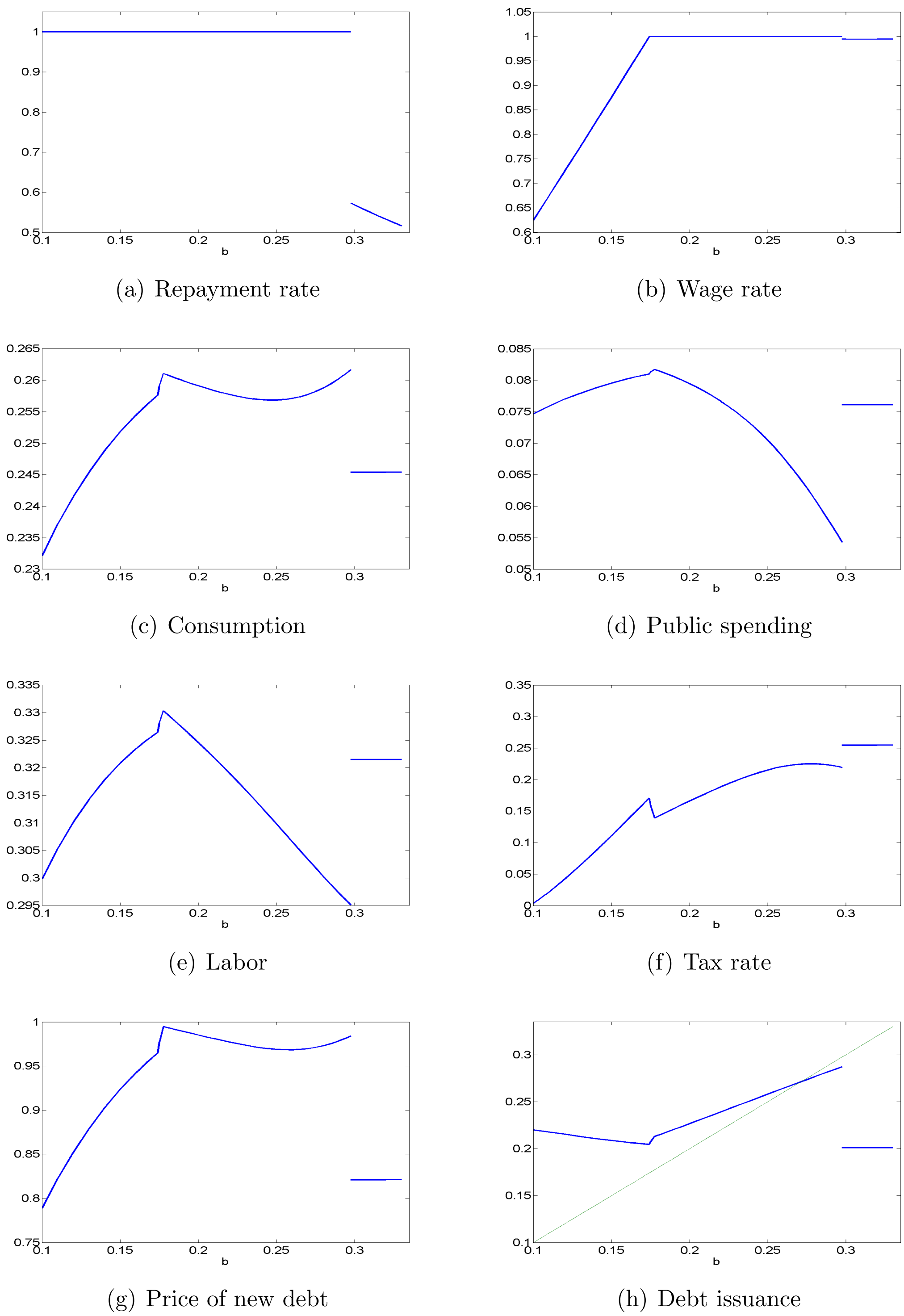
Figure 5 presents the policy functions under the option to default for the scenario of a fully operational bond market ($s = f$). For debt levels below the fiscal limit $\bar{b}_f^d$, these policy functions mirror the ones in the economy with commitment to full debt repayment (cf. Figure 1). At the fiscal limit, there is a discontinuity, and for debt levels exceeding $\bar{b}_f^d$ all policy functions are constant. The repayment policy $\hat{\rho}_f(b)$ is an exception; as predicted in Proposition 3, it is falling in b because the defaulting government always reduces effective debt to the same level, $\hat{\rho}_f(b)b = \underline{x}$, independent of the initial level of b. The optimal haircut at the fiscal limit is about 40%. Also in line with Proposition 3, when the government defaults the collateral constraint becomes strictly binding, such that the wage rate drops below labor productivity.

Government default affects the value of the assets held by private agents directly via the reduced repayment and indirectly via the loss in their liquidity. Households respond to the reduction in the value of their assets by increasing labor supply and reducing private consumption,[18] whereas the government responds to the reduction in its liabilities by increasing public consumption. The higher level of public spending is financed via increased taxes. This is optimal since the labor tax is hardly distortionary due to the binding collateral constraint, while the government has to pay high interest rates on its newly issued debt. The high interest rate emerges because, following default, the government is confined to finance itself via loans, which do not carry a liquidity premium and are also subject to significant default risk.

This risk is apparent in the policy functions when outstanding debt takes the form of loans. Figure 6 displays these policies, distinguishing between the situation when the government can re-access the bond market (the blue solid line) and when it cannot (the green dashed line). The top left panel shows the government's repayment policy. If the government can re-access the bond market, it partially defaults when debt exceeds the threshold $\bar{\ell}_a^d = 0.2630$. If it cannot, partial default occurs at all levels of debt exceeding $\bar{\ell}_e^d = 0.1705$. Given the government's debt issuance policy, which prescribes debt issuance beyond the threshold $\bar{\ell}_e^d$ for all levels of initial debt, extended periods of bond market exclusion are thus associated with *serial default*: the government always partially defaults when it remains excluded from the bond market. Loans therefore carry a significant default premium. However, once the government can re-access the bond market, the economy converges to its steady state without further defaults occurring during the transition.

[18]The wealth effect on labor supply associated with sovereign default is driven by the linear separable preferences in our calibrated economy. It may be overturned by considering GHH-preferences, as is often done in the sovereign debt literature.

Figure 5: Policy functions under the option to default – bonds



(a) Repayment rate

(b) Wage rate

(c) Consumption

(d) Public spending

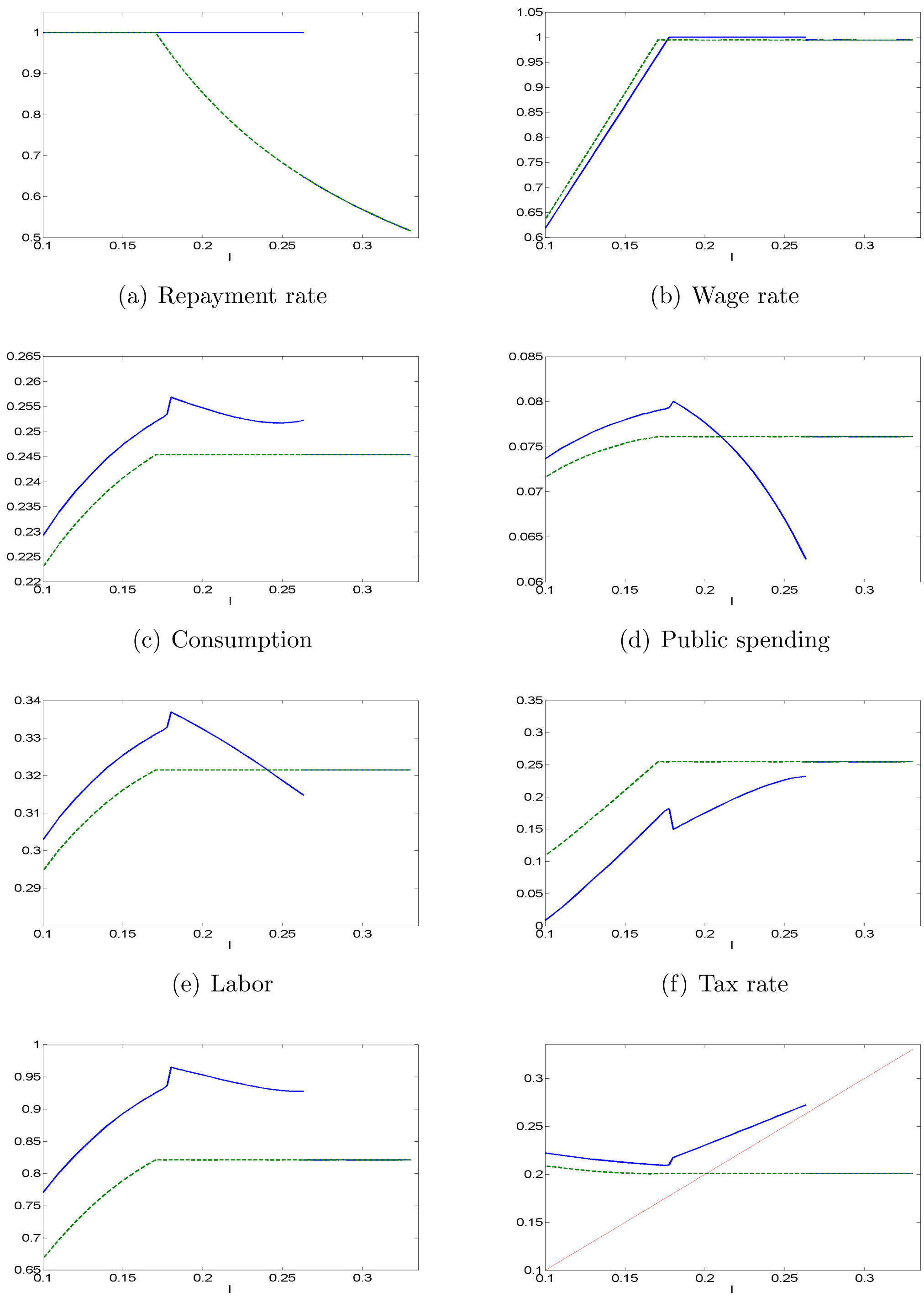
(e) Labor

(f) Tax rate

(g) Price of new debt

(h) Debt issuance

Figure 6: Policy functions under the option to default – loans



(a) Repayment rate (b) Wage rate

(c) Consumption (d) Public spending

(e) Labor (f) Tax rate

(g) Price of new debt (h) Debt issuance

Note: The blue solid line corresponds to the optimal policy functions when the government regains access to the primary bond market ($s = a$). The green dotted line corresponds to the optimal policy functions when the government remains excluded ($s = e$).

5 Conclusion

This paper has provided a quantitative framework to study the joint determinants of government debt and its sustainability in a closed economy subject to financial frictions. Fiscal policy is implemented under lack of commitment, which may extend also to the repayment of maturing government debt. Since debt is held domestically, it is valued as an instrument to smooth consumption, but also as a source of collateral and liquidity. Our particular interest is in three statistics for government debt: the steady state level, the maximum sustainable level (fiscal limit), and the optimal haircut rescaling the effective amount of liabilities in case of default. When default triggers the government's temporary exclusion from the bond market, the calibrated economy predicts empirically plausible outcomes for these three statistics. Another interesting feature is the prediction of possibly extended periods of serial default by governments without access to the bond market.

For our calibrated economy steady state debt is at approximately 84% of output, the default threshold is at 94% of output, and the haircut imposed at this threshold is about 40%. Notably, the full set of frictions invoked in our model is necessary to generate these empirically plausible statistics. Our calibration, particularly that of the collateral parameter ξ^c, implies a demand for collateral in the order of 50% of output. For higher levels of debt, the economy's collateral constraint is slack, which leaves the government facing a trade-off between the liquidity services of increased debt and the associated tax distortions. The liquidity role of government debt is therefore essential to generate a steady state with government liabilities in excess of the level satiating the economy's collateral constraint.

On the other hand, for sufficiently high levels of debt the liquidity value of government bonds tends to be dominated by the associated tax distortions, resulting in a downward-sloping value function, $V_b(b) < 0$. However, since default via *fractional repayment* of maturing debt amounts to rescaling the 'effective level' of debt, any level of debt such that $V_b(b) < 0$ is not sustainable, unless there is some additional fixed cost of defaulting. The loss in liquidity due to the government's exclusion from the bond market is therefore critically needed in order to sustain sizeable debt positions.[19]

[19] Debt-to-GDP ratios in the order of magnitude of 100% are sustainable under quite moderate average exclusion durations (cf. Figure 4).

Finally, under reasonable calibrations of the entrepreneurial return r but absent a demand for collateral ($\xi^c = 0$), the value function under default is flat already for very low levels of debt. Conditional on default occurring, this (counterfactual) scenario would induce the government to impose a close to 100% haircut. Hence, the costs for production and welfare due to scarce collateral are essential to generate haircuts in the empirically observed range.

In light of this discussion, our work differs from recent contributions that study the sustainability of sovereign debt in the absence of reputational costs in three-period economies (e.g., Brutti, 2011; Gennaioli, Martin, and Rossi, 2013). Similar to our approach, these papers invoke costs of default via repercussions for financial intermediation. However, given that the underlying model environment abstracts from intertemporal dynamics, they are silent about the determination and sustainability of steady state debt. By contrast, the present paper shows that the existence of a dynamic debt Laffer curve always induces the government to issue debt to a point where $V_b(b') < 0$; that is, marginal debt has negative welfare effects. As seen, equilibrium debt positions, including steady state debt, would thus not be sustainable without at least moderate reputational costs.

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A Appendix

A.1 First-order conditions

Recall the definition of $Q(u_c, b')$ via the bond pricing function (16),

$$Q(u_c, b') = \beta \frac{u_c(b')}{u_c} \left(1 + \hat{\pi}(b') + \hat{\phi}(b')\right),$$

and the associated partial derivatives,

$$\begin{aligned} Q_1(u_c, b') &= -\beta \frac{u_c(b')}{(u_c)^2} \left(1 + \hat{\pi}(b') + \hat{\phi}(b')\right), \\ Q_2(u_c, b') &= \beta \frac{u_c(b')}{u_c} \left\{ \frac{u_{cc}(b')}{u_c(b')} \hat{c}_b(b') \left(1 + \hat{\pi}(b') + \hat{\phi}(b')\right) + \left(\hat{\pi}_b(b') + \hat{\phi}_b(b')\right) \right\}. \end{aligned}$$

Similarly, from the definition of $\omega(b, n)$ in (17),

$$\omega(b, n) = \begin{cases} 1 & \text{if } (1-\theta)b > \xi^c n \\ \frac{(1-\theta)b}{\xi^c n} & \text{otherwise,} \end{cases}$$

with $\omega_1(b, n) = \omega_2(b, n) = 0$ when $\omega(b, n) = 1$ and otherwise

$$\begin{aligned} \omega_1(b, n) &= \frac{(1-\theta)}{\xi^c n}, \\ \omega_2(b, n) &= -\frac{(1-\theta)b}{\xi^c n^2}. \end{aligned}$$

The first-order conditions characterizing optimal government behavior under commitment to full debt repayment are given by

$$\begin{aligned} 0 &= u_c(1+\gamma) + \gamma u_{cc}(c - (1+\pi)b) + \gamma(u_{cc} Q(u_c, b')b' + u_c Q_1(u_c, b') u_{cc} b') - \gamma u_c \pi_c b - u_g \\ &= u_c(1+\gamma) + \gamma u_{cc}(c - (1+\pi)b) - \gamma u_c \pi_c b - u_g, \\ 0 &= u_l(1 + \gamma \frac{1}{\omega(b,n)}) - \gamma u_{ll} \frac{1}{\omega(b,n)} n + \gamma u_c \pi_n b - u_g - \gamma u_l \frac{1}{\omega(b,n)^2} n \omega_2(b,n), \\ 0 &= \beta V_b(b') + \gamma(u_c Q_2(u_c, b')b' + u_c Q(u_c, b')) \\ &= \beta V_b(b') + \gamma \beta u_c(b') \left(1 + \hat{\pi}(b') + \hat{\phi}(b')\right) \left\{1 + \varepsilon^q_{b'}\right\}, \end{aligned}$$

where $\varepsilon^q_{b'} = \frac{Q_2(u_c,b')b'}{Q(u_c,b')} = \frac{u_{cc}(b')}{u_c(b')}\hat{c}_b(b')b' + \frac{\left(\hat{\pi}_b(b')+\hat{\phi}_b(b')\right)b'}{\left(1+\hat{\pi}(b')+\hat{\phi}(b')\right)}$. The envelope condition for b is

$$V_b(b) \;\;=\;\; -\gamma\left\{u_c(1+\pi) - \frac{u_l n}{\omega(b,n)^2}\omega_1(b,n)\right\} + u_g r.$$

Substitution into the first-order condition with respect to b' yields the generalized Euler equation (19),

$$\gamma'\left\{u_c'(1+\pi') - \frac{u_l' n'}{\omega(b',n')^2}\omega_1(b',n')\right\} - u_g' r = \gamma u_c'(1+\pi'+\phi')\left\{1+\varepsilon^q_{b'}\right\}.$$

A.2 Recursive equilibrium under the option to default

The recursive equilibrium under the option to default is defined as follows:

Definition 2. *A recursive equilibrium under the option to default is a collection of consumption functions* $\{\hat{c}^o_s, \hat{c}^d_s, \hat{c}^{nd}_s\}_{s\in\{f,a,e\}}$*, labor supply functions* $\{\hat{n}^o_s, \hat{n}^d_s, \hat{n}^{nd}_s\}_{s\in\{f,a,e\}}$*, debt policy functions* $\{\hat{b}^o_f, \hat{b}^o_a, \hat{\ell}^o_e, \hat{\ell}^d_e, \hat{b}^{nd}_f, \hat{b}^{nd}_a, \hat{\ell}^{nd}_e\}$*, repayment policy functions* $\{\hat{\rho}_f, \hat{\rho}_a, \hat{\rho}_e\}$*, value functions* $\{V^o_f, V^{nd}_f, V^d_f, \tilde{V}^d, W^o, W^{nd}_a, W^{nd}_e, W^d, \tilde{W}^d\}$*, and pricing functions* $\{Q^b, Q^\ell\}$ *such that:*

(i) given V^o_f*,* W^o*,* Q^b*,* Q^ℓ*,* $\hat{\rho}_f$*,* $\hat{\rho}_a$ *and* $\hat{\rho}_e$*, the policy functions* $\{\hat{c}^{nd}_f, \hat{n}^{nd}_f, \hat{b}^{nd}_f\}$ *solve problem (26); the policy functions* $\{\hat{c}^d_f, \hat{n}^d_f, \hat{\ell}^d_f\}$ *solve problem (28); the policy functions* $\{\hat{c}^{nd}_a, \hat{n}^{nd}_a, \hat{b}^{nd}_a\}$ *solve problem (30); the policy functions* $\{\hat{c}^{nd}_e, \hat{n}^{nd}_e, \hat{\ell}^{nd}_e\}$ *solve problem (31); the policy functions* $\{\hat{c}^d_a, \hat{n}^d_a, \hat{\ell}^d_a\}$ *and* $\{\hat{c}^d_e, \hat{n}^d_e, \hat{\ell}^d_e\}$ *solve problem (33);*

(ii) given $\{\hat{\rho}_f, \hat{\rho}_a, \hat{\rho}_e\}$*, the consumption policy functions satisfy* $\hat{c}^o_s = \hat{c}^{nd}_s$ *when the government fully repays debt and* $\hat{c}^o_s = \hat{c}^d_s$ *otherwise; the labor and debt policies are constructed in the same way;*

(iii) given the value functions, the repayment policy functions $\{\hat{\rho}_f, \hat{\rho}_a, \hat{\rho}_e\}$ *solve (25), (27), (29) and (32);*

(iv) given V^{nd} *and* V^d*, the value function* V^o *satisfies (25); given* $\{W^{nd}_a, W^{nd}_e, W^d\}$*, the value function* W^o *satisfies (29);*

(v) given the policy functions, the pricing functions Q^b and Q^ℓ satisfy

$$
\begin{aligned}
Q^b(u_c, b') &= \beta \frac{u_c(\hat{c}_f^o(b'))}{u_c} \hat{\rho}_f(b') \left(1 + \hat{\pi}_f^o(b') + \hat{\phi}_f^o(b')\right), \\
Q^\ell(u_c, \ell') &= \alpha\beta \frac{u_c(\hat{c}_a^o(\ell'))}{u_c} \hat{\rho}_a(\ell') \left(1 + \hat{\phi}_a^o(\ell')\right) + (1-\alpha)\beta \frac{u_c(\hat{c}_e^o(\ell'))}{u_c} \hat{\rho}_e(\ell') \left(1 + \hat{\phi}_e^o(\ell')\right).
\end{aligned}
$$

(vi) given the policy functions and $\{V_f^o, W^o\}$, the value functions $\{V_f^{nd}, V_f^d, \tilde{V}^d, W_a^{nd}, W_e^{nd}, W^d, \tilde{W}^d\}$ satisfy (26), (27), (28), (30), (31), (32) and (33).

A.3 Proofs

Proof of Proposition 1. Under a collateral role of government debt, $\xi^c > 0$, zero debt is equivalent to zero production in our model economy, such that a positive steady state debt level emerges by construction. When there is only a liquidity role for government debt, $\xi^c = 0$ and $\xi^l > 0$, a positive steady state debt level emerges, too. This follows directly from the generalized Euler equation (19). When $\xi^c = 0$, the GEE, evaluated at the steady state, reads

$$
-{u_g}^* r = \gamma^* {u_c}^* (1 + \pi^*) {\varepsilon_{b'}^q}^*.
$$

Since ${u_g}^* > 0$, ${u_c}^* > 0$, $r > 0$, $\pi^* > 0$ and $\gamma^* > 0$, we have that ${\varepsilon_{b'}^q}^* < 0$. Since private agents cannot go short in government bonds, it follows that $b^* > 0$ and ${Q_2}^* < 0$. $\square$

Proof of Proposition 2. When $\xi^c > 0$, government debt is essential for production due to its collateral role. Thus, by construction, in the neighborhood of $b = 0$ welfare is increasing in debt. When $\xi^c = 0$ but $r > 0$, $\omega_1(b, n) = 0$ and the envelope condition for b is

$$
V_b(b) = -\gamma u_c (1 + \pi) + u_g r = -\gamma u_c \left(1 + \frac{u_l}{u_c} r\right) + u_g r,
$$

where the second equality follows from $\pi = \frac{u_l}{u_c} r$. The first-order condition with respect to n implies

$$
u_g - u_l = \gamma \left[u_l - u_{ll} n + u_c \pi_n b\right] = \gamma \left[u_l - u_{ll} \left(n + rb\right)\right],
$$

where the second equality follows from $\pi_n = -\frac{u_{ll}}{u_l}\pi = -\frac{u_{ll}}{u_c}r$. Solving for γ yields

$$\gamma = \frac{u_g - u_l}{u_l - u_{ll}\,(n + rb)} > 0.$$

Substituting into the envelope condition and evaluating at $b = 0$,

$$V_b(0) = -\frac{u_g - u_l}{u_l - u_{ll}n}u_c\left(1 + \frac{u_l}{u_c}r\right) + u_g r.$$

It follows that $V_b(0) > 0$ if and only if

$$\left(\frac{(u_l)^2 - u_g u_{ll} n}{u_l - u_{ll}n}\right) r \quad > \quad \frac{u_g - u_l}{u_l - u_{ll}n}u_c,$$

or equivalently,

$$r > \frac{u_g u_c - u_l u_c}{(u_l)^2 - u_g u_{ll} n} = \frac{\frac{u_c}{u_l} - \frac{u_c}{u_g}}{\frac{u_l}{u_g} - \frac{u_{ll}}{u_l}n} = \frac{u_c}{u_l}\frac{1 - \frac{u_l}{u_g}}{\frac{u_l}{u_g} - \frac{u_{ll}}{u_l}n}.$$

Finally, absent financial frictions, that is, when $\xi^c = 0$ and $r = 0$, the envelope condition for b is unambiguously negative, $V_b(b) = -\gamma u_c < 0$. □

Proof of Proposition 3. Problem (27) shows that, by choosing ρ, the government can effectively regulate the state ρb in the value function $\tilde{V}^d(\rho b)$, subject to the constraint $\rho b \leq b$. Accordingly, as ρ is chosen optimally, the value function $V_f^d(b)$ is *non-decreasing* over the entire state space. To see this formally, note that the first-order condition for ρ associated with problem (27) implies

$$\gamma\left[\frac{u_l n}{\omega(\rho b, n)^2}\omega_1(\rho b, n)b - u_c b\right] \geq 0, \tag{34}$$

with equality in case of an interior solution. But then the envelope condition associated with problem (28) implies

$$\tilde{V}_b^d(\rho b) = \gamma\left[\frac{u_l n}{\omega(\rho b, n)^2}\omega_1(\rho b, n)\rho - u_c\rho\right] = \gamma\frac{\rho}{b}\left[\frac{u_l n}{\omega(\rho b, n)^2}\omega_1(\rho b, n)b - u_c b\right] \geq 0, \tag{35}$$

where the weak inequality follows from (34), that is, under the optimal repayment policy associated with problem (27). It thus follows that $V_f^d(b)$ is non-decreasing.

Returning to (34), since $u_c b > 0$, it follows that an interior solution can only arise when $\omega_1(\rho b, n) > 0$. The same argument also implies that $\omega_1(\rho b, n) > 0$ is a necessary condition for a corner solution at $\rho = 1$.[20] Hence, the optimal repayment policy conditional on default, $\tilde{\rho}^d(b)$, ensures that the collateral constraint is strictly binding. Given $\omega_1(\rho b, n) > 0$, the envelope condition (35) implies

$$\tilde{V}_b^d(b) = \gamma \left[\frac{u_l n}{\omega(\rho b, n)^2} \omega_1(\rho b, n) \rho - u_c \rho \right] = \gamma \left[\frac{u_l n}{\omega(\rho b, n)} \frac{1}{b} - u_c \rho \right].$$

This expression is monotonically decreasing in b and ρ. Given some ρ, there is thus a unique b such that $\tilde{V}_b^d(b) = 0$. Let $\underline{b}^e$ denote the level of debt such that $\tilde{V}_b^d(\underline{b}^e) = 0$ when $\rho = 1$. When $\rho = 1$ and $b < \underline{b}^e$, $\tilde{V}^d(b)$ is increasing; a corner solution at full repayment, $\tilde{\rho}^d(b) = 1$, is thus indeed an optimizing choice, and $V_f^d(b)$ is increasing. Conversely, when $\rho = 1$ and $b > \underline{b}^e$, $\tilde{V}^d(b)$ is decreasing, which contradicts (35); the optimal repayment policy conditional on default is thus adjusted to an interior solution $\tilde{\rho}^d(b) < 1$, and $V_f^d(b)$ is flat. Finally, when $b = \underline{b}^e$, full repayment, $\tilde{\rho}^d(b) = 1$, is optimal.

Taking stock, when $b < \underline{b}^e$, $V_f^d(b)$ is strictly increasing. Moreover, at $\underline{b}^e$ the collateral constraint is strictly binding. When $b < \underline{b}^e$, the government always finds it optimal to fully repay its maturing bonds, $\tilde{\rho}^d(b) = 1$. However, due to the market exclusion costs of default, it follows that $V_f^{nd}(b) > V_f^d(b) = \tilde{V}^d(b)$ for all $b < \underline{b}^e$. By contrast, for any level of debt $b > \underline{b}^e$ such that the government finds it optimal to default, $V_f^d(b)$ is constant, that is, the value conditional on default is independent from initial debt. Denote this value by $\bar{V}^d$. Moreover, under the premise that the no-default value function $V_f^{nd}(b)$ is monotonically decreasing for large levels of debt and hence of an inverse U-shape, there exists a unique level of debt, $\bar{b}_f^d > \underline{b}^e$, such that $V_f^{nd}(\bar{b}_f^d) = \bar{V}^d$. By the same argument, $V_f^{nd}(b) \geq \bar{V}^d$ for $b \leq \bar{b}_f^d$, and $V^{nd}(b) < \bar{V}^d$ for $b > \bar{b}_f^d$. Accordingly, the government fully repays its outstanding bonds up to the threshold level $\bar{b}_f^d$ and partially defaults if inherited debt exceeds this threshold. This is the optimal (unconditional) repayment policy associated with problem (25); denote it by $\hat{\rho}_f(b)$.

In order to explicitly characterize the optimal (unconditional) repayment policy $\hat{\rho}_f(b)$, recall

[20] A corner solution at $\rho = 0$ can never occur because debt is essential for production.

first that $V_f^{nd}(b) \geq V_f^d(b)$ when $b \leq \bar{b}_f^d$; hence, $\hat{\rho}_f(b) = 1$ for all $b \leq \bar{b}_f^d$. Conversely, when $b > \bar{b}_f^d$, $V_f^{nd}(b) < V_f^d(b)$ and, since $\bar{b}_f^d > \underline{b}^e$, $\hat{\rho}_f(b) = \tilde{\rho}^d(b) < 1$. But this implies that, for $b \geq \underline{b}^e$, condition (34) holds at equality and $\omega_1(\rho b, n) > 0$. Since $\frac{u_l}{u_c} = (1-\tau)\omega(\rho b, n)$, condition (34) then implies

$$\rho b = \frac{\frac{u_l}{u_c} n}{\omega(\rho b, n)} = (1-\tau)n. \tag{36}$$

But for interior solutions $\hat{\rho}_f(b) = \tilde{\rho}^d(b) < 1$, $V_f^d(b) = \bar{V}^d$ is constant; that is, b does not matter for allocations and welfare, and $(1-\tau(b))n(b)$ is constant. It thus follows that the right-hand side in (36) is constant, implying that $\hat{\rho}_f(b)b$ must be constant and equal to $\underline{b}^e$ for all b that induce an interior solution for ρ. Since $\rho \leq 1$, we thus have $\hat{\rho}_f(b) = \underline{b}^e/b$ for $b > \underline{b}^e$.

Similar arguments are readily available for the case when the government's liabilities are in the form of loans.[21] In the region where $W^d(\ell)$ is increasing in ℓ, the optimal repayment policy conditional on default is given by $\tilde{\rho}^d(\ell) = 1$. $W^d(\ell)$ is constant for all loan levels exceeding a threshold $\underline{\ell}^e$. Comparison of problems (27)/(28) and (32)/(33) shows that $W^d(x) = V_f^d(x)$ for $x \in \{b, \ell\}$; hence, $\underline{\ell}^e = \underline{b}^e$. Moreover, when W_a^{nd} and W_e^{nd} are of an inverse U-shape, there exist unique thresholds $\bar{\ell}_a^d$ and $\bar{\ell}_e^d$ such that the government in state $s \in \{a, e\}$ fully repays its outstanding loans if and only if ℓ is below the threshold $\bar{\ell}_s^d$; otherwise, it partially defaults.

Finally, note that the no-default value functions satisfy $V_f^{nd}(x) > W_a^{nd}(x) > W_e^{nd}(x)$ globally. The first inequality follows because $u_g r - \gamma u_c \pi = r[u_g - \gamma u_l] > 0$; the second inequality follows because $Q^b(u_c, x') > Q^\ell(u_c, x')$ for all $x' \in \{b', \ell'\}$. Hence, the government's default thresholds satisfy $\bar{b}_f^d > \bar{\ell}_a^d > \bar{\ell}_e^d$. Accordingly, the economy's maximum sustainable level of debt is given by $\bar{b}_f^d$. Moreover, the default threshold under market exclusion is equal to $\bar{\ell}_e^d = \underline{\ell}^e$, which follows from the property that $W_e^{nd}(\ell) < \bar{V}^d$ if and only if $\ell > \underline{\ell}^e$. The optimal (unconditional) repayment policies when the government's liabilities are in the form of loans, $\hat{\rho}_s(\ell)$ for $s \in \{a, e\}$, can be constructed analogously to their counterpart $\hat{\rho}_f(b)$ when the maturing liabilities are bonds. When the government defaults, the optimal haircut reduces its effective liabilities (ρb or $\rho \ell$) to $\underline{b}^e = \underline{\ell}^e$, that is, to the lowest level that is consistent with the maximum default value $\bar{V}^d$. □

[21] Recall also that the relevant value functions in problems (32) and (33) are independent of the market access indicator $s \in \{a, e\}$.